The Class of Neat Reducts is Not Elementary

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Abstract . We solve a problem of Tarski, and his co-authors Andreka, Henkin, Monk, and Nemeti. We show that for any pair of ordinals $1 < \alpha < \beta$, the class of α-dimensional neat reducts of β-dimensional cylindric algebras is not elementary. The construction we use when α is infinite also settles a conjecture of Tarski in the affirmative.

§0. INTRODUCTION AND MAIN RESULTS

Subject

Neat reducts and related notions, like neat embeddings, play a central role in algebraic logic. One of the main representation methods used in the theory of cylindric algebras , and variants thereof, like polyadic algebras, is based on the Neat Embedding Theorem to be recalled below. (See also [HMT1] p.400 and [HMT2] Thm 3.2.10).

It is known (cf, for eg [S97] or [AGN77]) that the Neat Embedding Theorem, or the NET for short, proved by Henkin in the late fifties, is an algebraization of Henkin's celebrated proof of the completeness of first order logic.

The NET is also one of the earliest examples of what Hirsch and Hodkinson call "Step-by-step building representations in algebraic logic" (cf [HH97a]).

Ever since, variants of the NET have been applied successfully to prove (algebraically) completeness of other versions of quantifier logics (eg. Kieslers logics, cf [DM63]).

Other works addressing the notion of neat reducts, one way or another, include [A93],[S97] and [Sa98]. In [A93] and [S97], neat reducts are studied in connection with algebras of sentences of first order logic. In [Sa98] a NET is introduced (and proved) to demonstrate completeness of certain "finitary" fragments of Kieslers logics, that are also expansions of first order logic without equality. This provides a positive solution to one version of the so called finitization problem in algebraic logic. We refer the reader to [N96] for a survey on the finitization problem.

The connection of neat reducts to other metalogical properties like the Beth-definabilty property and omitting types will be discussed in section 6 below. This (and other) connection(s) are investigated more thoroughly in ([Pi71],[Ma77],[Ma78],[N83] [F92],[F98]), [ANS98] and [AS98].

History (Previous results)

Let $0 \leq \alpha < \beta$ be ordinals. Solving problem 2.11 in [HMT1], Nemeti proved in [N83], that the class $Nr_\alpha(CA_\beta)$, of α-dimensional neat reducts of β-dimensional cylindric algebras, though closed under homomorphic images, products and ultraproducts, is not a variety (i.e, not closed under forming subalgebras), iff $\alpha > 1$.

In II.8.6 of [HMTAN] (cf, p.266, therein), Andreka and Nemeti proved that the class $Nr_\alpha(CA_\beta)$ is not elementary for $\alpha = 2$ (i.e, not even closed under forming elementary subalgebras). [1]

Problem 18 on p.312 in [HMTAN] asks whether this result extends to higher dimensions and in [N83], Nemeti conjectures that it does. (cf, conjecture 8, p.408, therein).

Unsettled, this problem appears once again as problem 4.4 in [HMT2]. This paper is devoted to investigating this problem for various subclasses of cylindric algebras. More precisely the question we investigate is : For which ordinals $1 < \alpha < \beta$, and $L \subseteq CA_\beta$, is the class $Nr_\alpha L$ not elementary? Theorem 1 below formulates the answer for every pair $1 < \alpha < \beta$ of ordinals and various classes of cylindric algebras.

Basic notions and formulation of the main results

To formulate the main results, we review some basic notions. We try to use conventional notation. We introduce less usual notations at their first occurrence in the text.

ω denotes the least infinite ordinal, which is also the set of all finite ordinals. More generally an ordinal is the set of smaller ordinals. Cardinals, as usual, are identified with initial ordinals. If X, Y are sets then ${}^XY$ denotes the set

[1] Independently Roger Maddux [Ma78] showed that the class $Nr_\alpha(CA_\beta)$ is not closed under forming subalgebras for $\alpha = 3$ and $\beta > 4$. In [N83], the following is actually proved: For any pair of ordinals $1 < \alpha < \beta$ and any class L such that $Ws_\beta \subseteq L \subseteq CA_\beta$, the class $Nr_\alpha L$ is not closed under forming subalgebras, i.e is not a variety. When $\alpha = 2$, this result is strengthened in [HMTAN] to: The class $Nr_2 L$ is not closed under forming elementary subalgebras, hence not elementary. It is rather this (stronger) result that we generalize to higher dimensions (cf, Theorem 1). Here Ws_β stands for the class of weak set algebras of dimension β, to be recalled below.

of all functions from X into Y. Composition of two functions f and g, $f \circ g$ is defined so that $f \circ g(x) = f(g(x))$. Id_X, or simply Id when X is clear from context, denotes the identity function on X. $|X|$ denotes the cardinality of X, and $Sb(X)$ denotes the set of all subsets of X. The following notation will be used extensively: For a function f, and $X \subseteq domf$ (the domain of f), $f \upharpoonright X$ and $f(X)$ denote *the restriction of* f *to* X and *the image of* X *under* f, respectively.

Let K be a class of algebras. Then IK, HK, SK, and PK denote the classes of all isomorphic images, homomorphic images, subalgebras, and algebras isomorphic to direct products of members of K respectively. EqK, denotes the smallest variety containing K. That is $EqK = HSPK$. Given an algebra $\mathbf{A}$ and $X \subseteq \mathbf{A}$, then $Sg^{\mathbf{A}}X$ or simply SgX, when $\mathbf{A}$ is clear from context, denotes the subalgebra of $\mathbf{A}$ generated by X. Let $\mathbf{B}$ be an algebra having the same similarity type as $\mathbf{A}$. Then we write $f \in Ism(\mathbf{A}, \mathbf{B})$ to abbreviate that f is an isomorphism from $\mathbf{A}$ into $\mathbf{B}$. In boolean algebras (BA's), $\cup, \cap, -, 0$ and 1, stand for join, meet, complementation, least element and greatest element, respectively.

Let $\mathbf{A}$ be a cylindric algebra of dimension β, briefly a CA_β. Recall that $\mathbf{A}$ is of the form

$$\langle A, \cup, \cap, -, 0, 1, c_i, d_{ij} \rangle_{i,j \in \beta}$$

where $\langle A, \cup, \cap, -, 0, 1 \rangle$ is a boolean algebra, henceafter $Bl\mathbf{A}$, and for $i, j \in \beta$, c_i is a unary operation on A, and d_{ij} is a distinguished element of A. The class CA_β of all β dimensional cylindric algebras is a variety defined by a finite schema of equations (cf., [HMT1] Def.1.1.1) . We do not have to remember the specific form of these here. We recall from [HMT1], Def.2.6.28, a notion that prevails throughout this paper, that of of neat reducts:[2]

Let α and β be any ordinals with $\alpha \leq \beta$. Let $\mathbf{A} \in CA_\beta$. Then the *neat* $\alpha - $ *reduct of* $\mathbf{A}$, in symbols $\mathbf{N}r_\alpha\mathbf{A}$, is the following CA_α:

$\mathbf{N}r_\alpha\mathbf{A} = \langle Nr_\alpha A, \cup, \cap, -, c_i^A, d_{ij}^A \rangle_{i,j \in \alpha}$, where $Nr_\alpha A = \{x \in A : c_i^A x = x$, for all $\alpha \leq i < \beta\}$.

A subalgebra $\mathbf{B}$ of $\mathbf{N}r_\alpha\mathbf{A}$ is called a subneat reduct of $\mathbf{A}$. When further B generates $\mathbf{A}$, then $\mathbf{B}$ is called a generating subneat reduct of $\mathbf{A}$, or simply a generating subreduct of $\mathbf{A}$. For any class K of CA_β's, and $\alpha \leq \beta$, $Nr_\alpha K$ denotes $\{\mathbf{N}r_\alpha\mathbf{A} : \mathbf{A} \in K\}$.

Let α be an ordinal. Then Cs_α and Ws_α stand for the classes of cylindric set algebras and weak cylindric set algebras, of dimension α, respectively. (cf. [HMT2], Def.3.1.1). That is $\mathbf{A} \in Cs_\alpha$, (Ws_α) if A is a boolean field of sets with unit a (weak) cartesian space of dimension α. A cartesian space of dimension α is of the form ${}^\alpha U$, for some set U, while a weak one of dimension α is of the

[2]The notion of neat reducts was introduced by Henkin (cf [HMT1], footnote on p. 401). For a detailed and comprehensive treatment of the notion of neat reducts and related notions like neat embeddings we refer the reader to [HMT1] p.401-428.

form ${}^\alpha U^{(p)} = \{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$, for some $p \in {}^\alpha U$. In addition to this, A of course, has to be closed under the usual concrete interpretation of the cylindric operations c_i and diagonals d_{ij}, for all $i, j \in \alpha$. Note that when α is finite we have $Cs_\alpha = Ws_\alpha$. In any case, it is known (see e.g [N96]), that $SPCs_\alpha = SPWs_\alpha$, and that $SPCs_\alpha$ is a variety, i.e $SPCs_\alpha = EqCs_\alpha$. $EqCs_\alpha$ (which is the same as $EqWs_\alpha$), referred to as the class of representable CA_α's, is sometimes denoted by RCA_α. Every $\mathbf{A} \in RCA_\alpha, \alpha \geq 2$, is isomorphic to a generalized cylindric set algebra (a Gs_α): A Gs_α is a boolean field of sets with unit a disjoint union of cartesian spaces each of dimension α. A fundamental theorem of CA theory, the so called Neat Embedding Theorem (NET), states that $RCA_\alpha = SNr_\alpha CA_{\alpha+\omega}$. [3] The NET is known to hold for the most well known variants of CA's. These include: Pinter's substitution algebras and Halmos' quasi-polyadic algebras. For a survey on the NET and algebras other than CA's to which variations of the NET apply the reader can consult [Sa98].

In contrast to BA's, it is known that not every CA_α $\alpha > 1$ is representable (as algebras of *true* α ary relations).[4] We hasten to remark that all algebras considered herein though are weak set algebras, hence representable .

We are now ready to formulate the main results. Recall that a class K is elementary (or first order axiomatizable), if and only if it is closed under ultraproducts and elementary equivalence.

Theorem 1 (Main Theorem) . *For $K \in \{Ws, Cs, Gs, CA\}$ and $1 < \alpha < \beta$, the class $Nr_\alpha(K_\beta)$ is not elementary.*

Theorem 1 solves problem 18 on page 312 in [HMTAN] and 4.4 in [HMT2]. Theorem 1 also confirms a conjecture of Nemeti. As further conjectured by Nemeti in [N83] p.408 , Theorem 1 can be, and is indeed proved herein by methods similar to those used in the proof of II.8.6 in [HMTAN] and statement 1 in [N83]. Further refined by Andreka, such methods are more commonly referred to in the literature as Andreka's method of splitting elements, or simply "Andreka's splitting"[An98]. [5]

Using Andreka's splitting, we shall prove slightly more than what is stated in Theorem 1, namely, for any pair of ordinals $1 < \alpha < \beta$ and any L such

[3] Algebras in the class $SNr_\alpha CA_{\alpha+\omega}$, are referred to in [HMT1] as algebras having the neat embedding property. These, according to the NET, are isomorphic to Gs_α's. Ferenczi ([F92]- [F98]) investigates the question : For which proper superclasses K of $CA_{\alpha+\omega}$ is it still true that $IGs_\alpha = SNr_\alpha K$. This has implications in proof theory.

[4] Henkin showed that RCA_2 can be defined by adjoining two equations to the equations axiomatizing CA_2, cf. [HMT2] 3.2.65. In sharp contrast, when $\alpha > 2$ RCA_α is not even axiomatizable by any set of universal formulas using only finitely many variables. The latter, a result of Andreka [An98], considerably sharpens Monk's celebrated result [M69] of non finite schema axiomatizability of RCA_α's when $\alpha > 2$.

[5] The roots of this method, though, can be traced back to Henkin's method of splitting elements introduced on page 390 in [HMT1].

that $Ws_\beta \subseteq L \subseteq CA_\beta$, the class $Nr_\alpha L$ is not closed under forming elementary subalgebras, hence not elementary.

Theorem 2 . *Let $\beta > \alpha \geq \omega$. Then there exist $\mathbf{B} \in Ws_\alpha$, and $\mathbf{C} \in Ws_\beta$ such that B is a generating subreduct of $\mathbf{C}$, and $B \neq Nr_\alpha C$.*

Theorem 2, is actually an immediate consequence of Theorem 1 (when α is infinite). Theorem 2 contrasts Theorem 2.6.67 in [HMT1]. The latter states that if $\mathbf{B}$ is a *dimension complemented* CA_α, briefly a Dc_α[6], and B happens to be a generating subreduct of $\mathbf{C} \in CA_\beta$, $\beta > \alpha$, then $B = Nr_\alpha C$. In [HMT1], at the top of p.426, it was also stated that the class Dc_α in the hypothesis of Theorem 2.6.67 cannot be replaced by arbitrary CA_α's even if they are restricted to representable ones. The proof was postponed to appear in part 2. This statement appears again as item (v) in the introduction of [HMT2], this time among results of Tarski the proofs of which could not be reconstructed. Theorem 2, thus confirms what seems to have been a conjecture of Tarski. [7]

Organization of the paper

First we will give the proof of Theorem 1 for α finite, because in this case we use a fairly simpler construction (than that used in the infinite case) and so we believe that the idea of the proof will be easier to grasp. Sections 1-4 are devoted to this purpose. In Section 5 we give a separate proof to Theorem 1 with the necessary modifications to adapt the infinite case . Apparently more involved, the idea, in essence is the same as that of the finite dimensional case. We also obtain, as a byproduct, Theorem 2. In Section 6, a corollary dealing with certain reducts of CA's (namely Pinters substitution algebras), together with a conjecture dealing wih other variants of CA's (like Halmos qausipolyadic algebras) are stated. A few related results concerning neat reducts are also discussed in some detail.

§1. THE FINITE CASE

Throughout sections $1-4$, R denotes a fixed uncountable set, and $n = \{0, \cdots, n-1\}$ denotes a fixed finite ordinal with $1 < n$.

[6] $\mathbf{B} \in Dc_\alpha$ if for all $x \in B$, there is an $i \in \alpha$ such that $c_i x = x$. Dc_α ($\alpha \geq \omega$), contains properly the class of locally finite cylindric algebras of dimension α (Lf_α). The class Lf_α, introduced by Tarski, is the algebraic counterpart of first order logic with its provability and semantical notions. For more on these connections we refer the reader to the introduction of [HMT1] and [HMT2] sec 4.3.

[7] In the context of proving amalgamation results, Pigozzi conjectures in [Pi71], at the top of p. 331, that Theorem 2.6.67 does not even generalize to the classes of semisimple cylindric algebras (Sc_α) and the so called diagonal cylindric algebras (Di_α), respectively. (cf, footnote on p.416 in [HMT1]). Since $Ws_\alpha \subseteq Sc_\alpha \subseteq Di_\alpha$ for $\alpha \geq \omega$ (cf. [HMT2] Thm 2.6.50), Theorem 2 settles Pigozzi's conjecture as well.

Notation . *Let $k < \omega$. Then $S(n,k)$ denotes the set of sequences $\langle i_0, \cdots, i_{n-1}\rangle$ such that $i_0 \leq i_1 \cdots \leq i_{n-1} = k$. $Cof^+(R)$ denotes the set of all nonempty finite or cofinite subsets of R, i.e.*

$$Cof^+(R) = \{X \subseteq R : X \text{ is non empty and finite, or } R - X \text{ is finite}\}.$$

Let C_r be an n-ary relation symbol for all $r \in R$. For any $X \subseteq R$, X finite we define the formulas:

$$\eta(X) = \vee\{C_r(x_0, \cdots, x_{n-1}) : r \in X\}, \quad \text{and}$$

$$\eta(R - X) = \wedge\{\neg C_r(x_0, \cdots, x_{n-1}) : r \in X\}.$$

Lemma 1 . *There are a set W and n-ary relations $C_r \subseteq {}^nW$ on W for all $r \in R$ such that conditions (i)-(iv) below hold:*

(i) $C_r(w_0, \cdots, w_{n-1})$ implies $C_r(w_{\pi(0)}, \cdots, w_{\pi(n-1)})$ for all $r \in R$, $w_0, \cdots, w_{n-1} \in W$ and permutation π of n.

(ii) For all $w_0, \cdots, w_{n-2} \in W$ and $r \in R$ there is a $w_{n-1} \in W - \{w_0, \cdots, w_{n-2}\}$ such that $C_r(w_0, \cdots, w_{n-1})$.

(iii) For all $k \in \omega$, for all $w_0, \cdots, w_{k-1} \in W$ and for any function $f : S(n,k) \to Cof^+(R)$, there is a $w_k \in W - \{w_0, \cdots, w_{k-1}\}$ such that:

$$\wedge\{\eta(f(i))[w_{i_0}, \cdots, w_{i_{n-1}}] : \ i \in S(n,k)\}.$$

(iv) The C_r's are pairwise disjoint.

Proof. We shall construct the structure $\langle W, C_r\rangle_{r\in R}$ by a routine step by step fashion. We note that condition (ii) follows from (iii). Often, however, we will only need (and refer to) the weaker condition (ii), hence the redundancy in the formulation of Lemma 1. Let

$$Q = \cup\{{}^k(|R|) \times {}^{S(n,k)}\, Cof^+(R) : k < \omega\}$$

.

Then $|Q| = |R| = \mu$, say. We will construct a set W with cardinality μ. Q is intended to represent all the instances of condition (iii), as follows: An element of Q is of the form $\langle \alpha_0, \cdots, \alpha_{k-1}, f\rangle$, where $\alpha_0, \cdots, \alpha_{k-1} < \mu$ and $f : S(n,k) \to Cof^+(R)$. This represents the instance of (iii) where we take $k, w_{\alpha_0}, \cdots, w_{\alpha_{k-1}}, f$ as the concrete values of the quantified items in (iii). Let ρ be an enumeration of Q such that :

For all $l < \mu$, for all $q \in Q$, there exists j, with $l < j < \mu$, such that $\rho(j) = q$. Such a ρ clearly exists. Fix a well ordering $\prec$ of R. Let $l < \mu$, and suppose that for all $i < l$ we have already defined the element w_i, and the n-ary relation $C_r^i \subseteq {}^n W_i$, where $W_i = \{w_k : k < i\}$. In the l'th step we will make the $\rho(l)$'th instance of (iii) true. Assume that $\rho(l) = \langle \alpha_0, \cdots, \alpha_{k-1}, f \rangle$. Then $\alpha_0, \cdots, \alpha_{k-1} < \mu$ and $f : S(n,k) \to Cof^+R$, for some k. ($k = 0$ is allowed, too). Let w_l be an element not in $\cup\{W_i : i < l\}$. If there exists $i < k$ such that $l \leq \alpha_i$, then for all $r \in R$, we define $C_r^l = \cup\{C_r^i : i < l\}$. Else, $l > \alpha_i$ for all $i < k$. In this case, let $v_0 = w_{\alpha_0}, \cdots,\ v_{k-1} = w_{\alpha_{k-1}}$ and $v_k = w_l$. For all $r \in R$ we define:

$$X_r^l = \{\langle v_{i_0}, \cdots, v_{i_{n-1}} \rangle \ : \ i_0 \leq \cdots \leq i_{n-1} = k \text{ and } r \text{ is the } \prec\text{-least element of } f(\langle i_0, \cdots, i_{n-1} \rangle)\}$$

and

$$C_r^l = \cup\{C_r^i \cup \{\langle s_{\pi(0)}, \cdots, s_{\pi(n-1)} \rangle : s \in X_r^l, \pi \text{ is a permutation of } n\} \ : \ i < l\}.$$

Finally set

$$W = \cup\{W_l : l < \mu\} \quad \text{and } C_r = \cup\{C_r^l : l < \mu\}$$

Now we are going to check that the structure $\langle W, C_r \rangle_{r \in R}$, so defined, satisfies conditions (i)-(iv). To this end, for any $l < \mu$ and $r \in R$ let

$$Y_r^l = \{\langle s_{\pi(0)}, \cdots, s_{\pi(n-1)} \rangle : s \in X_r^l \text{ and } \pi \text{ is a permutation on } n\}$$

and

$$D_r^l = \cup\{C_r^j : j < l\}.$$

Then for all $r \in R$ we have $C_r^l = Y_r^l \cup D_r^l$. Also, the following are not difficult to check:

(1) $w_l \in Rgs$ (the range of s) if $s \in Y_r^l$, and $w_l \notin Rgs$ if $s \in D_r^l$.

(2) $C_r^j \subseteq C_r^m$ if $j \leq m < \mu$.

It is easy to show by induction on $l < \mu$ that for all $r \in R$, C_r^l is symmetric, thus C_r is symmetric, and so (i) holds.

Let $r, p \in R$ be distinct. We want to show by induction on $l < \mu$ that C_r^l and C_p^l are disjoint. Now D_r^l and D_p^l are disjoint by the induction hypothesis and by (2). By (1), it is therefore, enough to show that Y_r^l and Y_p^l are disjoint. Assume $s \in Y_r^l$, and let $\rho(l) = \langle \alpha_0, \cdots, \alpha_{k-1}, f \rangle$ and $v = \langle v_0, \cdots, v_k \rangle = \langle w_{\alpha_0}, \cdots, w_{\alpha_{k-1}}, w_l \rangle$. Then v is one to one and there are a unique $i \in S(n,k)$ and permutation π of n such that $s = \langle z_{\pi(0)}, \cdots, z_{\pi(n-1)} \rangle$, where $z = \langle v_{i_0}, \cdots, v_{i_{n-1}} \rangle \in X_r^l$. Thus r is the $\prec$- least element of $f(i)$, by $z \in X_r^l$. Since $p \neq r$, we get that $z \notin X_p^l$, and so $z \notin Y_p^l$. We have shown that C_r^l and C_p^l are disjoint. By (2) condition (iv) holds. Finally we check condition(s) (iii) (and (ii)): Let $k < \omega$, $w_{\alpha_0}, \cdots, w_{\alpha_{k-1}} \in W$ and $f : S(n,k) \to Cof^+R$. Let $l < \mu$ be such that $\rho(l) = \langle \alpha_0, \cdots, \alpha_{k-1}, f \rangle$ and $l > \alpha_0, \cdots, l > \alpha_{k-1}$. Such an l exists by the properties of ρ. Then it is not difficult to check that we constructed the w_l so that it satisfies $\phi = \wedge \{\eta(f(i))(w_{\alpha_{i_0}}, w_{\alpha_{i_1}}, \cdots, w_{\alpha_{i_{n-2}}}, x_{i_{n-1}}) : i \in S(n,k)\}$ in $\langle W_l, C_r^l \rangle_{r \in R}$. By (1) and (2) we have $C_r^l = {}^nW_l \cap C_r$ and so ϕ is satisfied in $\langle W, C_r \rangle_{r\in R}$, as well. By this the proof of Lemma 1 is complete. ■

Remarks .

(a) Notice that by the construction of W, $|W| = |R|$. In particular, W is also an uncountable set.

(b) We have excluded the empty set from Cof^+R in order that (iii) can be satisfied, because $\eta(\emptyset)$ is false for any relations C_r. Notice that condition (iii) in Lemma 1, is a "saturation condition" on W. It will be used in the proof of fact 2.1 below, to show that the structure $\langle W, C_r \rangle_{r\in R}$ admits elimination of quantifiers in a rather strong sense.

(c) We note that condition (iii) in Lemma 1, can be proved using a game theoretic approach, along the lines adopted in [HH97a].

Notation . *Let $\mathbf{W} = \langle W, C_r \rangle_{r\in R}$ with $C_r \subseteq {}^nW$, be the relational structure described in Lemma 1. Let $U = W \times \{0\} \cup W \times \{1\} \cdots \cup W \times \{n-1\}$. U is simply n disjoint copies of W. Let β be an ordinal, then $\mathbf{C}(\beta) = \langle Sb({}^\beta U), \cup, \cap, -, c_i, d_{ij} \rangle_{i,j\in\beta}$, denotes the full Cs_β, with unit ${}^\beta U$. S_n denotes the set of permutations on n . Let f be a function, then $f = \langle f_x : x \in domf \rangle$ and $ker(f) = \{(x,y) \in {}^2domf : \ f_x = f_y\}$.*

Defining our algebras

Let $u = \langle u_0, \cdots, u_{n-1} \rangle \in S_n$. Let $r \in R$. Then $p(u,r)$ is the following n-ary relation on U:

$$\{\langle (a_0, u_0), \cdots, (a_{n-1}, u_{n-1}) \rangle \in {}^nU : \langle a_0, \cdots, a_{n-1} \rangle \in C_r\}.$$

Intuitively, $p(u,r)$ is the relation C_r "diagonally embedded" in U, and then permuted along u.

$$P(u) = \{p(u,r) : r \in R\}.$$

Let N be a fixed countably infinite subset of R. [8] That is $N \subseteq R$ and $|N| = \omega$. Then $P_\omega(u) = \{p(u,r) : r \in N\}$.

[8]We point out that countability of N is not essential. In fact any infinite subset N of R with $|N| < |R|$ will do the job, just as well.

Now we define $\mathbf{A}(n) \in Cs_n$, and $\mathbf{B}(n) \subseteq \mathbf{A}(n)$ as follows:

$$\mathbf{A}(n) = Sg^{\mathbf{C}(n)}(\cup\{P(u) : u \in S_n\}).$$

$$\mathbf{B}(n) = Sg^{\mathbf{A}(n)}(\cup\{P(u) : u \in S_n - \{\langle 0, \cdots, n-1\rangle\} \cup P_\omega(\langle 0, \cdots, n-1\rangle)).$$

Plan of the proof

We will show that:

1. $\mathbf{A}(n) \in INr_nWs_\beta$, for all $\beta > n$.

   See fact 2.4 below.

2. $\mathbf{A}(n)$ is elementary equivalent to $\mathbf{B}(n)$.

   See fact 3.3 below.

3. $\mathbf{B}(n) \notin Nr_nCA_\beta$, for all $\beta > n$.

   See fact 4.3 below.

From (1),(2) and (3), Theorem 1 for $\alpha = n$ and $\beta > \alpha$ will immediately follow, since $Ws_\beta \subseteq Cs_\beta \subseteq Gs_\beta \subseteq CA_\beta$. Actually (1),(2) and (3) will show, as previously noted, that for any $\beta > n$ and any L such that $Ws_\beta \subseteq L \subseteq CA_\beta$, the class Nr_nL is not closed under forming elementary subalgebras, hence not elementary.

§2. $A(n)$ IS A NEAT REDUCT

In this section, we will show that $\mathbf{A}(n) \in INr_n(Ws_\beta)$, for all $\beta > n$. This will be done in two steps. First we will show that $\mathbf{A}(n) \in \cap_{n \leq k < \omega} INr_nCs_k$. Then we will show that $\mathbf{A}(n) \in INr_nWs_\beta$, for infinite β.

Convention . *As is standard practice, we do not distinguish between an algebra* $\mathbf{A}$*, and its universe* A*.*

Notation . L_ω *denotes the (usual) first order language of* $\mathbf{W}$*, the subscript* ω *indicating that all* ω *variables are available.* Fm^{L_ω} *stands for the set of all* L_ω *formulas. For* $\phi \in Fm^{L_\omega}$*,* $var(\phi)$ *denotes the set of all variables occurring in* ϕ*.*

To show that $\mathbf{A}(n) \in \cap_{n \leq k < \omega} INr_nCs_k$, we need:

Some more Definitions.

Let $n \le k < \omega, u = \langle u_0, u_1, \cdots, u_{k-1}\rangle \in {}^k n$ and $\phi \in Fm^{L_\omega}$ with $var(\phi) \subseteq \{x_0, \cdots, x_{k-1}\}$. Then $E(u,\phi)$ is the following k-ary relation on U:

$$\{\langle (a_0,u_0), \cdots, (a_{k-1},u_{k-1})\rangle \in {}^k U : \mathbf{W} \models \phi[a_0, \cdots, a_{k-1}]\}.$$

Notice that when $k = n$ and ϕ is $C_r(x_0, \cdots, x_{n-1})$, then $E(u, C_r(x_0, \cdots, x_{n-1}))$ is the same as $p(u,r)$.
$Eq(u)$ stands for the set $\{x_i = x_j,\ x_i \neq x_j\ : (i,j) \in ker(u)\}$. We identify the formulas $x_i = x_i$, and $x_i \neq x_i$ with the propositional constants T and F, standing for "True" and "False", respectively. In particular, $\{T, F\} \subseteq Eq(u)$, for any $u \in {}^k n$. Let $u \in {}^k n$. Then

$$\begin{aligned} F_k(u) = \{C_r(x_{i_0}, \cdots, x_{i_{n-1}}),\ \ \neg\ \ C_r(x_{i_0}, \cdots, x_{i_{n-1}}) : r \in R,\ i \in {}^n k, \\ \text{and } |\{u_{i_0}, \cdots, u_{i_{n-1}}\}| = n\} \cup Eq(u). \end{aligned}$$

Forming finite conjunctions of formulas in $F_k(u)$, set

$$F_k(u)^* = \{\wedge J : J \subseteq_\omega F_k(u)\}.$$

Here, and elsewhere, $\subseteq_\omega$ abbreviates "finite subset".

Forming finite disjunctions of formulas in $F_k(u)^*$, set

$$F_k(u)^{**} = \{\vee J : J \subseteq_\omega F_k(u)^*\}.$$

Now let

$$G(k) = \{E(u,\phi) : u \in {}^k n,\ \phi \in F_k(u)^{**}\} \quad \text{and}$$
$$G(k)^* = \{\cup J : J \subseteq_\omega G(k)\}.$$

We will show that $G(k)^*$ is (the universe of) a Cs_k and that $A(n) \cong Nr_n G(k)^*$ for all $n \le k < \omega$. Then we form a Ws_β for infinite β as a "limit" of the $G(k)^*$'s ($n \le k < \omega$); $A(n)$ will be the *neat* $n - reduct$ of this limit.

Fact 2.1. *$G(k)^* \in Cs_k$, for all $n \le k < \omega$.*

To prove fact 2.1 and fact 2.3 below, we shall need (and prove) the following rather easy consequence of Lemma 1 :

(*) Let $u \in {}^k n$ and $\phi \in F_k(u)^*$. Then for all $i < k$ and all $v \in {}^k n$ that differs from u in at most the ith place, there exists $\psi \in F_k(v)^*$, such that $x_i \notin var(\psi)$ and $\mathbf{W} \models \exists x_i \phi \equiv \psi$. Here, and elsewhere $\equiv$ abbreviates "(semantically) equivalent to".

Notation . *Let $\phi \in Fm^{L_\omega}$, and $i, j \in \omega$. Then $\phi[i|j]$ stands for the formula obtained from ϕ, by replacing all (free and bound) ocurrances of x_i in ϕ by x_j.*

Let $v, u \in {}^k n$. Then $v \equiv_i u$, abbreviates "v differs from u in at most the ith place".

Proof of (*). Fix $u \in {}^k n$, $\phi \in F_k(u)^*$, $i < k$, and $v \equiv_i u$. Then $\phi = \wedge J$, for some $J \subseteq_\omega F_k(u)$. Let $J^+ = \{\psi \in J : x_i \in var(\psi)\}$ and $J^- = J - J^+$. Then $\wedge J^- \in F_k(v)^*$, $x_i \notin var(\wedge J^-)$, and $\mathbf{W} \models \exists x_i \phi \equiv \exists x_i(\wedge J^+) \wedge (\wedge J^-)$.

It suffices, therefore, to show that $\mathbf{W} \models \exists x_i(\wedge J^+) \equiv \psi$ for some $\psi \in F_k(v)^*$.
Case (a): $x_i = x_j \notin J^+$ and $x_j = x_i \notin J^+$ for all $j < k$.
For $j \in S(n,k)$ put

$$\pi(j) = \{r \in R : \ C_r(x_{y_0}, \cdots, x_{y_{n-1}}) \in J^+ \text{ and } \{y_0, \cdots, y_{n-1}\} - \{i\} = \{j_0, \cdots, j_{n-2}\}\}$$

and

$$\tau(j) = \{r \in R : \neg C_r(x_{y_0}, \cdots, x_{y_{n-1}}) \in J^+ \text{ and } \{y_0, \cdots, y_{n-1}\} - \{i\} = \{j_0, \cdots, j_{n-2}\}\}.$$

If $x_l \neq x_l \in J^+$ for some $l < k$, or $\pi(j) \cap \tau(j) \neq \emptyset$ for some $j \in S(n,k)$, or $|\pi(j)| \geq 2$ for some $j \in S(n,k)$, then $\mathbf{W} \models \wedge J^+ \equiv F$ and $\mathbf{W} \models \exists x_i(\wedge J^+) \equiv F$. Assume therefore that for all $l < k$, $x_l \neq x_l \notin J^+$, that $|\pi(j)| < 2$ and that $\pi(j) \cap \tau(j) = \emptyset$, for all $j \in S(n,k)$. Let us define

$$f : S(n,k) \to Cof^+ R \quad \text{as follows: } f(j) = \begin{cases} \pi(j) & \text{if } \pi(j) \text{ is nonempty,} \\ R - \tau(j) & \text{otherwise} \end{cases}$$

Then applying Lemma 1 (iii) to f so defined, amounts to getting for any $w_0, \cdots, w_{k-1} \in W$ a new element, i.e $w_k \in W - \{w_0, \cdots, w_{k-1}\}$, such that

$$\mathbf{W} \models ((\wedge J^+)[i|k])[w_0, \cdots, w_k], i.e. \mathbf{W} \models \exists x_k((\wedge J^+)[i|k]) \equiv T.$$

But since x_k does not occur in $\wedge J^+$, we have $\mathbf{W} \models \exists x_k((\wedge J^+)[i|k])) \equiv \exists x_i(\wedge J^+) \equiv T$. Since $\{T, F\} \subseteq F_k(v)^*$, and x_i does not occur in either of them, we are done.
Case (b): there is an $m < k$ such that $x_i = x_m \in J^+$ or $x_m = x_i \in J^+$.
We can and will assume that $x_i = x_m \in J^+$ and that $i \neq m$. Thus $\wedge J^+$ is of the form $(\wedge I) \wedge x_i = x_m$.
Claim. $(\wedge I)[i|m] \in F_k(v)^*$, from which our conclusion readily follows, since $\mathbf{W} \models \exists x_i(\wedge J^+) \equiv (\wedge I)[i|m]$.

Proof of Claim: Let $\phi \in I$
Subcase (a): ϕ is of the form $x_j = x_i$, where j, i and m are pairwise distinct. Then $v(j) = v(m)$ because $u(i) = u(j) = u(m)$, by $(i,j), (j,m) \in ker(u)$ and since $u \equiv_i v$. By definition, we then have $x_j = x_m \in F_k(v)$.
Subcase (b): ϕ is of the form $D_r(x_{y_0}, \cdots, x_{y_{n-1}})$, where $D_r \in \{C_r, \neg C_r\}$. By definition of $F_k(u)$, we have $|\{u_{y_0}, \cdots, u_{y_{n-1}}\}| = n$. Assume that $y_l = i$, with $0 \leq l < n$. Then $|\{u_{y_0}, .u_{y_l}, \cdots, u_{y_{n-1}}\}| = |\{u_{y_0}, .u_i, \cdots, u_{y_{n-1}}\}| = n$. By

$u(i) = u(m)$ (since $(i, m) \in ker(u)$), and $v \equiv_i u$, we have $|\{u_{y_0}, .u_m, \cdots, u_{y_{n-1}}\}| = |\{v_{y_0}, .v_m, \cdots, v_{y_{m-1}}\}| = n$. But this implies that $D_r(x_{y_0}, \cdots, x_{y_{n-1}})[i|m] \in F_k(v)$. From the subcases (a) and (b) we get $(\wedge I)[i|m] \in F_k(v)^*$. ■

Let everything be as in (*) and suppose further, that $l \leq k$. For brevity let $v \in {}^l n$ denote $u \upharpoonright l$. Then an easy consequence of (*) is:

(**) There exists $\psi \in F_l(v)^*$, such that

$$\mathbf{W} \models \exists x_l \cdots \exists x_{k-1} \phi \equiv \psi.$$

Proof of (**). Applying (*) successively we obtain ψ in $F_k(u)^*$, such that $\{x_l, \cdots, x_{k-l}\} \cap var(\psi) = \emptyset$ and $\mathbf{W} \models \exists x_l \cdots \exists x_{k-l} \phi \equiv \psi$. By noting that $F_l(v)^* = \{\phi \in F_k(u)^* : \ var(\phi) \subseteq \{x_0, \cdots, x_{l-1}\}\}$ we get the desired. ■

Proof of fact 2.1.

(1) $G(k)^*$ is closed under the boolean set operations.

It is enough to show that for all $g_1, g_2 \in G(k)$, $g_1 \cap g_2$ and ${}^k U - g_1$ are in $G(k)^*$. To this end, let $u, v \in {}^k n$, $\phi \in F_k(v)^{**}$ and $\psi \in F_k(u)^{**}$. Then as easily checked $E(u, \phi) \cap E(v, \psi) = E(u, \phi \wedge \psi)$ if $u = v$ and is zero otherwise, while ${}^k U - E(u, \phi) = \cup\{E(v, T) : v \in {}^k n - \{u\}\} \cup E(u, \neg\phi)$. From the definition of $G(k)^*$, and by noting that for $u \in {}^k n$, ϕ and $\psi \in F_k(u)^{**}$, there exist η and $\theta \in F_k(u)^{**}$, such that $\mathbf{W} \models \phi \wedge \psi \equiv \eta$ and $\mathbf{W} \models \neg\phi \equiv \theta$, we get the desired.

(2) $G(k)^*$ contains the diagonals.

Let $i < j < k$. Then $d_{ij} = \cup\{E(u, x_i = x_j) : u \in {}^k n, u_i = u_j\}$ and so, by definition of $F_k(u)$, $d_{ij} \in G(k)^*$.

(3) $G(k)^*$ is closed under cylindrifications.

Let $g \in G(k)^*$, and $i < k$. We will show that $c_i g \in G(k)^*$. Since the $c_i' s(\exists)$ distribute(s) over joins $(\vee)$, we can assume that $g = E(u, \phi)$, with $u \in {}^k n$ and $\phi \in F_k(u)^*$. Now $c_i g = c_i E(u, \phi) = \cup\{E(v, \exists x_i \phi) : \ v \equiv_i u\}$. By (*), we get that $c_i g \in G(k)^*$.

■

Fact 2.2. $A(n) = G(n)^*$.

Proof. Since $P = \cup\{P(u) : u \in S_n\} \subseteq G(n)^*$, P generates $A(n)$ and $G(n)^* \in Cs_n$, we have $A(n) \subseteq G(n)^*$. For the other inclusion, it is obviously enough to show that for any $u \in {}^n n$ and $\phi \in F_n(u)$, we have $E(u, \phi) \in A(n)$. We shall start by showing that $E(u, T) \in A(n)$, for all $u \in {}^n n$.

Case (a): $u \in S_n$, i.e u is one to one.

Let $r \in R$. Then by Lemma 1(ii) we have

$$\cap_{i<n}\{c_i(E(u, C_r(x_0, \cdots, x_{n-1})))\} = \cup\{E(v, T) : \ v \equiv_i u, \text{ for all } i < n\} = E(u, T).$$

Case (b): u is not one to one.
Suppose then that $|Rgu| = l$, say with $l < n$. Let $y \in {}^l n$, be such that $y_0 < \cdots < y_{l-1}$ and $\{u_{y(0)}, \cdots, u_{y(l-1)}\} = Rgu$. For $i < l$, put $m_{y(i)} = \min u^{-1}(u_{y(i)})$, where min is short for minimum. Choose $v \in S_n$ such that $v(m_{y(i)}) = u_{y(i)}$. Such a v, clearly, exists. Let $J = n - \{m_{y(0)}, \cdots, m_{y(l-1)}\}$; suppose that $J = \{j_0, \cdots, j_{n-(l+1)}\}$ with $j_0 < \cdots < j_{n-(l+1)}$.
For $i, l \in n$, set $t_i^l x = c_l x \cap d_{il}$. A straightforward computation yields

$$t^{j(n-(l+1))}_{u_{j(n-(l+1))}} \circ \cdots \circ t^{j(0)}_{u_{j(0)}} E(v, T) = E(u, T).$$

From case (a), it follows that $E(u, T) \in A(n)$.
Now let $u \in {}^n n$, and suppose that $(i, j) \in ker(u)$ with $i < j$. Then

$$E(u, x_i = x_j) = E(u, T) \cap d_{ij} \in A(n).$$

Also, for all $u, v \in S_n$, we have by Lemma 1 (i) that

$$E(u, C_r(x_0, \cdots, x_{n-1})) = E(u, C_r(x_{v(0)}, \cdots, x_{v(n-1)})) = p(u, r).$$

Finally by noting that $E(u, \neg\phi) = E(u, T) - E(u, \phi)$ for all $u \in {}^n n$, and $\phi \in F_n(u)$; we get the desired. ■

Fact 2.3. *For $n \leq l \leq k < \omega$, we have $G(l)^* \cong Nr_l G(k)^*$. In particular $A(n) \in \cap_{n \leq k < \omega} INr_n(Cs_k)$.*

Proof. Let $n \leq l < k$. Recall that $C(k)$ denote the full Cs_k with unit ${}^k U$. Let $i(l, k)$ be the function with domain $G(l)^*$ such that

$$i(l, k)a = \{t \in {}^k U : (t \upharpoonright l) \in a\}.$$

Then $i(l, k) : G(l)^* \to C(k)$. Also, by [HMT2] 3.1.122, we have $i(l, k) \in Ism(G(l)^*, Nr_l C(k))$.
Now fix $n \leq l < k < \omega$. We will show that $i(l, k)G(l)^* \subseteq G(k)^*$. To this end let $v \in {}^l n$, and $\phi \in F_l(v)$. Then $i(l, k)E(v, \phi) = \cup\{E(u, \phi) : u \in {}^k n$, and $u \upharpoonright l = v\}$. The latter is in $G(k)^*$, because $F_l(v)^{**} \subseteq F_k(u)^{**}$, whenever $v = u \upharpoonright l$. But $G(k)^* \in Cs_k$, and so $i(l, k)E(v, \phi) \in G(k)^*$ for any $v \in {}^l n$ and $\phi \in F_l(v)^{**}$, whence the desired. By $Nr_l G(k)^* = G(k)^* \cap Nr_l C(k)$, we get $i(l, k) \in Ism(G(l)^*, Nr_l G(k)^*)$.
We will now show that $i(l, k)$ is actually onto $Nr_l G(k)^*$, i.e that $i(l, k)G(l)^* = Nr_l G(k)^*$. Since the c_i's are additive, it suffices to show that for all $g \in G(k)$, there exists $a \in G(l)^*$ such that $i(l, k)a = c_l c_{l+1} \cdots c_{k-1} g$. So let $g \in G(k)$ be as indicated above. Suppose that $g = E(u, \phi)$ say, with $u \in {}^k n$ and $\phi \in F_k(u)^{**}$. Suppose further, that $\phi = \phi_0 \vee \cdots \phi_m$, where $\phi_j \in F_k(u)^*$, for $0 \leq j \leq m$. For brevity, let v ($\in {}^l n$) denote $u \upharpoonright l$. By (**), for each j, with $0 \leq j \leq m$, there exists $\psi_j \in F_l(v)^*$ such that $i(l, k)E(v, \psi_j) = c_l c_{l+1} \cdots c_{k-1} E(u, \phi_j)$.

Now

$$c_l c_{l+1} \cdots c_{k-1} E(u, \phi) = c_l c_{l+1} \cdots c_{k-1} E(u, \phi_0 \vee \cdots \phi_m)$$
$$= c_l c_{l+1} \cdots c_{k-1} E(u, \phi_0) \cup \cdots \cup c_l c_{l+1} \cdots c_{k-1} E(u, \phi_m)$$
$$= i(l,k)(E(v, \psi_0) \cup \cdots \cup i(l,k)E(v, \psi_m)$$
$$= i(l,k)(E(v, \psi_0 \vee \cdots \psi_m).$$

Since by definition $\psi_0 \vee \cdots \vee \psi_l \in F_k(v)^{**}$, we are done. Taking $l = n$, we get, from fact 2.2, that $i(n,k)A(n) = Nr_n G(k)^*$, for all $n \leq k$.
Since $i(n,k)A(n) \cong A(n)$, we get that $A(n) \in INr_n Cs_k = INr_n Ws_k$, for all $n \leq k < \omega$. ■

Fact 2.4. *$A(n) \in INr_n Cs_\beta$, for all $\beta > n$.*

Proof. By fact 2.3, it suffices to show that $A(n) \in INr_n Cs_\beta$ for all $\beta \geq \omega$. To this end, fix $\beta \geq \omega$. Let $G(\beta) = Sg^{C(\beta)}\{i(k,\beta)G(k)^* : n \leq k < \omega\}$. Like in fact 2.3, here $C(\beta)$ denotes the full Cs_β with unit ${}^\beta U$ and for $a \in G(k)^*$, $i(k,\beta)a = \{s \in {}^\beta U : s \restriction k \in a\}$.
Let $a \in G(\beta)$. We will show that $a \in Nr_n G(\beta)$ iff $a \in i(n,\beta)A(n)$, i.e that $i(n,\beta)A(n) = Nr_n G(\beta)$.
First note that $\cup\{i(k,\beta)G(k)^* : n \leq k < \omega\}$ is a subuniverse of $G(\beta)$. This is straightforward. Thus $G(\beta) = \cup\{i(k,\beta)G(k)^* : n \leq k < \omega\}$. It then follows that $a = i(l,\beta)g$, for some $n \leq l < \omega$ and $g \in G(l)^*$. Now suppose that $a \in Nr_n G(\beta)$. Then $g = c_j g$ for all $n \leq j < l$, and so $g \in Nr_n G(l)^*$. By fact 2.3 we have $Nr_n G(l)^* = i(n,l)A(n)$, thus $g = i(n,l)g'$, say, for some $g' \in A(n)$. But $a = i(l,\beta)g = i(l,\beta) \circ i(n,l)g' = i(n,\beta)g'$. This shows that $a \in i(n,\beta)A(n)$. Conversely, if $a \in i(n,\beta)A(n)$ then again by fact 2.3 we have $a \in Nr_n G(\beta)$. We have shown that $i(n,\beta)A(n) = Nr_n G(\beta)$. Since $A(n) \cong i(n,\beta)A(n)$, we get $A(n) \in INr_n Cs_\beta$. ■

Notation . *Let β be an ordinal. Then Cs_β^{reg} and Lf_β stand for the classes of regular cylindric set algebras and locally finite cylindric algebras of dimension β, respectively.*

It is known (see eg [HMT2] sec 4.3) that for $\beta \geq \omega$, the class $ICs_\beta^{reg} \cap Lf_\beta$ relates CA theory to models of first order logic. [9]

Fact 2.5. *$A(n) \in INr_n Ws_\beta$, for all $\beta > n$*

[9] The class of the so called locally finite regular cylindric set algebras is introduced in [AN75], and [AGN77] to investigate connections between (models of) first order logic and CA's. More information about these connection can be found in [N83] and the references therein. In the above quoted papers this class was denoted by Lr. The notation we use here, is the more "official one" , namely that adopted in [HMT2] sec 4.3

Proof. It remains to show that $A(n) \in INr_nWs_\beta$ for all $\beta \geq \omega$. Fix $\beta \geq \omega$ and let $F(\beta) = Sg^{G(\beta)}A(n)$. Then as easily checked $F(\beta) \in Cs_\beta^{reg} \cap Lf_\beta$, and $A(n) \cong Nr_nF(\beta)$. From $ICs_\beta^{reg} \cap Lf_\beta \subseteq IWs_\beta$, (cf [HMT2] 3.1.70) and $INr_nWs_\beta = Nr_nIWs_\beta$, we get the desired. ■

§3. $A(n)$ AND $B(n)$ ARE ELEMENTARY EQUIVALENT

In this section we will show that $A(n)$ admits a particularly rich set of automorphisms, a property we use to show that $B(n)$ is elementary equivalent to $A(n)$.

Notation . *For $u = \langle u_0, ..u_{n-1}\rangle \in {}^nn$, we let $1_{u_0...u_{n-1}}$ and sometimes simply 1_u denote $E(u,T)$. For $a \in A(n)$ and $X \subseteq A(n)$, $Rl_aX = \{x \in X : x \leq a\}$.*

Fact 3.1. *Let $u \in S_n$. Then the following hold:*

(i) $Rl_{1_u}A(n) = \{E(u,\phi) : \phi \in F(u)^{**}\}$. Furthermore $Rl_{1_u}A(n)$ is atomic [10] *and its set of atoms is equal to $P(u)$.*

(ii) For all nonzero $a \in Rl_{1_u}A(n)$, for all $i < n$, $c_ia = c_i1_u$.

(iii) For all $a \in A(n)$, for all $i < n$, $c_ia \cap 1_u \in \{0, 1_u\}$.

Proof. (i) follows from fact 2.2, and Lemma 1 (iv).
Now let $a \in Rl_{1_u}A(n)$. Then $a \leq 1_u$. By fact 3.1 (i) $a = E(u,\phi)$, for some $\phi \in F_n(u)^{**}$. Suppose $0 \neq a$, and $i < n$. Then by fact 3.1 (ii) there exists an atom in $Rl_{1_u}A(n)$ below a, i.e an $r \in R$, such that $E(u, C_r(x_0, ..x_{n-1})) = p(u,r) \leq a$. By Lemma 1(ii), $c_i1_u = c_ip(u,r) \leq c_ia \leq c_i1_u$. This proves (ii).
(iii) easily follows from (ii). ■

Remark . $\{1_u : u \in {}^nn\}$ is a (finite) partition of the unit 1 of $A(n)$. That is $\cup\{1_u : u \in {}^nn\} = 1$, and $1_u \cap 1_v = 0$, for distinct u and v. When $u \in S_n$, then $P(u)$ constitutes further an (uncountable) partition of 1_u, that is also a *splitting* of 1_u, in the sense of [An98]. This means that every element in $P(u)$ is *cylindrically equivalent* to 1_u, i.e $c_ix = c_i(1_u)$ for every $i < n$ and every $x \in P(u)$. So in a sense (at least in the above one) , the elements in $P(u)$ are "big". Bieng atoms in $Rl_{1_u}A(n)$ (and even more, as easily checked, in $A(n)$), they are also "small". Andreka [An98] refers to such elements as *big atoms.* It is precisely the "duality" in the nature of such elements (the big atoms) that will enable us, in fact 3.2 below, to extend given permutations on the set $P = \cup\{P(u) : u \in S_n\}$, generating $A(n)$, to automorphisms on $A(n)$. $A(n)$ in this respect, resembles the free structure on P.

[10] A boolean algebra is atomic if every non zero element contains an atom. An atom is a minimal non zero element.

Fact 3.2. *Let $P = \cup\{P(u) : u \in S_n\}$. Let $L = 1_{01..n-1}$. Let p be a permutation of $P(\langle 0,..,n-1\rangle)$. Then there exists an automorphism f of A_n such that $p \subseteq f$ and $f \restriction (P - P(\langle 0,..n-1\rangle)) = Id$.*

Proof. Let p be a given permutation of $P(\langle 0,..,n-1\rangle)$. By fact 3.1 (i) , $Rl_L A(n)$ is generated as a BA, by the set $P(\langle 0,..,n-1\rangle)$ of atoms. It follows from BA theory, that there exists an automorphism p', say of $BlRl_L A(n)$, such that $p \subseteq p'$.

For $x \in A$, define $f(x) = p'(x \cap L) \cup (x \cap -L)$. We will show that f is the desired automorphism. Clearly f is a BA automorphism of $BlA(n)$, such that $p' \subseteq f$ and $f \restriction (P - P(\langle 0,..,n-1\rangle) = Id$. So all we have to do is check that f preserves the extra CA operations.

(1) Diagonals:

Let $i < j \in n$. Then

$$f(d_{ij}) = p(d_{ij} \cap L) \cup (d_{ij} \cap -L) = d_{ij} \cap -L = d_{ij}.$$

(2) Cylindrifications:

Let $x \in A(n)$ and $i \in n$. On the one hand,

$$\begin{aligned} c_i f(x) &= c_i((p'(x \cap L) \cup (x \cap -L)) = c_i p(x \cap L) \cup c_i(x \cap -L) \\ &= \text{by fact 3.1 (ii) } c_i(x \cap L) \cup c_i(x \cap -L) = c_i x. \end{aligned}$$

On the other hand,

$$f(c_i x) = p'(c_i x \cap L) \cup (c_i x \cap -L) = \text{by fact 3.1 (iii) } (c_i x \cap L) \cup (c_i x \cap -L) = c_i x.$$

We have shown , that for all $x \in A(n)$ and $i \in n$, we have $c_i f(x) = f(c_i x) = c_i x$. [11] By this the proof is complete. ■

Fact 3.3. *$A(n)$ is elementary equivalent to $B(n)$.*

Proof. We shall use the Tarski-Vaught criterion to show that $B(n)$ is an elementary submodel of $A(n)$. So let $\exists x_n \phi(x_0, ...x_n)$ be any first order formula in the language of CA_n. Suppose that $b_0, ...b_{n-1} \in B(n)$, and that $A(n) \models \phi(b_0, ..., b_{n-1}, a)$, for some $a \in A(n)$. We want to find $b_n \in B(n)$, such that $A(n) \models \phi(b_0, .., b_n)$.

Since $A(n)$ is generated by $P = \cup\{P(u) : u \in S_n\}$, there exists a finite subset H_0 of P, such that $a \in Sg^{A(n)} H_0$. Let $X = \cup\{P(u) : u \in S_n - \{\langle 0, 1..n-1\rangle\} \cup P_\omega(\langle 0, 1..n-1\rangle)$. Since $b_0, ...b_{n-1} \in B(n)$ and $B(n)$ is generated by X, then there exists a finite subset H_1 of X, such that $(H_0 \cap B) \cup \{b_0, ...b_{n-1}\} \subseteq Sg^{B(n)} H_1$.

[11] We used the "bigness" condition to show that f preserves cylindrifications. Loosely speaking, when a cylindric algebra is generated by a set of big atoms, then we can "forget about" cylindrifications when constructing automorphisms on A

Let p be a permutation of $P(\langle 0,..n-1\rangle)$ such that $p \restriction P(\langle 0,..n-1\rangle)\cap H_1 \subseteq Id$ and $p(H_0 - H_1) \subseteq P(\langle 0,1,..n-1\rangle) \cap B(n)$. Such a permutation, clearly, exists. By fact 3.2 p extends to an automorphism f of $A(n)$ such that $p \subseteq f$ and $f \restriction (P - P(\langle 0,..,n-1\rangle)) = Id$. But $A(n) \models \phi(b_0,..,b_{n-1},a)$ and so $A(n) \models \phi(f(b_0),..,f(b_{n-1}),f(a))$. By $f \restriction H_1 \subseteq Id$ and $\{b_0,..b_{n-1}\} \subseteq Sg^{B(n)}H_1$, it follows that $f(b_i) = b_i$, for all $i < n$. Since $f(H_0) \subseteq B$ and $a \in Sg^{A(n)}H_0$, we get $f(a) \in B$. This completes the proof. ■

§4. $B(n)$ IS NOT A NEAT REDUCT

In this section we will show that $B(n) \notin Nr_nCA_{n+1}$.

Since for $n < \alpha < \beta$, $Nr_nCA_\beta \subseteq Nr_nCA_\alpha$ (cf. [HMT1], 2.6.31) and CA_β is the largest of the classes mentioned in Theorem 1, this completes the proof of Theorem 1, when $\alpha = n$ and $\beta > n$.

Fact 4.1. *(i) For all $u \in {}^nn, 1_u = E(u,T) \in B(n)$.*

(ii) $|Rl_LB(n)| = \omega$

Proof.

(i) Let $u \in S_n$, and $r \in N$. Then by Lemma 1(ii),

$$\cap_{i<n}\{c_iE(u,C_r(x_0,...,x_{n-1})\} = E(u,T).$$

If u is not one to one, proceed as in the proof of case (b) in fact 2.2 above.

(ii) Let $X = \cup\{P(u) : u \in S_n - \{\langle 0,...n-1\rangle\} \cup P_\omega(\langle 0,...n-1\rangle)$.

Recall that $P_\omega(\langle 0,..,n-1\rangle) = \{p(\langle 0,...,n-1\rangle,r) : r \in N\}$ and that $B(n) = Sg^{A(n)}(X)$.

Let $Y = \{c_i^{A(n)}a : a \in A(n) \text{ and } i < n\} \cup \{d_{ij} : i,j \in n\}$ and let $D(n) = Sg^{BlA(n)}(X \cup Y)$. Then by lemma 1(ii) (see also the proof of fact 3.1 (iii)), $|Y| \leq \omega$, and $D(n)$ is a clearly a subuniverse of $A(n)$, containing X. Therefore $B(n) \subseteq D(n)$, (In fact, by fact 3.1 (i), $B(n) = D(n)$, but we do not need that much), and so $Rl_LB(n) \subseteq Rl_LD(n)$. It clearly suffices to show that $|Rl_LD(n)| \leq \omega$. Towards this, let rl_L denote the following endomorphism of $\langle A(n),\cup,\cap,-\rangle$:

$$rl_L(a) = a \cap L \quad \text{for } a \in A(n).$$

Then $Rl_LD(n) = Sg^{BlA(n)}rl_L(X \cup Y)$. But $rl_L(X \cup Y)$ is countable because $rl_L(X) = P_\omega(\langle 0,...n-1\rangle)$, and $|Y| \leq \omega$. Therefore $Rl_LD(n)$, generated as a BA by a countable set, is also countable.

$\blacksquare$

Notation . *Let $t(x)$ denote the CA_2 term $s^0_1 c_1 x \cap s^1_0 c_0 x$. Let $\alpha > 1$. Then ${}_\alpha s(0,1)$ denotes the $CA_{\alpha+1}$ substitution term as defined in 1.5.12 of [HMT1].*

Fact 4.2. *(i) $t^{B(n)}(1_{10\ldots n-1}) = 1_{01..n-1}$.*

(ii) For all $\beta > n$, $CA_\beta \models {}_n s(0,1) c_n x \leq t(c_n x)$.

(iii) For all $\beta > n$ and $A \in CA_\beta$, ${}_n s(0,1) \in Ism(BlNr_n A, BlNr_n A)$.

Proof.

(i) For brevity set $1_{10} = 1_{10..n-1}$, and $1_{01} = 1_{01\ldots n-1}$. By [HMT1], 1.5.1, we have

$$t^{B(n)}(1_{10}) = c_0(d_{01} \cap c_1 1_{10}) \cap c_1(d_{01} \cap c_0 1_{10}).$$

Computing we get:

$$\begin{aligned} c_0(d_{01} \cap c_1 1_{10}) &= c_0(d_{01} \cap (\cup\{1_u : u \equiv_1 1_{10}\}) \\ &= c_0(d_{01} \cap 1_{11,..,n-1}) = 1_{01} \cup 1_{11,..n-1}. \end{aligned}$$

A similar computation gives:

$$c_1(d_{01} \cap c_0 1_{01}) = 1_{00..n-1} \cup 1_{01}.$$

Therefore $t^{B(n)}(1_{10}) = 1_{01}$.

(ii) By [HMT1], 1.5.12,1.5.8 and 1.5.10(ii), we have:

$$\begin{aligned} {}_n s(0,1) c_n x &\leq {}_n s(0,1) c_1 c_n x = s^n_0 s^0_1 s^1_n c_1 c_n x = s^n_0 s^0_1 c_1 c_n x \\ &= s^n_0 s^0_1 c_n c_1 x = s^n_0 c_n s^0_1 c_1 x = c_n s^0_1 c_1 x = s^1_0 c_1 c_n x \end{aligned}$$

Similarly, ${}_n s(0,1) c_n x \leq s^0_1 c_0 c_n x$.

Therefore ${}_n s(0,1) c_n x \leq t(c_n x)$.

(iii) Let $C \in CA_\beta$, with $\beta > n$, and $0 < x \in Nr_n C$. By [HMT1], 1.5.12 and 1.5.1, we have:

$${}_n s(0,1) c_n x = s^n_0 s^0_1 s^1_n c_n x = c_n(d_{n0} \cap c_0(d_{01} \cap c_1(d_{01} \cap c_1(d_{1n} \cap c_n x)))).$$

By [HMT1], 1.3.8, $0 < x$, implies $0 < d_{ij} \cap c_j x$, for all $i, j \in \beta$.

We have shown that if $x > 0 \in Nr_n C$, then ${}_n s(0,1)x > 0$, i.e that ${}_n s(0,1)$ is one to one.

Since the s^j_i's are BA endomorphisms, it follows from [HMT1], 1.5.12, that ${}_n s(0,1) \in Ism(BlNr_n C, BlC)$.

It remains to show that ${}_n s(0,1)x \in Nr_n C$, for all $x \in Nr_n C$. But this follows easily from [HMT1], 1.6.13.

$\blacksquare$

Fact 4.3. $B(n) \notin Nr_n CA_{n+1}$.

Proof. Suppose, seeking a contradiction, that $B(n) = Nr_n C$, say with $C \in CA_{n+1}$. [12] Then $P(\langle 10,..,n-1\rangle) \subseteq Rl_{1_{10...n-1}} Nr_n C$, and so

$$|Rl_{1_{10..n-1}} Nr_n C| > \omega.$$

By fact 4.2 (iii), $|_n s(0,1)^C (Rl_{1_{10..n-1}} Nr_n C)| > \omega$.
On the other hand,

$$\begin{aligned} {}_n s(0,1)^C (1_{10..n-1}) &= {}_n s(0,1)^C c_n(1_{10..n-1}) \quad \text{(by fact 4.2(ii))} \\ &\leq t^C c_n(1_{10..n-1}) = t^B(1_{10..n-1}) \quad \text{(by fact 4.1(i))} \ = 1_{01..n-1}. \end{aligned}$$

This implies that ${}_n s(0,1)^C (Rl_{1_{10..n-1}} Nr_n C) \subseteq Rl_{1_{01..n-1}} Nr_n C = Rl_{1_{01..n-1}} B$.
By fact 4.1 (ii) we conclude that $|_n s(0,1)^C (Rl_{10..n-1} Nr_n C| \leq \omega$. Contradiction, implying that $B(n) \notin Nr_n CA_{n+1}$. By establishing fact 4.3, the proof of Theorem 1 for $\alpha = n$ (and $\beta > n$) is complete. $\blacksquare$

§5. THE INFINITE CASE

In this section we prove Theorem 1, for $\alpha \geq \omega$. We also prove Theorem 2.

Throughout this section α is a fixed arbitrary ordinal with $\alpha \geq \omega$ and R is an uncountable set with $|R| > |\alpha|$.

Notation . *Let s be an α-ary sequence, and $i < \alpha$. Then s^i_w denotes the α-ary sequence for which: $s^i_w(i) = w$ and $s^i_w(x) = s(x)$ for $x \neq i$. For a set W, $w \in W$ and β an ordinal, we let $\mathbf{w}$ denote the β-ary sequence for which $\mathbf{w}_i = w$, for all $i \in \beta$. The arity of $\mathbf{w}$, we hope, will be clear from context.*

We start by formulating infinite versions of conditions (ii) and (iv) of Lemma 1.

Lemma 1* . *There are a set W, $w \in W$, and α-ary relations $C_r \subseteq {}^\alpha W^{(\mathbf{w})}$ for all $r \in R$, such that conditions (i) and (ii) below hold:*

(i) For all $r \in R$, for all $i < \alpha$, for all $s \in {}^\alpha W^{(\mathbf{w})}$, there exists $y \in W - Rgs$ such that $s^i_y \in C_r$.

(ii) The C_r's ($r \in R$) are pairwise disjoint.

[12] Apart from being a CA_{n+1}, we impose no further restrictions on C. In particular, C may not be representable. It is only here and in fact 5.3 below that we deal with (possibly) non representable algebras

The proof can be recovered from that of Lemma 1^{**} below.

Notation . *Let* $\mathbf{W} = \langle W, C_r\rangle_{r\in R}$, *with* $C_r \subseteq {}^{\alpha}W^{(\mathbf{w})}$, *be the structure described in Lemma 1*. Let* $U = \cup_{i\in\alpha} W \times \{i\}$. U *is simply* $|\alpha|$ *disjoint copies of* W. *Let* $s = \langle (w,0), \cdots, (w,j)\cdots\rangle_{j<\alpha}$. *Then* $s \in {}^{\alpha}U$. *(Recall that* $\mathbf{w} = \langle w : i < \alpha\rangle$*). Let* V *be the weak space* ${}^{\alpha}U^{(s)}$ *and let* $C(\alpha) = \langle Sb(V), \cup, \cap, -, c_i, d_{ij}\rangle$ *be the full* Ws_α *with unit* V. S_α *denotes the set of one to one functions (equivalently permutations) which are almost everywhere identity, i.e, which are in* ${}^{\alpha}\alpha^{(Id)}$.

Defining our algebras

Let $u = \langle u_0, u_1, \cdots u_j\cdots\rangle_{j\in\alpha} \in S_\alpha$, and $r \in R$. Then $p(u,r)$ denotes the following α-ary relation on U:

$$\{\langle (a_0,u_0), \cdots, (a_j,u_j)\cdots\rangle_{j<\alpha} \in V : \quad \langle a_0, \cdots a_j, \cdots\rangle_{j<\alpha} \in C_r\}.$$

$$P(u) = \{p(u,r) : r \in R\}.$$

Let N be a fixed countably infinite subset of R.
Then $P_\omega(u) = \{p(u,r) : r \in N\}$.
Now we define $A(\alpha) \in Ws_\alpha$ and $B(\alpha) \subseteq A(\alpha)$ as follows:

$$A(\alpha) = Sg^{C(\alpha)}(\cup\{P(u) : u \in S_\alpha\})$$

$$B(\alpha) = Sg^{A(\alpha)}((\cup\{P(u) : u \in S_\alpha - \{Id\}) \cup P_\omega(Id)).$$

We shall start by showing that $A(\alpha)$ and $B(\alpha)$ are elementary equivalent, and that $B(\alpha)$ is not a neat reduct.
At this point we will be ready to prove Theorem 2 (where we shall only be using the fact that $B(\alpha)$ is *not* a neat reduct).
Then in Lemma 1** below, we shall impose further restrictions on the structure $\langle W, C_r\rangle_{r\in R}$, that will *force* $A(\alpha)$ to be a neat reduct (and not just a subneat reduct). These restrictions, are loosely speaking, the infinite versions of conditions (i) and (iii) in Lemma 1, way above.

Working with weak set algebras of infinite dimension, is a little more tedious but, as previously noted, the idea at heart is the same as that of the finite dimensional case. We start by the infinite analogue of facts 3.1, 3.2 and 3.3. The proofs we give are rather sketchy since they are almost identical to the finite case.

Notation . *Let* $u = \langle u_0, u_1, \cdots u_j\cdots\rangle_{j\in\alpha} \in {}^{\alpha}\alpha^{(Id)}$. *Then* 1_u *denote the* α*-ary relation on* U*:*

$$\{\langle (a_0,u_0), (a_1,u_1), \cdots (a_j,u_j)\cdots\rangle_{j<\alpha} \in V : \langle a_0, \cdots a_j\cdots\rangle_{j\in\alpha} \in {}^{\alpha}W^{(\mathbf{w})}\}.$$

L stands for 1_{Id}.

Fact 5.1. *(i) For all $u \in S_\alpha$, $1_u \in A(\alpha)$.*

(ii) Let $Rl_L A(\alpha) = \{x \in A(\alpha) : x \leq L\}$. Then $Rl_L A(\alpha)$ is atomic and its set of atoms is equal to $P(Id)$. Furthermore, $Rl_L A(\alpha)$ is generated as a BA by $P(Id)$.

(iii) For all non zero $a \in Rl_L A(\alpha)$, for all $i < \alpha$, $c_i a = c_i L$.

(iv) For all $a \in A(\alpha)$, for all $i < \alpha$, $c_i a \cap L \in \{0, L\}$.

(v) Let $P = \cup\{P(u) : u \in S_\alpha\}$. Let p be a permutation of $P(Id)$. Then there exists an automorphism f of $A(\alpha)$, such that $p \subseteq f$ and $f(x) = x$, for each $x \in P - P(Id)$.

Proof.

(i) Let $u \in S_\alpha$. Let $r_1, r_2 \in R$. Then as easily checked $c_0(p(u, r_1)) \cap c_1(p(u, r_2)) = 1_u$.

(ii) Since the C_r's are pairwise disjoint, we have $p(Id, r_1) \cap p(Id, r_2) = 0$ for distinct r_1 and r_2. Thus the set of atoms of $Rl_L A(\alpha)$ is precisely $P(Id)$. By $c_i(p(Id, r)) \cap 1_{Id} = 1_{Id}$ and $p(Id, r) \cap d_{ij} = 0$, for distinct $i, j < \alpha$, it follows that $Rl_L A(\alpha)$ is indeed generated as a BA by the set $P(Id)$.

(iii) Let $a \in Rl_L A(\alpha)$ be non zero. By (ii) there exists an atom $p(Id, r)$ below a. Let $i < \alpha$. Then $c_i(p(Id, r))$ =(by Lemma 1 (i)) $\cup\{1_u : u \in {}^\alpha\alpha^{(Id)}$ and $u(j) = j$ for all $j \neq i\} = c_i L$. From $p(Id, r) \leq a$, (iii) readily follows.

(iv) follows from (iii).

(v) We first, like we did in the finite case, extend p to a BA automorphism p', say, of $Rl_L A(\alpha)$. This is possible by (i). Then we define $f(x) = p'(x \cap L) \cup (x \cap -L)$. The rest is completely analogous to the proof of fact 3.1.

■

Fact 5.2. *$A(\alpha)$ is elementary equivalent to $B(\alpha)$.*

The proof, literally identical to the proof of fact 3.3, is left to the reader.

Now we show that $B(\alpha)$ is not a neat reduct. We proceed basically as we did in the finite case, the trick being "a(n) (infinite) cardinality twist" that first order logic cannot witness.

Fact 5.3. *Let $Rl_L B(\alpha) = \{x \in B(\alpha) : x \leq L\}$. Then $|Rl_L B(\alpha)| \leq \alpha$.*

Proof. First of all note that by 5.1 (i) we have $L \in B$. Now let $X = \cup\{P(u) : u \in S_\alpha - \{Id\}\} \cup P_\omega(Id)$, $Y = \{c_i a : a \in A\} \cup \{d_{ij} : i, j \in \alpha\}$ and $D(\alpha) = Sg^{Bl\mathbf{A}(\alpha)}(X \cup Y)$. Then $|Y| = \alpha$. This can be seen by noting that for every $x \in Y$, if x is non zero, and x is not a diagonal element, then by Lemma 1*(ii) there is an S with $S \subseteq {}^\alpha\alpha^{(Id)}$ such that $x = \cup\{1_u : u \in S\}$. Also $D(\alpha)$ is a subuniverse of $A(\alpha)$ because of the following: $D(\alpha)$ is closed under the boolean set operations , contains the diagonals, and for $a \in D(\alpha)$, and $i < \alpha$, we have $a \in A(\alpha)$ and so $c_i a \in D(\alpha)$, that is $D(\alpha)$ is closed under cylindrifications. Since $X \subseteq D(\alpha)$, X generates $B(\alpha)$ and $D(\alpha) \in CA_\alpha$, we have $B(\alpha) \subseteq D(\alpha)$. Let $rl_L(a) = a \cap L$. Then we have

$$Rl_L D(\alpha) = \{x \in D(\alpha) : x \leq L\} = Sg^{BlA(\alpha)} rl_L(X \cup Y).$$

But $|rl_L(X \cup Y)| = \alpha$, because $P_\omega(Id)$ is countable and $|Y| = \alpha$. Therefore $|rl_L(X \cup Y)| = \alpha$, and so $|Rl_L D(\alpha)| \leq \alpha$. By $Rl_L B(\alpha) \subseteq Rl_L D(\alpha)$ we are done. ■

Fact 5.4. *Let $p[0,1]$ be the transposition on α interchanging 0 and 1. Let $v = p[0,1] \circ Id = \langle 1023 \cdots \rangle$. Then $v \in S_\alpha$ and the following hold:*

(i) $t^{B(\alpha)}(1_v) = 1_{Id}$

(ii) For all $\beta > \alpha$, $CA_\beta \models {}_\alpha s(0,1) c_\alpha x \leq t(c_\alpha x)$.

(iii) For all $\beta > \alpha$ and $A \in CA_\beta$, ${}_\alpha s(0,1) \in Ism(BlNr_\alpha A, BlNr_\alpha A)$.

The proof is identical to the proof of fact 4.2. Therefore we omit it.

Fact 5.5. $B(\alpha) \notin Nr_\alpha CA_{\alpha+1}$.

Proof. Like the proof of fact 4.3, the idea bieng; if $B(\alpha)$ were a neat reduct then we would have $|Rl_L(B(\alpha)| = |R|$. But this contradicts fact 5.3, since by assumption we have $|R| > |\alpha|$. ■

Now we are ready to prove Theorem 2. [13]

Theorem 2 . *Let $\alpha \geq \beta$. Then there exist $B \in Ws_\alpha$ and $C \in Ws_\beta$ such that B is a generating subreduct of C, but $B \neq Nr_\alpha C$.*

Proof. Let $\beta > \alpha$. Then by [HMT2] 3.1.12 we have $B = B(\alpha) \in SNr_\alpha Ws_\beta$. Let $D \in Ws_\beta$ be such that $B \subseteq Nr_\alpha D$ and let $C = Sg^D B$. Then B and C are clearly as desired. ■

$A(\alpha)$ constructed so far may or may not be a neat reduct. We now impose further restrictions on the C_r's (or rather on the structure $\mathbf{W} = \langle W, C_r \rangle$),

[13] Though not explicitly stated, it is rather easy to derive Theorem 2, from statement 1 in [N83]. In more detail let B be the Ws_α constructed in statement 1 therein. Then $B \notin Nr_\alpha CA_{\alpha+1}$. Then proceed as in the proof of Theorem 2 above. Conversely Theorem 2 herein provides another solution to problem 2.11 in [HMT1].

described in Lemma 1*, that *force* $A(\alpha)$ to be in $\cap_{k\in\omega}Nr_\alpha Ws_{\alpha+k}$. These restrictions are described in Lemma 1** below. Forming a "limit" out of the $Ws_{\alpha+k}$'s $(k \in \omega)$, the $neat-\alpha$ reduct of which is $A(\alpha)$, we will show that $A(\alpha) \in Nr_\alpha Ws_{\alpha+\beta}$ for infinite β as well. This should be done , of course, in such a way that does not interfere with what we have already established, namely:
$B(\alpha)$ is *not* a neat reduct.
$A(\alpha)$ and $B(\alpha)$ are elementary equivalent.
To formulate Lemma 1** we fix some notation:

Notation . *Let* $1 \leq k < \omega$. *Then* $S(\alpha, \alpha+k)$ *or simply*

$$S(\alpha,k) = \{i \in {}^\alpha(\alpha+k) : \alpha+k-1 \in Rgi \quad \textit{and } \{m \in \alpha : |\{i(m) \neq m\}| < \omega\}.$$

That is $S(\alpha,k) = \{i \in {}^\alpha(\alpha+k)^{(Id)} : \alpha+k-1 \in Rgi\}$.
Cof^+R *denotes the set of all non empty finite or cofinite subsets of* R. *Let* C_r *be an* α*-ary relation symbol for all* $r \in R$.
For any $X \subseteq R$, X *finite we define the infinitary formulas:*

$$\eta(X) = \vee\{C_r(x_0, \cdots x_j \cdots)_{j<\alpha}\, r \in X\}, \quad \textit{and}$$

$$\eta(R-X) = \wedge\{\neg C_r(x_0, .x_j \cdots)_{j<\alpha} : r \in X\}^{14}.$$

LEMMA 1** . *There are a set* W, $w \in W$ *and* α*-ary relations* $C_r \subseteq {}^\alpha W^{(\mathbf{w})}$ *for all* $r \in R$ *such that, in addition to conditions (i) and (ii) of Lemma 1*, the following conditions hold:*

(i) For all $s \in {}^\alpha W^{(\mathbf{w})}$ *for all* $r \in R$, *for all permutation* $\pi \in S_\alpha$, *if* $s \in C_r$, *then* $s o \pi \in C_r$.

(ii) For all $1 \leq k < \omega$, *for all* $v \in {}^{\alpha+k-1}W^{(\mathbf{w})}$, *for any function* $f : S(\alpha,k) \to Cof^+R$ *for which* $\{i \in S(\alpha,k) : |\{f(i) \neq R\}| < \omega\}$, *there is a* $v_{\alpha+k-1} \in W - Rgv$ *such that*

$$\wedge\{\eta(f(i))[\langle v_{i_j}\rangle_{j<\alpha}] : i \in S(\alpha,k)\}.$$

Being very similar to the proof of Lemma 1, we only give a sketch:

[14] $\eta(X)$ and $\eta(R-X)$ are *restricted* formulas, in the sense of [HMT2] sec 4.3. Satisfiability for such formulas (by α-ary sequences) are defined the usual way. Below we shall have occasion to deal with (infinitary) formulas that are not restricted. These however can be obtained from restricted ones using the (logical interpretation of the) CA operations, i.e quantification on finitely many variables and equality.

Let $k \geq 1$. For brevity set

$$Y(\alpha, R, k) = \{f \in {}^{S(\alpha,k)}Cof^+R : \{i \in S(\alpha, k) : |\{f(i) \neq R\}| < \omega\}.$$

Let

$$Q = \cup\{{}^{\alpha+k-1}|R| \times Y(\alpha, R, k) : 1 \leq k < \omega\},$$

then $|Q| = |R| = \mu$, say. Let ρ be an enumeration of Q such that: for all $l < \mu$, for all $q \in Q$, there exists j with $l < j < \mu$ such that $\rho(j) = q$. Fix a well ordering $\prec$ of R. let $l < \mu$ and suppose that for all $i < l$ we have already defined the element v_i, and the α-ary relation $C_r^i \subseteq {}^{\alpha}W_i^{(\mathbf{w})}$, where $W_i = \{v_k : k < i\}$ and $w \in W_i$. Assume that $\rho(l) = \langle\langle\beta_j\rangle_{j<\alpha+k-1}, f\rangle$. Then $\beta_j \in |R|$ for all $j < \alpha + k - 1$, and $f \in Y(\alpha, R, k)$. Let v_l be an element not in $\cup\{W_i : i < l\}$. If there exists $i < \alpha + k - 1$ such that $l \leq \beta_i$, then for all $r \in R$ we define $C_r^l = \cup\{C_r^i : i < l\}$. Else, $l > \beta_i$ for all $i < \alpha + k - 1$. Let $u_i = v_{\beta_i}$ for $i < \alpha + k - 1$ and let $u_{\alpha+k-1} = v_l$. For all $r \in R$ and $i \in S(\alpha, k)$ we let

$$\langle u_{i_j}\rangle_{j<\alpha} \in X_r^l \text{ iff } r \text{ is the } \prec - \text{ least element of } f(i)$$

and

$$C_r^l = \cup\{C_r^i : i < l\} \cup \{\langle s_{\pi(0)}, s_{\pi(1)}, \cdots\rangle : s \in X_r^l \text{ and } \pi \in S_\alpha\}.$$

Finally set

$$W = \cup\{W_l : l < \mu\} \quad \text{and } C_r = \cup\{C_r^l : l < \mu\}.$$

It is not hard to check just like we did in the finite dimensional case, that $\mathbf{W} = \langle W, C_r\rangle_{r\in R}$ is as desired. ∎

With a slight abuse of notation we let $A(\alpha)$ and $B(\alpha)$ be the (new) algebras consructed out of the (new) structure $\mathbf{W} = \langle W, C_r\rangle_{r\in R}$, described in Lemma 1^{**}. It is easy to check that $A(\alpha)$ and $B(\alpha)$ are (still) elementary equivalent (cf., facts 5.1-2) and $B(\alpha)$ is (still) not a neat reduct (cf., facts 5.3-5.5). We will use the new condition of "saturation" or "elimination of quantifiers" expressed in Lemma 1^{**} (ii) to show that $A(\alpha) \in \cap_{k\in\omega} Nr_\alpha Ws_{\alpha+k}$. To "lift" $A(\alpha)$ to arbitrary finite extra dimensions, we shall further need:

Some more definitions.

Let $0 \leq k < \omega$. Then $U_k = U = \cup_{i\in\alpha} W \times \{i\}$. Let $s_k = s \cup \langle(w, 0) : \alpha \leq i < (\alpha + k)\rangle$. Then $s_k \in {}^{\alpha+k}U$. (When $k = 0$, then s_k is just s). Let V_k be the weak space ${}^{\alpha+k}U^{(s_k)}$ and let $C(\alpha + k)$ be the full $Ws_{\alpha+k}$ with unit V_k.
Let $u \in {}^{\alpha+k}\alpha^{(Id)}$. Then

$$F_k(u) = \{C_r(x_{i_0}, \cdots x_{i_j} \cdots)_{j<\alpha}, \neg C_r(x_{i_0}, \cdots x_{i_j} \cdots)_{j<\alpha} : r \in R,\ i \in {}^{\alpha}\alpha + k^{(Id)}$$
$$\text{and } \langle u_{i_0}, \cdots u_{i_j} \cdots\rangle_{j<\alpha} \quad \text{is one to one } \} \quad \cup Eq(u).$$

Recall that $Eq(u)$ stands for $\{x_i = x_j : x_i \neq x_j : (i,j) \in ker(u)\}$. Like the finite case we assume that $\{T, F\} \in F_k(u)$.

$$F_k(u)^* = \{\wedge J : J \subseteq_\omega F_k(u)\},$$

$$F_k(u)^{**} = \{\vee J : J \subseteq_\omega F_k(u)^*\}.$$

Let $u \in {}^{\alpha+k}\alpha^{(Id)}$ and $\phi \in F_k(u)^{**}$.
Then $E(u, \phi)$ denote the following $(\alpha + k)$-ary relation on U

$$\{\langle (a_0, u_0), \cdots (a_j, u_j) \cdots \rangle_{j<\alpha+k} \in V_k : \mathbf{W} \models \phi[\langle a_j \rangle_{j<\alpha+k}]\}.$$

As noted before, satisfiability is defined the usual way. In particular, when $k = 0$, $u \in S_\alpha$ and $r \in R$, then $E(u, C_r(x_0, x_1, \cdots)$ is the same as $p(u, r)$.
$ind(\phi)$ denotes the set $\{i \in \alpha + k : x_i \in var(\phi)\}$.
Let $G(k) = \{c_{(\Delta)}E(u, \phi), -c_{(\Delta)}E(u, \phi) : \Delta \subseteq_\omega \alpha + k,\ u \in {}^{\alpha+k}\alpha^{(Id)}, \phi \in F_k(u)^{**},\ \Delta \cap ind(\phi) = 0\} \cup \{d_{ij}^{V_k}, -d_{ij}^{V_k} : i, j \in \alpha + k\}$.
Here $c_{(\Delta)}$, the unary operation referred to as generalized cylindrification in [HMT1]. That is $c_\emptyset x = x$ and if $\Delta = \{k_0, \cdots k_{m-1}\}$ is a non-empty finite subset of $\alpha + k$, then $c_{(\Delta)}x = c_{k_0} \cdots c_{k_{m-1}}x$.
For $i, j \in \alpha + k$, $d_{ij}^{V_k}$, or simply d_{ij} is the diagonal element $\{s \in V_k :\ s_i = s_j\}$.
Finally set

$$G(k)^* = \{\cap J : J \subseteq_\omega G(k)\}$$

and

$$G(k)^{**} = \{\cup J : J \subseteq_\omega G(k)^*\}.$$

We will show that $G(k)^{**}$ is a $Ws_{\alpha+k}$ (with unit V_k), that $G(0)^{**} = Nr_\alpha G(k)^{**}$ for all $k \in \omega$, and finally that $G(0)^{**} = A(\alpha)$. This will show that $\mathbf{A}(\alpha) \in \cap_{k\in\omega} INr_\alpha Ws_{\alpha+k}$. To prove that $G(k)^{**}$ is a $Ws_{\alpha+k}$ we shall need:

Fact 5.6. *(i) Let $u \in {}^{\alpha+k}\alpha^{(Id)}$, $\Delta \subseteq_\omega \alpha + k$ and $\phi \in F_k(u)^{**}$. Then $c_{(\Delta)}E(u, \phi) \in G(k)^{**}$.*

(ii) Let $g \in G(k)^{**}$ and $i, j \in \alpha + k$ be distinct. Then $s_i^j(g) = c_j(g \cap d_{ij}) \in G(k)^{**}$.

(iii) Let $H(k) = \{g \in G(k) : g = c_{(\Delta)}E(u, \phi) : \Delta \subseteq_\omega \alpha + k, u \in {}^{\alpha+k}\alpha^{(Id)}$ and $\phi \in F_k(u)^{**}\}$. Then $H(k)$ is closed under finite intersections.

Proof.

(i) We first prove (i) when Δ consists of one element j, say. That is we will show that for u and ϕ as indicated above and $j < \alpha + k$ we have $c_jE(u, \phi) \in G(k)^{**}$. We can and will assume that $\phi \in F_k(u)^*$ since

$$c_jE(u, \phi_1 \vee \cdots \phi_n) = c_jE(u, \phi_1) \cup \cdots c_jE(u, \phi_n)$$

and $G(k)^{**}$ is closed under finite unions. Then $\phi = \wedge J$, where $J \subseteq_\omega F_k(u)$. Next, we break up ϕ into two parts just like we did in the finite dimensional case; that is let $J^+ = \{\psi \in J : x_j \in var(\psi)\}$ and $J^- = \{\psi \in J : x_j \notin var(\psi)\}$. For brevity set $\phi^+ = \wedge J^+$ and $\phi^- = \wedge J^-$, where by the empty conjunction we understand the formula T. Now $\phi = \phi^+ \wedge \phi^-$, $\phi^- \in F_k(u)^*$ and x_j occurs only in ϕ^-.

Notation . *For $i, l \in \alpha + k$ and $\phi \in F_k(u)^{**}$ $\phi[i|l]$ stands for the formula obtained from ϕ by replacing all (free and bound) occurances of x_i in ϕ by x_l.*

Case (a): $x_j = x_m \notin J^+$ and $x_j = x_m \notin J^+$ for all $m \in \alpha + k - \{j\}$.

Then either $\mathbf{W} \models \phi^+ \equiv F$, in which case $\mathbf{W} \models \exists x_j \phi^+ \equiv F$, and so $c_j E(u, \phi) = c_j E(u, F) = 0 \in G(k)^{**}$. Else ϕ^+ is satisfiable, in which case , by applying Lemma 1** (ii), we get

$$\mathbf{W} \models (\exists x_{\alpha+k-1} \phi^+[j|\alpha + k - 1]) \equiv T,$$

which is the same as $\mathbf{W} \models (\exists x_j \phi^+) \equiv T$, and so

$$c_j E(u, \phi) = c_j E(u, (\exists x_j \phi^+) \wedge \phi^-) = c_j E(u, \phi^-).$$

The latter is in $G(k)^{**}$, since now x_j does not occur in ϕ^-, and $\phi^- \in F_k(u)^*$. Note that $\phi^- \in F_k(v)^*$ for any v that differs from u in at most the j-th place; and for any such v we have $c_j E(u, \phi) = c_j E(v, \phi^-)$.

Case (b): $x_j = x_m \in J^+$ or $x_m = x_j \in J^+$ for some $m \in \alpha + k$ and $m \neq j$.

Assume that $x_j = x_m \in J^+$. Then , by definition of $F_k(u)$, we have $u_j = u_m$. Also $\phi = (\wedge I) \wedge x_j = x_m$, for some $I \subseteq_\omega F_k(u)$. For brevity let $\psi = \wedge I$. Then of course $\psi \in F_k(u)^*$. Computing we get

$$c_j E(u, \phi) = c_j E(u, \psi \wedge x_j = x_m) = c_j E(u, \psi[j|m]).$$

The latter is in $G(k)^{**}$ since x_j does not occur in $\psi[j|m]$, and $\psi[j|m] \in F_k(u)^*$ because $\psi \in F_k(u)^*$ and $u_j = u_m$. In fact, it is not hard to see that $\psi[j|m] \in F_k(v)^*$, for any v that differs from u in at most the j th place. The case when Δ is an arbitrary finite subset of $\alpha + k$, follows by using the above reasoning eliminating the variables occuring in ϕ, with indices in Δ, one by one. Alternatively one can use a straightforward induction on $|\Delta|$. Note that we have actually proved the following infinite analogue of (**) in fact 2.1 way above:

Let $u \in {}^\alpha\alpha + k^{(Id)}$ and $\phi \in F_k(u)^{**}$. Let $\Delta = \{i_0, \cdots i_{n-1}\}$ be a finite subset of $\alpha + k$. Let $v \in {}^\alpha\alpha + k^{(Id)}$ be such that $v(j) = u(j)$ for all $j \notin \Delta$.

Then there exists $\psi \in F_k(v)^{**}$ such that $var(\psi) \cap \{x_{i_0}, \cdots, x_{i_{n-1}}\} = \emptyset$. i.e., $ind(\psi) \cap \Delta = \emptyset$, and $\mathbf{W} \models \psi \equiv \exists x_{i_0} \cdots \exists x_{i_{n-1}} \phi$. In particular, $c_{(\Delta)} E(u, \phi) = c_{(\Delta)} E(v, \psi)$.

(ii) Let $g \in G(k)^* *$ and $i, j \in \alpha + k$ be distinct.

Case (a): $g = c_{(\Delta)} E(u, \phi)$ with $u \in {}^{\alpha+k}\alpha^{(Id)}$, $\phi \in F_k(u)^{**}$.

Subcase (a) : $j \in \Delta$.

Then by [HMT1] 1.5.8 (i) and 1.7.3 we have

$$s_i^j(g) = s_i^j c_j(c_{(\Delta - \{j\})})E(u, \phi) = c_j(c_{(\Delta - \{j\})})E(u, \phi) = g$$

Subcase (b) : $j \notin \Delta$ and $i \notin \Delta$.

Then by [HMT1], 1.5.8 (ii) we have

$$s_i^j g = s_i^j c_{(\Delta)} E(u, \phi) = c_{(\Delta)} s_i^j E(u, \phi).$$

Let $g^* = E(u, \phi) \cap d_{ij}$. Then g^* and hence $s_j^i g = c_{(\Delta)} c_j g^*$, is equal to zero if $u_i \neq u_j$. Else, as easily checked, $g^* = E(u, \phi \wedge x_i = x_j)$. By $u_i = u_j$ and $\phi \in F_k(u)^{**}$, we have $\phi \wedge x_i = x_j \in F_k(u)^{**}$. By fact 5.6 (i) we get that $s_j^i g = c_{(\Delta)} c_j g^* = c_{(\Delta)} c_j E(u, \phi \wedge x_i = x_j) \in G(k)^{**}$.

Subcase (c) : $j \notin \Delta$ and $i \in \Delta$.

By [HMT1] 1.7.3 and 1.5.8 (ii) we have

$$s_i^j c_{(\Delta)} E(u, \phi) = s_i^j c_{(\Delta - \{i\})} c_i E(u, \phi) = c_{(\Delta - \{i\})} s_i^j c_i E(u, \phi).$$

Let $g^* = s_i^j c_i E(u, \phi)$. Then by [HMT1] 1.5.1 we have $g^* = c_j(c_i(E(u, \phi) \cap d_{ij})$. Let $v = u^i_{u(j)} = uo[i|j]$. Here and elsewhere $[i|j]$ denotes the replacement that maps i to j and otherwise coincides with the identity. Then $v(j) = u(j) = v(i)$. Also, it is easy to see that

$$c_i(E(u, \phi) \cap d_{ij}) = E(v, \phi \wedge x_i = x_j).$$

Since $i \in \Delta$, we can assume by fact 5.6 (i) that x_i does not occur in ϕ. Since v differs from u in at most the ith place, we have $\phi \in F_k(v)^{**}$. But $v(i) = v(j)$, and so $x_i = x_j \in F_k(v)^{**}$, thus $\phi \wedge x_i = x_j \in F_k(v)^{**}$. By fact 5.6 (i) we get that g^* hence g is in $G(k)^{**}$.

Case (b) : $g = d_{kl}$, where $k, l \in \alpha + k$

By [HMT1], 1.5.4 we have $s_i^k(d_{kl}) = d_{il}$ if $k \neq l$ and $s_i^j d_{kl} = d_{kl}$ if $j \notin \{k, l\}$. In either case we have $s_i^j g \in G(k)^{**}$.

Case (c) : g is an element in $G(k)^{**}$.

Follows from the two previous cases, since the s_i^j's $(i, j \in \alpha + k)$ are BA endomorphisms of the boolean algebra $G(k)^{**}$ (cf., [HMT1] 1.5.3 (i)).

(iii) Let $g = c_{(\Delta_1)}E(u_1, \phi_1) \cap c_{(\Delta_2)}E(u_2, \phi_2)$ where $\Delta_i \cap ind(\phi_i) = 0$ for $i \in \{1, 2\}$. Then $g = 0$ if there exists $j \in (\alpha+k)-(\Delta_1 \cup \Delta_2)$ such that $u_1(j) \neq u_2(j)$. Else, it is not difficult to check that $g = c_{(\Delta_1 \cap \Delta_2)}E(w, \phi_1 \wedge \phi_2)$, where w is defined as follows:

$$w(j) = u_1(j) = u_2(j) \text{ for } j \in \alpha + k - (\Delta_1 \cup \Delta_2)$$

$$w(j) = u_1(j) \text{ if } j \in \Delta_2 - \Delta_1$$

$$w(j) = u_2(j) \text{ if } j \in \Delta_1 - \Delta_2, \text{ and } w \text{ is defined arbitrarily on } \Delta_1 \cap \Delta_2.$$

Also by $ind(\phi_i) \cap \Delta_i = 0$, for $i \in \{1, 2\}$ and $w(j) = u_1(j) = u_2(j)$ for all $j \in \alpha + k - (\Delta_1 \cup \Delta_2)$ we can assume by 5.6 (i) that $\phi_i \in F_k(w)^{**}$, hence $\phi_1 \wedge \phi_2 \in F_k(w)^{**}$.

■

To economise on notation we recall a few notions from [HMT1]. A generalized diagonal element is a finite intersection of diagonal elements. A co-diagonal element is the complement of a diagonal element. A generalized co-diagonal element is a finite intersection of co-diagonal elements. In particular the unit of an algebra can be viewed as both a generalized diagonal and a generalized co-diagonal.

Fact 5.7. *For all* $0 \leq k < \omega$, $G(k)^{**} \in Ws_{\alpha+k}$.

Proof. Clearly $G(k)^{**}$ is a BA field of sets with greatest element V_k. Also $G(k)^{**}$, by definition, contains all diagonal elements. We are thus left to check cylindrifications, by the additivity of which, it suffices to show that for every $g \in G(k)^*$ and $j < \alpha + k$ we have $c_j g \in G(k)^{**}$. By 5.6 (iii) a typical element g of $G(k)^*$ is of the form
$a_0 \cap b \cap c \cap -a_1 \cap -a_2 \cdots \cap -a_n$, where $n \in \omega$, $a_0, a_1, \cdots a_n \in H(k)$, b is a generalized diagonal element, and c is a generalized co-diagonal element. Now fix $j < \alpha + k$ and let g be as indicated above.
Case (a) : $b = d_{jl} \cap b'$ where b' is a generalized diagonal element and $j \neq l$. Let $g^* = a_0 \cap b' \cap c \cap -a_1 \cdots \cap -a_n$. Then, of course $g^* \in G(k)^*$ and $c_j g = s^j_l g^*$. By fact 5.6 (i) the latter is in $G(k)^{**}$.
Case (b) : $a_0 = b = V_k$.
In this case we have $g = -c_{(\Delta_1)}E(u_1, \phi_1) \cdots \cap -c_{(\Delta_n)}E(u_n, \phi_n) \cap -d_{jl_1} \cdots \cap -d_{jl_m}$, where $n, m \in \omega$, $\Delta_k \subseteq_\omega \alpha + k$ for $1 \leq k \leq n$ and since $c_j(x \cap c_j y) = c_j x \cap c_j y$, we can assume that $j \notin \Delta_1 \cup \cdots \Delta_n$. Assume that $g \neq 0$, for else there is nothing more to prove. In particular, $j, l_1, \cdots l_m$ are pairwise distinct. We claim that in this case we have $c_j g = V_k$.
Indeed, let $s = \langle (a_l, i_l) \rangle_{l \in \alpha+k}$ be an arbitrary element in V_k. We will show that $s \in c_j g$, by which we will be done. Choose $k_j \notin \{u_1(j), \cdots, u_n(j)\} \cup \{i_{l_1}, \cdots i_{l_m}\}$, and let $u = (a_j, k_j)$. Let $z = s^j_u$. Assume that $1 \leq i \leq n$. Then

$s_u^j \notin c_{(\Delta_i)}E(u_i,\phi_i)$ because $s_u^j(j)=(a_j,k_j)\neq(a_j,u_i(j))$. Assume now that $1\leq i\leq m$. Then $z(j)=(a_j,k_j)\neq(a_{l_i},i_{l_i})=z(l_i)$, that is $z\notin d_{jl_i}$. We have shown that $z\in g$. Since s differs fom z in at most the jth place we get that $s\in c_jg$. We have proved that $c_jg=V_k$.

Case (c) :

$$g=c_{(\Delta_0)}E(u_0,\phi_0)\cap -c_{(\Delta_1)}E(u_1,\phi)\cap\cdots c_{(\Delta_n)}E(u_n,\phi_n)\cap -d_{jl_1}\cdots\cap -d_{jl_m},$$

where $n,m\in\omega$, $\Delta_k\subseteq_\omega \alpha+k$ for $0\leq k\leq n$ and as in the previous case, we can assume that $j\notin\Delta_0\cup\cdots\Delta_n$ and that $j,l_1,\cdots l_m$ are pairwise distinct.
We can further assume that $\{l_1,\cdots l_m\}\subseteq\Delta_0$, for if $l_k\notin\Delta_0$ for some $1\leq k\leq n$, then we can simply "ignore" the co-diagonal element with indices j and l_k, i.e the element $-d_{jl_k}$, because of the following:

$$c_{(\Delta_0)}E(u_0,\phi_0)\cap -d_{jl_k}=c_{(\Delta_0)}E(u_0,\phi_0\wedge x_j\neq x_{l_k})$$

if $u_0(j)=u_0(l_k)$, and is equal to $c_{(\Delta_0)}E(u_0,\phi_0)$ otherwise. Note too, that $\phi_0\wedge x_j\neq x_{l_k}\in F_k(u_0)^{**}$, whenever $\phi_0\in F_k(u_0)^{**}$ and $u_0(j)=u_0(l_k)$. In either case, g can be written in the form:

$$c_{(\Delta_0)}E(u_0,\psi)\cap\cdots -c_{(\Delta_n)}E(u_n,\phi_n)\cap\cap\{-d_{jl}:l\in\{l_1,\cdots l_m\}-l_k\},$$

where $c_{(\Delta_0)}E(u_0,\psi)\in H(k)$.
Now let

$$\Gamma=\Delta_0-\{l_1,\cdots l_m\}\quad\text{and } \Gamma_1=\{l_1,\cdots l_m\}.$$

Then $\Delta_0=\Gamma\cup\Gamma_1$. Let $S=\{j\}\times\Gamma_1=\{(j,l):l\in\Gamma_1\}$ and let d_S be the co-diagonal element: $\cap\{-d_{jl}:(j,l)\in S\}$.
Let $g*=-c_{(\Delta_1)}E(u_i,\phi_1)\cap\cdots -c_{(\Delta_n)}E(u_n,\phi_n)$, $g_1=c_{(\Gamma)}E(u_0,\phi_0)\cap g*$ and $g_2=c_{(\Gamma_1)}E(u_0,\phi_0)\cap d_S\cap g*$. Then by [HMT1] 1.7.3, and the additivity of c_j we have $c_jg=c_jg_1\cup c_jg_2$.
We shall show that $c_jg_1\in G(k)^{**}$ and $c_jg_2\in G(k)^{**}$, by which we will be done.
Proof of $c_jg_1\in G(k)^{**}$.

Write $g_1=c_{(\Gamma)}E(u_0,\phi_0)\cap\cdots -c_{(\Delta_n)}E(u_n,\phi_n)$.
We can assume that for $1\leq k\leq n$ and $i\notin\Gamma\cup\Delta_k$, we have $u_0(i)=u_k(i)$, for if $u_0(i)\neq u_k(i)$ for some $1\leq k\leq n$, and some $i\notin\Gamma\cup\Delta_k$, then $c_{(\Gamma)}E(u_0,\phi_0)\cap -c_{(\Delta_k)}E(u_k,\phi_k)=c_{(\Gamma)}E(u_0,\phi_0)$.
For the time being assume that $\phi_1=\phi_2=\cdots\phi_n=T$.
Claim
$c_jg_1=c_jc_{(\Gamma)}E(u_0,\phi_0)\cap -c_jc_{(\Delta_1)}E(u_1,T)\cap\cdots -c_jc_{(\Delta_n)}E(u,T)$.
Proof of claim

It is straightforward to show that r.h.s is contained in c_jg. Now for the opposite inclusion: Let $s\in c_jg_1$. Then there exists $b=(b_j,l_j)\in\cup V_k$ such that $s_b^j\in c_{(\Gamma)}E(u_0,\phi_0)$ and $s_b^j\notin c_{(\Delta_k)}E(u_k,T)$ for all $1\leq k\leq n$. Thus

$s \in c_j c_{(\Gamma)} E(u_0, \phi_0)$. We will show that $s \notin c_j c_{(\Delta_k)} E(u_k, T)$ for all $1 \leq k \leq n$, by which case we will be done. Fix $1 \leq k \leq n$ and assume to the contrary that $s \in c_j c_{(\Delta_k)} E(u_k, T)$. Then there exists $c = (c_j, k_j) \in \cup V_k$ such that $s^j_c \in c_{(\Delta_k)} E(u_k, T)$. For $x = (x_0, x_1)$ we write $(x)_0 = x_0$ and $(x)_1 = x_1$. By $s^j_c \in c_{(\Delta_k)} E(u_k, T)$ and $j \notin \Delta_k$ we get $(c)_1 = u_k(j)$. Similarly, by $s^j_b \in c_{(\Gamma)} E(u_0, \phi_0)$ and $j \notin \Gamma$ we get $(b)_1 = u_0(j)$. Recall that we assumed that $u_0(j) = u_k(j)$ since $j \notin \Gamma \cup \Delta_k$, thus $(b)_1 = (c)_1$. Since s^j_b and s^j_c differ in at most the jth place , $(b)_1 = (c)_1$ and $s^j_c \in c_{(\Delta_k)} E(u_k, T)$ we get $s^j_b \in c_{(\Delta_k)} E(u_k, T)$. Contradiction. Now the general case follows from this by noting that

$$-c_{(\Delta)} E(u, \phi) = c_{(\Delta)} E(u, \neg\phi) \cup -c_{(\Delta)} E(u, T),$$

and that $H(k)$ is closed under finite intersections.
We now turn to showing that $c_j g_2 \in G(k)^{**}$. This will be done by showing that $a = c_{(\Gamma_1)} E(u_0, \phi_0) \cap d_S$, hence $g_2 = a \cap g*$, can be written as a finite union of elements of the form $a_0 \cap -a_1 \cdots \cap -a_n$ where $a_i \in H(k)$. Then by the additivity of c_j, and the previous case we get the desired. We start off when Γ_1 consists of a single element m say. We compute $a = c_m E(u_0, \phi_0) \cap -d_{jm}$. For brevity set $u = u_0$. Let $v = u^m_{u(j)} = u \circ [m|j]$. Then $v(m) = v(j) = u(j)$. Moreover it is not hard to check that

$$a = (c_m E(u, \phi_0) \cap -E(v, T)) \cup E(v, \phi_0 \wedge x_j \neq x_m).$$

Since x_m does not occur in ϕ_0, and v differs from u in at most the m-th place, we infer from fact 5.6 (i) that $\phi_0 \in F_k(v)^{**}$. By $v(m) = v(j)$ we get that $x_j \neq x_m$ is also in $F_k(v)$ hence in $F_k(v)^{**}$. It follows that $\phi_0 \wedge x_j \neq x_m \in F_k(v)^{**}$. We have shown that $E(v, \phi_0 \wedge x_j \neq x_m) \in H(k)$. Thus a is as desired, since $c_m E(u, \phi_0)$ and $E(v, T)$ are in $H(k)$, too. Now for the general case. Assume that $\Gamma_1 = \{l_1, \cdots l_m\}$. Having treated the case when $m = 1$ we now assume that $m > 1$. Then

$$a = c_{(\Gamma_1)} E(u, \phi) \cap -d_S = \cap\{a_k : 1 \leq k \leq m\}$$

where $a_k = c_{(\Gamma_1 - \{l_k\})} c_{l_k} E(u, \phi) \cap -d_{jl_k}$. Therefore a, hence $g_2 = a \cap g^*$, is indeed the finite union of elements of the $a_0 \cap -a_1 \cdots \cap -a_n$ where $a_i \in H(k)$. Thus $c_j g_2$ is in $G(k)^{**}$. By this the proof of fact 5.7 is complete. ■

Fact 5.8. *$G(0)^{**} \cong Nr_\alpha G(k)^{**}$, for all $0 \leq k < \omega$*[15]

Proof. Let $k \geq 0$. Define $i(k)$ as in fact 2.3. That is for $a \in G(0)^{**}$, let $i(k)(a) = \{t \in V_k : t \upharpoonright \alpha \in a\}$. Then $i(k) \in Ism(A(\alpha), Nr_\alpha C(\alpha + k))$, where

[15] We should point out that the complete analogue of fact 2.3 is true: That is for $0 \leq l < k < \omega$ we have $G(l)^{**} \cong Nr_{\alpha+l} G(k)^{**}$. Fact 5.8 is the special case when $l = 0$, which is all what we need to show that $\mathbf{A}(\alpha) \in Nr_\alpha Ws_{\alpha+k}$, for all $k < \omega$.

$C(\alpha+k)$ is the full $Ws_{\alpha+k}$ with unit V_k. We will show that $i(k)G(0)^{**} \subseteq G(k)^{**}$. For that it clearly suffices to show that for all $u \in {}^{\alpha}\alpha^{(Id)}$, and $\phi \in F_0(u)$, we have $i(k)E(u,\phi) \in G(k)^{**}$. Let u and ϕ be as indicated: Then $i(k)E(u,\phi) = \cup\{E(v,\phi) : v \in {}^{\alpha+k}\alpha^{(Id)} : v \restriction \alpha = u\} = c_\alpha \cdots c_{\alpha+k-1}E(u*,\phi)$ where $u* = u \cup \langle 0 : \alpha \leq i < \alpha+k\rangle$. Since $F_0(u) \subseteq F_k(v)$ whenever $v \in {}^{\alpha+k}\alpha^{(Id)}$ is such that $v \restriction \alpha = u$, we get $\phi \in F_k(u*)^{**}$, hence $i(k)E(u,\phi) \in G(k)^{**}$. We have shown that $i(k) \in Ism(G(0)^{**}, Nr_\alpha G(k)^{**})$. We will now show that $i(k)$ is actually onto $Nr_\alpha G(k)^{**}$. Since the c_i's are additive, it suffices to show that for all $g \in G(k)^*$, there exists $a \in G(0)^{**}$, such that $i(k)a = c_\alpha \cdots c_{\alpha+k-1}g$.
Case (a): $g = E(v,\phi)$, $v \in {}^{\alpha+k}\alpha^{(Id)}$, and $\phi \in F_k(v)^{**}$.
Then by fact 5.6 (i) there exists $\psi \in F_k(v)^*$ such that $var(\psi) \cap \{x_\alpha, \cdots x_{\alpha+k-1}\} = \emptyset$ and $\mathbf{W} \models \exists x_\alpha \cdots \exists x_{\alpha+k-1}\phi \equiv \psi$. Let $u = v \restriction \alpha \in {}^{\alpha}\alpha^{(Id)}$. By noting that $F_0(u)^* = \{\phi \in F_k(v)^* : var(\phi) \subseteq \alpha\}$, we get that $\psi \in F_0(u)^*$. Moreover we have $i(k)E(u,\psi) = c_\alpha \cdots c_{\alpha+k-1}E(u,\phi)$.
Case (b): $g = a \cap c$, where $a = -a_0 \cap \cdots - a_n$, $a_i \in H(k)$ and c is a generalized codiagonal.
If $c_\alpha \cdots c_{\alpha+k-1}g = g$, then $g \in G(0)^{**}$. Else $c_\alpha \cdots c_{\alpha+k-1}g$ is either 0 or V_k, pending on whether $g = 0$ or not. The choice of $a \in G(0)^{**}$ in this case is also obvious.
Case (c): $g = d_0 \cap -d_1 \cap \cdots - d_n \cap b \cap c$, where $d_i \in H(k)$, b is a generalized diagonal, and c a generalized codiagonal.
Assume that $1 \leq k < \omega$; else there is nothing to prove.
Let $\Gamma = \{\alpha, \cdots \alpha+k-1\}$. By the proof of fact 5.7 , we have $c_{(\Gamma)}g$ is the finite union of elements of the form $a_0 \cap -a_1 \cdots \cap -a_n \cap b \cap c$ where $a_0, \cdots a_n \in H(k)$, b is a generalized diagonal and c a generalized codiagonal. Further we can assume that if $b = \cap\{d_{ij}^{V_k} : i,j \in \Delta_1\}$ and $c = \cap\{-d_{ij}^{V_k} : i,j \in \Delta_2, i \neq j\}$ then $\Delta_i \subseteq_\omega \alpha$, for $i \in \{1,2\}$. By the additivity of generalized cylindrifications (namely of $c_{(\Gamma)}$), we can and will assume that

$$c_{(\Gamma)}g = c_{(\Gamma_0)}E(v_0,\phi_0) \cap c_{(\Gamma_1)}E(v_1,\phi_1) \cdots \cap -c_{(\Gamma_n)}E(v_n,\phi_n) \cap b \cap c,$$

where for each $i \leq n$ we have $\Gamma \subseteq \Gamma_i \subseteq_\omega \alpha+k$, $\phi_i \in F_k(v_i)^{**}$, b is a generalized diagonal and c is a generalized codiagonal as indicated above. By fact 5.6 (i) we can assume that $var(\phi_i) \cap \{x_\alpha, \cdots x_{\alpha+k-1}\} = \emptyset$. For $i \leq n$ let $u_i = v_i \restriction \alpha$. Then $u_i \in {}^{\alpha}\alpha^{(Id)}$, and by fact 5.6 (i) $\phi_i \in F_0(u_i)^{**}$. Let $b' = \cap\{d_{ij}^{V_0} : i,j \in \Delta_1\}$ and $c' = \cap\{d_{ij}^{V_0} : i,j \in \Delta_2, i \neq j\}$.
Let $a = c_{(\Gamma_0-\Gamma)}E(u_0,\phi_0) \cap \cdots c_{(\Gamma_n-\Gamma)}E(u_n,\phi_n) \cap b' \cap c'$. Then $a \in G(0)^{**}$ and an easy checked $i(k)a = c_{(\Gamma)}g$. ∎

Fact 5.9. $G(0)^* = A(\alpha)$.

Proof. Let $u \in {}^{\alpha}\alpha^{(Id)}$ be one to one and $r \in R$. Then by definition we have

$$p(u,r) = E(u, C_r(x_0, x_1, \cdots)).$$

Hence $p(u,r) \in F_0(u)$. Since $G(0)^{**}$ is a Ws_α containing the generators of $A(\alpha)$ it follows that that $A(\alpha) \subseteq G(0)^{**}$. For the other inclusion it is enough to show that for $u \in {}^\alpha\alpha^{(Id)}$ and $\phi \in F_0(u)$, we have $E(u,\phi) \in A(\alpha)$. We start by showing that $E(u,T) \in A(\alpha)$ for all such u.

Case (a): u is one to one.

Let $r \in R$. Then $c_0(p(u,r)) \cap c_1(p(u,r)) = E(u,T) \in A(\alpha)$.

Case (b): u is not one to one.

Let J be a finite subset of α, such that $u(J) \subseteq J$ and $u(i) = i$ whenever $i \notin J$. Such a J exists, but of course is not unique. Fix one such J. Let $u' = u|J$. Then $u' : J \to J$. Let $n = |Rgu'|$. Then $n < |J|$, because u hence u' is not one to one. We next procced as in the finite dimensional case; correlating a permuation v to u' as follows: Let $y \in {}^nJ$ such that $y(0) < \cdots < y(n-1)$ and $\{u'_{y(0)}, \cdots u'_{y(n-1)}\} = Rgu'$.

For $i < l$ put $m_{y(i)} = \min(u'^{-1}(u'_{y(i)}))$. Choose $v \in S_J$ (a permutation on J) such that $v(m_{y(i)}) = u'_{y(i)}$. Such a v exists. Let $I = J - \{m_{y(0)}, \cdots, m_{y(l-1)}\}$; suppose that $I = \{j_0, \cdots j_k\}$, $k \in \omega$ and $j_0 < j_1 \cdots < j_k$. Let $t_i^l(x) = c_l x \cap d_{il}$. Let $v* = v \cup Id_{\alpha - J}$. Then $v*$ is a permutation on α, and like the finite case, we have

$$t_{j(k)}^{u_{j(k)}} \circ \cdots \circ t_{j(0)}^{u_{j(0)}} E(v*,T) = E(u,T).$$

From case (a) we get that $E(u,T) \in A(\alpha)$. Now let $u \in {}^\alpha\alpha^{(Id)}$ and $i,j \in ker(u)$. Then $E(u, x_i = x_j) = E(u,T) \cap d_{ij} \in A(\alpha)$. Also for all $u, v \in S_\alpha$, we have by Lemma 1**(i), $E(u, C_r(x_0, x_1, \cdots)) = E(u, C_r(x_{v(0)}, \cdots) = p(u,r)$. Finally by noting that $E(u, \neg\phi) = E(u,T) - E(u,\phi)$ we get the desired. ■

Fact 5.10. *$A(\alpha) \in INr_\alpha Ws_\beta$, for all $\beta > \alpha$.*

Proof. By fact 5.9, it suffices to show that $A(\alpha) \in Nr_\alpha Ws_{\alpha+\beta}$ for infinite β. So let $\beta \geq \omega$. Let $V_0 = {}^\alpha U^{(s)}$ be the unit of $A(\alpha)$.

Let $s_\beta = s \cup \{(w, 0) : \alpha \leq i < \beta\}$. Then $s_\beta \in {}^\beta U$. Let V_β be the weak space ${}^{\alpha+\beta}U^{(s_\beta)}$ and let $C(\alpha+\beta)$, or simply $C(\beta)$, be the full $Ws_{\alpha+\beta}$ with unit V_β. For $k < \omega$ let $i(k,\beta)$ be the function with domain $G(k)^{**}$, such that $i(k,\beta)a = \{s \in V_\beta : s \restriction \alpha + k \in a\}$, $a \in G(k)^{**}$. Then $i(k,\beta) : G(k)^{**} \to C(\beta)$. Let $G(\beta) = Sg^{C(\beta)}\{i(k,\beta)G(k)^{**} : 0 \leq k < \omega\}$. Then $G(\beta) \in Ws_{\alpha+\beta}$. Moreover, it is easy to show that $G(\beta) = \cup\{i(k,\beta)G(k)^{**} : 0 \leq k < \omega\}$. From this together with facts 5.8 and 5.9, we get (using the same ideas of the proof of fact 2.4) that $A(\alpha) \cong Nr_\alpha G(\beta)$, thus $A(\alpha) \in INr_\alpha Ws_{\alpha+\beta}$. ■

By establishing fact 5.10, the proof of Theorem 1 is complete.

Remarks .

(a) Let K be a class of algebras. Then UpK and UfK denote the classes of ultraproducts and ultraroots of members of K, respectively. ElK denotes the smallest elementary class containing K. That is (see eg [HMT1]),

$ElK = UfUpK$. Let $\beta > \alpha > 1$. Since $B(\alpha)$ is elementary equivalent to $A(\alpha)$, then by the Keisler-Shelah ultrapower Theorem (cf [CK], Thm 6.1.15) we have $B(\alpha) \in UfUp\{A(\alpha)\}$. It then follows from

$$INr_\alpha Gs_\beta = Nr_\alpha IGs_\beta = Nr_\alpha UpGs_\beta = UpNr_\alpha Gs_\beta$$

(cf [N83], Cor.2), and from fact 4.3 and fact 5.3 that $B(\alpha) \in UfNr_\alpha Ws_\beta - Nr_\alpha CA_\beta$. In particular, some ultrapower of $B(\alpha)$ is in $Nr_\alpha CA_{\alpha+\omega}$.

(b) From now on, for undefined categorial terms, we refer the reader to [AHS]. Let K be a class of algebras. Then we identify K with the (concrete) catogory whose objects are the members of K, and whose morphisms are the homomorphisms between members of K. Let $\beta > \alpha > 1$. Assume that $Ws_\beta \subseteq L \subseteq CA_\beta$. Then $K = Nr_\alpha L$ is not a complete category , i.e not closed under all limits. We will show that in fact K is not closed w.r.t taking equalizers (which are special limits). To see this, let $A(\alpha)$ and $B(\alpha)$ be the algebras dealt with so far. Let $P = \cup\{P(u) : u \in S_\alpha\}$. Let $g : P(Id) \to P(Id)$ be any function such that $g(p(Id, r)) = p(Id, r)$ iff $r \in N$. Let $f \in Aut(A(\alpha))$ be such that $g \subseteq f$ and $f(x) = x$ for all $x \in P - P(Id)$. Such an f exists by fact 3.2 (the finite dimensional case) and fact 5.1 (v) (the infinite dimensional case). It is now easy to see that $B(\alpha)$ is the equalizer of f and the identity map on $A(\alpha)$. That is K is not closed under forming equalisers, since $A(\alpha)$ is in K while $B(\alpha)$ is not.

In particular, $Nr_\alpha RCA_\beta$ and $Nr_\alpha CA_\beta$, though closed under products, are not complete categories (It is known that a category is complete iff it is closed under products and equalizers).

§6. DISCUSSION AND RELATED RESULTS

(1) SC_α, QA_α and QEA_α stand for the classes of Pinters substitution, Halmos'quasipolyadic and quasipolyadic equality algebras of dimension α, respectively. We refer the reader to the appendix of [N96] for an explicit (finite schema) axiomatisation of such varieties.

Let $K \in \{SC, QA, QEA\}$.

Generalizing the [N83] result cited in the introduction, it is shown in [ANS98] that for $1 < \alpha < \beta$, the class of α-dimensional neat reducts of algebras in K_β, or $Nr_\alpha K_\beta$ for short, is not closed under forming subalgebras. This result is announced in [ANS96]. In [S98a], it is shown that $Nr_2 K_\beta$, $(\beta > 2)$ is not closed under forming elementary subalgebras (equivalently, not closed under ultraroots). It is absolutely straightforward to check that the construction adopted herein generalizes the [S98a] result to higher dimensions for SC's. More precisely:

Corollary 1 . *Let $1 < \alpha < \beta$. Then $Nr_\alpha SC_\beta$ is not elementary.*

Sketch of proof. Passing to reducts; that is let $Rd_{sc}A(\alpha)$ [16] be the SC reduct of $A(\alpha)$ and let $Rd_{sc}B(\alpha)$ be the SC reduct of $B(\alpha)$. Then $Rd_{sc}A(\alpha) \in Nr_\alpha SC_{\alpha+\omega}$ and $Rd_{sc}B(\alpha) \notin Nr_\alpha SC_{\alpha+1}$

It is also straightforward to show that the complete analogue of Theorem 1 continue to hold for SC's. (Since reducts of weak set algebras are weak set algebras). ■

However the construction adopted herein *does not* settle the QA and QEA cases. This follows from the fact that in these algebras the term ${}_\alpha s(0,1)$ (that played the key role in showing that $B(\alpha)$ is not a neat reduct, cf. facts 4.3 and 5.3) is actually a basic operation in these algebras, whereas the the proof adopted herein very much depended on the fact that this term *is not* (even) term definable in CA's, a result of Andreka and Nemeti (cf., [AN83]). We should mention , though that, $A(\alpha)$ is actually the reduct of a QEA_α, [17] and even more this expansion of $A(\alpha)$ is in $Nr_\alpha QEA_\beta$.

Therefore it seems very plausible though, that the [S98a] result, can be generalized to higher dimensions for QA's and QEA's using a combination of the ideas presented herein, and those in [S98a]. [18]

More precisely:

Conjecture 1 . *Let $\beta > \alpha > 1$, and $K \in \{QA, QEA\}$. Then $UfNr_\alpha K_\beta \neq Nr_\alpha K_\beta$.*

(2) Let $1 < \alpha < \beta$. Let $K \in \{SC, CA, QA, QEA\}$.

It is proved in [N83] and [ANS98] that $SNr_\alpha K_\beta \not\subseteq ElNr_\alpha K_\beta$. In particular, in view of Theorem 1 and Corollary 1, the following inclusions are

[16] Let $A \in CA_\alpha$. Then $Rd_{sc}A = \langle A, \cup, \cap, -, c_i, s_i^j \rangle_{i,j\in\alpha}$, where s_i^j is the unary term as defined in 1.5.1 [HMT1]. Pinters algebars, are thus (generalized) reducts of CA's

[17] A QEA_α is an algebra of the form $\langle A, p_{ij} \rangle_{i,j\in\alpha}$ where $A \in CA_\alpha$ and the p_{ij}'s are unary operations on A, abstracting the familiar metalogical operation of interchanging the (free) occurances of the variables x_i and x_j in a formula ϕ. The p_{ij}'s are BA automorphisms of A. $A(\alpha)$ can be expanded to a representable QEA_α by defining $p_{ij}X = \{s \in V : so[i,j] \in X\}(X \in (A(\alpha))$. Here $[i,j]$ is the transposition on α interchanging i, j. and leaving everything else fixed. We note that $p_{01} = {}_\alpha s(0,1)$.

[18] In [S98a], instead of ${}_2(s(0,1))$ a "composition like" term (that *is not* term definable in QEA's) was used.

proper for $L \in \{SC, CA\}$:

$$Nr_\alpha L_\beta \subset ElNr_\alpha L_\beta \subset SNr_\alpha L_\beta.$$

In passing we note that $SNr_\alpha K_\beta$ is a variety (cf, [HMT1], 2.6.33). [19]

In [ANS98], the QA and QEA analogue of Theorem 2 are also proved.

(3) Let $K = \langle K_\alpha : \alpha \geq \omega \rangle$ be a system of varieties, not necessarily definable by schemes in the sense of [HMT2] Def.,5.6.12, but for which the notion of neat reducts make sense, so that in particular , we can talk about the class $SNr_\alpha K_{\alpha+\omega}$. For a "concrete" K, and in the presence of a NET, this class is just the class of representable algebras. (See for eg [S97]). In addition to cylindric algebras (CA's), typical examples include the most well known varients of CA's namely $K \in \{SC, QA, QEA, PA\}$. Here PA denotes Halmos' infinite dimensional polyadic algebras in the sense of [HMT2], Def., 5.4.1.

We say that K has the S neat reduct property if no new α- dimensional elements are generated from α dimensional elements even in the presence of ω extra operations. Formally; for all $\alpha \geq \omega$, for all $C \in K_{\alpha+\omega}$, and for all $X \subseteq Nr_\alpha C$, we have

$$Sg^{Nr_\alpha C} X = Nr_\alpha (Sg^C X).$$

The following is investigated in [ANS98]: Under what conditions on (rather abstract) K are (any two of) the following three conditions equivalent:

(a) For all α, the class $SNr_\alpha K_{\alpha+\omega}$ has the strong amalgamation property .

(b) In $SNr_\alpha K_{\alpha+\omega}$ epis (in the categorial sense) are surjective.

(c) K has the S neat reduct property.

It is known (see for eg [N96]) that (a) and (b), are the algebraic counterparts of the familiar metalogical properties of Craig and Beth, respectively. By Theorem 2 CA's do not have the S neat reduct property. In more detail, taking $X = B(\alpha)$, and $C = C(\alpha + \omega)$ we get:

[19] In fact K_α is the first member of a descending sequence of varieties: $SNr_\alpha K_{\alpha+k}$, $k \in \omega$, approximating RK_α. It is easy to check that $SNr_\alpha K_{\alpha+\omega} \subseteq .....SNr_\alpha K_{\alpha+k+1} \subseteq SNr_\alpha K_{\alpha+k} \subseteq SNr_\alpha K_\alpha = K_\alpha$. It is known, see for eg [N96] that RK_α (the first member of the sequence) $\neq SNr_\alpha K_{\alpha+k}$ for $\alpha > 2$ and finite k. Pigozzi showed (unpublished) that the above inclusions are proper when $K = CA$ and α is infinite. Recently, Hirsch, Hodkinson and Maddux proved that, in the CA case, the above inclusions are also proper when $\alpha = n$ is finite ($n > 2$). That is $SNr_n CA_{n+k+1} \neq SNr_n CA_{n+k}$, for all finite k. Besides generalizing Monks result quoted above (that $RCA_n \neq SNr_n CA_{n+k}$ for all finite k), this solves problem 2.12 in [HMT1].

$B(\alpha) = Sg^{Nr_\alpha C}B(\alpha) \neq Nr_\alpha(Sg^C B(\alpha)) = A(\alpha)$. As previously noted, the above (counterexample) also settles item (v) in the introduction of [HMT2]. (See also [M91], p.724, and [ANS98] for further elaboration). We should also mention, that the connection of (a) and (c) was studied, in the context of CA's in [Pi71]. (See 2.2.11-2.2.20, therein). [20] The logical implications of (c) not holding for CA's are hinted upon in [N83].p 406. Logics corresponding to $SNr_\alpha K_{\alpha+\omega}$, $K \in \{SC, CA\}$ were also investigated in [AGN77]. That PA has the S neat reduct property, was used by Johnson in [J70], to show that (a) and (b) hold for (representable) PA_α's, $(\alpha \geq \omega)$, equivalently that the properties of Beth and Craig hold for Kieslers logics.

In (4) and (5) below, n stands for a finite ordinal.

(4) In [S97], the class Nr_nCA_ω is extensively studied in connection to first order logic. For example it is shown in [S97], that countable algebras in Nr_nCA_ω satisfy the algebraic version of Henkin-Orey's omitting types theorem, (cf, [CK], Theorem 2.2.9), or OTT, for short. This provides a partial answer to the following question: When does the OTT, hold for L_n theories, i.e first order theories restricted to the first n variables?.

(5) In [AS98], the OTT for L_n theories is further investigated in its own right, and in connection to the notion of complete representations in algebraic logic. (Cf., [HH97a],and [HH97b]). It is shown therein, that atomic countable algebras in Nr_nCA_ω are completely representable, or CR for short. This is generally not true, for atomic countable algebras in SNr_nCA_ω, if $n > 2$. Recall that by the NET, $SNr_nCA_\omega = RCA_n$. We refer the reader to [HH97b] and [An97] for examples of countable atomic $RCA_n's$, $n > 2$ that are not CR. These counterexamples are used in [AS98] to show that a weak version of the OTT fails for L_n, if $n > 2$. The condition $n > 2$ is important here, since it is known that every atomic (not necessarily countable) RCA_n, $n \leq 2$ is CR. The latter can be easily recovered from the proof of Thm 3.2.65 in [HMT2].

The class of countable completely representable CA_n's, or CR_n for short, is further characterised via special neat embeddings in [AS98]. In more detail, let A and B be boolean algebras, such that $A \subseteq B$. B is said to be a faithful extension of A, if for all $X \subseteq A$, whenever $supX = 1$ in A, then $supX = 1$ in B.

[20] In [Pi71], the following was proved: Let $L \in RCA_\alpha$. If L has the amalgamation property , briefly AP and whenever $A \in L$ is a generating subreduct of $B \in CA_{\alpha+\omega}$, (here the NET is actually used) then $A = Nr_\alpha B$; then L has strong AP, briefly SAP. In [ANS98], it is shown that conversely when generating subreducts in L are not neat reducts, then SAP fails for L, even if it already has AP. The following stronger result is also proved: If $Ws_\alpha \subseteq L \subseteq RCA_\alpha$, then L does not have SAP even w.r.t RCA_α. The reader is refered to [Pi71] for definitions.

$A \in CA_n$ is said to be strongly representable, or SR for short, if there exists $B \in CA_\omega$ such that $A \subseteq Nr_nB$, and B is a faithful extension of A. The following is proved in [AS98]:

Let SR_n denote the class of strongly representable CA_n's. Let $A \in CA_n$ be atomic and countable. Then $A \in CR_n$ iff $A \in SR_n$.

The proof is a Baire category argument at heart. We do not know, how much "countability" is needed in the above result when $n > 2$, though. That is:

QUESTION 1 . *Let $n > 2$. Let $A \in SR_n$, with $|A| > \omega$ be atomic. Is A completely representable?*

In particular are atomic (not necessarily countable) algebras in Nr_nCA_ω, $n > 2$ completely representable?

We point out that the converse of the above is true, regardless of the cardinality of the algebra involved. That is if $A \in CA_n$ is completely representable, then $A \in SR_n$.

In this connection, we note that $A(n)$ and $B(n)$ are both in SR_n, since, as easily checked, they are completely representable. Also Nr_nCA_ω is a proper subclass of SR_n; since $B(n) \notin Nr_nCA_\omega$. We do not know whether there are *countable* algebras in SR_n, that are not in Nr_nCA_ω. A candidate would be a countable elementary subalgebra of $B(n)$. We should mention that countable algebras in SR_n [21], still admit an OTT, in the following sense: Let T be a countable L_n theory. If the RCA_n of formulas corresponding to T (in the sense of [HMT2] sec 4.3), is in SR_n, then any countable family of non principal types in T are *omissable*. Here types, non principal types and omissable are defined as one might expect; i.e the usual way. [22] Also it is shown therein that the omission of $< {}^\omega 2$ (not necessarily countably) many L_n types, of a countable L_n theory, lead to propositions that are independent of ZFC. In fact, the statement that countable algebras in SR_n admit omission of $< {}^\omega 2$ non principal types, can be regarded as an instance of the famous (independent) *Martins axiom* in set theory, restricted to countable BA's.

A different characterisation of CR_n, using game-theoretic techniques is given in [HH97b], to show that CR_n is not elementary, iff $n > 2$. It seems

[21] It is shown in [AS98], using the characterisation of countable CR_n's, how SR_n can be regarded as the best possible class of RCA_n's admitting an OTT

[22] Let T be an L_n theory. A type is simply a consistent set of L_n formulas. A type Γ is non principal if no L_n formula consistent with T isolates it: i,e there is no ϕ with $var(\phi) \subseteq \{x_0, ...x_{n-1}\}$ such that ϕ is consistent over T and $T \models \phi \to \psi$, for every $\psi \in \Gamma$. Γ is omissable if there exists a countable model $M \models T$ such such that no assighnment in M satisfies all formulas in Γ.

that one can make the translation from the “Baire category” approach adopted in [AS98] to Hirsch-Hodkinsons games, both being instances of Robinsons finite forcing, by way of the Banach-Mazur Theorem (cf [H] p. 358).

(6) In contrast to $K \in \{SC, CA, QA, QEA\}$, it is shown in [S98b] that if K_α, with $\omega \leq \alpha \leq \omega + \omega$, is the variety (of algebras of α-ary relations) introduced and investigated in [Sa98], then $K_\omega = Nr_\omega K_{\omega+\omega}$. In particular, neat reducts are closed under forming subalgebras. Using the ideas presented in [AS98] and [ANS98] an OTT and a Beth-definability Theorem is also proved for the corresponding logics. These, as noted in the introduction, are expansions of first order logic without equality.

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