APPLYING ALGEBRAIC LOGIC; A GENERAL METHODOLOGY

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Abstract

Connections between Algebraic Logic and (ordinary) Logic. Algebraic counterpart of model theoretic semantics, algebraic counterpart of proof theory, and their connections. The class **Alg**(**L**) of algebras associated to any logic **L**. Equivalence theorems stating that **L** has a certain logical property iff **Alg**(**L**) has a certain algebraic property. (E.g. **L** admits a strongly complete Hilbert-style inference system iff **Alg**(**L**) is a finitely axiomatizable quasi-variety. Similarly, **L** is compact iff **Alg**(**L**) is closed under taking ultraproducts; **L** has the Craig interpolation property iff **Alg**(**L**) has the amalgamation property, etc.)

Contents

*Research supported by Hungarian National Foundation for Scientific Research grant Nos ...

1 Introduction

The idea of solving problems in logic by first translating them to algebra, then using the powerful methodology of algebra for solving them, and then translating the solution back to logic, goes back to Leibnitz and Pascal. Papers on the history of Logic (e.g. Anellis–Houser [10], Maddux [33]) point out that this method was fruitfully applied in the 19^{th} century not only to propositional logics but also to quantifier logics (De Morgan, Peirce, etc. applied it to quantifier logics too). The number of applications grew ever since. (Though some of these remained unnoticed, e.g. the celebrated Kripke–Lemmon completeness theorem for modal logic w.r.t. Kripke models was first proved by Jónsson and Tarski in 1948 using algebraic logic.)

For brevity, we will refer to the above method or procedure as "applying Algebraic Logic (AL) to Logic". This expression might be somewhat misleading since AL itself happens to be a part of logic, and we do not intend to deny this. We will use the expression all the same, and hope, the reader will not misunderstand our intention.

In items (i) and (ii) below we describe two of the main motivations for applying AL to Logic.

(i) This is the more obvious one: When working with a relatively new kind of problem, it is often proved to be useful to "transform" the problem into a well understood and streamlined area of mathematics, solve the problem there and translate the result back. Examples include the method of Laplace Transform in solving differential equations (a central tool in Electrical Engineering).

At this point we should dispell a misunderstanding: In certain circles of logicians there seems to be a belief that AL applies only to syntactical problems of logic and that semantical and model-theoretic problems are not treated by AL or at least not in their original model theoretic form. Nothing can be as far from the truth as this belief, as e.g. looking into the present work should reveal. A variant of this belief is that the main bulk of AL is about offering a cheap pseudo semantics to Logics as a substitute for intuitive, model theoretic semantics. Again, this is very far from being true. (This is a particularly harmful piece of misinformation, because, this "slander" is easy to believe if one looks only superficially into a few AL papers.) To illustrate how far this belief is from truth, the semantical-model theoretic parts of the present work emphasize that they start out from a logical system $\mathcal{L}$ whose semantics is as intuitive and as non-algebraic as it wants to be, and then we transform $\mathcal{L}$ into algebra, paying special attention to not distorting its semantics in the process; and anyway, finally we translate the solutions back to the very original non-algebraic framework (including model theoretical semantics).

In the present paper we define the algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ of a logic $\mathcal{L}$

together with the algebraic counterpart $\mathsf{Alg}_m(\mathcal{L})$ of the semantical-model theoretical ingredients of $\mathcal{L}$. Then we prove equivalence theorems, which to essential logical properties of $\mathcal{L}$ associate natural and well investigated properties of $\mathsf{Alg}(\mathcal{L})$ such that if we want to decide whether $\mathcal{L}$ has a certain property, we will know what to ask from our algebraician colleague about $\mathsf{Alg}(\mathcal{L})$. The same devices are suitable for finding out what one has to change in $\mathcal{L}$ if we want to have a variant of $\mathcal{L}$ having a desirable property (which $\mathcal{L}$ lacks). To illustrate these applications we include several exercises (which deal with various concrete Logics). For all this, first we have to define what we understand by a logic $\mathcal{L}$ in general (because otherwise it is impossible to define e.g. the function Alg associating a class $\mathsf{Alg}(\mathcal{L})$ of algebras to each logic $\mathcal{L}$.

(ii) With the rapidly growing variety of applications of logic (in diverse areas like computer science, linguistics, AI, law, etc.) there is a growing number of new logics to be investigated. In this situation AL offers us a tool for economy and a tool for unification in various ways. One of these is that $\mathsf{Alg}(\mathcal{L})$ is always a class of algebras, therefore we can apply the same machinery namely Universal Algebra to study all the new logics. In other words we bring all the various logics to a kind of "normal form" where they can be studied by uniform methods. Moreover, for most choices of $\mathcal{L}$, $\mathsf{Alg}(\mathcal{L})$ tends to appear in the same "area" of Universal Algebra, hence specialized powerful methods lend themselves to studying $\mathcal{L}$. There is a fairly well understood "map" available for the landscape of Universal Algebra. By using our algebraization process and equivalence theorems we can project this "map" back to the (far less understood) landscape of possible logics.

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The approach reported here is strongly related to works of Blok and Pigozzi cf. e.g. [14], [16], [15], [41], Czelakowski [21], Font–Jansana [22]. A more ambitious version of the present approach is in Andréka–Németi–Sain–Kurucz [9] (cf. also Henkin–Monk–Tarski [27] §5.6). However, the paper [9] is harder to read than the present work, therefore it is advisable to read the present one before seriously studying [9]. On the other hand, the investigation of Hilbert-style inference systems done herein are not yet pushed through in that (more ambitious) setting. Anyway, it is advisable to consider [9] as a kind of second part of this work. But after having read a reasonably large part of the present work, it might be a good idea to experiment with looking into [9] in a parallel manner.

The present paper grew out of course materials used mainly (but not solely) at the Logic Graduate School, Budapest. Therefore the style is often that of a lecturer writing to her/his students. E.g. there is a large number of exercises written in an informal imperative style, and there are explanations which would be omitted from any research paper or research monograph. We hope, this style will not offend the reader.

In the paper we follow the notation of Sain [49]. A summary of prerequisites from universal algebra, Boolean and cylindric algebras, logic and naive set theory can also be found therein.

ACKNOWLEDGEMENT. The authors are grateful to Wim Blok, Josep M. Font, Ramon Jansana and Don Pigozzi for most valuable suggestions. Special thanks are due to Font and Jansana for carefully reading the paper, suggesting improvements, and writing a paper on the connection between the presently reported approach and the Blok–Pigozzi approach. Thanks are due to Johan van Benthem, Ágoston Eiben and Yde Venema for careful reading and helpful remarks. Also thanks are due to the following students (listed in alphabetical order) of Logic Graduate School Budapest for helpful remarks, suggestions, solving exercises: Viktor Gyuris, Ben Hansen, Maarten Marx, Szabolcs Mikulás and András Simon.

2 General framework for studying logics

2.1 Defining the framework

Those readers who are interested only in the technical parts of this paper, and who do not care for intuitive motivation, mathematical motivation, applicability, connections with the various more traditional concepts of a logic, may skip the beginning of this section and start reading with Definition 2.1.3.

Defining a logic is an experience similar to defining a language. (This is no coincidence if you think about the applications of logic in e.g. theoretical linguistics.) So how do we define a language, say a programming language like PASCAL. First one defines the *syntax* of PASCAL. This amounts to defining the set of all PASCAL programs. This definition tells us which strings of symbols count as PASCAL programs and which do not. But this information in itself is not very useful, because having only this information enables the user to write programs but the user will have no idea what his programs will do. (This is more sensible if instead of PASCAL we take a more esoteric language like ALGOL 68.) Indeed, the second, and more important step in defining PASCAL amounts to describing what the various PASCAL programs will do when executed. In other words, we have to define the meaning, or *semantics* of the language, e.g. of PASCAL. Defining semantics can be done in two steps, (i) we define the class M of *possible machines* that understand PASCAL, and then (ii) to each machine $\mathfrak{M}$ and each string φ of symbols that counts as a PASCAL program we tell what $\mathfrak{M}$ will do if we "ask" it to execute φ. In other words we define the *meaning* $mng(\varphi, \mathfrak{M})$ of program φ in machine $\mathfrak{M}$.

The procedure remains basically the same if the language in question is not a programming language but something like a natural language or a simple declarative language like first-order logic. When teaching a foreign language, e.g. German, one has to explain which strings of symbols are German sentences and which are not (e.g. "Der Tisch ist rot" is a German sentence while "Das Tisch ist rot" is not). This is called explaining the syntax of German. Besides this, one has to explain what the German sentences mean. This amounts to defining the semantics of German. If we want to formalize the definition of semantics (for, say, a fragment of German) then one again defines a class M of possible situations or with other words, "possible worlds" in which our German sentences are interpreted, and then to each situation $\mathfrak{M}$ and each sentence φ we define the *meaning* or *denotation* $mng(\varphi, \mathfrak{M})$ of φ in situation (or possible world) $\mathfrak{M}$.

At this point we could discuss the difference between a language and a logic, but we do not do that. For our present purposes it is enough to say that the two things are very-very similar.[1]

Soon (in Definition 2.1.3 below) we will define what we mean by a logic. (A more carefully chosen word would be "logical system".) Roughly speaking, a *logic* $\mathcal{L}$ is a five-tuple

$$\mathcal{L} = \langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle \,,$$

where

- $F_{\mathcal{L}}$ is a set, called the set of all *formulas* of $\mathcal{L}$;
- $\vdash_{\mathcal{L}}$ is a binary relation between sets of formulas and individual formulas, that is, $\vdash_{\mathcal{L}} \subseteq \mathcal{P}(F_{\mathcal{L}}) \times F_{\mathcal{L}}$ (for any set X, $\mathcal{P}(X)$ denotes the powerset of X); $\vdash_{\mathcal{L}}$ is called the *provability relation* of $\mathcal{L}$;
- $M_{\mathcal{L}}$ is a class, called the class of all *models* (or *possible worlds*) of $\mathcal{L}$;
- $mng_{\mathcal{L}}$ is a function with domain $F_{\mathcal{L}} \times M_{\mathcal{L}}$, called the *meaning function* of $\mathcal{L}$;
- $\models_{\mathcal{L}}$ is a binary relation, $\models_{\mathcal{L}} \subseteq M_{\mathcal{L}} \times F_{\mathcal{L}}$, called the *validity relation* of $\mathcal{L}$;
- there is some connection between $\models_{\mathcal{L}}$ and $mng_{\mathcal{L}}$, namely for all $\varphi, \psi \in F_{\mathcal{L}}$ and $\mathfrak{M} \in M_{\mathcal{L}}$ we have

$$(\star) \qquad [mng_{\mathcal{L}}(\varphi, \mathfrak{M}) = mng_{\mathcal{L}}(\psi, \mathfrak{M}) \text{ and } \mathfrak{M} \models_{\mathcal{L}} \varphi] \Longrightarrow \mathfrak{M} \models_{\mathcal{L}} \psi \,.$$

Intuitively, $F_{\mathcal{L}}$ is the collection of "texts" or "sentences" or "formulas" that can be "said" in the language $\mathcal{L}$. For $\Gamma \subseteq F_{\mathcal{L}}$ and $\varphi \in F_{\mathcal{L}}$, the intuitive meaning of $\Gamma \vdash_{\mathcal{L}} \varphi$ is that φ is provable (or derivable) from Γ with the syntactic inference system (or deductive mechanism) of $\mathcal{L}$. In all important cases, $\vdash_{\mathcal{L}}$ is subject to certain (well-known) conditions like $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathcal{L}} \psi$ imply $\Gamma \vdash_{\mathcal{L}} \psi$ for any $\Gamma \subseteq F_{\mathcal{L}}$ and $\varphi, \psi \in F_{\mathcal{L}}$. The meaning function tells us what the texts belonging to $F_{\mathcal{L}}$ mean in the possible worlds from $M_{\mathcal{L}}$.[2] The validity relation tells us which texts are "true" in which possible worlds (or models) under what conditions. In all the interesting cases from $mng_{\mathcal{L}}$ the relation $\models_{\mathcal{L}}$ is definable. A typical possible definition of $\models_{\mathcal{L}}$ from $mng_{\mathcal{L}}$ is the following.

$$\mathfrak{M} \models_{\mathcal{L}} \varphi \quad \text{iff} \quad (\forall \psi \in F_{\mathcal{L}})\big[mng_{\mathcal{L}}(\psi, \mathfrak{M}) \subseteq mng_{\mathcal{L}}(\varphi, \mathfrak{M})\big],$$

[1] The philosophical minded reader might enjoy looking into the book [42], cf. e.g. B.Partee's paper therein. More elementary ones are: Sain [?] and [?].

[2] For fixed $\varphi \in F_{\mathcal{L}}$ and $\mathfrak{M} \in M_{\mathcal{L}}$, $mng_{\mathcal{L}}(\varphi, \mathfrak{M})$ is called *meaning* or *denotation* or *intension* of expression φ in model (or "possible environment" or "possible interpretation") $\mathfrak{M}$. The literature makes subtle distinctions between these words. We deliberately ignore these distinctions, because on the present level of abstraction they are not relevant yet.

for all $\varphi \in F_{\mathcal{L}}$, $\mathfrak{M} \in M_{\mathcal{L}}$. However, in general, definability of $\models_{\mathcal{L}}$ from $mng_{\mathcal{L}}$ is *not* required (condition $(\star)$ above is *not* a definition).

When no confusion is likely, we omit the subscripts $\mathcal{L}$ from $F_{\mathcal{L}}$, $\vdash_{\mathcal{L}}$ etc.

Usually $F_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}$ are defined by what is called a grammar in mathematical linguistics. $\langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$ together with the grammar defining them is called the *syntactical part* of $\mathcal{L}$, while $\langle M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$ is the *semantical part* of $\mathcal{L}$.

When defining a logic, a typical definition of F has the following recursive form. Two sets, P and $Cn(\mathcal{L})$ are given; P is called the set of primitive or *atomic formulas* and $Cn(\mathcal{L})$ is called the set of *logical connectives* of $\mathcal{L}$ (these are operation symbols with finite or infinite ranks). Then we require F to be the smallest set H satisfying

(1) $P \subseteq H$, and

(2) for every $\varphi_1, \ldots, \varphi_n \in H$ and $f \in Cn(\mathcal{L})$ of rank n, $f(\varphi_1, \ldots, \varphi_n) \in H$.

For example, in propositional logic, if p is some propositional variable (atomic formula according to our terminology), then $(\neg p)$ is defined to be a formula (where $\neg$ is a logical connective of rank 1).

For formulas $\varphi \in F$ and models $\mathfrak{M} \in M$, $mng(\varphi, \mathfrak{M})$ and $\mathfrak{M} \models \varphi$ are defined in uniform ways (by some finite "schemas").

Given a logic $\mathcal{L}$, for $\varphi \in F_{\mathcal{L}}$ we say that φ is *valid* (in $\mathcal{L}$), in symbols $\models_{\mathcal{L}} \varphi$, iff $(\forall \mathfrak{M} \in M_{\mathcal{L}})\mathfrak{M} \models \varphi$. For φ as above and $\Gamma \subseteq F_{\mathcal{L}}$ we say that φ is a *semantical consequence* of Γ, in symbols $\Gamma \models_{\mathcal{L}} \varphi$, iff $(\forall \mathfrak{M} \in M_{\mathcal{L}})[(\forall \psi \in \Gamma)\mathfrak{M} \models_{\mathcal{L}} \psi \Longrightarrow \mathfrak{M} \models_{\mathcal{L}} \varphi]$. (We hope that the traditional double use of symbol $\models$ does not cause real ambiguity.) One of the important topics of Logic is the study of the connection between semantic consequence $\Gamma \models_{\mathcal{L}} \varphi$ and the syntactic consequence $\Gamma \vdash_{\mathcal{L}} \varphi$. If the two coincide then $\vdash_{\mathcal{L}}$ is said to be strongly complete and sound (for $\mathcal{L}$).

Figure 2.1.1 below illustrates the general pattern of a logic.

Exercises 2.1.1 below are designed to illuminate the intuitive content of the concept of a logic as outlined above, and to show how familiar logics are special cases of our general concept.

Exercises 2.1.1.

(1) Create an illustration for the above outlined concept of a logic, that is, for $\mathcal{L} = \langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$, by formalizing *classical sentential logic* in this spirit; and do this in the following way. Let P be a set, called the set of *atomic formulas* of $\mathcal{L}_S$. Let $\{\wedge, \neg\} = Cn(\mathcal{L}_S)$ be a set disjoint from P, called the set of *logical connectives* of $\mathcal{L}_S$ (usually called *Boolean connectives*). Define the set F_S (of formulas) to be the smallest set H satisfying the two conditions:

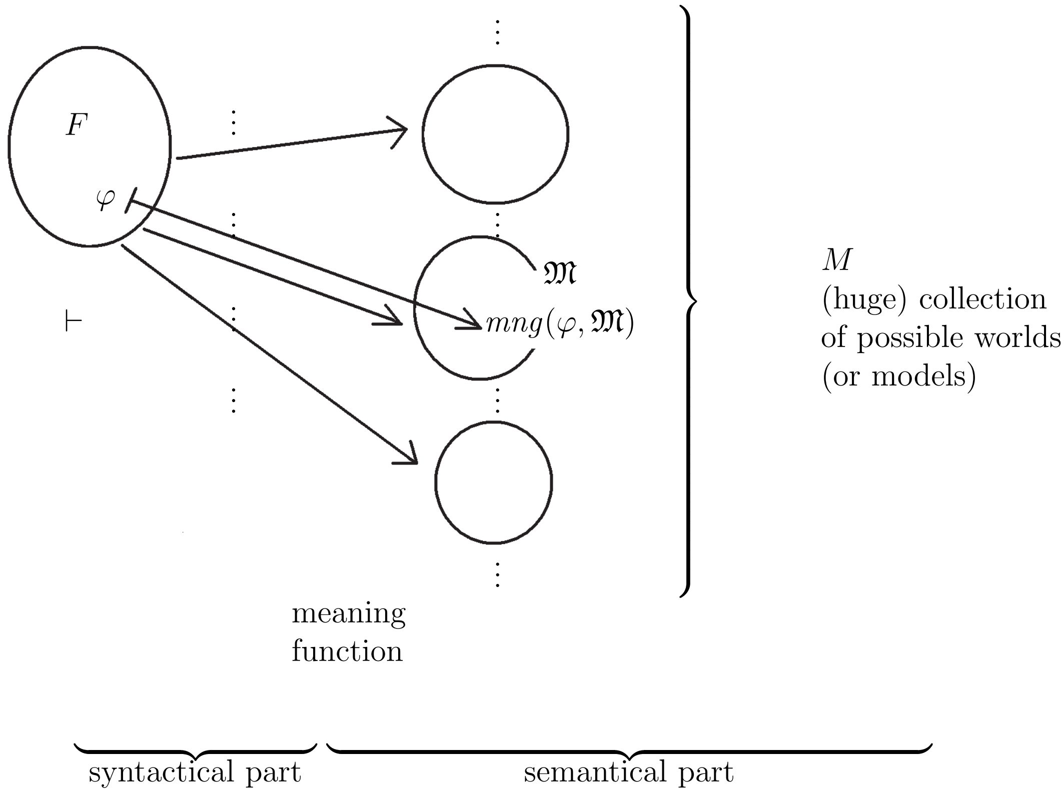

Figure 2.1.1

$P \subseteq H$ and $[\varphi, \psi \in H \Longrightarrow (\varphi \wedge \psi), \neg\varphi \in H]$. Further, define the class M_S (of models) as $M_S \stackrel{\text{def}}{=} \{\langle W, v\rangle : W$ is a non-empty set, and $v : P \longrightarrow \mathcal{P}(W)\}$. Now, you want to recast sentential logic $\mathcal{L}_S$ in the form $\mathcal{L}_S^0 = \langle F_S, \vdash_S^0, M_S, mng_S^0, \models_S^0\rangle$ such that it could be a concrete example of our general ideas outlined above. For this, F_S and M_S are already defined. We leave $\vdash_S^0$ to the end. Let $Sets$ denote the class of all sets. Define $mng_S^0 : F_S \times M_S \to Sets$ in the following way. Let $\mathfrak{M} = \langle W, v\rangle \in M_S$ be arbitrary but fixed. For any $p \in P$ define $mng_S^0(p, \mathfrak{M}) \stackrel{\text{def}}{=} v(p)$. For any $\varphi, \psi \in F_S$ define $mng_S^0\big((\varphi \wedge \psi), \mathfrak{M}\big) \stackrel{\text{def}}{=} mng_S^0(\varphi, \mathfrak{M}) \cap mng_S^0(\psi, \mathfrak{M})$ and $mng_S^0(\neg\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} W \smallsetminus mng_S^0(\varphi, \mathfrak{M})$. For any $\mathfrak{M} = \langle W, v\rangle \in M_S$, $\varphi \in F_S$ let $\mathfrak{M} \models_S^0 \varphi$ iff $mng_S^0(\varphi, \mathfrak{M}) = W$. Check that you indeed defined (the set of formulas together with) the "semantical part" $\langle F_S, M_S, mng_S^0, \models_S^0\rangle$ of a logic in the sense outlined above these exercises.

Let us turn to defining a possible choice of $\vdash_S^0$.

Throughout, we use $(\varphi \to \psi)$ as an abbreviation for $\neg(\varphi \wedge \neg\psi)$ and $(\varphi \leftrightarrow \psi)$ as that for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. List a set Ax of valid formulas of $\mathcal{L}_S$ and call these *logical axioms*.[3] Possible elements of this list are $(\varphi \to \varphi)$ for all $\varphi \in F_S$, $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$, $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \neg\varphi) \to (\psi \wedge \neg\psi)$, $\varphi \to (\psi \to \varphi)$, for all $\varphi, \psi \in F_S$. Having defined your set Ax of logical axioms, add the inference rule $\{\varphi, (\varphi \to \psi)\} \vdash \psi$ (for all $\varphi, \psi \in F_S$) which is called Modus Ponens. If you wish, you may add similar rules like $\{\varphi, \psi\} \vdash (\varphi \wedge \psi)$ (but they are not really needed). For $\Gamma \subseteq F_S$, define $\Gamma \vdash^0_S \varphi$ to hold iff $\varphi \in H$ for the smallest set $H \subseteq F_S$ such that $\Gamma \cup Ax \subseteq H$ and H is closed under your inference rules, e.g. whenever $\psi, (\psi \to \rho) \in H$ then also $\rho \in H$. With this, you defined your choice of $\vdash^0_S$ for $\mathcal{L}^0_S$. If $(\Gamma \vdash^0_S \varphi \Longrightarrow \Gamma \models^0_S \varphi)$ for all Γ, φ then $\vdash^0_S$ is called *sound*. If the other direction "$\Longleftarrow$" holds, then $\vdash^0_S$ is called *strongly complete*. Spend a little time with trying to guess whether your $\vdash^0_S$ has one of these properties. Now, check that you indeed defined a logic

$$\mathcal{L}^0_S = \langle F_S, \vdash^0_S, M_S, mng^0_S, \models^0_S \rangle$$

in the sense outlined above the present exercises.

(2) Compare the just defined version $\mathcal{L}^0_S$ of sentential logic with the ideas outlined above.

(3) Compare $\mathcal{L}^0_S$ with your own previous concept of sentential logic, and try to prove that they are the same thing (perhaps in different forms).

(4) Change the logic $\mathcal{L}^0_S$ obtaining $\mathcal{L}^1_S$ in the following way. Leave F_S and $\vdash^0_S$ unchanged. Define the new M^1_S by postulating that its elements are functions $\mathfrak{M} : P \to \{0, 1\}$. (Identify 0 with *False* and 1 with *True*.) Define $mng^1_S : F_S \times M^1_S \to \{0, 1\}$ and $\models^1_S$ in the natural way. (Hint: If $p \in P$ then $mng^1_S(p, \mathfrak{M}) \stackrel{\text{def}}{=} \mathfrak{M}(p)$, and $mng^1_S(\neg\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} 1 - mng^1_S(\varphi, \mathfrak{M})$, etc.) Check that what you obtained, $\mathcal{L}^1_S = \langle F_S, \vdash^0_S, M^1_S, mng^1_S, \models^1_S \rangle$, is again an example of our general concept of a logic.

(5) Try to compare logics $\mathcal{L}^0_S$ and $\mathcal{L}^1_S$. Try to find ways in which they could be called equivalent. (Hint: Prove e.g. that they have the same semantic consequence relation, i.e. $(\forall \Gamma \cup \{\varphi\} \subseteq F_S)\ \Gamma \models^0_S \varphi \Leftrightarrow \Gamma \models^1_S \varphi$.)

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[3]If the instructions below would be too vague for the non-logician reader then s/he has three options: (i) Consult Definitions 3.1.12–3.1.15 together with the 11 lines preceding Definition 3.1.12 in Section 3 herein. There we define and discuss inference systems $\vdash_{\mathcal{L}}$ in detail, so that should suffice. (ii) Recall any of the known inference systems for propositional logic from the literature. (iii) Ignore this "$\vdash$-part" of this exercise, since we will not rely on it later.

(6) Let $\varphi \in F_S$ be arbitrary. Prove that φ is valid in every model of $\mathcal{L}^0_S$ iff it is valid in every model of $\mathcal{L}^1_S$. That is, the validities of $\mathcal{L}^0_S$ and $\mathcal{L}^1_S$ coincide. Try to find further similar "equivalence properties".

Instead of the general concept of a logic outlined above, in many cases we will consider only four of its five components: $F_{\mathcal{L}}$, $M_{\mathcal{L}}$, $mng_{\mathcal{L}}$ and $\models_{\mathcal{L}}$. Namely, we found that we can simplify the theory *without loss of generality* by not dragging $\vdash_{\mathcal{L}}$ along with us for the following reason.[4] The validity relation $\models_{\mathcal{L}}$ (or the function $mng_{\mathcal{L}}$ if you like) induces the *semantical consequence relation* $\models_{\mathcal{L}} \subseteq \mathcal{P}(F_{\mathcal{L}}) \times F_{\mathcal{L}}$, given above Exercises 2.1.1. There is a natural temptation to try to replace $\vdash_{\mathcal{L}}$ with $\models_{\mathcal{L}}$ in the theory, though at several places (e.g. at completeness theorems) this would be a grave oversimplification. Surprisingly enough, we found that all the theorems we prove for $\models_{\mathcal{L}}$ carry over to $\vdash_{\mathcal{L}}$, whenever the theorems are not about connections between $\models_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}$ (see explanation below). Therefore we decided to drop $\vdash_{\mathcal{L}}$ for the time being and introduce it only where we must say something about $\vdash_{\mathcal{L}}$ which cannot be said about $\models_{\mathcal{L}}$ in itself.

The *reader interested in logics in the purely syntactical sense* $\langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$ is invited to read our paper in the way described as follows.

Let $\mathcal{L}_{syn} = \langle F, \vdash\rangle$ be a logic in the syntactical sense. To simplify the arguments below, we assume that $\mathcal{L}_{syn}$ has a derived logical connective "$\leftrightarrow$" just as classical logics do, see Ex. 2.1.1 (1) above. Of course, we assume the usual properties of "$\leftrightarrow$", e.g. $\{\varphi, (\varphi \leftrightarrow \psi)\} \vdash \psi$ etc. (cf. the $\vdash^0_S$ part of Ex. 2.1.1 (1)). Intuitively, $(\varphi \leftrightarrow \psi)$ expresses that φ and ψ are equivalent. In Remark 2.1.2 below the present discussion, we discuss how to eliminate the assumption of the expressibility of "$\leftrightarrow$". (However, the reader may safely skip Remark 2.1.2, since we will not rely on it later.)

Assume we want to study the "syntactical logic" $\mathcal{L}_{syn} = \langle F, \vdash\rangle$. To be able to apply the theorems of the present paper, we will associate a class $M_\vdash$ of pseudo-models, a $mng_\vdash$ etc. to $\mathcal{L}_{syn}$. The class of *pseudo-models* is

$$M_\vdash \stackrel{\text{def}}{=} \{T \subseteq F \ : \ T \text{ is closed under } \vdash\}.$$

For any pseudo-model $T \in M_\vdash$ and formula $\varphi \in F$,

$$mng_\vdash(\varphi, T) \stackrel{\text{def}}{=} \{\psi \in F : T \vdash (\varphi \leftrightarrow \psi)\} \ .$$

Further, validity in pseudo-models $T \in M_\vdash$ is defined as

$$T \models_\vdash \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \varphi \in T\,.$$

[4]The following considerations, together with Remark 2.1.2, grew out from discussions with Wim Blok, Joseph M. Font, Ramon Jansana and Don Pigozzi. In particular, Remark 2.1.2 is due to Font and Jansana.

Now, if we want to investigate the "syntactic logic" $\langle F, \vdash\rangle$, we apply our theorems to the logic

$$\mathcal{L}_\vdash \stackrel{\text{def}}{=} \langle F, M_\vdash, mng_\vdash, \models_\vdash\rangle\,.$$

Then condition ($\star$) above holds for $\mathcal{L}_\vdash$ and the semantical consequence relation induced by $\models_\vdash$ coincides with the original syntactical one $\vdash$. (These are easy to check.) Hence, applying the theorems to the logic $\mathcal{L}_\vdash$ yields results about $\langle F, \vdash\rangle$ as was desired. In other words, $\mathcal{L}_\vdash$ is an equivalent reformulation of the "syntactic logic" $\langle F, \vdash\rangle$, hence studying $\mathcal{L}_\vdash$ is the same as studying $\langle F, \vdash\rangle$.

Remark 2.1.2 (Eliminating the assumption of expressibility of "$\leftrightarrow$"). Here we show that in the above argument showing that our results can be applied to a wider class of syntactical logics $\mathcal{L}_{syn} = \langle F, \vdash\rangle$, the assumption of expressibility of "$\leftrightarrow$" in $\mathcal{L}_{syn}$ is not needed. It will turn out in Definition 3.1.1 in Section 3 that for any logic $\mathcal{L}$, the set F of formulas has an algebraic structure, that is F is the universe of an algebra $\mathfrak{F}$. (The operations of $\mathfrak{F}$ are the logical connectives of $\mathcal{L}$ collected in $Cn(\mathcal{L})$.) Let

$$M_\vdash \stackrel{\text{def}}{=} \{\langle T, h\rangle \,:\, T \subseteq F,\ T \text{ is closed under } \vdash,\ h \text{ is a homomorphism from } \mathfrak{F} \text{ into } \mathfrak{F}\}.$$

For any $\varphi \in F$, $\langle T, h\rangle \in M_\vdash$, let

$$mng_\vdash(\varphi, \langle T, h\rangle) \stackrel{\text{def}}{=} h(\varphi)$$
$$\langle T, h\rangle \models_\vdash \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad h(\varphi) \in T.$$

Then $\mathcal{L}_\vdash \stackrel{\text{def}}{=} \langle F, M_\vdash, mng_\vdash, \models_\vdash\rangle$ is a logic such that for all $\Gamma \cup \{\varphi\} \subseteq F$, $(\Gamma \models_\vdash \varphi$ iff $\Gamma \vdash \varphi)$ holds. Moreover, if $\vdash$ satisfies some natural conditions then $\mathcal{L}_\vdash$ is a "structural" logic (cf. Def. 3.1.1), therefore all the theorems of this paper can be applied to it. For more information in this line see [23].

Summing up, for a while we will concentrate our attention on the simplified form

$$\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L}\rangle$$

of a logic. For the reasons outlined above, this temporary restriction of attention will not result in any loss of generality.

To conclude this section, we turn to nailing down our definitions formally in the form we will use them.

For any set X, we let X^* denote the set of all finite sequences ("words") over X. That is, $X^* \stackrel{\text{def}}{=} \bigcup_{n\in\omega} ({}^nX)$ (cf. [49]).

Definition 2.1.3 (logic). By a *logic* $\mathcal{L}$ we mean an ordered quadruple

$$\mathcal{L} \stackrel{\text{def}}{=} \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle,$$

where (i)–(v) below hold.

(i) $F_{\mathcal{L}}$ (called the set of *formulas*) is a set of finite sequences (called *words*) over some set X (called the *alphabet* of $\mathcal{L}$) that is, $F_{\mathcal{L}} \subseteq X^*$.

(ii) $M_{\mathcal{L}}$ is a class (called the class of *models*).

(iii) $mng_{\mathcal{L}}$ is a function with domain $F_{\mathcal{L}} \times M_{\mathcal{L}}$ (called the *meaning function*).

(iv) $\models_{\mathcal{L}}$ (called the *validity relation*) is a relation between $M_{\mathcal{L}}$ and $F_{\mathcal{L}}$ that is, $\models_{\mathcal{L}} \subseteq M_{\mathcal{L}} \times F_{\mathcal{L}}$. (According to the tradition, instead of "$\langle \mathfrak{M}, \varphi \rangle \in \models_{\mathcal{L}}$" we write "$\mathfrak{M} \models_{\mathcal{L}} \varphi$".)

(v) For all $\varphi, \psi \in F_{\mathcal{L}}$ and $\mathfrak{M} \in M_{\mathcal{L}}$ we have

$$(\star) \qquad [mng_{\mathcal{L}}(\varphi, \mathfrak{M}) = mng_{\mathcal{L}}(\psi, \mathfrak{M}) \text{ and } \mathfrak{M} \models_{\mathcal{L}} \varphi] \Longrightarrow \mathfrak{M} \models_{\mathcal{L}} \psi\,. \qquad \blacktriangleleft$$

Remark 2.1.4. In the above definition, we nailed down the expression "model of $\mathcal{L}$" instead of the more suggestive one "possible world of $\mathcal{L}$" only for purely technical reasons, namely, to avoid a danger of potential ambiguity with the literature.

Definition 2.1.5 (semantical consequence, valid formulas). Let $\mathcal{L} = \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ be a logic. For every $\mathfrak{M} \in M_{\mathcal{L}}$ and $\Sigma \subseteq F_{\mathcal{L}}$,

$$\mathfrak{M} \models_{\mathcal{L}} \Sigma \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall \varphi \in \Sigma)\, \mathfrak{M} \models_{\mathcal{L}} \varphi,$$
$$Mod_{\mathcal{L}}(\Sigma) \stackrel{\text{def}}{=} \{\mathfrak{M} \in M_{\mathcal{L}} \ : \ \mathfrak{M} \models_{\mathcal{L}} \Sigma\}.$$

$Mod_{\mathcal{L}}(\Sigma)$ is called the class of models of Σ.

A formula φ is said to be *valid*, in symbols $\models_{\mathcal{L}} \varphi$, iff $Mod_{\mathcal{L}}(\{\varphi\}) = M_{\mathcal{L}}$.

For any $\Sigma \cup \{\varphi\} \subseteq F_{\mathcal{L}}$,

$$\Sigma \models_{\mathcal{L}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad Mod_{\mathcal{L}}(\Sigma) \subseteq Mod_{\mathcal{L}}(\{\varphi\}),$$
$$Csq_{\mathcal{L}}(\Sigma) \stackrel{\text{def}}{=} \{\varphi \in F_{\mathcal{L}} \ : \ \Sigma \models_{\mathcal{L}} \varphi\}\,.$$

If $\varphi \in Csq_{\mathcal{L}}(\Sigma)$ then we say that φ is a *semantical consequence* of Σ (in logic $\mathcal{L}$). Csq abbreviates "$\underline{c}$on$\underline{s}$e$\underline{q}$ence".

Definition 2.1.6 (theory, set of validities). Let $\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L}\rangle$ be any logic. For any $K \subseteq M_\mathcal{L}$ let the *theory of* K *in* $\mathcal{L}$ be defined as

$$Th_\mathcal{L}(K) \stackrel{\text{def}}{=} \{\varphi \in F_\mathcal{L} \ : \ (\forall \mathfrak{M} \in K)\ \mathfrak{M} \models_\mathcal{L} \varphi\}.$$

If $K = \{\mathfrak{M}\}$ for some $\mathfrak{M} \in M_\mathcal{L}$ then instead of $Th_\mathcal{L}(\{\mathfrak{M}\})$ we write $Th_\mathcal{L}(\mathfrak{M})$.

The set $Th_\mathcal{L}(M_\mathcal{L})$ is called the *set of validities* of $\mathcal{L}$.

For any set X^* of "strings of symbols", the notion of a *decidable* subset $H \subseteq X^*$ is introduced in almost any introductory book on logic or on the theory of computation (see e.g. [37]). The same applies to $H \subseteq X^*$ being *recursively enumerable* (*r.e.*).

Definition 2.1.7 (decidability of logics). We say that a logic $\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L}\rangle$ is *decidable* iff the set $Th_\mathcal{L}(M_\mathcal{L})$ of validities of $\mathcal{L}$ is a decidable subset of the set $F_\mathcal{L}$ of formulas.

2.2 Distinguished logics

Now we define some basic logics. Some of them are well-known, but we recall their definitions for illustrating that they are special cases of the concept defined in Definition 2.1.3 above, and also for fixing our notation.

Definition 2.2.1 (Propositional or sentential logic $\mathcal{L}_S$). Let P be a set, called the set of *atomic formulas* of $\mathcal{L}_S$. Let $\{\wedge, \neg\}$ be a set disjoint from P, called the set of *logical connectives* of $\mathcal{L}_S$ (usually called *Boolean connectives*).

Propositional (or *sentential*) *logic* (corresponding to P) is defined to be a quadruple

$$\mathcal{L}_S \stackrel{\text{def}}{=} \langle F_S, M_S, mng_S, \models_S \rangle,$$

for which conditions (i)–(iii) below hold.

(i) The set F_S of formulas is the smallest set H satisfying

- $P \subseteq H$
- $\varphi, \psi \in H \Longrightarrow (\varphi \wedge \psi) \in H$ and $(\neg\varphi) \in H$.

(That is, the alphabet of this logic is $\{\wedge, \neg\} \cup P$.)

(ii) The class M_S of *models* of $\mathcal{L}_S$ is defined by

$$M_S \stackrel{\text{def}}{=} \{\langle W, v\rangle : W \text{ is a non-empty set and } v : P \to \mathcal{P}(W)\}.$$

If $\mathfrak{M} = \langle W, v\rangle \in M_S$ then W is called the set of *possible states* (or *worlds*[5] or *situations*) of $\mathfrak{M}$.

(iii) Let $\langle W, v\rangle \in M_S$, $w \in W$, and $\varphi \in F_S$. We define the binary relation $w \Vdash_v \varphi$ by recursion on the complexity of the formulas:

- if $p \in P$ then $\big(w \Vdash_v p \stackrel{\text{def}}{\Longleftrightarrow} w \in v(p)\big)$
- if $\psi_1, \psi_2 \in F_S$, then

$$\begin{aligned} w \Vdash_v \neg\psi_1 \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad w \not\Vdash_v \psi_1 \\ w \Vdash_v (\psi_1 \wedge \psi_2) \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad w \Vdash_v \psi_1 \text{ and } w \Vdash_v \psi_2. \end{aligned}$$

[5]It is *important* to keep the two senses in which "possible world" can be used separate. The elements $\langle W, v\rangle$ of M_S can be called possible worlds since we inherit this usage from the general concept of a logic. At the same time, the elements $w \in W$ can be called "possible states or worlds" as a technical expression of modal logic. So there is a potential confusion here, which has to be kept in mind.

If $w \Vdash_v \varphi$ then we say that φ is *true in* w, or w *forces* φ.

Now $mng_S(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$.

$\langle W, v\rangle \models_S \varphi$ (φ is *valid in* $\langle W, v\rangle$), iff for every $w \in W$, $w \Vdash_v \varphi$.

It is important to note that the set P of atomic formulas is a parameter in the definition of $\mathcal{L}_S$. Namely, in the definition above, P is a fixed but *arbitrary* set. So in a sense $\mathcal{L}_S$ is a function of P, and we could write $\mathcal{L}_S(P)$ to make this explicit. However, the choice of P has only limited influence on the behaviour of $\mathcal{L}_S$, therefore, following the literature we write simply $\mathcal{L}_S$ instead of $\mathcal{L}_S(P)$. From time to time, however, we will have to remember that P is a freely chosen parameter because in certain investigations the choice of P does influence the behaviour of $\mathcal{L}_S = \mathcal{L}_S(P)$.

Exercises 2.2.2.

(1) Think of $P = \emptyset$, of $P = \{p\}$ a singleton, or of infinite P. Write up explicitly what $\mathcal{L}_S$ is like in each of these three cases. What is the cardinality $|F_S|$ of the formulas in each case? What is the cardinality $|\,\{Mod_{\mathcal{L}_S}(\Sigma) : \Sigma \subseteq F_S\}\,|$ of axiomatizable model classes in each case?

(2) Let $(\varphi \to \psi) \stackrel{\text{def}}{\iff} \neg(\varphi \wedge \neg\psi)$ and $(\varphi \leftrightarrow \psi) \stackrel{\text{def}}{\iff} \big((\varphi \to \psi) \wedge (\psi \to \varphi)\big)$. Prove that

- $\{\varphi\} \models_S \psi \iff \models_S (\varphi \to \psi)$
- $\big(\{\varphi\} \models_S \psi$ and $\{\psi\} \models_S \varphi\big) \iff \models_S (\varphi \leftrightarrow \psi)$.

Exercises 2.2.3.

(1) Prove that $\mathcal{L}_S$ is a decidable logic (cf. Def. 2.2.1).

(2) (Important!) Let $Ax \subseteq F_S$ be an arbitrary but finite set of formulas. Prove that the set $Csq_{\mathcal{L}_S}(Ax)$ of consequences of Ax (cf. Def. 2.1.7) is decidable.

(3) (This might be too hard. Then ignore it.) Show that (2) becomes false if we generalize it to all decidable sets Ax. (Hint: Use an infinite set P.)

(4) Assume that P is finite. Prove that then (2) becomes true for any set Ax. (Might be too hard; then come back to this after doing the next Ex.(5).)

(5) (Important!) Assume P is finite. Let $\mathfrak{M} \in M_S$ be arbitrary. Prove that $Th_{\mathcal{L}_S}(\mathfrak{M})$ is decidable. (Hint: Let $\varphi \equiv \psi$ iff $\mathfrak{M} \models_S (\varphi \leftrightarrow \psi)$. Prove that $F_S/\!\equiv$ is finite (use that P is finite). But then $F_S/\!\equiv$ together with the logical connectives is a finite algebra. Show that in such a finite algebra we can always compute the "meaning" of any formula.)

As Ex's. 2.2.3 show, logic $\mathcal{L}_S$ has a lot of "nice" properties. On the other hand, $\mathcal{L}_S$ is a very "weak" logic. It is well-known that e.g. first-order logic $\mathcal{L}_{\mathrm{FOL}}$ (cf. Def. 2.2.23 below) is much stronger than $\mathcal{L}_S$. However, to build up $\mathcal{L}_{\mathrm{FOL}}$ from $\mathcal{L}_S$ we have to modify the notion of a model, of an atomic formula, etc. in the usual way. We do not want to "throw out" $\mathcal{L}_S$ so drastically, we want to increase the expressive power without changing the class of models or without any other "major surgery". Is it possible to leave M_S unchanged *and* to obtain some significantly stronger (and more interesting) logic (e.g. by adding some new connectives)? The answer is affirmative according to Def. 2.2.4 and Ex's. 2.2.6 below. However, we are also interested in how far we can push this procedure of obtaining stronger and stronger logics without changing the models (or other parts) of $\mathcal{L}_S$. What is the price of this increasing expressive power? How far do the nice properties of $\mathcal{L}_S$ remain true?

Definition 2.2.4 (Modal logic $S5$). The set of connectives of *modal logic* $S5$ is $\{\wedge, \neg, \Diamond\}$.

The set of formulas (denoted as F_{S5}) of $S5$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi \in F_{S5} \implies \Diamond\varphi \in F_{S5}.$$

Let $M_{S5} \stackrel{\mathrm{def}}{=} M_S$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\Diamond$:

$$w \Vdash_v \Diamond\varphi \quad \stackrel{\mathrm{def}}{\iff} \quad (\exists w' \in W)\ w' \Vdash_v \varphi.$$

Then $mng_{S5}(\varphi, \langle W, v\rangle) \stackrel{\mathrm{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{S5}$ is defined as follows.

$$\langle W, v\rangle \models_{S5} \varphi \qquad \stackrel{\mathrm{def}}{\iff} \qquad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, modal logic $S5$ is $S5 \stackrel{\mathrm{def}}{=} \langle F_{S5}, M_{S5}, mng_{S5}, \models_{S5}\rangle$.

Remark 2.2.5. According to a rather respectable (and useful) tradition, an extra-Boolean connective is called a *modality* iff it distributes over disjuction. This will not be true for all of our connectives that we will call modalities. (Exercise: check for which ones is it true). Thus, regrettably, we sometimes ignore this useful tradition. For this tradition cf. e.g. Venema [55, Appendix A (pp. 143–152)].

Exercises 2.2.6.

(1) Prove that $S5$ is a decidable logic. (Hint: Prove that if $\langle W, v\rangle \not\models_{S5} \varphi$ then $\langle W_0, v\rangle \not\models_{S5} \varphi$ for some finite $W_0 \subseteq W$ in the following way. Let P_0 be the set of atomic formulas occurring in φ. Define an equivalence relation $\sim$ on W by stipulating that $w_1 \sim w_2$ iff they agree on every element of P_0. Then from each equivalence class of $W/{\sim}$ keep only one element in W_0.)

Note that this amounts to repeating Ex's. 2.2.3 (1) above for $S5$ in place of $\mathcal{L}_S$.

(2) Repeat Ex's. 2.2.3 (2) above for $S5$ in place of $\mathcal{L}_S$.

(3) (Important!) Repeat Ex's. 2.2.3 (5) above for $S5$ in place of $\mathcal{L}_S$.

(4) Try doing Ex's. 2.2.3 (4) for $S5$.

The following logic is discussed e.g. in Sain [**?**, **?**], Venema [55], Roorda [43], but see also Segerberg [50] who traces this logic back to von Wright.

Definition 2.2.7 (Difference logic $\mathcal{L}_D$). The set of connectives of *difference logic* $\mathcal{L}_D$ is $\{\wedge, \neg, D\}$.

The set of formulas (denoted as F_D) of $\mathcal{L}_D$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi \in F_D \implies D\varphi \in F_D.$$

Let $M_D \stackrel{\text{def}}{=} M_{S5}(= M_S)$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of D:

$$w \Vdash_v D\varphi \quad \stackrel{\text{def}}{\iff} \quad (\exists w' \in W \smallsetminus \{w\})\ w' \Vdash_v \varphi.$$

Then $mng_D(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_D$ is defined as follows.

$$\langle W, v\rangle \models_D \varphi \qquad \stackrel{\text{def}}{\iff} \qquad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, difference logic $\mathcal{L}_D$ is $\mathcal{L}_D \stackrel{\text{def}}{=} \langle F_D, M_D, mng_D, \models_D\rangle$. We note that $\mathcal{L}_D$ is also called *"Some-other-time logic"* (cf. Sain [47], Segerberg [50]).

Exercises 2.2.8.

(1) The definition above defines difference logic $\mathcal{L}_D$ indirectly via earlier definitions. Write up a self-contained definition of $\mathcal{L}_D$ without referring back to earlier texts.

(2) (Important!) Try to guess whether Ex's. 2.2.3 (1), (4), (5) extend to $\mathcal{L}_D$. Try hard, do not give up too soon and remember that you are required to *guess* only. Try to formulate some reasons why you are guessing the outcome you do. Try to guess the same for Ex's. 2.2.3 (2) and (3).

(3) Prove that Ex's. 2.2.3 (1), (4), (5) do generalize to $\mathcal{L}_D$! (Hint: Use the same equivalence relation $\sim$ defined on W as in Ex's. 2.2.6 (1). But now, from each equivalence class keep two elements (if there are more than one there) in W_0.)

(4) Prove that the connective $\Diamond$ of $S5$ is expressible in $\mathcal{L}_D$. Prove that D is not expressible in $S5$. (Hint: If the second one is too hard, postpone it to the end of this section.)

The logics $\mathcal{L}_{\kappa\text{-times}}$ to be introduced below play quite an essential rôle in Artificial Intelligence in the theory what is called there "stratified logic", cf. e.g. works of H. J. Ohlbach, see e.g. [19].

Definition 2.2.9 (κ-times logic $\mathcal{L}_{\kappa\text{-times}}$, twice logic Tw). Let κ be any cardinal. The set of connectives of *κ-times logic* $\mathcal{L}_{\kappa\text{-times}}$ is $\{\wedge, \neg, \Diamond_\kappa\}$.

The set of formulas (denoted as $F_{\Diamond_\kappa}$) of $\mathcal{L}_{\kappa\text{-times}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi \in F_{\Diamond_\kappa} \implies \Diamond_\kappa \varphi \in F_{\Diamond_\kappa}.$$

Let $M_{\Diamond_\kappa} \stackrel{\text{def}}{=} M_{S5} (= M_S)$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\Diamond_\kappa$:

$$w \Vdash_v \Diamond_\kappa \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists H \subseteq W)\big(|H| = \kappa \text{ and } (\forall w' \in H)\ w' \Vdash_v \varphi\big).$$

Then $mng_{\Diamond_\kappa}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\Diamond_\kappa}$ is defined as follows.

$$\langle W, v\rangle \models_{\Diamond_\kappa} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, κ-times logic $\mathcal{L}_{\kappa\text{-times}}$ is $\mathcal{L}_{\kappa\text{-times}} \stackrel{\text{def}}{=} \langle F_{\Diamond_\kappa}, M_{\Diamond_\kappa}, mng_{\Diamond_\kappa}, \models_{\Diamond_\kappa}\rangle$. We note that if $\kappa = 2$ then logic $\mathcal{L}_{2\text{-times}}$ is also called *Twice logic* and is denoted as Tw.

Exercises 2.2.10.

(1) Write up a self-contained definition of the logic $\mathcal{L}_{\kappa\text{-times}}$ without referring back to earlier texts.

(2) Prove that $\mathcal{L}_{0\text{-times}}$ is equivalent to $\mathcal{L}_S$ and that $\mathcal{L}_{1\text{-times}}$ is equivalent to $S5$. Prove that $\Diamond_2$ is expressible in $\mathcal{L}_D$. (What do you think of the other direction of expressing D in $\mathcal{L}_{n\text{-times}}$, for some $n \in \omega$?)

(3) Try to guess whether Ex's 2.2.3 (1), (4), (5) extend to $\mathcal{L}_{n\text{-times}}$ for finite n (that is, for $\kappa = n \in \omega$). How about $n = 0$?? How about $n = 1$?

(4) (Probably too hard. May be ignored.) Try to guess how the logics introduced so far, especially the various $\mathcal{L}_{\kappa\text{-times}}$ logics for different cardinals κ, relate to each other in terms of expressive power. (Do *not* spend all your time on this!) Is the connective $\Diamond$ of $S5$ expressible in $\mathcal{L}_{2\text{-times}}$?

(5) Prove that Ex's. 2.2.3 (1), (4), (5) generalize to $\mathcal{L}_{2\text{-times}}$. (Hint: The same as given for $\mathcal{L}_D$ in (the hints of) Ex's. 2.2.8 (3), 2.2.6 (1).)

(6) Can you generalize Ex's. 2.2.3 (1), (4), (5) to $\mathcal{L}_{3\text{-times}}$? If yes, how about $\mathcal{L}_{n\text{-times}}$, for finite n? (Hint: Keep n elements from each equivalence class of $\sim$.)

(7) What do you think, does the method of Ex's. 2.2.6 (1), 2.2.8 (3) and 2.2.10 (5), (6) above generalize to $\mathcal{L}_{\kappa\text{-times}}$ when κ is infinite? (Hint: Look at the hint of Ex. 2.2.36 below. Do not spend all your time with this exercise.)

(8) Think about the logic with extra-Boolean logical connectives $\Diamond_2$ *and* $\Diamond_3$. Is it equivalent to $\mathcal{L}_{2\text{-times}}$ or to $\mathcal{L}_{3\text{-times}}$? (Hint: No.) Is it decidable?

(9) Think about the logic $\mathcal{L}_{\text{count}}$ with extra-Boolean connectives $\{\Diamond_n : n \in \omega\}$. It can "count" up to any natural number. Is it decidable? (Hint: Yes.)

So far the *extra-Boolean connectives* $\Diamond$, D, $\Diamond_\kappa$ were all unary ones. Next we will see examples when the extra-Booleans are binary.

Definition 2.2.11 ($\mathcal{L}_{\text{bin}}$). The set of connectives of $\mathcal{L}_{\text{bin}}$ is $\{\wedge, \neg, \blacklozenge\}$, where $\blacklozenge$ is a new *binary* modality.

The set of formulas (denoted as F_{bin}) of $\mathcal{L}_{\text{bin}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi, \psi \in F_{\text{bin}} \implies \blacklozenge(\varphi, \psi) \in F_{\text{bin}}.$$

Let $M_{\text{bin}} \stackrel{\text{def}}{=} M_S$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\blacklozenge$:

$$w \Vdash_v \blacklozenge(\varphi, \psi) \quad \stackrel{\text{def}}{\iff} \quad (\exists u, z \in W)\big[w \neq u \neq z \neq w \text{ and } u \Vdash_v \varphi \text{ and } z \Vdash_v \psi\big] .$$

As usual, $mng_{\text{bin}}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\text{bin}}$ is defined as follows.

$$\langle W, v\rangle \models_{\text{bin}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, let $\mathcal{L}_{\text{bin}} \stackrel{\text{def}}{=} \langle F_{\text{bin}}, M_{\text{bin}}, mng_{\text{bin}}, \models_{\text{bin}}\rangle$.

Exercises 2.2.12.

(1) Compare $\mathcal{L}_{\text{bin}}$ with the previous logics. E.g. show that $\Diamond$ and D are expressible in $\mathcal{L}_{\text{bin}}$. Is $\Diamond_3$ expressible in $\mathcal{L}_{\text{bin}}$? (Hint: $\blacklozenge\big(\varphi \wedge \blacklozenge(\varphi, \varphi)\big)$.)

(2) Try to guess whether Ex's. 2.2.3 (1), (4), (5) extend to $\mathcal{L}_{\text{bin}}$. (Hint: The method of extending Ex's. 2.2.3 (1) to $\mathcal{L}_D$ should be adaptable to the present case, cf. hint of Ex's. 2.2.8 (3). So validity in $\mathcal{L}_{\text{bin}}$ should be decidable. To attack Ex's. 2.2.3 (5) in this case, recall the equivalence $\equiv$ on formulas in the hint for Ex's. 2.2.3 (5). Check whether $F_{\text{bin}}/\equiv$ is still finite!)

Definition 2.2.13 ($\mathcal{L}_{\text{more}}$). The set of connectives of $\mathcal{L}_{\text{more}}$ is $\{\wedge, \neg, \blacklozenge_{\text{more}}\}$, where $\blacklozenge_{\text{more}}$ is a new *binary* modality.

The set of formulas (denoted as F_{more}) of $\mathcal{L}_{\text{more}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi, \psi \in F_{\text{more}} \implies \blacklozenge_{\text{more}}(\varphi, \psi) \in F_{\text{more}}.$$

Let $M_{\text{more}} \stackrel{\text{def}}{=} M_S$. The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\blacklozenge_{\text{more}}$:

$$w \Vdash_v \blacklozenge_{\text{more}}(\varphi, \psi) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad |\{u \in W : u \Vdash_v \varphi\}| \geq |\{u \in W : u \Vdash_v \psi\}|.$$

As usual, $mng_{\text{more}}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\text{more}}$ is defined as follows.

$$\langle W, v\rangle \models_{\text{more}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, $\mathcal{L}_{\text{more}} \stackrel{\text{def}}{=} \langle F_{\text{more}}, M_{\text{more}}, mng_{\text{more}}, \models_{\text{more}}\rangle$.

Exercises 2.2.14.

(1) Show that the connective $\Diamond$ of $S5$ is expressible in $\mathcal{L}_{\text{more}}$.

(2) Compare $\mathcal{L}_{\text{more}}$ with the previous logics (concerning their expressive power).

(3) Try to guess whether Ex's. 2.2.3 (1) or (5) extend to $\mathcal{L}_{\mathrm{more}}$. (Hint: Recall the hint given for Ex's. 2.2.3 (5). Try to prove that for any fixed $\mathfrak{M}$, assuming that P is finite, the set $F_{\mathrm{more}}/{\equiv}$ is still finite.)

(4) (If too hard, might be postponed to the end of this paper, but give it a few hours first, and then look at the detailed hints at the end of section 2.2.) Prove that Ex's. 2.2.3 (1) does extend to $\mathcal{L}_{\mathrm{more}}$ (i.e. $\mathcal{L}_{\mathrm{more}}$ is decidable). (Hint: If you followed the hints given for Ex's. 2.2.6 (1), 2.2.8 (3), etc. then you proved for those logics the so called *finite model property (fmp)*. ("fmp" says that a formula is valid [in $\mathcal{L}$] iff it is valid in all finite models [of $\mathcal{L}$]. The cardinality of a model $\langle W, v\rangle$ is that of W.) Decide whether $\mathcal{L}_{\mathrm{more}}$ has the fmp. You will see, it does not. Thus the hint given for Ex's. 2.2.6 (1), 2.2.8 (3), etc. has to be refined in order to make it applicable here. See the end of section 2.2 for a detailed hint.)

(5) Define $\Diamond_{\max}$ to be $\blacklozenge(\varphi, \mathit{True})$, where *True* abbreviates $(\varphi \vee \neg\varphi)$. Define $\mathcal{L}_{\max}$ by replacing $\Diamond_\kappa$ with $\Diamond_{\max}$ in $\mathcal{L}_{\kappa\text{-times}}$. What are the basic properties of $\mathcal{L}_{\max}$? Write up an explicit definition for $\mathcal{L}_{\max}$ without referring to $\mathcal{L}_{\mathrm{more}}$. Is $\Diamond_{\max}$ expressible in one of the logics in Defs. 2.2.1–2.2.13?

Beginning with Definition 2.2.15 below, we start discussing various Arrow Logics. The field of Arrow Logics grew out of application areas in Logic, Language and Computation, and plays an important rôle there, cf. e.g. van Benthem [12, 13], and the proceedings of the Arrow Logic day at the conference "Logic at Work" (December 1992, Amsterdam [CCSOM of Univ. of Amsterdam]).

So far we strengthened $\mathcal{L}_S$ without modifying the class M_S of models. The mildest way of modifying M_S is to take a subclass (i.e. the models themselves do not change, only some of them are excluded).

Definition 2.2.15 (Arrow logic $\mathcal{L}_{\mathrm{PAIR}}$). The set of connectives of $\mathcal{L}_{\mathrm{PAIR}}$ is $\{\wedge, \neg, \circ\}$, where $\circ$ is a binary connective.

The set of formulas (denoted as F_{PAIR}) of $\mathcal{L}_{\mathrm{PAIR}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clause:

$$\varphi, \psi \in F_{\mathrm{PAIR}} \implies \varphi \circ \psi \in F_{\mathrm{PAIR}}.$$

Let $M_{\mathrm{PAIR}} \stackrel{\mathrm{def}}{=} \{\langle W, v\rangle \in M_S : W \subseteq U \times U \text{ for some set } U\}$.

The definition of $w \Vdash_v \varphi$ is the same as in the propositional case but we also have the case of $\circ$:

$$\langle a, b\rangle \Vdash_v \varphi \circ \psi \iff \exists c\big[\langle a, c\rangle, \langle c, b\rangle \in W \text{ and } \langle a, c\rangle \Vdash_v \varphi \text{ and } \langle c, b\rangle \Vdash_v \psi\big]\,.$$

As usual, $mng_{\mathrm{PAIR}}(\varphi, \langle W, v\rangle) \stackrel{\mathrm{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\mathrm{PAIR}}$ is defined as follows.

$$\langle W, v\rangle \models_{\mathrm{PAIR}} \varphi \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, arrow logic $\mathcal{L}_{\mathrm{PAIR}}$ is $\mathcal{L}_{\mathrm{PAIR}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{PAIR}}, M_{\mathrm{PAIR}}, mng_{\mathrm{PAIR}}, \models_{\mathrm{PAIR}}\rangle$.

Exercises 2.2.16.

(1) Write up a self-contained definition of the logic $\mathcal{L}_{\mathrm{PAIR}}$ without referring back to earlier texts.

(2) (Important!) Try to guess whether Ex's. 2.2.3 (1), (4), (5) extend to $\mathcal{L}_{\mathrm{PAIR}}$. Guess separately (the answers need not be uniform). Concentrate first only on Ex's. 2.2.3 (1). This will be very hard but spend some considerable time with guessing each of the exercises. Do not spend all your time on this, but 8 hours is reasonable. Do not worry if you cannot prove anything in this connection, the insight gained by trying is enough. The solutions will be given at the end of section 2.2, *but* wait one week at least before reading them!!

(3) Assume that the set P of atomic formulas is *finite*. Is there a model $\mathfrak{M}$ of $\mathcal{L}_{\mathrm{PAIR}}$ such that $Th_{\mathcal{L}_{\mathrm{PAIR}}}(\mathfrak{M})$ is not even recursively enumerable? Note that this is a generalization of Ex's. 2.2.3 (5). (Why?) (Hint: A set X is called *transitive* if $(\forall y \in X)\ y \subseteq X$. A set Y is called *hereditarily finite* if $Y \subseteq X$ for some finite transitive set X. Let $\mathfrak{M} = \langle W, v\rangle$ be defined as follows.

$$\begin{aligned}
W &\stackrel{\mathrm{def}}{=} \text{"all hereditarily finite sets"} \\
P &\stackrel{\mathrm{def}}{=} \{p_0, p_1, p_2\} \\
v(p_0) &\stackrel{\mathrm{def}}{=} \{\langle a, b\rangle \in W \ :\ a \in b\} \\
v(p_1) &\stackrel{\mathrm{def}}{=} \{\langle a, b\rangle \in W \ :\ b \in a\} \\
v(p_2) &\stackrel{\mathrm{def}}{=} \{\langle a, b\rangle \in W \ :\ a = b\}\ .
\end{aligned}$$

Show first that many relations definable in the model $\mathfrak{W} = \langle W, \in\rangle$ of Finite Set Theory (using first-order logic) are also definable in $\mathfrak{M}$ using $\mathcal{L}_{\mathrm{PAIR}}$. Define first the relation $\{\langle \emptyset, \emptyset\rangle\}$. (Hint: $p_2 \wedge \neg(\mathit{True} \circ p_0)$.) Then the relation $\{\langle X, Y\rangle : Y \subseteq X \in W\}$. Next try to define the relations $\{\langle X, \cup X\rangle : X \in W\}$, and $\{\langle X, \mathcal{P}(X)\rangle : X \in W\}$. Eventually you will have to use the well known fact that the set of first-order formulas involving only 3 variables (free or bound) and valid in $\mathfrak{W}$ is not recursively enumerable.

(This exercise is not easy if you are not experienced with first-order logic and Gödel's incompleteness theorem, so you may postpone doing it after having spent about 7 hours with it.)

(4) Compare the answer to the previous exercise with the fact that $Th(\mathfrak{M})$ is decidable for all the logics discussed so far. Observe the contrast! Try to find a reason for the sudden change of behaviour (of the logics we are looking at)!

(5) Try to guess the answer (yes or no) to Ex's. 2.2.3 (2), (3) when applied to $\mathcal{L}_{\mathrm{PAIR}}$. Is there e.g. a finite set $Ax \subseteq F_{\mathrm{PAIR}}$ such that $Csq_{\mathcal{L}_{\mathrm{PAIR}}}(Ax)$ would not be decidable? (Do not spend all your time here. But spend a few hours.)

Definition 2.2.17 (Arrow logic $\mathcal{L}_{\mathrm{REL}}$). The set of connectives of $\mathcal{L}_{\mathrm{REL}}$ is $\{\wedge, \neg, \circ\}$.

Let $F_{\mathrm{REL}} \stackrel{\mathrm{def}}{=} F_{\mathrm{PAIR}}$.

Let $M_{\mathrm{REL}} \stackrel{\mathrm{def}}{=} \{\langle W, v\rangle \in M_S : W = U \times U \text{ for some set } U\}$.

The definition of $w \Vdash_v \varphi$ is the same as in the case of $\mathcal{L}_{\mathrm{PAIR}}$.

As usual, $mng_{\mathrm{REL}}(\varphi, \langle W, v\rangle) \stackrel{\mathrm{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\mathrm{REL}}$ is defined as follows.

$$\langle W, v\rangle \models_{\mathrm{REL}} \varphi \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Now, arrow logic $\mathcal{L}_{\mathrm{REL}}$ is $\mathcal{L}_{\mathrm{REL}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{REL}}, M_{\mathrm{REL}}, mng_{\mathrm{REL}}, \models_{\mathrm{REL}}\rangle$.

Exercises 2.2.18.

(1) The logics $\mathcal{L}_{\mathrm{REL}}$ and $\mathcal{L}_{\mathrm{PAIR}}$ are among the most important ones discussed in the whole material. So think about $\mathcal{L}_{\mathrm{REL}}$ and compare it with the previous ones!

(2) Show that the connective $\Diamond$ of $S5$ is expressible in $\mathcal{L}_{\mathrm{REL}}$.

(Hint: $\Diamond\varphi$ is $(\mathit{True} \circ \varphi) \circ \mathit{True}$.)

Show that "$\circ$" is associative in $\mathcal{L}_{\mathrm{REL}}$ (i.e.

$$\models_{\mathrm{REL}} \big[(\varphi_1 \circ \varphi_2) \circ \varphi_3\big] \longleftrightarrow \big[\varphi_1 \circ (\varphi_2 \circ \varphi_3)\big].$$

(Hence omitting brackets and writing "$\mathit{True} \circ \varphi \circ \mathit{True}$" [for $\Diamond\varphi$] is justified.)

(3) (Important!) Try to guess whether some of Ex's. 2.2.3 (1)–(5) generalizes to $\mathcal{L}_{\mathrm{REL}}$ (give yes or no answers). (This is very hard, so concentrate on only one item for a while. Do not spend all your time, but spend 6–8 hours. Solutions will be at the end of section 2.2, but wait a few weeks before looking at them.)

(4) Try to prove that the set $Th_{\mathcal{L}_{\mathrm{REL}}}(M_{\mathrm{REL}})$ of validities of $\mathcal{L}_{\mathrm{REL}}$ is recursively enumerable. (Hint: To $\varphi \in F_{\mathrm{REL}}$ associate a first-order formula $f(\varphi)$ such that

$$\models_{\mathrm{REL}} \varphi \quad \Longleftrightarrow \quad \models f(\varphi).$$

Then use the recursive enumerability of the validities of first-order logic (e.g. via Gödel's completeness theorem). If this would be too hard, you may postpone it to the end of the section, but do not postpone it forever.)

Definition 2.2.19 (Arrow logics $\mathcal{L}_{\mathrm{ARW0}},\ \mathcal{L}_{\mathrm{ARROW}},\ \mathcal{L}_{\mathrm{RA}}$). The set of connectives of *arrow logics* $\mathcal{L}_{\mathrm{ARW0}},\ \mathcal{L}_{\mathrm{ARROW}},\ \mathcal{L}_{\mathrm{RA}}$ is $\{\wedge, \neg, \circ, \smile, Id\}$, where $\circ$ is a binary, $\smile$ is a unary, and Id is a zero-ary modality.

- The set of formulas (denoted as F_{ARW0}) of $\mathcal{L}_{\mathrm{ARW0}}$ is defined as that of propositional logic $\mathcal{L}_S$ together with the following clauses:

$$\begin{aligned}
&\varphi, \psi \in F_{\mathrm{ARW0}} \implies (\varphi \circ \psi),\ \varphi^{\smile} \in F_{\mathrm{ARW0}}\\
&Id \in F_{\mathrm{ARW0}}
\end{aligned}$$

The models are those of propositional logic $\mathcal{L}_S$ enriched with three relations, called *accessibility relations*. That is,

$$\begin{aligned}
M_{\mathrm{ARW0}} \stackrel{\mathrm{def}}{=} \{\big\langle \langle W, v\rangle, C_1, C_2, C_3\big\rangle \ :\ \langle W, v\rangle \in M_S,\ & C_1 \subseteq W \times W \times W,\\
& C_2 \subseteq W \times W,\ C_3 \subseteq W\}.
\end{aligned}$$

For propositional connectives $\neg$ and $\wedge$ the definition of $w \Vdash_v \varphi$ is the same as in the propositional case. For the new connectives we have:

$$\begin{aligned}
w \Vdash_v (\varphi \circ \psi) \quad &\stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\exists w_1, w_2 \in W)\\
&\qquad\qquad \big(C_1(w, w_1, w_2) \text{ and } w_1 \Vdash_v \varphi \text{ and } w_2 \Vdash_v \psi\big)\\
w \Vdash_v \varphi^{\smile} \quad &\stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\exists w' \in W)\big(C_2(w, w') \text{ and } w' \Vdash_v \varphi\big)\\
w \Vdash_v Id \quad &\stackrel{\mathrm{def}}{\Longleftrightarrow} \quad C_3(w).
\end{aligned}$$

As usual, $mng_{\mathrm{ARW0}}(\varphi, \langle W, v\rangle) \stackrel{\mathrm{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\mathrm{ARW0}}$ is defined as follows.

$$\langle W, v\rangle \models_{\mathrm{ARW0}} \varphi \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

Then arrow logic $\mathcal{L}_{\mathrm{ARW0}}$ is $\mathcal{L}_{\mathrm{ARW0}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{ARW0}}, M_{\mathrm{ARW0}}, mng_{\mathrm{ARW0}}, \models_{\mathrm{ARW0}}\rangle$.

- $F_{\text{ARROW}} \stackrel{\text{def}}{=} F_{\text{ARW0}}$. $M_{\text{ARROW}} \stackrel{\text{def}}{=} M_{\text{PAIR}}$.

  For connectives $\neg$, $\wedge$ and $\circ$ the definition of $w \Vdash_v \varphi$ is the same as in the case of $\mathcal{L}_{\text{PAIR}}$. For the new connectives we have:

$$\langle a, b\rangle \Vdash_v \varphi^{\smile} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \big[\langle b, a\rangle \in W \text{ and } \langle b, a\rangle \Vdash_v \varphi\big],$$
$$\langle a, b\rangle \Vdash_v Id \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad a = b.$$

  As usual, $mng_{\text{ARROW}}(\varphi, \langle W, v\rangle) \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and the validity relation $\models_{\text{ARROW}}$ is defined by

$$\langle W, v\rangle \models_{\text{ARROW}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall w \in W)\ w \Vdash_v \varphi.$$

  Arrow logic $\mathcal{L}_{\text{ARROW}}$ is defined by

$$\mathcal{L}_{\text{ARROW}} \stackrel{\text{def}}{=} \langle F_{\text{ARROW}}, M_{\text{ARROW}}, mng_{\text{ARROW}}, \models_{\text{ARROW}}\rangle.$$

- $F_{\text{RA}} \stackrel{\text{def}}{=} F_{\text{ARROW}}$. $M_{\text{RA}} \stackrel{\text{def}}{=} M_{\text{REL}}$. The definitions of $w \Vdash_v \varphi$, mng_{RA} and $\models_{\text{RA}}$ are the same as in the case of $\mathcal{L}_{\text{ARROW}}$.

  Arrow logic $\mathcal{L}_{\text{RA}}$ is $\mathcal{L}_{\text{RA}} \stackrel{\text{def}}{=} \langle F_{\text{RA}}, M_{\text{RA}}, mng_{\text{RA}}, \models_{\text{RA}}\rangle$. $\mathcal{L}_{\text{RA}}$ is also called as the *logic of relation algebras.*

Exercises 2.2.20.

(1) Define the arrow logics $\mathcal{L}_{\text{ARW0}}$, $\mathcal{L}_{\text{ARROW}}$, $\mathcal{L}_{\text{RA}}$ without referring back to earlier texts.

(2) Consider the fragment $\mathcal{L}^0_{\text{ARW0}} = \langle F^0_{\text{ARW0}}, M^0_{\text{ARW0}}, mng^0_{\text{ARW0}} \models^0_{\text{ARW0}}\rangle$ of arrow logic $\mathcal{L}_{\text{ARW0}}$ defined above which differ from the original version only in that it does not contain the logical connectives $^{\smile}$ and Id. Prove that $\mathcal{L}^0_{\text{ARW0}}$ is equivalent to $\mathcal{L}_{\text{PAIR}}$ in the sense that they have the same semantical consequence relation that is, for all $\Sigma \cup \{\varphi\} \subseteq F^0_{\text{ARW0}} = F_{\text{PAIR}}$

$$\Sigma \models^0_{\text{ARW0}} \varphi \quad \Longleftrightarrow \quad \Sigma \models_{\text{PAIR}} \varphi.$$

Prove that $\mathcal{L}_{\text{ARW0}}$ is not equivalent, in the above sense, to $\mathcal{L}_{\text{ARROW}}$.

Definition 2.2.21 (First-order logic with n variables $\mathcal{L}_n$). Let $V \stackrel{\text{def}}{=} \{v_0, \ldots, v_{n-1}\}$ be a set, called the set of *variables* of $\mathcal{L}_n$. Let the set P of *atomic formulas* of $\mathcal{L}_n$ be defined as $P \stackrel{\text{def}}{=} \{r_i(v_0 \ldots v_{n-1}) : i \in I\}$ for some set I.

(i) The set F_n of formulas is the smallest set H satisfying

- $P \subseteq H$
- $(v_i = v_j) \in H \quad$ for each $i, j < n$
- $\varphi, \psi \in H,\ v_i \in V \implies (\varphi \wedge \psi),\ \neg\varphi,\ \exists v_i \varphi \in H.$

(ii) The class M_n of *models* of $\mathcal{L}_n$ is defined by

$$M_n \stackrel{\text{def}}{=} \{\langle M, R_i \rangle_{i \in I} \ : \ M \text{ is a non-empty set and for all } i \in I,\ R_i \subseteq {}^nM\}\,.$$

If $\mathfrak{M} = \langle M, R_i \rangle_{i \in I} \in M_n$ then M is called the *universe* (or *carrier*) of $\mathfrak{M}$.

(iii) Let $\mathfrak{M} = \langle M, R_i \rangle_{i \in I} \in M_n$, $q \in {}^nM$ and $\varphi \in F_n$. We define the ternary relation $\mathfrak{M} \models \varphi[q]$ by recursion on the complexity of φ as follows.

- $\mathfrak{M} \models r_i(v_0 \ldots v_{n-1})[q] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad q \in R_i \quad (i \in I)$
- $\mathfrak{M} \models (v_i = v_j)[q] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad q_i = q_j \quad (i, j < n)$
- if $\psi_1, \psi_2 \in F_n$, then

$$\begin{aligned}
\mathfrak{M} \models \neg\psi_1[q] \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad \text{not } \mathfrak{M} \models \psi_1[q] \\
\mathfrak{M} \models (\psi_1 \wedge \psi_2)[q] \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad \mathfrak{M} \models \psi_1[q] \text{ and } \mathfrak{M} \models \psi_2[q] \\
\mathfrak{M} \models \exists v_i \psi_1[q] \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists q' \in {}^nM)(\forall j < n)\big[j \neq i \ \Rightarrow \\
&\qquad\qquad \big(q'_j = q_j \text{ and } \mathfrak{M} \models \psi_1[q']\big)\big].
\end{aligned}$$

If $\mathfrak{M} \models \varphi[q]$ then we say that the *evaluation q satisfies φ in the model $\mathfrak{M}$*.

Now we define mng_n as follows.

$$mng_n(\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} \{q \in {}^nM \ : \ \mathfrak{M} \models \varphi[q]\}.$$

(iv) Validity is defined by

$$\mathfrak{M} \models_n \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall q \in {}^nM) \quad \mathfrak{M} \models \varphi[q].$$

First-order logic with n variables

$$\mathcal{L}_n \stackrel{\text{def}}{=} \langle F_n, M_n, mng_n, \models_n \rangle$$

has been defined.

Intuitive explanation

Our $\mathcal{L}_n$ might look somewhat unusual because we do not allow substitution of variables in atomic formulas $r_i(v_0 \dots)$. This does not restrict generality, because substitution is expressible by using quantifiers and equality. This is explained in more detail in Remark 2.2.24 (2) below.

Exercises 2.2.22.

(1) Write up a detailed definition of $\mathcal{L}_n$ as a modal logic. (Hint: Define the class of models by

$$M_n \stackrel{\text{def}}{=} \{\langle W, v\rangle \in M_S \ : \ W = {}^nU \text{ for some set } U\} \ .$$

The extra-Boolean connectives are "$\exists v_i$" and "$v_i = v_j$" for $i, j < n$. Here $(\exists v_i)$ is a unary modality while $(v_i = v_j)$ is a zero-ary modality.)

(2) Show that in some sense $\mathcal{L}_1$ is equivalent to modal logic S5. (In what sense? Try to define!)

(3) Show that in some sense $\mathcal{L}_D$ and $\mathcal{L}_{\text{2-times}}$ are comparable with $\mathcal{L}_2$. Show that $\mathcal{L}_D$ and $\mathcal{L}_{\text{2-times}}$ are strictly weaker than $\mathcal{L}_2$.

Next we define first-order logic in a non-traditional form. Therefore, below the definition, we will give intuitive explanations for our present definition.

Definition 2.2.23 (First-order logic $\mathcal{L}_{\mathbf{FOL}}$, rank-free formulation). Recall that ω is the set of natural numbers.

Let $V \stackrel{\text{def}}{=} \{v_i : i \in \omega\}$ be a set, called the set of *variables* of $\mathcal{L}_{\text{FOL}}$. As before, let P be an arbitrary set, called the set of *atomic formulas* of $\mathcal{L}_{\text{FOL}}$. (Now, we will think of atomic formulas as relation symbols, hence we will use the letter R for elements of P rather than p as in case of $\mathcal{L}_S$.)

(i) The set F_{FOL} is the smallest set H satisfying

- $P \subseteq H$
- $(v_i = v_j) \in H \quad$ for each $i, j \in \omega$
- $\varphi, \psi \in H,\ i \in \omega \implies (\varphi \wedge \psi),\ \neg\varphi,\ \exists v_i\varphi \in H.$

(ii) The class M_{FOL} of *models* of $\mathcal{L}_{\text{FOL}}$ is

$$M_{\text{FOL}} \stackrel{\text{def}}{=} \Big\{\, \mathfrak{M} \ : \ \mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle_{R\in P},\ M \text{ is a non-empty set and} \\ \text{for all } R \in P,\ R^{\mathfrak{M}} \subseteq {}^nM \text{ for some } n \in \omega \Big\} \ .$$

If $\mathfrak{M} \in M_{FOL}$ then M and $R^{\mathfrak{M}}$ denote parts of $\mathfrak{M}$ determined by the convention $\langle M, R^{\mathfrak{M}} \rangle = \mathfrak{M}$.[6]

(iii) Validity relation $\models_{FOL}$.

In $\mathcal{L}_{S5}$ the "basic semantical units" were the possible situations $w \in W$. In FOL the basic semantical units are the evaluations of individual variables into models $\mathfrak{M}$, where $q \in {}^{\omega}M$ and q evaluates variables v_i as element $q_i \in M$ in the model $\mathfrak{M}$. To follow model theoretic tradition, instead of $\mathfrak{M}, q \Vdash \varphi$ we will write $\mathfrak{M} \models \varphi[q]$ (though the former would be more in the line with our definitions of $\mathcal{L}_{S5}$ etc.).

Let $\mathfrak{M} = \langle M, R^{\mathfrak{M}} \rangle_{R \in P} \in M_{\mathrm{FOL}}$, $q \in {}^{\omega}M$ and $\varphi \in F_{\mathrm{FOL}}$. We define the ternary relation "$\mathfrak{M} \models \varphi[q]$" by recursion on the complexity of φ as follows:

- $\mathfrak{M} \models R[q] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \langle q_0, \ldots, q_{n-1} \rangle \in R^{\mathfrak{M}} \quad \text{for some } n \in \omega \quad (R \in P)$
- $\mathfrak{M} \models (v_i = v_j)[q] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad q_i = q_j \quad (i, j \in \omega)$
- if $\psi_1, \psi_2 \in F_{\mathrm{FOL}}$, then

$$\begin{aligned} \mathfrak{M} \models \neg\psi_1[q] \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad \text{not } \mathfrak{M} \models \psi_1[q] \\ \mathfrak{M} \models (\psi_1 \wedge \psi_2)[q] \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad \mathfrak{M} \models \psi_1[q] \text{ and } \mathfrak{M} \models \psi_2[q] \\ \mathfrak{M} \models \exists v_i \psi_1[q] \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists q' \in {}^{\omega}M)(\forall j \in \omega) \\ &\qquad\qquad \big[j \neq i \Rightarrow (q'_j = q_j \mathfrak{M} \models \psi_1[q'])\big]. \end{aligned}$$

If $\mathfrak{M} \models \varphi[q]$ holds then we say that q *satisfies* φ in $\mathfrak{M}$.

Now we define mng_{FOL} as follows.

$$mng_{\mathrm{FOL}}(\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} \{q \in {}^{\omega}M : \mathfrak{M} \models \varphi[q]\}.$$

(iv) Validity is defined by

$$\mathfrak{M} \models_{\mathrm{FOL}} \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall q \in {}^{\omega}M)\ \mathfrak{M} \models \varphi[q].$$

(v) *First-order logic* (in rank-free form) is

$$\mathcal{L}_{\mathrm{FOL}} \stackrel{\text{def}}{=} \langle F_{\mathrm{FOL}}, M_{\mathrm{FOL}}, mng_{\mathrm{FOL}}, \models_{\mathrm{FOL}} \rangle . \qquad \blacktriangleleft$$

For more on $\mathcal{L}_{\mathrm{FOL}}$ see e.g. Henkin–Tarski [?], Simon [51], Venema [55], Henkin–Monk–Tarski [27, §4.3].

[6] That is, if $\mathfrak{M}$ is given, then M denotes the universe of $\mathfrak{M}$. Further, for $R \in P$, $R^{\mathfrak{M}}$ denotes the meaning of R in $\mathfrak{M}$.

Intuitive explanations for $\mathcal{L}_{\mathbf{FOL}}$

There are two kinds of explanations needed. Namely,

(i) Why does the definition go as it does?

and

(ii) Why do we say that $\mathcal{L}_{\mathrm{FOL}}$ is first-order logic? That is, what are the connections between $\mathcal{L}_{\mathrm{FOL}}$ and the more traditional formulations of first-order logic?

We discuss (ii) in Remark 2.2.24 below. Let us first turn to (i).

Let R be a relation symbol, that is $R \in P$. Then instead of the traditional formula $R(v_0, v_1, v_2, \dots)$ we simply write R. That is, we treat R as a shorthand for $R(v_0, v_1, v_2, \dots)$.

So this is why R is an atomic formula. The next part of the definition which may need intuitive explanation is the definition of the satisfaction relation's behaviour on R. That is, the definition of $\mathfrak{M} \models R[k]$. So let $R^{\mathfrak{M}} \subseteq {}^nM$ be given. Recall that R abbreviates $R(v_0, v_1, v_2, \dots)$ here. Clearly we want $\mathfrak{M} \models R[k]$ to hold if in the traditional sense $\mathfrak{M} \models R(v_0, v_1, v_2, \dots)[k]$ holds. But by the traditional definition this holds iff $\langle k_0, \dots, k_{n-1}\rangle \in R^{\mathfrak{M}}$. Which agrees with our definition. The rest of the definition of $\mathcal{L}_{\mathrm{FOL}}$ coincides with the definition of the most traditional version of first-order logic.

Remark 2.2.24 (Connections between $\mathcal{L}_{\mathbf{FOL}}$ and the more traditional form of first-order logic). (1) The logic $\mathcal{L}_{\mathrm{FOL}}$ is slightly more general than the more traditional forms of first-order logic in that here the *logic* does not tell us in advance which relation symbol has what rank (that is why it is called *rank-free*). This information is postponed slightly, because it is not considered to be purely logical. The information about the ranks of the relation symbols will be provided by the models, or equivalently, by the non-logical axioms of some theory. However, we can simulate the more traditional form of first-order logic in $\mathcal{L}_{\mathrm{FOL}}$ as follows.

Any language (or similarity type) of traditional first-order logic is a *theory* of our $\mathcal{L}_{\mathrm{FOL}}$. Namely, such a language includes the rank $\varrho(R)$ of each relation symbol $R \in P$. So, a traditional language is given by a pair $\langle P, \varrho\rangle$. To such a language we associate the following theory T_ϱ (given as a set of formulas):

$$T_\varrho \stackrel{\text{def}}{=} \left\{\forall v_i\big((\exists v_i R) \leftrightarrow R\big) \ : \ R \in P \text{ and } i \geq \varrho(R)\right\}.$$

The theory T_ϱ spells out for each $R \in P$ that the rank of R is $\varrho(R)$. After T_ϱ has been postulated, whenever one sees R as a formula, one can read it as an abbreviation of $R(v_0 \dots v_{\varrho(R)-1})$. To *any* theory T it is usual to associate a "sublogic" of $\mathcal{L}_{\mathrm{FOL}}$ as follows:

$$\mathcal{L}_T \stackrel{\text{def}}{=} \langle F_{\mathrm{FOL}}, Mod(T), mng_{\mathrm{FOL}}, \models_{\mathrm{FOL}}\rangle.$$

For our T_ϱ, the sublogic $\mathcal{L}_{T_\varrho}$ is strongly equivalent with the most traditional first-order logic of language $\langle P, \varrho\rangle$.[7]

(2) The other feature of traditional first-order logic which might seem to be missing from $\mathcal{L}_{\mathrm{FOL}}$ is *substitution* of individual variables, that is, $\mathcal{L}_{\mathrm{FOL}}$ includes atomic formulas with a fixed order of variables only. The reason for this is that Tarski discovered in the 40's that substitution can be expressed with quantification and equality. Namely, if we want to substitute v_1 for v_0 in formula φ then the resulting formula is equivalent to $\exists v_0(v_0 = v_1 \wedge \varphi)$. E.g. $R(v_1, v_1, v_2)$ is equivalent to

$$\exists v_0\big(v_0 = v_1 \wedge R(v_0, v_1, v_2)\big).$$

What happens if we want to *interchange* v_0 and v_1, i.e. we want to express $R(v_1, v_0, v_2)$. Then write

$$\exists v_3 \exists v_4\big[v_3 = v_0 \wedge v_4 = v_1 \wedge \exists v_0 \exists v_1\big(v_0 = v_4 \wedge v_1 = v_3 \wedge R(v_0, v_1, v_2)\big)\big].$$

Someone might object that *before* writing up the theory T_ϱ (cf. item (1) above) one cannot interchange variables. There are two answers: *(i)* This does not really matter if we want to simulate traditional first-order logic. *(ii)* This can be easily done by adding extra unary connectives p_{ij} $(i, j \in \omega)$ to those of $\mathcal{L}_{\mathrm{FOL}}$. The semantics of p_{ij} is given by

$$\mathfrak{M} \models p_{ij}\varphi[q] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathfrak{M} \models \varphi[\langle q_0, \ldots, q_{i-1}, q_j, q_{i+1}, \ldots, q_{j-1}, q_i, q_{j+1}, \ldots\rangle],$$

if $i \leq j$, and similarly otherwise. Adding such connectives does not change the basic properties of the logic.

For more on the properties of $\mathcal{L}_{\mathrm{FOL}}$ see e.g. the Appendix of Blok–Pigozzi [14], Andréka–Gergely–Németi [3] and reference Henkin–Tarski [**?**] of [27] Part I.

Exercises 2.2.25.

(1) Write up a detailed definition of $\mathcal{L}_{\mathrm{FOL}}$ as a multi-modal logic.

Hint: Define the modal models as

$$M_m \stackrel{\text{def}}{=} \{\langle W, v\rangle \in M_S : W \subseteq {}^{\omega}U \text{ for some set } U\text{, and for each } R \in P,$$
$$(\exists n \in \omega)(\exists R_1 \subseteq {}^{n}M) v(R) = \{s \in {}^{\omega}U : \langle s_0, \ldots, s_n\rangle \in R_1\}\}\,.$$

The rest of the hint is in Ex. 2.2.22 (1).

[7] This equivalence is the strongest possible one. The models are practically the same, and the formulas are alphabetical variants of each other in the following sense. To each "traditional" formula ψ of $\langle P, \varrho\rangle$ there is $\varphi \in F_{\mathrm{FOL}}$ such that their meanings coincide in every model. (Same holds in the other direction: for every $\varphi \in F_{\mathrm{FOL}}$ there is a "traditional" ψ, etc.)

(2) Take the multi-modal form of $\mathcal{L}_{\mathrm{FOL}}$ obtained in (1) above. Consider the "modality" $(\exists v_i)$. Can you write down its meaning definition in the $\Vdash$-style of modal logics, that is, the logics studied before $\mathcal{L}_n$?

Hint: Let $s \in W$. (Recall that $W = {}^{\omega}U$.) Then

$$s \Vdash \exists v_i \varphi \quad \text{iff} \quad (\exists q \in W)\forall j(j \neq i \Rightarrow s_i = q_i \text{ and } q \Vdash \varphi)\,.$$

What is the $\Vdash$-style definition of the zero-ary modality $(v_i = v_j)$?

(3) Consider the modal forms of $\mathcal{L}_n$ and $\mathcal{L}_{\mathrm{FOL}}$. Prove that D is expressible in $\mathcal{L}_n$. Prove that $\Diamond_2$ is expressible in $\mathcal{L}_n$ if $n > 3$. Is D expressible in $\mathcal{L}_{\mathrm{FOL}}$? Is $\Diamond_2$ expressible in $\mathcal{L}_{\mathrm{FOL}}$?

(4) Prove that the following are expressible in $\mathcal{L}_{\mathrm{FOL}}$ about its models $\mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle_{R\in P} \in M_{\mathrm{FOL}}$.

(4.1) $|M| > 1$.

(4.2) $|M| = 2$.

(4.3) $|M| > n$ for any fixed number n.

(4.4) $|M| < n$ for any fixed number n.

(5) What part of (4) above carries over to $\mathcal{L}_n$?

(6) Prove that $\mathcal{L}_1$ is decidable. Do you think that $\mathcal{L}_2$ is decidable? Do you think that $\mathcal{L}_{\mathrm{FOL}}$ is decidable??

(7) Do you think that the valid formulas of $\mathcal{L}_{\mathrm{FOL}}$ are recursively enumerable?

Exercises 2.2.26.

(1) Write up a detailed definition of $\mathcal{L}_{\mathrm{FOL}}$ as a modal logic. (Hint: See Ex. 2.2.25 (1) above.)

(2) Prove that $\mathcal{L}_{\mathrm{FOL}}$ is as expressive as the traditional form of first-order logic. Prove that traditional first-order logic with a language $\langle P, \rho\rangle$ is strongly equivalent with the sublogic $\mathcal{L}_{T_\rho}$ as described in Remark 2.2.24.

(3) Assume $\mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle \in M_{\mathrm{FOL}}$ with $R^{\mathfrak{M}} \subseteq M \times M$. Express that R is a transitive relation. (This means that you are asked to write up a formula $\varphi \in F_{\mathrm{FOL}}$ such that for every $\langle M, R\rangle$ with $R \subseteq M \times M$, if $\langle M, R\rangle \models \varphi$ then R is transitive.)

Express that R is a partial ordering (transitive, reflexive and antisymmetric).

Express that R is a dense ordering (density is the property $\forall x, y(xRy \Rightarrow \exists z(xRz$ and $zRy)).)$

Express that R is an equivalence relation.

(4) Think of $\mathcal{L}_{\mathrm{FOL}}$ again as a multi-modal logic as in the previous list of exercises. Are there two models $\mathfrak{M}, \mathfrak{N}$ such that they are not distinguishable in $\mathcal{L}_{\mathrm{FOL}}$ but they are distinguishable in any of $\mathcal{L}_D$, $\mathcal{L}_{n\text{-times}}$ for $n \in \omega$? (Hint: no.) What is the answer for $\mathcal{L}_{\kappa\text{-times}}$ with some infinite κ (say $\kappa > 2^\omega$)?

Exercises 2.2.27.

(1) Let $\mathcal{L}_i = \langle F_i, M_i, mng_i, \models_i \rangle$ with $i \leq 2$ be two logics. Call $\mathcal{L}_0$ and $\mathcal{L}_1$ *weakly equivalent* iff

$$F_0 = F_1 \text{ and } (\forall \Gamma \subseteq F_0)(\forall \varphi \in F_0)(\Gamma \models_0 \varphi \Leftrightarrow \Gamma \models_1 \varphi)\,.$$

Prove that the following logics are weakly equivalent: $\mathcal{L}_S$ and $\mathcal{L}^0_S$ from Ex. 2.1.1.

(2) Let $\mathcal{L}_i$, $i \leq 2$ be as above. Assume that $F_i \subseteq Z_i$ for some set of "symbols" Z_i. That is, we are assuming that the formulas are finite sets of symbols. For a function $f : Z_0 \longrightarrow Z_1$ define its natural extension $\tilde{f} : Z_0^* \longrightarrow Z_1^*$ the usual way: $f\langle a+1, \ldots, a_n \rangle = \langle f(a_1), \ldots, f(a_n) \rangle$. Call $\mathcal{L}_0$ and $\mathcal{L}_1$ *reasonably equivalent* iff there is a function $f : Z_0 \longrightarrow Z_1$ such that $\tilde{f}(F_0) = F_1$ and

(i) $(\forall \Gamma \cup \{\varphi\} \subseteq F_0)[\Gamma \models_0 \varphi$ iff $\tilde{f}(\Gamma) \models_1 \tilde{f}(\varphi)]$,

(ii) $(\forall \Gamma \cup \{\varphi\} \subseteq F_1)[\Gamma \models_1 \varphi$ iff $\tilde{f}^{-1}(\Gamma) \models_0 \tilde{f}^{-1}(\varphi)]$, and

(iii) $(\forall \varphi, \psi \in F_0)[\tilde{f}(\varphi) = \tilde{f}(\psi) \Rightarrow (\varphi \models_0 \psi$ and $\psi \models_0 \varphi)]$.

Prove that any two weakly equivalent logics are reasonably equivalent.

(3) Consider propositional logic with logical connectives $\{\wedge, \vee, \neg\}$ and another version of the same logic with $\{\wedge, \rightarrow, \text{false}\}$. Clearly these two versions of propositional logic are equivalent in some natural sense. Prove that they are not equivalent in the sense of (1), (2) above. Try to broaden the scope of equivalence such that these two versions of $\mathcal{L}_S$ become equivalent.

(4) Let $\mathcal{L}_i$ be as in (1). Consider the existence of two "semantical" functions

$$m_{01} : M_0 \longrightarrow (\text{Subsets of } M_1) \text{ and}$$
$$m_{10} : M_1 \longrightarrow (\text{Subsets of } M_0)\,.$$

Call $\mathcal{L}_0$ and $\mathcal{L}_1$ *semantically equivalent* iff $F_1 = F_2$ and there are m_{01}, m_{10} as above such that

$$(\forall\varphi \in F_0)(\forall\mathfrak{M} \in M_0)(\forall\mathfrak{N} \in M_1)$$
$$[\mathfrak{M} \models \varphi \Leftrightarrow m_{01}(\mathfrak{M}) \models \varphi] \text{ and } [\mathfrak{N} \models \varphi \Leftrightarrow m_{10}(\mathfrak{N}) \models \varphi]\,.$$

Prove that $\mathcal{L}_S$ and $\mathcal{L}_S^0$ (in Ex. 2.1.1) are strongly semantically equivalent.

(5) Combine the equivalences defined in (2) and (4) above. Call this combined concept semantical equivalence. Find logics which are semantically equivalent.

(6) Try to combine (5) and (3) above!

* * *

SUMMARY (of the logics defined so far):

$\mathcal{L}_S$	propositional logic
$S5$	modal logic, where the accessibility relation is $W \times W$ for a set W
$\mathcal{L}_D$	difference logic (or "some-other-time" logic)
Tw	twice logic
$\mathcal{L}_{\kappa\text{-times}}$	κ-times logic (κ is any cardinal)
$\mathcal{L}_{\mathrm{bin}}$	
$\mathcal{L}_{\mathrm{more}}$	
$\mathcal{L}_{\mathrm{PAIR}}$	set of worlds is arbitrary $W \subseteq U \times U$ for some U, extra-Boolean is $\circ$
$\mathcal{L}_{\mathrm{REL}}$	set of worlds is $U \times U$ for some U, extra-Boolean is $\circ$
$\mathcal{L}_{\mathrm{ARROW}}$	set of worlds is arbitrary $W \subseteq U \times U$ for some U, extra-Booleans are $\circ$, $\smile$, Id
$\mathcal{L}_{\mathrm{RA}}$	(logic of relation algebras) set of worlds is $U \times U$, extra-Booleans are $\circ$, $\smile$, Id
$\mathcal{L}_n$	first-order logic restricted to the first n variables ($n \in \omega$)
$\mathcal{L}_{\mathrm{FOL}}$	(rank-free) first-order logic

DISTINGUISHED PROPERTIES to be checked for every logic $\mathcal{L}$:

(The reason for looking at these properties is that they distinguish first-order like logics from propositional like logics.)

dec The set of all valid formulas of $\mathcal{L}$ is decidable. (Briefly: $\mathcal{L}$ is decidable.)

r.e. The set of all valid formulas of $\mathcal{L}$ is recursively enumerable. (Briefly: $\mathcal{L}$ is r.e.)

fmp $\mathcal{L}$ has the *finite model property* that is,

$$(\forall\varphi \in F_{\mathcal{L}})\big[\models_{\mathcal{L}} \varphi \quad\Longleftrightarrow\quad (\forall\mathfrak{M} \in M_{\mathcal{L}})(\mathfrak{M} \text{ is finite}^{8} \;\Rightarrow\; \mathfrak{M} \models_{\mathcal{L}} \varphi)\big].$$

Gip $\mathcal{L}$ has *Gödel's incompleteness property* that is,

$$(\exists\varphi \in F_{\mathcal{L}})(\forall T \subseteq F_{\mathcal{L}})\big[(\varphi \in T \text{ and } T \text{ is consistent}) \implies \\ \implies Csq_{\mathcal{L}}(T) \text{ is undecidable})\big].$$

clm We say that the *distinction between set-models and class-models counts* in $\mathcal{L}$ ($\mathcal{L}$ has clm for short) iff (roughly speaking)[9] even in the case when the set P of atomic formulas of $\mathcal{L}$ is finite, we have

$$(\exists \text{ class-model } \mathfrak{M})\big[Th_{\mathcal{L}}(\mathfrak{M}) \text{ is not definable without parameters} \\ \text{in our Set Theory}\big].$$

unm Assuming again that the set P of atomic formulas of $\mathcal{L}$ is finite, there is some $\mathfrak{M} \in M_{\mathcal{L}}$ such that $Th_{\mathcal{L}}(\mathfrak{M})$ is undecidable (unm abbreviates existence of <u>un</u>decidable <u>m</u>odel).

Exercise 2.2.28. Prove that if $\mathcal{L}$ is r.e. and $\mathcal{L}$ has the fmp the $\mathcal{L}$ is decidable.

COMPARISON OF LOGICS w.r.t. the distinguished properties above:

(An arrow points to the place where the property in question becomes true "moving from left to right". Hence in principle it should always point to a gap between two logics.)

Exercise 2.2.29. Check which claims represented on Figure 2.2.1 were asked as an exercise in the text. Try to prove (and claim, if necessary) the missing ones too.

[8] $\mathfrak{M} = \langle W, v\rangle$ is called *finite* if W is a finite set.

[9] Recall that for fixed $\mathfrak{M}$, $mng_{\mathfrak{M}}(\varphi)$ was defined by recursion on the complexity of φ in case of each of our distinguished logics discussed so far. (This was so in $\mathcal{L}_S, \ldots$, in $\mathcal{L}_n$, and also in $\mathcal{L}_{FOL}$ to mention only a few.) Saying that $Th(\mathfrak{M})$ is undefinable implies that our recursive definition of $mng_{\mathfrak{M}}$ *becomes incorrect as a definition* if we permit $\mathfrak{M}$ to be a class model. Roughly, $\mathcal{L}$ has *clm* of Tarski's Undefinability of Truth Theorem is applicable to $\mathcal{L}$. For more on this property of logics see [9, Appendix B].

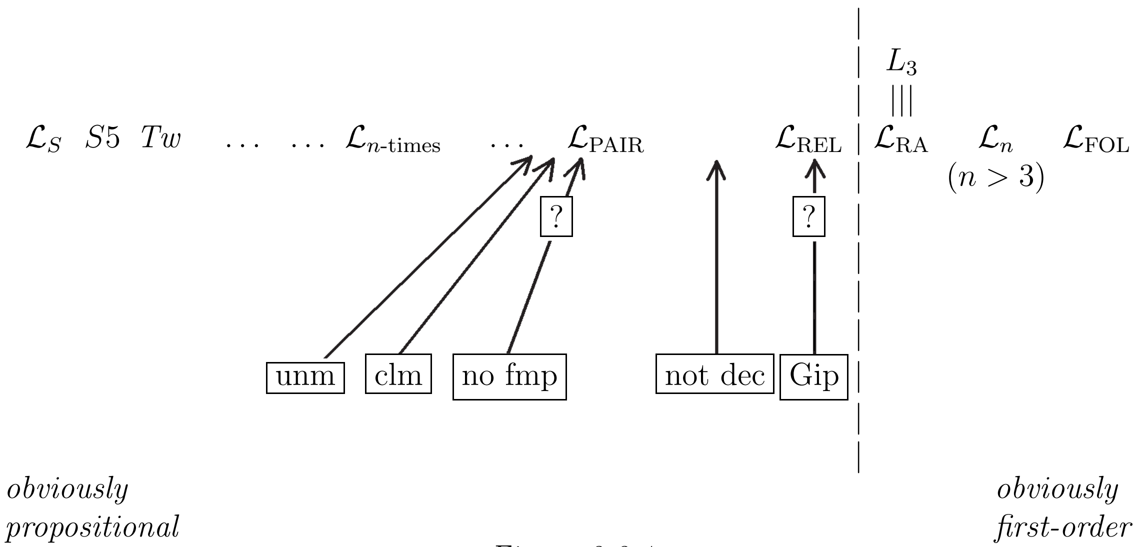

Figure 2.2.1

* * *

The following logics are of a different "flavor" than the ones seen so far. They include Lambek Calculus, some fragments of Linear Logic, Pratt's Action Logic, Dynamic Logic, different kinds of semantics than seen so far. The main purpose of giving them is to indicate that the methods of algebraic logic are applicable almost to any unusual logic coming from completely different paradigms of logical or linguistic or computer science research areas, and are not restricted to the kinds of logics discussed so far. If the reader is already convinced, then he may safely skip Definitions 2.2.30–2.2.33.

Some *further logics*, which are even *less similar* to the ones discussed so far, are collected in Appendix A. It is advisable to look into Appendix A because our theorems apply to all the logics discussed there. The only reason why those logics are postponed to an appendix is that we did not want to postpone the main theorems too much. For example, infinite valued logics, relevant logics and partial logics are in Appendix A.

Definition 2.2.30 (Lambek Calculus [slightly extended]). Recall the logic $\mathcal{L}_{\mathrm{RA}}$ (Def. 2.2.19). The connectives of Lambek calculus $\mathcal{L}_{\mathrm{LC}}$ are $\{\wedge, \circ, \backslash, /, \rightarrow\}$. This defines the formulas F_{LC} of Lambek Calculus. Now,

$$\mathcal{L}_{\mathrm{LC}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{LC}}, M_{\mathrm{RA}}, mng_{\mathrm{LC}}, \models_{\mathrm{LC}} \rangle \,,$$

where for all $\varphi, \psi \in F_{\mathrm{LC}}$ and all $\mathfrak{M} \in M_{\mathrm{RA}}$

$$\begin{aligned}
mng_{\mathrm{LC}}(\varphi \backslash \psi, \mathfrak{M}) &\stackrel{\mathrm{def}}{=} mng_{\mathrm{RA}}\big(\neg(\varphi^{\smile} \circ \neg\psi), \mathfrak{M}\big),\\
mng_{\mathrm{LC}}(\varphi / \psi, \mathfrak{M}) &\stackrel{\mathrm{def}}{=} mng_{\mathrm{RA}}\big(\neg(\neg\varphi \circ \psi^{\smile}), \mathfrak{M}\big),\\
mng_{\mathrm{LC}}(\varphi \to \psi, \mathfrak{M}) &\stackrel{\mathrm{def}}{=} mng_{\mathrm{RA}}(\neg\varphi \vee \psi, \mathfrak{M}),
\end{aligned}$$

and $\models_{\mathrm{LC}}$ is defined analogously to $\models_{\mathrm{RA}}$.

Remark 2.2.31. Original Lambek Calculus is only a fragment of $\mathcal{L}_{\mathrm{LC}}$ because in the original case the use of "$\to$" is restricted. (In any formula, "$\to$" can be used only once, and it is the outer most connective.)

The methods of the present work yielded quite a few results for Lambek Calculus and for some further fragments of Linear Logic, cf. Andréka–Mikulás [4].

Definition 2.2.32 (Language model for Lambek Calculus and other logics [e.g. arrow logic]).

(1) *Notation:* Recall that U^* denotes the set of all finite sequences over the set U. A set $X \subseteq U^*$ is called a language (in the syntactic sense). Let $X, Y \subseteq U^*$. Then $X * Y = \{s^\frown q : s \in X \text{ and } q \in Y\}$, where $s^\frown q$ is the concatenation of s and q.

$$M_L \stackrel{\mathrm{def}}{=} \{\langle U, f\rangle : U \text{ is a set and } f : P \longrightarrow U^*\}\,.$$

We write $mng(\varphi)$ instead of $mng_L(\varphi, \langle U, f\rangle)$.

$$\begin{aligned}
mng(p_i) &\stackrel{\mathrm{def}}{=} f(p_i) \quad \text{for } p_i \in P,\\
mng(\varphi \wedge \psi) &\stackrel{\mathrm{def}}{=} mng(\varphi) \cap mng(\psi),\\
mng(\varphi \circ \psi) &\stackrel{\mathrm{def}}{=} mng(\varphi) * mng(\psi),\\
mng(\varphi \to \psi) &\stackrel{\mathrm{def}}{=} [U^* \smallsetminus mng(\varphi)] \cup mng(\psi),\\
mng(\varphi \backslash \psi) &\stackrel{\mathrm{def}}{=} \{q : \big(\forall s \in mng(\varphi)\big)\ s^\frown q \in mng(\psi)\},\\
mng(\varphi / \psi) &\stackrel{\mathrm{def}}{=} \{s : \big(\forall q \in mng(\psi)\big)\ s^\frown q \in mng(\varphi)\}\,.
\end{aligned}$$

Now, $\models_L$ is defined as before.

(2) *Lambek calculus with language models* is

$$\mathcal{L}_{\mathrm{LCL}} \stackrel{\mathrm{def}}{=} \langle F_{\mathrm{LC}}, M_L, mng_L, \models_L\rangle\,.$$

This is quite a well investigated logic, and in some respects behaves slightly differently from $\mathcal{L}_{\mathrm{LC}}$.

Now we can extend the definition of mng_L to the connectives $\neg$, $\smile$ and Id as follows:

$$\begin{aligned} mng(\neg\varphi) &\stackrel{\text{def}}{=} U^* mng(\varphi), \\ mng(\varphi^{\smile}) &\stackrel{\text{def}}{=} \{\langle s_n, \dots, s_1\rangle : \langle s_1, \dots, s_n\rangle \in mng(\varphi)\}, \\ mng(Id) &\stackrel{\text{def}}{=} \{\langle\rangle\}, \end{aligned}$$

where $\langle\rangle$ denotes the sequence of length 0.

(3) *Extended Lambek calculus with language models*: F^+_{LC} has all the Booleans as connectives in addition to F_{LC}, and the semantics described in (1) above.

$$\mathcal{L}^+_{\mathrm{LCL}} = \langle F^+_{\mathrm{LC}}, M_L, mng_L, \models_L\rangle \,.$$

(4) *Arrow Logic with language models* is

$$\mathcal{L}_{\mathrm{ARROWL}} = \langle F^+_{\mathrm{ARROW}}, M_L, mng_L, \models_L\rangle \,. \qquad \blacktriangleleft$$

Definition 2.2.33 (Dynamic Arrow Logic). Recall the definition of $\mathcal{L}_{\mathrm{RA}}$. Add the unary connective * sending φ to φ^*. The set of formulas (denoted as F_{DL}) of Dynamic Arrow Logic is defined as that of $\mathcal{L}_{\mathrm{RA}}$ together with the following clause:

$$\varphi \in F_{\mathrm{DL}} \implies \varphi^* \in F_{\mathrm{DL}} \,.$$

The semantics of this connective is defined by

$$mng_{DL}(\varphi^*, \mathfrak{M}) \stackrel{\text{def}}{=} \text{“reflexive and transitive closure of the relation } mng_{DL}(\varphi, \mathfrak{M})\text{”}.$$

This defines $\models^*$ from $\models_{\mathrm{RA}}$. Now, Dynamic Arrow Logic is

$$\mathcal{L}_{\mathrm{DL}} = \langle F_{\mathrm{DL}}, M_{\mathrm{RA}}, mng_{DL}, \models^*\rangle \,.$$

Pratt's original dynamic logic can easily and naturally be interpreted into $\mathcal{L}_{\mathrm{DL}}$. For more on Dynamic Arrow Logic cf. e.g. van Benthem [13], Marx [35].

Answers/solutions for important and hard exercises of Section 2.2

Exercises 2.2.34.

(2) $\mathcal{L}_{\mathrm{PAIR}}$ is decidable.

There is a model $\mathfrak{M} \in M_{\mathrm{PAIR}}$ such that $Th_{\mathcal{L}_{\mathrm{PAIR}}}(\mathfrak{M})$ is not even recursively enumerable. See the hint for Ex's. 2.2.34 (3).

(5) A. Simon proved that for finite Ax, $Csq_{\mathcal{L}_{\mathrm{PAIR}}}(Ax)$ is decidable. He proved that the logic $\mathcal{L}_{\mathrm{PAIR}}$+"$\Diamond$ of $S5$" is still decidable; then, using $\Diamond$, $Ax \models \varphi$ is equivalent to validity of a single formula (see Simon [53]).

Exercises 2.2.35.

(3) $\mathcal{L}_{\mathrm{REL}}$ is undecidable. This hint is for the case you know that the word problem of semigroups [or equivalently, the quasi-equational theory of semigroups] is undecidable. Define a computable function f which to every quasi-equation q in the language of semigroups associates $f(q) \in F_{\mathrm{REL}}$ such that

$$\models_{\mathrm{REL}} f(q) \quad \iff \quad \text{Semigroups} \models q.$$

Conclude that $\mathcal{L}_{\mathrm{REL}}$ cannot be decidable because that would provide a decision algorithm for the quasi-equations of semigroups. There are other ways of handling this problem besides the "semigroup" one, cf. e.g. the important book Tarski–Givant [54].

There is a formula $\varphi \in F_{\mathrm{REL}}$ such that $Csq_{\mathcal{L}_{\mathrm{REL}}}(\{\varphi\})$ is undecidable. Moreover, $\mathcal{L}_{\mathrm{REL}}$ has the *Gödel's incompleteness property* that is,

$$(\exists\varphi \in F_{\mathrm{REL}})(\forall T \subseteq F_{\mathrm{REL}})\big[(\varphi \in T \text{ and } T \text{ is consistent}) \implies \\ \implies Csq_{\mathcal{L}_{\mathrm{REL}}}(T) \text{ is undecidable})\big].$$

Observe the contrast between $\mathcal{L}_{\mathrm{PAIR}}$ and $\mathcal{L}_{\mathrm{REL}}$!

(4) The others (Ex's. 2.2.3 (3)–(5) for $\mathcal{L}_{\mathrm{REL}}$) follow from the corresponding answers for Ex's. 2.2.34 above.

Exercises 2.2.36 (4). Here we give a very detailed hint for solving this exercise, i.e. for proving that $\mathcal{L}_{\mathrm{more}}$ is decidable.

Let $\mathfrak{A} = \langle A, \leq, +, O, I\rangle$ be a structure where $\leq$ is a binary relation on A, $+$ is a partial binary operation on A (i.e. $Dom(+) \subseteq A \times A$), $I \subseteq A$ and $O \in I$. The *diagram of* $\mathfrak{A}$, in symbols $\Delta(\mathfrak{A})$, is defined as follows. Let $a_0, \ldots, a_n$ be a repetition-free enumeration of $A \setminus I$. Let $x_0, \ldots, x_n$ be variables. For any $i, j \leq n$ let

$$\pi(x_i, x_j) \stackrel{\text{def}}{=} \begin{cases} x_i + x_j = x_k & \text{if } \ a_i + a_j = a_k \text{ in } \mathfrak{A} \\ x_i = x_j & \text{if } \ a_i + a_j \text{ is not defined in } \mathfrak{A}, \end{cases}$$

$$\varrho(x_i, x_j) \stackrel{\text{def}}{=} \begin{cases} x_i \leq x_j & \text{if } a_i \leq a_j \text{ in } \mathfrak{A} \\ x_i \not\leq x_j & \text{if } a_i \not\leq a_j \text{ in } \mathfrak{A}, \end{cases}$$

$$\delta(x_i, x_j) \stackrel{\text{def}}{=} \pi(x_i, x_j) \wedge \varrho(x_i, x_j).$$

$$\Delta(\mathfrak{A}) \stackrel{\text{def}}{=} \exists x_0 \ldots x_n \Big(\bigwedge \{\delta(x_i, x_j) \ : \ i, j \leq n\} \Big).$$

We note that $\Delta(\mathfrak{A})$ is a (first-order) formula containing only $+$ and $\leq$, therefore it is decidable whether this formula is valid in standard arithmetic or not.

We say that $\mathfrak{A}$ is a *cardinality structure* iff the following hold for all $a, b \in A$:

$\leq$ is a linear ordering on A;

O is the smallest element, i.e. $O \leq a$ for every $a \in A$;

I is an end segment, i.e. $a \in I$ and $a \leq b$ imply $b \in I$;

$O + a = a + O = a, \quad a + b = b$ if $a \leq b$ and $b \in I$;

$a + b \in I$ implies ($a \in I$ or $b \in I$);

$+$ is commutative and associative in the sense that
if $a + b$ exists then $b + a$ exists and $a + b = b + a$;
$a + b,\ (a + b) + c$ exist iff $b + c,\ a + (b + c)$ exist and $(a + b) + c = a + (b + c)$;

$\langle \mathbf{N}, \leq, + \rangle \models \Delta(\mathfrak{A})$.

We say that $(\mathfrak{A}, \kappa)$ is an *abstract cardinality model*, in symbols $(\mathfrak{A}, \kappa) \in ACMod$, iff

$\mathfrak{A}$ is a cardinality structure;

$\kappa : \mathcal{P}(P) \to A$ (where P is the set of atomic formulas of $\mathcal{L}_{\text{more}}$);

$\sum \langle \kappa(H) : H \in \mathcal{H} \rangle$ exists for all $\mathcal{H} \subseteq \mathcal{P}(P)$, where $\sum$ refers to addition in $\mathfrak{A}$.

Now let $(\mathfrak{A}, \kappa) \in ACMod$ and $\chi \in F_{\text{more}}$. We define $\sigma(\chi) \subseteq \mathcal{P}(P)$ by induction on the complexity of the formula χ as follows.

$\sigma(p) \stackrel{\text{def}}{=} \{H \in \mathcal{P}(P) : p \in H\}$ if $p \in P$;

$\sigma(\varphi \wedge \psi) \stackrel{\text{def}}{=} \sigma(\varphi) \cap \sigma(\psi)$;

$\sigma(\neg\varphi) \stackrel{\text{def}}{=} \mathcal{P}(P) \setminus \sigma(\varphi)$;

$$\sigma\big(\blacklozenge(\varphi, \psi)\big) \stackrel{\text{def}}{=} \begin{cases} \mathcal{P}(P) & \text{if } \sum\langle\kappa(H) : H \in \sigma(\varphi)\rangle \geq \sum\langle\kappa(H) : H \in \sigma(\psi)\rangle \\ \emptyset & \text{otherwise.} \end{cases}$$

$$(\mathfrak{A}, \kappa) \models \varphi \quad \stackrel{\text{def}}{\iff} \quad \sigma(\varphi) = \mathcal{P}(P).$$

Show that the following gives an algorithm for deciding validity of φ:

φ is valid in $\mathcal{L}_{\text{more}}$
iff
$(\mathfrak{A}, \kappa) \models \varphi$ for all $(\mathfrak{A}, \kappa) \in ACMod$ such that $|A| \leq 2^{2^{|P|}}$.

3 Bridge between the world of logics and the world of algebras

The algebraic counterpart of classical sentential logic $\mathcal{L}_S$ is the variety BA of Boolean algebras. Why is this so important? The answer lies in the general experience that it is usually much easier to solve a problem concerning $\mathcal{L}_S$ by translating it to BA, solving the algebraic problem, and then translating the result back to $\mathcal{L}_S$ (than solving it directly in $\mathcal{L}_S$).

In this section we extend applicability of BA to $\mathcal{L}_S$ to applicability of algebra in general to logics in general. We will introduce a standard translation method from logic to algebra, which to each logic $\mathcal{L}$ associates a class of algebras $\mathsf{Alg}_{\models}(\mathcal{L})$. (Of course, $\mathsf{Alg}_{\models}(\mathcal{L}_S)$ will be BA.) Further, this translation method will tell us how to find the algebraic question corresponding to a logical question. If the logical question is about $\mathcal{L}$ then its algebraic equivalent will be about $\mathsf{Alg}_{\models}(\mathcal{L})$. For example, if we want to decide whether $\mathcal{L}$ has the proof theoretic property called Craig's interpolation property, then it is sufficient to decide whether $\mathsf{Alg}_{\models}(\mathcal{L})$ has the so called amalgamation property (for which there are powerful methods in the literature of algebra). If the logical question concerns connections between several logics, say between $\mathcal{L}_1$ and $\mathcal{L}_2$, then the algebraic question will be about connections between $\mathsf{Alg}_{\models}(\mathcal{L}_1)$ and $\mathsf{Alg}_{\models}(\mathcal{L}_2)$. (The latter are quite often simpler, hence easier to investigate.)

3.1 Fine-tuning the framework

The definition of a logic in Section 2.1 is very wide. Actually, it is too wide for proving interesting theorems about logics. Now we will define a subclass of logics which we will call *nice logics*. Our notion of a nice logic is wide enough to cover the logics mentioned in the previous section, moreover, it is broad enough to cover almost all logics investigated in the literature. (Certain quantifier logics might need a little reformulation for this, but that reformulation does not effect the essential aspects of the logic in question as we will see.) On the other hand, the class of nice logics is narrow enough for proving interesting theorems about them, that is, we will be able to establish typical logical facts that hold for most logics studied in the literature.

Before reading Def. 3.1.1 below, it might be useful to contemplate the common features of the logics studied so far, e.g. $\mathcal{L}_S$, $S5$, $\mathcal{L}_{\mathrm{ARW0}}$, $\mathcal{L}_n$ (cf. Section 2.2).

In all the logics studied so far the biconditional $\leftrightarrow$ is available as a derived connective. In condition (3) of Def. 3.1.1 below new symbols $\Delta_0, \ldots, \Delta_{n-1}$ will occur, denoting derived connectives of the logic in question. Certainly, condition (3) is a weaker assumption than expressibility of $\leftrightarrow$ ($n = 1$, $\Delta_0 =\leftrightarrow$), thus all theorems remain true for this simpler case.

We also note that the theorems of Section 3.2 below (based on the next definition) can be proved in a more general setting (cf. [9]). Here we do restrictions in order to make the methodology more transparent. The reader who would find the definition below too restrictive is asked to consult Section 4 "Generalizations", where several conditions are either eliminated or it is explained how to eliminate them, and references are given where the elimination is done.

Definition 3.1.1 (nice logic, strongly nice logic, structural logic). Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a logic in the sense of Definition 2.1.3.

We say that $\mathcal{L}$ is a *nice logic* if conditions (1–4) below hold for $\mathcal{L}$.

(1) A set $Cn(\mathcal{L})$, called the set of *logical connectives* of $\mathcal{L}$, is fixed. Every $c \in Cn(\mathcal{L})$ has some rank $rank(c) \in \omega$. The set of all logical connectives of rank k is denoted by $Cn_k(\mathcal{L})$.

There is a set P, called the set of *atomic formulas* (or *parameters* or *propositional variables*), such that F is the smallest set satisfying conditions (a–b) below.

(a) $P \subseteq F$,

(b) if $c \in Cn_k(\mathcal{L})$ and $\varphi_1, \ldots, \varphi_k \in F$ then $c(\varphi_1, \ldots, \varphi_k) \in F$.

The word-algebra generated by P using the logical connectives from $Cn(\mathcal{L})$ as algebraic operations is denoted by $\mathfrak{F}$, that is, $\mathfrak{F} = \langle F, c\rangle_{c\in Cn(\mathcal{L})}$. $\mathfrak{F}$ is called the *formula algebra* of $\mathcal{L}$.

(2) The function $mng_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle mng(\varphi, \mathfrak{M}) : \varphi \in F\rangle$ is a homomorphism from $\mathfrak{F}$, for every $\mathfrak{M} \in M$.

(3) There are *"derived" connectives* $\varepsilon_0, \ldots, \varepsilon_{m-1}$ and $\delta_0, \ldots, \delta_{m-1}$ (unary) and $\Delta_0, \ldots, \Delta_{n-1}$ (binary) $(m, n \in \omega)$ of $\mathcal{L}$ with the following properties:

(i) $(\forall \mathfrak{M} \in M)(\forall \varphi, \psi \in F)\big[mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi) \iff (\forall i < n)\, \mathfrak{M} \models \varphi\Delta_i\psi\big]$.

(ii) $(\forall \mathfrak{M} \in M)(\forall \varphi \in F)\big[\mathfrak{M} \models \varphi \iff (\forall j < m)(\forall i < n)\, \mathfrak{M} \models \varepsilon_j(\varphi)\Delta_i\delta_j(\varphi)\big]$.

(By "derived" we mean that ε_j, δ_j and Δ_i are not necessarily members of $Cn(\mathcal{L})$. They are only "built up" from elements of $Cn(\mathcal{L})$. But we do not know from which elements of $Cn(\mathcal{L})$ they are built up, or how. We do not care!)

(4) $(\forall\psi,\varphi_0,\dots,\varphi_k\in F)(\forall p_0,\dots,p_k\in P)\big[\models\psi(\overline{p}) \implies \models\psi(\overline{p}/\overline{\varphi})\big]$,
where $\overline{p}=\langle p_0,\dots,p_k\rangle$, $\overline{\varphi}=\langle\varphi_0,\dots,\varphi_k\rangle$, and $\psi(\overline{p}/\overline{\varphi})$ denotes the formula that we get from ψ after simultaneously substituting φ_i for every occurrence of p_i $(i\leqslant k)$ in ψ. We refer to this condition as '$\mathcal{L}$ *has the substitution property*'.

$\mathcal{L}$ is called *strongly nice* iff it is nice and satisfies condition (5) below.

(5)
$$(+)\quad (\forall s\in {}^{P}F)(\forall\mathfrak{M}\in M)(\exists\mathfrak{N}\in M)(\forall\varphi(p_{i_0},\dots,p_{i_k})\in F)$$
$$mng_{\mathfrak{N}}(\varphi)=mng_{\mathfrak{M}}\big(\varphi(p_{i_0}/s(p_{i_0}),\dots,p_{i_k}/s(p_{i_k}))\big)\,.$$

Let $\hat{s}\in {}^{F}F$ be the natural extension of s to $\mathfrak{F}$. Then (+) says $mng_{\mathfrak{N}}(\varphi)=mng_{\mathfrak{M}}(\hat{s}(\varphi))$. If this property holds, then we say that the logic '$\mathcal{L}$ *has the semantical substitution property*' (the model $\mathfrak{N}$ is the substituted version of $\mathfrak{M}$ along substitution s).

Following the terminology of Blok and Pigozzi (cf. e.g. [14]), logics satisfying conditions (1), (2) and (5) above are called *structural logics*.

Recall that if $\mathfrak{A}$ and $\mathfrak{B}$ are two similar algebras, then $Hom(\mathfrak{A},\mathfrak{B})$ denotes the set of all homomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$ (cf. [49]).

Remark 3.1.2.

(i) Item (2) of Definition 3.1.1 above is a purely logical criterion. Namely, it is Frége's principle of compositionality.

(ii) Conditions (1–3) of Def. 3.1.1 above imply condition ($\star$) of Def. 2.1.3.

(iii) A special case of condition (3) of Def. 3.1.1 above is the case of $m=1$, $\varepsilon_0(\varphi)=True$, $\delta_0(\varphi)=\varphi$. This simplification implies the following connection between $\models$ and mng:

$$(\forall\varphi,\psi\in F)\big[\models\varphi \text{ and } \models\psi \quad\implies\quad (\forall\mathfrak{M}\in M)\ mng_{\mathfrak{M}}(\varphi)=mng_{\mathfrak{M}}(\psi)\big].$$

This does not follow from condition (3) of Def. 3.1.1.

(iv) An equivalent form of (+) above is the very natural condition

$$(\forall h\in Hom(\mathfrak{F},\mathfrak{F}))\,(\forall\mathfrak{M}\in M)(\exists\mathfrak{N}\in M)\ mng_{\mathfrak{N}}=mng_{\mathfrak{M}}\circ h\,.$$

Since h is just a substitution, this form makes it explicit that $\mathfrak{N}$ is the h-substituted version of $\mathfrak{M}$. Another equivalent version is the following.

$$(\forall\mathfrak{M}\in M)\big(\forall h\in Hom(\mathfrak{F},mng_{\mathfrak{M}}(\mathfrak{F}))\big)(\exists\mathfrak{N}\in M)\ mng_{\mathfrak{N}}=h.$$

(v) In the presence of (3) of Definition 3.1.1 above, semantical substitution property (5) implies substitution property (4).

Remark 3.1.3 (Connections with the Blok–Pigozzi approach). Here we mention only a small part of these connections.

The $\langle F_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ part [10] of a strongly nice, consequence compact (see Def. 3.2.13 below) logic $\mathcal{L}$ is always an *algebraizable deductive system* in the sense of Blok–Pigozzi [14] (which is an algebraizable 1-deductive system in [16]). The other way round, if $\langle F, \vdash \rangle$ is an algebraizable deductive system then $\mathcal{L}_{\vdash}$, as defined in Remark 2.1.2 above, is always a strongly nice consequence compact logic in our sense. Sructural logics and the connections between the two approaches are discussed in more detail in Font–Jansana [23].

A small sample of references of the Blok–Pigozzi approach is [14], [16], [15], [41], Czelakowski [21], Font–Jansana [22].

Exercises 3.1.4.

(1) (*Important!*) Show that all the logics introduced in Defs. 2.2.1–2.2.21 above are strongly nice logics. It is especially important to do it for $\mathcal{L}_n$!

(2) Show that $\mathcal{L}_{\mathrm{FOL}}$ (cf. Def. 2.2.23) is a nice logic.

Exercises 3.1.5. Show logics where $n = 1$ but Δ_0 is not our old biconditional $\leftrightarrow$. (E.g., in $S5$ we can also take $\Box(\Phi_1 \leftrightarrow \Phi_2)$ as $\Phi_1 \Delta_0 \Phi_2$.) Show logics where $n > 1$.

For any class K of similar algebras, $\mathbf{I}K \stackrel{\text{def}}{=} \{\mathfrak{M} : (\exists \mathfrak{N} \in K)\, \mathfrak{M} \text{ is isomorphic to } \mathfrak{N}\}$ (cf. [49]).

Definition 3.1.6 (algebraic counterpart of a logic). Let $\mathcal{L} = \langle F, M, mng, \models \rangle$ be a logic satisfying conditions (1),(2) of Definition 3.1.1.

(i) Let $K \subseteq M$. Then for every $\varphi, \psi \in F$

$$\varphi \sim_K \psi \stackrel{\text{def}}{\iff} (\forall \mathfrak{M} \in K)\ mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi).$$

Then $\sim_K$ is an equivalence relation, which is a congruence on $\mathfrak{F}$ by condition (2) of Def. 3.1.1. $\mathfrak{F}/{\sim_K}$ denotes the factor-algebra of $\mathfrak{F}$, factorized by $\sim_K$. Now,

$$\mathsf{Alg}_{\models}(\mathcal{L}) \stackrel{\text{def}}{=} \mathbf{I}\, \{\mathfrak{F}/{\sim_K} :\ K \subseteq M\}\,.$$

(ii)

$$\mathsf{Alg}_m(\mathcal{L}) \stackrel{\text{def}}{=} \{mng_{\mathfrak{M}}(\mathfrak{F})\ :\ \mathfrak{M} \in M\}\,,$$

[10] Here $\models_{\mathcal{L}}$ denotes the semantical consequence relation induced by the validity relation of $\mathcal{L}$.

where $mng_{\mathfrak{M}}$ was defined in item (2) of Definition 3.1.1, and for any homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, $h(\mathfrak{A})$ is the homomorphic image of $\mathfrak{A}$ along h i.e., $h(\mathfrak{A})$ is the smallest subalgebra of $\mathfrak{B}$ such that $h : \mathfrak{A} \longrightarrow h(\mathfrak{A})$ (cf. [49]).

Remark 3.1.7. In the definition of $\mathsf{Alg}_m(\mathcal{L})$ above, it is important that $\mathsf{Alg}_m(\mathcal{L})$ is not an abstract class in the sense that it is not closed under isomorphisms. The reason for defining $\mathsf{Alg}_m(\mathcal{L})$ in such a way is that since $\mathsf{Alg}_m(\mathcal{L})$ is the class of algebraic counterparts of the *models* of $\mathcal{L}$, we need these algebras as concrete algebras and replacing them with their isomophic copies would lead to loss of information (about semantic-model theoretic matters). See e.g. Thm. 5.12 in Appendix B about the algebraic characterization of the *weak Beth definability property*.

Fact 3.1.8. *Let $\mathcal{L}$ be a logic satisfying conditions (1–3(i)) of Def. 3.1.1. Then*

$$\mathsf{Alg}_{\models}(\mathcal{L}) = \mathbf{I}\left\{\mathfrak{F}/{\sim_{Mod_{\mathcal{L}}(\Gamma)}} \ : \ \Gamma \subseteq F\right\}.$$

Proof. For every $K \subseteq M$, $\quad \mathfrak{F}/{\sim_K} = \mathfrak{F}/{\sim_{Mod_{\mathcal{L}}(Th_{\mathcal{L}}(K))}}$ holds (cf. Defs. 2.1.5 and 2.1.6). $\square$

Exercises 3.1.9. Show that for any logic $\mathcal{L}$ satisfying conditions (1), (2) of Def. 3.1.1

- $\mathsf{Alg}_m(\mathcal{L}) \subseteq \mathsf{Alg}_{\models}(\mathcal{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$
- $\mathbf{SP}\mathsf{Alg}_{\models}(\mathcal{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$.

Exercises 3.1.10. Prove that

(i) $\mathsf{Alg}_m(\mathcal{L}_S) \subseteq$"class of all Boolean set algebras"

(ii) $\mathsf{Alg}_m(S5) \subseteq$"class of all one-dimensional cylindric set algebras"

Theorem 3.1.11. *For any logic $\mathcal{L} = \langle F, M, mng, \models\rangle$ satisfying conditions (1),(2) of Definition 3.1.1, (i) and (ii) below hold.*

- *"(i)"* $\mathsf{Alg}_{\models}(\mathcal{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$.
- *"(ii)"* $\mathbf{SP}\mathsf{Alg}_{\models}(\mathcal{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathcal{L})$.

Proof of (i). Let $\mathfrak{A} \in \mathsf{Alg}_{\models}(\mathcal{L})$ that is, let $\mathfrak{A} \cong \mathfrak{F}/{\sim_K}$ for some class $K \subseteq M$. Then there is a subset $K' \subseteq K$ such that $\mathfrak{F}/{\sim_K} = \mathfrak{F}/{\sim_{K'}}$ (this holds because F is always a set). Now we can define an embedding h from $\mathfrak{F}/{\sim_{K'}}$ into $\mathbf{P}_{\mathfrak{M}\in K'}\, mng_{\mathfrak{M}}(\mathfrak{F})$ as follows. For each $\varphi \in F$

$$h\left(\varphi/{\sim_{K'}}\right) \stackrel{\text{def}}{=} \langle mng_{\mathfrak{M}}(\varphi) \ : \ \mathfrak{M} \in K'\rangle.$$

$\square$

Proof of (ii). To be filled in later. □

* * *

Next we turn to *inference systems*. Inference systems (usually denoted as $\vdash$) are syntactical devices serving to recapture (or at least to approximate) the semantical consequence relation of the logic $\mathcal{L}$. The idea is the following. Suppose $\Sigma \models_{\mathcal{L}} \varphi$. This means that, in the logic $\mathcal{L}$, the assumptions collected in Σ semantically imply the conclusion φ. (In any possible world $\mathfrak{M}$ of $\mathcal{L}$ that is, in any $\mathfrak{M} \in M_{\mathcal{L}}$, whenever Σ is valid in $\mathfrak{M}$, then also φ is valid in $\mathfrak{M}$.) Then we would like to be able to reproduce this relationship between Σ and φ by purely syntactical, "finitistic" means. That is, by applying some formal rules of inference (and some axioms of the logic $\mathcal{L}$) we would like to be able to derive φ from Σ by using "paper and pencil" only. In particular, such a derivation will always be a finite string of symbols. If we can do this, that will be denoted by $\Sigma \vdash \varphi$.

Definition 3.1.12 (formula scheme). Let $\mathcal{L}$ be a logic satisfying condition (1) of Def. 3.1.1, with the set $Cn(\mathcal{L})$ of logical connectives. Fix a countable set $A = \{A_i : i < \omega\}$, called the set of *formula variables*. The set $Fms_{\mathcal{L}}$ of *formula schemes* of $\mathcal{L}$ is the smallest set satisfying conditions (a–b) below.

(a) $A \subseteq Fms_{\mathcal{L}}$,

(b) if $c \in Cn_k(\mathcal{L})$ and $\Phi_1, \dots, \Phi_k \in Fms_{\mathcal{L}}$ then $c(\Phi_1, \dots, \Phi_k) \in Fms_{\mathcal{L}}$.

An *instance of a formula scheme* is given by substituting formulas for the formula variables in it.

Definition 3.1.13 (Hilbert-style inference system). Let $\mathcal{L}$ be a logic satisfying condition (1) of Def. 3.1.1. An *inference rule* of $\mathcal{L}$ is a pair $\langle\langle B_1, \dots, B_n\rangle, B_0\rangle$, where every B_i ($i \leqslant n$) is a formula scheme. This inference rule will be denoted by

$$\frac{B_1, \dots, B_n}{B_0}\,.$$

An *instance of an inference rule* is given by substituting formulas for the formula variables in the formula schemes occurring in the rule.

A *Hilbert-style inference system* (or *calculus*) for $\mathcal{L}$ is a **finite** set of formula schemes (called *axiom schemes* or *axioms*) together with a **finite** set of inference rules.

Definition 3.1.14 (derivability). Let $\mathcal{L}$ be a logic satisfying condition (1) of Def. 3.1.1 and let $\vdash$ be a Hilbert-style inference system for $\mathcal{L}$. Assume $\Sigma \cup \{\varphi\} \subseteq F_{\mathcal{L}}$. We say that φ is $\vdash$-*derivable* (or $\vdash$-*provable*) from Σ iff there is a finite sequence $\langle\varphi_1, \dots, \varphi_n\rangle$ of formulas (an $\vdash$-*proof of* φ *from* Σ) such that φ_n is φ and for every $1 \leqslant i \leqslant n$

- $\varphi_i \in \Sigma$ or
- φ_i is an instance of an axiom scheme (an *axiom* for short) of $\vdash$ or
- there are $j_1, \ldots, j_k < i$, and there is an inference rule of $\vdash$ such that $\frac{\varphi_{j_1}, \ldots, \varphi_{j_k}}{\varphi_i}$ is an instance of this rule.

We write $\Sigma \vdash \varphi$ if φ is $\vdash$-provable from Σ. (We will often identify an inference system $\vdash$ with the corresponding derivability relation.)

Definition 3.1.15 (complete and sound Hilbert-type inference system). Let $\mathcal{L}$ be a logic satisfying condition (1) of Def. 3.1.1 and let $\vdash$ be a Hilbert-type inference system for $\mathcal{L}$. Then

- $\vdash$ is *weakly complete* for $\mathcal{L}$ iff
$$(\forall \varphi \in F_{\mathcal{L}})\,(\models_{\mathcal{L}} \varphi \implies \vdash \varphi)\,;$$
- $\vdash$ is *finitely complete* for $\mathcal{L}$ iff
$$(\forall \Sigma \subseteq_\omega F_{\mathcal{L}})(\forall \varphi \in F_{\mathcal{L}})\,(\Sigma \models_{\mathcal{L}} \varphi \implies \Sigma \vdash \varphi)\,;$$
that is, we consider only finite Σ's;
- $\vdash$ is *strongly complete* for $\mathcal{L}$ iff
$$(\forall \Sigma \subseteq F_{\mathcal{L}})(\forall \varphi \in F_{\mathcal{L}})\,(\Sigma \models_{\mathcal{L}} \varphi \implies \Sigma \vdash \varphi)\,;$$
- $\vdash$ is *weakly sound* for $\mathcal{L}$ iff
$$(\forall \varphi \in F_{\mathcal{L}})\,(\vdash \varphi \implies \models_{\mathcal{L}} \varphi)\,;$$
- $\vdash$ is *strongly sound* for $\mathcal{L}$ iff
$$(\forall \Sigma \subseteq F_{\mathcal{L}})(\forall \varphi \in F_{\mathcal{L}})\,(\Sigma \vdash \varphi \implies \Sigma \models_{\mathcal{L}} \varphi)\,. \qquad \blacktriangleleft$$

3.2 Algebraic characterizations of completeness and compactness properties via Alg_m and $\mathsf{Alg}_{\models}$ (main theorems)

Theorem 3.2.1. *(i) Let $\mathcal{L} = \langle F, M, mng, \models_{\mathcal{L}} \rangle$* [11] *be a strongly nice logic. Let m be as in Def. 3.1.1 (3). Then for any formulas $\varphi_0, \varphi_1, \ldots, \varphi_k$,*

$$\{\varphi_1, \ldots, \varphi_k\} \models_{\mathcal{L}} \varphi_0 \quad \Longleftrightarrow \quad \text{for each } j < m$$

$$\mathsf{Alg}_m(\mathcal{L}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big).$$

(ii) Let $\mathcal{L}$ be a strongly nice logic in the sense of Def. 3.1.1. Let n be as in Def. 3.1.1 (3). Then for any quasi-equation q of form $(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k' \Rightarrow \tau_0 = \tau_0')$,

$$\mathsf{Alg}_m(\mathcal{L}) \models q \quad \Longleftrightarrow \quad \{\tau_s \Delta_j \tau_s' : 1 \le s \le k,\ j < n\} \models_{\mathcal{L}} \tau_0 \Delta_i \tau_0' \quad \text{for each } i < n.$$

(iii) For proving the "$\Longleftarrow$" direction of (i), it is enough to assume that $\mathcal{L}$ satisfies conditions (1–3) of Def. 3.1.1. For proving the "$\Longrightarrow$" direction of (ii), it is enough to assume that $\mathcal{L}$ satisfies conditions (1–3(i)) of Def. 3.1.1. However, there exist logics satisfying (1–3(i)) of Def. 3.1.1 for which direction "$\Longleftarrow$" of (ii) does not hold. For proving this direction we do not have to assume condition (3)(ii) of Def. 3.1.1.

Proof of (i). Direction "$\Longrightarrow$": Assume $p_0, \ldots, p_\ell$ are the only atomic formulas occurring in $\varphi_0, \ldots, \varphi_k$ and assume that

$$\{\varphi_1(p_0, \ldots, p_\ell), \ldots, \varphi_k(p_0, \ldots, p_\ell)\} \models_{\mathcal{L}} \varphi_0(p_0, \ldots, p_\ell).$$

Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$. Then $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M$. Let $a \in {}^P A$ be arbitrary. For every $i \le \ell$ we denote $a_i \stackrel{\text{def}}{=} a(p_i)$. Clearly for every $i \le \ell \quad a_i = mng_{\mathfrak{M}}(\gamma_i)$ for some $\gamma_i \in F$. For every $s \le k$

$$\varphi_s[a_0, \ldots, a_\ell]^{\mathfrak{A}} = \varphi_s[mng_{\mathfrak{M}}(\gamma_0), \ldots, mng_{\mathfrak{M}}(\gamma_\ell)]^{\mathfrak{A}} = mng_{\mathfrak{M}}\big(\varphi_s(\gamma_0, \ldots, \gamma_\ell)\big),$$

since $mng_{\mathfrak{M}}$ is a homomorphism by (2) of Def. 3.1.1.

[11] During the proofs of the main theorems we make a careful distinction between $\models_{\mathcal{L}}$ and $\models$, using the former symbol for the validity (and semantical consequence) relation of logic $\mathcal{L}$ and $\models$ for the usual first-order validity relation.

Assume that for every $1 \le s \le k$ and $j < m$, $\mathfrak{A} \models \big(\varepsilon_j(\varphi_s) = \delta_j(\varphi_s)\big)[a]$.

$\iff mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_s(\gamma_0, \dots, \gamma_\ell))\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_s(\gamma_0, \dots, \gamma_\ell))\big) \quad (1 \le s \le k,\ j < m)$

$\overset{\text{(by Def. 3.1.1 (5))}}{\Longrightarrow}$ There is $\mathfrak{N}$ as described in condition (5) of Def. 3.1.1 for s sending p_0 to γ_0, ..., p_k to γ_k an for $\mathfrak{M}$. Let this $\mathfrak{N}$ be fixed.

$\Longrightarrow mng_{\mathfrak{N}}\big(\varepsilon_j(\varphi_s)\big) = mng_{\mathfrak{N}}\big(\delta_j(\varphi_s)\big) \quad (1 \le s \le k,\ j < m)$

$\overset{\text{(by Def. 3.1.1 (3))}}{\iff} \mathfrak{N} \models_{\mathcal{L}} \varphi_s \quad (1 \le s \le k)$

$\overset{\text{(by our assumption)}}{\Longrightarrow} \mathfrak{N} \models_{\mathcal{L}} \varphi_0$

$\overset{\text{(by Def. 3.1.1 (3))}}{\iff} mng_{\mathfrak{N}}\big(\varepsilon_j(\varphi_0)\big) = mng_{\mathfrak{N}}\big(\delta_j(\varphi_0)\big) \quad (j < m)$

$\overset{\text{(by Def. 3.1.1 (5))}}{\Longrightarrow} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_0(\gamma_0, \dots, \gamma_\ell))\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_0(\gamma_0, \dots, \gamma_\ell))\big) \quad (j < m)$

$\iff \mathfrak{A} \models \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big)[a], \quad (j < m)$

proving Thm. 3.2.1 (i) direction "$\Longrightarrow$", since a was chosen arbitrarily.

Direction "$\Longleftarrow$": Assume that

$$\mathsf{Alg}_m(\mathcal{L}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big).$$

Let $\mathfrak{M} \in M$. Assume that for every $1 \le s \le k$ $\mathfrak{M} \models_{\mathcal{L}} \varphi_s$.

$$\overset{\text{(by Def. 3.1.1 (3))}}{\Longrightarrow} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_s)\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_s)\big) \quad (1 \le s \le k,\ j < m)$$
$$\overset{\text{(by our assumption)}}{\iff} mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi_0)\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi_0)\big) \quad (j < m)$$
$$\overset{\text{(by Def. 3.1.1 (3))}}{\Longrightarrow} \mathfrak{M} \models_{\mathcal{L}} \varphi_0$$

proving Thm. 3.2.1 (i) direction "$\Longleftarrow$". □

Proof of (ii). Direction "$\Longrightarrow$": Assume that for every $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$ and for every valuation $a \in {}^P A$

$$\mathfrak{A} \models q[a].$$

Let $\mathfrak{M} \in M$ such that $\mathfrak{M} \models_{\mathcal{L}} \{\tau_s \Delta_i \tau'_s : 1 \le s \le k,\ i < n\}$. Then by Def. 3.1.1 (3)(i) $mng_{\mathfrak{M}}(\tau_s) = mng_{\mathfrak{M}}(\tau'_s)$ for each $1 \le s \le k$. Now let $\mathfrak{A} \overset{\text{def}}{=} mng_{\mathfrak{M}}(\mathfrak{F})$ and let $a \in {}^P A$ be such that for each $p \in P$ $\quad a(p) \overset{\text{def}}{=} mng_{\mathfrak{M}}(p)$. Then

$$\mathfrak{A} \models (\tau_1 = \tau'_1 \wedge \dots \wedge \tau_k = \tau'_k)[a],$$

which implies by our assumption that $\mathfrak{A} \models (\tau_0 = \tau_0')[a]$. This is the same as $mng_{\mathfrak{M}}(\tau_0) = mng_{\mathfrak{M}}(\tau_0')$, thus again by Definition 3.1.1 (3)(i), $\mathfrak{M} \models_{\mathcal{L}} \tau_0 \Delta_i \tau_0'$ for each $i < n$, which proves direction "$\Longrightarrow$" of Thm. 3.2.1 (ii).

Direction "$\Longleftarrow$": Assume $\{\tau_s \Delta_j \tau_s' : 1 \leq s \leq k,\ j < n\} \models_{\mathcal{L}} \tau_0 \Delta_i \tau_0'$ for each $i < n$. Assume $p_0, \ldots, p_\ell$ are the only atomic formulas occurring in $\tau_0, \tau_0', \ldots, \tau_k, \tau_k'$. Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$. Then $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M$. Let $a \in {}^P A$ be arbitrary. For every $i \leq \ell$ we denote $a_i \stackrel{\text{def}}{=} a(p_i)$. Clearly for every $i \leq \ell$ $a_i = mng_{\mathfrak{M}}(\gamma_i)$ for some $\gamma_i \in F$. For every $\varphi \in F$

$$\varphi[a_0, \ldots, a_\ell]^{\mathfrak{A}} = \varphi[mng_{\mathfrak{M}}(\gamma_0), \ldots, mng_{\mathfrak{M}}(\gamma_\ell)]^{\mathfrak{A}} = mng_{\mathfrak{M}}\big(\varphi(\gamma_0, \ldots, \gamma_\ell)\big)$$

since $mng_{\mathfrak{M}}$ is a homomorphism by (2) of Def. 3.1.1. Assume that for every $1 \leq s \leq k$, $\mathfrak{A} \models \tau_s = \tau_s'[a]$.

$\Longleftrightarrow mng_{\mathfrak{M}}\big(\tau_s(\gamma_0, \ldots, \gamma_\ell)\big) = mng_{\mathfrak{M}}\big(\tau_s'(\gamma_0, \ldots, \gamma_\ell)\big)$

$\stackrel{\text{(by Def. 3.1.1 (5))}}{\Longrightarrow}$ There is $\mathfrak{N}$ as described in condition (5) of Def. 3.1.1 for s sending p_0 to γ_0, …, p_k to γ_k and for $\mathfrak{M}$. Let this $\mathfrak{N}$ be fixed.

$\Longrightarrow mng_{\mathfrak{N}}(\tau_s) = mng_{\mathfrak{N}}(\tau_s') \quad (1 \leq s \leq k)$

$\stackrel{\text{(by Def. 3.1.1 (3)(i))}}{\Longleftrightarrow} \mathfrak{N} \models_{\mathcal{L}} \tau_s \Delta_i \tau_s' \quad (1 \leq s \leq k,\ i < n)$

$\stackrel{\text{(by our assumption)}}{\Longrightarrow} \mathfrak{N} \models \tau_0 \Delta_i \tau_0' \quad (i < n)$

$\stackrel{\text{(by Def. 3.1.1 (5))}}{\Longrightarrow} mng_{\mathfrak{M}}\big(\tau_0(\gamma_0, \ldots, \gamma_\ell)\big) = mng_{\mathfrak{M}}\big(\tau_0'(\gamma_0, \ldots, \gamma_\ell)\big)$

$\Longleftrightarrow \mathfrak{A} \models \big(\tau_0 = \tau_0'\big)[a]$,

proving Thm. 3.2.1 (ii) direction "$\Longleftarrow$", since a was chosen arbitrarily. □

Proof of (iii). To be filled in later. □

Corollary 3.2.2. *Let $\mathcal{L}$ be a nice logic. Let $\varepsilon, \delta, \Delta, m, n$ be as in Def. 3.1.1 (3). Then (i) and (ii) below hold.*

(i) For any formula φ,

$$\models_{\mathcal{L}} \varphi \quad \Longleftrightarrow \quad \mathsf{Alg}_m(\mathcal{L}) \models \varepsilon_j(\varphi) = \delta_j(\varphi) \quad \textit{for each } j < m.$$

(ii) For any equation $\tau = \tau'$,

$$\mathsf{Alg}_m(\mathcal{L}) \models \tau = \tau' \quad \Longleftrightarrow \quad \models_{\mathcal{L}} \tau \Delta_i \tau' \quad \textit{for each } i < n.$$

Proof. Item (ii) is a special case of item (ii) of Thm. 3.2.1, but now we have to prove (i) for *nice* logics, cf. Def. 3.1.1.

Assume $\models_{\mathcal{L}} \varphi(p_0, \ldots, p_\ell)$. Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$. Then $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M$. Let $a \in {}^P A$ be arbitrary. We denote $a_0 \stackrel{\text{def}}{=} a(p_0), \ldots, a_\ell \stackrel{\text{def}}{=} a(p_\ell)$. Clearly $(\forall s \leq \ell)$ $(a_s = mng_{\mathfrak{M}}(\gamma_s)$ for some $\gamma_s \in F)$.

$$\varphi[a_0, \ldots, a_\ell]^{\mathfrak{A}} = \varphi[mng_{\mathfrak{M}}(\gamma_0), \ldots, mng_{\mathfrak{M}}(\gamma_\ell)]^{\mathfrak{A}} = mng_{\mathfrak{M}}\left(\varphi(\gamma_0, \ldots, \gamma_\ell)\right),$$

since $mng_{\mathfrak{M}}$ is a homomorphism.

$\models_{\mathcal{L}} \varphi(p_0, \ldots, p_\ell)$ implies, by Def. 3.1.1 (4), that $\models_{\mathcal{L}} \varphi(\gamma_0, \ldots, \gamma_\ell)$. Thus by Def. 3.1.1 (3), for each $j < m$

$$mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big) = mng_{\mathfrak{M}}\big(\delta_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big).$$

But

$$\begin{aligned} &mng_{\mathfrak{M}}\big(\varepsilon_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big) = \varepsilon_j(\varphi)[a]^{\mathfrak{A}} \text{ and} \\ &mng_{\mathfrak{M}}\big(\delta_j(\varphi(\gamma_0, \ldots, \gamma_\ell))\big) = \delta_j(\varphi)[a]^{\mathfrak{A}} \qquad (j < m). \end{aligned}$$

Thus we have $\mathfrak{A} \models \big(\varepsilon_j(\varphi) = \delta_j(\varphi)\big)[a]$ for each $j < m$, completing the proof since a was chosen arbitrarily. $\square$

In Theorem 3.2.3 below, we will give a sufficent and necessary condition for a strongly nice logic to have a finitely complete Hilbert-style inference system.

Theorem 3.2.3. *Assume $\mathcal{L}$ is a strongly nice logic and $Cn(\mathcal{L})$ is finite*[12]. *Then*

$$\begin{gathered} \mathsf{Alg}_m(\mathcal{L}) \textit{ generates a finitely axiomatizable quasi-variety} \\ \Longleftrightarrow \\ (\exists \textit{ Hilbert-style } \vdash)(\vdash \textit{ is finitely complete and strongly sound for } \mathcal{L}). \end{gathered}$$

Proof of ($\Longrightarrow$). **Notation** Let $\Phi_0, \Phi_1, \ldots$ denote formula variables, $\tau_0, \tau_1, \ldots$ denote formula schemes, $\overline{\Phi}$ denote sequence of formula variables and $\overline{x}$ denote sequence of variables. Let m and n $(m, n \in \omega)$ denote the number of ε_j's and Δ_i's, respectively. For any formula schemes τ, τ', let $\tau\boldsymbol{\Delta}\tau'$ abbreviate the system $\tau\Delta_0\tau', \ldots, \tau\Delta_{n-1}\tau'$ of formula schemes.

Now assume that Ax is a finite set of quasi-equations axiomatizing the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L})$ and define a Hilbert-style inference system $\vdash_{Ax}$ as follows:

[12]One can eliminate the assumption of $Cn(\mathcal{L})$ being finite. Then the finitary character of a Hilbert-style inference system has to be ensured in a more subtle way. Also, "finitely axiomatizable quasi-variety" must be replaced by "finite schema axiomatizable quasi-variety" in the second clause, cf. e.g. Monk [36], Németi [39].

Axiom schemes: $\Phi_0\Delta_i\Phi_0 \qquad (i<n)$.
Inference rules: If $\big[\big(\tau_1(\overline{x})=\tau_1'(\overline{x})\wedge\cdots\wedge\tau_k(\overline{x})=\tau_k'(\overline{x})\big)\Rightarrow\tau_0(\overline{x})=\tau_0'(\overline{x})\big]\in Ax$, then

$$\frac{\tau_1(\overline{\Phi})\mathbf{\Delta}\tau_1'(\overline{\Phi}),\ldots,\tau_k(\overline{\Phi})\mathbf{\Delta}\tau_k'(\overline{\Phi})}{\tau_0(\overline{\Phi})\Delta_i\tau_0'(\overline{\Phi})}$$

is a rule for each $i<n$. Other rules are:

$$(\forall i<n)\quad\frac{\Phi_0\mathbf{\Delta}\Phi_1,\ \Phi_1\mathbf{\Delta}\Phi_2}{\Phi_0\Delta_i\Phi_2},$$

$$(\forall i<n)\quad\frac{\Phi_0\mathbf{\Delta}\Phi_1}{\Phi_1\Delta_i\Phi_0},$$

$$(\forall c\in Cn_\ell(\mathcal{L}))(\forall i<n)\quad\frac{\Phi_1\mathbf{\Delta}\Phi_1',\ldots,\Phi_\ell\mathbf{\Delta}\Phi_\ell'}{c(\Phi_1,\ldots,\Phi_\ell)\Delta_i c(\Phi_1',\ldots,\Phi_\ell')},$$

$$\frac{\varepsilon_0(\Phi_0)\mathbf{\Delta}\delta_0(\Phi_0),\ldots,\varepsilon_{m-1}(\Phi_0)\mathbf{\Delta}\delta_{m-1}(\Phi_0)}{\Phi_0},$$

$$(\forall i<n)(\forall j<m)\quad\frac{\Phi_0}{\varepsilon_j(\Phi_0)\Delta_i\delta_j(\Phi_0)}\,.$$

We will show that the inference system $\vdash_{Ax}$ is finitely complete and strongly sound for $\mathcal{L}$.

For any set Σ of formulas we define

$$\psi\sim_\Sigma\psi'\qquad\stackrel{\text{def}}{\Longleftrightarrow}\qquad\Sigma\vdash_{Ax}\{\psi\Delta_i\psi':i<n\}.$$

Note that, by the definition of $\vdash_{Ax}$ and by the definition of derivability (Def. 3.1.14), $\sim_\Sigma$ is a congruence relation on $\mathfrak{F}$ for any Σ. □

Claim 3.2.4. *For any* $\Sigma\subseteq F$, $\quad(\mathfrak{F}/{\sim_\Sigma})\models Ax$.

Proof of Claim 3.2.4. Let $q\in Ax$ and assume that q is of form

$$\big(\tau_1(\overline{x})=\tau_1'(\overline{x})\wedge\cdots\wedge\tau_k(\overline{x})=\tau_k'(\overline{x})\big)\Rightarrow\tau_0(\overline{x})=\tau_0'(\overline{x})\,.$$

Let $\mathfrak{A}\stackrel{\text{def}}{=}(\mathfrak{F}/{\sim_\Sigma})$. We want to prove that, for every valuation a of the variables into $\mathfrak{A}$, $\mathfrak{A}\models q[a]$.

So let a be an arbitrary valuation into $\mathfrak{A}$. Then $(\forall i\in\omega)$ $a(x_i)=\varphi_i/{\sim_\Sigma}$ for some $\varphi_i\in F$. Assume that

$$\mathfrak{A}\models\tau_1\big[\overline{\varphi/{\sim_\Sigma}}\big]=\tau_1'\big[\overline{\varphi/{\sim_\Sigma}}\big]\wedge\cdots\wedge\tau_k\big[\overline{\varphi/{\sim_\Sigma}}\big]=\tau_k'\big[\overline{\varphi/{\sim_\Sigma}}\big].$$

Then
$$\big(\tau_1(\overline{\varphi})\big)/{\sim_\Sigma} = \big(\tau_1'(\overline{\varphi})\big)/{\sim_\Sigma}, \ldots, \big(\tau_k(\overline{\varphi})\big)/{\sim_\Sigma} = \big(\tau_k'(\overline{\varphi})\big)/{\sim_\Sigma},$$
since $\sim_\Sigma$ is a congruence on $\mathfrak{F}$. Then
$$\tau_1(\overline{\varphi}) \sim_\Sigma \tau_1'(\overline{\varphi}), \ldots, \tau_k(\overline{\varphi}) \sim_\Sigma \tau_k'(\overline{\varphi})$$
that is,
$$\Sigma \vdash_{Ax} \{\tau_j(\overline{\varphi})\, \Delta_i \tau_j'(\overline{\varphi}) : 1 \le j \le k,\ i < n\}$$
by the definition of $\sim_\Sigma$. In $\vdash_{Ax}$, we have the following rule for each $i < n$ (corresponding to quasiequation q):
$$\frac{\tau_1(\overline{\Phi})\boldsymbol{\Delta}\tau_1'(\overline{\Phi}), \ldots, \tau_k(\overline{\Phi})\boldsymbol{\Delta}\tau_k'(\overline{\Phi})}{\tau_0(\overline{\Phi})\Delta_i\tau_0'(\overline{\Phi})}\,.$$
By these rules, we get that $\Sigma \vdash_{Ax} \tau_0(\overline{\varphi})\,\Delta_i\tau_0'(\overline{\varphi})$ for each $i < n$. Then $\tau_0(\overline{\varphi}) \sim_\Sigma \tau_0'(\overline{\varphi})$, whence $\big(\tau_0(\overline{\varphi})\big)/{\sim_\Sigma} = \big(\tau_0'(\overline{\varphi})\big)/{\sim_\Sigma}$ that is, $\mathfrak{A} \models \tau_0\big[\overline{\varphi/{\sim_\Sigma}}\big] = \tau_0'\big[\overline{\varphi/{\sim_\Sigma}}\big]$ which implies $\mathfrak{A} \models \big(\tau_0(\overline{x}) = \tau_0'(\overline{x})\big)[a]$. By this we proved Claim 3.2.4. $\square$

Now let $\Sigma \overset{\text{def}}{=} \{\varphi_1, \ldots, \varphi_k\}$ and assume $\Sigma \models_{\mathcal{L}} \varphi_0$. Then, by Thm. 3.2.1 (i),

$$\begin{aligned}
&\mathsf{Alg}_m(\mathcal{L}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big) \quad (j < m)\\
&\Longrightarrow Ax \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big) \quad (j < m)\\
&\overset{\text{(Claim 3.2.4)}}{\Longrightarrow} (\mathfrak{F}/{\sim_\Sigma}) \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow\\
&\qquad\qquad\qquad\qquad \Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big) \quad (j < m)\\
&\Longrightarrow \big[\text{if } \big((\forall i < m)(\forall 1 \le s \le k)\ \varepsilon_i(\varphi_s) \sim_\Sigma \delta_i(\varphi_s)\big) \text{ then } (\forall j < m)\ \varepsilon_j(\varphi_0) \sim_\Sigma \delta_j(\varphi_0)\big]\\
&\Longleftrightarrow \big[\text{if } \big((\forall \ell < n)(\forall i < m)(\forall 1 \le s \le k)\ \Sigma \vdash_{Ax} \varepsilon_i(\varphi_s)\Delta_\ell\delta_i(\varphi_s)\big)\\
&\qquad\qquad \text{then } (\forall \ell < n)(\forall j < m)\ \Sigma \vdash_{Ax} \varepsilon_j(\varphi_0)\Delta_\ell\delta_j(\varphi_0)\big]. \qquad (\bullet)
\end{aligned}$$

By the rules $\frac{\Phi_0}{\varepsilon_i(\Phi_0)\Delta_\ell\delta_i(\Phi_0)}$ we have $\Sigma \vdash_{Ax} \varepsilon_i(\varphi_s)\Delta_\ell\delta_i(\varphi_s)$ for every $i < m$, $\ell < n$, $1 \le s \le k$. Thus, by $(\bullet)$, $\Sigma \vdash_{Ax} \varepsilon_j(\varphi_0)\Delta_\ell\delta_j(\varphi_0)$ holds for each $\ell < n$, $j < m$. Now using the rule $\frac{\varepsilon_0(\Phi_0)\boldsymbol{\Delta}\delta_0(\Phi_0), \ldots, \varepsilon_{m-1}(\Phi_0)\boldsymbol{\Delta}\delta_{m-1}(\Phi_0)}{\Phi_0}$ we get $\Sigma \vdash_{Ax} \varphi_0$, proving the finite completeness of $\vdash_{Ax}$.

The strong soundness of $\vdash_{Ax}$ can be proved by induction on the length of the $\vdash_{Ax}$-proof of φ_0 from $\{\varphi_1, \ldots, \varphi_k\}$. We only show one part of the induction step, namely the case when φ_0 is 'obtained' by one of the inference rules corresponding to a quasi-equation $q \in Ax$. Say q has the form
$$\big(\tau_1(\overline{x}) = \tau_1'(\overline{x}) \wedge \cdots \wedge \tau_r(\overline{x}) = \tau_r'(\overline{x})\big) \Rightarrow \tau_0(\overline{x}) = \tau_0'(\overline{x})\,,$$

where $\overline{x} = \langle x_1, \ldots, x_z \rangle$. Then a corresponding inference rule is

$$\frac{\tau_1(\overline{\Phi})\boldsymbol{\Delta}\tau_1'(\overline{\Phi}), \ldots, \tau_r(\overline{\Phi})\boldsymbol{\Delta}\tau_r'(\overline{\Phi})}{\tau_0(\overline{\Phi})\Delta_i\tau_0'(\overline{\Phi})},$$

for some $i < n$. Assume that φ_0 is obtained with the help of this rule by substituting the members of the sequence $\overline{\gamma} = \langle \gamma_1, \ldots, \gamma_z \rangle$ of formulas for the members of the sequence $\overline{\Phi} = \langle \Phi_1, \ldots, \Phi_z \rangle$ of formula variables, i.e. φ_0 has the form $\tau_0(\overline{\gamma})\Delta_i\tau_0'(\overline{\gamma})$.

Now fix a model $\mathfrak{M}$ and assume that

$$\mathfrak{M} \models_{\mathcal{L}} \tau_1(\overline{\gamma})\boldsymbol{\Delta}\tau_1'(\overline{\gamma}), \ldots, \mathfrak{M} \models_{\mathcal{L}} \tau_r(\overline{\gamma})\boldsymbol{\Delta}\tau_r'(\overline{\gamma}).$$

We have to show that $\mathfrak{M} \models_{\mathcal{L}} \tau_0(\overline{\gamma})\Delta_i\tau_0'(\overline{\gamma})$.

Let $\mathfrak{A} \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\mathfrak{F}) \in \mathsf{Alg}_m(\mathcal{L})$. and let a be a valuation of $\mathfrak{A}$ such that for every $1 \leq v \leq z \quad a(x_v) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\gamma_v)$. Then by Definition 3.1.1 (3)(i)

$$\begin{array}{rcl}
(\forall 1 \leq j \leq r) \;\; mng_{\mathfrak{M}}\big(\tau_j(\overline{\gamma})\big) = mng_{\mathfrak{M}}\big(\tau_j'(\overline{\gamma})\big) & & \\
& \iff & \mathfrak{A} \models \big(\tau_1(\overline{x}) = \tau_1'(\overline{x}) \wedge \cdots \wedge \tau_r(\overline{x}) = \tau_r'(\overline{x})\big)[a] \\
(\text{by } \mathsf{Alg}_m(\mathcal{L}) \models Ax) & \Longrightarrow & \mathfrak{A} \models \big(\tau_0(\overline{x}) = \tau_0'(\overline{x})\big)[a] \\
(\text{by Def. 3.1.1 (3)(i)}) & \Longrightarrow & \mathfrak{M} \models_{\mathcal{L}} \tau_0(\overline{\gamma})\Delta_i\tau_0'(\overline{\gamma}).
\end{array}$$

This completes the proof of direction "$\Longrightarrow$" of Theorem 3.2.3. □

Proof of ($\Longleftarrow$). Let $\Phi_1, \ldots, \Phi_z$ denote formula variables, $\tau_0, \tau_1, \ldots, \tau_k$ denote formula schemes, let $\overline{\Phi} \stackrel{\text{def}}{=} \langle \Phi_1, \ldots, \Phi_z \rangle$, and let $\overline{x} \stackrel{\text{def}}{=} \langle x_1, \ldots, x_z \rangle$ be a sequence of variables. Assume that $\vdash$ is a finitely complete and strongly sound Hilbert-type inference system for the logic $\mathcal{L}$, and define the finite set Ax of quasi-equations as follows:

(1) If $\tau_0(\overline{\Phi})$ is an axiom scheme of $\vdash$ then let "$\varepsilon_j\big(\tau_0(\overline{x})\big) = \delta_j\big(\tau_0(\overline{x})\big)$" belong to Ax for each $j < m$.

(2) If $\frac{\tau_1(\overline{\Phi}), \ldots, \tau_k(\overline{\Phi})}{\tau_0(\overline{\Phi})}$ is an inference rule of $\vdash$ then let

$$\text{“}\bigwedge\{\varepsilon_i\big(\tau_s(\overline{x})\big) = \delta_i\big(\tau_s(\overline{x})\big) : 1 \leq s \leq k,\; i < m\} \Rightarrow \varepsilon_j\big(\tau_0(\overline{x})\big) = \delta_j\big(\tau_0(\overline{x})\big)\text{”}$$

belong to Ax for each $j < m$.

(3) Let "$\varepsilon_j(x_0\Delta_i x_0) = \delta_j(x_0\Delta_i x_0)$" belong to Ax for each $i < n,\; j < m$.

(4) Let "$\bigwedge\{\varepsilon_j(x_0\Delta_i x_1) = \delta_j(x_0\Delta_i x_1) : j < m,\ i < n\} \Rightarrow (x_0 = x_1)$" belong to Ax.

We will show that Ax axiomatizes the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L})$.

Claim 3.2.5. $\mathsf{Alg}_m(\mathcal{L}) \models Ax$.

Proof of Claim 3.2.5. Quasi-equations of type (3) and (4) above obviously hold in $\mathsf{Alg}_m(\mathcal{L})$ by Definition 3.1.1 (3).

Now consider a quasi-equation of type (2). Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$ and let a be an arbitrary valuation of the variables into $\mathfrak{A}$. Let $\mathfrak{M}$ be such that $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$. Then for every $i \in \omega$ $\quad a(x_i) = mng_{\mathfrak{M}}(\varphi_i)$ for some $\varphi_i \in F$. Assume that

$$\mathfrak{A} \models \bigwedge\{\varepsilon_i\big(\tau_s(\overline{x})\big) = \delta_i\big(\tau_s(\overline{x})\big) : 1 \le s \le k,\ i < m\}[a].$$

Then by Definition 3.1.1 (3)

$$(\bullet\bullet) \qquad \mathfrak{M} \models_{\mathcal{L}} \tau_s(x_1/\varphi_1, \dots, x_z/\varphi_z) \quad \text{(for each } 1 \le s \le k).$$

But $\frac{\tau_1(\overline{\Phi}),\dots,\tau_k(\overline{\Phi})}{\tau_0(\overline{\Phi})}$ is an inference rule of $\vdash$, therefore $\{\tau_1(\overline{\varphi}), \dots, \tau_k(\overline{\varphi})\} \vdash \tau_0(\overline{\varphi})$. This implies by the strong soundness of $\vdash$ that $\{\tau_1(\overline{\varphi}), \dots, \tau_k(\overline{\varphi})\} \models_{\mathcal{L}} \tau_0(\overline{\varphi})$. Now, by $(\bullet\bullet)$ above, $\mathfrak{M} \models \tau_0(\overline{\varphi})$, hence again by Definition 3.1.1 (3), $\mathfrak{A} \models \big(\varepsilon_j(\tau_0(\overline{x})) = \delta_j(\tau_0(\overline{x}))\big)[a]$ for each $j < m$, which was desired. $\square$

The case of equations of type (1) can be proved similarly. $\square$

Claim 3.2.6. *For any formulas $\varphi_0, \varphi_1, \dots, \varphi_k$,*

$$\{\varphi_1, \dots, \varphi_k\} \vdash \varphi_0 \quad \Longrightarrow$$
$$\Longrightarrow \quad Ax \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow \big(\varepsilon_j(\varphi_0) = \delta_j(\varphi_0)\big)$$

for each $j < m$.

Proof of Claim 3.2.6. It can be proved by induction on the length of the $\vdash$-proof of φ_0 from $\{\varphi_1, \dots, \varphi_k\}$. We only show one part of the induction step, namely the case when φ_0 is 'obtained' by an inference rule $\frac{\tau_1(\overline{\Phi}),\dots,\tau_r(\overline{\Phi})}{\tau_0(\overline{\Phi})}$, where $\overline{\Phi} = \langle \Phi_1, \dots, \Phi_z\rangle$. Then there are formulas $\gamma_1, \dots, \gamma_z$ such that $\varphi_0 = \tau_0(\gamma_1, \dots, \gamma_z)$ and for every $1 \le \ell \le r$ $\{\varphi_1, \dots, \varphi_k\} \vdash \tau_\ell(\overline{\gamma})$. Then by the induction hypothesis

$$(\natural) \qquad \begin{aligned} Ax \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \le s \le k,\ i < m\} \Rightarrow \\ \Rightarrow\ \varepsilon_j\big(\tau_\ell(\overline{\gamma})\big) = \delta_j\big(\tau_\ell(\overline{\gamma})\big) \qquad \text{(for each } j < m,\ 1 \le \ell \le r). \end{aligned}$$

By the definition of Ax

$$(\natural\natural) \qquad Ax \models \bigwedge\{\varepsilon_i\big(\tau_\ell(\overline{x})\big) = \delta_i\big(\tau_\ell(\overline{x})\big) : 1 \leq \ell \leq r,\ i < m\} \Rightarrow \\ \Rightarrow\ \varepsilon_j\big(\tau_0(\overline{x})\big) = \delta_j\big(\tau_0(\overline{x})\big) \qquad \text{(for each } j < m).$$

Let $\mathfrak{B}$ be an algebra with $\mathfrak{B} \models Ax$ and let b be any valuation of the variables into B. Now we can define a valuation b' with $b'(x_v) \overset{\text{def}}{=} \gamma_v[b]^{\mathfrak{B}}$ $(1 \leq v \leq z)$. Then for every $0 \leq \ell \leq r$ $\tau_\ell(\overline{x})[b']^{\mathfrak{B}} = \tau_\ell(\overline{\gamma})[b]^{\mathfrak{B}}$. Thus, by ($\natural$) and ($\natural\natural$),

$$\mathfrak{B} \models \bigwedge\{\varepsilon_i(\varphi_s) = \delta_i(\varphi_s) : 1 \leq s \leq k,\ i < m\} \Rightarrow \big(\varepsilon_j(\tau_0(\overline{\gamma})) = \delta_j(\tau_0(\overline{\gamma}))\big)[b]$$

for each $j < m$, which was desired. □

Now we can prove that each quasi-equation which holds in $\mathsf{Alg}_m(\mathcal{L})$ is a consequence of Ax. Assume that

$$\mathsf{Alg}_m(\mathcal{L}) \models \big(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k'\big) \Rightarrow \tau_0 = \tau_0'.$$

$$\overset{\text{(Thm. 3.2.1 (ii))}}{\Longrightarrow} \{\tau_s \Delta_j \tau_s' : 1 \leq s \leq k,\ j < n\} \models_{\mathcal{L}} \tau_0 \Delta_i \tau_0' \quad \text{for each } i < n$$

$$\overset{\text{(finite completeness)}}{\Longrightarrow} \{\tau_s \Delta_j \tau_s' : 1 \leq s \leq k,\ j < n\} \vdash \tau_0 \Delta_i \tau_0' \quad \text{for each } i < n$$

$$\overset{\text{(Claim 3.2.6)}}{\Longrightarrow} Ax \models \bigwedge\{\varepsilon_\ell(\tau_s \Delta_j \tau_s') = \delta_\ell(\tau_s \Delta_j \tau_s') : \ell < m,\ 1 \leq s \leq k,\ j < n\} \Rightarrow \\ \Rightarrow\ \big(\varepsilon_p(\tau_0 \Delta_i \tau_0') = \delta_p(\tau_0 \Delta_i \tau_0')\big) \quad \text{for all } p < m,\ i < n.$$

But, since quasi-equations of type (3) and (4) belong to Ax, this implies to

$$Ax \models \big(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k'\big) \Rightarrow \tau_0 = \tau_0',$$

completing the proof of direction "$\Longleftarrow$" of Theorem 3.2.3. □

Having found the algebraic counterpart of "finitely complete", let us try to characterize "weakly complete". Since weak completeness is slightly weaker than finite completeness, we have to weaken the algebraic counterpart of finite completeness for characterizing weak completeness. This way we obtain condition $(*)$ below, where $Eq_{\mathcal{L}}$ and $Qeq_{\mathcal{L}}$ denote the set of all equations and the set of all quasi-equations, respectively, of the language of $\mathsf{Alg}_m(\mathcal{L})$ (cf. [49]).

$$(*) \qquad (\exists Ax \subseteq_\omega Qeq_{\mathcal{L}})\left[(\forall e \in Eq_{\mathcal{L}})\,(\mathsf{Alg}_m(\mathcal{L}) \models e \Longrightarrow Ax \models e)\,\mathsf{Alg}_m(\mathcal{L}) \models Ax\right].$$

That is, the equational theory of $\mathsf{Alg}_m(\mathcal{L})$ is finitely axiomatizable by quasi-equations valid in $\mathsf{Alg}_m(\mathcal{L})$.

Theorem 3.2.7. *Assume that $\mathcal{L}$ is nice and $Cn(\mathcal{L})$ is finite*[13]*. Then*

$$(*) \quad \Longleftrightarrow \quad (\exists \text{ Hilbert-style} \vdash)(\vdash \text{ is weakly complete and strongly sound for } \mathcal{L}).$$

In particular, if the equational theory of $\mathsf{Alg}_m(\mathcal{L})$ is finitely axiomatizable, then $\mathcal{L}$ admits a weakly complete Hilbert-style inference system.

Proof. It is similar to the proof of Theorem 3.2.3. The only important difference is that Theorem 3.2.7 already holds for *nice* logics. However, the only part of the proof of Theorem 3.2.3 which used the additional criterion for strong niceness (Definition 3.1.1 (5)) was Thm. 3.2.1 (i). Here one has to use Cor. 3.2.2 (i) instead. ☐

Exercise 3.2.8. Give weakly complete and sound calculi for the logics $\mathcal{L}_S$ and $S5$. (Hint: Use that the **SP**-closure of the Alg_m-image of these logics are finitely axiomatizable varieties, so $(*)$ is satisfied.)

Definition 3.2.9 (deduction theorem, deduction term). Let $\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L} \rangle$ be a logic satisfying condition (1) of Def. 3.1.1. We say that $\mathcal{L}$ *has a deduction theorem*, iff

$$(\exists(\Phi_1 \nabla \Phi_2) \in Fms_\mathcal{L}))\,(\forall \Sigma \subseteq F_\mathcal{L})(\forall \varphi, \psi \in F_\mathcal{L})\,(\Sigma \cup \{\varphi\} \models_\mathcal{L} \psi \Longleftrightarrow \Sigma \models_\mathcal{L} \varphi \nabla \psi)\ ,$$

where "$\varphi \nabla \psi$" denotes an instance of scheme "$\Phi_1 \nabla \Phi_2$". Such a "$\Phi_1 \nabla \Phi_2$" is called a *deduction term* for $\mathcal{L}$.

Proposition 3.2.10. *$\mathcal{L}_S$ and $S5$ have deduction terms.*

Proof. It is not hard to show that "$\Phi_1 \to \Phi_2$" and "$\Box\Phi_1 \to \Box\Phi_2$" (where $\Box$ is the abbreviation of $\neg\Diamond\neg$) are suitable deduction terms for $\mathcal{L}_S$ and $S5$, respectively. ☐

The following theorem states that for any nice logic the existence of a deduction term and that of a weakly complete Hilbert-style calculus provides a finitely complete inference system.

Theorem 3.2.11. *Assume $\mathcal{L}$ is a logic satisfying (1) of Def. 3.1.1. Assume $\mathcal{L}$ has a deduction theorem, and there is some Hilbert-style inference system which is weakly complete and strongly sound for $\mathcal{L}$. Then*

$$(\exists \text{ Hilbert-style } \vdash)(\vdash \text{ is finitely complete and strongly sound for } \mathcal{L}).$$

First we note the following fact (its proof is straightforward by the assumptions on ∇).

[13] cf. the footnote of Theorem 3.2.3.

Fact 3.2.12. *The inference rule modus ponens w.r.t. ∇ (MP_∇) that is,*

$$(\mathrm{MP}_\nabla) \qquad \frac{\Phi_1, \quad \Phi_1 \nabla \Phi_2}{\Phi_2}$$

is strongly sound for $\mathcal{L}$. □

Proof of Theorem 3.2.11. Assume that there is some Hilbert-style inference system which is weakly complete and strongly sound for $\mathcal{L}$. Let such an inference system be fixed and let us add (MP_∇) to it. We denote this (extended) inference system by $\vdash$.

To prove finite completeness, assume $\{\varphi_0, \ldots, \varphi_n\} \models \psi$. Then, applying the deduction theorem $n+1$ times, we get:

$$\begin{aligned}
\{\varphi_0, \ldots, \varphi_{n-1}\} &\models (\varphi_n \nabla \psi) \\
\{\varphi_0, \ldots, \varphi_{n-2}\} &\models (\varphi_{n-1} \nabla (\varphi_n \nabla \psi)) \\
&\vdots \\
&\models \underbrace{(\varphi_0 \nabla (\varphi_1 \nabla \ldots (\varphi_n \nabla \psi) \ldots)}_{\gamma_0}.
\end{aligned}$$

Then $\vdash \gamma_0$ by weak completeness of $\vdash$. Then, using (MP_∇) $n+1$ times, we get:

$$\begin{aligned}
\{\varphi_0\} \vdash \{\varphi_0, \gamma_0\} &\vdash \underbrace{\varphi_1 \nabla (\varphi_2 \nabla \ldots (\varphi_n \nabla \psi) \ldots)}_{\gamma_1} \\
\{\varphi_0, \varphi_1\} \vdash \{\varphi_1, \gamma_1\} &\vdash \underbrace{\varphi_2 \nabla (\varphi_2 \nabla \ldots (\varphi_n \nabla \psi) \ldots)}_{\gamma_2} \\
\vdots \qquad\qquad & \vdots \\
\{\varphi_0, \varphi_1, \ldots, \varphi_n\} \vdash \{\varphi_n, \gamma_n\} &\vdash \psi \quad , \text{ where } \gamma_n = (\varphi_n \nabla \psi).
\end{aligned}$$

Thus we received the following $\vdash$-proof of ψ from $\{\varphi_0, \ldots, \varphi_n\}$:

$$\langle \gamma_0, \varphi_0, \gamma_1, \varphi_1, \gamma_2, \varphi_2, \ldots, \gamma_n \varphi_n, \psi \rangle,$$

which proves Theorem 3.2.11. □

We will study strong completeness in item 3.2.27. As a preparation, first we study compactness.

Definition 3.2.13 (compactness of a logic). Let $\mathcal{L} = \langle F_\mathcal{L}, M_\mathcal{L}, mng_\mathcal{L}, \models_\mathcal{L} \rangle$ be a logic. We say that

(i) $\mathcal{L}$ is *satisfiability compact* (sat. compact for short), if

$$(\forall \Gamma \subseteq F_{\mathcal{L}})\big[(\forall \Sigma \subseteq_\omega \Gamma)\,(\Sigma \text{ has a model}) \implies (\Gamma \text{ has a model})\big]\,, \text{ and}$$

(ii) $\mathcal{L}$ is *consequence compact* (cons. compact), if for every $\Gamma \cup \{\varphi\} \subseteq F_{\mathcal{L}}$

$$\Gamma \models_{\mathcal{L}} \varphi \quad \implies \quad (\exists \Sigma \subseteq_\omega \Gamma)\ \Sigma \models_{\mathcal{L}} \varphi. \quad \blacktriangleleft$$

Exercise 3.2.14. Prove that even for nice logics we have

- sat. compact $\not\Longrightarrow$ cons. compact;
- sat. compact $\not\Longleftarrow$ cons. compact.

(Hint for (1): Let the logical connectives be Δ (binary), and $\mathit{True}, k_0, \ldots, k_n, \ldots$ all zero-ary. A model $\mathfrak{M}$ is a function $\mathfrak{M} : \{\mathit{True}, p_i, k_i : i \in \omega\} \to \{0, 1\}$. $mng_{\mathfrak{M}}(\mathit{True}) = 1$ for every $\mathfrak{M}$ and meaning of Δ is the standard meaning of the biconditional $\leftrightarrow$. Exclude those models from M in which $(\forall\, i > 0)\ \mathfrak{M}(k_i) = 1$ but $\mathfrak{M}(k_0) = 0$. [This logic is not strongly nice!] Observe that for $\mathfrak{M} = \{\mathit{True}, p_i, k_i : i \in \omega\} \times \{1\}$ we have $\mathfrak{M} \models_{\mathcal{L}} F_{\mathcal{L}}$. Hence sat. completeness trivially holds.)

(Hint for (2): Let $\mathcal{L}$ have True and Δ as the only logical connectives. Exclude the models $\mathfrak{M}$ with $\mathfrak{M} \models_{\mathcal{L}} F_{\mathcal{L}}$. Then sat. completeness fails (we have infinitely many propositional variables). Show that cons. completeness remains true.)

Exercise 3.2.15. Find natural conditions under which "$\Longrightarrow$" and/or "$\Longleftarrow$" of Exercise 3.2.14 above hold.

- We say that $\mathcal{L}$ *has weak false* if $(\exists \varphi \in F_{\mathcal{L}})$ such that $(\forall \mathfrak{M} \in M_{\mathcal{L}})\ \mathfrak{M} \not\models_{\mathcal{L}} \varphi$. Show that under this assumption

$$\text{cons. compact} \implies \text{sat. compact}.$$

- We say that $\mathcal{L}$ *has negation* if

$$(\forall \varphi \in F_{\mathcal{L}})(\exists \psi \in F_{\mathcal{L}})(\forall \mathfrak{M} \in M_{\mathcal{L}})[\mathfrak{M} \models_{\mathcal{L}} \psi \iff \mathfrak{M} \not\models_{\mathcal{L}} \varphi]\,.$$

  Show that under this assumption

$$\text{sat. compact} \implies \text{cons. compact}.$$

- Try to find weaker sufficient conditions.

- Show that for nice logics

$$\mathcal{L} \text{ has weak false } \iff \mathcal{L} \text{ has negation.}$$

For more information about the two notions of compactness, see [9].

* * *

Recall that in Definition 3.1.1 (and also in the logics studied so far), there was a parameter P, which was the set of atomic formulas. The choice of P influenced what the set F of formulas would be. Thus in fact, our old definition of a logic yields a family

$$\big\langle \langle F^P, M^P, mng^P, \models^P \rangle : \; P \text{ is a set} \big\rangle$$

of logics.

Definition 3.2.16 (general logic). A *general logic* is a class

$$\mathbf{L} \stackrel{\text{def}}{=} \langle \mathcal{L}^P \; : \; P \text{ is a set} \rangle,$$

where for each set P, $\quad \mathcal{L}^P = \langle F^P, M^P, mng^P, \models^P \rangle$ is a logic in the sense of Def. 2.1.3.

$\mathbf{L}$ is called a *nice [strongly nice, structural] general logic* iff conditions (1–4) below hold for $\mathbf{L}$.

(1) $\mathcal{L}^P$ is a nice [strongly nice, structural] logic (cf. Def. 3.1.1) for each set P, and P is the set of atomic formulas of logic $\mathcal{L}^P$.

(2) For any sets P and Q, $Cn(\mathcal{L}^P) = Cn(\mathcal{L}^Q) \stackrel{\text{def}}{=} Cn(\mathbf{L})$. The "special" connectives ε_j, δ_j $(j < m)$ and Δ_i $(i < n)$ are the same for any logic $\mathcal{L}^P$ (cf. Def. 3.1.1 (3)).

(3) For any sets P, Q, if there is a bijection $f : P \to Q$ then logic $\mathcal{L}^Q$ is an "isomorphic copy" of logic $\mathcal{L}^P$, i.e. there are bijections $f^F : F^P \to F^Q$ and $f^M : M^P \to M^Q$ such that

(a) f^F is an isomorphism from $\mathfrak{F}^P$ onto $\mathfrak{F}^Q$ extending f;

(b) for all $\varphi \in F^P$, $\mathfrak{M} \in M^P$

$$mng^P(\varphi, \mathfrak{M}) = mng^Q\big(f^F(\varphi), f^M(\mathfrak{M})\big)$$
$$\mathfrak{M} \models^P \varphi \quad \iff \quad f^M(\mathfrak{M}) \models^Q f^F(\varphi).$$

(4) For all sets $P \subseteq Q$,

$$\left\{ mng^P_{\mathfrak{M}} \ : \ \mathfrak{M} \in M^P \right\} = \left\{ (mng^Q_{\mathfrak{M}}) \restriction F^P \ : \ \mathfrak{M} \in M^Q \right\}.$$

(Intuitively, condition (4) says that $\mathcal{L}^P$ is the "natural" restriction of $\mathcal{L}^Q$.)

Remark 3.2.17. We note that if $\mathbf{L}$ is a nice general logic then $\mathbf{L}$ has the following property. For all sets $P \subseteq Q$,

$$\{\{\varphi \in F^P : \mathfrak{M} \models^P \varphi\} : \mathfrak{M} \in M^P\} = \{\{\varphi \in F^P : \mathfrak{N} \models^Q \varphi\} : \mathfrak{N} \in M^Q\}.$$

Moreover, for all $\Gamma \cup \{\varphi\} \subseteq F^P$,

$$\Gamma \models^P \varphi \iff \Gamma \models^Q \varphi.$$

However, (5) below does not automatically hold for all strongly nice logics.

(5) For each $P \subseteq Q$ there is a "reduct-function" $r : M^Q \longrightarrow M^P$ with $Rng(r) = M^P$ such that

$$(\forall \mathfrak{M} \in M^Q)(\forall \varphi \in F^P)[(\mathfrak{M} \models^Q \varphi \iff r(\mathfrak{M}) \models^P \varphi) \text{ and } mng^Q_{\mathfrak{M}}(\varphi) = mng^P_{r(\mathfrak{M})}(\varphi)].$$

We will not assume and use condition (5), but it can be useful for investigations of the Beth definability properties (and related issues) which are (partially) treated in Appendix B.

Definition 3.2.18 (algebraic counterpart of a general logic). Let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set} \rangle$ be a nice or structural general logic. Then

$$\mathsf{Alg}_{\models}(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup \left\{ \mathsf{Alg}_{\models}(\mathcal{L}^P) \ : \ P \text{ is a set} \right\},$$
$$\mathsf{Alg}_{m}(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup \left\{ \mathsf{Alg}_{m}(\mathcal{L}^P) \ : \ P \text{ is a set} \right\}$$

(cf. Def. 3.1.6).

Exercise 3.2.19. Prove that

(i) $\mathsf{Alg}_m(\mathbf{L}_S) =$"class of all Boolean set algebras"

(ii) $\mathsf{Alg}_m(\mathbf{L}_{\mathrm{S5}}) =$"class of all one-dimensional cylindric set algebras"

(cf. Defs. 2.2.1 and 2.2.4). (Hint: The part "$\subseteq$" will be easy. If you would encounter cardinality difficulties in the other direction, e.g. a Boolean algebra $\mathfrak{A}$ with $|A|$ too big, then choose the set P of atomic formulas to be bigger than $|A|$.)

Theorem 3.2.20. *For structural general logics*

$$\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathbf{L}).$$

Proof. Proof First we not that, by Thm. 3.1.11, $\mathsf{Alg}_{\models}(\mathcal{L}^P) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathcal{L}^P)$ for any set P, thus $\mathsf{Alg}_{\models}(\mathbf{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ holds. $\square$

To prove $\mathbf{SP}\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathsf{Alg}_{\models}(\mathbf{L})$ we need Claims 3.2.21 and 3.2.22 below.

Claim 3.2.21. *For any sets P, Q, algebra $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L}^Q)$ and homomorphism $h : \mathfrak{F}^P \to \mathfrak{A}$,*

$$(\exists \mathfrak{N} \in M^P)(\forall \varphi \in F^P)\ h(\varphi) = mng^P_{\mathfrak{N}}(\varphi).$$

Proof of Claim 3.2.21. Let $\mathfrak{M} \in M^Q$ be such that $\mathfrak{A} = mng^Q_{\mathfrak{M}}$. Then

$$(*) \qquad (\forall p \in P)(\exists \varphi_p \in F^Q)\ h(p) = mng^Q_{\mathfrak{M}}(\varphi_p).$$

Because of condition (3) of Def. 3.2.16 without loss of generality we can assume that either $P \subseteq Q$ or $Q \subseteq P$ hold.

1^{st} *case* : $Q \subseteq P$

Then, by (4) of Def. 3.2.16,

$$(**) \qquad (\exists \mathfrak{M}' \in M^P)(\forall p \in P)\ mng^P_{\mathfrak{M}'}(\varphi_p) = mng^Q_{\mathfrak{M}}(\varphi_p) = h(p).$$

Let $s : P \to F^P$ be defined by $s(p) \stackrel{\text{def}}{=} \varphi_p$, for any $p \in P$. Then, by (5) of Def. 3.1.1,

$$(***) \qquad (\exists \mathfrak{N} \in M^P)(\forall p \in P)\ mng^P_{\mathfrak{N}}(p) = mng^P_{\mathfrak{M}'}(\varphi_p).$$

Now $(*)$, $(**)$ and $(***)$ together imply that $(\forall p \in P)\ h(p) = mng^P_{\mathfrak{N}}(p)$.

2^{nd} *case* : $P \subseteq Q$

Let $s : Q \to F^Q$ be defined by

$$s(p) \stackrel{\text{def}}{=} \begin{cases} \varphi_p, & \text{if } p \in P \\ \text{any element of } F^Q, & \text{else.} \end{cases}$$

Then, by (5) of Def. 3.1.1,

$$(\dagger) \qquad (\exists \mathfrak{M}' \in M^Q)(\forall p \in P)\ mng^Q_{\mathfrak{M}'}(p) = mng^Q_{\mathfrak{M}}(\varphi_p).$$

By (4) of Def. 3.2.16,

$$(\dagger\dagger) \qquad (\exists \mathfrak{N} \in M^P)(\forall p \in P)\ mng^P_{\mathfrak{N}}(p) = mng^Q_{\mathfrak{M}'}(p).$$

Then, by $(*)$, $(\dagger)$ and $(\dagger\dagger)$, $(\forall p \in P)\ mng^P_{\mathfrak{N}}(p) = h(p)$ holds again. $\square$

Claim 3.2.22. *Let $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ and let $h : \mathfrak{F}^P \twoheadrightarrow \mathfrak{A}$ be a surjective homomorphism for some set P. Then*

$$(\exists K \subseteq M^P)\ (ker(h) = \sim_K \ \ \textit{that is, } \mathfrak{A} \cong \mathfrak{F}^P/\sim_K).$$

Proof of Claim 3.2.22. Let $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Then there are some sets I and Q_i ($i \in I$) and $\mathfrak{A}_i \in \mathsf{Alg}_m(\mathcal{L}^{Q_i})$ such that $\mathfrak{A} \subseteq \mathbf{P}_{i\in I}\mathfrak{A}_i$. For each $i \in I$ let π_i denote the projection function into $\mathfrak{A}_i$. Then, by Claim 3.2.21, $(\forall i \in I)(\exists \mathfrak{N}_i \in M^P)(\forall p \in P)$ $(\pi_i \circ h)(p) = mng^P_{\mathfrak{N}_i}(p)$. Let $K \stackrel{\text{def}}{=} \{\mathfrak{N}_i : i \in I\}$. Then it is easy to check that for any $\varphi, \psi \in F^P$,

$$h(\varphi) = h(\psi) \quad \text{iff} \quad \varphi \sim_K \psi$$

that is, $\mathfrak{A} \cong \mathfrak{F}^P/\sim_K$. □

Now, to prove $\mathbf{SP}\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathsf{Alg}_\models(\mathbf{L})$, assume $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Let $h : \mathfrak{F}^A \twoheadrightarrow \mathfrak{A}$ be the usual extension of the identity map of A to a homomorphism. Then, by Claim 3.2.22, $(\exists K \subseteq M^A)$ $\mathfrak{A} \cong \mathfrak{F}^A/\sim_K$ that is, $\mathfrak{A} \in \mathsf{Alg}_\models(\mathbf{L})$, completing the proof of Theorem 3.2.20. □

Definition 3.2.23 (compactness of a general logic). A general logic $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ is *satisfiability (consequence) compact* if for each set P the logic $\mathcal{L}^P$ is satisfiability (consequence) compact.

Recall that for an arbitrary class K of algebras,

$$\mathbf{Up}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\left\{\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F} : \mathcal{F} \text{ is an ultrafilter over the set } I, \text{ and } (\forall i \in I)\ \mathfrak{A}_i \in \mathsf{K}\right\}.$$

We say that K is **Up**-*closed* if $\mathbf{Up}\mathsf{K} \subseteq \mathsf{K}$, in other words, K is **Up**-closed if it is closed under taking ultraproducts (cf. [49]).

Our next theorem gives a sufficent condition for sat. compactness of a strongly nice general logic.

Theorem 3.2.24. *Let* $\mathbf{L}$ *be a strongly nice general logic. Then*

$$(\mathsf{Alg}_\models(\mathbf{L}) \textit{ is } \mathbf{Up}\textit{-closed}) \quad \Longrightarrow \quad (\mathbf{L} \textit{ is sat. compact}).$$

Proof. We let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ We give a proof for the case of $P = \omega$ that is, for the compactness of $\mathcal{L}^\omega = \langle F^\omega, M^\omega, mng^\omega, \models^\omega\rangle$. For other sets the proof is similar and is left to the reader. Assume $\Gamma \subseteq F^\omega$ and

$$(\forall \Sigma \subseteq_\omega \Gamma)\ \Sigma \text{ has a model}.$$

Then we may assume that Γ is countable, say $\Gamma = \{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\}_{n\in\omega}$ and

$$(\forall k \in \omega)(\exists \mathfrak{M}_k \in M^\omega)\ \mathfrak{M}_k \models^\omega \{\varphi_0, \ldots, \varphi_k\}\,.$$

Let such $\mathfrak{M}_k$'s be fixed. Let $\mathfrak{A}_k \stackrel{\text{def}}{=} mng^\omega_{\mathfrak{M}_k}(\mathfrak{F}^\omega) \in \mathsf{Alg}_m(\mathbf{L})$. Then $\mathfrak{A}_k \in \mathsf{Alg}_\models(\mathbf{L})$ also holds (by Ex. 3.1.9). Let $mng_k \stackrel{\text{def}}{=} mng^\omega_{\mathfrak{M}_k} \restriction \omega$. Since ω is the set of atomic formulas of $\mathcal{L}^\omega$, the function $mng_k : \omega \longrightarrow A_k$ is a valuation of the (propositional) variables into $\mathfrak{A}_k$. Let $\mathcal{F}$ be a non-principal ultrafilter over ω, and let $\mathfrak{A} \stackrel{\text{def}}{=} \mathbf{P}_{k\in\omega}\mathfrak{A}_k/\mathcal{F}$ denote the ultraproduct of algebras $\mathfrak{A}_k$ w.r.t. $\mathcal{F}$. We define the function $v : \omega \longrightarrow A$ as follows:

$$v(i) \stackrel{\text{def}}{=} \langle mng_k(i) : k \in \omega\rangle/\mathcal{F}\,.$$

See Figure 3.2.1 below.

$$\begin{tikzcd}
 & \omega \arrow[dl, "mng_0"'] \arrow[dr, "mng_k"] \arrow[drrrr, "v"] & & & & \\
\mathfrak{A}_0 & \cdots & \mathfrak{A}_k & \cdots & \xRightarrow[\mathbf{lim}]{} & \mathfrak{A}
\end{tikzcd}$$

Figure 3.2.1

By assumption, $\mathfrak{M}_k \models^\omega \varphi_i$ for every $i \le k$. Thus, for every $i \le k \in \omega$, we have the following:

$$\begin{array}{rl}
\mathfrak{M}_k & \models^\omega \varphi_i \\
 & \Updownarrow \quad \text{by Definition 3.1.1 (3)(ii)} \\
\mathfrak{M}_k & \models^\omega \varepsilon_j(\varphi_i)\Delta_\ell\delta_j(\varphi_i) \quad \text{for each } j < m,\ \ell < n \\
 & \Updownarrow \quad \text{by Definition 3.1.1 (3)(i)} \\
mng^\omega_{\mathfrak{M}_k}\big(\varepsilon_j(\varphi_i)\big) & = mng^\omega_{\mathfrak{M}_k}\big(\delta_j(\varphi_i)\big) \quad \text{for each } j < m \\
 & \Updownarrow \\
\mathfrak{A}_k & \models \big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[mng_k] \quad \text{for each } j < m.
\end{array}$$

We derived that $(\forall k \in \omega)(\forall i \le k)\ \mathfrak{A}_k \models \bigwedge_{j<m}\big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[mng_k]$, i.e. for every $i \in \omega$, $\{k \in \omega : \mathfrak{A}_k \models \bigwedge_{j<m}\big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[mng_k]\} \in \mathcal{F}$. Using Łos's theorem (cf. [49]), we have that

$$(\forall i \in \omega) \quad \mathfrak{A} \models \bigwedge_{j<m} \big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[v]\,.$$

Since by our assumption $\mathsf{Alg}_{\models}(\mathbf{L})$ is $\mathbf{Up}$-closed, $\mathfrak{A} \in \mathsf{Alg}_{\models}(\mathbf{L})$. Thus, Def. 3.2.16 (3), $(\exists \text{ set } P \supseteq \omega)$ $(\exists K \subseteq M^P)$ $\mathfrak{A} \cong \mathfrak{F}^P/{\sim_K}$. Let iso denote this isomorphism. Let $\mathfrak{B} \overset{\text{def}}{=} \mathfrak{F}^P/{\sim_K}$, and let $w \overset{\text{def}}{=} iso \circ v$. Then

$$(\forall i \in \omega) \quad \mathfrak{B} \models \bigwedge_{j<m} \big(\varepsilon_j(\varphi_i) = \delta_j(\varphi_i)\big)[w]$$

that is,

$$(\forall i \in \omega)(\forall j < m) \quad \varepsilon_j(\varphi_i)[w(p_{i_0}), \ldots, w(p_{i_z})]^{\mathfrak{B}} = \delta_j(\varphi_i)[w(p_{i_0}), \ldots, w(p_{i_z})]^{\mathfrak{B}},$$

where all the atomic formulas (elements of ω) occurring in φ_i are among $\{p_{i_0}, \ldots, p_{i_z}\}$. Let $s : P \longrightarrow F^P$ be such that for all $p \in \omega$ $\ s(p)$ is an element of the congruence class $w(p)$. For every $i \in \omega$, let $\hat{\varphi}_i \in F^P$ be $\varphi_i(p_{i_0}/s(p_{i_0}), \ldots, p_{i_z}/s(p_{i_z}))$. Then for every $i \in \omega$, $j < m$ we have,

$$\begin{aligned}
&\varepsilon_j(\varphi_i)[s(p_{i_0})/{\sim_K}, \ldots, s(p_{i_z})/{\sim_K}]^{\mathfrak{B}} = \delta_j(\varphi_i)[s(p_{i_0})/{\sim_K}, \ldots, s(p_{i_z})/{\sim_K}]^{\mathfrak{B}}\\
&\qquad\qquad\Downarrow \quad (\sim_K \text{ is a congruence on } \mathfrak{F}^P)\\
(\dagger)\qquad &\varepsilon_j\big(\varphi_i(s(p_{i_0}), \ldots, s(p_{i_z}))\big)/{\sim_K} = \delta_j\big(\varphi_i(s(p_{i_0}), \ldots, s(p_{i_z}))\big)/{\sim_K}\\
&\qquad\qquad\Updownarrow\\
&\varepsilon_j(\hat{\varphi}_i) \sim_K \delta_j(\hat{\varphi}_i).
\end{aligned}$$

We have that $(\forall \mathfrak{M} \in K)(\forall i \in \omega)(\forall j < m)$

$$mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\varphi}_i)\big).$$

Let $\mathfrak{M}$ be any model belonging to K. Then, by (5) of Def. 3.1.1, $(\exists \mathfrak{N}' \in M^P)$ $(\forall i \in \omega)$ $(\forall j < m)$

$$mng^P_{\mathfrak{N}'}\big(\varepsilon_j(\varphi_i)\big) = mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{N}'}\big(\delta_j(\varphi_i)\big).$$

Since for each $i \in \omega$, φ_i belongs to F^ω, by (3) of Def. 3.2.16,

$$(\exists \mathfrak{N} \in M^\omega)(\forall i \in \omega)(\forall j < m)\ mng^\omega_{\mathfrak{N}}\big(\varepsilon_j(\varphi_i)\big) = mng^\omega_{\mathfrak{N}}\big(\delta_j(\varphi_i)\big).$$

Then, by (3) of Definition 3.1.1,

$$(\forall i \in \omega)\ \mathfrak{N} \models^\omega \varphi_i,$$

which proves Theorem 3.2.24. $\square$

Our next theorem states that the condition of Theorem 3.2.24 above is sufficient and also necessary for cons. compactness, and so for strong completeness (cf. Theorem 3.2.27 below).

Theorem 3.2.25 (cf. [9] Thm. 2.8). *Assume* $\mathbf{L}$ *is a strongly nice general logic. Then*

$$(\mathsf{Alg}_{\models}(\mathbf{L}) \textit{ is } \mathbf{Up}\textit{-closed}) \quad \Longleftrightarrow \quad (\mathbf{L} \textit{ is cons. compact})\,.$$

Proof of ($\Longrightarrow$). One can push through the proof of Thm. 3.2.24 for this case, as follows. Now we want to prove $\{\varphi_i : i \in \omega\} \not\models^\omega \psi$ from the assumption $\{\varphi_0, \dots, \varphi_k\} \not\models^\omega \psi$ for each $k \in \omega$. Change $\mathfrak{M}_k$ in the above proof such that $\mathfrak{M}_k \models^\omega \{\varphi_0, \dots, \varphi_k\}$ *and* $\mathfrak{M}_k \not\models^\omega \psi$. Drag this "$\not\models^\omega \psi$" part through the whole argument in exactly the same style as "$\models^\omega \varphi_k$" was treated in the original proof. Then in line (†) of the proof above we have

$$(\ddagger) \qquad \begin{array}{rll} (\forall i \in \omega)(\forall j < m) & \varepsilon_j(\hat{\varphi}_i) \sim_K \delta_j(\hat{\varphi}_i) & \text{and} \\ (\exists j < m) & \varepsilon_j(\hat{\psi}) \not\sim_K \delta_j(\hat{\psi}). & \end{array}$$

Now we cannot choose an arbitrary $\mathfrak{M} \in K$ but we can infer that there exists some $\mathfrak{M} \in K$ such that $(\forall i \in \omega)$ $(\forall j < m)$

$$mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\varphi}_i)\big) = mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\varphi}_i)\big)$$

and $(\exists j < m)$

$$mng^P_{\mathfrak{M}}\big(\varepsilon_j(\hat{\psi})\big) \neq mng^P_{\mathfrak{M}}\big(\delta_j(\hat{\psi})\big).$$

Thus, again by (5) of Def. 3.1.1 and by (3) of Def. 3.2.16, there is an $\mathfrak{N} \in M^\omega$ with $\mathfrak{N} \models^\omega \{\varphi_i : i \in \omega\}$ and $\mathfrak{N} \not\models^\omega \psi$, as was desired. □

Proof of ($\Longleftarrow$). Fix any set I and assume that for each $i \in I$, $\mathfrak{A}_i \in \mathsf{Alg}_{\models}(\mathbf{L})$.

We let $\mathfrak{P} \stackrel{\text{def}}{=} \mathbf{P}_{i\in I}\mathfrak{A}_i$. For each $X \subseteq I$ define the congruence R_X of $\mathfrak{P}$ as follows.

$$R_X \stackrel{\text{def}}{=} \{(a,b) \in P \times P \;:\; a \restriction X = b \restriction X\}\,.$$

Then for each $X \subseteq I$, $\mathfrak{P}/R_X \cong \mathbf{P}_{i\in X}\mathfrak{A}_i$ obviously holds. Therefore

$$\mathfrak{P}/R_X \in \mathbf{P}\mathsf{Alg}_{\models}(\mathbf{L}) \stackrel{\text{Thm. 3.1.11}}{\subseteq} \mathbf{PSP}\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$$

(cf. e.g. [49] for $\mathbf{PSP} \subseteq \mathbf{SP}$). Let $h : \mathfrak{F}^P \twoheadrightarrow \mathfrak{P}$ be the natural extension of the identity map on P to a homomorphism and let $g_X : \mathfrak{P} \twoheadrightarrow \mathfrak{P}/R_X$ be the quotient map corresponding to R_X. Then, by Claim 3.2.22, for each $X \subseteq I$ there is some class $K_X \subseteq M^P$ such that $ker(g_X \circ h) = \sim_{K_X}$ that is,

$$(*) \qquad (\forall \varphi, \psi \in F^P)\ \big[\big(h(\varphi), h(\psi)\big) \in R_X \quad \Longleftrightarrow \quad \varphi \sim_{K_X} \psi\big].$$

Moreover, an inspection of the proof of Claim 3.2.22 shows, that

$$(**) \qquad X \subseteq Y \subseteq I \quad \Longrightarrow \quad K_X \subseteq K_Y.$$

For each $X \subseteq I$, let $\Gamma_X \overset{\text{def}}{=} Th(K_X)$. Recall (cf. Fact 3.1.8) that $\sim_{K_X} = \sim_{Mod(\Gamma_X)}$ holds.

Claim 3.2.26. *Let $\mathcal{F}$ be any filter on I and let $\Gamma \overset{\text{def}}{=} \bigcup\{\Gamma_X : X \in \mathcal{F}\}$. Then for every $\varphi, \psi \in F^P$*

$$\varphi \sim_{Mod(\Gamma)} \psi \quad \Longleftrightarrow \quad (\exists X \in \mathcal{F})\ \varphi \sim_{Mod(\Gamma_X)} \psi.$$

Proof of Claim 3.2.26. First, assume that $(\exists X \in \mathcal{F})\ \varphi \sim_{Mod(\Gamma_X)} \psi$. Then, since $\Gamma_X \subseteq \Gamma$, $\varphi \sim_{Mod(\Gamma)} \psi$ obviously holds.

On the other hand, assume $\varphi \sim_{Mod(\Gamma)} \psi$. Then, by (3) of Def. 3.1.1, $(\forall i < n)$ $\Gamma \models^P \varphi \Delta_i \psi$. Then, by the cons. compactness of $\mathcal{L}^P$, for each $i < n$ there is some $\Sigma_i \subseteq_\omega \Gamma$ with $\Sigma_i \models^P \varphi\Delta_i\psi$. Then there is some $\Sigma \subseteq_\omega \Gamma$ such that for each $i < n$ $\Sigma \models^P \varphi\Delta_i\psi$. Say, $\Sigma = \{\chi_0, \ldots, \chi_{z-1}\}$. Since $\Sigma \subseteq \Gamma$, $(\forall j < z)(\exists X_j \in \mathcal{F})\ \chi_j \in \Gamma_{X_j}$. Let $X \overset{\text{def}}{=} \bigcap\{X_j : j < z\}$. Then $X \in \mathcal{F}$, since $\mathcal{F}$ is a filter. Now $\Sigma \subseteq \Gamma_{X_0} \cup \cdots \cup \Gamma_{X_{z-1}} \subseteq \Gamma_X$ holds by $(**)$ above, thus for each $i < n$, $\Gamma_X \models^P \varphi\Delta_i\psi$, which implies $\varphi \sim_{Mod(\Gamma_X)} \psi$. □

Now we want to prove that $\mathfrak{P}/\mathcal{F} \in \mathsf{Alg}_\models(\mathbf{L})$. We show that $\mathfrak{P}/\mathcal{F} \cong \mathfrak{F}^P/{\sim_{Mod(\Gamma)}}$ (cf. Claim 3.2.26 above for the definition of Γ). That is,

$$(\forall \varphi, \psi \in F^P)\ \big[h(\varphi) \sim_{\mathcal{F}} h(\psi) \quad \Longleftrightarrow \quad \varphi \sim_{Mod(\Gamma)} \psi\big]$$

holds. Indeed,

$$\begin{aligned} & h(\varphi) \sim_{\mathcal{F}} h(\psi) \\ \Longleftrightarrow \quad & (\exists X \in \mathcal{F})\ \big(h(\varphi), h(\psi)\big) \in R_X \\ \overset{(*)}{\Longleftrightarrow} \quad & (\exists X \in \mathcal{F})\ \varphi \sim_{Mod(\Gamma_X)} \psi \\ \overset{\text{Cl. 3.2.26}}{\Longleftrightarrow} \quad & \varphi \sim_{Mod(\Gamma)} \psi, \end{aligned}$$

which completes the proof of Theorem 3.2.25. We note that we proved that $\mathsf{Alg}_\models(\mathbf{L})$ is closed under taking arbitrary *reduced* products (not only ultraproducts). □

Theorem 3.2.27. *Assume $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ is a strongly nice general logic. Then*

$$\mathsf{Alg}_{\models}(\mathbf{L}) \text{ is a finitely axiomatizable quasi-variety}$$
$$\Longleftrightarrow$$
$$(\exists \text{ Hilbert-style } \vdash)(\forall \text{ set } P)(\vdash \text{ is strongly complete and strongly sound for } \mathcal{L}^P).$$

Proof. To prove Theorem 3.2.27 we need the following lemma.

Lemma 3.2.28. *For every infinite set P and for every quasi-equation q*

$$\mathsf{Alg}_m(\mathcal{L}^P) \models q \quad \Longrightarrow \quad \mathsf{Alg}_m(\mathbf{L}) \models q.$$

Proof of Lemma 3.2.28. Fix an infinite set P and a quasi-equation q such that $\mathsf{Alg}_m(\mathcal{L}^P) \models q$. Let $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L}^Q)$ for some set Q. Then there is some $\mathfrak{M} \in M^Q$ with $\mathfrak{A} = mng^Q_{\mathfrak{M}}(\mathfrak{F}^Q)$. By (3) of Def. 3.2.16, without loss of generality we can assume that either $P \subseteq Q$ or $Q \subseteq P$ hold.

First assume that $Q \subseteq P$. Then, by (4) of Def. 3.2.16, $(\exists \mathfrak{N} \in M^P)\ mng^P_{\mathfrak{N}} \restriction \mathfrak{F}^Q = mng^Q_{\mathfrak{M}}$. Then $\mathfrak{A} \subseteq mng^P_{\mathfrak{N}}(\mathfrak{F}^P) \in \mathsf{Alg}_m(\mathcal{L}^P)$, thus $\mathfrak{A} \models q$, since quasi-equations are preserved under taking subalgebras.

Now let $Q \supseteq P$ and assume that $\mathfrak{A} \not\models q[k]$ for some evaluation k of the variables. Say, let $k(x_i) \stackrel{\text{def}}{=} mng^Q_{\mathfrak{M}}(\gamma_i))$ $(1 \leq i \leq n)$, assuming that $x_1, \ldots, x_n$ are the only variables occurring free in q. Assume that the atomic formulas occurring in the formulas $\gamma_1, \ldots, \gamma_n$ are among $p_{i_1}, \ldots, p_{i_m}$ and let s be the following substitution:

$$(\forall 1 \leq j \leq m)\ s(p_j) \stackrel{\text{def}}{=} p_{i_j}.$$

Then, by (5) of Definition 3.1.1,

$$(\exists \mathfrak{N} \in M^Q)(\forall 1 \leq i \leq n)\ mng^Q_{\mathfrak{M}}(\gamma_i) = mng^Q_{\mathfrak{N}}\big(\gamma_i(p_{i_1}/p_1, \ldots, p_{i_m}/p_m)\big).$$

By (4) of Definition 3.2.16, $(\exists \mathfrak{N}' \in M^P)\ mng^Q_{\mathfrak{N}} \restriction \mathfrak{F}^P = mng^P_{\mathfrak{N}'}$. Now, let $\mathfrak{B} \stackrel{\text{def}}{=} mng^P_{\mathfrak{N}'}$ and let $k'(x_i) \stackrel{\text{def}}{=} mng^P_{\mathfrak{N}'}\big(\gamma_i(p_1, \ldots, p_m)\big)$. Then $\mathfrak{A} \not\models q[k]$ implies $\mathfrak{B} \not\models q[k']$, which contradicts to $\mathfrak{B} \in \mathsf{Alg}_m(\mathcal{L}^P)$. □

□

Proof of ($\Longrightarrow$) of Theorem 3.2.27. Assume that Ax is a finite set of quasi-equations axiomatizing $\mathsf{Alg}_{\models}(\mathbf{L})$. Since $\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ (cf. Theorem 3.2.20), by Lemma 3.2.28 above, Ax also axiomatizes the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L}^P)$ for each infinite set P. Thus, by Theorem 3.2.3, for each infinite P there is a finitely complete and strongly sound Hilbert-style inference system $\vdash$ for $\mathcal{L}^P$. Moreover, checking the proof of Theorem 3.2.3 one can observe that the same inference system $\vdash$ works for all infinite sets P.

We show that for any set Q, $\vdash$ is strongly complete for $\mathcal{L}^Q$. Assume that for some $\Gamma \cup \{\varphi\} \subseteq F^Q \quad \Gamma \models^Q \varphi$. Then there is some infinite set P such that $\Gamma \cup \{\varphi\} \subseteq F^P$ and $\Gamma \models^P \varphi$ (cf. Remark 3.2.17 above). Since quasi-varieties are **Up**-closed, $\mathcal{L}^P$ is cons. compact by Theorem 3.2.25. Therefore there is a finite subset Σ of Γ such that $\Sigma \models^P \varphi$. Thus, by finite completeness $\Sigma \vdash \varphi$, which implies $\Gamma \vdash \varphi$ by the definition of derivability (Def. 3.1.14). $\square$

Proof of ($\Longleftarrow$) *of Theorem 3.2.27.* If $\vdash$ is strongly complete then it is also finitely complete. Thus, by Theorem 3.2.3, the quasi-variety generated by $\mathsf{Alg}_m(\mathcal{L}^P)$ is finitely axiomatizable for each set P.

On the other hand, strong completeness implies cons. compactness, as follows. Assume that for some P, $\Gamma \cup \{\varphi\} \subseteq F^P \quad \Gamma \models^P \varphi$. Then $\Gamma \vdash \varphi$, which implies by Definition 3.1.14 that there is a finite subset Σ of Γ such that $\Sigma \vdash \varphi$. Then, by strong soundness, $\Sigma \models^P \varphi$. Now, by Theorem 3.2.25, $\mathsf{Alg}_{\models}(\mathbf{L})$ is **Up**-closed. But by Theorem 3.2.20, it is also closed under **S** and **P**, thus it is a quasi-variety (cf. "quasi-variety characterization" in [49]). This and the fact that the quasi-varieties generated by $\mathsf{Alg}_m(\mathcal{L}^P)$ are finitely axiomatizable (with the same set Ax of quasi-equations, as the proof of Theorem 3.2.3 shows) imply that $\mathsf{Alg}_{\models}(\mathbf{L})$ is a finitely axiomatizable quasi-variety. $\square$

Exercise 3.2.29. Show that $\mathcal{L}_S$ and $S5$ have strongly complete and sound Hilbert-style inference systems. Give such calculi. (Hint: Use that the corresponding classes of algebras ($\mathsf{Alg}_m(\mathbf{L}_S) = \mathsf{BA}$ and $\mathsf{Alg}_m(\mathbf{L}_{S5}) = \mathsf{Cs}_1$) generate finitely axiomatizable varieties.)

In all the above we investigated only some logical properties, e.g. completeness and compactness. However, the literature contains similar theorems for a very large number of *further logical properties*. Such are e.g. Craig's interpolation property, the various definability properties (e.g. Beth's), the property of having a deduction theorem, the property of admitting Gabbay-style inference systems, to mention only a few. Some of these are discussed in Appendix B below.

4 Generalizations

First we relax the assumption on our logic having derived connectives "ε_j", "δ_j" ($j < m$) (cf. Def. 3.1.1). We will omit condition (3)(ii) from the definition of a nice logic obtaining the notion of a semi-nice logic.

Definition 4.1 ((strongly) semi-nice (general) logic). Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a logic in the sense of Def. 2.1.3. Then

(i) $\mathcal{L}$ is said to be *semi-nice* if it satisfies conditions (1), (2), (3)(i), (4) of Def. 3.1.1.

(ii) $\mathcal{L}$ is said to be *strongly semi-nice* if $\mathcal{L}$ is semi-nice and it also satisfies condition (5) of Def. 3.1.1.

(iii) A *(strongly) semi-nice general logic* is obtained by replacing "nice logic" with "semi-nice logic" in condition (1) of Def. 3.2.16 (i.e. by doing the natural change in the definition of a (strongly) nice general logic).

Semi-nice logics, even without condition (4) of Def. 3.1.1, were investigated in [9] but investigation of the $\models$ relation was restricted to the case of one Δ_i and to formulas of the form $(\varphi\Delta_0\psi)$. Below we indicate how to extend investigation to all formulas, i.e. how to extend the theory described in the present work to semi-nice logics.

To algebraize (in a reversible way) these more general logics, we add a new unary operation symbol "c" to (the language of) our algebras. So the new version $\mathsf{Alg}_i^+(\mathcal{L})$ of $\mathsf{Alg}_i(\mathcal{L})$ ($i \in \{\models, m\}$) will consist of algebras which have an extra operation "C" not available in $\mathsf{Alg}_i(\mathcal{L})$. However, in order to make our approach work, we have to permit "c" to be a *partial operation*. This means that for certain elements of our algebras "c" may not be defined. (A classical example of a partial operation is inversion $x \to x^{-1}$ in the field of real numbers. Zero has no inverse, so $^{-1}$ is undefined at argument 0.) Universal algebra for partial algebras (i.e. algebras with partial operations) is well defined, cf. e.g. Burmeister [17], Andréka–Németi [6]. Therefore generalizing our previous theorems to the new algebras causes no real difficulty. Those readers who would prefer avoiding partial algebras are asked to consult Remark 4.3 below. It is shown there how to eliminate the partial operation symbol "c".

Definition 4.2 ($\mathsf{Alg}_{\models}^+(\mathcal{L})$, $\mathsf{Alg}_m^+(\mathcal{L})$). Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a logic. Assume $\mathcal{L}$ satisfies conditions (1), (2) of Def. 3.1.1.

Let $K \subseteq M$. Then we define the partial function $c_K : F \to F$ in the following way. For any $\varphi \in F$,

$$\begin{array}{llll} \text{if} & K \models \varphi & \text{then} & c_K(\varphi) \text{ is defined and } c_K(\varphi) = \varphi; \quad \text{while} \\ \text{if} & K \not\models \varphi & \text{then} & c_K(\varphi) \text{ is undefined.} \end{array}$$

Clearly, $\langle \mathfrak{F}, c_K \rangle$ is a partial algebra for every $K \subseteq M$. The equivalence relation $\sim_K$ (defined in Def. 3.1.6) is a congruence not only on $\mathfrak{F}$ but also on $\langle \mathfrak{F}, c_K \rangle$ (c_K was defined in a way to ensure this). Now,

$$\mathsf{Alg}^+_{\models}(\mathcal{L}) \stackrel{\text{def}}{=} \mathbf{I}\left\{\langle \mathfrak{F}, c_K \rangle / {\sim_K} : \ K \subseteq M\right\}.$$

Let us turn to defining $\mathsf{Alg}^+_m(\mathcal{L})$. First we define a new partial function c on the algebra $\mathfrak{A}(\mathfrak{M}) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\mathfrak{F})$ as follows. For every $\varphi \in F$,

$$\begin{array}{l} c\big(mng_{\mathfrak{M}}(\varphi)\big) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(\varphi) \quad \text{if} \quad \mathfrak{M} \models \varphi; \quad \text{else} \\ c\big(mng_{\mathfrak{M}}(\varphi)\big) \quad \text{is undefined.} \end{array}$$

The new partial algebra $\mathfrak{A}^+(\mathfrak{M})$ associated to $\mathfrak{M}$ is

$$\mathfrak{A}^+(\mathfrak{M}) \stackrel{\text{def}}{=} \langle \mathfrak{A}(\mathfrak{M}), c \rangle.$$

Now

$$\mathsf{Alg}^+_m(\mathcal{L}) \stackrel{\text{def}}{=} \left\{\mathfrak{A}^+(\mathfrak{M}) \ : \ \mathfrak{M} \in M\right\}. \blacktriangleleft$$

As we mentioned, universal algebra for partial algebras is well developed (cf. op cit). For completeness we recall those notions which are most needed. Since any partial algebra $\langle \mathfrak{A}, c \rangle$ is a model in the model theoretic sense (consider "c" as a binary relation), the model theoretic operations like direct products ($\mathbf{P}$), ultraproducts ($\mathbf{Up}$), reduced products ($\mathbf{P^r}$) need not de defined. Subalgebras are submodels closed under "c", i.e. to each element x of our subalgebra, if c is defined on x then $c(x)$ is also in our subalgebra.

Let τ be a term, $\mathfrak{A}$ a partial algebra and $k \in {}^{\omega}A$ an evaluation of the variables. Then, τ is said to be *defined* at evaluation k (in $\mathfrak{A}$) iff every subterm of τ is defined at k.[14]

Now, $\mathfrak{A} \models (\tau = \sigma)[k]$ (i.e. evaluation k satisfies the equation $\tau = \sigma$) iff both τ and σ are defined at evaluation k and their values coincide.

[14] Variables are always defined and $c(\tau)$ is defined if $\big[\tau$ is defined and c is defined at the value of $\tau\big]$.

With this we defined the satisfaction for atomic formulas (i.e. equations) of the language of partial algebras. The logical connectives are interpreted the usual way, hence satisfaction (and thus validity) is defined for all formulas of partial algebras. In particular, quasi-equations $(\tau_1 = \sigma_1 \wedge \cdots \wedge \tau_n = \sigma_n) \to \tau_0 = \sigma_0$ are defined and interpreted in the usual way. A class K is said to be a *quasi-variety* iff it is definable by a set of quasi-equations. It is a *variety* iff it is definable by equations. The usual theorems carry over, e.g.

$$\mathsf{K} \text{ is a quasi-variety} \quad \text{iff} \quad \mathsf{K} = \mathbf{SPUp}\,\mathsf{K} = \mathbf{SP^r}\,\mathsf{K}.$$

For more cf. [17], [6].

With the above in mind, it seems reasonable to repeat for semi-nice logics and $\mathsf{Alg}_i^+(\mathcal{L})(i \in \{\models, m\})$ what we did in section 3 for nice logics and $\mathsf{Alg}_i(\mathcal{L})(i \in \{\models, m\})$.

We note that Blok and Pigozzi (cf. [14], [16] and the references therein) have strong results in this direction (in perhaps a slightly different formulation). Before turning to generalizing section 3 to the present more general setting, we should mention an equivalent form of what we are doing.

Remark 4.3. If the reader would like to avoid using partial algebras, then the following equivalent more natural approach works. Instead of "c" we add a new unary predicate "$T(x)$" (T for truth). Imitating the definition of "c_K", we let $T_K \stackrel{\text{def}}{=} \{\varphi \in F : K \models \varphi\}$ for any $K \subseteq M$. Similarly, the algebraic counterpart of a model $\mathfrak{M}$ looks like $\langle \mathfrak{A}, T\rangle$, where $\mathfrak{A} \in \mathsf{Alg}_m(\mathcal{L})$ and $T \subseteq A$ such that

$$(\forall \varphi \in F)\big(\mathfrak{M} \models \varphi \quad \Longleftrightarrow \quad mng_{\mathfrak{M}}(\varphi) \in T\big);$$

holds for T.

This approach is practically equivalent to the one using "c" instead of "T". Further, this is very-very closely related to what is called "matrix semantics" in Blok–Pigozzi [14], [16], Czelakowski [20] and in the papers quoted in these works. In these papers there are several strong results about the presently outlined approach.

Now, many of the results proved for nice logics so far, can be pushed through for semi-nice logics (with $\mathsf{Alg}_{\models}^+$, Alg_m^+ in place of $\mathsf{Alg}_{\models}$, Alg_m).

For example, the proof of

$$(\mathsf{Alg}_{\models}^+(\mathbf{L}) \text{ is } \mathbf{Up}\text{-closed}) \quad \Longrightarrow \quad (\mathbf{L} \text{ is sat. compact})$$

(cf. Thm. 3.2.24) should go through with the natural modifications for semi-nice logics.

For some of the results the formulation of the result needs a minor modification. E.g. the algebraic equation corresponding to logical formula φ is now $c(\varphi) = \varphi$ (instead of $\varepsilon_j(\varphi) = \delta_j(\varphi)$ for all $j < m$). But again we have

$$\models_{\mathcal{L}} \varphi \quad \Longleftrightarrow \quad \mathsf{Alg}_m^+(\mathcal{L}) \models c(\varphi) = \varphi$$

(cf. Thm. 3.2.1).

Exercises 4.4. (1) Replace the definition of the validity relation $f \models_\infty \varphi$ of logic $\mathcal{L}_\infty$ (cf. Def. 2.2.30) by

$$f \models'_\infty \varphi \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad mn_f(\varphi) > 0.9$$

and show that the resulting logic is not nice but semi-nice.

(2) Push through the proof of Thm. 3.2.24 for strongly semi-nice general logics.

(3) Check what is needed for the other direction, i.e. for Thm. 3.2.25 to go through.

(4) Repeat the proof of Fact 3.1.8 in the new (semi-nice) setting.

(5) Look at the major theorems in section 3.2 one by one and check if their proofs can be pushed through in the new setting. Where it does not seem to go through, check whether some change in the formulation of the result permits you to push the proof through.

(6) Try to find out whether we could use a total operation instead of our partial one "c". E.g. try to define $c_K(\varphi) = \varphi$ if $K \models \varphi$ else $(\varphi\Delta_0\varphi)$ (assume that only Δ_0 is available as "special" connective). Now our algebra is not partial! Can this approach work? Show that the validity relation $\models$ can be recovered from the new total "c", so the coding is faithful. But do the results go through? Check them! Show that Ex. 3.1.9 fails. Show that Thm. 3.2.3 does not want to go through even with modifications.

* * *

If we want to drop condition (3) of the definition of nice logic (Def. 3.1.1) altogether, then a possibility is to restrict the validity relation $\models$ to *sequents* $(\varphi \Rightarrow \psi)$ of formulas (instead of having it for all formulas). Here "$\Rightarrow$" is not a logical connective, but rather a metalevel symbol. If $\varphi, \psi \in F$ then $(\varphi \Rightarrow \psi)$ is a sequent (sequents are *not* formulas). Further,

$$\mathfrak{M} \models (\varphi \Rightarrow \psi) \quad \text{iff} \quad mng_{\mathfrak{M}}(\varphi) \subseteq mng_{\mathfrak{M}}(\psi).$$

This approach is applicable to more logics, hence more kinds of algebras show up in $\mathsf{Alg}_i(\mathcal{L})$ ($i \in \{\models, m\}$). However, similarly to the way we had to introduce "c" above to code validity in a model, now we have to introduce a pre-ordering "$\leq$" on our algebras to code "$\Rightarrow$". However, this is not needed if we restrict the validity relation $\models$ a little bit more, namely to pairs $\{(\varphi \Rightarrow \psi), (\psi \Rightarrow \varphi)\}$ of sequents. Then we do not need new symbols like "$\leq$" in our algebras. This approach is investigated e.g. in [9] to quite some extent. See also investigations on k-deductive systems in Blok–Pigozzi [16]. For a general method using sequents see Guzman–Verdu [26], Font–Verdu [24].

We could also try to drop conditions (4), (5) of Def. 3.1.1, i.e. permutability of atomic formulas. This would enable us to treat traditional first-order logic more comfortably (with less preparation to do). This can be done, the only thing needed is the universal algebraic concept of a *free algebra over some defining relations*. The details are available in [9].

5 Further equivalence results

In this section we give algebraic characterizations for further logical properties, such as *decidability of the validity problem*, various *Beth's definability properties* and *Craig's interpolation properties*.

First recall that a logic is called *decidable* iff the set of its validities is a decidable subset of the set of all formulas (cf. Definition 2.1.7).

Theorem 5.1. *Assume that $\mathcal{L}$ is a nice logic. Then*

(i) $\mathcal{L}$ is decidable $\iff$ the equational theory of $\mathsf{Alg}_{\models}(\mathcal{L})$ is decidable.

(ii) The validities of $\mathcal{L}$ are recursively enumerable $\iff$ the equational theory of $\mathsf{Alg}_{\models}(\mathcal{L})$ is recursively enumerable.

Proof. It is a straightforward corollary of Cor. 3.2.2 way above. $\square$

Let $\mathcal{L}$ be a nice logic. An inference rule $B_1, \ldots, B_n \vdash B_0$ is called *admissible* for $\mathcal{L}$ iff it is strongly sound for $\mathcal{L}$. We note that, in the style of Theorem 5.1, the set of admissible rules of $\mathcal{L}$ is decidable iff the quasi–equational theory of $\mathsf{Alg}_{\mathcal{L}}$ is decidable.

Next we turn to the algebraic characterization of some definability properties. Beth definability properties of logics were introduced, e.g., in Barwise–Feferman [11] and in Sain [48]. Here we give the definitions in the framework of the present paper. The proofs of Theorems 5.6 and 5.12 below can be found in Németi [38] and in Hoogland [28].[15]

Definition 5.2 (implicit definition, explicit definition, local explicit definition). Let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ be a general logic. Let $P \subsetneqq Q$ be sets with $F^P \neq \emptyset$, and let $R \stackrel{\text{def}}{=} Q \setminus P$.

A set $\Sigma \subseteq F^Q$ of formulas *defines R implicitly in Q* iff

$$\big(\forall \mathfrak{M}, \mathfrak{N} \in Mod^Q(\Sigma)\big)\big(mng^Q_{\mathfrak{M}} \restriction F^P = mng^Q_{\mathfrak{N}} \restriction F^P \implies mng^Q_{\mathfrak{M}} = mng^Q_{\mathfrak{N}}\big) .$$

Σ *defines R explicitly in Q* iff

$$(\forall r \in R)(\exists \varphi_r \in F^P)(\forall \mathfrak{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathfrak{M}}(r) = mng^Q_{\mathfrak{M}}(\varphi_r).$$

Σ *defines R local-explicitly in Q* iff

$$(\forall \mathfrak{M} \in Mod^Q(\Sigma))(\forall r \in R)(\exists \varphi_r \in F^P)\ mng^Q_{\mathfrak{M}}(r) = mng^Q_{\mathfrak{M}}(\varphi_r). \quad \blacktriangleleft$$

[15] Actually, Theorem 5.12 is not in [38], an early version of Theorem 5.12 is in [48] and the full version is in [28].

Definition 5.3 ((strong) Beth definability property). Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *(strong) Beth definability property* iff for all P, Q, R and Σ as in Def. 5.2 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q \quad \Longrightarrow \quad \Sigma \text{ defines } R \text{ explicitly in } Q). \quad \blacktriangleleft$$

Definition 5.4 (patchwork property of models). Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *patchwork property of models* iff

$$(\forall \text{sets } P, Q)(\forall \mathfrak{M} \in M^P)(\forall \mathfrak{N} \in M^Q)$$
$$\big(F^{P\cap Q} \neq \emptyset \ \text{ and } \ mng^P_{\mathfrak{M}} \restriction F^{P\cap Q} = mng^Q_{\mathfrak{N}} \restriction F^{P\cap Q}\big) \quad \Longrightarrow$$
$$\Longrightarrow \ (\exists \mathfrak{P} \in M^{P\cup Q})\big(mng^{P\cup Q}_{\mathfrak{P}} \restriction F^P = mng^P_{\mathfrak{M}} \ \text{ and } \ mng^{P\cup Q}_{\mathfrak{P}} \restriction F^Q = mng^Q_{\mathfrak{N}}\big)\,. \quad \blacktriangleleft$$

Definition 5.5 (morphism, epimorphism). Let K be a class of algebras. By a *morphism of* K we understand a triple $\langle \mathfrak{A}, h, \mathfrak{B}\rangle$, where $\mathfrak{A}, \mathfrak{B} \in K$ and $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism.

A morphism $\langle \mathfrak{A}, h, \mathfrak{B}\rangle$ is an *epimorphism of* K iff for every $\mathfrak{C} \in K$ and every pair $f : \mathfrak{B} \to \mathfrak{C}$, $k : \mathfrak{B} \to \mathfrak{C}$ of homomorphisms we have $(f \circ h = k \circ h \Longrightarrow f = k)$.

Typical examples of epimorphisms are the surjections. But for certain choices of K there are epimorphisms of K which are not surjective. This is the case, e.g., when K is the class of distributive lattices.

Theorem 5.6 ([38], [8, sec. II.2], [28]). *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$\mathbf{L}$ *has the (strong) Beth definability property*
$$\Longleftrightarrow$$
all the epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

The **proof** is in [38] and Hoogland [28]. A less general version of this theorem is proved in [27, Thm.5.6.10]. $\square$

Definition 5.7 ((strong) local Beth definability property). Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *(strong) local Beth definability property* iff for all P, Q, R and Σ as in Definition 5.2 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q \quad \Longrightarrow \quad \Sigma \text{ defines } R \text{ local-explicitly in } Q). \quad \blacktriangleleft$$

Theorem 5.8. *J. X. Madarász] Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$\mathbf{L}$ *has the (strong) local Beth definability property*
$$\Longleftrightarrow$$
all the epimorphisms of $\mathsf{Alg}_m(\mathbf{L})$ *are surjective.* $\square$

Definition 5.9 (strong implicit definition). Let $\mathbf{L}$ be a general logic. Let P, Q, R and Σ be as in Def. 5.2 above. Σ *defines R implicitly in Q in the strong sense* iff

$$\begin{aligned}&\Sigma \text{ defines } R \text{ implicitly in } Q \quad \text{and}\\ &\big(\forall \mathfrak{M} \in Mod^P(Th^Q\mathsf{Mod}^Q(\Sigma) \cap F^P)\big)(\exists \mathfrak{N} \in Mod^Q(\Sigma))\ mng^Q_{\mathfrak{N}} \restriction F^P = mng^P_{\mathfrak{M}}. \quad \blacktriangleleft\end{aligned}$$

Definition 5.10 (weak Beth definability property). [16] Let $\mathbf{L}$ be a general logic. $\mathbf{L}$ has the *weak Beth definability property* iff for all P, Q, R and Σ as in Def. 5.2 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q \text{ in a strong sense} \implies \Sigma \text{ defines } R \text{ explicitly in } Q). \quad \blacktriangleleft$$

Definition 5.11 (K-extensible). Let $K_0 \subseteq K$ be two classes of algebras. Let $\langle \mathfrak{A}, h, \mathfrak{B}\rangle$ be a morphism of K. h is said to be K_0-*extensible* iff for every algebra $\mathfrak{C} \in K_0$ and every homomorphism $f : \mathfrak{A} \to \mathfrak{C}$ there exists some $\mathfrak{N} \in K_0$ and $g : \mathfrak{B} \to \mathfrak{N}$ such that $\mathfrak{C} \subseteq \mathfrak{N}$ and $g \circ h = f$.

It is important to emphasize that $\mathfrak{C}$ is a concrete subalgebra of $\mathfrak{N}$ and *not* only is embeddable into $\mathfrak{N}$.

Theorem 5.12 (Hoogland [28], Sain [48]). *Let $\mathbf{L}$ be a strongly nice general logic which has the patchwork property of models. Then*

$$\begin{gathered}\mathbf{L} \textit{ has the weak Beth definability property}\\ \Longleftrightarrow\\ \textit{every } \mathsf{Alg}_m(\mathbf{L})\textit{-extensible epimorphism of } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ is surjective.}\end{gathered} \quad \square$$

In the formulation of Theorem 5.12 above, it was important that $\mathsf{Alg}_m(\mathbf{L})$ is not an abstract class in the sense that it is not closed under isomorphisms, since the definition of K-extensibility strongly differentiates isomorphic algebras.

Theorem 5.12 and Theorem 5.14 below are solutions for Problem 14 in [48]. On the other hand, Theorem 5.15 together with Definition 5.13 aims for being a possible solution for Problem 15 of [48].

Definition 5.13 (full algebras of $\mathsf{Alg}_m(\mathbf{L})$). Let $\mathbf{L}$ be a nice general logic. The class $\mathsf{FullAlg}_m(\mathbf{L})$ of algebras is defined as follows.

$$\mathsf{FullAlg}_m(\mathbf{L}) := \{\mathfrak{A} \in \mathsf{Alg}_m(\mathbf{L}) : (\forall \mathfrak{B} \in \mathsf{Alg}_m(\mathbf{L}))(\mathfrak{A} \subseteq \mathfrak{B} \implies \mathfrak{A} = \mathfrak{B})\}. \quad \blacktriangleleft$$

We will use the notions of "reflective subcategory" and "limits of diagrams of algebras" as in Mac Lane [29]. We will not recall these.

Throughout, by a reflective subcategory we will understand a full and isomorphism closed one.

[16] The weak Beth definability property was introduced in Friedman [25] and has been investigated since then, cf. e.g. [11, pp. 73–76, 689–716].

Theorem 5.14 (Sain–Madarász–Németi (cf. [48, item(9) on p. 223])). *Assume the conditions of Theorem 5.12. Assume* $\mathsf{Alg}_m(\mathbf{L}) \subseteq \mathsf{SFullAlg}_m(\mathbf{L})$*. Then*

$$\mathbf{L} \text{ has the weak Beth definability property}$$
$$\Longleftrightarrow$$
$$\mathsf{Alg}_{\models}(\mathbf{L}) \text{ is the smallest full reflective subcategory } K \text{ of } \mathsf{Alg}_{\models}(\mathbf{L}) \text{ with}$$

$\mathsf{FullAlg}_m(\mathbf{L}) \subseteq K$. □

Theorem 5.15. *Assume the conditions of Theorem 5.14. Then*

$$\mathbf{L} \text{ has the weak Beth definability property}$$
$$\Longleftrightarrow$$
$$\mathsf{FullAlg}_m(\mathbf{L}) \text{ generates } \mathsf{Alg}_{\models}(\mathbf{L}) \text{ by taking limits of diagrams of algebras.}$$[17]

On the proof

The proof is based on Theorem 5.14 and on the simple lemma denoted as (†) below.

(†) Assume $K_0 = \mathbf{SP}K_0$ and $K_1 \subseteq K_0$ is a set of algebras in K_0. Then the smallest full reflective subcategory K of K_0 containing K_1 exists and coincides with the smallest limit-closed class containing K_1.[18]

Next one uses the fact that

$$(\dagger\dagger)\qquad \begin{array}{l}(\exists\kappa \in Card)\big(\forall\mathfrak{A} \in \mathsf{FullAlg}_m(\mathbf{L})\big)(\forall H \subseteq A)\\ \big(|H| < \kappa \Longrightarrow (\exists\mathfrak{B} \subseteq \mathfrak{A})(H \subseteq B \;\&\; \mathfrak{B} \in \mathsf{FullAlg}_m(\mathbf{L}) \;\&\; |B| < \kappa)\big)\,.\end{array}$$

(††) follows from the assumption that $\mathbf{L}$ is a structural nice general logic; cf. in particular item (4) in the definition of "general logic". □

$\mathbf{L}_n$ denotes the general logic which we get from $\mathcal{L}_n$ (cf. Definition 2.2.21).

Remark 5.16. Note that $\mathsf{FullAlg}_m(\mathbf{L}_n) = \mathsf{FullCs}_n$. ◄

Conjecture 5.1. *We conjecture that item (4) in the definition of general logic is essential for Theorem 5.15. Indeed, we conjecture that without this condition Theorem 5.15 might become independent of ZFC set theory.* ◄

[17] I.e., there is no limit-closed class separating these two classes of algebras.

[18] We conjecture that (†) might become independent of set theory if the restriction that K_1 is a set is omitted. Clearly, (†) becomes false if K_0 is permitted to be an arbitrary complete and co-complete category.

Terminology

Let $\mathbf{L}$ be a nice general logic. Then $\mathrm{Mod}(\mathbf{L}) := \bigcup\{M^P : P \text{ is a set}\}$ is the class of all models of $\mathbf{L}$.

Let $\mathfrak{M}, \mathfrak{M}_1 \in \mathrm{Mod}(\mathbf{L})$. Then: $\mathfrak{M}_1$ is an *expansion* of $\mathfrak{M}$ iff $\mathfrak{M}$ is a *reduct* of $\mathfrak{M}_1$ iff $\exists P(\mathfrak{M} = \mathfrak{M}_1 \restriction P)$. Further:

$\mathrm{Mng}(\mathfrak{M})$:= set of meanings of $\mathfrak{M}$ = universe of the meaning-algebra $mng_{\mathfrak{M}}(\mathcal{F})$ of $\mathfrak{M}$.

$\mathrm{Alg}(\mathfrak{M}) := mng_{\mathfrak{M}}(\mathcal{F})$ = the meaning-algebra of $\mathfrak{M}$.

Conjecture 5.2. *We conjecture that the characterizations of weak Beth in items 5.14–5.15 can be made more "logic oriented" (or more intuitive) the following way: Let $\mathbf{L}$ be a nice general logic and $\mathfrak{M} \in Mod(\mathbf{L})$. Then $\mathfrak{M}$ is called* full *iff ($\forall$ expansion $\mathfrak{M}_1$ of $\mathfrak{M}$) $Mng(\mathfrak{M}_1) \subseteq Mng(\mathfrak{M})$. Now we define*

$$\mathsf{FuAlg}_m(\mathbf{L}) := \{Alg(\mathfrak{M}) : \mathfrak{M} \textit{ is a full model of } \mathbf{L}\}$$

Now, the assumption that

$$(*) \qquad \mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{S}\mathsf{FullAlg}_m(\mathbf{L})$$

in items 5.14–5.15 can be replaced with the more intuitive assumption that

$$(**) \qquad \textit{every } \mathfrak{M} \in Mod(\mathbf{L}) \textit{ has a full expansion.}$$

We conjecture that the characterizations of weak Beth property in items 5.14–5.15 remain true if we replace $(*)$ with $(**)$ and "Full" with "Fu" in them. In particular, $(**) \Longrightarrow \mathsf{Alg}_m(\mathbf{L}) \subseteq \mathbf{S}\mathsf{FuAlg}_m(\mathbf{L})$ holds for structural general logics with the patchwork property. For such a logics we also have $\mathsf{FuAlg}_m(\mathbf{L}) = \mathsf{FullAlg}_m(\mathbf{L})$, hence we conclude that full meaning algebras are exactly the meaning algebras of full models.

The purpose of the present conjecture is to find a natural (or logic-oriented) characterization of $\mathsf{FullAlg}_m(\mathbf{L})$, which in turn, might be a kind of solution of Problem 15 from [48] ◀

For the origins of our characterizations of weak Beth property (in items 5.12, 5.14, 5.15) see items (8), (9) below Problem 14 in [48]. (In this connection it is useful to read [48] beginning with Problem 12 to the end.)

Next we turn to characterizing Craig's interpolation property.

Definition 5.17 (($\models$ interpolation) property). Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a nice logic. For each formula $\varphi \in F$ let $atf(\varphi)$ denote the set of atomic formulas occurring in φ. Then $\mathcal{L}$ has the *($\models$ interpolation) property* iff

$$(\forall\varphi,\psi \in F)\big(\{\varphi\} \models \psi \;\Rightarrow\; (\exists\chi \in F)(atf(\chi) \subseteq atf(\varphi) \cap atf(\psi) \text{ and } \{\varphi\} \models \chi \text{ and } \{\chi\} \models \psi)\big)\,. \quad \blacktriangleleft$$

Recall that for any class K of algebras $\mathbf{I}K$ denotes the class of all isomorphic copies of members of K.

Definition 5.18 (amalgamation property). Let K be a class of algebras. We say that K *has the amalgamation property* iff for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbf{I}K$ with $\mathfrak{B} \supseteq \mathfrak{A} \subseteq \mathfrak{C}$, there are $\mathfrak{N} \in K$ and injective homomorphisms (embeddings) $f : \mathfrak{B} \rightarrowtail \mathfrak{N}$ $h : \mathfrak{C} \rightarrowtail \mathfrak{N}$ such that $f \restriction A = h \restriction A$.

Theorem 5.19 (J. Czelakowski). *Let $\mathcal{L}$ be a strongly nice and consequence compact logic. Assume that usual conjunction "$\wedge$" is in $Cn(\mathcal{L})$. Assume that $\mathcal{L}$ has a deduction theorem. Then*

$\mathcal{L}$ has the ($\models$ interpolation) property $\iff$ $\mathsf{Alg}_{\models}(\mathcal{L})$ has the amalgamation property.

Proof. It can be found in Czelakowski [20], cf. Thm.3 therein. $\square$

Definition 5.20 (($\to$ interpolation) property). Let $\mathbf{L}$ be a general logic satisfying condition (1) in Definition 3.1.1, and let $\to$ be a binary connective of $\mathbf{L}$. We say that $\mathbf{L}$ *has the ($\to$ interpolation) property* if

$$(\forall\varphi,\psi \in F)\big(\models \varphi \to \psi \;\Rightarrow\; (\exists\chi \in F)(atf(\chi) \subseteq atf(\varphi) \cap atf(\psi) \text{ and } \models \varphi \to \chi \text{ and } \models \chi \to \psi)\big)\,. \quad \blacktriangleleft$$

By a *partially ordered algebra* we mean a structure $(\mathfrak{A}, \leq)$ where $\mathfrak{A}$ is an algebra and $\leq$ is a partial ordering on the universe A of $\mathfrak{A}$.

Definition 5.21 (super–amalgamation property (cf. Maksimova [34])**).** A class K of partially ordered algebras has the *super–amalgamation property* if for any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{K}$ and for any embeddings
$i_1 : \mathfrak{A}_0 \longrightarrow \mathfrak{A}_1$ and $i_2 : \mathfrak{A}_0 \longrightarrow \mathfrak{A}_2$ there exist an $\mathfrak{A} \in \mathsf{K}$ and embeddings
$m_1 : \mathfrak{A}_1 \longrightarrow \mathfrak{A}$ a nd $m_2 : \mathfrak{A}_2 \longrightarrow \mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$ and

$$(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \Rightarrow (\exists z \in A_0)(x \leq i_j(z) \quad \text{and} \quad i_k(z) \leq y)),$$

where $\{j,k\} = \{1,2\}$. $\blacktriangleleft$

Theorem 5.22 (Madarász [32]). *Let* $\mathbf{L}$ *be a strongly nice general logic such that* $\mathbf{L}$ *contains the classical propositional logic as a fragment (i.e.* $\mathsf{Alg}_{\models}(\mathbf{L})$ *has a Boolean reduct). Assume that* $\mathsf{Alg}_{\models}(\mathbf{L})$ *forms a variety. Let* $\rightarrow$ *be the usual Boolean implication. Assume that* $\mathbf{L}$ *has the local deduction property in the following sense:*[19]

For all $\varphi, \psi \in F$ *there is a unary derived connective, say,* $\Box$ *of* $\mathbf{L}$*, such that*

$$(\varphi \models \psi \implies \models \Box(\varphi) \rightarrow \psi) \quad \textit{and} \quad \varphi \models \Box(\varphi).$$

Then

$$\mathbf{L} \textit{ has the } (\rightarrow \textit{interpolation}) \textit{ property}$$
$$\Longleftrightarrow$$
$$\mathsf{Alg}_{\models}(\mathbf{L}) \textit{ has the super–amalgamation property,}$$

where super–amalgamation is understood via the following partial ordering: $a \leq b \Leftrightarrow a \rightarrow b = True$. □

Further investigations concerning the ($\rightarrow$ interpolation) property, its algebraic characterizability and related algebraic results are in [30], [32] and [31].

$\mathbf{L}_2$ and $\mathbf{L}_{\mathrm{ARROW}}$ denotes the general logic which we get from $\mathcal{L}_2$ (cf. Definition 2.2.21) and $\mathcal{L}_{\mathrm{ARROW}}$ (cf. Definition 2.2.19), respectively.

Definition 5.23. $\mathbf{L}_2^+$ is $\mathbf{L}_2$ expanded with atomic formulas of the form $R(v_1, v_0)$. Equivalently we could add the connective $\smile$ of $\mathbf{L}_{\mathrm{ARROW}}$ to $\mathbf{L}_2$ and have the atomic formulas unchanged. ◀

Open problems:

(1) Are all the conditions of Theorem 5.19 needed? Try to characterize ($\models$ interpolation) property with fewer assumptions on the logic.

(2) What is the logical counterpart of the algebraic property that "$\mathsf{Alg}_{\models}(\mathcal{L})$ has the *strong amalgamation property*" (i.e., we also require $f(B \setminus A) \cap h(C \setminus A) = \emptyset$ in Definition 5.18 above)?

(3) Does $\mathbf{L}_2^+$ have the weak Beth property? Does $\mathbf{L}_2$ have it? Does $\mathbf{L}_2^+$ without equality have weak Beth property (or even (strong) Beth property)? We note that the $\mathsf{Alg}_{\models}\big(\mathbf{L}_2^+$ without equality$\big) = \mathsf{RPA}_2$ where RPA_n is the class of representable polyadic algebras of dimension n.

We note that $\mathbf{L}_2^+$ restricted to models of cardinality ≤ 10 has the weak Beth property but not the Beth property. Hence this logic "($\mathbf{L}_2^+ \restriction \leq 10$)" separates the Beth property from the weak Beth property, showing that Theorems 5.12, 5.14, 5.15 above are not superfluous.

[19] The usual deduction property is also sufficient for the conclusion of this theorem.

A Appendix. New kinds of logics

In this appendix we collect a few logics which are of a different "flavor" than the ones listed in section 2.2. The main purpose of these examples is showing that the present Algebraic Logic framework is suitable for handling all sorts of unusual logics coming from completely different paradigms of logical or linguistic or computer science research areas.

Definition A.1 (infinite valued logic $\mathcal{L}_\infty$). Let P be any set, the set of atomic formulas of $\mathcal{L}_\infty$. The logical connectives of $\mathcal{L}_\infty$ are $\wedge$, $\neg$, $\vee$ and $\rightarrow$. The set F_∞ of formulas is defined the usual way. Recall that $P \subseteq F_\infty$ is the set of atomic formulas.

$$M_\infty \stackrel{\text{def}}{=} \{f \;:\; (f : P \rightarrow [0,1])\}\ ,$$

where $[0,1]$ denotes the usual interval of real numbers.

Let $f \in M_\infty$. First we define $mn_f(\varphi)$:

$$\begin{aligned}
mn_f(p) &\stackrel{\text{def}}{=} f(p) \quad \text{for } p \in P\\
mn_f(\varphi \wedge \psi) &\stackrel{\text{def}}{=} min\,\{mn_f(\varphi), mn_f(\psi)\}\\
mn_f(\neg\varphi) &\stackrel{\text{def}}{=} 1 - mn_f(\varphi)\\
mn_f(\varphi \vee \psi) &\stackrel{\text{def}}{=} max\,\{mn_f(\varphi), mn_f(\psi)\}\\
mn_f(\varphi \rightarrow \psi) &\stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } mn_f(\varphi) \leq mn_f(\psi)\\ 1 - \big(mn_f(\varphi) - mn_f(\psi)\big), & \text{else.}\end{cases}
\end{aligned}$$

For any $f \in M_\infty,\ \varphi \in F_\infty$,

$$\begin{aligned}
mng_\infty(\varphi, f) &\stackrel{\text{def}}{=} \{x \in [0,1] \;:\; x \leq mn_f(\varphi)\}\,;\\
f \models_\infty \varphi \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad mng_f(\varphi) = [0,1].
\end{aligned}$$

With this the logic

$$\mathcal{L}_\infty \stackrel{\text{def}}{=} \langle F_\infty, M_\infty, mng_\infty, \models_\infty\rangle$$

is defined.

Even in intuitionistic logic we have $\models \neg(\varphi \wedge \neg\varphi)$. However, in $\mathcal{L}_\infty$ this is not so, the truth value of $(\varphi \wedge \neg\varphi)$ can be as high as $1/2$. So in a sense, $\mathcal{L}_\infty$ tolerates contradictions (and by a cheap joke, we could call it "dialectical" because of this, but we will not do so). Also $(\varphi \leftrightarrow \neg\varphi)$ can be valid in some of our models. This again cannot happen even in intuitionistic logic. Further, $mn_f(\varphi) \geq 1/2$ is expressible as

$(\neg\varphi \to \varphi)$, hence if we would want to have a new validity relation $\models_1$, where $f \models_1 \varphi$ iff $mn_f(\varphi) \geq 1/2$, then we can express this new $\models_1$ by $f \models_1 \varphi$ iff $mng_\infty(\neg\varphi \to \varphi) = mng_\infty(\varphi \to \varphi)$. We do not look into this new $\models_1$ any more, we only use it as an example of definability of $\models_1$ from *mng* without identifying truth with a greatest meaning or even with a single meaning.

$\mathcal{L}_\infty$ is strongly nice since we can define $\varepsilon_0(\varphi) \stackrel{\text{def}}{=} (\varphi \to \varphi)$, $\delta_0(\varphi) \stackrel{\text{def}}{=} \varphi$ and $(\varphi \Delta_0 \psi) \stackrel{\text{def}}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Remark A.2. If we omit $\wedge$ from the connectives then we will need $\Delta_0 \stackrel{\text{def}}{=}$ " $\to$ " and $\Delta_1 \stackrel{\text{def}}{=}$ " $\leftarrow$ ". If we replaced $f \models_\infty \varphi \Leftrightarrow mn_f(\varphi) = 1$ by $f \models_\infty \varphi \Leftrightarrow mn_f(\varphi) > 0.9$ then we would loose niceness. However, our logic would still remain semi-nice as described in Section 4.

Exercises A.3. (1) Try to define logics similar to $\mathcal{L}_\infty$ but perhaps with more intuitive appeal to you.

(2) Prove that the intuitionistic tautology $\big(\varphi \wedge (\varphi \to \psi)\big) \to \psi$ is not valid in $\mathcal{L}_\infty$. Change the semantics in order to make this valid.

(3) Show that $\mathcal{L}_\infty$ is strongly nice.

(4) Obtain a new logic $\mathcal{L}_\mathbb{Q}$ from $\mathcal{L}_\infty$ by executing the following modifications in the definition. Replace [0,1] with the set $\mathbb{Q}$ of rational numbers everywhere. Define $mn_f(\neg\varphi) \stackrel{\text{def}}{=} -mn_f(\varphi)$. Redefine the meaning of "$\to$" as follows:

$$mn_f(\varphi \to \psi) \stackrel{\text{def}}{=} mn_f(\psi) - mn_f(\varphi)\,,$$

and let $mng_\mathbb{Q}(\varphi, f) \stackrel{\text{def}}{=} \{x \in \mathbb{Q} : x \leq mn_f(\varphi)\}$. Change the definition of $f \models_\infty \varphi$ to the following:

$$f \models_\mathbb{Q} \varphi \stackrel{\text{def}}{\iff} 0 \in mng_{\shortparallel}(\varphi, f)\,.$$

The rest remains unchanged.

(4.1) Investigate the logic $\mathcal{L}_\mathbb{Q}$! Compare it with $\mathcal{L}_\infty$.

(4.2) Prove that $mng_\mathbb{Q}(\neg\varphi \vee \psi, f) \neq mng_\mathbb{Q}(\varphi \to \psi, f)$, for some model f. Prove that $\models_\mathbb{Q} (p_1 \vee \neg p_1)$.

(4.3) Prove that $\not\models_\mathbb{Q} p_0 \to (p_1 \vee \neg p_1)$. (This property is aimed at by relevance logic, the idea being, roughly, that the formulas p_0 and $(p_1 \vee \neg p_1)$ have no common atomic formulas, hence they are not relevant to each other, so they cannot "relevantly imply" each other.)

(4.4) Prove that $\models_\mathbb{Q} (\varphi \to \varphi)$.

(4.5) Prove that $\models_{\mathbb{Q}} \varphi$ iff $(\forall f \in M_{\mathbb{Q}})$ $mng_{\mathbb{Q}}(\varphi, f) = mng_{\mathbb{Q}}\big((\varphi \to \varphi) \vee \varphi, f\big)$. ◀

Definition A.4 (Relevance Logic $\mathcal{L}_r$). We obtain a new logic $\mathcal{L}_r$ from $\mathcal{L}_\infty$ by executing the following modifications in the definition. Replace $[0,1]$ with the set $\mathbb{Q}$ of rational numbers everywhere. Define $mn_f(\neg\varphi) \stackrel{\text{def}}{=} -mn_f(\varphi)$. Redefine the meaning of "$\to$" as follows:

$$mn_f(\varphi \to \psi) \stackrel{\text{def}}{=} \begin{cases} \max\{-mn_f(\varphi), mn_f(\psi)\}, & \text{if } mn_f(\varphi) \le mn_f(\psi) \\ min\{mn_f(\varphi), -mn_f(\psi)\}, & \text{else.} \end{cases}$$

The rest is exactly as in Ex. A.3 (4).

Now, Relevance Logic is

$$\mathcal{L}_r = \langle F_r, M_r, mng_r, \models_r \rangle \,. \qquad ◀$$

We note that logic $\mathcal{L}_r$ is also called *R-Mingle (RM)* in the literature.

Exercises A.5. (1) Compare $\mathcal{L}_r$, $\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{L}_\infty$! Compare them with $\mathcal{L}_S$. What are the most striking differences?

(2) Prove that $\models_r (\varphi \to \varphi)$, and $\models_r (\varphi \vee \neg\varphi)$.

(3) Prove that $\not\models_r (p_0 \to (p_1 \vee \neg p_1))$. Compare with what we said about Relevance Logic in Ex. A.3 (4)!

(4) Prove that $(\models_r \varphi) \Longleftrightarrow [mng_{\mathbb{Q}}(\varphi, f) = mng_{\mathbb{Q}}(\varphi \to \varphi, f)$, for all $f \in M_{\mathbb{Q}}]$.

(5) Check what happens if we replace $\mathbb{Q}$ with $\mathbb{Z}$ (the set of integers) or with the interval $[-n, n]$ for some $n \in \omega$.

(6) Prove that in $\mathcal{L}_r$ we have

$$[f \models \varphi \text{ and } f \models \psi] \not\Longrightarrow mng_r(\varphi, f) = mng_r(\psi, f).$$

Compare with Def. 2.1.3!

(7) Prove that $\models_r (\varphi \to \psi) \to (\neg\varphi \vee \psi)$ but $\not\models_r (\neg\varphi \vee \psi) \to (\varphi \to \psi)$.

(8) Compare the $\{\wedge, \vee, \neg\}$-fragment of $\mathcal{L}_r$ with that of $\mathcal{L}_S$! (Prove e.g. that for φ of this fragment, $(\models_r \varphi \Rightarrow \models_S \varphi)$). ... Go on comparing!)

Next we define Partial Logics ($\mathcal{L}_P$). Partial logics are designed to express the fact that in certain situations, certain statements may be meaningless. For example, the statement "the integer 2 is of pink color" may be meaningless in certain situations. If φ is meaningless then so is $\neg\varphi$. Also, according to the Copenhagen interpretation of quantum mechanics, in certain situations certain statements are meaningless, e.g. asking for the exact location of a particle in a situation where the particle has only a probability distribution of locations is meaningless.

Definition A.6 (Partial Logic, $\mathcal{L}_P$). Connectives of $\mathcal{L}_P$ are: $\wedge$, $\vee$, $\neg$, N, where the new kind of formula $N(\varphi)$ intends to express that φ is either meaningless or false ("It is not the case that φ" or perhaps "It is not the fact that φ"). (N is a very strong negation.)

- The set of formulas F_P is obtained from F_S by adding the new unary connective N.
- The class M_P of models is

$$M_P \stackrel{\text{def}}{=} \left\{ f : f \in {}^P\{0,1,2\} \right\} .$$

  Here 0, 1, 2 are intended to denote the truthvalues "false", "true" and "undefined", respectively.

- If $2 \notin \{mng_P(\varphi, f), mng_P(\psi, f)\}$, then mng_P of $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi$ is defined as in the case of $\mathcal{L}_S$. Else (if 2 is one of the meanings) then mng_P of $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi$ is 2 (so all three are the same and they all are 2).

$$mng_P(N\varphi, f) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } mng_P(\varphi, f) = 1 \\ 1, & \text{otherwise.} \end{cases}$$

$$f \models_P \quad \text{iff} \quad mng_P(\varphi, f) = 1 \,.$$

With this, $\mathcal{L}_P = \langle F_P, M_P, mng_P, \models_P \rangle$ is defined.

$\mathcal{L}_P$ above is a quite important logic. It was introduced by Prior and was further investigated by I. Ruzsa (cf. e.g. [44]).

Exercises A.7. (1) Prove that $\mathcal{L}_P$ is a nice logic. (Hint: Use $\varepsilon_0(\varphi) \stackrel{\text{def}}{=} N(\varphi \wedge \neg\varphi)$, $\delta_0(\varphi) \stackrel{\text{def}}{=} \varphi$. Then use $\varphi\Delta_0\psi \stackrel{\text{def}}{=} N\neg(\varphi \leftrightarrow \psi) \wedge (u(\varphi) \leftrightarrow u(\psi))$, where $u(\varphi) \stackrel{\text{def}}{\iff} N(\varphi) \wedge N(\neg\varphi)$. Here $u(\varphi)$ means that φ is undefined [or meaningless]).

(2) Try to characterize $\mathsf{Alg}_{\models}(\mathcal{L}_P)$ and $\mathsf{Alg}_m(\mathcal{L}_P)$. How many non-isomorphic algebras are there in $\mathsf{Alg}_m(\mathcal{L}_P)$?

(3) Try to invent the partial version of our more sophisticated logics, e.g. of $\mathcal{L}_{S5}$ (or the others). (Warning: This might take too much time, because there are too many logics. So try one or two [if you are interested] and then try to develop an "intuition" that you probably could do the rest.) ◀

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