Conceptual structure of spacetimes
and
Category of concept algebras

Hajnal Andréka, Judit Madarász, István Németi, Gergely Székely

MTA Rényi Institute

Categorical Philosophy of Science, Munich, April 26, 2019.

Plan of the talk

1. The concept algebra of a concrete physical theory: Special relativistic spacetime.
2. The category of concept algebras of a(n arbitrary) language: Universal algebraic logic.

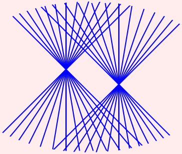
PART I

Concept algebra of a concrete physical theory:
special relativistic spacetime

What is special relativistic spacetime $\mathfrak{SR}$?

Definition (Relativistic Spacetime $\mathfrak{SR}$)

$\mathfrak{SR}$ is the system of timelike straight lines:



$$\mathfrak{SR} = \left\langle \mathbb{R}^4, \mathrm{col}^t \right\rangle$$

$\mathrm{col}^t(p, q, r) \iff p, q, r$ are on a timelike straight line.

We will show that from timelike collinearity one can define the full-fledged scale-invariant Minkowski spacetime: lightlike connectedness, Minkowski-equidistance, Minkowski-orthogonality, etc.

What is the concept algebra of $\mathfrak{SR}$?

The concept algebra of $\mathfrak{SR}$

Definition (Concept)

A concept in $\mathfrak{SR}$ is the extension of any open formula.
If $\varphi(x_1, \ldots, x_n)$ is a formula in the language of $\mathfrak{SR}$ with free variables $x_1, \ldots, x_n$, its extension in $\mathfrak{SR}$ is

$$\varphi(x_1, \ldots, x_n)^{\mathfrak{SR}} = \{\langle a_1, \ldots, a_n\rangle : \mathfrak{SR} \models \varphi(a_1, \ldots, a_n)\}.$$

Definition (Concept Algebra)

The concept algebra of $\mathfrak{SR}$ is the natural algebra of these concepts, where the operations are defined by the connectives of our language:

$$\mathrm{CA}(\mathfrak{SR}) = \left\langle \left\{ \varphi^{\mathfrak{SR}} : \varphi \text{ is in the language of } \mathfrak{SR} \right\}, \wedge, \neg, \exists x_n \right\rangle_{n \in \mathbb{N}},$$

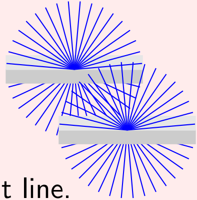
where

$$\varphi^{\mathfrak{SR}} \wedge \psi^{\mathfrak{SR}} = (\varphi \wedge \psi)^{\mathfrak{SR}}, \neg\varphi^{\mathfrak{SR}} = (\neg\varphi)^{\mathfrak{SR}}, \exists x_n \varphi^{\mathfrak{SR}} = (\exists x_n \varphi)^{\mathfrak{SR}}.$$

You get two in one

Definition (Classical non-Relativistic Spacetime $\mathfrak{NT}$)

$\mathfrak{NT}$ is the system of non-horisontal straight lines:

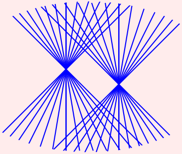


$$\mathfrak{NT} = \left\langle \mathbb{R}^4, \mathrm{col}^\infty \right\rangle$$

$\mathrm{col}^\infty(p, q, r) \iff p, q, r$ are on a slanted straight line.

Definition (Relativistic Spacetime $\mathfrak{SR}$)

$\mathfrak{SR}$ is the system of timelike straight lines:



$$\mathfrak{SR} = \left\langle \mathbb{R}^4, \mathrm{col}^t \right\rangle$$

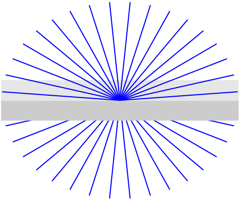
$\mathrm{col}^t(p, q, r) \iff p, q, r$ are on a timelike straight line.

Speed limit

Summing up:

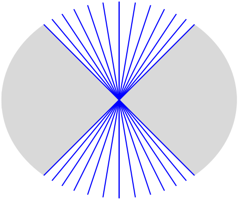
Newton spacetime

$$\mathfrak{NT} = \left\langle \mathbb{R}^4, \mathsf{col}^{\infty} \right\rangle$$



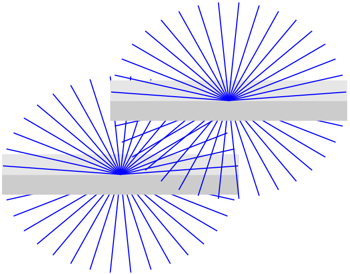
Einstein spacetime

$$\mathfrak{SR} = \left\langle \mathbb{R}^4, \mathsf{col}^{t} \right\rangle$$



Newton spacetime

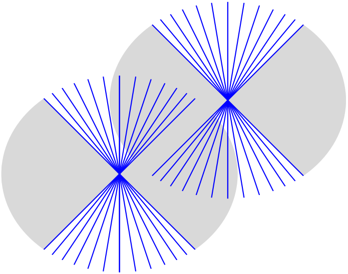
$$\mathfrak{NT} = \langle \mathbb{R}^4, \mathrm{col}^\infty \rangle$$



The grey parts form an equivalence relation

Einstein spacetime

$$\mathfrak{SR} = \langle \mathbb{R}^4, \mathrm{col}^t \rangle$$



Transitive closure of the gray parts is everything

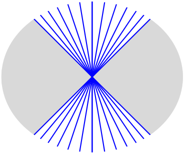
Theorem

No nontrivial equivalence relation can be defined in $\mathfrak{SR}$.

Theorem

No nontrivial equivalence relation can be defined in $\mathfrak{S}\mathfrak{R}$.

Proof. First show that any two timelike connected pairs of events can be taken to each other by an automorphism of $\mathfrak{S}\mathfrak{R}$.



Einstein spacetime

$$\mathfrak{S}\mathfrak{R} = \left\langle \mathbb{R}^4, \mathsf{col}^t \right\rangle$$

Lorenz transformations are automorphisms of $\mathfrak{S}\mathfrak{R}$.
Key players in relativity theory.

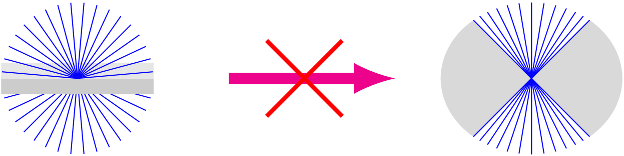
Do the same for spacelike and lightlike connected pairs of events. Then show that the transitive closure of each of these relations have one block. Q.E.D.

Newton spacetime

$$\mathfrak{NT} = \left\langle \mathbb{R}^4, \mathrm{col}^\infty \right\rangle$$

Einstein spacetime

$$\mathfrak{SR} = \left\langle \mathbb{R}^4, \mathrm{col}^t \right\rangle$$



Theorem (Corollary)

$\mathfrak{NT}$ *cannot be interpreted in* $\mathfrak{SR}$.

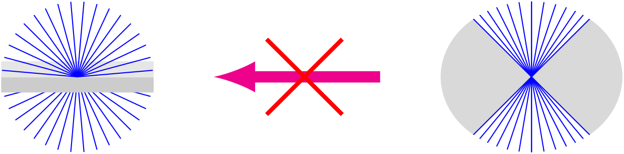
Interpretations are homomorphisms between the concept algebras.

Newton spacetime

$$\mathfrak{NT} = \left\langle \mathbb{R}^4, \mathrm{col}^{\infty} \right\rangle$$

Einstein spacetime

$$\mathfrak{SR} = \left\langle \mathbb{R}^4, \mathrm{col}^{t} \right\rangle$$



Theorem

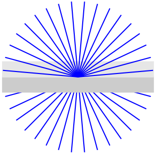
$\mathfrak{SR}$ cannot be interpreted in $\mathfrak{NT}$, either.

Reason: $\mathfrak{SR}$ is conceptually richer, more relations can be defined in $\mathfrak{SR}$ than in $\mathfrak{NT}$ (even when $\mathfrak{NT}$ is enriched with more structure).

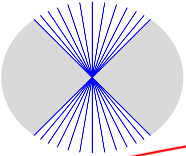
Let's see.

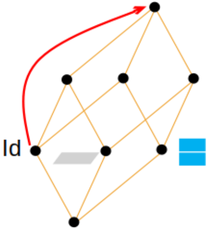
Structure of binary relations

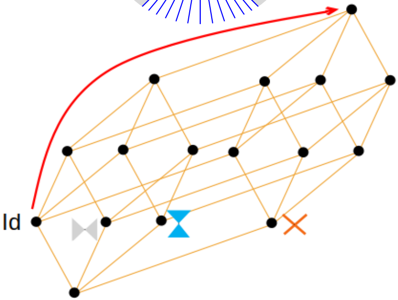
Newton spacetime
$\mathfrak{NT} = \langle \mathbb{R}^4, \mathsf{col}^\infty \rangle$



Einstein spacetime
$\mathfrak{SR} = \langle \mathbb{R}^4, \mathsf{col}^t \rangle$





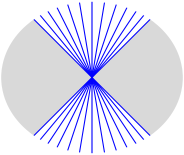


We get the same picture if we use a stronger language, e.g., SOL.
Except that we have concepts concerning subsets of $\mathbb{R}^4$, too!

Theorem

Lightlike connectedness can be defined from timelike connectedness in $\mathfrak{SR}$ by using 4 variables.

Proof.



Einstein spacetime

$$\mathfrak{SR} = \left\langle \mathbb{R}^4, \mathsf{col}^t \right\rangle$$

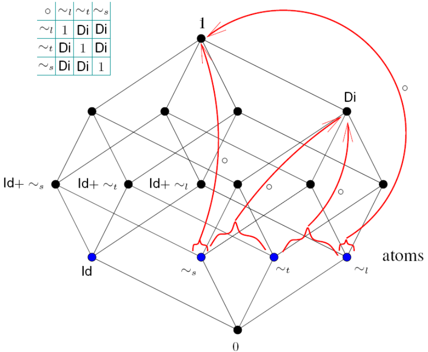
Theorem

Lightlike connectedness cannot be defined from timelike connectedness in $\mathfrak{S}\mathfrak{R}$ by using only 3 variables.

Proof.

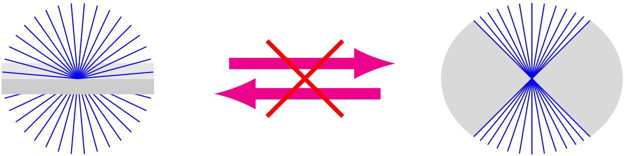
In the relation algebra of the binary definable relations, timelike connectedness does not generate lightlike connectedness. By a theorem from algebraic logic, this implies that lightlike connectedness cannot be defined with a FOL-formula using only 3 variables.

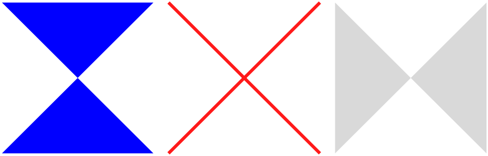
$\circ$	$\sim_l$	$\sim_t$	$\sim_s$
$\sim_l$	1	Di	Di
$\sim_t$	Di	1	Di
$\sim_s$	Di	Di	1



Newton spacetime
$\mathfrak{NT} = \langle \mathbb{R}^4, \mathsf{col}^\infty \rangle$

Einstein spacetime
$\mathfrak{SR} = \langle \mathbb{R}^4, \mathsf{col}^t \rangle$





these are all definitionally equivalent

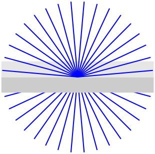
Theorem

$\mathfrak{NT}$ is not definitionally equivalent to any structure that has only binary relations.

Structure of ternary relations

Newton spacetime

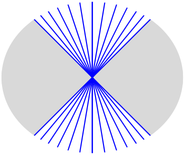
$\mathfrak{NT} = \langle \mathbb{R}^4, \mathsf{col}^\infty \rangle$



Eucl circles *cannot* be defined.

Einstein spacetime

$\mathfrak{SR} = \langle \mathbb{R}^4, \mathsf{col}^t \rangle$



Mink circles *can* be defined

Both:

Col can be defined in the grey part.

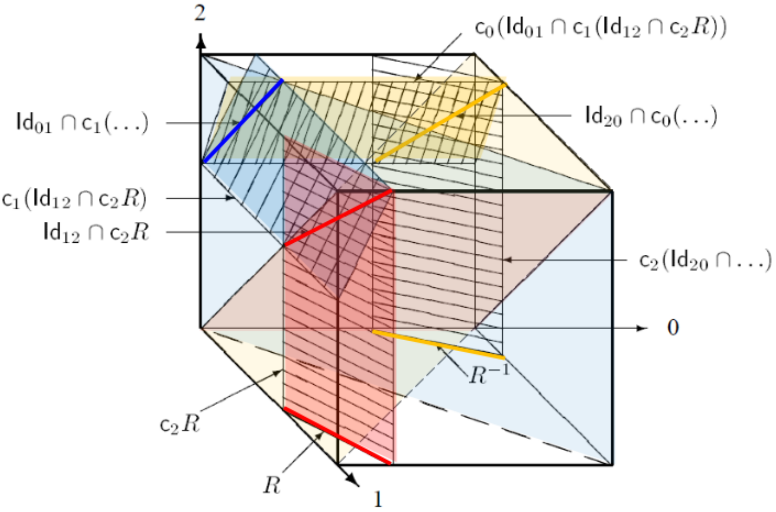
Have infinitely many atoms.

For reals:

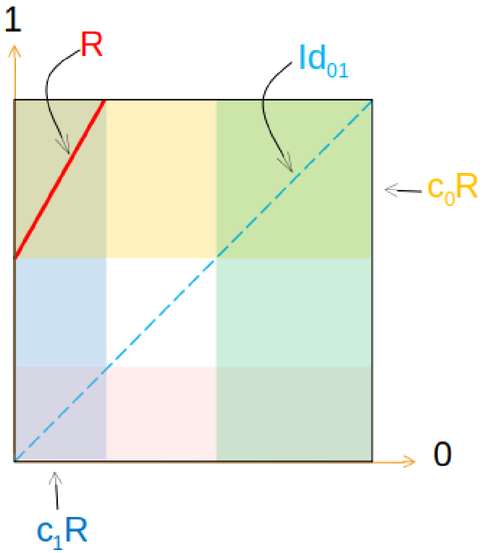
Atomic, we know the atoms.

There are non-trivial subalgebras.

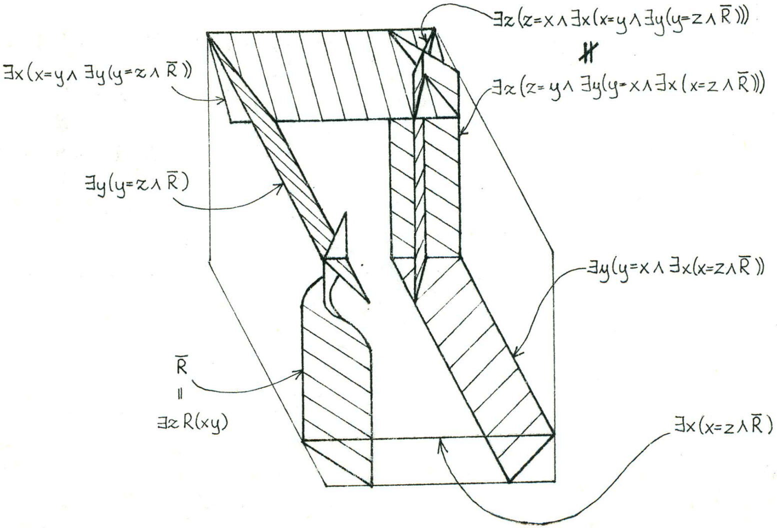
Converse can be defined with 3 variables



Converse cannot be defined with 2 variables



Some properties of converse cannot be proved with 3 variables



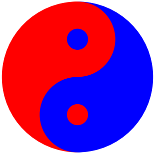
Concept algebras have geometric aspects.

Concept algebras have algebraic aspects.

Concept algebras have logical aspects.

They have categorical aspects, too!

PART II



Duality between algebra and category theory

Universal Algebraic Logic

We depart from first-order logic: we deal with any logic, second-order logic, many-sorted logic, modal logics,. . .

Category of concept algebras

Concept algebras for an arbitrary language in the framework of general language theory

Definition (Algebraizable language)

Algebraizable language: $L = \langle F, M, \mathrm{mng} \rangle$

(1.) $F = W(P, \mathrm{Cn})$ is a context-free language.

(2.) Compositionality: the meaning of a compound term depends only on the meanings of the compounds.

Examples: FOL, SOL, modal logic, propositional logic, ...

Non-examples: equational logic, injectivity logic, ...

These two conditions are exactly what are needed for forming concept algebras!

Algebraic version of L

$\mathrm{F} = \langle W(P, \mathrm{Cn}), c\rangle_{c\in\mathrm{Cn}}$ is the word-algebra $c(w_1, w_2) = cw_1w_2$.

Definition (concept algebra)

$$\mathrm{CA}(\mathfrak{M}) = \langle \mathrm{CA}_{\mathfrak{M}}, c\rangle_{c\in\mathrm{Cn}}$$

is the concept algebra of $\mathfrak{M}$, where

$$\mathrm{CA}_{\mathfrak{M}} = \{\mathrm{mng}(\varphi, \mathfrak{M}) : \varphi \in F\}$$

the set of concepts (meanings) of $\mathfrak{M}$ and

$$c(\mathrm{mng}(\varphi, \mathfrak{M}), \mathrm{mng}(\psi, \mathfrak{M})) = \mathrm{mng}(c\varphi\psi, \mathfrak{M}).$$

Forgetting P! Category theoretic logic

Definition (class of concept algebras)

$\mathsf{Alg}_m(L) = \{\mathrm{CA}(\mathfrak{M}) : \mathfrak{M} \in M\}$, the class of concept algebras.

Definition (class of Lindenbaum-Tarski algebras)

$$\mathsf{Alg}(L) = I\{\mathrm{F}/{\sim_K} : K \subseteq M\}, \text{where}$$

$$\varphi \sim_K \psi \iff \mathsf{mng}(\varphi, \mathfrak{M}) = \mathsf{mng}(, \mathfrak{M}) \text{ for all } \mathfrak{M} \in K,$$

the class of Lindenbaum-Tarski algebras, up to isomorphism.

Introducing truth: $L = \langle F, M, \mathsf{mng}, \models \rangle$, language becomes logic

Definition

(3.) We can "code" $\mathsf{mng}(\varphi) = \mathsf{mng}(\psi)$ with formulas, by using derived connectives $\leftrightarrow$, $\top$ as

$$\mathfrak{M} \models \varphi \iff \mathfrak{M} \models \varphi \leftrightarrow \top, \text{ and}$$

$$\mathfrak{M} \models \varphi \leftrightarrow \psi \iff \mathsf{mng}(\varphi, \mathfrak{M}) = \mathsf{mng}(\psi, \mathfrak{M}).$$

Algebraic logic is bridge between Logic and Algebra:

Logic L		Algebra $\mathrm{Alg}(L)$
formulas		equations
$\mathrm{Taut}(L)$		$\mathrm{Eq}(\mathrm{Alg}(L))$
L is complete		$\mathrm{Alg}(L)$ is axiomatizable by ...
L is compact		$\mathrm{Alg}(L)$ is closed under ultraproducts
Definability properties of L		category theoretic properties of $\mathrm{Alg}(L)$

Definability theory of general languages and Category of concept algebras

We need varying the vocabulary/signature P:

Definition (general logic)

A general logic is $L = \{L^P : P \in \mathcal{P}\}$ where

$L^P = \langle F, M, \mathrm{mng}, \models \rangle$ satisfies (1.)-(3.) for all $P \in \mathcal{P}$, and

(4.) Cn, $\leftrightarrow$, $\top$ are the same for all L^P, $P \in \mathcal{P}$.

(5.) Some conditions between L^P and L^Q for $P, Q \in \mathcal{P}$ such as:

- there are arbitrary large signatures in $\mathcal{P}$;
- if $P \subseteq Q$, then L^P is the natural restriction of L^Q to L^P;
- all meanings can be chosen to be the meanings of atomic formulas in P for some $P \in \mathcal{P}$.

FOL, SOL, modal logic, propositional logic are general logics.

Algebraic version of a general logic

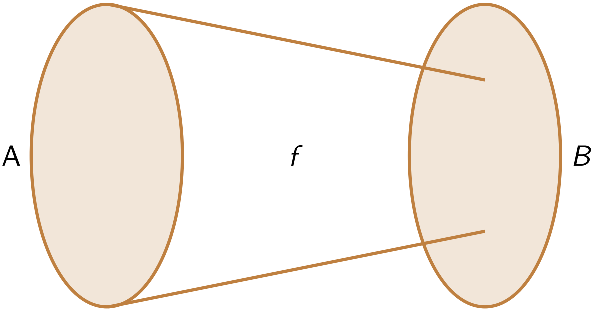
Let $L = \{L^P : P \in \mathcal{P}\}$ be a general logic.

$$\mathsf{Alg}_m(L) = \bigcup_{P \in \mathcal{P}} \mathsf{Alg}_m\left(L^P\right)$$

$$\mathsf{Alg}(L) = \bigcup_{P \in \mathcal{P}} \mathsf{Alg}\left(L^P\right)$$

Category of $\mathsf{Alg}(L)$

Objects: elements of $\mathsf{Alg}(L)$,
Morphisms: homomorphisms f between $A, B \in \mathsf{Alg}(L)$, (A, f, B).



Objects correspond to theories of L, and Morphisms correspond to interpretations between these theories.

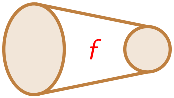
Categories of Algebras

Objects: all algebras of some similarity type,

Morphisms: homomorphisms between these algebras.

Internal properties	External properties

Onto maps

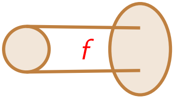


Epimorphisms: left cancellative morphisms

$$\begin{tikzcd} {} \arrow[r, "f"] & {} \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & {} \end{tikzcd}$$

$$fg = fh \implies g = h$$

One-to-one maps

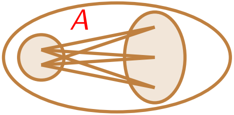


Monomorphisms: right cancellative ones

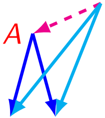
$$\begin{tikzcd} {} \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & {} \arrow[r, "f"] & {} \end{tikzcd}$$

$$gf = hf \implies g = h$$

Direct product



Universal (smallest) cone

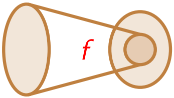


Categories of Algebras

Objects: some algebras of some similarity type,
Morphisms: homomorphisms between these algebras.

External properties are sensitive to context!

???



Epimorphisms: left cancellative morphisms

$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & \cdot \end{tikzcd}$$

$$fg = fh \implies g = h$$

In many well-behaved classes of algebras, there are epimorphisms that are not surjective.

Well investigated question in algebra: In which classes of algebras are epimorphisms surjective.

Beth definability property of general logics

Let $L = \{L^P : P \in \mathcal{P}\}$ be a general logic.
Let $P, Q \in \mathcal{P}$, and $R = Q \setminus P$. Let $\Sigma \subseteq L^Q$.

Definition (implicit definition)

Σ defines R implicitly in Q iff
for all Q-models $\mathfrak{M}, \mathfrak{N}$ of Σ if all P-formulas have the same meanings in $\mathfrak{M}, \mathfrak{N}$ then all Q-formulas have the same meanings in them.

Definition (explicit definition)

Σ defines R explicitly in Q iff
for all $r \in R$, there is $\varphi_r \in F^P$ such that $\Sigma \models r \leftrightarrow \varphi_r$.

Definition

L has the Beth definability property iff
for all P Q, R and Σ as above, if Σ defines R implicitly, then Σ defines R also explicitly.

Theorem

L has Beth definability property $\iff$ Epis are surjective in $\mathrm{Alg}(L)$ if L has the patchwork property for models.

(FOL, SOL,... all have the patchwork property for models).

Logic  Category

Beth Definability Property | Epimorphisms are surjective

Idea of proof. Morphisms correspond to definitions.
Epimorphisms correspond to implicit definitions.
Surjections correspond to explicit definitions.

weak Beth definability property of general logics

Let $L = \{L^P : P \in \mathcal{P}\}$ be a general logic.
Let $P, Q \in \mathcal{P}$, and $R = Q \setminus P$. Let $\Sigma \subseteq L^Q$.

Definition (strongly implicit definition)

Σ defines R strongly implicitly in Q iff
Σ defines R implicitly in Q, and in addition for all P-models $\mathfrak{M}$ of the P-consequences of Σ there are Q-models $\mathfrak{N}$ of Σ such that all P-formulas have the same meanings in $\mathfrak{M}$ and $\mathfrak{N}$.

Definition

L has the weak Beth definability property iff
for all P Q, R and Σ as above, if Σ defines R strongly implicitly, then Σ defines R also explicitly.

K-injective morphism, Full model

Definition (K-injectivity)

Let C be a category, K be a subcategory and f a morphism in C. f is K-injective iff all morphisms from the domain of f into an object of K factor through f. (Validity in injectivity logic.)

Definition

Let L be a general logic. $\mathrm{CA}(\mathfrak{M}) \in \mathrm{Alg}_m(L)$ is maximal iff it is not a proper subalgebra of any member of $Alg_m(L)$.

Full(L) denotes the class of all maximal members of Algm(L) in this sense.

Theorem

Let L be a general logic in which each model can be expanded to a maximal one (FOL, SOL,... are like this). Then (i) - (iii) below are equivalent, and perhaps with (?) also.

(i) *L has weak Beth definability property*

(ii) $\mathrm{Full}(L)$*-injective epis are surjective in* $\mathrm{Alg}(L)$.

(iii) $\mathrm{Alg}(L)$ *has no proper reflective subcategory containing* $\mathrm{Full}(L)$.

(?) $\mathrm{Alg}(L)$ *has no proper limit-closed subcategory containing* $\mathrm{Full}(L)$.

Equivalence of (i) and (?) may be independent of set theory, but it cannot be false!

Conclusion

- What have we learnt?
- What have we learnt about the world?
- What have we learnt about the physical world?

Logic, categories and philosophy of mathematics
Budapest 20 June - 21 June 2019

Michael Makkai is turning 80 in 2019. We are pleased to announce that the Alfréd Rényi Institute of Mathematics, the Department of Logic,Institute of Philosophy, Eötvös University, and the Faculty of Science, Eötvös University are organizing a conference celebrating this occasion. The main topics of the conference are logic, category theory, model theory, philosophy of mathematics.

Thank you for your attention!