On the equational theory of representable polyadic equality algebras

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incomplete, undebugged, draft version, please comment (!)

Abstract

Among others we will prove that the equational theory of ω dimensional representable polyadic equality algebras is not schema axiomatizable. This result is in interesting contrast with the Daigneault-Monk representation theorem which states that the class of representable polyadic algebras is finite schema-axiomatizable (and hence the equational theory of this class is finite schema-axiomatizable, too). We will also show that the complexity of the equational theory of $RPEA_\omega$ is also extremely high in the recursion theoretic sense. Finally, comparing our negative results with the positive ones of Ildikó Sain and Viktor Gyuris [8], we will draw the following methodological conclusions. The negative properties of polyadic (equality) algebras can be removed by switching from the so called "polyadic algebraic paradigm" to the "cylindric algebraic" one.

Contents

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1 Introduction

It is well known that the class of (ω dimensional) representable polyadic equality algebras ($RPEA_\omega$ for short) is not closed under taking ultraproducts, and hence this class is not axiomatizable. This result leaves the question of axiomatizability of the equational theory of $RPEA_\omega$ open. The equational theory of $RPEA_\omega$ ($\mathbf{Eq}(RPEA_\omega)$ for short) is not finitely axiomatizable, simply because the language of $RPEA_\omega$ contains continuum many operation symbols. So the next interesting question is axiomatizability by some kind of schemas. Indeed, "schema axiomatization" has been the traditional approach in polyadic algebra theory, namely, Halmos's axiom schemas appeared in 1960's (see [3]). Our goal in §§2-3 is to prove that the equational theory of $RPEA_\omega$ is not axiomatizable by schemas similar to the usual defining schemas of (abstract) polyadic equality algebras. This result is in interesting contrast with the theorem of Daigneault and Monk [5] which states, that the class of representable polyadic algebras (without equality) is finite schema axiomatizable.

Before turning to the goal of §§2-3, let us look briefly to the goal of §4. In §4 we look at the equational theory of $RPEA_\omega$ from the point of view of recursion theoretic degrees of unsolvability. We have two aims with this:

(i) To shed some more light on the possibility or impossibility of finding some notion of a schema with which $\mathbf{Eq}(RPEA_\omega)$ and related theories would be axiomatizable.

(ii) To study the complexity properties of theories like $\mathbf{Eq}(RPEA_\omega)$ and its finite reducts on their own right.

We will show that $\mathbf{Eq}(RPEA_\omega)$ is highly non-computable, and even its restrictions to finite sublanguages are very far from being recursively enumerable. More concretely, in Theorem 4.1 we will claim that

$$\mathbf{Eq}(RPEA_\omega) \text{ is } \Pi^1_1 \text{ hard (in the recursion theoretic sense).}$$

This means the following. Let $\mathcal{N}$ denote the standard model of arithmetic. There is a computable function (e.g. by a Turing machine)

$$Tr : (\Pi^1_1 \text{ formulas }^1 \text{ of arithmetic}) \to (\text{equations of } RPEA_\omega)$$

such that for any Π^1_1 sentence φ:

[1] Π^1_1 formulas are second order formulas in which the second order variables are bounded universally (in prenex form). For precise definitions we refer to [7]. We note that Π^1_1 hardness automatically implies non-recursive-enumerability, but Π^1_1 hardness is a much stronger negative property than this.

$$\mathcal{N} \models \varphi \Leftrightarrow RPEA_\omega \models Tr(\varphi).$$

We will see that the above is meaningful, because the range of Tr contains only finitely many operation symbols (so the cardinality of $\mathbf{Eq}(RPEA_\omega)$ causes no problem from the point of view of computability of Tr).

We will prove Theorem 4.1 in §4. On the intuitive consequences of Theorem 4.1.:

If $\mathbf{Eq}(RPEA_\omega)$ were finitely axiomatizable by some kind of schemas, then the equational consequences of these schemas would form a Π^1_1 hard set (by Theorem 4.1). Usually, finite (schema) axiomatizability of some theory implies recursively enumerability of this theory. So we conclude, that $\mathbf{Eq}(RPEA_\omega)$ can not be finitely (or recursively) axiomatizable by any kind of reasonable schemas. Further investigations and discussions in this line are in §4.

It is interesting to contrast non-computability of $\mathbf{Eq}(RPEA_\omega)$ (i.e. Theorem 4.1) with the facts that $\mathbf{Eq}(RCA_\omega)$ and $\mathbf{Eq}(RQPEA_\omega)$ are recursively enumerable (cf. e.g. Németi, [6]). Here RCA_ω and $RQPEA_\omega$ denote the class of representable cylindric algebras and the class of representable quasi-polyadic algebras, respectively. For more on this contrast cf. the end of §5 herein. Let us now return to formulating the results of sections 2-3.

Definition 1.1 *Throughout, α is an ordinal. L_{PEA_α} denotes the language of α dimensional polyadic equality algebras (for details see [4], p.225).*

Let us turn to the difference (or distance) between PEA_ω and $\mathbf{HSP}RPEA_\omega$. Here, PEA_ω is an abstract class defined by equation schemas ($P_0 - P_{11},\ E_1 - E_3$) originating with Halmos cf. [3], [4]. We will see that $PEA_\omega \neq \mathbf{HSP}RPEA_\omega$ because an equation separates them. We will construct such an equation. One could hope that adding similar, new schemas to $P_0 - E_3$ might alleviate the situation. We will see below that this cannot be done.

Motivating Example 1.
Consider the axiom schema of polyadic algebras $s_\tau s_\sigma(x) = s_{\tau\circ\sigma}(x)$, as an example. For simplicity let us write this as $s_\tau s_\sigma = s_{\tau\circ\sigma}$. Let us notice τ, σ are "meta-variables" ranging over ${}^\omega\omega$. We can write the same schema into the form

$$(*) \quad \tau \circ \sigma = f \Rightarrow s_\tau s_\sigma = s_f.$$

When studying schemas like $(*)$, sometimes it will be convenient to treat τ, σ, f in $(*)$ as function symbols of some first order language L_T. At other times we will treat τ, σ, f as second order variables ranging over functions.

Definition 1.2 *L_T is the following first order language: L_T consists of countably many unary relational symbols, and countably many unary function symbols and countably many constants (and nothing more), which are $r_1, r_2, \ldots$ and $f_1, f_2, \ldots$ and $n_1, n_2, \ldots$ respectively.*

Definition 1.3 *Let L_{PT} be the following similarity type (in the algebraic sense). In L_{PT} $C_{r_1}, C_{r_2}, \ldots, S_{f_1}, S_{f_2}, \ldots$ are unary operation symbols, $D_{n_1,n_1}, D_{n_1,n_2}, D_{n_2,n_1} \ldots$ are constants. Here the indices of the symbols C, S and D are the same as the corresponding relation, function and constant symbols in the previous definition. In addition L_{PT} contains a boolean (type) reduct, too.*

Definition 1.4 *By a Halmos schema we mean a pair $\langle s, e \rangle$ where s is a sentence in the language L_T, and e is an equation in L_{PT}.*

We will denote $\langle s, e \rangle$ as $s \Rightarrow e$ in order to keep the notation closer to intuition.

Motivating Example 2.
The form $(*)$ will help us in making certain things explicit about these kinds of schemas. Let now $f_1, f_2, f_3 \in {}^{\omega}\omega$. Then the equation $s_{f_1} s_{f_2}(x) = s_{f_3}(x)$ will be called an instance of $(*)$ iff indeed $f_1 \circ f_2 = f_3$. Thus, the single schema $(*)$ has continuum many instances (if the meta-variables τ, σ, f are ranging over ${}^{\omega}\omega$).

Definition 1.5 *Let $s \Rightarrow e$ be a Halmos schema, let g be an equation in L_{PEA_α}. We say that g is an (α dimensional) instance of $s \Rightarrow e$ iff there exist sets, functions and constants*

$$r_1^M, r_2^M, \ldots, \subseteq \alpha, \qquad f_1^M, f_2^M, \ldots \in {}^{\alpha}\alpha \qquad \textit{and} \qquad n_1^M, n_2^M, \ldots \in \alpha \qquad \textit{such that}$$

$$M = \langle \alpha; r_1^M, r_2^M, \ldots, f_1^M, f_2^M, \ldots n_1^M, n_2^M, \ldots \rangle \models s;$$

and g is obtained from e by replacing r_i, f_i and n_i by r_i^M, f_i^M and n_i^M respectively. In this case we say that g is an instance of e with respect to M.

Definition 1.6 *Let A be an L_{PEA_α} type algebra. A Halmos schema h is valid in A iff every α dimensional instance of h is valid in A.*

We note that one can easily translate the usual defining schemas of polyadic (equality) algebras into Halmos schemas. Consider, for example $s_{\sigma \circ \tau}(x) = s_\sigma s_\tau(x)$, where $\sigma, \tau \in {}^{\alpha}\alpha$. This schema is axiom P_7 in [4], on page 225, which is a typical example of the usual defining schemas of polyadic algebras. Let h be the following Halmos schema:

$$(\forall x)(f_1(f_2(x)) = f_3(x)) \quad \Rightarrow \quad s_{f_3}(y) = s_{f_1} s_{f_2}(y).$$

Now one can easily see that the instances of P_7 and the α dimensional instances of h are the same. In the Appendix we give Halmos schemas, which correspond to the usual definition of PEA_α. That is, we give a complete list of the translations of the usual defining schemas of polyadic (equality) algebras. Now we can state our first main result.

Theorem 1.7
(i) The set $\mathbf{Eq}(RPEA_\omega)$ *of equations valid in* $RPEA_\omega$ *is not axiomatizable by any set of Halmos schemas. More formally:*

(ii) Let Σ *be a set of Halmos schemas valid in* $RPEA_\omega$*. Then for the class* $Mod(\Sigma)$ *of all models of all* ω *dimensional instances of* Σ *we have*

$$Mod(\Sigma) \neq \mathbf{HSP}(RPEA_\omega), \text{ equivalently } Cons(\Sigma) \neq \mathbf{Eq}(RPEA_\omega)$$

where $Cons(\Sigma)$ *denotes the set of equational consequences of* Σ*.*

Before proving this theorem we need some further definitions and "preparations".

Definition 1.8 PEA^+_α *denotes the class of all models of all* α *dimensional instances of those Halmos schemas which are valid in* $RPEA_\alpha$*.*

Note that PEA^+_α is the smallest variety which contains $RPEA_\alpha$ and defined by Halmos schemas. Our goal is to prove that the variety generated by $RPEA_\omega$ is not axiomatizable by Halmos schemas, as it was stated in Theorem 1.7. For this, it is enough to prove the following equivalent form of Theorem 1.7.

Theorem 1.7* $PEA^+_\omega \neq \mathbf{HSP}(RPEA_\omega)$.

To show this, we will prove that there is an equation e_{CM} which is valid in $RPEA_\omega$ but not in PEA^+_ω. This equation is based on the method of W. Craig and R. McKenzie (cf. [2]), hence the notation.

On the structure of the paper: as we mentioned, the above formulated results will be proved and elaborated sections 2-3. In section 2 we will construct an equation e_{CM} and will show $RPEA_\omega \models e_{CM}$. In section 3 we will show $PEA^+_\omega \not\models e_{CM}$. In section 4 the recursion theoretic (degrees of unsolvability) results are presented, while in section 5 we will compare our negative results with the positive results occurring in [8].

2 The equational theory of $RPEA_\omega$.

In this section we will construct and discuss the equation e_{CM} mentioned above. First we introduce some notation. $t^i_j(x)$ abbreviates the term $d_{i,j} \wedge c_j(x)$. As usual,

$[i,j] \in {}^\omega\omega$ denotes the transposition which interchanges i and j and leaves every other element fixed. Here i and j are distinct elements of ω. The symbol *suc* denotes the successor function on ω. We will introduce some equations (see Definitions 2.1 and 2.6 below) and will prove some properties of these equations. After this, we will construct the equation e_{CM}, and in Theorem 2.9 we will show $RPEA_\omega \models e_{CM}$.

Definition 2.1 *Throughout* e_1, e_2, e_3 *denote the following equations:*
e_1 *is* $x - d_{3,4} \leq d_{2,3}$.
e_2 *is* $(x \wedge d_{3,4} - c_0c_1t_3^4t_2^3s_{suc}(x)) \ \wedge \ (c_0c_2s_{[1,2]}(x) - c_0c_2t_3^4t_1^3s_{suc}(x)) \leq d_{1,2}$.
e_3 *is* $x \wedge d_{3,4} \leq c_0c_1c_2(x \wedge d_{3,4} - c_0c_1t_3^4t_2^3s_{suc}(x))$.

If s is a finite sequence, and h is an other (possibly infinite) sequence, then $s{}^\frown h$ denotes the concatenation of s and h.

Definition 2.2 *Let $A \in RPEA_\omega$, with base U, let $a \in A$, and let $h \in {}^\omega U$. Then $(a)_h$ denotes the "projection" of a with respect to h to the second coordinate, that is $(a)_h = \{u \in U : \ (\exists v_0, v_1 \in U) \ \ \langle v_0, v_1, u\rangle{}^\frown h \in a\}$.*

Lemma 2.2.1 *Suppose $A \in RPEA_\omega$, with a nonempty base U, and $a \in A$ such that*
(0) $C_1^A(a) = a$. *Then the following hold.*
(1) *If* $A \models e_1[a]$ *then* $(\forall h \in {}^\omega U, h_0 \neq h_1)(\mid (a)_h \mid \ \leq \ 1)$.
(2) *If* $A \models e_2[a]$ *then* $(\forall h \in {}^\omega U)(\mid (a)_{h_0{}^\frown h} - (a)_h \mid \ \leq \ 1)$.
(3) *If* $A \models e_3[a]$ *then* $(\forall h \in {}^\omega U)(\mid (a)_{h_0{}^\frown h} \mid \neq 0 \ \Rightarrow \mid (a)_{h_0{}^\frown h} - (a)_h \mid \ \geq \ 1)$.

Proof. To show (1) suppose $A \models e_1[a]$ and $h \in {}^\omega U$, such that $h_0 \neq h_1$. Suppose $g \in (a)_h$. That is, there exist $v_0, v_1 \in U$ such that $\langle v_0, v_1, g\rangle{}^\frown h \in a$. Of course, $\langle v_0, v_1, g\rangle{}^\frown h \in -D_{3,4}^A$ and therefore by e_1, $\langle v_0, v_1, g\rangle{}^\frown h \in D_{2,3}^A$ so $g = h_0$. This means that $(a)_h \subseteq \{h_0\}$, since g were chosen arbitraryly. So $\mid (a)_h \mid \leq 1$ therefore (1) is proved.

To show (2) suppose indirectly $A \models e_2[a]$, $h \in {}^\omega U$, $v_0, v_1 \in (a)_{h_0{}^\frown h} - (a)_h$ and $v_0 \neq v_1$. This means the following:
(a) $(\exists u_0, u_1 \in U)(\langle u_0, u_1, v_0\rangle{}^\frown h_0{}^\frown h \in a)$
(b) $(\exists w_0, w_1 \in U)(\langle w_0, w_1, v_1\rangle{}^\frown h_0{}^\frown h \in a)$
(c) $(\nexists t_0, t_1 \in U)(\langle t_0, t_1, v_0\rangle{}^\frown h \in a)$
(d) $(\nexists z_0, z_1 \in U)(\langle z_0, z_1, v_1\rangle{}^\frown h \in a)$

Observe $\langle u_0, v_1, v_0\rangle{}^\frown h_0{}^\frown h \notin D_{1,2}^A$. So, to derive a contradiction we have to show that this sequence is an element of the left hand side of $e_2[a]$. To do this we need the following claim.

Claim 2.3.1 $C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a) = \{\langle x_0, x_1, u_2, u_3, u_3, u_4, u_5 ...\rangle \in {}^\omega U :$ $(\exists u_0, u_1 \in U)(\langle u_0, u_1, u_2, u_3, u_4, u_5, ...\rangle \in a)\}$.

Similarly, $C_0^A C_2^A t_3^{4A} t_1^{3A} S_{suc}^A(a) = \{\langle x_0, u_2, x_2, u_3, u_3, u_4, u_5 ...\rangle \in {}^\omega U :$ $(\exists u_0, u_2 \in U)(\langle u_0, u_1, u_2, u_3, u_4, u_5, ...\rangle \in a)\}$.

Proof of the claim.
We only show the first part of the statement, because the proof of the second part is a similar direct computation.

$$\begin{array}{ll}
\langle x_0, x_1, u_2, u_3, u_3, u_4, u_5 ...\rangle \in C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a) & \text{iff} \\
(\exists u_{-1}, u_0 \in U)\langle u_{-1}, u_0, u_2, u_3, u_3, u_4, u_5 ...\rangle \in t_3^{4A} t_2^{3A} S_{suc}^A(a) & \text{iff} \\
(\exists u_{-1}, u_0 \in U)\langle u_{-1}, u_0, u_2, u_3, u_3, u_4, u_5 ...\rangle \in C_3^A t_2^{3A} S_{suc}^A(a) & \text{iff} \\
(\exists u_{-1}, u_0 \in U)\langle u_{-1}, u_0, u_2, u_2, u_3, u_4, u_5 ...\rangle \in C_2^A S_{suc}^A(a) & \text{iff} \\
(\exists u_{-1}, u_0, u_1 \in U)\langle u_{-1}, u_0, u_1, u_2, u_3, u_4, u_5 ...\rangle \in S_{suc}^A(a) & \text{iff} \\
(\exists u_0, u_1 \in U)\langle u_0, u_1, u_2, u_3, u_4, u_5 ...\rangle \in a. &
\end{array}$$

End of proof of claim 2.3.1.

Now we are ready to show that $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h$ is an element of the left hand side of $e_2[a]$.

(i)
$\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h \in D_{3,4} \wedge C_1^A(a)$ from (a). Hence from (0) $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h \in D_{3,4} \wedge a$.

(ii)
Now we want to prove $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h \notin C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a)$. Suppose indirectly $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h \in C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a)$. Then, by claim 2.3.1, $(\exists t_0, t_1 \in U)(\langle t_0, t_1, v_0\rangle ^\frown h \in a$ which is impossible because of (c).
Similarly, $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h \notin C_0^A C_2^A t_3^{4A} t_1^{3A} S_{suc}^A(a)$.

(iii)
From (b) $\langle w_0, w_1, v_1\rangle ^\frown h_0 ^\frown h \in a$ so $\langle w_0, v_1, w_1\rangle ^\frown h_0 ^\frown h \in S_{[1,2]}^A(a)$ so $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h \in C_0^A C_2^A S_{[1,2]}^A(a)$.

Computations (i)-(iii) show that $\langle u_0, v_1, v_0\rangle ^\frown h_0 ^\frown h$ is an element of the left hand side of $e_2[a]$.

Lastly we have to show (3).
Suppose $A \models e_3[a]$, $h \in {}^\omega U$ and $v_0 \in (a)_{h_0 ^\frown h}$. So there exist $u_0, u_1 \in U$ such that $\langle u_0, u_1, v_0\rangle ^\frown h_0 ^\frown h \in a \wedge D_{3,4}^A$. By e_3 we have

(2.3.2) $\langle u_0, u_1, v_0\rangle^\frown h_0^\frown h \in C_0^A C_1^A C_2^A(a \wedge D_{3,4}^A - C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a))$.

Therefore there exist $u_2, u_3, u_4 \in U$ such that

(2.3.3) $\langle u_2, u_3, u_4\rangle^\frown h_0^\frown h \in (a \wedge D_{3,4}^A - C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a))$.

Of course $u_4 \in (a)_{h_0^\frown h}$. On the other hand, suppose $u_4 \in (a)_h$, so $(\exists t_0, t_1 \in U)(\langle t_0, t_1, u_4\rangle^\frown h \in a)$. But then, by claim 2.3.1, $\langle u_2, u_3, u_4\rangle^\frown h_0^\frown h \in C_0^A C_1^A t_3^{4A} t_2^{3A} S_{suc}^A(a)$. By (2.3.3), this is impossible, so $u_4 \in (a)_{h_0^\frown h} - (a)_h$ therefore (3) is proved. ∎

Theorem 2.3 *Suppose $A \in RPEA_\omega$, with base U, and $a \in A$ such that $C_1^A(a) = a$, and e_1, e_2, e_3 hold for a. Then* $(\forall h \in {}^\omega U)(\mid (a)_h \mid < \aleph_0)$.

Proof. First note that if U is empty, then the statement is true. Otherwise (0),(1),(2),(3) of Lemma 2.3 hold. Choose an arbitrary $h \in {}^\omega U$. If h is a constant sequence, then $h_0^\frown h = h$ so $(a)_{h_0^\frown h} = (a)_h$ hence by (3) of Lemma 2.3 $\mid (a)_{h_0^\frown h} \mid = \mid (a)_h \mid = 0 < \aleph_0$.

Now suppose that h is not a constant sequence, so there exists $n \in \omega$ such that $h_0 = h_1 = ... = h_n \neq h_{n+1}$ We will use induction to show that in this case

$(*) \quad \mid (a)_h \mid \leq n + 1 < \aleph_0$.

If $n = 0$ then by (1) of the previous theorem we have $\mid (a)_h \mid \leq 1$. Assume second, that $(*)$ holds for those sequences which are not constant on the first n natural number, and let $h' = h \circ suc = \langle h_1, h_2, h_3, ...\rangle$
Now $\mid (a)_h \mid \leq \mid (a)_h - (a)_{h'} \mid + \mid (a)_{h'} \mid \leq 1 + n$, by (2) of Lemma 2.3, and the induction hypothesis $(*)$. ∎

Definition 2.4 *Suppose $A \in RPEA_\omega$, with base U, $a \in A$ and $h \in {}^\omega U$. Then $f_{a,h}$ denotes the following binary relation:*

$$f_{a,h} = \{\langle x, y\rangle \in {}^2U : (\exists z)(\langle y, z, x\rangle^\frown h \in a)\}.$$

Of course, $dom(f_{a,h}) = (a)_h$.

Definition 2.5 *Throughout e_4, e_5, e_6, e_7 denote the following equations:*
e_4 *is* $s_{[0,1]}(x) \wedge x \leq d_{0,1}$,
e_5 *is* $s_{[1,2]}(x) \wedge x \leq d_{1,2}$,
e_6 *is* $s_{[0,2]}(x) \leq c_0(x)$,
e_7 *is* $s_{[0,2]}(x) \leq c_2(x)$.

Lemma 2.5.1 *Suppose $A \in RPEA_\omega$, with base U, $a \in A$ and $h \in {}^\omega U$. If $C_1^A(a) = a$ then the following statements hold:*
(4) If $A \models e_4[a]$ then $f_{a,h}$ is a function.
(5) If $A \models e_5[a]$ then $f_{a,h}$ is an injective relation.
(6) If $A \models e_6[a]$ then $range(f_{a,h}) \subseteq dom(f_{a,h})$.
(7) If $A \models e_7[a]$ then $dom(f_{a,h}) \subseteq range(f_{a,h})$.

Proof. First we want to prove (4). Suppose $v_0, v_1, z \in U$ such that $\langle z, v_0\rangle \in f_{a,h}$, and $\langle z, v_1\rangle \in f_{a,h}$. Hence, by the definition of $f_{a,h}$, there exists $t \in U$ such that $\langle v_0, t, z\rangle^\frown h \in a$. So $\langle v_0, v_1, z\rangle^\frown h \in C_1^A(a) = a$. Similarly, $\langle v_1, v_0, z\rangle^\frown h \in a$. But then $\langle v_0, v_1, z\rangle^\frown h \in S_{[0,1]}^A(a)$. Using e_4 we conclude $\langle v_0, v_1, z\rangle^\frown h \in D_{0,1}^A$. This means $v_0 = v_1$ so $f_{a,h}$ is a function.

The proof of (5) is absolutely similar.

To prove (6), assume $z \in range(f_{a,h})$. That is, there exists $v_0 \in U$ such that $\langle v_0, z\rangle \in f_{a,h}$, so there exists $v_1 \in U$ such that $\langle z, v_1, v_0\rangle^\frown h \in a$, hence $\langle v_0, v_1, z\rangle^\frown h \in S_{[0,2]}^A(a)$. Using e_6, we conclude there exists $v_2 \in U$ such that $\langle v_2, v_1, z\rangle^\frown h \in a$, so $\langle z, v_2\rangle \in f_{a,h}$ which means $z \in dom(f_{a,h})$, as desired.

The proof of (7) is absolutely similar. ∎

The variety PEA_ω^+ is a discriminator variety because $C_{(\omega)}$ is a complemented normal closure operator which majorates all other extra-boolean operation. These properties are expressible by Halmos schemas. It is well known from the theory of discriminator varieties that the quasi-equations over a discriminator variety are translatable into equivalent equations. That is, for any discriminator variety V and for any quasi-equation q over V, there is a translation $tr(q)$ which is an equation and $V \models q$ iff $tr(q) \in \mathbf{Eq}(V)$. So below we will write quasi equations and will refer to them as equations.

Definition 2.6 *Throughout, e_{CM} is defined as follows:*

$$e_{CM} \textit{ is the (quasi-)equation } e_0 \& e_1 \& ... \& e_6 \Rightarrow e_7.$$

Theorem 2.7 $RPEA_\omega \models e_{CM}$.

Proof. Let $A \in RPEA_\omega$, with base U, let $a \in A$ such that $A \models (e_0 \& e_1 \& ... \& e_6)[a]$. We want to show $A \models e_7[a]$ that is, $\langle v_0, v_1, v_2\rangle^\frown h \in C_2^A(a)$ whenever $h \in {}^\omega U, v_0, v_1, v_2 \in U$ such that $\langle v_0, v_1, v_2\rangle^\frown h \in S_{[0,2]}^A(a)$.

By the previous theorems $(a)_h$ is finite, and $f_{a,h}$ is an injective function such that $range(f_{a,h}) \subseteq dom(f_{a,h})$. It is easy to see that $v_0 \in (a)_h = dom(f_{a,h})$. Using other terminology, we have a permutation $f_{a,h}$ on a finite set $(a)_h$, and v_0 is in the domain of this permutation. Of course, in this situation $v_0 \in range(f_{a,h})$ that is, there exists $z_0 \in U$ such that $\langle z_0, v_0\rangle \in f_{a,h}$. By the definition of $f_{a,h}$ there exists $z_1 \in U$ such that $\langle v_0, z_1, z_0\rangle^\frown h \in a$. But then $\langle v_0, v_1, z_0\rangle^\frown h \in C_1^A(a) = a$, so $\langle v_0, v_1, v_2\rangle^\frown h \in C_2^A(a)$. This means that $S^A_{[0,2]}(a) \leq C_2^A(a)$ so the conclusion of e_{CM} hold, as desired. ∎

3 The equational theory of PEA_ω^+.

In this section we prove that e_{CM} is not valid in PEA_ω^+.

Theorem 3.1 *$PEA_\omega^+ \not\models e_{CM}$.*

Proof. Assume the opposite. Then there exist a proof of e_{CM} from the (ω dimensional) instances of certain Halmos schemas. Let p such a proof. Suppose that exactly the following sets, functions and numbers occur in p as indices of certain operations: $r_1^M, ..., r_k^M, f_1^M, ..., f_n^M, i_1^M, ...i_m^M$.

Let $M = \langle \omega; \leq^\omega, r_1^M, ..., r_k^M, f_1^M, ..., f_n^M, i_1^M, ...i_m^M\rangle$. Let $\mathcal{F}$ be a nonprincipal ultrafilter over ω. Let M' be the ultrapower of M with respect to $\mathcal{F}$, so $M' = ({}^\omega M)/\mathcal{F}$. Let M'' be an elementary part of M' which has a countable universe, and $\langle 0, 1, 2, 3, 4...\rangle/\mathcal{F}$ is an element of the universe of M''. Let h be a bijection, which maps the universe of M'' onto ω such that

$$\begin{gathered} h(\langle 0,0,0...\rangle/\mathcal{F}) = 0,\\ h(\langle 1,1,1...\rangle/\mathcal{F}) = 1,\\ h(\langle 2,2,2...\rangle/\mathcal{F}) = 2,\\ h(\langle 3,3,3...\rangle/\mathcal{F}) = 3,\\ h(\langle 4,4,4...\rangle/\mathcal{F}) = 4. \end{gathered}$$

Let $\leq^h = h[\leq^{M''}]$, $\quad r_i^h = h[r_i^{M''}]$, $\quad f_j^h = h \circ f_j^{M''} \circ h^{-1}$ $\quad$ and let $i_l^h = h(i_l^{M''})$ ($1 \leq i \leq k,\ \ 1 \leq j \leq n,\ \ 1 \leq l \leq m$).
Finally let $H = \langle \omega; \leq^h, r_1^h, ..., r_k^h, f_1^h, ..., f_n^h, i_1^h, ...i_m^h\rangle$.

Proposition 3.1.1 *M and H are elementarily equivalent.*

Proof. M and M' and M'' are elementarily equivalent, furthermore, by definition H and M'' are isomorphic - h is an isomorphism. ∎

Let p' be the sequence of equations obtained from p by replacing r_i^M, f_j^M, i_l^M with r_i^h, f_j^h, i_l^h respectively. ($1 \leq i \leq k,\ \ 1 \leq j \leq n,\ \ 1 \leq l \leq m$)

Proposition 3.1.2 *p' is a proof (using Birkhoff's calculus for equational logic).*

Proof. We will use induction to show that the initial segments of p' form a proof. We will ignore some obvious case. Suppose the $k-th$ equation in p is g_k, and the $k-th$ equation in p' is g'_k.

Case (0).
By the definition of the proof, we may assume that g_1 (the first element of p) is an ω dimensional) instance of some Halmos schema $s \Rightarrow e$. That is, s is valid in M, and g_1 is an instance of e. But then by the previous claim, s is valid in H, and g'_1 is an instance of e with respect to H (by the definition of p'). This means, that g'_1 is also an instance of $s \Rightarrow e$.

Now suppose that the first $k-1$ elements of p' form a proof. We have 2 cases.

Case (1).
g_k is an (ω dimensional) instance of some Halmos schema $s \Rightarrow e$.
Then, similarly to Case (0), g'_k is also an instance of $s \Rightarrow e$. Therefore, by the induction hypothesis, the first k element of p' form a proof.

Case (2).
g_k is a consequence of the preceding equations of p. One can easily see, that in this situation g'_k is a consequence of the preceding equations of p'. This completes the proof. ∎

Throughout, e'_{CM} denotes the last equation of p'. This is the substituted version of e_{CM}.

Proposition 3.1.3 $RPEA_\omega \not\models e'_{CM}$.

Proof. We will construct a concrete $A \in RPEA_\omega$ and $a \in A$ such that $A \not\models e'_{CM}[a]$.

Let the base of A be ω, and let $a \in A$ be the following:

$a = \{s \in {}^\omega\omega : (\exists k, n \in \omega)(k$ is even or $k \leq 2n+1)$ and

$$s_i = \begin{cases} \text{2k} & \text{if } i = 0, \\ \text{k} & \text{if } i = 2, \\ 0 & \text{if } h^{-1}(3) \ \leq \ h^{-1}(i) \ \leq \ suc^n(\langle 0,1,2,3...\rangle/\mathcal{F}) \\ 1 & \text{if } h^{-1}(i) \ >^{M''} \ suc^n(\langle 0,1,2,3...\rangle/\mathcal{F})\} \end{cases}$$

}, where "$\leq$","$>$" and "suc" are understood in M''. The same convention applies in further definitions of s_i or s'_i in this proof.

Note that $e'_1 = e_1, e'_4 = e_4, e'_5 = e_5, e'_6 = e_6, e'_7 = e_7$ (as strings of symbols) because of the construction of h. Therefore one can easily see that $C_1^A(a) = a$ and e_1, e_4, e_5, e_6 hold for a in A. On the other hand $A \not\models e_7[a]$ because $(\exists h \in {}^{\omega}U)(1 \in dom(f_{a,h}), 1 \notin range(f_{a,h}))$, since the range of $f_{a,h}$ consists of even numbers. So the remaining is to prove that $A \models e'_2 \& e'_3[a]$.

To see $A \models e'_2[a]$ choose an arbitrary element s of the left hand side of $e'_2[a]$. Clearly, $s \in a$. That is, there exist $k, l, n \in \omega$ such that s has the following form:

$$s_i = \begin{cases} \text{2k} & \text{if } i = 0, \\ \text{l} & \text{if } i = 1, \\ \text{k} & \text{if } i = 2, \\ 0 & \text{if } h^{-1}(3) \ \le\ h^{-1}(i) \ \le\ suc^n(\langle 0,1,2,3...\rangle/\mathcal{F}) \\ 1 & \text{if } h^{-1}(i) \ >\ suc^n(\langle 0,1,2,3...\rangle/\mathcal{F}) \end{cases}$$

k is not even and $k \not< 2n+1$ because otherwise direct calculations show $s \in C_0^A C_1^A t_3^{4A} t_2^{3A} S^A_{suc^H}(a)$. (These calculations are similar to the proof of claim 2.3.1) So $k = 2n+1$. Analogously, $s \in C_0^A C_2^A S^A_{[1,2]}(a) - C_0^A C_2^A t_3^{4A} t_1^{3A} S^A_{suc^H}(a)$ implies $l = 2n+1$. This means $l = k$, other words $s \in D^A_{1,2}$, which is a right hand side of e'_2.

To show $A \models e'_3[a]$ assume $s \in (a \wedge D^A_{3,4})$, so it is enough to prove $s \in C_0^A C_1^A C_2^A (a \wedge D^A_{3,4} - C_0^A C_1^A t_3^{4A} t_2^{3A} S^A_{suc^H}(a))$.
Suppose that $s_{h((suc^{M''})^n(\langle 0,1,2,3,...\rangle/\mathcal{F})} = 0$ and $s_{h((suc^{M''})^{n+1}(\langle 0,1,2,3,...\rangle/\mathcal{F})} = 1$. Such an $n \in \omega$ exists because $s \in a$. Let $s' \in {}^{\omega}\omega$ be the following sequence.

$$s'_i = \begin{cases} \text{4n+2} & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \text{2n+1} & \text{if } i = 2, \\ 0 & \text{if } h^{-1}(3) \ \le\ h^{-1}(i) \ \le\ suc^n(\langle 0,1,2,3...\rangle/\mathcal{F}) \\ 1 & \text{if } h^{-1}(i) \ >\ suc^n(\langle 0,1,2,3...\rangle/\mathcal{F}) \end{cases}$$

Observe s and s' differs at most in the $0th$, $1th$ and the $2nd$ coordinate. Using this, direct calculations would lead to conclude $s' \in a \wedge D^A_{3,4} - C_0^A C_1^A t_3^{4A} t_2^{3A} S^A_{suc^H}(a)$. Thus $s \in C_0^A C_1^A C_2^A (a \wedge D^A_{3,4} - C_0^A C_1^A t_3^{4A} t_2^{3A} S^A_{suc^H}(a))$. This completes the proof.

∎

Now the proof of Theorem 3.1 is as follows. Suppose $PEA_\omega^+ \models e_{CM}$. Then e'_{CM} is provable in PEA_ω^+ (by proposition 3.1.2), $RPEA_\omega$ is a subclass of PEA_ω^+, and e'_{CM} is not valid in $RPEA_\omega$ (by proposition 3.1.3). This contradiction completes the proof.

Finally, Theorems 1.7 and 1.7* hold, because by Theorem 3.1 $PEA_\omega^+ \not\models e_{CM}$ but by Theorem 2.9 $RPEA_\omega \models e_{CM}$.

∎

4 Complexity of the equational theories.

As we mentioned in the introduction, in [5] Daigneault and Monk proved a representation theorem for (ω dimensional) polyadic algebras (PA_ω for short). However, this result does not imply that the equational theory of PA_ω is recursively enumerable, one simple reason is that the language of PA_ω consists of continuum many operation symbols. Actually, in [10] we proved the following. There is a strictly finite reduct L_G of the language of PA_ω, such that the following set of equations is not recursively enumerable: $\{e \in L_G : (PA_\omega\ axioms) \vdash e\}$. Moreover, the indices of the substitutions and cylindrifications occurring in L_G are recursive functions and sets, respectively.

In this section we prove that the set $\mathbf{Eq}(RPEA_\omega)$ i.e. the equational theory of $RPEA_\omega$ is complex in the recursion theoretic sense too. That is, $\mathbf{Eq}(RPEA_\omega)$ is strictly more complex than the full first order theory of standard arithmetic.

To prove this, we will recursively reduce the second order universal theory of standard arithmetic to $\mathbf{Eq}(RPEA_\omega)$. Recall that $\mathcal{N}$ is the standard model of arithmetic.

Theorem 4.1 *$\mathbf{Eq}(RPEA_\omega)$ is Π^1_1 hard, which means that there is a strictly finite reduct L of the language of $RPEA_\omega$ and a recursive function Tr such that Tr maps the Π^1_1 formulas of arithmetic to equations written in L, and for any Π^1_1 sentence σ*

$$\mathcal{N} \models \sigma \Leftrightarrow RPEA_\omega \models Tr(\sigma).$$

(For the second order logical and recursion theoretic details see the next remark and [7].)

Remark. By a Π^1_1 formula of arithmetic we mean a second order formula in which every second order variable is bounded universally (in prenex form). We say, that a set A of natural numbers is Π^1_1, iff A is definable by a Π^1_1 formula in Arithmetic. The set S of all (Gödel numbers of) true Π^1_1 sentences of arithmetic is not Π^1_1, by Tarski's theorem of undefinability of truth. That is, S is more complex than any Π^1_1 set. So Theorem 4.1 really means, that $\mathbf{Eq}(RPEA_\omega)$ is more complex than any Π^1_1 set, so it is Π^1_1 hard.

Proof. The proof based on a systematic translation of the Π^1_1 formulas of arithmetic into polyadic equations. First we introduce some notation. Throughout t, y, z

denote the following terms.

$t(a)$ is $s_{pred}s_{pred}s_{pred}c_0c_1c_2(a)$.
$y(a)$ is $s_{pred}s_{pred}(-c_0c_1(a)) \wedge s_{pred}s_{pred}s_{pred}(c_0c_1(s_{[3/4]}(a)))$.
$z(a)$ is $c_{\omega-\{0,1\}}(s_{[0,1]}s_{suc}(y(a)) \wedge s_{[0/1]}s_{[1/2]}(y(a)))$.

Proposition 4.1.1 *If $A \in RPEA_\omega$ with base U and $a \in A$ then the preceding terms describe the following sets:*

$$t^A(a) = \{h \in {}^\omega U : (\exists v_0v_1v_2)(\langle v_0v_1v_2\rangle^\frown h \in a)\}.$$
$$y^A(a) = \{\langle u\rangle^\frown h \in {}^\omega U : h \in t^A(a),\ u \notin (a)_h,\ u \in (a)_{h_0h}\}.$$
$$z^A(a) = \{\langle u, v\rangle^\frown h \in {}^\omega U : (\exists h' \in {}^\omega U)(u^\frown h' \in y^A(a), v^\frown h'_0h' \in y^A(a)\}.$$

Proof. The proof is a simple computation. ∎

Throughout $e_8 - e_{12}$ denote the following equations:

$e_8 = c_\omega(z) = 1$.
$e_9 = c_{\omega-\{0,1\}}(z) = z$.
$e_{10} = s_{[0/3]}(c_1(z) - c_1s_{[0,1]}(z)) \wedge (c_1(z) - c_1s_{[0,1]}(z)) \leq d_{0,3}$.
$e_{11} = y - d_{1,2} \leq c_1(z) - c_1s_{[0,1]}(z)$.
$e_{12} = y \wedge d_{1,2} \leq s_{suc}c_0(y)$.

Proposition 4.1.2 *If $A \in RPEA_\omega$ with a nonempty base U and $a, b \in A$ then the preceding equations mean the following:*

$A \models e_8[a]$ iff a is not empty.
$A \models e_{10}[a]$ iff the following set contains at most one element:
$\{u \in U : (\exists h \in {}^\omega U)(u^\frown h \in a)\} - \{u \in U : (\exists v \in U, h \in {}^\omega U)(v^\frown u^\frown h \in a)\}$.
$A \models e_{11}[a, b]$ iff
$\langle v, h\rangle \in a - D^A_{1,2} \Rightarrow v \in \{u \in U : (\exists h \in {}^\omega U)(u^\frown h \in b)\} - \{u \in U : (\exists v \in U, h \in {}^\omega U)(v^\frown u^\frown h \in b)\}$.
$A \models e_{12}[a]$ iff $\langle v, h\rangle \in a \wedge D^A_{1,2} \Rightarrow \langle v, h_1, h_2, ...\rangle \in a$.

Proof. Direct computation. ∎

The requirement that a given element b of A is an injective function on its first two coordinates such that $range(b) \subseteq dom(b)$, is also expressible by an equation. Let e_{13} be such an equation, so

$e_{13} = c_{\omega-2}(b) = 1$ & $s_{[1,2]}(b) \wedge s_{[1,3]}(b) \leq d_{2,3}$ &$s_{[0,2]}(b) \wedge s_{[0,3]}(b) \leq d_{2,3}$ &

$s_{[0,1]}(b) \leq c_1(b)$.

If for some $a \in A$, $A \models e_9[z(a)]$, then we may assume that $z(a)$ is a binary relation.

Proposition 4.1.3 *If $A \in RPEA_\omega$ with a nonempty base U and $a \in A$ such that $A \models e_1\&e_2\&e_3[a]$ and $A \models e_8\&...\&e_{13}[a, t^A(a), y^A(a), z^A(a)]$ then $\langle \omega, suc\rangle \cong \langle dom(z^A(a)), z^A(a)\rangle$.*

Proof. We will define an isomorphism $\hat{\ } : \omega \to dom(z^A(a))$.

By proposition 4.1.2 and e_8, e_{10} there is an uniqe $u \in U$ such that $u \in dom(z^A(a)) - range(z^A(a))$. Let $0\hat{\ } = u$. Now suppose $n\hat{\ }$ is defined for some $n \in \omega$. Let $(n{+}1)\hat{\ } = v$ such that $(\exists h \in {}^\omega U)(\langle n\hat{\ }, v\rangle^\frown h \in z^A(a))$. Such a v is exists and uniqe because of $e_{13}[z^A(a)]$.

The function $\hat{\ }$ is injective because of $e_{13}[z^A(a)]$. We have to show that surjectivity is also true. Choose an element $u \in dom(z^A(a))$. By $e_{13}[z^A(a)]$ there is an uniqe $v \in U$ such that $(\exists h \in {}^\omega U)(\langle u, v\rangle^\frown h \in z^A(a))$. By proposition 4.1.1 and the definition of $z(a)$ this means $(\exists h' \in {}^\omega U)(u^\frown h' \in y^A(a))$, $v^\frown h'_0 h' \in y^A(a)$. By the definition of $y(a)$, $h' \in t^A(a)$. This implies $(a)_{h'} \neq 0$. By $e_1\&...\&e_3[a]$, h' is not a constant sequence. (This is proved in the first paragraph of the proof of theorem 2.4.) So $(\exists n \in \omega)(h'_0 = ... = h'_n \neq h'_{n+1})$.

If $n = 0$ then $u^\frown h' \in y^A(a) - D^A_{1,2}$ so by $e_{11}[y^A(a), z^A(a)]$ and proposition 4.1.2, $u \in dom(z^A(a)) - range(z^A(a))$ which is uniqe, so $u = 0\hat{\ }$.

If $n > 0$ then $u^\frown h' \in y^A(a) \wedge D^A_{1,2}$ thus by $e_{12}[y^A(a), z^A(a)]$ and proposition 4.1.2, $u^\frown\langle h'_1 h'_2 ...\rangle \in C^A_0 y^A(a)$. That is, $(\exists u_1 \in U)(u_1^\frown\langle h'_1 h'_2 ...\rangle \in y^A(a)$. Observe $\langle u_1, u\rangle^\frown\langle h'_1 h'_2 ...\rangle \in z^A(a)$ (by the definition of $z(a)$). Repeating the previous argument n times, we have $u_1, u_2, ..., u_n \in U$ such that $(\forall\, 2 \leq k \leq n)(\langle u_k, u_{k-1}\rangle^\frown\langle h'_k h'_{k+1} ...\rangle \in z^A(a)$ and $u_k^\frown\langle h'_k h'_{k+1} ...\rangle \in y^A(a)$. Observe $u_n^\frown\langle h'_n h'_{n+1} ...\rangle \in y^A(a) - D^A_{1,2}$. Using the previous case, this implies $u = n\hat{\ }$.

It is easy to see that $\hat{\ }$ preserves the successor function (according to $z^A(a)$). ∎

Now we systematically translate the Π^1_1 formulas of arithmetic into polyadic equations. We will do this in several steps. If φ is a Π^1_1 formula of arithmetic, then l_φ denotes the number of different individuum variables occurring in φ. First we translate the terms, then the first order formulas then the Π^1_1 formulas. The idea is the following. We will define an $RPEA_\omega$ term $Un^{l_\varphi}(z)$ which simulates all interpretations over ω. That is, the first l_φ coordinates of $Un^{l_\varphi}(z)$ can be identified by ω.

After this we can postulate the meaning of a term, first order formula, Π^1_1 formula of arithmetic, as usual.

Translation of the terms.

Definition 4.2 *Let $l \in \omega$. $Un^l(z)$ denotes the following term: $Un^l(z) = s_{suc}c_1(z) \wedge s^2_{suc}c_1(z) \wedge ... \wedge s^l_{suc}c_1(z)$.*

To translate addition and multiplication we need a definition for them in the polyadic environment. In $\langle \omega; 0, suc, +, \cdot \rangle$ addition and multiplication is definable form suc as follows.

$(\forall n \in \omega)(n + 0 = n)$.
$(\forall n, m \in \omega)(n + suc(m) = suc(n + m))$.

$(\forall n \in \omega)(n \cdot 0 = 0)$.
$(\forall n, m \in \omega)(n \cdot suc(m) = (n \cdot m) + n)$.

Using induction it is easily seen that the previous formulas define addition and multiplication. We will express these formulas in the language of $RPEA_\omega$.

Definition 4.3 *Throughout $e_{14} - e_{18}$ denote the following equations: (here Plus, Mul and z are variables in the equational language of $RPEA_\omega$).*

$$e_{14} = c_{\omega-3}(Plus) = Plus, \quad c_{\omega-3}(Mul) = Mul, \quad Plus \leq c_1(z) \wedge s_{suc}c_1(z) \wedge s^2_{suc}c_1(z).$$
$$e'_{14} = Mul \leq c_1(z) \wedge s_{suc}c_1(z) \wedge s^2_{suc}c_1(z), \quad c_1c_2(Plus) = c_1(z),$$
$$s_{[0,1]}c_1c_2(Plus) = c_1(z), \quad c_1c_2(Mul) = c_1(z), \quad s_{[0,1]}c_1c_2(Mul) = c_1(z).$$
$$e_{15} = s_{[0,1]}c_1(z) - c_0(z) \wedge Plus \leq d_{0,2}.$$
$$e_{16} = s_{[0,3]}(z) \wedge Plus \wedge s_{[1,3]}s_{[2,4]}(Plus) \leq s_{[0,4]}s_{[1,2]}(z).$$
$$e_{17} = s_{[0,1]}c_1(z) - c_0(z) \wedge Mul \leq s_{[0,2]}c_1(z) - s_{[1,2]}c_0(z).$$
$$e_{18} = s_{[0,3]}(z) \wedge Mul \wedge s_{[1,3]}s_{[2,4]}(Mul) \leq s_{[1,4]}(Plus).$$

Proposition 4.3.1 *If $A \in RPEA_\omega$ with base U and $a, Plus, Mul \in A$ such that $A \models e_1 \& e_2 \& e_3[a]$ and $A \models e_8 \& ... \& e_{18}[a, t^A(a), y^A(a), z^A(a), Plus, Mul]$ then The following hold:*

$Plus = \{\langle u, v, w\rangle ^\frown h \in {}^\omega U :$
$u, v, w \in dom(z^A(a))$ and $(\exists n_1, n_2 \in \omega)(n_1\hat{} = u, n_2\hat{} = v, (n_1 + n_2)\hat{} = w)\}$.

$Mul = \{\langle u, v, w\rangle ^\frown h \in {}^\omega U :$
$u, v, w \in dom(z^A(a))$ and $(\exists n_1, n_2 \in \omega)(n_1\hat{} = u, n_2\hat{} = v, (n_1 \cdot n_2)\hat{} = w)\}$.

Proof. By e_{14} the first 2 coordinates of $Plus \in A$ consists of all (the images of the) natural numbers, (and hence Plus is not empty) so there is an $h \in Plus$. Suppose

$h = \langle u_0\hat{}, u_1\hat{}, u_2\hat{}, u_3, u_4 ...\rangle$ so the first 3 elements of h are the images of the natural numbers u_0, u_1, u_2, respectively. If $u_1\hat{} \in dom(z(a)) - range(z(a))$ then $u_1 = 0$ and h is an element of the left hand side of e_{15}, thus $h \in D^A_{0,2}$ hence $u_0\hat{} = u_2\hat{}$. This means $(\forall n \in \omega)(n + 0 = n)$ is satisfied.

Now suppose h is in the left hand side of e_{16}. This means $(\exists v_3, v_4 \in \omega)(v_3\hat{} = u_3, v_4\hat{} = u_4$ and $u_1 = v_3 + 1,\ u_0 + u_1 = u_2,\ u_0 + v_3 = v_4)$. Using the truth of e_{16}, $v_4{+}1 = u_2$ also hold in this case, which means $(\forall n, m \in \omega)(n{+}suc(m) = suc(n{+}m))$ is satisfied.

So $Plus$ satisfies the definition of addition, thus the first statement is true.

The proof of the statement about Mul is similar. ∎

Now we can define the translation function Tr on the terms of arithmetic. Before reading carefully Definition 4.4, probably useful to read proposition 4.4.1.

Definition 4.4 *Let t be a term of arithmetic, which contains at most l distinct variables. Then $Tr(t)$, the translation of t is an $RPEA_\omega$ term, computable by the following recursive rules.*

$$\begin{array}{lcl}
Tr(0) & = & Un^l \wedge c_1(z) - c_1 s_{[0,1]}(z). \\
Tr(v_i) & = & t^i_0(Un^l). \\
Tr(suc(t)) & = & s_{pred} c_0(z \wedge s_{[0,1]} s_{suc}(Tr(t)). \\
Tr(t_1 + t_2) & = & s_{pred} s_{pred} c_0 c_1(Plus \wedge s_{[0,2]} s_{suc} s_{suc}(Tr(t_1)) \wedge s_{[1,2]} s_{suc} s_{suc}(Tr(t_2))). \\
Tr(t_1 \cdot t_2) & = & s_{pred} s_{pred} c_0 c_1(Mul \wedge s_{[0,2]} s_{suc} s_{suc}(Tr(t_1)) \wedge s_{[1,2]} s_{suc} s_{suc}(Tr(t_2))).
\end{array}$$

Here $z, Plus, Mul$ are variables in the equational language of $RPEA_\omega$.

Proposition 4.4.1 *If $a, Un, Plus, Mul \in A$ such that*

$$A \models e_1 \& ... \& e_{18}[a, t(a), y(a), z(a), Un, Plus, Mul]$$

then for every term t of arithmetic which contains at most l distinct variable, hold the following:

$Tr(t) = \{\langle u_0, ...u_l\rangle^\frown h \in {}^\omega U : (\exists n_0, ...n_l \in \omega)(n_0\hat{} = u_0, ..., n_l\hat{} = u_l,$ *and* $n_0 = (t^\omega[n_1, ...n_l])\hat{})\}$.

Proof. Obvious. ∎

Translation of the first order formulas.

Definition 4.5 *Let φ be a first order formula of arithmetic. Then $Tr(\varphi)$, the translation of φ is an $RPEA_\omega$ term, computable by the following recursive rules.*

$$
\begin{array}{lcl}
Tr(t_1 = t_2) & = & c_0(Tr(t_1) \wedge Tr(t_2)). \\
Tr(\varphi_1 \& \varphi_2) & = & Tr(\varphi_1) \wedge Tr(\varphi_2). \\
Tr(\neg\varphi) & = & c_0(Un^{l_\varphi}) - Tr(\varphi). \\
Tr(\exists v_i \varphi) & = & c_{i+1}(Tr(\varphi) \wedge c_0(Un^{l_\varphi})).
\end{array}
$$

Proposition 4.5.1
(i) If $a, Un, Plus, Mul \in A$ such that

$$A \models e_1 \& ... \& e_{18}[a, t(a), y(a), z(a), Un, Plus, Mul]$$

then for every first order formula φ of arithmetic, the following hold:

$Tr(\varphi) = \{\langle u_0, ...u_{l_\varphi}\rangle^\frown h \in {}^\omega U : (\exists n_1, ...n_{l_\varphi} \in \omega)(n_1\hat{} = u_1, ..., n_{l_\varphi}\hat{} = u_{l_\varphi},$ *and* $\langle \omega;\ 0, suc, +, \cdot\rangle \models \varphi[n_1, ...n_{l_\varphi}]\}$.

(ii) $\langle \omega;\ 0, suc, +, \cdot\rangle \models \varphi$ *iff* $RPEA_\omega \models (e_1 \& ... \& e_{18}) \Rightarrow Un^{l_\varphi} = Tr(\varphi)$.

Proof. Obvious. ∎

Translation of the Π^1_1 formulas.

We can use such a second order language, which contains relation variables, all of them having rank 1. This is enough, because in the the arithmetic we can encode the pairs of natural numbers by one number, using a first order formula. Similarly, we may assume that our second order language does not contain function variables, because functions can be identified by their graphs, which is a binary relation, so encodable by an unary one. However, when convenient, we will write function variables, but one can see this as only short forms of some second order formulas.

For every relational variable X_i we associate a new variable x_i, which is in the equational language of $RPEA_\omega$. Of course, all the relational variables have arity 1, see the remarks in the beginning of this section.

Definition 4.6 *Let φ be a first order formula of arithmetic, which contains a relation variable X_i. Then $Tr(\varphi)$, the translation of φ is an $RPEA_\omega$ term, computable by the following recursive rules.*

$Tr(X_i(t)) = x_i \wedge Tr(t)$. *The other rules are similar to the preceding case.*

Definition 4.7 *Throughout, e_{19}, e_{20} denote the following equations:*

$$\begin{aligned} e_{19} &= c_{\omega-1}(x_i) = x_i. \\ e_{20} &= x_i \leq c_1(z). \end{aligned}$$

Proposition 4.7.1 *If $a, Un, Plus, Mul, b_1, ...b_k \in A$ such that*

$$A \models e_1\&...\&e_{20}[a, t(a), y(a), z(a), Un, Plus, Mul, b_1, ...b_k]$$

then for any $X_1, ..., X_k$ relation variables and for every first order formula φ of arithmetic containing $X_1, ..., X_k$, hold the following:

$\langle\omega;\ 0, suc, +, \cdot\rangle \models (\forall X_1, ..., X_k)\varphi$ *iff*
$RPEA_\omega \models e_1\&...\&e_{20} \Rightarrow Un^{l(\forall X_1,...,X_k)(\varphi)} = Tr(\ (\forall X_1, ...X_k)\varphi\ ).$

Note that, we may assume $(\forall X_1, ...X_k)\varphi$ is an arbitrary Π^1_1 formula of arithmetic.

Proof. Obvious. ∎

Now we can finish the proof of Theorem 4.1 as follows. Observe that in e_1, e_2, e_3 and in $e_8...e_{20}$ are exactly the following operation symbols can occur:

the Booleans, $c_i, d_{i,j}, i, j \in \omega,$
$s_{suc}, s_{pred}, s_{[0,1]}, s_{[0,2]}, s_{[0,3]}, s_{[0,4]}, s_{[1,2]}, s_{[1,3]}, s_{[1,4]}, s_{[2,4]}, s_{[0/3]},$
$c_\omega, c_{\omega-1}, c_{\omega-2}, c_{\omega-3}.$

Let L_1 be this reduct of $RPEA_\omega$.

So the remaining part is finding a strictly finite reduct L of the language of $RPEA_\omega$ in which all the preceding operations are term definable.

Definition 4.8 *Let f, g, h be the following $\omega \to \omega$ functions:*

$$\forall x \in \omega, \quad f(x) = 2 \cdot x$$

.

$$\forall x \in \omega, \quad g(x) = \begin{cases} x - 2 & \textit{if } x > 1 \textit{ and } x \textit{ is even} \\ 1 & \textit{if } x = 0 \\ x + 2 & \textit{if } x \textit{ is odd} \end{cases}$$

$$\forall x \in \omega, \quad h(x) = \begin{cases} k & \textit{if } x = 2k \\ k & \textit{if } x = 2k + 1 \end{cases}$$

Note that g is a bijection.

Definition 4.9 *Let L be the following strictly finite reduct of the language of* $RPEA_\omega$*: The booleans,* $c_0, d_{0,1}, s_f, s_g, s_{g^{-1}}, s_h, s_{suc}, s_{pred}, s_{[0,1]}, s_{[0,2]}, s_{[0,3]}, s_{[0,4]}, s_{[1,2]}, s_{[1,3]}, s_{[1,4]}, s_{[2,4]}, s_{[0/3]}, c_{\omega-3}$.

Proposition 4.9.1 *For every* $i, j \in \omega$ *the following operations are term definable in L:* $c_{i+1}, d_{i,j}, c_\omega, c_{,\omega-1}, c_{\omega-2}$.

Proof. It is easy to see that the following definitions are work:

$$
\begin{aligned}
c_{i+1}(x) &= s_h s_g^{i+1} c_0 s_{g^{-1}}^{i+1} s_f(x).\\
d_{0,j+1} &= (s_{[0,1]} s_{suc})^j (d_{0,1}).\\
d_{i+1,j+1} &= s_{suc}(d_{i,j}).\\
c_\omega(x) &= c_0 c_1 c_2 c_{\omega-3}(x).\\
c_{\omega-1}(x) &= c_1 c_2 c_{\omega-3}(x).\\
c_{\omega-2}(x) &= c_2 c_{\omega-3}(x).
\end{aligned}
$$

∎

This completes the proof of Theorem 4.1.

∎

As we mentioned in the introduction, if $\mathbf{Eq}(RPEA_\omega)$ were finitely axiomatizable by some kind of schemas, then the equational consequences of these schemas written in the above defined language L would form a Π^1_1 hard set (by Theorem 4.1). Usually, finite (schema) axiomatizability of some theory implies recursively enumerability of this theory. However, in [10] there is an other strictly finite reduct of the language of PA_ω such that the equational consequences of the PA_ω axioms written in this language, form a not recursively enumerable set. Moreover, the indices of the substitutions and cylindrifications occurring in this reduct, are recursive functions and sets, respectively.

The intuitive meaning of these results is that $\mathbf{Eq}(RPEA_\omega)$ is not computable. This is not a precise mathematical statement, because the cardinality of this set is continuum, so too big to handle by a computer. But suppose one generalizes the concept of computability in a meaningful way to continuum big sets. The only thing we assume about the new, generalized concept of computability is the following. The intersection of two computable set is computable, and the old, countable noncomputable sets are not computable in the new sense. Now $\mathbf{Eq}(RPEA_\omega)$ is not computable in the new, generalized sense, because its intersection with a decidable (in the old sense) set L, forms a Π^1_1 hard set, as Theorem 4.1 states. Summing up, the set $\mathbf{Eq}(RPEA_\omega)$ is really not computable.

Let us turn briefly to trying to give upper bounds for complexity. What we do below are only the first steps in exploring this direction.

Definition 4.10 *If Σ is a set of Halmos schemas then*

$$Ded(\Sigma) = \{e : e \textit{ is an equation derivable from the } \omega \textit{ dimensional instances of } \Sigma\}.$$

We will single out certain kinds of reducts L' of the language of $RPEA_\omega$ such that the axiomatic approximations of $RPEA_\omega$ have the following property. Roughly speaking, if the basic operations of L' are neither too numerous nor too complicated, then the set $Ded(\Sigma) \cap L'$ of the L' consequences of a finite schema Σ of axioms will be "of relatively low complexity". More formally: if Σ is a finite set of Halmos schemas valid in $RPEA_\omega$ then $Ded(\Sigma) \cap L'$ i.e. the equational consequences of Σ written in the language L' form a Σ^1_1 enumerable set. That is, there exists an algorithm, which uses an oracle deciding the Σ^1_1 theory of standard arithmetic, and enumerates $Ded(\Sigma) \cap L'$.

Remark. We do not claim that the set $Ded(\Sigma) \cap L'$ is Σ^1_1 i.e. is definable by some Σ^1_1 formula of arithmetic.

Definition 4.11 *If L is a reduct of the language of $RPEA_\omega$ such that*

– L contains at most countably many operation symbol, and
– in L only finitely many operation symbol occur which is not a CA_ω operation symbol, and
– whenever c_Γ or s_τ is an operation symbol in L then Γ, τ are first order definable over $\langle \omega; 0, suc\rangle$, respectively,

then we say that L is a simple language.

The language L defined in 4.9 is not simple because of s_f, s_g, s_h but L_1 is simple. (L_1 is defined after proposition 4.7.1.)

Let $\mathcal{A} = \langle \omega; 0, suc\rangle$. Note that the Σ^1_1 theory of $\mathcal{A}$ is as hard as the Σ^1_1 theory of standard arithmetic, because addition and multiplication are definable by a Σ^1_1 formula.

Definition 4.12 *Let L be a simple language, Σ be a set of Halmos schemas, and e be an equation written in L and derivable from (the ω dimensional instances of the elements of) Σ. Then $w(e)$ is a witness sentence of e iff the following hold.*

– *$w(e)$ is a Σ_1^1 formula of $\mathcal{A}$, and*
– *$w(e) = (\exists R_1, ..., R_k)(\exists f_1, ..., f_l)(\varphi_1 \& ... \& \varphi_n \& \psi_1 \& ... \& \psi_m)$ such that*
– *$\varphi_1, ..., \varphi_n$ are the left hand side of some Halmos schemas occurring in Σ, and*
– *$\psi_1, ..., \psi_m$ are the definitions of the indices of the operation symbols of L, that is, if c_{Γ_p} is the $p-th$ operation symbol in L then $\psi_p = (\forall x)(R_p(x) \Leftrightarrow$ the definition of $\Gamma_p)$ (similarly for substitutions), and*
– *e is provable from Σ_0 where $\Sigma_0 = \{f : f$ is an equation in L such that f is an ω dimensional instance of some $\langle \varphi_m, g \rangle$ with respect to $\langle \omega, \Gamma_1, ..., \Gamma_{p_1}, \tau_1, ..., \tau_{p_2} \rangle$.*
$(p_1 + p_2 = m.)$

Obviously, if all conditions hold, then there exists a witness sentence for e.

Note that if $\Sigma \vdash e$ and $w(e)$ is a witness sentence of e then $\mathcal{A} = \langle \omega; 0, suc \rangle \models w(e)$.

Theorem 4.13 *Let L' be a simple language. Let Σ be any finite (or recursively enumerable) set of Halmos schemas. Then $Ded(\Sigma) \cap L'$ is recursively enumerable over the Σ_1^1 theory of $\mathcal{A}$.*

Proof. We will construct an algorithm (see alg_3 below), which enumerates $Ded(\Sigma) \cap L'$. Let alg_0 be an algorithm (oracle) which enumerates the Σ_1^1 theory of $\mathcal{A}$. As we mentioned, the complexity of Σ_1^1 theory of $\mathcal{A}$ is the same as the complexity of the Σ_1^1 theory of standard arithmtic. For a given valid Σ_1^1 sentence σ we will systematically enumerate the equations having witness sentence σ.

Let alg_1 be the following algorithm. alg_1 maps the Σ_1^1 formulas σ of $\mathcal{A}$ to the finite subsets of Σ such that if $\sigma = (\exists X_1 ... X_k)(\varphi_1 \& ... \& \varphi_n)$ where $\varphi_1, ... \varphi_n$ are first order formulas containing the definitions of the indices of operation symbols of L' then $alg_1(\sigma) = \{h \in \Sigma : (\exists 1 \leq m \leq n)(\exists e_{\varphi_m} \in L_{PT})(h = \langle \varphi_m, e_{\varphi_m} \rangle)\}$. If σ has different form, then $alg_1(\sigma) = 0$ (the empty set).

Let alg_2 be an algorithm, which enumerates the consequences of a recursively enumerable set of equations written in L_{PT}, using logic of equations as inference system.

Let alg_3 be the following algorithm. Combining alg_0, alg_1 first alg_3 enumerates the (finite) sets of Halmos schemas H produced by alg_1 on the output of alg_0. Secondly alg_3 between two such steps, using alg_2, enumerates the consequences of the set $\{e \in L_{PT} : (\exists \varphi)(\langle \varphi, e \rangle \in H\}$. More precisely, alg_3 enumerates all pairs of natural numbers, and if $\langle i, j \rangle$ are computed, then alg_2 runs j steps on the $i - th$ output of alg_0, alg_1. Now the derived equation is in L_{PT}, so finally, of course, alg_3 substitute every relation and function variables with the corresponding set or function (using the current output of alg_0 and the definitions of the indices of the operation symbols of L') so the result equation is written in L'. After this, alg_3 take the next pair.

we claim that alg_3 do the job.

Proposition 4.13.1 *alg_3 is sound, that is, if e is listed by alg_3, then $e \in Ded(\Sigma) \cap L'$.*

Proof. Assume e is the $i-th$ output of alg_3. By the definition of alg_3 $e \in L'$. Suppose e was obtained from $e' \in L_{PT}$ by substitutions, and the $i-th$ output of alg_0 is φ. (This is a Σ^1_1 sentence of $\mathcal{A}$.) There exist a proof p' of e' such that p' uses the second components of the $i-th$ output of alg_1 as hypotheses. By the definition of alg_0, alg_1 this proof is translatable into an $RPEA_\omega$ proof for e (using the same substitutions). ∎

Proposition 4.13.2 *alg_3 is complete, that is, if $e \in Ded(\Sigma) \cap L'$ then e is listed by alg_3.*

Proof. Assume $e \in Ded(\Sigma) \cap L'$. Let $w(e)$ be a witness sentence for e. This is an element of the Σ^1_1 theory of $\mathcal{A}$, so alg_0 will enumerate $w(e)$. Then, by the definition of witness sentence, e is provable from the corresponding output of alg_1 so alg_3 really enumerates e. ∎

This completes the proof of theorem 4.5. ∎

5 Related Results

Here we summarize some results about the possibility of axiomatizability of some reducts of $RPEA_\omega$. The complexity of the equational theories of these reducts are also discussed.

Case of algebras without equality

By the theorem of Daigneault and Monk every PA_ω is representable, so the "complexity" of the equational theory of PA_ω is the same as the "complexity" of the set of all equational consequences of the PA_ω axiom schemas. But this representation theorem does not imply recursively enumerability, as the following theorem claims.

Theorem 5.1 *There is a strictly finite reduct L_G of the language of PA_ω, such that the following set of equations is not recursively enumerable:*
$\{e \in L_G : (Axioms\ of\ PA_\omega) \vdash e\}$. Moreover, the indices of the substitutions and cylindrifications are recursive functions and sets, respectively.

For the proof see [10].

Definition 5.2 *Throughout $L_{suc,pred}$ denote the following language. $L_{suc,pred}$ consists of the Boolean operations, and $c_0, s_{suc}, s_{pred}, s_{[0,1]}, s_{[i/j]},\ i,j \in \omega$.*

Note that in $L_{suc,pred}$ the following operations are term definable: $c_i, s_{[i,j]},\ i,j \in \omega$. The following theorem is due to I. Sain and V. Gyuris, (see [8]) which is a possible solution of (some variant of) the finitization problem.

Theorem 5.3 *Let $L_{suc,pred,\omega} = L_{suc,pred} \cup \{c_\omega\}$. Then the reduct of RPA_ω to $L_{suc,pred,\omega}$ generates a variety, which is axiomatizable by a recursive set $\Sigma_{suc,pred,\omega}$ of equations. This set is explicitly given by finitely many schemas (these schemas are essentially simpler than Halmos schemas in general).*

This theorem is a representation theorem, but the axiom system is recursive, so $\mathbf{Eq}(RPA_\omega) \cap L_{suc,pred,\omega} = Ded(\Sigma_{suc,pred,\omega}) \cap L_{suc,pred,\omega}$ is recursively enumerable.

Case of algebras with equality

The following theorem is also in [8].

Theorem 5.4 *Let $L_{suc,pred,D} = L_{suc,pred} \cup \{d_{0,1},\ i,j \in \omega\}$. Note that in this language all the $D_{i,j}$'s are term definable, $i,j \in \omega$. Then the reduct of $RPEA_\omega$ to $L_{suc,pred,D}$ generates a variety, which is axiomatizable by a recursive set $\Sigma_{suc,pred,D}$ of equations. This set is explicitly given by finitely many schemas (these schemas are essentially simpler than Halmos schemas in general).*

All the preceding remarks hold for this case, too. The following theorem is due to R. McKenzie.

Theorem 5.5 *Let $L_{suc,pred,\omega,D} = L_{suc,pred} \cup \{c_\omega, d_{i,j},\ i,j \in \omega\}$. Then the set $\mathbf{Eq}(RPEA_\omega) \cap L_{suc,pred,\omega,D}$ is not recursively enumerable.*

Finally about the complexity of the first order theory of $RPEA_\omega$. One can continue the proof of Theorem 4.1 as follows. Tr is extendable in such a way, that Tr maps the full second order theory of arithmetic into the first order theory of (a finite reduct of) $RPEA_\omega$. This means that $\mathbf{Th}(RPEA_\omega) \cap L$ does not form an analytical set (so at least as complex as the full second order theory of standard arithmetic.) Here $\mathbf{Th}(RPEA_\omega)$ denotes the first order theory of $RPEA_\omega$.

Methodological conclusions

Comparing the negative and positive results above, we found the following. There are two paradigms. In the first paradigm (called polyadic algebraic oriented) the indices or names of the operation symbols have a complex and fixed structure. For example the indices of the substitutions are functions, and one can compose them. The properties and the complexity of these objects are implicitly mirroring in the equational theories that we are studied, and possibly this is the reason of our negative results.

The other paradigm (called cylindric algebraic oriented) is the opposite. In this case the indices of operation symbols have no essentially structure. There are no operations and relations defined between the names of the operation symbols. We observed, that all the positive results follow the cylindric paradigm.

These paradigms not are rigid things, for example the results about quasi polyadic algebras in [9] follow the cylindric paradigm (and the results are positive).

The main methodological conclusion is the following. Suppose one wants to apply polyadic algebras to the certain area, but the negative complexity-properties of polyadic algebras are undesirable for the applications. Then it is possible to remove these properties by switching from the polyadic paradigm to the cylindric paradigm.

6 Appendix

Here we give the translations of the usual defining schemas of α dimensional polyadic (equality) algebras into Halmos schemas. First of all we recall from [4] the original defining schemas:

(P0) The Boolean Algebra axioms,
(P1) $c_\Gamma(0) = 0$,
(P2) $x \leq c_\Gamma(x)$,
(P3) $c_\Gamma(x \wedge c_\Gamma(y)) = c_\Gamma(x) \wedge c_\Gamma(y)$,
(P4) $c_0(x) = x$,
(P5) $c_\Gamma c_\Delta(x) = c_{\Gamma \cup \Delta}(x)$,
(P6) $s_{Id_\alpha}(x) = x$,
(P7) $s_{\sigma \circ \tau}(x) = s_\sigma s_\tau(x)$,
(P8) $s_\sigma(x \vee y) = s_\sigma(x) \vee s_\sigma(y)$,
(P9) $s_\sigma(-x) = -s_\sigma(x)$,

(P10) If $(\alpha - \Gamma) \mid \sigma = (\alpha - \Gamma) \mid \tau$, then $s_\sigma c_\Gamma(x) = s_\tau c_\Gamma(x)$,
(P11) If $(\tau^{-1}(\Gamma)) \mid \tau$ is one-one and $\Delta = \tau^{-1}(\Gamma)$ then $c_\Gamma s_\tau(x) = s_\tau c_\Delta(x)$,

(E1) $d_{\kappa,\kappa} = 1$,
(E2) $x \wedge d_{\kappa,\lambda} \leq s_{[\kappa/\lambda]}(x)$,
(E3) $s_\tau(d_{\kappa,\lambda}) = d_{\tau(\kappa),\tau(\lambda)}$.

Here $\Gamma, \Delta \subseteq \alpha, \quad \sigma, \tau \in {}^{\alpha}\alpha, \quad \kappa, \lambda \in \alpha.$

The translations are:
(P0') $\langle true,$ The Boolean Algebra axioms $\rangle$,
(P1') $\langle true,\ c_{r_1}(0) = 0\rangle,$
(P2') $\langle true,\ x \leq c_{r_1}(x)\rangle,$
(P3') $\langle true,\ c_{r_1}(x \wedge c_{r_1}(y)) = c_{r_1}(x) \wedge c_{r_1}(y)\rangle,$
(P4') $\langle(\forall y)(\neg r_1(y)),\ c_{r_1}(x) = x\rangle,$
(P5') $\langle(\forall y)(r_1(y) \vee r_2(y) \Leftrightarrow r_3(y)),\ c_{r_1}c_{r_2}(x) = c_{r_3}(x)\rangle,$
(P6') $\langle(\forall y)(f_1(y) = y),\ s_{f_1}(x) = x\rangle,$
(P7') $\langle(\forall y)(f_1(f_2(y)) = f_3(y)),\ s_{f_3}(x) = s_{f_1}s_{f_2}(x)\rangle,$
(P8') $\langle true,\ s_{f_1}(x \vee y) = s_{f_1}(x) \vee s_{f_1}(y)\rangle,$
(P9') $\langle true,\ s_{f_1}(-x) = -s_{f_1}(x)\rangle,$

(P10') $\langle(\forall y)(\neg r_1(y) \Rightarrow f_1(y) = f_2(y)),\ s_{f_1}c_{r_1}(x) = s_{f_2}c_{r_1}(x)\rangle,$
(P11') $\langle(\forall y)((r_2(y) \Leftrightarrow\ (\exists z)(r_1(z)\ \&\ y = f_1(z))\ \&\ (\forall y, z)(r_2(y)\ \&\ r_2(z)\ \&\ y \neq z \Rightarrow f_1(y) \neq f_1(z))),\ c_{r_1}s_{f_1}(x) = s_{f_1}c_{r_2}(x)\rangle,$

(E1') $\langle true,\ d_{n_1,n_1} = 1\rangle,$
(E2') $\langle(\forall y)(y \neq n_1 \Rightarrow f_1(y) = y,\ \&\ f_1(n_1) = n_2),\ x \wedge d_{n_1,n_2} \leq s_{f_1}(x)\rangle,$
(E3') $\langle(f_1(n_1) = n_3, \&\ f_1(n_2) = n_4),\ s_{f_1}(d_{n_1,n_2}) = d_{n_3,n_4}\rangle.$

It is easy to see that the instances of (P0)-P(11), (E1)-(E3) are the same as the α dimensional instances of (P0')-P(11'), (E1')-(E3').

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