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Weakly higher order cylindric algebras and finite axiomatization of the representables

Abstract. We show that the variety of n-dimensional weakly higher order cylindric algebras, introduced in Németi [9], [8], is finitely axiomatizable when $n > 2$. Our result implies that in certain non-well-founded set theories the finitization problem of algebraic logic admits a positive solution; and it shows that this variety is a good candidate for being the cylindric algebra theoretic counterpart of Tarski's quasi-projective relation algebras.

Keywords: algebraic logic, cylindric algebra, quasi-projective relation algebra, non-well-founded set theory, finitization problem

1. Introduction

In this paper we give a finite axiomatization for the variety of n-dimensional weakly higher order cylindric algebras ($\mathsf{RCA}_n^{\uparrow}$, $3 \leq n \in \omega$), introduced in Németi [9], [8]. Weakly higher order cylindric algebras are natural expansions of representable cylindric algebras. They have extra operations that correspond to a kind of bounded (existential) quantification along a binary relation R (called an *accessibility relation*) and its converse. The relation R is most conveniently thought of as being the "element of" relation in a model of some set theory. In this interpretation, the logical constants corresponding to the new operations are second order quantifiers. See Definition 1.1 below for the details. Our result may be of some interest to (algebraic) logicians, because

1. it implies that in certain non-well-founded set theories the *finitization problem* of algebraic logic admits a positive solution; and
2. it shows that $\mathsf{RCA}_n^{\uparrow}$ is a good candidate for being the cylindric algebra theoretic counterpart of Tarski's quasi-projective relation algebras (QRAs).

As for 1, we note that, as oulined above and explained in detail in [9] and [8], $\mathsf{RCA}_n^{\uparrow}$ can be thought of as the algebraization of an untyped higher order logic $\mathcal{L}$. See [9], [8], [3] and [11] for details, and for an explanation of how our result implies that in some non-well-founded set theories $\mathcal{L}$ is a finitizable extension of first-order logic. In (non-algebraic) logical terms, this means that some versions of higher order logic (with standard semantics) admit a truly finite* axiomatization in some set theories, and, for some extensions $\mathcal{L}$ of first order logic, both the valid formula-schemas of $\mathcal{L}$ (in the sense of e.g. Rybakov [10]) and the propositional modal version of $\mathcal{L}$ (cf. e.g. Venema [14]) are finitely axiomatizable.

Item 2 above seems to be of some interest, e.g. because the results in Németi [7], [6] and Andréka–Németi [1] indicate that finding the CA-theoretic counterpart of QRAs may not be very easy. The problem of finding such a counterpart appeared implicitly in Maddux–Tarski [5] (last sentence) and Maddux [4].

Definition 1.1 (cf. Németi [9], [8]). *Let R be a binary relation on a set U. We say that $\langle U, R\rangle$ is* pairing *if*

$$\langle U, R\rangle \models \forall xy\exists w[xRw \wedge yRw \wedge \forall z(zRw \to (z = x \vee z = y))].$$

For α an ordinal,

$$\mathsf{Cs}^{\uparrow}_{\alpha} = \mathbf{S}\{\langle \mathcal{P}({}^{\alpha}U), \cup, \setminus, \mathsf{C}^{\uparrow R}_{i}, \mathsf{C}^{\downarrow R}_{i}, \mathsf{D}^{U}_{ij}\rangle_{i,j\in\alpha} : \\ U \textit{ is a set and } \langle U, R\rangle \textit{ is pairing}\},$$

where

$\mathsf{C}^{\uparrow R}_{i} X = \{f \in {}^{\alpha}U : \exists g \in X(g_i R f_i$ *and* $g_j = f_j$ *for all* $j \in \alpha,\ j \neq i)\}$ *for* $X \subseteq {}^{\alpha}U$,

$\mathsf{C}^{\downarrow R}_{i} X = \{f \in {}^{\alpha}U : \exists g \in X(f_i R g_i$ *and* $g_j = f_j$ *for all* $j \in \alpha,\ j \neq i)\}$ *for* $X \subseteq {}^{\alpha}U$, *and*

$\mathsf{D}^{U}_{ij} = \{f \in {}^{\alpha}U : f_i = f_j\}$.

We will usually omit the superscripts R and U from the operations. We let $\mathsf{RCA}^{\uparrow}_{\alpha} = \mathbf{I}\,\mathbf{S}\,\mathbf{P}\,\mathsf{Cs}^{\uparrow}_{\alpha}$.

Theorem 1.2. $\mathsf{RCA}^{\uparrow}_{n}$ *is a finitely axiomatizable variety (if $2 < n < \omega$).*

Outline of proof. First we show (Lemma 2.5) that every QRA (see Definition 2.1 below) $\mathfrak{B}$ with $R \in B$ has an $\mathsf{RCA}^{\uparrow}_{n}$ reduct (which we denote by $\mathfrak{Rd}_{\mathsf{CA}^{\uparrow}_{n},R}\,\mathfrak{B}$). The proof of this result is based on Tarski's representation theorem for QRAs (recalled in Lemma 2.2 below). By imposing finitely many axioms on $\mathsf{RCA}^{\uparrow}_{n}$-type algebras we get a variety $\mathsf{CA}^{\uparrow}_{n}$ (Definition 3.2) such that for any $\mathfrak{A} \in \mathsf{CA}^{\uparrow}_{n}$

*as opposed to e.g. finite schema

- the RA-reduct (taken in the last two coordinates, cf. Definition 3.1) $\mathfrak{B}$ of $\mathfrak{A}$ is a QRA
- there is a term-definable operation in $\mathfrak{A}$ that embeds $\mathfrak{A}$ in $\mathfrak{Rd}_{\mathsf{CA}_n^{\uparrow}, R^{\mathfrak{A}}}\, \mathfrak{B} \in \mathsf{RCA}_n^{\uparrow}$. Here $R = R_{n-2,n-1}$ is a constant term of $\mathfrak{A}$ such that $R^{\mathfrak{A}}$ is in $\mathfrak{B}$.

$\square$

In other words, by finding $\mathsf{RCA}^{\uparrow}$-like algebras inside QRAs, we can use an RA-theoretic result (Tarski's representation theorem for QRAs) to prove a CA-theoretic result. We note that in Simon [12] the definability of CA-like algebras in QRAs was used to derive the representation theorem for QRAs from a CA-theoretic result (Henkin's neat embedding theorem).

2. Finding $\mathsf{RCA}^{\uparrow}$s in QRAs

We start by recalling the definition of quasi-projective relation algebras from Tarski–Givant [13].

Definition 2.1 (Tarski–Givant [13]). *Let* $\mathfrak{B} \in \mathsf{RA}$. $\mathfrak{B}$ *is a* QRA *if there are elements* $\mathsf{p}, \mathsf{q} \in B$ *such that the following equations hold in* $\mathfrak{B}$*:*

(e1) $\mathsf{p}^{\smile} \circ \mathsf{p} \leq \mathrm{Id}$ *and* $\mathsf{q}^{\smile} \circ \mathsf{q} \leq \mathrm{Id}$

(e2) $\mathsf{p}^{\smile} \circ \mathsf{q} = 1$

p *and* q *are called* quasi-projections *in* $\mathfrak{B}$.

RRA denotes the class of representable RAs.

Lemma 2.2 (Tarski–Givant [13, Theorem 8.4(iii)]). $\mathsf{QRA} \subseteq \mathsf{RRA}$.

The following definition and lemma are from Tarski–Givant [13].

Definition 2.3. *For* $\mathfrak{B} \in RA$ *and* $R \in B$*, define*

$$\mathsf{p}_R = A \wedge -(A \circ -\mathrm{Id}) \qquad\qquad \mathsf{q}_R = B \wedge -[(B \wedge -\mathsf{p}_R) \circ -\mathrm{Id}],$$

where

$$A = R^{\smile} \circ (R^{\smile} \wedge -(R^{\smile} \circ -\mathrm{Id})) \qquad B = R^{\smile} \circ R^{\smile}$$

This definition of quasi-projections from R is an RA-rewrite of Kuratowski's construction of (unordered) pairs.

Lemma 2.4. *Let $\mathfrak{B} \in \mathsf{RRA}$ be simple and represented on the (base) set U, and let $R \in B$. Then $\mathfrak{B}$ is a QRA with quasi-projections p_R and q_R iff $\langle U, R\rangle$ is pairing.*

Proof. This is a straightforward calculation, cf. [13, 4.6(ii)]. ■

Let $\mathfrak{B} \in \mathsf{QRA}$ with quasi-projections p and q, let $R \in B$ and let $n \in \omega$. We define the $\mathsf{CA}_n^{\uparrow}$-reduct[†] of $\mathfrak{B}$ as follows. The idea is that the unit of the $\mathsf{CA}_n^{\uparrow}$ will consist of pairs whose second coordinate is an n-tuple, and we use the RA operations (together with the quasi-projections and R) to express the $\mathsf{CA}^{\uparrow}$-operations.

Define

$$\epsilon^{(n)} = \mathrm{Id} \wedge (\mathsf{q}^{n-1} \circ \mathsf{q}^{\smile n-1})$$
$$\pi_i^{(n)} = \epsilon^{(n)} \circ \mathsf{q}^i \circ \mathsf{p} \text{ if } i < n-1 \text{ and } \pi_{n-1}^{(n)} = \mathsf{q}^{n-1}$$
$$\delta_{ij}^{(n)} = 1 \circ (\pi_i^{(n)\smile} \wedge \pi_j^{(n)\smile})$$
$$\chi_j^{(n)} = \pi_j^{(n)} \circ \pi_j^{(n)\smile}$$
$$\mathsf{t}^{(n)} = \textstyle\prod_{j<n} \chi_j^{(n)}$$
$$\rho_i^{\uparrow(n)} = \pi_i^{(n)} \circ R \circ \pi_i^{(n)\smile} \text{ and } \rho_i^{\downarrow(n)} = \pi_i^{(n)} \circ R^{\smile} \circ \pi_i^{(n)\smile}$$
$$\mathsf{t}_i^{\uparrow(n)} = \rho_i^{\uparrow(n)} \wedge \textstyle\prod_{i\neq j<n} \chi_j^{(n)} \text{ and } \mathsf{t}_i^{\downarrow(n)} = \rho_i^{\downarrow(n)} \wedge \textstyle\prod_{i\neq j<n} \chi_j^{(n)}$$
$$\tau_i^{\uparrow(n)} x = x \circ \mathsf{t}_i^{\uparrow(n)} \text{ and } \tau_i^{\downarrow(n)} x = x \circ \mathsf{t}_i^{\downarrow(n)}$$
$$\tau^{(n)} x = x \circ \mathsf{t}^{(n)}$$
$$1^{(n)} = 1 \circ \epsilon^{(n)}$$

and let
$$\mathfrak{Rd}_{\mathsf{CA}_n^{\uparrow},R}\mathfrak{B} = \langle B_n, \vee, -_{1^{(n)}}, \tau_i^{\uparrow(n)}, \tau_i^{\downarrow(n)}, \delta_{ij}^{(n)}\rangle_{i,j<n},$$
where $B_n = \{x \in B : x = 1 \circ \tau^{(n)} x\}$, and $-_{1^{(n)}}$ is complementation with respect to $1^{(n)}$. We leave it to the reader to verify that $\mathfrak{Rd}_{\mathsf{CA}_n^{\uparrow},R}\mathfrak{B}$ is closed under the indicated operations.

Intuitively, $\epsilon^{(n)}$ corresponds to the set of n-tuples, $\pi_i^{(n)}$ is the ith projection (restricted to $\epsilon^{(n)}$), $\chi_i^{(n)}$ connects n-tuples whose ith coordinates are the same, and two n-tuples are $\rho_i^{\uparrow(n)}$-related if their ith coordinates are R-related. $\mathsf{t}^{(n)}$ connects n-tuples that agree in all their coordinates; the requirement $\tau^{(n)} x \leq x$ in the definition of B_n is needed to ensure that such

[†]Actually, it is a relativization of a (generalized) reduct.

n-tuples can not be distinguished (that is, if an element of the CA_n reduct contains (the code of) a sequence, it contains all possible codes of that sequence).

We will omit the superscript $^{(n)}$ when n is clear from the context.

LEMMA 2.5. *Let $n \in \omega$, $n \geq 3$ and let $\mathfrak{B} \in \mathsf{RA}$ be simple. Suppose that $R \in B$ is such that $\mathfrak{B}$ is a QRA with quasi-projections $\mathsf{p} = \mathsf{p}_R$ and $\mathsf{q} = \mathsf{q}_R$. Then $\mathfrak{Rd}_{\mathsf{CA}^{\uparrow}_{n},R}\mathfrak{B} \in \mathsf{Cs}^{\uparrow}_{n}$ with accessibility relation R.*

PROOF. By Lemma 2.2 $\mathfrak{B}$ is representable, and since $\mathfrak{B}$ is simple, we may assume that $1^{\mathfrak{B}} = U \times U$ for some set U. We claim that the function

$$\mathrm{rep}(x) = \{\langle u_0, \ldots, u_{n-1}\rangle \in {}^nU : (\exists\langle s,t\rangle \in x)(\forall k < n)\langle t, u_k\rangle \in \pi_k\}$$

embeds $\mathfrak{C} = \mathfrak{Rd}_{\mathsf{CA}^{\uparrow}_{n},R}\mathfrak{B}$ in the full $\mathsf{Cs}^{\uparrow}_{n}$ with base U and accessibility relation R. (We note that $\langle U, R\rangle$ is pairing because of Lemma 2.4.)

First we claim that

$$\mathrm{rep}(1^{(n)}) = {}^nU. \tag{1}$$

To see that (1) holds, it suffices to show that

$$\begin{aligned}\forall 0 < i < n \forall u_{n-1-i}, \ldots, u_{n-1} \in U \exists v \in U[\langle v, u_{n-1}\rangle \in \mathsf{q}^i \\ \text{and } \forall 0 < j \leq i(\langle v, u_{n-1-j}\rangle \in \mathsf{p} \circ \mathsf{q}^{i-j})]\end{aligned} \tag{2}$$

Indeed, taking $i = n-1$ in (2) we get that for all $u_0, \ldots, u_{n-1} \in U$ there is a $v \in U$ such that $\langle v, u_{n-1}\rangle \in \mathsf{q}^{n-1}$ (whence $\langle v, v\rangle \in \epsilon^{(n)}$) and if $k \leq n-2$, then setting $j = n-1-k$ we have $\langle v, u_k\rangle = \langle v, u_{n-1-j}\rangle \in \mathsf{p}\circ\mathsf{q}^{n-1-j} = \mathsf{p}\circ\mathsf{q}^k$. Hence $\langle v, u_k\rangle \in \pi_k$ for all $k < n$, so $\langle u_0, \ldots, u_{n-1}\rangle \in \mathrm{rep}(\epsilon^{(n)}) \subseteq \mathrm{rep}(1^{(n)})$.

We prove (2) by induction on i. For $i = 1$ we have $\langle u_{n-2}, u_{n-1}\rangle \in 1^{\mathfrak{B}} = \mathsf{p}^{\smile}\circ q$ (by (e2)) so there is a $v \in U$ such that $\langle v, u_{n-1},\rangle \in \mathsf{q}$ and $\langle v, u_{n-2}\rangle \in \mathsf{p}$.

Suppose now that (2) holds for some $i < n-1$ and let $v_i \in U$ witness this for $u_{n-1-i}, \ldots, u_{n-1} \in U$. Given $u_{n-1-(i+1)}$, $\langle u_{n-1-(i+1)}, v_i\rangle \in 1^{\mathfrak{B}} = \mathsf{p}^{\smile} \circ \mathsf{q}$, so there is a $v \in U$ such that $\langle v, u_{n-1-(i+1)}\rangle \in \mathsf{p}$ and $\langle v, v_i\rangle \in \mathsf{q}$. This v works for i+1 because $\langle v, u_{n-1}\rangle \in \{\langle v, v_i\rangle\} \circ \{\langle v_i, u_{n-1}\rangle\} \subseteq \mathsf{q}^i \circ \mathsf{q} = \mathsf{q}^{i+1}$, for $j \leq i$ we have $\langle v, u_{n-1-j}\rangle \in \{\langle v, v_i\rangle\} \circ \{\langle v_i, u_{n-1-j}\rangle\} \subseteq \mathsf{q} \circ (\mathsf{q}^{i-j} \circ \mathsf{p}) = \mathsf{q}^{i+1-j} \circ \mathsf{p}$ and for $j = i+1$ we have $\langle v, u_{n-1-j}\rangle \in \mathsf{p} = \mathsf{p} \circ \mathsf{q}^{i+1-j}$. This completes the proof of (2).

Next we claim that

$$\mathrm{rep}(-_{1^{(n)}}x) = {}^nU \setminus \mathrm{rep}(x)$$

if $x \in B_n$. To show that the inclusion ($\subseteq$) holds, suppose for contradiction that $\langle u_0, \ldots, u_{n-1}\rangle \in \mathrm{rep}(x)\cap\mathrm{rep}(-_{1^{(n)}}x)$, i.e. there are $\langle s,t\rangle \in 1^{(n)} \wedge -x$ and

$\langle v,w\rangle \in x$ such that $\langle t,u_k\rangle, \langle w,u_k\rangle \in \pi_k$ for all $k<n$. Then $\langle w,t\rangle \in \mathsf{t}^{(n)}$ and hence $\langle s,t\rangle \in \{\langle s,v\rangle\} \circ \{\langle v,w\rangle\} \circ \{\langle w,t\rangle\} \subseteq 1 \circ x \circ \mathsf{t}^{(n)} = x$, a contradiction.

The other direction is an obvious consequence of (1).

It is clear that rep respects $\vee$, so rep is a Boolean homomorphism, and is one-to-one because $\mathrm{rep}(x) \neq 0$ if $0 \neq x \in B_n$.

Now let $i,j<n$. To prove that $\mathrm{rep}(\delta_{ij}) = \mathsf{D}_{ij}$, it is enough to show that for all $u_0,\ldots,u_{n-1} \in U$

$$(\exists \langle s,t\rangle \in 1 \circ (\pi_i{}^{\smile} \wedge \pi_j{}^{\smile}))(\forall k<n)\langle t,u_k\rangle \in \pi_k \iff u_i = u_j.$$

($\Rightarrow$) Let $\langle s,t\rangle$ be as in the left-hand side. Then there is an $s' \in U$ such that $\langle s',t\rangle \in \pi_i{}^{\smile} \wedge \pi_j{}^{\smile}$, whence $\langle s',u_i,\rangle \in \{\langle s',t\rangle\} \circ \{\langle t,u_i\rangle\} \subseteq \pi_i{}^{\smile} \circ \pi_i \subseteq \mathrm{Id}$ by (e1) and $\langle s',u_j,\rangle \in \{\langle s',t\rangle\} \circ \{\langle t,u_j\rangle\} \subseteq \pi_j{}^{\smile} \circ \pi_j \subseteq \mathrm{Id}$. Thus $u_i = s' = u_j$ as desired.

($\Leftarrow$) By (1) there is $\langle s,t\rangle \in 1^{(n)}$ such that $\langle t,u_k\rangle \in \pi_k$ for all $k<n$. It is enough to show that $\langle s,t\rangle \in 1 \circ (\pi_i{}^{\smile} \wedge \pi_j{}^{\smile})$. But this is clear, since $\langle s,t\rangle \in \{\langle s,u_i\rangle\} \circ \{\langle u_i,t\rangle\} \subseteq 1 \circ (\pi_i{}^{\smile} \wedge \pi_j{}^{\smile})$; the last inequality holds because $u_i = u_j$.

It remains to show that rep respects the $\tau_i^{\uparrow}$'s and the $\tau_i^{\downarrow}$'s. We will only prove $\mathrm{rep}(\tau_i^{\uparrow}\, x) = \mathsf{C}_i^{\uparrow}\, \mathrm{rep}(x)$ for $x \in B_n$ and $i<n$, since the proof of the other statement is analogous.

So let $x \in B_n$ and $i<n$. We have to show that for all $u_0,\ldots,u_{n-1} \in U$, the following statements are equivalent:

1. $\exists \langle s,t\rangle \in \tau_i^{\uparrow}\, x$ such that $\langle t,u_k\rangle \in \pi_k$ for all $k<n$
2. $\exists \langle v,w\rangle \in x$ and $\exists u' \in U$ such that $u' R u_i$, $\langle w,u'\rangle \in \pi_i$ and $\langle w,u_k\rangle \in \pi_k$ for all $i \neq k < n$.

($1 \Rightarrow 2$) Let $\langle s,t\rangle$ be as in 1. Then there is an $s' \in U$ such that $\langle s,s'\rangle \in x$ and $\langle s',t\rangle \in \mathsf{t}_i^{\uparrow}$. We claim that $\langle s,s'\rangle$ is a good choice for $\langle v,w\rangle$ in 2. Indeed, $\langle s,s'\rangle \in x$ and if $i \neq k < n$, then $\langle s',u_k\rangle \in \{\langle s',t\rangle\} \circ \{\langle t,u_k\rangle\} \subseteq \chi_k \circ \pi_k \subseteq \pi_k$ by (e1). Finally, since $\langle s',t\rangle \in \rho_i^{\uparrow}$, there are $u',u'' \in U$ such that $\langle s',u'\rangle \in \pi_i$, $\langle u',u''\rangle \in R$ and $\langle u'',t\rangle \in \pi_i{}^{\smile}$, whence $u'' = u_i$ by (e1), and we have $u' R u_i$ and $\langle s',u'\rangle \in \pi_i$, as desired.

($2 \Rightarrow 1$) Let $\langle v,w\rangle$ be as in 2. By (1), there is a pair $\langle s,t\rangle \in 1^{(n)}$ such that $\langle t,u_k\rangle \in \pi_k$ for all $k<n$. It suffices to show that $\langle s,t\rangle \in \tau_i^{\uparrow}\, x$. Now $\langle s,t\rangle \in \{\langle s,w\rangle\} \circ \{\langle w,t\rangle\}$ and $\langle s,w\rangle \in \{\langle s,v\rangle\} \circ \{\langle v,w\rangle\} \subseteq 1 \circ x \subseteq x$, so we are done if we can show that $\langle w,t\rangle \in \mathsf{t}_i^{\uparrow}$. But this is clear, since $\langle w,t\rangle \in \{\langle w,u'\rangle\} \circ \{\langle u',u_i\rangle\} \circ \{\langle u_i,t\rangle\} \subseteq \pi_i \circ R \circ \pi_i{}^{\smile} = \rho_i^{\uparrow}$, and if $i \neq k < n$, then $\langle w,t\rangle \in \{\langle w,u_k\rangle\} \circ \{\langle u_k,t\rangle\} \subseteq \pi_k \circ \pi_k{}^{\smile} = \chi_k$. ∎

3. The representation theorem

The purpose of this section is to define (by finitely many equations) the class $\mathsf{CA}_n^{\uparrow}$ and prove that $\mathsf{CA}_n^{\uparrow} = \mathsf{RCA}_n^{\uparrow}$. First we define some operations on $\mathsf{RCA}_n^{\uparrow}$-type algebras; the important ones below are R, p and q which make it possible to define the QRA-reduct and use Lemma 2.5, and suc, because it will be suc^{n-1} which embeds the algebra to be represented in the $\mathsf{RCA}_n^{\uparrow}$ reduct of its QRA-reduct.

First, we need the following variant of the standard method of associating an RA-type algebra to a cylindric algebra, cf. eg. Henkin–Monk–Tarski [2, Definition 5.3.7.].

Definition 3.1. *Let $\mathfrak{A}$ be an algebra with a CA_n-type reduct, $n \geq 3$. The RA-reduct of $\mathfrak{A}$ is the algebra*

$$\mathfrak{Ra}\,\mathfrak{A} = \langle B, \vee, -, \circ, {}^{\smile}, \mathrm{Id}\rangle,$$

where $B = \{a \in A : \mathsf{c}_0 \ldots \mathsf{c}_{n-3}\, a \leq a\}$, $\vee$ and $-$ are the restrictions of the corresponding operations of $\mathfrak{A}$, $\mathrm{Id} = \mathsf{d}_{n-2,n-1}$, $x \circ y = \mathsf{c}_0(\mathsf{s}_0^{n-1}\, x \wedge \mathsf{s}_0^{n-2}\, y)$ and $x^{\smile} = \mathsf{s}_{n-2}^{0}\, \mathsf{s}_{n-1}^{n-2}\, \mathsf{s}_0^{n-1}\, x$ for $x, y \in B$.

Now let $\mathfrak{A}$ be an $\mathsf{RCA}_n^{\uparrow}$-type algebra, $n \geq 3$. Define

$\mathsf{c}_i\, x = \mathsf{c}_i^{\downarrow}\, \mathsf{c}_i^{\uparrow}\, x$

$R_{ij} = \mathsf{c}_j^{\uparrow}(\mathsf{d}_{ij})$ and $R = R_{n-2,n-1}$

$\mathsf{p} = \mathsf{p}_R$ and $\mathsf{q} = \mathsf{q}_R$ (computed in $\mathfrak{Ra}\,\mathfrak{A}$)

$\mathrm{singl} = -\,\mathsf{c}_0\,\mathsf{c}_1(R_{0,n-1} \wedge R_{1,n-1} \wedge -\,\mathsf{d}_{01})$,

and

$$\begin{aligned}\mathrm{suc}\, x = \mathsf{s}_1^0\, \mathsf{s}_2^1 \ldots \mathsf{s}_{n-2}^{n-3}\, \mathsf{c}_{n-2}(\mathsf{p}^{\smile} \wedge [(\mathrm{singl} \wedge \mathsf{s}_{n-2}^{n-1}\, x)\\ \vee (-\,\mathrm{singl} \wedge \mathsf{c}_{n-1}^{\uparrow}\, \mathsf{c}_{n-1}^{\uparrow}(x \wedge -\,\mathsf{d}_{n-2,n-1}))]),\end{aligned}$$

where $\mathsf{p}^{\smile}$ is computed in $\mathfrak{Ra}\,\mathfrak{A}$.

To see the intuition behind these definitions, note the following: Suppose that $\mathfrak{A} \in \mathsf{RCA}_n^{\uparrow}$ is represented with base set U and accessibility relation R'. Then c_i is ordinary cylindrification (this is where we use the assumption that $\langle U, R'\rangle$ is pairing), $R_{ij}^{\mathfrak{A}} = \{s \in {}^nU : \langle s_i, s_j\rangle \in R'\}$, $\mathrm{singl}^{\mathfrak{A}} = \{s \in {}^nU : \neg\exists u, v \in U(u \neq v \text{ and } \langle u, s_{n-1}\rangle, \langle v, s_{n-1}\rangle \in R')\}$, and $\mathrm{suc}\, x = \{s \in {}^nU : \langle s_1, \ldots, s_{n-2}, \mathsf{p}(s_{n-1}), \mathsf{q}(s_{n-1})\rangle \in x\}$.

Now we are ready to define the class $\mathsf{CA}_n^{\uparrow}$.

Definition 3.2. $\mathfrak{A} = \langle A, \vee, -, \mathsf{c}_i^\uparrow, \mathsf{c}_i^\downarrow, \mathsf{d}_{ij}\rangle_{i,j\in n} \in \mathsf{CA}_n^\uparrow$ *iff* $\mathfrak{A}$ *satisfies the following axioms. (In axioms C10—C17 below, composition* $(\circ)$ *is meant to be computed in* $\mathfrak{Ra}\,\mathfrak{A}$*, and* $\tau, \tau_i^\uparrow$ *etc. are the operations of* $\mathfrak{Rd}_{\mathsf{CA}_n^\uparrow, R^{\mathfrak{A}}}\,\mathfrak{Ra}\,\mathfrak{A}$*, cf. axiom C8 below.)*

(C0) $\langle A, \vee, -, \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{i,j\in n} \in \mathsf{CA}_n$

(C8) $\mathfrak{Ra}\,\mathfrak{A}$ *is a* QRA *with quasi-projections* p *and* q

(C9) $\mathsf{c}_0 \ldots \mathsf{c}_{n-3}\,\mathrm{suc}^{n-1} x \leq \mathrm{suc}^{n-1} x$

(C10) $1 \circ \tau(\mathrm{suc}^{n-1} x) = \mathrm{suc}^{n-1} x$

(C11) $\mathrm{suc}^{n-1}(1) = 1^{(n)}$

(C15) $\mathrm{suc}^{n-1}(x \vee y) = \mathrm{suc}^{n-1}(x) \vee \mathrm{suc}^{n-1}(y)$ *and* $\mathrm{suc}^{n-1}(-x) = -_{1^{(n)}}\,\mathrm{suc}^{n-1}(x)$

(C16) $\mathrm{suc}^{n-1}(\mathsf{c}_i^\uparrow(x)) = \tau_i^\uparrow(\mathrm{suc}^{n-1}(x))$ *and* $\mathrm{suc}^{n-1}(\mathsf{c}_i^\downarrow(x)) = \tau_i^\downarrow(\mathrm{suc}^{n-1}(x))$ *for all* $i < n$

(C17) $\mathrm{suc}^{n-1}(\mathsf{d}_{ij}) = \delta_{ij}$ *for all* $i, j < n$.

(C18) $x \leq \mathsf{c}_0 \ldots \mathsf{c}_{n-1}\,\mathrm{suc}\, x$

(C21) $\mathsf{c}_i^\uparrow x \vee \mathsf{c}_i^\downarrow x \leq \mathsf{c}_i\, x$

(C22) $\mathsf{c}_i^\uparrow$ *and* $\mathsf{c}_i^\downarrow$ *are normal operators*

Remark. If $n \geq 4$, then (C8) can probably be replaced by

(C8') $\mathsf{c}_{n-1}(R_{0,n-1} \wedge R_{1,n-1} \wedge -\mathsf{c}_{n-2}(-\mathsf{d}_{0,n-2} \wedge -\mathsf{d}_{1,n-2} \wedge R_{n-2,n-1})) = 1.$

□

Open problem 3.3. *Find axioms for* $\mathsf{CA}_n^\uparrow$ *that do not mention the derived operations (except perhaps the* c_i*s and* R*) but which have clear intuitive content.*

Theorem 3.4. *For all* $3 \leq n < \omega$, $\mathsf{CA}_n^\uparrow = \mathsf{RCA}_n^\uparrow$.

Proof. We leave the soundness proof (i.e. the proof of the inclusion $\supseteq$) to the reader.

First we note that $\mathsf{CA}_n^\uparrow$ is a discriminator variety (this follows easily from (C0), (C21) and (C22), using the fact that CA_n is a discriminator variety). So it is enough to represent the simple members of $\mathsf{CA}_n^\uparrow$. Now let $\mathfrak{A} \in \mathsf{CA}_n^\uparrow$ be simple. It follows from elementary CA-theory that $\mathfrak{B} = \mathfrak{Ra}\,\mathfrak{A}$ is also simple, so by (C8) and Lemma 2.5 $\mathfrak{C} = \mathfrak{Rd}_{\mathsf{CA}_n^\uparrow, R^{\mathfrak{A}}}\,\mathfrak{B} \in \mathsf{Cs}_n^\uparrow$. By (C9) and (C10), suc^{n-1} maps $\mathfrak{A}$ into $\mathfrak{C}$, and it is one-to-one because of (C18). The rest of the axioms (C11)—(C17) ensures that suc^{n-1} is an embedding. ■

Open problem 3.5. *Is* $\mathsf{RCA}_2^\uparrow$ *a finitely axiomatizable variety?*

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