On generalizing the logic-approach to space-time towards general relativity: first steps[1]

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1 Introduction

In this and related work, we study relativity theory, or theory of space-time, as a theory of first-order logic. It is important for our approach that we work in the framework of (mathematical) logic and within that in (many-sorted) first-order logic (FOL). The reasons for the latter choice can be found in, e.g., [2, Appendix], [3], [22], [23].[2] The aims of our project are summarized in the introduction of [2] available on the Internet (cf. also [3], [1]), here we briefly mention only aims (i) and (ii) below; (i) to do work on the logical foundation of space-time theories, and (ii) to elaborate a logic based conceptual analysis of relativity theories. For both of these goals, we want to start out with the so-called observational (in the sense of, e.g., Reichenbach [19]) or "bottom-up"

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[2]This work is a self-contained part of a larger project, cf. e.g., [4], [2], [13], [3]. In the present introduction, "we" occasionally refers to the larger project and occasionally to the present paper only.

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versions of (kinematics of) relativity theories as opposed to the "monolithic", theoretically oriented "top-down" approaches. Of course, in due time we arrive at the theoretical versions, too, e.g., in [13], but by that time they will be well motivated, cf. e.g., [13], [14].

First we build up (the kinematics of) special relativity theory in FOL obtaining the finitely axiomatized FOL-theory Specrel. We put emphasis on making the axioms of Specrel streamlined, transparent, and intuitively convincing. Then we elaborate a conceptual analysis of special relativity, its variants, and its generalizations. This analysis is based on the FOL axiom system Specrel, on variants and fragments of Specrel and their generalizations. Among other things, we analyze Specrel both from the logical point of view (model theory, proof theory, "reverse mathematics" etc.) and from the physico-philosophical relativity theoretic point of view. Much of this is done in [2], [13], [1], [4]. As a natural continuation, we also experiment with generalizing Specrel in the direction of general relativity.

The first two steps in this generalization are (I) and (II) below. (I) We extend Specrel to accommodate accelerated observers, which, via Einstein's equivalence principle, enables us to discuss some features of gravity. E.g., the Twin Paradox and the Tower Paradox (gravity slows time down) become provable in the accelerated observers version Acc(Specrel) of Specrel, cf. e.g., [3]. (II) As a second step in this direction, we make Acc(Specrel) *local*, where local is understood in the sense of general relativity. We do this via the so-called method of localization which can be applied basically to any version of Specrel and Acc(Specrel). The localized versions of these theories are also built up in FOL (we make special efforts to ensure this) for methodological reasons mentioned earlier. Since localization turns out to be such a general procedure, we can denote the thus obtained theories as Loc(Specrel), Loc(Acc(Specrel)) etc.[3]

It is explained in the classic textbook [17, pp.163-5] on general relativity that by suitably combining accelerated observers and localization one can safely move towards general relativity by starting out from special relativity, cf. e.g., Box 6.1 on p.164 therein. This motivates our study of the FOL-theory Loc(Acc(Specrel)) and its variants. The investigation of Loc(Acc(Specrel)) is analogous with that of Specrel, i.e. after introducing the theory and proving theorems about its basic properties comes a fine-scale conceptual analysis

[3] So, Loc(-) can be regarded as some kind of a general "operator" applicable to theories which are variants of Specrel.

both from the logic point of view and from the relativity-theoretic point of view.

In the present paper we concentrate on illustrating the process of localization, i.e. step (II) towards general relativity mentioned above. To make the essential ideas stand out more, we concentrate on describing Loc(Specrel), since extending the localization procedure from Specrel to Acc(Specrel) goes the natural way. All the same, we would like to emphasize that the FOL-theory which brings us closer to having a FOL-based version of general relativity is Loc(Acc(Specrel)) and not Loc(Specrel) in itself. But if we keep this fact in mind, it is more useful to study the method and effects of localization first on the example of Loc(Specrel). For Acc(Specrel) and the definition of Loc(Acc(Specrel)) we refer to [3] available on the net. Besides the research school represented here, localization was used for moving towards general relativity in, e.g., Latzer [11], and Buseman [6].

Here, we introduce the theory Loc(Specrel) and prove some theorems about it. E.g., it turns out that already a small fragment of Loc(Specrel) proves distinguished predictions of relativity theory in the local setting. For lack of space the present paper gives only a small sample from the theory. More on Loc(Specrel) can be found in [14] where Loc(Specrel) is called partial domain relativity theory and is denoted as LocStd. Cf. also [3]. Here, a formal definition of Loc(Specrel) is given in §2 above Theorem 3, where Loc(Specrel) is denoted as **LocRel**. Its fragments and versions are denoted as $\mathbf{LocRel}^{-}$, $\mathbf{LocRel}_0^{-}$ etc.

In passing we note that the process of localization is related to that of relativization used in areas of logic related to algebraic logic, cf. e.g., [3], [5] or the volume [16], [18].

Intuitively, Loc(Specrel) is obtained from Specrel in two steps. These are: (A) We relax the condition in Specrel that all events "seen" by one observer are "seen" by the others. This is implemented by permitting observers not to put any event to points of their coordinate-systems too far from the origin (of the coordinate-system). I.e. we allow observers to use subsets of their coordinate-systems for coordinatizing events instead of using the whole coordinate-system. (B) Axioms of Specrel are rephrased in the local spirit (in the topological sense) which is something like the following: If Axi is an axiom of Specrel, then instead of Axi we say that for each point p of space-time (whatever this may mean) there is a small enough open neighborhood D of p such that Axi is true in D.

Latzer [11] pointed out a problem with (this kind of) localization of special relativity, as follows. In studying global theories like Specrel, one can rely on the so-called Alexandrov-Zeeman type theorems, e.g., in the style followed in the book of Goldblatt [9], or in [1], [4]. Because of their just mentioned usefulness, the Alexandrov-Zeeman theorems have been thoroughly generalized in various directions, cf. e.g., [3], [12], [10]. Latzer [11, p.237 lines 8-12, p.255 lines 5-8] writes that some kind of a generalization of the Alexandrov-Zeeman type theorems to the local approach to relativity would be needed for studying local versions of relativity, hence for moving towards general relativity. However, Lester [12, p.929] points out that the Alexandrov-Zeeman theorem does not generalize to the local setting. This fact slowed down progress with the logical analysis of local relativity theories. We address this problem in Theorems 1 and 2 way below. Namely, we prove two theorems in fragments of Loc(Specrel), which can be regarded as Alexandrov-Zeeman type results in the local setting. To illustrate their usefulness in analyzing local relativity, we state a theorem to the effect that a quite small fragment of Loc(Specrel) already proves the nonexistence of faster-than-light observers (NoFTL) in the local sense, cf. Theorems 3–5 in this connection.[4] This is proved via Theorems 1 and 2. In related work we also use our localized Alexandrov-Zeeman type results for establishing various distinguished predictions of local relativity, e.g., predicting the behavior of fast moving clocks, meter-rods, etc. We also indicate some of the global peculiarities of the localized theory, cf. [14].

2 The FOL-theory Loc(Specrel) of localized relativity

In this paper we will deal with kinematics of relativity, i.e. we will deal with motion of *bodies* (e.g., of *test-particles*). The motivation for our choice of vocabulary (for special relativity and its generalizations) is summarized as follows. We will represent motion as changing spatial location in time. To do so, we will have reference-frames for coordinatizing events and, for simplicity, we will associate reference-frames with special bodies which we will call *observers*. There will be another special kind of bodies which we will call *photons*. For coordinatizing events we will use an arbitrary *ordered field* in place of the field of the real numbers. Thus the elements of this field

[4]This can be interpreted as follows. In Theorems 1 and 2 we propose a kind of solution to the problem (mentioned, e.g., by Latzer) of extending the Alexandrov-Zeeman style approach to local versions of relativity theory. Then by Theorems 3–5 and ones in [14] we illustrate that this solution really works, in some sense.

will be the "*quantities*" which we will use for marking time and space.

Let us fix a natural number $n > 1$. n will be the number of space-time dimensions. In most works $n = 4$, i.e. one has 3 space dimensions and one time dimension.

Motivated by the above, our first-order language contains the following symbols:

- unary relation symbols $\mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{F}$ (for bodies, observers, photons, and quantities, i.e. elements of the field, respectively),
- binary function symbols $+, \cdot$ and a binary relation symbol $<$ (for the field-operations and ordering on F),
- a $2+n$-ary relation symbol W (for coordinatizing events, i.e. for the world-view relation).

We will read "$\mathsf{B}(x), \mathsf{Ob}(x), \mathsf{Ph}(x), \mathsf{F}(x)$" as "$x$ is a body", "x is an observer", "x is a photon", "x is a field-element", and we will read "$\mathsf{W}(x, y, z_1, z_2, \ldots, z_n)$" as "observer x sees (or observes) the body y at time z_1 at spatial location $(z_2, \ldots, z_n)$". This "seeing" or "observing" has nothing to do with seeing via photons or observing via experiments, it simply means that, according to x's coordinate-system or reference frame, y is present at coordinates $(z_1, \ldots, z_n)$.

The following axiom will *always* be assumed and will be part of every axiom system we propose, without mentioning.

AxFrame $\mathsf{Ob} \cup \mathsf{Ph} \subseteq \mathsf{B}$, $+$ and $\cdot$ are binary operations on F, $<$ is a binary relation on F, and $\langle \mathsf{F}, +, \cdot, < \rangle$ is a Euclidean ordered field, i.e. an ordered field in which positive elements have square roots.

In pure first-order logic the above axiom would look like $(\mathsf{Ob}(x) \vee \mathsf{Ph}(x)) \rightarrow \mathsf{B}(x)$ etc. We do not write out the purely first-order logic translations of our axioms since they are straightforward to obtain.

Let $\mathbf{M}$ be a model of **AxFrame**. Let $\mathbf{F} = \langle \mathsf{F}, +, \cdot \rangle$ denote the field reduct of $\mathbf{M}$. We will use the following notation and terminology:

$-, /, 0, 1$ are the usual field operations. ${}^n\mathsf{F}$ denotes the set of all n-tuples of elements of F. If a is an n-tuple, then we will assume that $a = \langle a_1, \ldots, a_n \rangle$, i.e. a_i denotes the i-th member of the n-tuple a (for $0 < i \leq n$). We will use the vector-space structure of ${}^n\mathsf{F}$. I.e. if $p, q \in {}^n\mathsf{F}$ and $\lambda \in \mathsf{F}$, then $p + q, -p, \lambda p \in {}^n\mathsf{F}$, and $\bar{0} = \langle 0, \dots, 0\rangle$ is the *null vector*. Let $p \in {}^n\mathsf{F}$. Then $p_t := p_1$ denotes the time component of p and $p_s := \langle 0, p_2, \dots, p_n\rangle$ denotes the space component of p. $|p| := p_1^2 + \ldots + p_n^2$ is the (square of the) length of p. The (square of the) *speed* of p is defined as $\mathsf{speed}(p) := |p_s|/p_t^2$ if $p_t \neq 0$, $\mathsf{speed}(p) := \infty$ otherwise. Here we require that $\infty \notin \mathsf{F}$. We extend the ordering $<$ on F to an ordering on $\mathsf{F} \cup \{\infty\}$ in the usual way, i.e. $(\forall x \in \mathsf{F})$ $x < \infty$.

Let $q, v \in {}^n\mathsf{F}, v \neq \bar{0}$. The (straight) *line* going through q and $q + v$ is $\{q + \lambda v : \lambda \in \mathsf{F}\}$. The set of lines is then

$$\mathsf{Lines} := \{\, \{\, q + \lambda v : \lambda \in \mathsf{F}\} : q, v \in {}^n\mathsf{F}, v \neq \bar{0}\,\}.$$

If ℓ is a subset of a line and has at least two elements, then

$$\mathsf{speed}(\ell) := \mathsf{speed}(p - q) \text{ for some (and then for all) } p, q \in \ell, p \neq q.$$

We say that a line ℓ is slower than $\lambda \in {}^+\mathsf{F}$ iff $\mathsf{speed}(\ell) < \lambda$.
$\|$ is the binary relation of *parallelism* on the set Lines, i.e.

$$\ell \parallel \ell' \qquad \Leftrightarrow \qquad (\exists p, q \in \ell)(\exists p', q' \in \ell')\ p - q = p' - q' \neq \bar{0}.$$

Coll is the ternary relation of *collinearity* on ${}^n\mathsf{F}$, i.e.

$$\mathsf{Coll}(p, q, r) \qquad \Leftrightarrow \qquad (\exists \ell \in \mathsf{Lines})\ \{p, q, r\} \subseteq \ell.$$

Let $q, u, v \in {}^n\mathsf{F}$, $\neg\mathsf{Coll}(q, q+u, q+v)$. The *plane* that contains $q, q+u, q+v$ is $\{\, q + \lambda u + \mu v \ :\ \lambda, \mu \in \mathsf{F}\,\}$. The set of planes is then

$$\mathsf{Planes} := \{\, \{\, q + \lambda u + \mu v \ :\ \lambda, \mu \in \mathsf{F}\,\} \ :\ q, u, v \in {}^n\mathsf{F}, \neg\mathsf{Coll}(q, q + u, q + v)\}.$$

${}^+\mathsf{F} := \{\lambda \in \mathsf{F} \ :\ \lambda > 0\}$ is the set of positive elements of F.

The (open) *ball* with center $p \in {}^n\mathsf{F}$ and radius $\varepsilon \in {}^+\mathsf{F}$ is

$$S(p, \varepsilon) := \{q \in {}^n\mathsf{F} \ :\ |p - q| < \varepsilon^2\}.$$

A set $N \subseteq {}^n\mathsf{F}$ is a *neighborhood* of $p \in {}^n\mathsf{F}$ iff $(\exists \varepsilon \in {}^+\mathsf{F})\ S(p, \varepsilon) \subseteq N$. A set $D \subseteq {}^n\mathsf{F}$ is *open* iff $(\forall p \in D)(\exists \varepsilon \in {}^+\mathsf{F})\ S(p, \varepsilon) \subseteq D$.

$\mathcal{A} := \langle {}^n\mathsf{F}, \mathsf{Coll}\rangle$ is the n-dimensional *affine structure* over the field $\mathbf{F}$. $\mathcal{A}$ can be extended to an n-dimensional *projective structure* $\mathcal{P} = \langle \mathsf{P}^n\mathsf{F}, \mathsf{PColl}\rangle$ over $\mathbf{F}$ in the usual way, i.e. as follows: The relation of parallelism $\|$ is an equivalence relation on the set Lines. For every $\ell \in \mathsf{Lines}$ let ℓ^∞ denote the equivalence class of ℓ under the relation $\|$. Intuitively ℓ^∞ is the point of line ℓ at infinity. For every $P \in \mathsf{Planes}$ let $P^\infty := \{\ell^\infty \ :\ \ell \in \mathsf{Lines},\ \ell \subseteq P\}$. Intuitively P^∞ is the line in plane P at infinity. The set of *points* of the projective structure is defined as

$$\mathsf{P}^n\mathsf{F} := {}^n\mathsf{F} \cup \{\ell^\infty \ :\ \ell \in \mathsf{Lines}\}.$$

The set of *(projective) lines* is defined as

$$\mathsf{PLines} := \{\, \ell \cup \{\ell^\infty\} \;:\; \ell \in \mathsf{Lines}\,\} \cup \{\, P^\infty \;:\; P \in \mathsf{Planes}\,\}.$$

Finally, the ternary relation PColl of *collinearity* on $\mathsf{P}^n\mathsf{F}$ is defined as

$$\mathsf{PColl}(a,b,c) \qquad \Leftrightarrow \qquad (\exists \ell \in \mathsf{PLines})\ \{a,b,c\} \subseteq \ell.$$

By the above, the *n-dimensional projective structure*

$$\mathcal{P} := \langle\, \mathsf{P}^n\mathsf{F},\ \mathsf{PColl}\,\rangle$$

over the field **F** has been defined. We note that the affine structure $\mathcal{A}$ is a strong sub-model of the projective structure $\mathcal{P}$.

An $\underline{\mathcal{A}\textit{-collineation}}$ is an automorphism of the affine structure $\mathcal{A}$. In other words, an $\mathcal{A}$-collineation is a permutation of ${}^n\mathsf{F}$ that takes lines onto lines. A $\underline{\mathcal{P}\textit{-collineation}}$ is an automorphism of the projective structure $\mathcal{P}$.

${}^n\mathsf{F}$ is the *coordinate-system* (of each observer) and we will refer to its elements as *coordinate-points*.

The *life-line*, or the *trace* of a body b in observer m's coordinate-system, or as seen by m, is the set of coordinate-points at which m sees b:

$$\mathsf{tr}_m(b) := \{\, p \in {}^n\mathsf{F} \;:\; \mathsf{W}(m,b,p)\,\}.$$

The set of bodies observer m sees at a given coordinate-point $p \in {}^n\mathsf{F}$ is the *event* happening for m at p:

$$\mathsf{ev}_m(p) := \{\, b \in \mathsf{B} \;:\; \mathsf{W}(m,b,p)\,\}.$$

The *coordinate-domain* of observer m is the set of the coordinate-points p where m sees non-empty events:

$$\mathsf{cd}(m) := \{\, p \in {}^n\mathsf{F} \;:\; \mathsf{ev}_m(p) \neq \emptyset\,\}.^5$$

The *world-view transformation* between the coordinate-systems of observers m and k is defined as:

$$\mathsf{f}_{mk} := \{\langle p,q\rangle \in {}^n\mathsf{F} \times {}^n\mathsf{F} \;:\; \mathsf{ev}_m(p) = \mathsf{ev}_k(q) \neq \emptyset\}.$$

Note that f_{mk} is a binary relation. f_{mk} will turn out to be an injective partial function assuming axiom **Ax∃Ob** below, cf. Proposition 1.

[5] $\mathsf{ev}_m(p) = \emptyset$ does not mean that space-time would be empty at point p (as seen by m). Instead, it means that observer m does not use point p for coordinatization. I.e. the world-view function $\mathsf{ev}_m : {}^n\mathsf{F} \longrightarrow \mathsf{Events}$ is partial. Part of the explanation of this meaning of $\mathsf{ev}_m(p) = \emptyset$ is that our bodies are only potential bodies. Hence $b \in \mathsf{ev}_m(p)$ means that, potentially, a body b could be present at point p for m. Hence $\mathsf{ev}_m(p) = \emptyset$ implies that nothing could be present, even in principle, at p for m.

If $R \subseteq A \times B$ is a binary relation, then $\mathsf{Dom}(R)$ and $\mathsf{Rng}(R)$ denote the *domain* and *range* of R, respectively, i.e. $\mathsf{Dom}(R) := \{a \in A \ : \ (\exists b \in B)\ \langle a, b\rangle \in R\}$ and $\mathsf{Rng}(R) := \{b \in B \ : \ (\exists a \in A)\ \langle a, b\rangle \in R\}$.

Now everything is ready to state further axioms.

AxLine The traces of observers and photons are subsets of lines, but they must be restrictions of lines to the coordinate-domain (or empty), i.e.

$$(\forall m \in \mathsf{Ob})(\forall k \in \mathsf{Ob} \cup \mathsf{Ph})(\exists \ell \in \mathsf{Lines})$$
$$(\, \mathsf{tr}_m(k) = \ell \cap \mathsf{cd}(m) \quad \text{or} \quad \mathsf{tr}_m(k) = \emptyset\,).$$

The above axiom motivates the definition: if $m \in \mathsf{Ob}$ and $\ell \in \mathsf{Lines}$, then ℓ is called an *observer line for* m iff $(\exists k \in \mathsf{Ob})\ \mathsf{tr}_m(k) = \ell \cap \mathsf{cd}(m)$; and ℓ is called a *photon line for* m iff $(\exists \mathsf{ph} \in \mathsf{Ph})\ \mathsf{tr}_m(\mathsf{ph}) = \ell \cap \mathsf{cd}(m)$.

Ax∃Ob Each point in the coordinate-domain has a neighborhood and a "speed threshold" λ such that each line slower than λ that intersects the neighborhood is an observer line, i.e.

$$(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{cd}(m))(\exists \varepsilon, \lambda \in {}^{+}\mathsf{F})(\forall \ell \in \mathsf{Lines})$$
$$\Big((\mathsf{speed}(\ell) < \lambda \ \wedge\ \ell \cap S(p,\varepsilon) \neq \emptyset) \quad \rightarrow \quad \ell \text{ is an observer line for } m\Big).$$

AxOpen $(\forall m, k \in \mathsf{Ob})(\mathsf{Dom}(\mathsf{f}_{mk})$ is an *open* subset of ${}^{n}\mathsf{F})$.

AxOpen implies that $\mathsf{cd}(m)$ is an open set for every observer m since $\mathsf{cd}(m) = \mathsf{Dom}(\mathsf{f}_{mm})$.

The theorem below says that, locally, the world-view transformations are $\mathcal{P}$-collineations in models of **AxLine**, **Ax∃Ob**, **AxOpen**. This, we think, is a rather strong Alexandrov-Zeeman type theorem. For the Alexandrov-Zeeman theorem cf. Goldblatt [9] or Lester [12] or [4] in the present volume. It says that any bijection from ${}^{n}\mathsf{F}$ to ${}^{n}\mathsf{F}$ that takes lines of speed 1 onto lines of speed 1 is an $\mathcal{A}$-collineation if we assume that $\mathbf{F}$ is the field of reals and $n = 4$. Lester shows in [12, p.929] that this statement does not hold for partial injections in place of bijections.[6]

[6] The theorem on p.929 of Lester [12] and the discussion preceding it are strongly relevant to our Theorems 1 and 2 herein. The general relativistic generalization of Zeeman's theorem in Malament [15] is also relevant. Cf. also Guts [10, §26].

As usual, functions are binary relations. Thus $\mathcal{P}$-collineations are binary relations. We say that the binary relations R and R' *agree on a set* D iff $R \cap (D \times \mathsf{Rng}(R)) = R' \cap (D \times \mathsf{Rng}(R'))$.

Theorem 1. *Assume* **AxLine**, **Ax∃Ob**, **AxOpen**. *Then (i), (ii) below hold.*

(i) For every $m, k \in \mathsf{Ob}$ and $p \in \mathsf{Dom}(\mathsf{f}_{mk})$ there is a unique $\mathcal{P}$-collineation, denoted by C^p_{mk}, that agrees with f_{mk} on some neighborhood of p.

(ii) For every $m, k \in \mathsf{Ob}$ and $\langle p, q\rangle \in \mathsf{f}_{mk}$ the $\mathcal{P}$-collineations C^p_{mk} and C^q_{km} are inverses of each other.

The proof of Theorem 1 is in §3.

By item (i) of the above theorem, the world-view transformations preserve Coll and $\neg\mathsf{Coll}$ locally in models of **AxLine**, **Ax∃Ob**, **AxOpen**.

Conjecture 1. *Theorem 1 above remains true if we omit the assumption that our ordered field is Euclidean. Furthermore, the proof given in §3 works for the non-Euclidean case if we use cubes instead of balls in the proof.*

Question 1 *Does Theorem 1 above remain true if we replace the assumption* **Ax∃Ob** *by the weaker* **Ax∃Ob$^-$** *below?*

Ax∃Ob$^-$ For each point p in the coordinate-domain there is a "speed threshold" λ such that each line slower than λ that passes through point p is an observer line, i.e.

$$(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{cd}(m))(\exists \lambda \in {}^{+}\mathsf{F})(\forall \ell \in \mathsf{Lines})$$
$$\Big((\mathsf{speed}(\ell) < \lambda \ \wedge\ p \in \ell) \quad \rightarrow \quad \ell \text{ is an observer line for } m\Big).$$

If we replace **AxOpen** by the much stronger **AxFull** below, then **Ax∃Ob** can be replaced by **Ax∃Ob$^-$** in Theorem 1. Beyond that, the world-view transformations will turn out to be $\mathcal{A}$-collineations in models of **AxLine**, **Ax∃Ob$^-$**, **AxFull**.

AxFull below is a typical example of a potential assumption which does *not* have the status of an axiom in the present work. It is a typical postulate which distinguishes special relativity from our more general theories studied herein and in [3]. We formulate **AxFull** to make it sure that we do not assume it in our generalized theories, not even by chance or even implicitly.

AxFull $(\forall m, k \in \mathsf{Ob})\ \mathsf{Dom}(\mathsf{f}_{mk}) = {}^n\mathsf{F}$.

Roughly speaking, **AxFull** says that every observer sees all the events and sees something everywhere in his coordinate-system. From the point of view of general relativity theory, **AxFull** is a too strong assumption, therefore we will not include **AxFull** in our localized relativity theories.

In Theorem 1 above, $\mathcal{P}$-collineation cannot be replaced by $\mathcal{A}$-collineation, see [14]. We get $\mathcal{A}$-collineation, however, if we add axioms about photons. Notice that so far, nothing has been used about photons. We will assume that the photon traces form an "upright" cone, called *light-cone*, at each point, however the angle (or "openness" or "width") of the light-cone may differ from point to point. We are going to formalize this, the result of which will be axiom **AxPh** below. We note that, assuming **AxLine**, **AxPh** is equivalent with $(\star)$ below.

$(\star)$ $(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{cd}(m))(\exists \lambda \in {}^+\mathsf{F})$
$(\forall \ell \in \mathsf{Lines})\,[\, p \in \ell \quad \to \quad (\ell \text{ is a photon line for } m \ \leftrightarrow\ \mathsf{speed}(\ell) = \lambda)\,]$.

Therefore we could have stated Theorem 2 below here without introducing any further definitions and axioms.

The set of *(spatial) directions* dir is defined as

$$\mathsf{dir} := \{\, d \in {}^n\mathsf{F} \ :\ d_t = 0,\ |d| = 1 \}\,.$$

Assume $m \in \mathsf{Ob}$, $b \in \mathsf{B}$, $d \in \mathsf{dir}$. We say that b *moves in direction* d *as seen by* m iff $\quad (\forall p, q \in \mathsf{tr}_m(b))(\exists \lambda \in \mathsf{F})\ [p_s - q_s = \lambda d \ \wedge\ (p_t > q_t \ \to\ \lambda \geq 0)]$.

The *speed* of a body b as seen by an observer m is $\mathsf{speed}_m(b) := \mathsf{speed}(\mathsf{tr}_m(b))$ if $\mathsf{tr}_m(b)$ is a subset of a line and it has at least two elements, otherwise $\mathsf{speed}_m(b)$ is undefined. Note that $\mathsf{speed}_m(b) = \infty$ is possible. Furthermore, assuming **AxLine** and **AxOpen**, for any $h \in \mathsf{Ob} \cup \mathsf{Ph}$, $\mathsf{speed}_m(h)$ is defined or $\mathsf{tr}_m(h) = \emptyset$. Whenever we use $\mathsf{speed}_m(b)$ in an axiom, we will assume that the axiom states the existence of $\mathsf{speed}_m(b)$, too. Cf. e.g., **AxIstr** below.

Assume $m \in \mathsf{Ob}$, $b \in \mathsf{B}$ and $\mathsf{speed}_m(b)$ is defined. We note the following. If $\mathsf{speed}_m(b) \in {}^+\mathsf{F}$, then b moves in exactly one direction; if $\mathsf{speed}_m(b) = \infty$, then b moves in exactly two directions, i.e. b moves both in d and $-d$ for some direction d; and if $\mathsf{speed}_m(b) = 0$, then b moves in every direction as seen by m.

Ax∃Ph From any point $p \in \mathsf{cd}(m)$ in any direction there is a photon moving in that direction, i.e.

$$(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{cd}(m))(\forall d \in \mathsf{dir})(\exists \mathsf{ph} \in \mathsf{Ph})$$
$$\Big(p \in \mathsf{tr}_m(\mathsf{ph}) \ \wedge\ (\mathsf{ph} \text{ moves in direction } d \text{ as seen by } m)\Big).$$

AxIstr below abbreviates Axiom of Isotropy.

AxIstr The speed of light is direction-independent at each point $p \in \mathsf{cd}(m)$, i.e.

$$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph}, \mathsf{ph'} \in \mathsf{Ph})$$
$$(\mathsf{tr}_m(\mathsf{ph}) \cap \mathsf{tr}_m(\mathsf{ph'}) \neq \emptyset \quad \to \quad \mathsf{speed}_m(\mathsf{ph}) = \mathsf{speed}_m(\mathsf{ph'})).$$

AxFin The speed of each photon is nonzero and finite, i.e.

$$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph})\,(0 < \mathsf{speed}_m(\mathsf{ph}) < \infty \ \text{ or } \ \mathsf{tr}_m(\mathsf{ph}) = \emptyset).$$

AxPh := Ax∃Ph ∧ AxIstr ∧ AxFin.

In effect, the photon traces that cross a given $p \in \mathsf{cd}(m)$ show an "upright" cone-like shape, called *light-cone*. Notice that the speed of light—the angle of the light-cone—may differ from point to point.

We note that assuming **AxLine**, **AxOpen**, **AxPh**, the speed of light is constant locally, i.e. $(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{cd}(m))(\exists \varepsilon, \lambda \in {}^{+}\mathsf{F})(\forall \mathsf{ph} \in \mathsf{Ph})$
$$(\mathsf{tr}_m(\mathsf{ph}) \cap S(p,\varepsilon) \neq \emptyset \quad \to \quad \mathsf{speed}_m(\mathsf{ph}) = \lambda).$$

The theorem below says that, locally, the world-view transformations are $\mathcal{A}$-collineations in models of **AxLine**, **Ax∃Ob**, **AxOpen**, **AxPh**.

Theorem 2. *Assume* **AxLine**, **Ax∃Ob**, **AxOpen**, **AxPh**. *Then for every* $m, k \in \mathsf{Ob}$ *and* $p \in \mathsf{Dom}(\mathsf{f}_{mk})$ *there is a unique* $\mathcal{A}$*-collineation that agrees with* f_{mk} *on some neighborhood of* p.

The proof of Theorem 2 is in §3.

By the above theorem, the f_{mk}'s preserve parallelism, Coll and ¬Coll locally under certain assumptions.

If, in Theorem 2, $n > 2$ and we replace the assumption **AxOpen** by the much stronger **AxFull**, the assumption **Ax∃Ob** becomes superfluous. Moreover, the world-view transformations are $\mathcal{A}$-collineations in models of

AxLine, **AxFull**, **AxPh** if $n > 2$ by the proof of the Alexandrov-Zeeman theorem. Despite of this fact, the assumption **Ax∃Ob** cannot be omitted from Theorem 2 even if $n > 2$ is assumed. This is so because the Alexandrov-Zeeman theorem does not generalize to the local approach pursued herein, as it is shown in Lester [12, p.929].

Question 2 *Does Theorem 2 above remain true if we replace the assumption* **Ax∃Ob** *by the much weaker* **Ax∃Ob^{--}** *below?*

Ax∃Ob^{--} Each line of speed 0 that intersects the coordinate-domain is an observer line, i.e.

$$(\forall m \in \mathsf{Ob})(\forall \ell \in \mathsf{Lines})$$
$$\Big((\mathsf{speed}(\ell) = 0 \ \wedge\ \ell \cap \mathsf{cd}(m) \neq \emptyset) \rightarrow\ \ell \text{ is an observer line for } m\Big).$$

We note that, in Theorem 1, **Ax∃Ob** cannot be replaced by **Ax∃Ob^{--}**. We conjecture that the assumption **Ax∃Ob** can be replaced by **Ax∃Ob^{-}** in Theorem 2.

Finally, we are going to state theorems concerning faster than light observers. To this end we introduce further axioms.

AxSelf Observers can see themselves only on the time-axis, i.e.

$$(\forall m \in \mathsf{Ob})\ \mathsf{tr}_m(m) \subseteq \{\langle t, 0, \dots, 0\rangle \ :\ t \in \mathsf{F}\}.$$

There may be points on the time-axis where an observer can see nothing. Intuitively, such a point may be after the "Big Crunch"; or for an observer falling into a Schwarzschild black hole, it may be the point (measured by his own clock, i.e. his proper time) where his life-line intersects the singularity.

Assume $k, h \in \mathsf{Ob}$. Then we say that k is a *brother* of h iff $(\forall m \in \mathsf{Ob})\ \mathsf{tr}_m(k) = \mathsf{tr}_m(h)$.

AxEvent If m sees an event happening to k, some brother of k sees it, too,

$$(\forall m, k \in \mathsf{Ob})(\forall p \in \mathsf{tr}_m(k))(\exists h \in \mathsf{Ob})$$
$$[h \text{ is a brother of } k \text{ and } p \in \mathsf{Dom}(\mathsf{f}_{mh})].$$

Intuitive motivation for **AxEvent** above: Consider the life-line of Earth in general relativity. It is an infinitely long spiral. Therefore we cannot approximate the world-view of Earth by a single, long *inertial* frame (such does not exist).[7] On the other hand, we can hope for approximating the world-view of Earth by an infinite sequence of relatively small (hence also "short") inertial frames. Formally, this amounts to decomposing Earth to infinitely many observers whose body part is the same, namely Earth, but whose coordinate-domains correspond to different bounded pieces of Earth's history, so to speak. These versions "... Earth_{-2}, Earth_{-1}, Earth_0, Earth_1, Earth_2,..." of Earth will be brothers in our sense with different, small, coordinate-domains. The union of these domains covers the whole life-line of Earth. This is why in our **AxEvent** above we had to talk about some brother h of k instead of k itself.

Our axiom system **LocRel**, roughly speaking, consists of all axioms introduced so far, except **AxFull**. We note that **LocRel** is a concretely specified version of the axiom system Loc(Specrel) promised in the introduction.[8] **LocRel** excludes faster than light observers if $n > 2$ by Theorem 3 below.

**LocRel := {AxLine, Ax∃Ob, AxOpen,
Ax∃Ph, AxIstr, AxFin, AxSelf, AxEvent}.**

FTL abbreviates "faster than light". Let $k, m \in \mathsf{Ob}$. We call k FTL w.r.t. m iff there is a $\mathsf{ph} \in \mathsf{Ph}$ such that k and ph move in the same direction as seen by m, they meet, i.e. $\mathsf{tr}_m(k) \cap \mathsf{tr}_m(\mathsf{ph}) \neq \emptyset$, and $\mathsf{speed}_m(k) > \mathsf{speed}_m(\mathsf{ph})$. **noFTL** abbreviates the formula saying that no observer k can move *faster than light* relative to any other observer, i.e. it abbreviates the formula $\neg(\exists m, k \in \mathsf{Ob})[k \text{ FTL w.r.t. } m]$.

Theorem 3. **LocRel** $\setminus$ **{AxFin}** $\models$ **noFTL** *if* $n > 2$.

The proof of Theorem 3 is in §3.

[7] In general relativity, it is the so-called local inertial frames (LIF's) that correspond, roughly, to the world-views of inertial observers in our present local version of relativity, cf. Rindler [20] for LIF's and their such role.

[8] To be precise, **LocRel** is a streamlined, slightly generalized version of the result Loc(Specrel) of applying the localization procedure Loc(-), described in the introduction, to Specrel mechanically. As illustrated in [2], general procedures like Loc(-) are always meant to be applied in this way: we first apply the procedure Loc(-) "mechanically" and then streamline the thus obtained theory.

For $n = 2$, FTL observers do become possible even in models of axiom system $\mathsf{Specrel}$ mentioned in the introduction.

We are going to replace axiom **AxIstr** in **LocRel** by the much weaker **AxP1** below.

AxP1 The speed of light is unique and well-defined in each direction at each point $p \in \mathsf{cd}(m)$. In particular, it does not depend on the movement of the source: photons are unlike bullets. Basically, this is the first-order logic formalization of Friedman's principle (P1) in [8, p.159],

$$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph}, \mathsf{ph'} \in \mathsf{Ph})$$
$$\Big(\mathsf{ph} \text{ and } \mathsf{ph'} \text{ move in the same direction as seen by } m \quad \rightarrow$$
$$(\mathsf{speed}_m(\mathsf{ph}) = \mathsf{speed}_m(\mathsf{ph'}) \text{ or } \mathsf{tr}_m(\mathsf{ph}) \cap \mathsf{tr}_m(\mathsf{ph'}) = \emptyset)\Big).$$

Let the axiom system $\mathbf{LocRel^-}$ be obtained from **LocRel** by replacing **AxIstr** by **AxP1**, i.e.

$$\mathbf{LocRel^- := \{AxLine, Ax\exists Ob, AxOpen,}$$
$$\mathbf{Ax\exists Ph, AxP1, AxFin, AxSelf, AxEvent\}.}$$

Question 3 *Assume $n > 2$. Does* $\mathbf{LocRel^-} \models \mathbf{noFTL}$ *hold?*

Let $\mathbf{LocRel^{--}}$ be obtained from $\mathbf{LocRel^-}$ by replacing $\mathbf{Ax\exists Ob}$ by $\mathbf{Ax\exists Ob^{--}}$. The following theorem is due to Gergely Székely [21].

Theorem 4. $\mathbf{LocRel^{--}} \cup \{\mathbf{AxFull}\} \not\models \mathbf{noFTL}$ *if* $n \in \{3, 4\}$.

We will weaken-and-strengthen $\mathbf{Ax\exists Ob}$ in $\mathbf{LocRel^-}$ to requiring that the observer-traces "fill" the light-cones. The thus obtained axiom system will exclude FTL observers if $n > 2$.

AxOb There are observers on lines which are slower than light, i.e.

$$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph}, p \in \mathsf{tr}_m(\mathsf{ph}))(\forall 0 \le \lambda < \mathsf{speed}_m(\mathsf{ph}))(\exists k \in \mathsf{Ob})$$
$$[p \in \mathsf{tr}_m(k),\ \mathsf{speed}_m(k) = \lambda, \text{ and } \mathsf{ph}, k \text{ move in the same direction as seen by } m].$$

Let $\mathbf{LocRel_0^-}$ be obtained from $\mathbf{LocRel^-}$ by replacing $\mathbf{Ax\exists Ob}$ by **AxOb**, i.e.

$$\mathbf{LocRel_0^- := \{AxLine, AxOb, AxOpen,}$$
$$\mathbf{Ax\exists Ph, AxP1, AxFin, AxSelf, AxEvent\}.}$$

Theorem 5. $\mathbf{LocRel_0^-} \models \mathbf{noFTL}$ *if* $n > 2$.

The proof of Theorem 5 is in §3.

The assumption **AxFin** cannot be omitted from $\mathbf{LocRel_0^-}$ in the above theorem.

3 Proofs

The proof of Theorem 1 is based on Desargues' theorem and on Propositions 1, 2 below.

$f : A \longrightarrow B$ denotes that f is a function from A to B, i.e. $\mathsf{Dom}(f) = A$ and $\mathsf{Rng}(f) \subseteq B$.

$f : A \overset{\circ}{\longrightarrow} B$ denotes that f is a *partial function* from A to B; this means that f is a function $\mathsf{Dom}(f) \subseteq A$ and $\mathsf{Rng}(f) \subseteq B$.

Proposition 1. *Assume* $\mathbf{Ax\exists Ob^-}$.

Then for every $m, k \in \mathsf{Ob}$, $\mathsf{f}_{mk} : {}^n\mathsf{F} \overset{\circ}{\longrightarrow} {}^n\mathsf{F}$ *is an injective partial function.*

Proof: Assume the assumptions. Due to the definition of the world-view transformation, it is enough to prove that for every $m \in \mathsf{Ob}$ and distinct $p, q \in \mathsf{cd}(m)$, $\mathsf{ev}_m(p) \neq \mathsf{ev}_m(q)$. Let $m \in \mathsf{Ob}$ and $p, q \in \mathsf{cd}(m)$, $p \neq q$. Let $\ell \in \mathsf{Lines}$ and $k \in \mathsf{Ob}$ be such that $p \in \ell$, $q \notin \ell$, and $\mathsf{tr}_m(k) = \ell \cap \mathsf{cd}(m)$. They exist by $\mathbf{Ax\exists Ob^-}$. Now, $k \in \mathsf{ev}_m(p)$ but $k \notin \mathsf{ev}_m(q)$. Thus $\mathsf{ev}_m(p) \neq \mathsf{ev}_m(q)$.

QED (Prop.1)

In the remaining part of the present paper we use the following notation and definitions.

- If $a, b \in {}^n\mathsf{F}$ with $a \neq b$, then ab denotes the unique element of Lines that contains a and b.

- If $a, b \in \mathsf{P}^n\mathsf{F}$ with $a \neq b$, then $\mathsf{P}ab$ denotes the unique element of PLines that contains a and b.

- Bw is the ternary relation of *strict betweenness* on ${}^n\mathsf{F}$, i.e. $\mathsf{Bw}(p, q, r)$ iff p, q, r are distinct collinear points and q is between p and r. This can be formalized as
  $\mathsf{Bw}(p, q, r) \quad \Leftrightarrow \quad (\exists \lambda \in {}^+\mathsf{F})\,(q = p + \lambda(r - p) \ \wedge\ \lambda < 1)$.

- If $a, b \in {}^n\mathsf{F}$, then $[a, b]$ denotes the closed segment determined by a and b, i.e. $[a, b] := \{\, c \in {}^n\mathsf{F} \;:\; \mathsf{Bw}(a, c, b) \;\vee\; c \in \{a, b\} \,\}$.

- Points $p, q, r, s \in {}^n\mathsf{F}$ are *coplanar* iff $(\exists P \in \mathsf{Planes})\, p, q, r, s \in P$.

- $P \in \mathsf{Planes}$ is a *vertical plane* iff $(\exists \ell \in \mathsf{Lines})(\ell \subseteq P \;\wedge\; \mathsf{speed}(\ell) = 0)$.

Next we recall Desargues' theorem from the literature, cf. e.g., Goldblatt [9]. To do so we need the following definitions:

Consider the projective structure $\mathcal{P} = \langle \mathsf{P}^n\mathsf{F}, \mathsf{PColl} \rangle$. A *triangle* is a triple of non-collinear points from $\mathsf{P}^n\mathsf{F}$. These points are the *vertices*, and the (projective) lines connecting two of the vertices are the *sides* of the triangle.

Triangles a', b', c' and a'', b'', c'' are *centrally perspective* iff there is $p \in \mathsf{P}^n\mathsf{F}$ such that $\mathsf{PColl}(p, a', a'')$, $\mathsf{PColl}(p, b', b'')$ and $\mathsf{PColl}(p, c', c'')$, see Fig.1. Triangles a', b', c' and a'', b'', c'' are *axially perspective* iff there are $a, b, c \in$

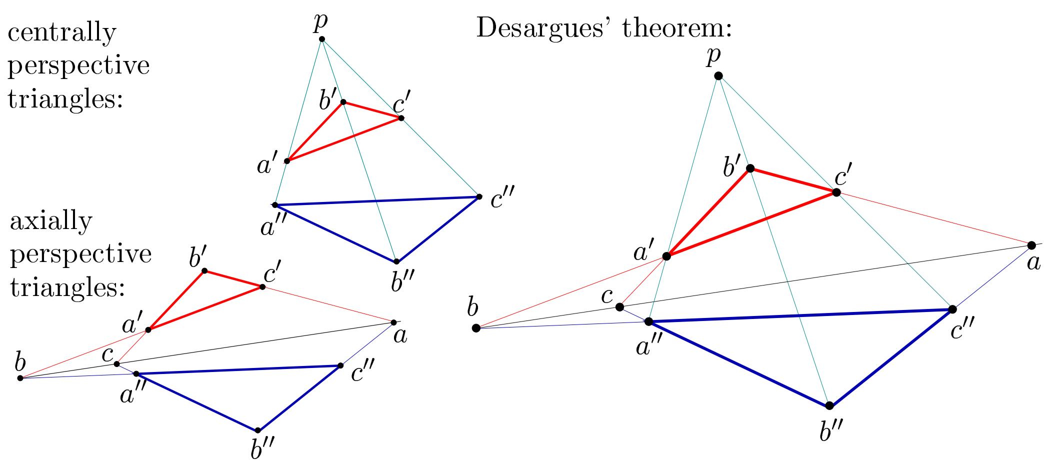

Figure 1: Desargues' theorem

$\mathsf{P}^n\mathsf{F}$ such that $\mathsf{PColl}(a, b, c)$, $\mathsf{PColl}(a, b', c')$, $\mathsf{PColl}(a, b'', c'')$, $\mathsf{PColl}(b, a', c')$, $\mathsf{PColl}(b, a'', c'')$, $\mathsf{PColl}(c, a', b')$, and $\mathsf{PColl}(c, a'', b'')$, see Fig.1.

Desargues' Theorem *Two triangles are centrally perspective if and only if they are axially perspective. Cf. Fig.1.*

<u>*Definition:*</u> Let $f : A \overset{\circ}{\longrightarrow} A$ be a partial function and let R be a ternary relation on A. We say that f *preserves* R *on a set* H iff $H \subseteq \mathsf{Dom}(f)$ and $(\forall x, y, z \in H)[R(x, y, z) \ \rightarrow \ R(f(x), f(y), f(z))]$. Furthermore, f *preserves* R (or f is *R-preserving*) iff f preserves R on $\mathsf{Dom}(f)$.

Proposition 2. *Let* $f : {}^n\mathsf{F} \overset{\circ}{\longrightarrow} {}^n\mathsf{F}$ *be a partial function. Assume* f *preserves* Coll *and* $\neg\mathsf{Coll}$*, and* $\mathsf{Dom}(f)$ *is a ball.*

Then there is a unique PColl*-preserving function* $g : \mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ *extending* f $(f \subseteq g)$.[9] *Furthermore, this unique* g *is injective.*

Question Does g in Prop.2 above preserve $\neg\mathsf{PColl}$?

Proof of Prop.2: Assume $f : {}^n\mathsf{F} \overset{\circ}{\longrightarrow} {}^n\mathsf{F}$ satisfies the assumptions.

Let $\mathsf{L} := \{\, \ell \in \mathsf{PLines} \ : \ \ell \cap \mathsf{Dom}(f) \neq \emptyset \,\}$. For every $\ell \in \mathsf{L}$ there is a unique element of PLines that contains the f-image of ℓ. We will denote this unique element of PLines by $f(\ell)$.

<u>*Definition:*</u> Lines ℓ_1, ℓ_2, ℓ_3 are *concurrent* iff $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$.

<u>*Claim:*</u> For any distinct and concurrent $\ell_1, \ell_2, \ell_3 \in \mathsf{L}$, the lines $f(\ell_1), f(\ell_2), f(\ell_3)$ are distinct and concurrent.

<u>*Proof:*</u> Assume $\ell_1, \ell_2, \ell_3 \in \mathsf{L}$ are distinct and concurrent. Since f preserves $\neg\mathsf{Coll}$, the lines $f(\ell_1), f(\ell_2), f(\ell_3)$ are distinct. It remains to prove that they are concurrent. We will prove this by Desargues' theorem.

Let $a', a'' \in \ell_1 \cap \mathsf{Dom}(f)$, $b', b'' \in \ell_2 \cap \mathsf{Dom}(f)$, $c', c'' \in \ell_3 \cap \mathsf{Dom}(f)$ be distinct points such that a', b', c' and a'', b'', c'' are triangles and the points of intersection of the corresponding sides of these triangles are in $\mathsf{Dom}(f)$, see Fig.2. It is explained in the caption of Fig.2 why such triangles exist. These triangles are centrally perspective. Thus, by Desargues' theorem, they are axially perspective, i.e. the points of intersection of the corresponding sides are collinear. Since f preserves Coll and $\neg\mathsf{Coll}$, $f(a'), f(b'), f(c')$ and $f(a''), f(b''), f(c'')$ are axially perspective triangles. But then, by Desargues' theorem, they are centrally perspective. Thus the lines $f(\ell_1), f(\ell_2)$ and $f(\ell_3)$ are concurrent. QED (Claim)

We are going to define a function $g : \mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$. Let $p \in \mathsf{P}^n\mathsf{F}$. By the above claim, there is a unique $p' \in \mathsf{P}^n\mathsf{F}$ such that $(\forall \ell \in \mathsf{L})\,(p \in \ell \ \rightarrow \ p' \in$

[9] In particular, $\mathsf{Dom}(g) = \mathsf{P}^n\mathsf{F}$.

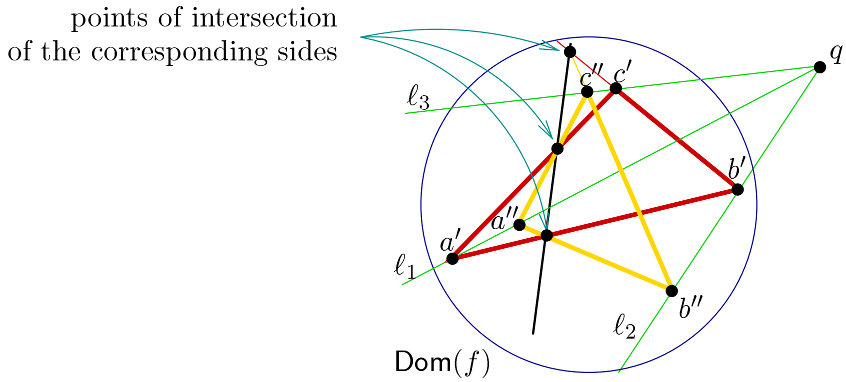

Figure 2: Let $q \in \ell_1 \cap \ell_2 \cap \ell_3$. Choose triangle a', b', c' arbitrarily. Choose a'', b'' such that $\mathsf{Bw}(a', a'', q)$ and $\mathsf{Bw}(b'', b', q)$. Then $a'b' \cap a''b''$ is a point in $\mathsf{Dom}(f)$. Choose c'' so "close" to c' that $a'c' \cap a''c''$ and $b'c' \cap b''c''$ are points in $\mathsf{Dom}(f)$.

$f(\ell))$. We define $g(p)$ to be this unique p'. Clearly, g extends f. Note that g is the unique $\mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ function with the property

$$(\forall \ell \in \mathsf{L})\,(p \in \ell \;\rightarrow\; g(p) \in f(\ell)). \tag{1}$$

If $g' : \mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ is a PColl-preserving function extending f, then g' satisfies (1) above, hence $g' = g$.

It remains to prove that g preserves PColl and that g is injective.

To prove that g is injective let $a, b \in \mathsf{P}^n\mathsf{F}$ be distinct points. Since g extends f and f preserves $\neg\mathsf{Coll}$, g is injective on $\mathsf{Dom}(f)$. Thus, there is $c \in \mathsf{Dom}(f)$ such that $g(a) \neq g(c) \neq g(b)$. Fix such a c. By (1), $g(a), g(c) \in f(\mathsf{P}ac)$ and $g(b), g(c) \in f(\mathsf{P}bc)$. But $f(\mathsf{P}ac), f(\mathsf{P}bc)$ are distinct because $\mathsf{P}ac, \mathsf{P}bc$ were such and f preserves $\neg\mathsf{Coll}$. Hence $g(a) \neq g(b)$.

We will use Desargues' theorem to prove that g preserves PColl. By (1), g preserves collinearity on elements of L, i.e. for any $\ell \in \mathsf{L}$ and $a, b, c \in \ell$, $\mathsf{PColl}(g(a), g(b), g(c))$.

To prove that g preserves PColl, let $a, b, c \in \mathsf{P}^n\mathsf{F}$ be such that $\mathsf{PColl}(a, b, c)$. We can assume that a, b, c are distinct. Let $a', b', c' \in \mathsf{Dom}(f)$ and $a'', b'', c'' \in \mathsf{Dom}(f)$ be triangles such that the corresponding sides meet in a, b, c, respectively, see Fig.3. It is explained in the caption of Fig.3 why such triangles exist. The two triangles are axially perspective. By Desargues' theorem, they are centrally perspective. Furthermore, the lines connecting the corresponding vertices are in L. Therefore, since g preserves $\neg\mathsf{Coll}$ on $\mathsf{Dom}(f)$ and

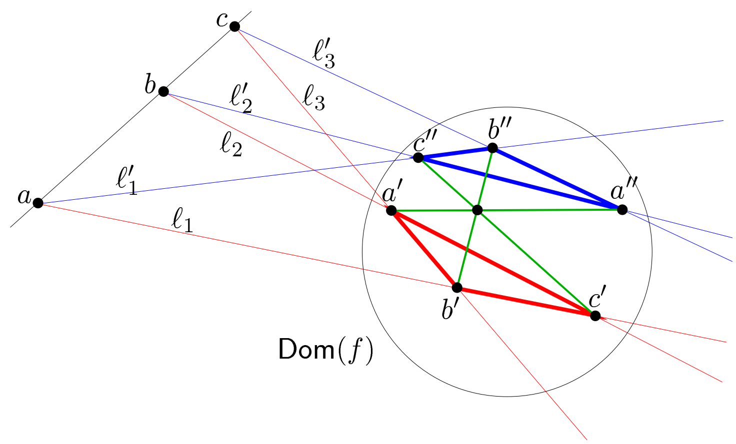

Figure 3: Let $\ell_1, \ell_2, \ell_3 \in \mathsf{L}$ be non-concurrent lines such that $a \in \ell_1,\ b \in \ell_2,\ c \in \ell_3$ and pairwise they meet in $\mathsf{Dom}(f)$. Let $c' \in \ell_1 \cap \ell_2,\ a' \in \ell_2 \cap \ell_3,\ b' \in \ell_1 \cap \ell_3$. Then a', b', c' is a triangle in $\mathsf{Dom}(f)$. The triangle a'', b'', c'' in $\mathsf{Dom}(f)$ is obtained analogously by using $\ell_1', \ell_2', \ell_3'$ distinct from ℓ_1, ℓ_2, ℓ_3.

g preserves PColl on elements of L, $g(a'), g(b'), g(c')$ and $g(a''), g(b''), g(c'')$ are centrally perspectivive triangles and the corresponding sides meet in $g(a), g(b), g(c)$, respectively. But then, by Desargues' theorem, the two triangles are axially perspective, which means that $\mathsf{PColl}(g(a), g(b), g(c))$.

QED (Prop.2)

Proof of Theorem 1:

Assume **AxLine**, **Ax∃Ob**, **AxOpen**. Recall that the world-view transformations (f_{mk}) are injective partial functions by Proposition 1. Furthermore, for every $m, k \in \mathsf{Ob}$, f_{mk} and f_{km} are inverses of each other. We will use these facts tacitly throughout the present proof.

Notation: Assume $m, k \in \mathsf{Ob}$. Then for every $a \in \mathsf{f}_{mk}$, a_m denotes the first component of a, while a_k denotes the second component of a, i.e. $a = \langle a_m, a_k \rangle$. Furthermore, if $a_m \in \mathsf{Dom}(\mathsf{f}_{mk})$, then a_k denotes $\mathsf{f}_{mk}(a_m)$ and if $a_k \in \mathsf{Rng}(\mathsf{f}_{mk}) = \mathsf{Dom}(\mathsf{f}_{km})$, then a_m denotes $\mathsf{f}_{km}(a_k)$.

Claim 1: Assume $m, k \in \mathsf{Ob}$ and $a, b \in \mathsf{f}_{mk}$, $a \neq b$. Then (i), (ii) below hold.

(i) $(a_m b_m$ is an observer line for $m) \Leftrightarrow (a_k b_k$ is an observer line for $k)$.

(ii) f_{mk} preserves Coll and $\neg\mathsf{Coll}$ between three points if the line determined by two of the points is an observer line. Formally: Assume $a_m b_m$ is an observer line for m. Then for every $c \in \mathsf{f}_{mk}$,
$\mathsf{Coll}(a_m, b_m, c_m) \Leftrightarrow \mathsf{Coll}(a_k, b_k, c_k)$, or equivalently $c_m \in a_m b_m \Leftrightarrow c_k \in a_k b_k$.

We omit the easy *proof.*

Claim 2: The world-view transformations preserve Coll locally, i.e. for every $m, k \in \mathsf{Ob}$ and $p \in \mathsf{Dom}(\mathsf{f}_{mk})$, there is a ball S with center p such that f_{mk} preserves Coll on S.

Proof: Let $m, k \in \mathsf{Ob}$. To prove that f_{mk} preserves Coll locally, let $p \in \mathsf{Dom}(\mathsf{f}_{mk})$. We need a ball with center p such that f_{mk} preserves Coll on that ball.

Let $\varepsilon, \lambda \in {}^{+}\mathsf{F}$ be such that $S := S(p, \varepsilon) \subseteq \mathsf{Dom}(\mathsf{f}_{mk})$ and any line slower than λ that intersects S is an observer line for m. Such ε, λ exist by **Ax∃Ob** and **AxOpen**.

Let S' be a ball with center p such that S' is a proper subset of S. For any $H \subseteq {}^{n}\mathsf{F}$, the "vertical cylinder" $c(H)$ of H is defined as
$c(H) := \{\, q \in {}^{n}\mathsf{F} \;:\; (\exists r \in H)\; q_s = r_s \,\}$.

Let $S'' \subseteq S'$ be a ball with center p such that S'' is small enough to satisfy $(*)$ below. See Figure 4.

$(*)$ Any line that intersects both S'' and $c(S'') \setminus S'$ is slower than λ.

Let $X := (S \setminus S') \cap c(S'')$, $X^{+} := \{\, q \in X \;:\; q_t > p_t \,\}$, and $X^{-} := \{\, q \in X \;:\; q_t < p_t \,\}$.

We will use Desargues' theorem and Claim 1 to prove that f_{mk} preserves Coll on S''. Let $a_m, b_m, c_m \in S''$ be such that $\mathsf{Coll}(a_m, b_m, c_m)$. We will prove that $\mathsf{Coll}(a_k, b_k, c_k)$. We can assume that a_m, b_m, c_m are distinct. Let $a'_m, b'_m, c'_m \in X^{+} \subseteq S$ and $a''_m, b''_m, c''_m \in X^{-} \subseteq S$ be triangles such that the corresponding sides meet in a_m, b_m, c_m, respectively, and $b'_m b''_m \cap c'_m c''_m$ is a point in S, see Fig.4. It is explained in the caption of Fig.4 why such triangles exist.

By $(*)$ above, all the sides and the lines connecting the corresponding vertices of these triangles are slower than λ. Thus all these lines are observer lines for m. Furthermore, these triangles are axially perspective. By Desargues' theorem, they are centrally perspective. Moreover, $a'_m a''_m \cap b'_m b''_m \cap c'_m c''_m$ is a point in S. Therefore, by Claim 1 (ii), a'_k, b'_k, c'_k and a''_k, b''_k, c''_k are centrally perspective triangles and the corresponding sides of these triangles meet in

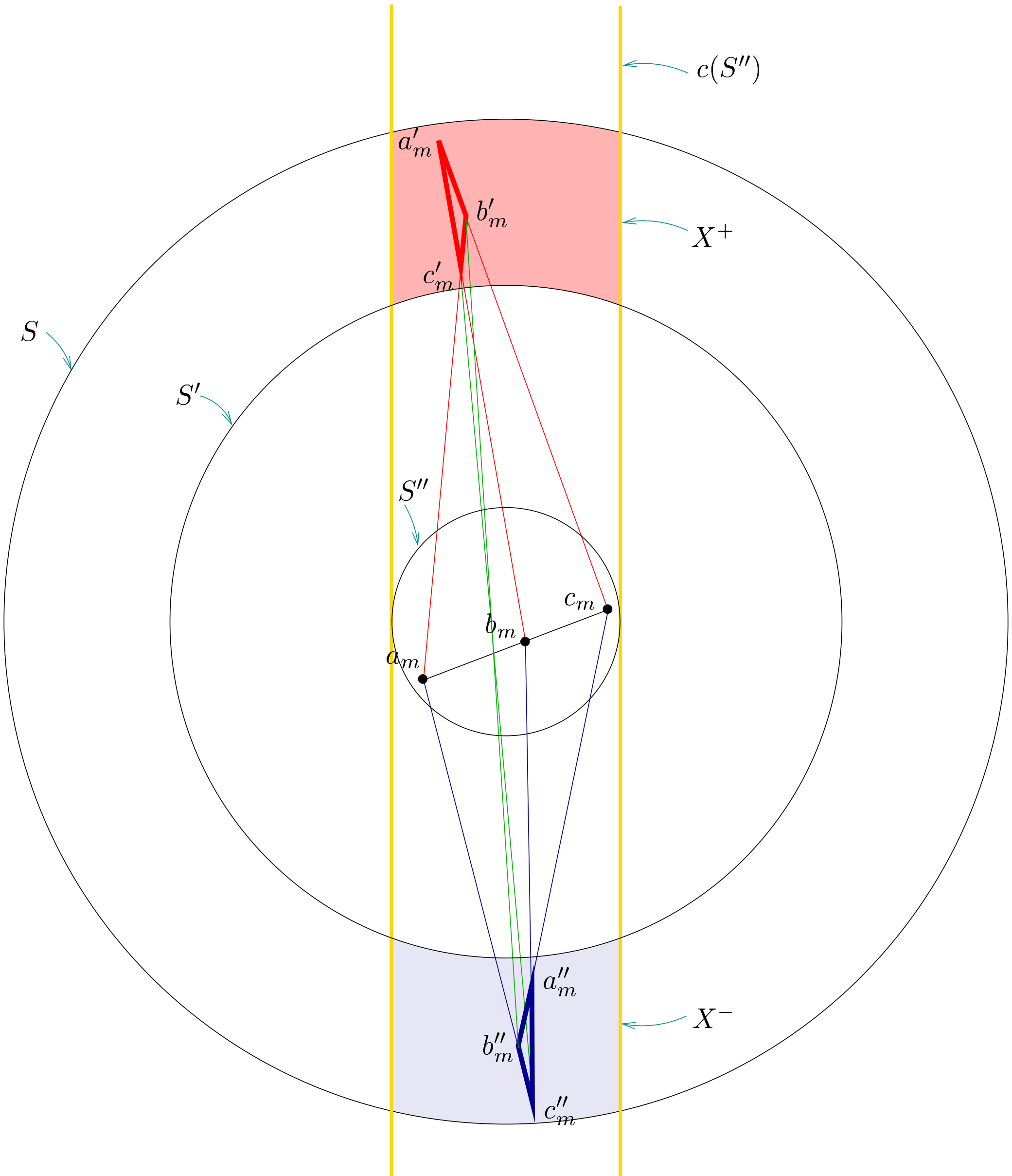

Figure 4: Choose $c'_m, b'_m \in X^+$ such that $\mathsf{Bw}(a_m, c'_m, b'_m)$ and $b_m c'_m \cap c_m b'_m$ is a point in X^+. The latter can be achieved by choosing c'_m and b'_m "close" to each other. Let a'_m be $b_m c'_m \cap c_m b'_m$. Choose $c''_m, b''_m \in X^-$ such that $\mathsf{Bw}(a_m, b''_m, c''_m)$, $b_m c''_m \cap c_m b''_m$ is a point in X^-, $a_m b'_m \neq a_m b''_m$, $b_m c'_m \neq b_m c''_m$ and $c_m b'_m \neq c_m b''_m$. Let a''_m be $b_m c''_m \cap c_m b''_m$. Then, by $\mathsf{Bw}(a_m, c'_m, b'_m)$ and $\mathsf{Bw}(a_m, b''_m, c''_m)$, $c'_m c''_m \cap b'_m b''_m$ is a point in S.

a_k, b_k, c_k, respectively. By Desargues' theorem, we conclude $\mathsf{Coll}(a_k, b_k, c_k)$.
QED (Claim 2)

<u>*Claim 3:*</u> Assume $m, k \in \mathsf{Ob}$, $a, b, c, d \in \mathsf{f}_{mk}$, $d \notin \{a, b, c\}$ and $a_m d_m$, $b_m d_m$, $c_m d_m$ are observer lines for m. Then

$$a_m, b_m, c_m, d_m \text{ are coplanar} \quad \Leftrightarrow \quad a_k, b_k, c_k, d_k \text{ are coplanar.}$$

<u>*Proof:*</u> Assume m, k, a, b, c, d satisfy the assumptions. By Claim 1 (i), it is enough to prove one direction of "$\Leftrightarrow$" in the present claim. We will prove, e.g., the "$\Leftarrow$" direction. Assume a_k, b_k, c_k, d_k are coplanar. Let $S \subseteq \mathsf{Dom}(\mathsf{f}_{km})$ be a ball with center d_k such that f_{km} preserves Coll on S. S exists by Claim 2. See the left hand side of Figure 5. Let $a'_k \in S \cap a_k d_k$, $b'_k \in S \cap b_k d_k$

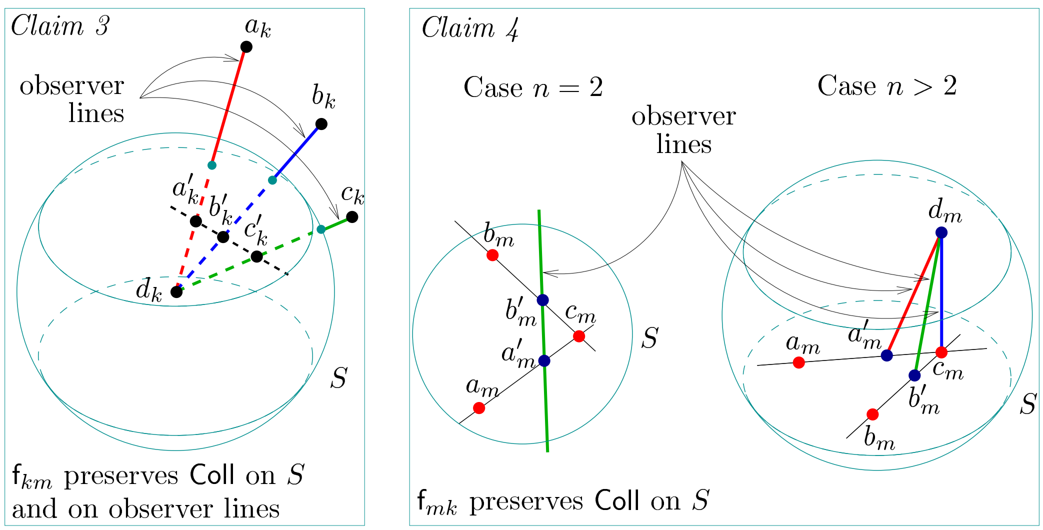

Figure 5: Illustrations for the proofs of Claims 3,4.

and $c'_k \in S \cap c_k d_k$ be such that $\mathsf{Coll}(a'_k, b'_k, c'_k)$ and $d_k \notin \{a'_k, b'_k, c'_k\}$, cf. Fig.5. Clearly, $d_m \notin \{a'_m, b'_m, c'_m\}$. Since f_{km} preserves Coll on S, we have $\mathsf{Coll}(a'_m, b'_m, c'_m)$. By Claim 1 (ii), $a'_m \in a_m d_m$, $b'_m \in b_m d_m$ and $c'_m \in c_m d_m$, since $a_m d_m$, $b_m d_m$, $c_m d_m$ are observer lines. Therefore a_m, b_m, c_m, d_m are coplanar.
QED (Claim 3)

<u>*Claim 4:*</u> Assume $m, k \in \mathsf{Ob}$ and S is a ball such that f_{mk} preserves Coll on S. Then f_{mk} preserves $\neg\mathsf{Coll}$ on S.

<u>*Proof:*</u> Assume m, k, S satisfy the assumptions. Let $a_m, b_m, c_m \in S$ be such that $\neg\mathsf{Coll}(a_m, b_m, c_m)$. We want to prove $\neg\mathsf{Coll}(a_k, b_k, c_k)$. We distinguish two cases, the case of $n = 2$ and the case of $n > 2$. See the right hand side of Fig.5.

Case of $n = 2$: Assume $n = 2$. Let $a'_m \in a_m c_m \cap S$ and $b'_m \in b_m c_m \cap S$ be such that $c_m \notin \{a'_m, b'_m\}$ and $a'_m b'_m$ is an observer line for m. a'_m, b'_m exist by **Ax∃Ob**. Clearly, $\neg\mathsf{Coll}(a'_m, b'_m, c_m)$. Then, by Claim 1 (ii), $\neg\mathsf{Coll}(a'_k, b'_k, c_k)$ since $a'_m b'_m$ is an observer line. By $\neg\mathsf{Coll}(a'_k, b'_k, c_k)$ and the assumption that f_{mk} preserves Coll on S, we have $\neg\mathsf{Coll}(a_k, b_k, c_k)$.

Case of $n > 2$: Assume $n > 2$. Let $a'_m \in a_m c_m \cap S$, $b'_m \in b_m c_m \cap S$ and $d_m \in S$ be such that a'_m, b'_m, c_m, d_m are not coplanar and $a'_m d_m$, $b'_m d_m$, $c_m d_m$ are observer lines for m, cf. Fig.5. a'_m, b'_m, d_m exist by **Ax∃Ob**. Then, by Claim 3, we have that a'_k, b'_k, c_k, d_k are not coplanar. But then $\neg\mathsf{Coll}(a'_k, b'_k, c_k)$. By this and by the assumption that f_{mk} preserves Coll on S, we have $\neg\mathsf{Coll}(a_k, b_k, c_k)$.

QED (Claim 4)

Let $m, k \in \mathsf{Ob}$, $p \in \mathsf{f}_{mk}$ be fixed until the proof is complete. Furthermore, let a ball S with center p_m be fixed such that f_{mk} preserves Coll on S. S exists by Claim 2. Then f_{mk} preserves $\neg\mathsf{Coll}$ on S by Claim 4.

Now, by Prop.2, there is a unique PColl-preserving $\mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ function that agrees with f_{mk} on S. Denote this function by g. g is injective by Prop.2. We will prove that g is a $\mathcal{P}$-collineation.

<u>*Claim 5:*</u> Assume $H \subseteq \mathsf{Dom}(\mathsf{f}_{mk})$ is an open set and g agrees with f_{mk} on H. Assume $e \in \mathsf{f}_{mk}$ and ℓ, ℓ' are observer lines for m such that $\ell \cap \ell' = \{e_m\}$ and $\ell \cap H \neq \emptyset \neq \ell' \cap H$. Then $e \in g$.

<u>*Proof:*</u> Note that for any line ℓ and open set H, we have
$(\ell \cap H \neq \emptyset) \Rightarrow (\ell \cap H$ is an infinite set$)$. Now, assume H, e, ℓ, ℓ' satisfy the assumptions. Let $a_m, b_m \in H \cap \ell$ and $c_m, d_m \in H \cap \ell'$ be distinct points, cf. the left hand side of Figure 6. Then, by Claim 1 (ii), $a_k b_k \cap c_k d_k = \{e_k\}$. g takes a_m, b_m, c_m, d_m to a_k, b_k, c_k, d_k, respectively, by the assumption that g agrees with f_{mk} on H. Since g preserves PColl, it takes e_m to e_k. Thus $e \in g$.

QED (Claim 5)

Let $c(S) := \{\, q \in {}^n\mathsf{F} \ : \ (\exists r \in S)\ q_s = r_s \,\}$ be the "vertical cylinder" of S, cf. the right hand side of Fig.6.

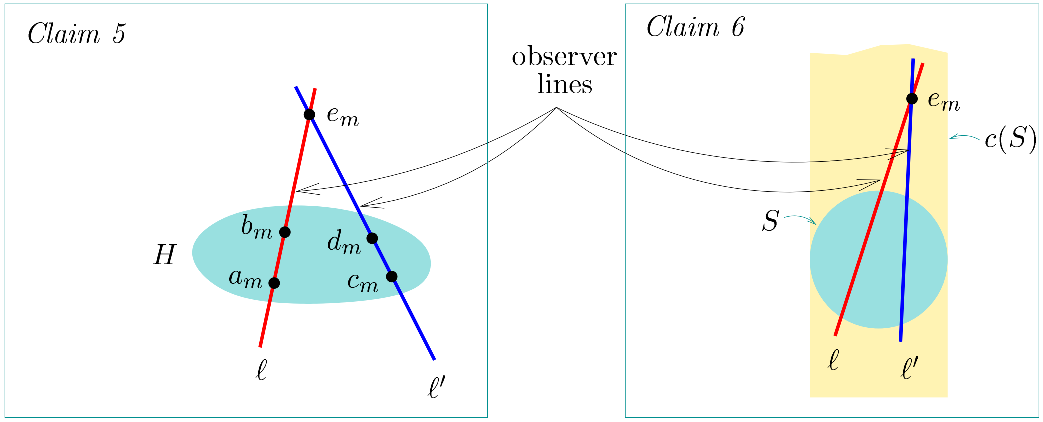

Figure 6: Illustrations for proofs of Claims 5,6.

Claim 6: g agrees with f_{mk} on $c(S) \cap \mathsf{Dom}(\mathsf{f}_{mk})$, i.e. for every $e \in \mathsf{f}_{mk}$, $e_m \in c(S) \;\Rightarrow\; e \in g$.

Proof: Let $e \in \mathsf{f}_{mk}$ be such that $e_m \in c(S)$. By **Ax∃Ob**, there are two observer lines ℓ, ℓ' for m such that they meet in e_m and both of them intersect S, cf. the right hand side of Fig.6. Then, by Claim 5 and by the fact that g agrees with f_{mk} on S, we have $e \in g$. QED (Claim 6)

Let a ball S_k with center p_k and $\lambda \in {}^{+}\mathsf{F}$ be fixed until the proof is complete such that

- f_{km} preserves Coll on S_k, and
- each line slower than λ that intersects S_k is an observer line for k.

Such S_k exists by **Ax∃Ob** and Claim 2.

Claim 7: Assume $e \in \mathsf{f}_{mk}$ is such that $e_k \neq p_k$ and $\mathsf{speed}(e_k p_k) < \lambda$. Then $e \in g$.

Proof: Assume e satisfies the assumptions. See Figure 7. Let $q_m \in S$ be such that $p_m \neq q_m$ and $\mathsf{speed}(p_m q_m) = 0$. Note that $p_m q_m \subseteq c(S)$ and, by **Ax∃Ob**, $p_m q_m$ is an observer line for m. Choose $a_k \in p_k q_k \cap S_k$ such that $a_k \neq p_k$ and $\mathsf{speed}(e_k a_k) < \lambda$. $\mathsf{speed}(e_k a_k) < \lambda$ can be achieved by choosing a_k "very close" to p_k.

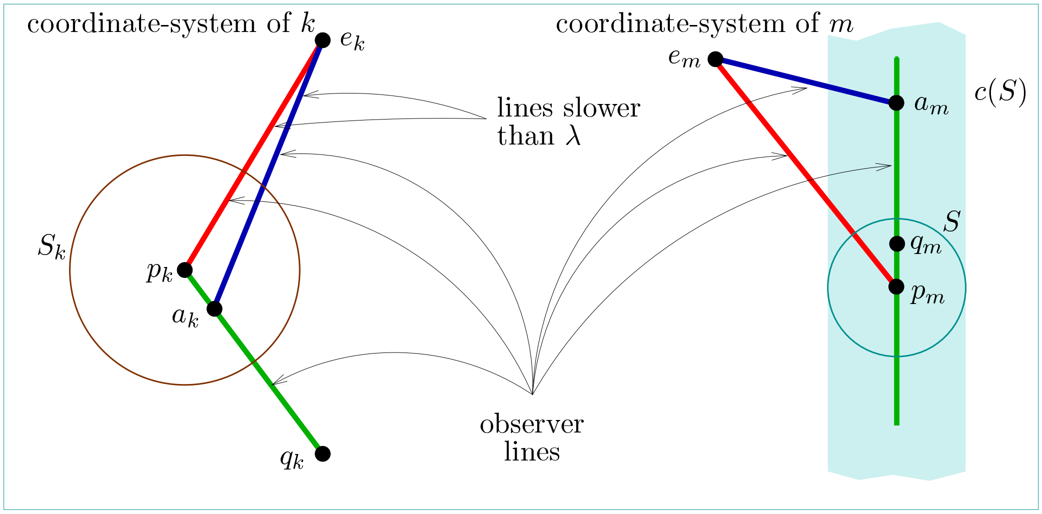

Figure 7: Illustration for the proof of Claim 7.

Now, by our choice of S_k and λ, both $e_k a_k$, $e_k p_k$ are observer lines for k. Thus, by Claim 1 (i), $e_m a_m$ and $e_m p_m$ are observer lines for m. Furthermore, by Claim 1 (ii), $a_m \in p_m q_m$.

Assume $e_m \in p_m q_m$. Then $e_m \in c(S)$. Thus, by Claim 6, $e \in g$.

Assume $e_m \notin p_m q_m$. Then the two observer lines $e_m a_m$, $e_m p_m$ meet in e_m and both of them intersect the open set $c(S) \cap \mathsf{Dom}(\mathsf{f}_{mk})$. But g agrees with f_{mk} on this open set by Claim 6. Therefore, by Claim 5, $e \in g$. QED (Claim 7)

g^{-1} denotes the inverse of g. We note that $g^{-1} : \mathsf{P}^n\mathsf{F} \xrightarrow{\circ} \mathsf{P}^n\mathsf{F}$ is a partial function.

<u>*Claim 8:*</u> $S_k \subseteq \mathsf{Rng}(g)$ and f_{km} and g^{-1} agree on S_k, i.e.

$$(e \in \mathsf{f}_{mk} \land e_k \in S_k) \;\Rightarrow\; e \in g.$$

<u>*Proof:*</u> Assume $e \in \mathsf{f}_{mk}$ and $e_k \in S_k$. Let lines ℓ, ℓ' be slower than λ such that $\ell \cap \ell' = \{p_k\}$ and e_k is in the plane determined by ℓ, ℓ'. See the left hand side of Figure 8.

If $e_k \in \ell \cup \ell'$, then $e \in g$ by Claim 7. So we can assume $e_k \notin \ell \cup \ell'$. Let a_k, b_k, c_k, d_k be distinct points such that $a_k, b_k \in \ell$, $c_k, d_k \in \ell'$ and

$$a_k c_k \cap b_k d_k = \{e_k\}.$$

Note that, by our choice of S_k and Claim 4, f_{km} preserves Coll and $\neg\mathsf{Coll}$ on

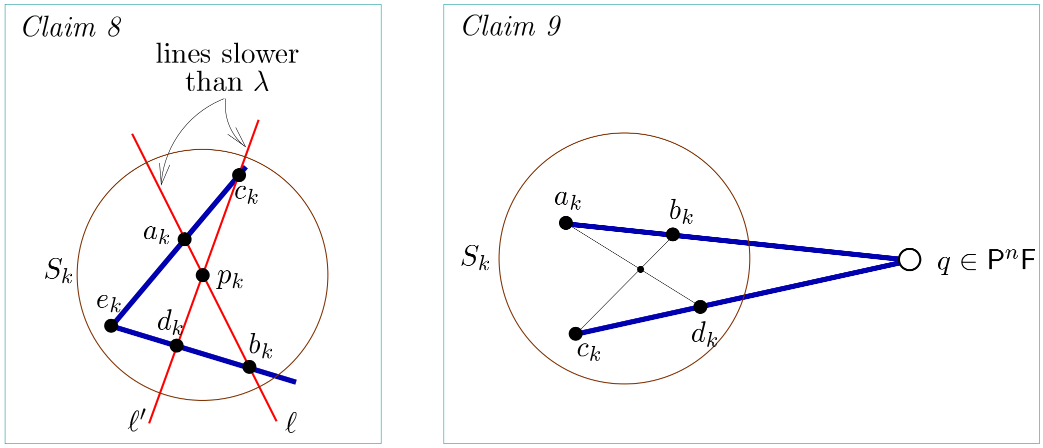

Figure 8: Illustrations for the proofs of Claims 8,9

S_k. Thus

$$a_m c_m \cap b_m d_m = \{e_m\}.$$

By Claim 7, g takes a_m, b_m, c_m, d_m to a_k, b_k, c_k, d_k, respectively. Since g preserves PColl, g takes e_m to e_k, i.e. $e \in g$. QED (Claim 8)

<u>*Claim 9:*</u> g is surjective, i.e. $\mathsf{Rng}(g) = \mathsf{P}^n\mathsf{F}$. Hence g is a bijection.

<u>*Proof:*</u> Let $q \in \mathsf{P}^n\mathsf{F}$. Let $a_k, b_k, c_k, d_k \in S_k$ be distinct points such that

$$\mathsf{P}a_k b_k \cap \mathsf{P}c_k d_k = \{q\},$$

cf. the right hand side of Figure 8. Note that the points a_k, b_k, c_k, d_k are coplanar and the lines $a_k b_k$, $c_k d_k$ are distinct. Since f_{km} preserves Coll and $\neg\mathsf{Coll}$ on S_k, a_m, b_m, c_m, d_m are coplanar and the lines $a_m b_m$ and $c_m d_m$ are distinct. But then there is $r \in \mathsf{P}^n\mathsf{F}$ such that

$$\mathsf{P}a_m b_m \cap \mathsf{P}c_m d_m = \{r\}.$$

By Claim 8, g takes a_m, b_m, c_m, d_m to a_k, b_k, c_k, d_k respectively. Since g preserves PColl it takes r to q. Thus $q \in \mathsf{Rng}(g)$. QED (Claim 9)

<u>*Claim 10:*</u> g is a $\mathcal{P}$-collineation.

<u>*Proof:*</u> By Claim 9, $g : \mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ is a PColl-preserving bijection. But any PColl-preserving bijection $f : \mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ is a $\mathcal{P}$-collineation.

QED (Claim 10)

Claim 11: Assume g' is a $\mathcal{P}$-collineation and S' is a neighborhood of p_m, and g' agrees with f_{mk} on S'. Then $g' = g$.

Proof: Assume g', S' satisfy the assumptions. Let S'' be a ball with center p_m such that $S'' \subseteq S \cap S'$. Then both g, g' agree with f_{mk} on S'', and f_{mk} preserves Coll and $\neg\mathsf{Coll}$ on S''. But then, by the "uniqueness" part of Prop.2, $g = g'$. QED (Claim 11)

At this point, item (i) of the theorem has been proven. Item (ii) of the theorem follows from Claims 8, 10 above and from item (i) of the theorem.
QED (Theorem 1)

Definition: Assume **Ax∃Ph**, **AxIstr**. Assume $m \in \mathsf{Ob}$ and $p \in \mathsf{cd}(m)$. Then there is a unique $\lambda \in {}^+\mathsf{F} \cup \{0, \infty\}$ such that

$$(\forall \mathsf{ph} \in \mathsf{Ph})(p \in \mathsf{tr}_m(\mathsf{ph}) \;\rightarrow\; \mathsf{speed}_m(\mathsf{ph}) = \lambda).$$

This unique λ is called the *speed of light at p for m.*

Proof of Theorem 2: Assume the assumptions. Recall that the f_{mk}'s are injective partial functions by Proposition 1.

Let $m, k \in \mathsf{Ob}$ and $p \in \mathsf{Dom}(\mathsf{f}_{mk})$ be fixed. Let g be a $\mathcal{P}$-collineation and let S be a ball with center p such that f_{mk} agrees with g on S. Such g and S exist by Theorem 1. Note that $S \subseteq \mathsf{Dom}(\mathsf{f}_{mk}) \subseteq \mathsf{cd}(m)$. We will prove that the restriction of g to ${}^n\mathsf{F}$ is an $\mathcal{A}$-collineation.

Claim 1: Assume $x, y \in \mathsf{Dom}(\mathsf{f}_{mk})$, $x \neq y$. Then
(xy is a photon line for m) $\Leftrightarrow$ ($\mathsf{f}_{mk}(x)\mathsf{f}_{mk}(y)$ is a photon line for k).

Here we omit the easy *proof.* If one wants to obtain a proof, one has to use **AxLine**.

Claim 2: Assume $h \in \mathsf{Ob}$, $\ell \in \mathsf{Lines}$ and $x \in \ell \cap \mathsf{cd}(h) \neq \emptyset$. Let η be the speed of light at x for h. Then

$$\mathsf{speed}(\ell) = \eta \quad \Leftrightarrow \quad (\ell \text{ is a photon line for } h).$$

Here we omit the easy *proof.* If one wants to obtain a proof one has to use **AxLine** and **Ax∃Ph**.

Let λ be the speed of light at p for m. By **AxFin**, $\lambda \in {}^+\mathsf{F}$.

Definition: We call the elements of $\mathsf{P}^n\mathsf{F} \setminus {}^n\mathsf{F}$ *infinite points* and the elements of ${}^n\mathsf{F}$ *finite points*.

Recall that if $\ell \in \mathsf{Lines}$, then $\ell^\infty \in \mathsf{P}^n\mathsf{F}$ is "the point of ℓ at infinity".

Claim 3: Assume $\ell \in \mathsf{Lines}$ is such that $p \in \ell$ and $\mathsf{speed}(\ell) = \lambda$. Then $g(\ell^\infty)$ is an infinite point.

Proof: Assume ℓ satisfies the assumptions. Let $a, b, c \in S$ be such that $a \notin \ell$,

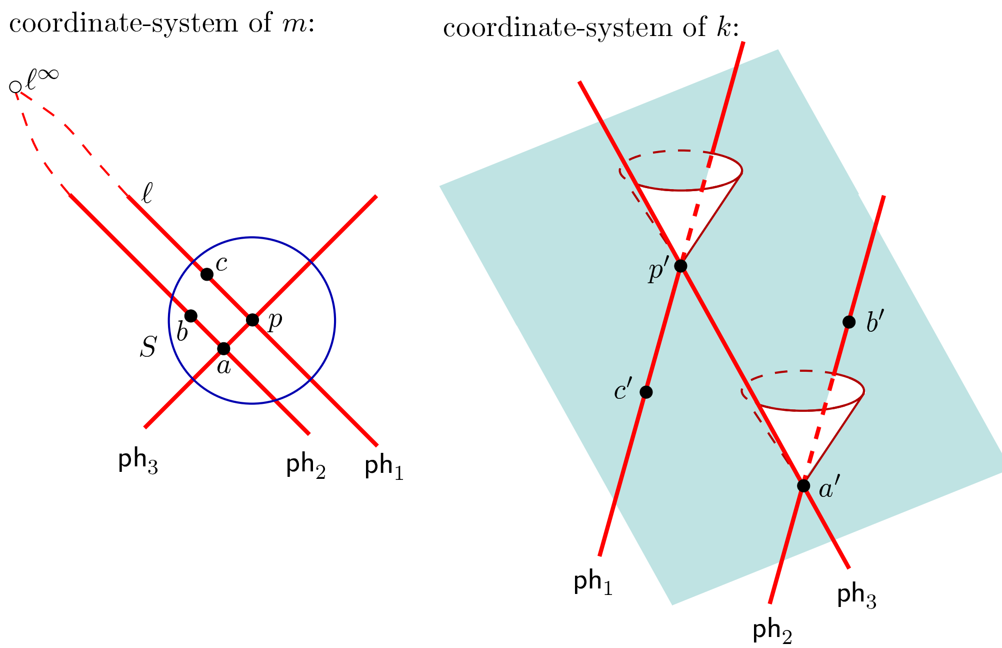

Figure 9: Illustration for the proof of Claim 3.

$\mathsf{speed}(pa) = \lambda$, $b \neq a$, $ab \parallel \ell$, $c \in \ell$ and $c \neq p$. See Figure 9. Then $\mathsf{speed}(ab) = \lambda$, too. Since the speed of light is λ at p for m, pc and pa are photon lines for m, by Claim 2. Since pa is a photon line, the speed of light at a is $\mathsf{speed}(pa) = \lambda$ for m. Thus, since $\mathsf{speed}(ab) = \lambda$, ab is a photon line for m by Claim 2. Note that a, b, c, p are coplanar and no three of them are collinear.

Let a', b', c', p' be the f_{mk} images of a, b, c, p, respectively. See Figure 9. Recall that f_{mk} agrees with a $\mathcal{P}$-collineation on S. Thus a', b', c', p' are coplanar and no three of them are collinear since a, b, c, p are such. By Claim

1, $p'c', p'a', a'b'$ are photon lines for k since pc, pa, ab are photon lines for m. Thus the speed of light is the same at p' and a' for k, which is $\mathsf{speed}(p'a')$; the speed of light at p' is $\mathsf{speed}(p'c')$; and the speed of light at a' is $\mathsf{speed}(a'b')$. Hence $\mathsf{speed}(p'c') = \mathsf{speed}(a'b')$. But then, $p'c' \parallel a'b'$. Now, since g preserves PColl and agrees with f_{mk} on S, it takes $\mathsf{P}pc \cap \mathsf{P}ab = \{\ell^\infty\}$ to $\mathsf{P}p'c' \cap \mathsf{P}a'b'$. Hence g takes ℓ^∞ to an infinite point. QED (Claim 2)

Claim 4: g takes infinite points to infinite points.

Proof: Let q be an infinite point. Then $q = \ell^\infty$ for some $\ell \in \mathsf{Lines}$ with $p \in \ell$. Let such an ℓ be fixed. Let P be a vertical plane that contains ℓ. See Figure 10. Let ℓ_1, ℓ_2 be lines of speed λ such that $\ell_1 \cap \ell_2 = \{p\}$ and

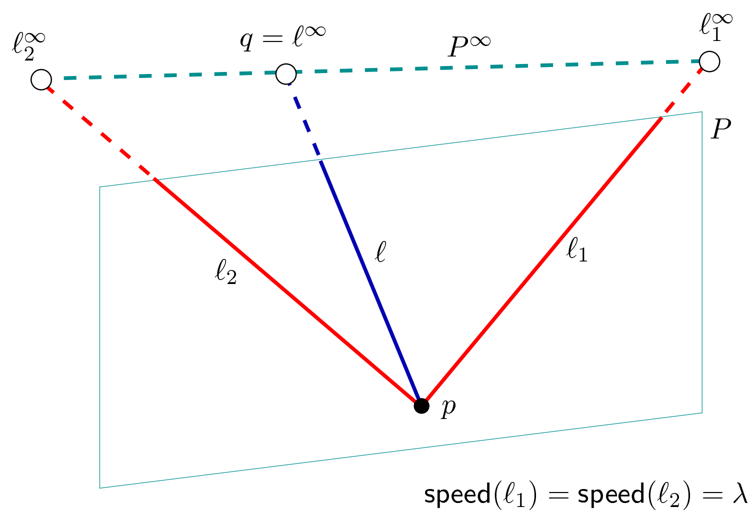

Figure 10: Illustration for the proof of Claim 4.

$\ell_1, \ell_2 \subseteq P$. Then, since $\ell^\infty, \ell_1^\infty, \ell_2^\infty \in P^\infty$, we have $\mathsf{PColl}(\ell^\infty, \ell_1^\infty, \ell_2^\infty)$. But then $\mathsf{PColl}(g(\ell^\infty), g(\ell_1^\infty), g(\ell_2^\infty))$. By Claim 3, $g(\ell_1^\infty)$ and $g(\ell_2^\infty)$ are infinite points. Furthermore, they are distinct. Since no projective line contains two infinite points and a finite point, we conclude that $g(\ell^\infty)$ is an infinite point. QED (Claim 4)

Claim 5: $g \cap ({}^n\mathsf{F} \times {}^n\mathsf{F})$ is an $\mathcal{A}$-collineation.

Proof: Since g is a $\mathcal{P}$-collineation it is enough to prove that $g \cap ({}^n\mathsf{F} \times {}^n\mathsf{F})$ is a permutation. By Claim 4, it is enough to prove that g takes finite points to finite points since g is a permutation on $\mathsf{P}^n\mathsf{F}$. g takes p to a finite point, i.e. to $\mathsf{f}_{mk}(p)$. To prove that g takes finite points to finite points let q be a finite point, $p \neq q$. Since g preserves PColl, we have $\mathsf{PColl}(g(p), g(q), g(pq^\infty))$. Since $g(pq^\infty)$ is an infinite point and $g(p)$ is a finite point, we conclude that $g(q)$ is a finite point. QED (Claim 5)

By this the "existence" part of our theorem has been proven. The "uniqueness" part of the theorem follows from Theorem 1 and from the fact that any $\mathcal{A}$-collineation can be extended to a $\mathcal{P}$-collineation.

QED (Theorem 2)

We will use Lemma 1 in the proof of Theorem 3.

Lemma 1. *Assume* **LocRel $\setminus$ {AxIstr, AxFin}**. *If there is an observer trace in a plane passing through a point, then there is a photon trace in the plane passing through the point.*

Formally: Assume $m, k \in \mathsf{Ob}$ *and* $p \in \mathsf{tr}_m(k) \subseteq P \in \mathsf{Planes}$. *Then there is a* $\mathsf{ph} \in \mathsf{Ph}$ *such that* $p \in \mathsf{tr}_m(\mathsf{ph}) \subseteq P$.

Proof: Assume $m, k \in \mathsf{Ob}$ and $p \in \mathsf{tr}_m(k) \subseteq P \in \mathsf{Planes}$. f_{mk} is an injective partial function by Proposition 1. We can assume $p \in \mathsf{Dom}(\mathsf{f}_{mk})$ since, by **AxEvent**, k has a brother h such that $p \in \mathsf{Dom}(\mathsf{f}_{mh})$. Let $p' := \mathsf{f}_{mk}(p)$. By Theorem 1, there are a $\mathcal{P}$-collineation g and balls S, S' with centers p, p', respectively, such that f_{mk} agrees with g on S and f_{km} agrees with g^{-1} on S', where g^{-1} denotes the inverse of g. Let such g, S, S' be fixed. See Figure 11. By **AxLine** and $S \subseteq \mathsf{Dom}(\mathsf{f}_{mk}) \subseteq \mathsf{cd}(m)$, $\ell \cap S = \mathsf{tr}_m(k) \cap S$, for some $\ell \in \mathsf{Lines}$. Thus there is $q \in \mathsf{tr}_m(k) \cap S$ such that $p \neq q$. Let such a q be fixed and let $r \in S \cap P$ be such that p, q, r are non-collinear points. Let q' and r' be the f_{mk} images of q and r, respectively. The $\mathcal{P}$-collineation g takes p, q, r to p', q', r', respectively. Thus p', q', r' are non-collinear, too. Let P' be the plane that contains p', q', r'. We have $p', q' \in \mathsf{tr}_k(k)$ since $p, q \in \mathsf{tr}_m(k)$. By **AxSelf**, $\mathsf{speed}(p'q') = 0$. Thus P is a vertical plane. Hence, by **Ax∃Ph**, there is a photon ph such that $p' \in \mathsf{tr}_k(\mathsf{ph}) \subseteq P'$. Let such a ph be fixed. Let $a' \in \mathsf{tr}_k(\mathsf{ph}) \cap S'$ be such that $a' \neq p'$. Such an a' exists by **AxLine**. Let $a := \mathsf{f}_{km}(a')$. Since g^{-1} agrees with f_{km} on S' it takes a' to a. Since $\mathcal{P}$-collineation g^{-1} takes p', q', r', a' to p, q, r, a, respectively, we

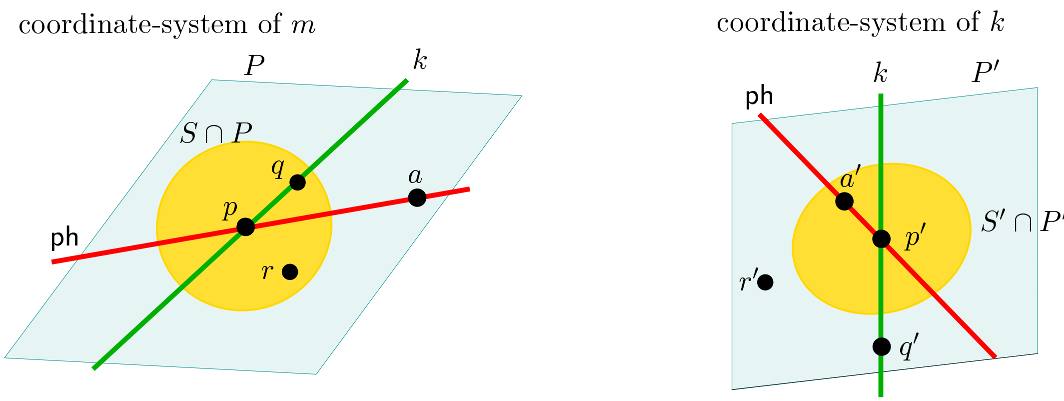

Figure 11: Illustration for the proof of Lemma 1.

conclude that p, q, r, a are coplanar, i.e. $a \in P$. Furthermore, $a, p \in \mathsf{tr}_m(\mathsf{ph})$ since $a', p' \in \mathsf{tr}_k(\mathsf{ph})$. But then, by **AxLine**, $\mathsf{tr}_m(\mathsf{ph}) \subseteq P$.

QED (Lemma 1)

Proof of Theorem 3: Assume $n > 2$ and **LocRel** $\setminus$ **{AxFin}**. Assume there is an FTL observer, i.e. there are $k, m \in \mathsf{Ob}$ such that k is FTL w.r.t. m. Let such m, k be fixed. Then there is $p \in \mathsf{tr}_m(k)$ such that $\mathsf{speed}_m(k) >$ (speed of light at p for m). Let such a p be fixed and let λ be the speed of light at p for m. Let P be a plane such that $\mathsf{tr}_m(k) \subseteq P$ and

$$(\forall \ell \in \mathsf{Lines})(\ell \subseteq P \;\rightarrow\; \mathsf{speed}(\ell) > \lambda). \tag{2}$$

See Figure 12. Such a plane exists since $\mathsf{speed}_m(k) > \lambda$ and $n > 2$. Now, by Lemma 1, there is a photon ph such that $p \in \mathsf{tr}_m(\mathsf{ph}) \subseteq P$. For this $\mathsf{ph} \in \mathsf{Ph}$, by (2), we have $\mathsf{speed}_m(\mathsf{ph}) > \lambda$. This contradicts the fact that the speed of light at p for m is λ.

QED (Theorem 3)

Now, we turn to the proof of our last "**noFTL**" theorem, Theorem 5. Propositions 3–6 and Lemmas 2–4 below are needed for the proof of this theorem.

Proposition 3. *Assume* **AxLine, Ax∃Ph, AxFin**.
Then for every $m, k \in \mathsf{Ob}$, $\mathsf{f}_{mk} : {}^n\mathsf{F} \xrightarrow{\circ} {}^n\mathsf{F}$ *is an injective partial function.*

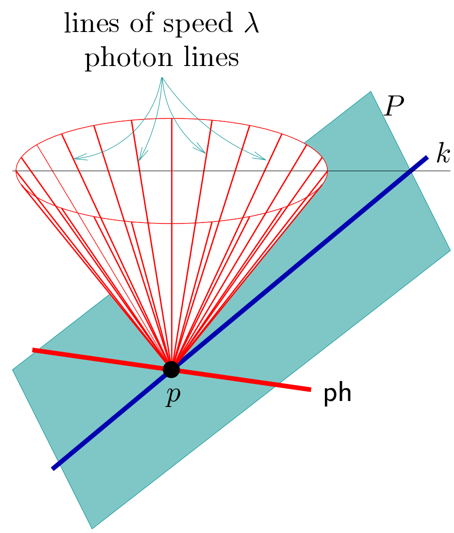

Figure 12: Illustration for the proof of Theorem 3.

Outline of proof: The proof of the proposition is similar to that of Proposition 1. Instead of observer m in that proof one has to use a photon to obtain a proof for the present proposition.

QED (Prop.3)

Ax∃Ob* below is a weaker version of **Ax∃Ob**.

Ax∃Ob* For every vertical plane P, each point in the coordinate-domain has a neighborhood and a "speed threshold" λ such that each line in plane P slower than λ and intersecting the neighborhood is an observer line, i.e.

$$(\forall m \in \mathsf{Ob})(\forall \text{ vertical plane } P)(\forall p \in \mathsf{cd}(m))(\exists \varepsilon, \lambda \in {}^{+}\mathsf{F})$$
$$(\forall \ell \in \mathsf{Lines})\Big((\ell \subseteq P \ \wedge\ \mathsf{speed}(\ell) < \lambda \ \wedge\ \ell \cap S(p,\varepsilon) \neq \emptyset) \qquad \rightarrow$$
$$\ell \text{ is an observer line for } m\Big).$$

Proposition 4. $\mathbf{LocRel}_0^- \setminus \{\mathbf{AxSelf}, \mathbf{AxEvent}\} \models$ **Ax∃Ob***.

Proof:

Claim 1: Assume P is a vertical plane, $p \in P$, S' is a ball with center p, $\lambda \in {}^{+}\mathsf{F}$, and ℓ is a line such that $p \in \ell \subseteq P$ and $\lambda < \mathsf{speed}(\ell)$. Then there is a ball $S \subseteq S'$ with center p such that each line ℓ' in plane P slower than λ and intersecting S meets ℓ within S' (i.e. $\emptyset \neq \ell \cap \ell' \subseteq S'$).

Proof: Assume P, λ, p, S', ℓ satisfy the assumptions. Let $a, b \in \ell \cap S'$ be such that $\mathsf{Bw}(a,p,b)$. See the left hand side of Figure 13. Let $\ell_a, \ell'_a, \ell_b, \ell'_b$ be lines in P of speed λ such that $\{a\} = \ell_a \cap \ell'_a$ and $\{b\} = \ell_b \cap \ell'_b$. Let S be a ball with center p such that circle $S \cap P$ is inside the parallelogram determined by $\ell_a, \ell'_a, \ell_b, \ell'_b$, i.e. such that $S \cap (\ell_a \cup \ell'_a \cup \ell_b \cup \ell'_b) = \emptyset$. This S has the desired properties, cf. Figure 13. QED (Claim 1)

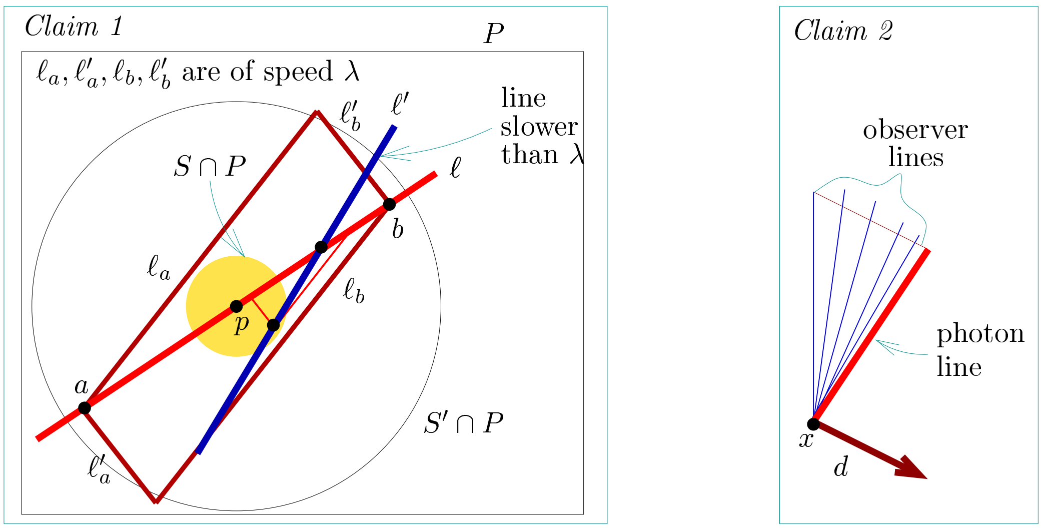

Figure 13: Illustrations for Claims 1 and 2.

Definition: Let $\ell \in \mathsf{Lines}$ and $d \in \mathsf{dir}$. We say that ℓ *is in direction* d iff $\mathsf{speed}(\ell) = 0$ or $(\exists p, q \in \ell)(p_t > q_t \ \wedge \ p_s - q_s = d)$.

Now, assume $\mathbf{LocRel_0^-} \setminus \{\mathbf{AxSelf, AxEvent}\}$.

Claim 2: Assume $m \in \mathsf{Ob}$, $x \in \mathsf{cd}(m)$ and $d \in \mathsf{dir}$. Then there is exactly one photon line for m in direction d passing through x. Let ℓ be this photon line. Then $0 < \mathsf{speed}(\ell) < \infty$ and any line slower than $\mathsf{speed}(\ell)$ in direction

d passing through x is an observer line for m. See the right hand side of Figure 13.

We omit the easy *proof.*

To prove that $\mathbf{Ax\exists Ob^*}$ holds, let $m \in \mathsf{Ob}$, let P be a vertical plane and $p \in \mathsf{cd}(m)$. We need a ball S with center p and a "speed threshold" λ such that each line in plane P slower than λ and intersecting S is an observer line for m. We can assume that $p \in P$. Let $d \in \mathsf{dir}$ be such that each line in P is in direction d or $-d$. Let ℓ_1, ℓ_2 be photon lines for m passing through p in directions $d, -d$, respectively. ℓ_1, ℓ_2 exist by Claim 2. $\mathsf{speed}(\ell_1), \mathsf{speed}(\ell_2) \in {}^+\mathsf{F}$ by Claim 2. Furthermore, $\ell_1, \ell_2 \subseteq P$. Let $\lambda \in {}^+\mathsf{F}$ be such that $\lambda <$ $\mathsf{speed}(\ell_1), \mathsf{speed}(\ell_2)$ and let $S' \subseteq \mathsf{cd}(m)$ be a ball with center p. By Claim 1, there is a ball $S \subseteq S'$ with center p such that any line in P slower than λ and intersecting S meets both ℓ_1 and ℓ_2 within S'. Let such an S be fixed.

Claim 3: Any photon line ℓ for m in P intersecting S is of speed $\geq \lambda$.

Proof: Assume ℓ is a line in P slower than λ and intersecting S. Then ℓ is in direction d or $-d$. Assume, e.g., ℓ is in direction d. See Figure 14. By our choice of S, $\emptyset \neq \ell \cap \ell_1 \subseteq S' \subseteq \mathsf{cd}(m)$. Let $q \in \ell \cap \ell_1$. By Claim 2, there is exactly one photon line in direction d passing through $q \in \mathsf{cd}(m)$. This photon line is ℓ_1. Thus ℓ cannot be a photon line. QED (Claim 3)

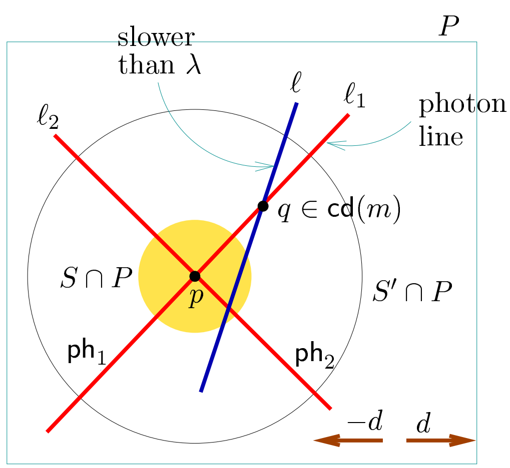

Figure 14: Illustration for the proof of Claim 3.

Claim 4: Any line in plane P slower than λ and intersecting S is an observer line for m.

Proof: The claim follows from Claims 2 and 3. For completeness we include a detailed proof. Let ℓ' be a line in plane P slower than λ and intersecting S. Let $q \in \ell' \cap S \subseteq \mathsf{cd}(m)$. ℓ' is in direction d or $-d$. Assume, e.g., ℓ' is in direction d. Let ℓ be the photon line passing through q in direction d. ℓ exists by Claim 2. Clearly $\ell \subseteq P$. By Claim 3, $\mathsf{speed}(\ell) \geq \lambda$. Thus, $\mathsf{speed}(\ell') < \mathsf{speed}(\ell)$. Hence, ℓ' is an observer line by Claim 2.

QED (Prop.4)

Proposition 5. *Assume* $\mathbf{LocRel_0^-} \setminus \{\mathbf{AxSelf, AxEvent}\}$ *or* $\{\mathbf{AxLine, Ax\exists Ob^*, AxOpen}\}$. *Then for every* $m, k \in \mathsf{Ob}$*, vertical plane* P *and* $p \in \mathsf{Dom}(\mathsf{f}_{mk}) \cap P$ *there is a neighborhood* N *of* p *and a* $\mathcal{P}$*-collineation that agrees with* f_{mk} *on* $N \cap P$.

Proof: Assume the assumptions. Then, by Proposition 4, $\mathbf{Ax\exists Ob^*}$ holds.

Notation: Assume $m, k \in \mathsf{Ob}$. Then for every $a \in \mathsf{f}_{mk}$, a_m denotes the first component of a, while a_k denotes the second component of a, i.e. $a = \langle a_m, a_k \rangle$. Furthermore, if $a_m \in \mathsf{Dom}(\mathsf{f}_{mk})$, then a_k denotes $\mathsf{f}_{mk}(a_m)$ and if $a_k \in \mathsf{Rng}(\mathsf{f}_{mk}) = \mathsf{Dom}(\mathsf{f}_{km})$, then a_m denotes $\mathsf{f}_{km}(a_k)$.

Let $m, k \in \mathsf{Ob}$, a vertical plane P and $p_m \in \mathsf{Dom}(\mathsf{f}_{mk}) \cap P$ be fixed until the proof is complete.

Claim 1: Assume $m, k \in \mathsf{Ob}$ and $a, b \in \mathsf{f}_{mk}$, $a \neq b$. Then (i), (ii) below hold.
(i) ($a_m b_m$ is an observer line for m) $\Leftrightarrow$ ($a_k b_k$ is an observer line for k).
(ii) f_{mk} preserves Coll and $\neg\mathsf{Coll}$ between three points if the line determined by two of the points is an observer line. Formally: Assume $a_m b_m$ is an observer line for m. Then for every $c \in \mathsf{f}_{mk}$,
$\mathsf{Coll}(a_m, b_m, c_m) \Leftrightarrow \mathsf{Coll}(a_k, b_k, c_k)$, or equivalently $c_m \in a_m b_m \Leftrightarrow c_k \in a_k b_k$.
We omit the easy *proof.*

Claim 2: There is a ball $S \subseteq \mathsf{Dom}(\mathsf{f}_{mk})$ with center p_m such that f_{mk} preserves Coll on $S \cap P$.

Proof: A proof can be obtained from the proof of Claim 2 for the proof of Theorem 1 (p.244), in the following way: One uses $\mathbf{Ax\exists Ob^*}$ in place of $\mathbf{Ax\exists Ob}$.

QED (Claim 2)

Let $S \subseteq \mathsf{Dom}(\mathsf{f}_{mk})$ be a ball with center p_m such that f_{mk} preserves Coll on $S \cap P$.

Claim 3: f_{mk} preserves $\neg\mathsf{Coll}$ on $S \cap P$.

Proof: A proof can be obtained from the proof of Claim 4 for the case $n = 2$ in the proof of Theorem 1 (p.247), in the following way: One uses **Ax∃Ob*** in place of **Ax∃Ob**. QED (Claim 3)

Recall that for $B \in \mathsf{Planes}$, B^∞ denotes the "line of B at infinity".

f_{mk} preserves Coll and $\neg\mathsf{Coll}$ on $S \cap P$ by the choice of S and by Claim 3. Thus, by the proof of Proposition 2, there is a unique PColl-preserving function $g : (P \cup P^\infty) \longrightarrow \mathsf{P}^n\mathsf{F}$ that agrees with f_{mk} on $S \cap P$. Furthermore, this g is injective. Let g be fixed. By **Ax∃Ob***, there is a neighborhood of p_m and a "speed threshold" λ such that each line in P slower than λ and intersecting the neighborhood is an observer line for m. Let such a λ be fixed.

Claim 4: g agrees with f_{mk} on the set

$$\{\, e_m \in \mathsf{Dom}(\mathsf{f}_{mk}) \cap P \;:\; \mathsf{speed}(p_m e_m) < \lambda \,\}.$$

Proof: Let $e_m \in \mathsf{Dom}(\mathsf{f}_{mk}) \cap P$ be such that $\mathsf{speed}(p_m e_m) < \lambda$. Then by the choice of λ, $p_m e_m$ is an observer line for m. Let $q_m \in S \cap P$ be such that $\neg\mathsf{Coll}(q_m, p_m, e_m)$ and $e_m q_m$ is an observer line for m, too. q_m exists by the choice of λ. See the left hand side of Figure 15. Let $a_m \in e_m p_m \cap S$ and $b_m \in q_m e_m \cap S$ be such that $b_m \neq q_m$ and $a_m \neq p_m$. Hence $p_m a_m \cap q_m b_m = \{e_m\}$. By Claim 1 (ii), $p_k a_k \cap q_k b_k = \{e_k\}$. Since g agrees with f_{mk} on $S \cap P$, it takes p_m, a_m, b_m, q_m to p_k, a_k, b_k, q_k, respectively. Since g preserves PColl, it takes e_m to e_k. QED (Claim 4)

Since $g : (P \cup P^\infty) \longrightarrow \mathsf{P}^n\mathsf{F}$ preserves PColl, there is $Q \in \mathsf{Planes}$ such that $\mathsf{Rng}(g) \subseteq Q \cup Q^\infty$. Let such a Q be fixed.

Claim 5: $\mathsf{Rng}(g) = Q \cup Q^\infty$. Thus g is a bijection between projective planes $P \cup P^\infty$ and $Q \cup Q^\infty$.

Proof: Let $a_m, b_m, c_m \in S \cap P$ be such that $p_m \notin \{a_m, b_m, c_m\}$ and $p_m a_m$, $p_m b_m$, $p_m c_m$ are distinct lines slower than λ. Then by our choice of λ, $p_m a_m$, $p_m b_m$, $p_m c_m$ are observer lines for m. See the right hand side of Figure 15. Since g agrees with f_{mk} on $S \cap P$ and $\mathsf{Rng}(g) \subseteq Q \cup Q^\infty$, $\mathsf{P}p_k a_k, \mathsf{P}p_k b_k, \mathsf{P}p_k c_k \subseteq Q \cup Q^\infty$.

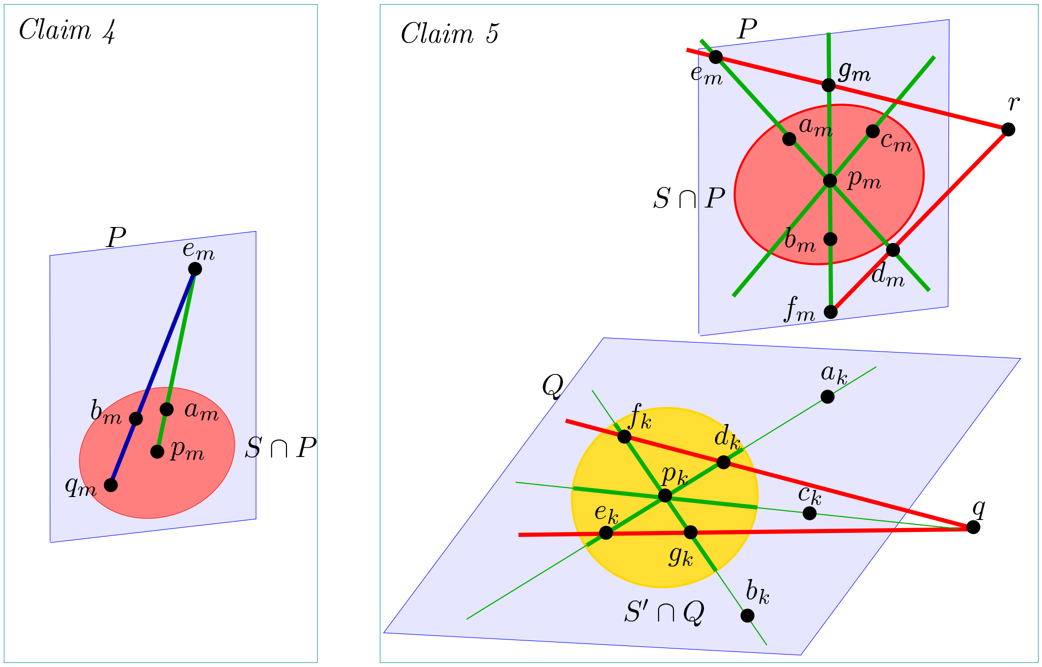

Figure 15: Illustration for the proofs of Claims 4 and 5

To prove $\mathsf{Rng}(g) = Q \cup Q^\infty$, let $q \in Q \cup Q^\infty$. By Claim 3, $\mathsf{P}p_ka_k$, $\mathsf{P}p_kb_k$, $\mathsf{P}p_kc_k$ are distinct. Thus two of these lines do not contain q. We can assume $q \notin \mathsf{P}p_ka_k \cup \mathsf{P}p_kb_k$. Let S' be a ball with center p_k such that $S' \subseteq \mathsf{Dom}(\mathsf{f}_{km}) = \mathsf{Rng}(\mathsf{f}_{mk})$. S' exists by **AxOpen**. Let $d_k, e_k \in p_ka_k \cap S'$ and $f_k, g_k \in p_kb_k \cap S'$ be such that $\mathsf{P}d_kf_k \cap \mathsf{P}e_kg_k = \{q\}$. See the right hand side of Figure 15. By Claim 1, $d_m, e_m \in p_ma_m$ and $f_m, g_m \in p_mb_m$ since p_ma_m, p_mb_m are observer lines. d_m, e_m, f_m, g_m are distinct because d_k, e_k, f_k, g_k were such. Let $r \in \mathsf{P}^n\mathsf{F}$ be such that $\{r\} = \mathsf{P}d_mf_m \cap \mathsf{P}e_mg_m$. By Claim 4, g takes d_m, e_m, f_m, g_m to d_k, e_k, f_k, g_k, respectively. Since g preserve PColl, it takes r to q. Thus $q \in \mathsf{Rng}(g)$. QED (Claim 5)

Any PColl preserving bijection between two projective planes can be extended to a $\mathcal{P}$-collineation, cf. e.g. [7, 4.4.11, p.40]. Thus, there is a $\mathcal{P}$-collineation f such that $f \supseteq g$. Now, f_{mk} agrees with such a $\mathcal{P}$-collineation of $S \cap P$.

QED (Prop.5)

Proposition 6. *Assume f is a $\mathcal{P}$-collineation and $a, b \in {}^n\mathsf{F}$ are such that $(\forall p \in [a,b]) f(p) \in {}^n\mathsf{F}$. Then f takes $[a,b]$ onto $[f(a), f(b)]$.*

Outline of proof: Consider the $(n+1)$-dimensional vector space ${}^{n+1}\mathbf{F} := \langle {}^{n+1}\mathsf{F}, \bar{0}, +, \ldots \rangle$ over the field $\mathbf{F}$. Let us introduce an equivalence relation on the set ${}^{n+1}\mathsf{F} \setminus \{\bar{0}\}$ as follows. $u, v \in {}^{n+1}\mathsf{F} \setminus \{\bar{0}\}$ are equivalent iff there exists $\lambda \in \mathsf{F} \setminus \{0\}$ such that $u = \lambda v$. The set of equivalence classes is called the projective space associated with the vector space ${}^{n+1}\mathbf{F}$ and is denoted by $\mathbf{F}\mathsf{P}^n$ according to [7]. We will denote the equivalence class of a vector $\bar{0} \neq v \in {}^{n+1}\mathsf{F}$ by $[v]$. The collinearity relation Pcoll on $\mathbf{F}\mathsf{P}^n$ is defined as follows. $\mathsf{Pcoll}([u],[v],[z])$ iff $\bar{0}, u, v, z$ are coplanar. The following is known from projective geometry.

<u>*Fact 1:*</u> Structures $\mathcal{P} = \langle \mathsf{P}^n\mathsf{F}, \mathsf{PColl} \rangle$ and $\langle \mathbf{F}\mathsf{P}^n, \mathsf{Pcoll} \rangle$ are isomorphic. Moreover, there is a unique isomorphism i between the two structures such that
$(*)$ $\qquad (\forall p \in {}^n\mathsf{F})\ i(p) = [\langle 1, p_1, p_2, \ldots, p_n \rangle]$.

Let $f : \mathbf{F}\mathsf{P}^n \longrightarrow \mathbf{F}\mathsf{P}^n$ be a function. We say that f is induced by a bijective linear transformation iff there is a bijective linear transformation A of the vector space ${}^{n+1}\mathbf{F}$ such that $f([v]) = [Av]$ for all $v \in {}^{n+1}\mathsf{F} \setminus \{\bar{0}\}$. We say that f is induced by a field automorphism iff there is an automorphism ψ of the field $\mathbf{F}$ such that $f([v]) = [\langle \psi(v_1), \psi(v_2), \ldots, \psi(v_{n+1}) \rangle]$ for all $v \in {}^{n+1}\mathsf{F} \setminus \{\bar{0}\}$.

From now on we identify $\mathcal{P} = \langle \mathsf{P}^n\mathsf{F}, \mathsf{PColl} \rangle$ with $\langle \mathbf{F}\mathsf{P}^n, \mathsf{Pcoll} \rangle$ by the unique isomorphism i that satisfies $(*)$ in Fact 1 above, consequently, we treat $\mathsf{P}^n\mathsf{F}$ and $\mathbf{F}\mathsf{P}^n$ as they were identical. Then, $\mathsf{P}^n\mathsf{F} \longrightarrow \mathsf{P}^n\mathsf{F}$ functions induced by bijective linear transformations and the ones induced by automorphisms are $\mathcal{P}$-collineations. The following is known from projective geometry, cf. [7, 6.3, p.60].

<u>*Fact 2:*</u> Any $\mathcal{P}$-collineation is a composition of a $\mathcal{P}$-collineation induced by a bijective linear transformation and a $\mathcal{P}$-collineation induced by a field automorphism.

<u>*Claim 1:*</u> Assume f is a $\mathcal{P}$-collineation induced by a field automorphism. Then f takes ${}^n\mathsf{F}$ onto ${}^n\mathsf{F}$ and $(\forall a, b \in {}^n\mathsf{F})$ (f takes $[a,b]$ onto $[f(a), f(b)]$).

Proof: Since **F** is a Euclidean field, any automorphism ψ of **F** preserves $<$, i.e. $x < y \Rightarrow \psi(x) < \psi(y)$. By this fact, one can easily check that the statement holds. QED (Claim 1)

Claim 2: Assume f is a $\mathcal{P}$-collineation induced by a bijective linear transformation and $a, b \in {}^n\mathsf{F}$ are such that $(\forall p \in [a,b]) f(p) \in {}^n\mathsf{F}$. Then f takes $[a,b]$ onto $[f(a), f(b)]$.

We omit the easy *proof.*

Now, the proposition follows from Fact 2 and Claims 1 and 2.

QED (Prop.6)

Lemma 2. *Assume* $\mathbf{LocRel}_0^- \setminus \{\mathbf{AxSelf, AxEvent}\}$. *Assume* $m, k \in \mathsf{Ob}$. *Then for every* $\ell \in \mathsf{Lines}$ *and* $p \in \ell \cap \mathsf{Dom}(\mathsf{f}_{mk})$ *there is* $q \in \ell \cap \mathsf{Dom}(\mathsf{f}_{mk})$ *such that* $p \neq q$ *and* f_{mk} *takes* $[p, q]$ *onto* $[\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q)]$.

Proof: The lemma follows from Propositions 5 and 6. QED (Lemma 2)

Lemma 3. *Assume* $\mathbf{LocRel}_0^-$. *Then the conclusion of Lemma 1 holds, i.e. if there is an observer trace in a plane passing through a point, then there is a photon trace in the plane passing through the point.*

Formally: Assume $m, k \in \mathsf{Ob}$ *and* $p \in \mathsf{tr}_m(k) \subseteq P \in \mathsf{Planes}$. *Then there is a* $\mathsf{ph} \in \mathsf{Ph}$ *such that* $p \in \mathsf{tr}_m(\mathsf{ph}) \subseteq P$.

Proof: Assume $\mathbf{LocRel}_0^-$. Let $m, k \in \mathsf{Ob}$, $p \in \mathsf{tr}_m(k)$ and $P \in \mathsf{Planes}$ be such that $\mathsf{tr}_m(k) \subseteq P$. Then f_{mk} is an injective partial function by Proposition 3. We can assume $p \in \mathsf{Dom}(\mathsf{f}_{mk})$ since, by **AxEvent**, k has a brother h such that $p \in \mathsf{Dom}(\mathsf{f}_{mh})$.

Let $q \in P \cap \mathsf{Dom}(\mathsf{f}_{mk})$ be such that $q \notin \mathsf{tr}_m(k)$ and f_{mk} takes $[p, q]$ onto $[\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q)]$. Such a q exists by Lemma 2. See Figure 16. Let p' and q' be the f_{mk} images of p and q, respectively. Then $p' \in \mathsf{tr}_k(k)$ and $q' \notin \mathsf{tr}_k(k)$. Let P' be the plane determined by $\mathsf{tr}_k(k)$ and q'. P' is vertical by **AxSelf** (and **AxLine**, **AxOpen**).

Let S be a ball with center p' and let g be a $\mathcal{P}$-collineation such that g agrees with f_{km} on $S \cap P'$. They exist by Proposition 5. Let $\mathsf{ph} \in \mathsf{Ph}$ be such that $p' \in \mathsf{tr}_k(\mathsf{ph}) \subseteq P'$. Such a ph exists by **Ax∃Ph**. Let $a' \in S \cap \mathsf{tr}_k(\mathsf{ph})$, $b' \in S \cap [p', q']$ and $c' \in \mathsf{tr}_k(k) \cap S$ be such that $p' \notin \{a', b', c'\}$. Such a', c' exist by **AxLine** and by $S \cap P' \subseteq \mathsf{Dom}(\mathsf{f}_{mk}) \subseteq \mathsf{cd}(m)$. Let a, b, c be the

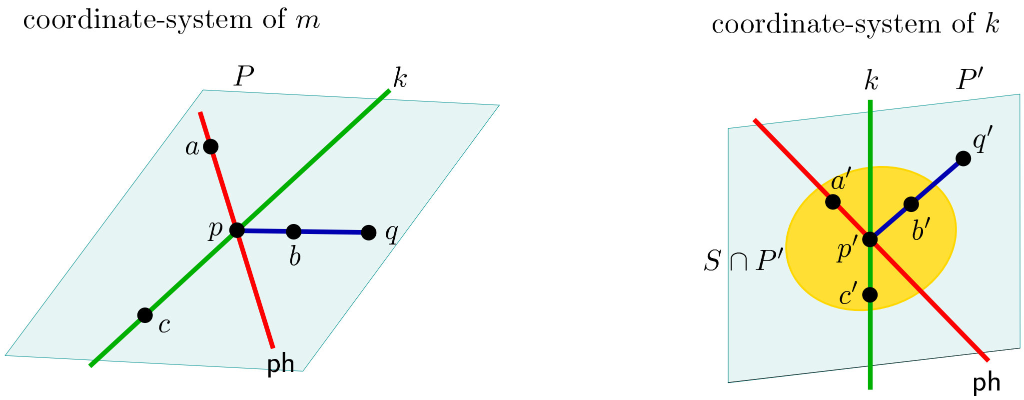

Figure 16: Illustration for the proof of Lemma 3.

f_{km} images of a', b', c', respectively. Then $p \notin \{a, b, c\}$, $c \in \mathsf{tr}_m(k)$, $b \in [p, q]$ and $a, p \in \mathsf{tr}_m(\mathsf{ph})$. Since $\mathcal{P}$-collineation g agrees with f_{km} on $S \cap P$, g takes p', a', b', c' to p, a, b, c, respectively. But then p, a, b, c are coplanar since p', a', b', c' are such. Thus $a \in P$. But then, by **AxLine** and $a, p \in \mathsf{tr}_m(\mathsf{ph})$, we have $\mathsf{tr}_m(\mathsf{ph}) \subseteq P$. QED (Lemma 3)

Lemma 4. *Assume* $\mathbf{LocRel_0^-}$. *Assume* $m \in \mathsf{Ob}$, $p, a, b \in {}^n\mathsf{F}$ *are non-collinear points,* $p \in \mathsf{cd}(m)$, *the plane that contains* p, a, b *is vertical,* pa *is an observer line for* m *and* $(\nexists q \in [a, b])(pq$ *is a photon line for* $m)$.

Then pb *is an observer line for* m.

Proof: Assume $\mathbf{LocRel_0^-}$. Assume m, p, a, b satisfy the assumptions. Then $pa \cap \mathsf{cd}(m) = \mathsf{tr}_m(k)$, for some $k \in \mathsf{Ob}$. Let such a k be fixed. We are in the coordinate-system of m. Let P be the vertical plane that contains p, a, b. Let S be a ball with center p such that f_{mk} agrees with a $\mathcal{P}$-collineation on $S \cap P$. Such an S exists by Proposition 5.

Let $c, d \in S \cap P$ be such that $\mathsf{Bw}(p, c, a)$ and $\mathsf{Bw}(p, d, b)$. See Figure 17. Then

$$(\nexists q \in [c, d])(pq \text{ is a photon line for } m). \tag{3}$$

By **AxLine** and $S \cap P \subseteq \mathsf{Dom}(\mathsf{f}_{mk}) \subseteq \mathsf{cd}(m)$, $p, c \in \mathsf{tr}_m(k)$.

Let us switch over to the coordinate-system of k. Let p', c', d' be, respectively, the f_{mk} images of p, c, d. Since f_{mk} agrees with a $\mathcal{P}$-collineation on

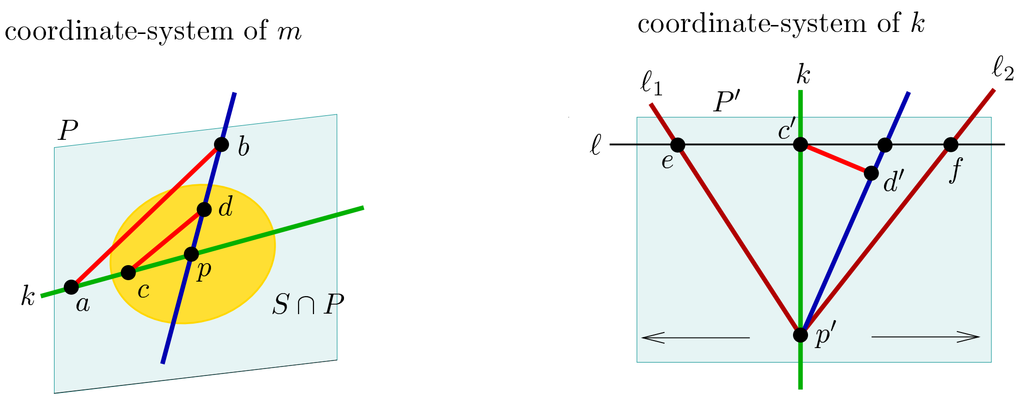

Figure 17: Illustration for the proof of Lemma 4.

$S \cap P$, p', c', d' are non collinear. Furthermore, f_{mk} takes $[c, d]$ onto $[c', d']$ by Proposition 6. Therefore, by (3),

$$(\nexists q \in [c', d'])(p'q \text{ is a photon line for } k). \tag{4}$$

Let P' be the plane that contains p', d', c'. $p', c' \in \mathsf{tr}_k(k)$ by $p, c \in \mathsf{tr}_m(k)$. By **AxSelf**, $\mathsf{speed}(p'c') = 0$. Thus P' is a vertical plane. Let $\ell \in \mathsf{Lines}$ be such that $\mathsf{speed}(\ell) = \infty$ and $c' \in \ell \subseteq P'$. By **Ax∃Ph**, **AxP1**, **AxFin**, there are exactly two photon-lines for k in plane P' passing through p, cf. Claim 2 on p.257. Let ℓ_1, ℓ_2 be these photon lines. $\mathsf{speed}(\ell_1), \mathsf{speed}(\ell_2) \in {}^{+}\mathsf{F}$ by **AxFin**. Let $e \in \ell \cap \ell_1$ and $f \in \ell \cap \ell_2$. Then $\mathsf{Bw}(e, c', f)$ by **AxP1**. By (4), neither ℓ_1 nor ℓ_2 intersects $[c', d']$. But then, by **AxOb**, $p'd'$ is an observer line for k, cf. Claim 2 on p.257. Thus, $pd = pb$ is an observer line for m, too.

QED (Lemma 4)

Proof of Theorem 5: Assume $n > 2$ and $\mathbf{LocRel}_0^-$. We will show that if there is an FTL observer, then there is a photon with infinite speed. This will contradict **AxFin**.

Assume there is an FTL observer, i.e. there are $m, k \in \mathsf{Ob}$, $\mathsf{ph} \in \mathsf{Ph}$, $d \in \mathsf{dir}$ and $p \in {}^{n}\mathsf{F}$ such that k and ph move in direction d as seen by m, $p \in \mathsf{tr}_m(k) \cap \mathsf{tr}_m(\mathsf{ph})$ and $\mathsf{speed}_m(k) > \mathsf{speed}_m(\mathsf{ph})$. Let such m, k, ph, d, p be fixed. See Figure 18.

By **AxLine** and **AxOpen**, there are unique $\ell_k, \ell_{\mathsf{ph}} \in \mathsf{Lines}$ such that $\mathsf{tr}_m(k) = \ell_k \cap \mathsf{cd}(m)$ and $\mathsf{tr}_m(\mathsf{ph}) = \ell_{\mathsf{ph}} \cap \mathsf{cd}(m)$. Let such ℓ_k, ℓ_{ph} be fixed. Let

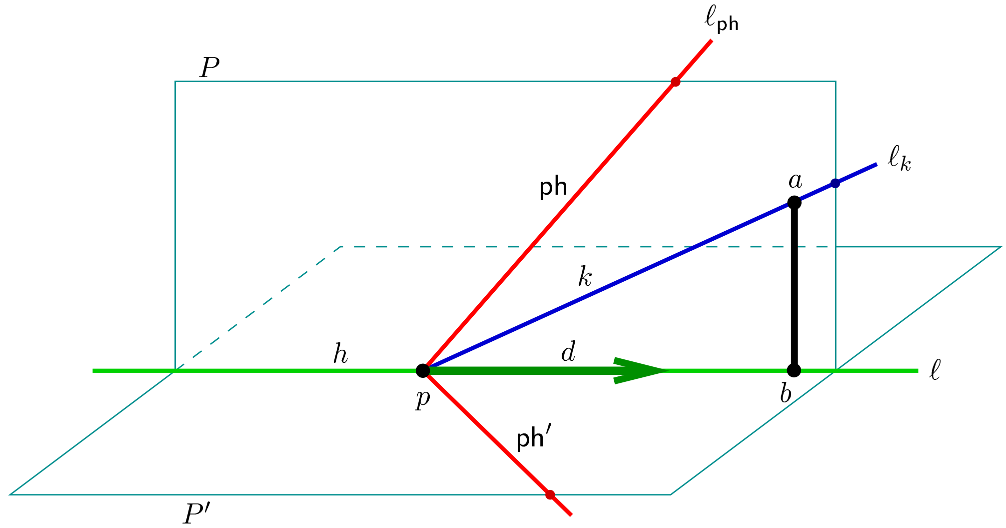

Figure 18: Illustration for the proof of Theorem 5.

P be the plane that contains ℓ_k and ℓ_{ph}. P is vertical since k and ph move in the same direction. Let $\ell \in \mathsf{Lines}$ be such that $p \in \ell \subseteq P$ and $\mathsf{speed}(\ell) = \infty$. We will show that ℓ is an observer line for m. Since ℓ_k is an observer line, we can assume that $\ell \neq \ell_k$. Since k is an FTL observer (w.r.t. m), $\mathsf{speed}(\ell_k) \neq 0$. Let $a \in \ell_k$ and $b \in \ell$ be such that $p \notin \{a, b\}$ and $\mathsf{speed}(ab) = 0$. Since k moves faster than ph (as seen by m), $\ell_{\mathsf{ph}} \cap [a, b] = \emptyset$. Hence, by **AxP1**, for any photon line ℓ' for m that passes through p, we have $\ell' \cap [a, b] = \emptyset$. Thus, by Lemma 4, $\ell = pb$ is an observer line for m.

Let $h \in \mathsf{Ob}$ be such that $\mathsf{tr}_m(h) = \ell \cap \mathsf{cd}(m)$. Let P' be a plane such that $\ell \subseteq P'$ and any line contained in P' has infinite speed. There is such a plane by $n > 2$. Clearly, $p \in \mathsf{tr}_k(h) \subseteq P'$. By Lemma 3, there is a photon ph' such that $p \in \mathsf{tr}_m(\mathsf{ph}') \subseteq P'$. Then, by **AxLine** and **AxOpen**, $\mathsf{speed}_m(\mathsf{ph}') = \infty$ for this ph'. But this contradicts **AxFin**.

QED (Theorem 5)

4 Bibliography

[1] H. Andréka, J. X. Madarász, and I. Németi. Logical analysis of special relativity. In J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, editors, *JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday*, http://www.wins.uva.nl/˜j50/cdrom, 1999. Vossiuspers, Amsterdam University Press. CD-ROM, ISBN: 90 5629 104 1, 23pp.

[2] H. Andréka, J. X. Madarász, and I. Németi. On the logical structure of relativity theories. Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, July 5, 2002. With contributions from A. Andai, G. Sági, I. Sain and Cs. Tőke. http://www.math-inst.hu/pub/algebraic-logic/Contents.html. 1312 pp.

[3] H. Andréka, J. X. Madarász, and I. Németi. Logical axiomatizations of space-time. Samples from the literature. In: *Non-Euclidean geometries (János Bolyai Memorial Volume).* Kluwer Scientific Publishers, to appear. http://www.math-inst.hu/pub/algebraic-logic/Contents.html

[4] H. Andréka, J. X. Madarász, and I. Németi. Logical analysis of relativity theories. In the present volume.

[5] H. Andréka, J. van Benthem, and I. Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 65:217–274, 1998.

[6] H. Busemann. Time-like spaces. *Dissertationes Math. (Rozprawy Math.)*, 53, 1967.

[7] B. Csikós. Projective geometry. Lecture notes. http://www.cs.elte.hu/geometry/csikos/proj/proj.html.

[8] M. Friedman. *Foundations of Space-Time Theories. Relativistic Physics and Philosophy of Science.* Princeton University Press, 1983.

[9] R. Goldblatt. *Orthogonality and Spacetime Geometry*. Springer-Verlag, 1987.

[10] A. K. Guts. Axiomatic relativity theory. *Russian Math. Survey*, 37(2):41–89, 1982.

[11] R. W. Latzer. Nondirected light signals and the structure of time. *Synthese*, 24:236–280, 1972.

[12] J. A. Lester. Distance preserving transformations. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 921–944. Elseview Science B. V., 1995.

[13] J. X. Madarász. *Logic and Relativity (in the light of definability theory)*. PhD thesis, ELTE, Budapest, http://www.math-inst.hu/pub/algebraic-logic/Contents.html., 2002.

[14] J. X. Madarász and Cs. Tőke. Partial relativity theories. Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, July, 2003.

[15] D. B. Malament. The class of continuous timelike curves determines the topology of space-time. *J. Mathematical Phys.* 18 (1977), 1399-1404.

[16] M. Marx, L. Pólos, and M. Masuch, editors. *Arrow logic and multi-modal logic.* CSLI Publications and FOLLI, Stanford, 1996.

[17] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* W. H. Freeman, San Francisco, 1973.

[18] I. Németi, A fine-structure analysis of first-order logic. In [16], pp.221-247.

[19] H. Reichenbach. *Axiomatization of the theory of relativity.* University of California Press, Berkeley, 1969. Translated by M. Reichenbach. Original German edition published in 1924.

[20] W. Rindler. *Essential relativity. Special, general and cosmological.* Springer-Verlag, Berlin, 1969, 1977.

[21] G. Székely. Personal communication. 2003 November.

[22] J. Väänänen. Second-order logic and foundations of mathematics. *The Bulletin of Symbolic Logic*, 7(4):504–520, 2001.

[23] J. Woleński. First-order logic: (philosophical) pro and contra. In the present volume.