A finite axiomatization of locally square cylindric-relativized set algebras*

by

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Abstract

We give a finite set of equations axiomatizing the class G_n of locally square cylindric-relativized set algebras of dimension n, if n is finite. For infinite n, we give an axiomatization of the equational theory of G_n. Here G_n denotes the class of all cylindric-relativized set algebras of dimension n with unit a union of Cartesian spaces.

Let n be an ordinal. We will deal with algebras of n-ary relations. A cylindric-relativized set algebra of dimension n is an algebra of n-ary relations. In more detail, an algebra $\mathfrak{A} = \langle A, +, -, c_i, d_{ij} \rangle_{i,j<n}$ is a *cylindric-relativized set algebra* of dimension n (a Crs_n) if the following (i)-(ii) hold.

(i) $\langle A, +, - \rangle$ is a Boolean set algebra whose elements are n-ary relations. Let V denote the greatest element of this algebra. V is called the *unit* of $\mathfrak{A}$. Thus $+, -$ denote the operations of taking union and taking complement w.r.t. V.

(ii) The additional operations are as follows. Let $i, j < n$. Then d_{ij} is a constant and c_i is a unary operation defined as follows.

$$d_{ij} \stackrel{\text{def}}{=} \{z \in V : z_i = z_j\} .$$

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Let $x \in A$. Then x is an n-ary relation, i.e. x is a set of n-sequences. Now $c_i(x)$ is the set of those n-sequences from V which agree everywhere but on i with a sequence in x; formally

$$c_i(x) \stackrel{\text{def}}{=} \{z \in V : (\exists s \in x)(\forall k < n, k \neq i) z_k = s_k\}.$$

Thus d_{ij} and $c_i(x)$ are n-ary relations. We require that $d_{ij} \in A$ and $c_i(x) \in A$ for all $i, j < n$ and $x \in A$.

Crs_n's are Boolean algebras with additional unary operators c_i and constants d_{ij} (Bo_n's, cf. [14]). In some sense, these are the most natural algebras of n-ary relations.

Crs_n denotes the class of all cylindric-relativized set algebras of dimension n, and $Gs_n \subseteq G_n \subseteq D_n \subseteq Crs_n$ are subclasses of Crs_n defined by making restrictions on the biggest element (unit) of the algebra. We briefly recall their definitions below.

By a *Cartesian space* we mean the set of all U-termed n-sequences for some set U, where s is a *U-termed n-sequence* if $s = \langle s_i \rangle_{i<n}$ and $s_i \in U$ for all $i < n$. We will consider sequences to be functions, and an ordinal to be the set of all smaller ordinals. Thus, an n-sequence $s = \langle s_i \rangle_{i<n}$ is a function with domain $n = \{i : i < n\}$. A U-termed n-sequence s is then a function $s : n \to U$. If $\tau : n \to n$ is a transformation on n, then $s \circ \tau = \langle s_{\tau(i)} \rangle_{i<n}$ is a new sequence, namely the sequence we obtain from s by "rearranging" it according to τ. $\tau : n \to n$ is called *permutational* if it is a bijection. We call τ *finite* if τ moves only finitely many elements, i.e. if $\{i < n : \tau(i) \neq i\}$ is finite.

Let $\mathfrak{A}$ be a cylindric-relativized set algebra of dimension n with unit V. Then

$\mathfrak{A} \in D_n$ iff for every $s \in V$ and every finite nonpermutational transformation τ of n, also $s \circ \tau$ is in V.

$\mathfrak{A} \in G_n$ iff V is a union of Cartesian spaces.

$\mathfrak{A} \in Gs_n$ iff V is a disjoint union of Cartesian spaces.

An equivalent definition of G_n is that $s \circ \tau \in V$ for every transformation τ of n and sequence $s \in V$.

For a class K of algebras, IK denotes the class of all isomorhic copies of elements of K. Now, $ICrs_n, ID_n, IG_n, IGs_n$ are all distinct classes, equations distinguishing them are given in [19, 20].

Let n be finite. In this paper we give a finite equational axiom system for G_n. It was already known that IG_n is axiomatizable by equations, i.e. it is a *variety* (Németi [19]), moreover it is a canonical variety, i.e. it is closed under perfect extensions (Andréka-Goldblatt-Németi [3]). These will also follow from the theorem in the present paper because the axiomatization we give contains only positive equations (i.e. Boolean negation $-$ does not occur in the extra-Boolean equations, but Boolean product $\cdot$ may occur in them).

If n is infinite, then we do not know whether IG_n is a variety or not. We do not even know whether IG_n is axiomatizable with first-order formulas or not. In this paper we give a finite scheme of equations axiomatizing the equational theory of G_n. This finite scheme is a natural generalization of the axiom system given for the finite case. More on IG_n can be found in [3, 4, 13, 15, 19, 20, 21].

Let now $n \geq 3$ be arbitrary. To contrast the above results, we note that $ICrs_n, ID_n, IGs_n$ are all varieties, $ICrs_n$ and IGs_n are not finitely axiomatizable, and a finite axiomatization for ID_n is given in [6]. There is a difference between non-finite (schema) axiomatizability of Gs_n and Crs_n, however: Gs_n cannot be axiomatized with any axiom system that contains only finitely many variables ([1]), while an (infinite) equational axiomatization of Crs_n which uses only two variables is given in [16] (this system is due to D. Resek and R. Thompson). Axiomatizations for Gs_n are surveyed in [9], a recent new axiom system for Gs_n is in [11]. A rich material on these classes can be found, besides the above cited references, e.g. in [7, 5, 10, 9, 17, 18].

We now turn to giving the equational axiomatization of IG_n.

Let n be a finite number and let $i, j < n$. Let Ax_{ij} be the following equation:

$$x \leq c_i c_j (s^i_j c_j x \cdot s^j_i c_i x \cdot \prod_{k<n, k \neq i,j} s^k_i s^i_j s^j_k c_k x).$$

In the above, $\prod$ is the grouped version of $\cdot$, and $s^i_j(x) = c_i(d_{ij} \cdot x)$ if $i \neq j$ and $s^i_i(x) = x$. We will see from the proof of the next theorem that Ax_{ij} intuitively says that for any sequence in the unit, we can also put into the unit the sequence we obtain from it by interchanging its i'th and j'th elements. From the equation Ax_{ij} we will only use its following consequence:

$$x \neq 0 \quad \rightarrow \quad s^i_j c_j x \cdot s^j_i c_i x \cdot \prod_{k<n, k \neq i,j} s^k_i s^i_j s^j_k c_k x \neq 0.$$

Intuitively, this ensures that if a sequence is in x, then its permuted version can be put into $s^i_j c_i x \cdot \ldots$.

To extend these equations for infinite n, assume now that n is any ordinal, possibly infinite. For all $i, j < n$ and all finite subsets Γ of n let $Ax_{ij\Gamma}$ denote the following equation:

$$x \leq c_i c_j (s^i_j c_j x \cdot s^j_i c_i x \cdot \prod_{k \in \Gamma, k \neq i,j} s^k_i s^i_j s^j_k c_k x).$$

Let Ax_n be the set of all Ax_{ij}, for $i, j < n$, and let $Ax'_n = \{Ax_{ij\Gamma} : i, j < n, \Gamma \subseteq n, \Gamma \text{ finite }\}$. We will simply write Ax and Ax' if there is no danger of confusion. IG_n, HG_n denote the classes of all isomorphic copies and all homomorphic images of elements of G_n, respectively. We note that a simple finite set of equations (equation schemes in the case when n is infinite) which axiomatizes ID_n is given in [6]. Therefore Theorem 1 below gives axiom systems for IG_n and HG_n; and these axiom systems are finite if n is finite.

Theorem 1 (Axiomatizations of IG_n and HG_n)

(i) $IG_n = \{\mathfrak{A} \in ID_n : \mathfrak{A} \models Ax\}$, *if n is finite, $n \geq 3$.*

(ii) $HG_n = \{\mathfrak{A} \in ID_n : \mathfrak{A} \models Ax'\}$ *for all $n \geq 3$.*

(iii) $IG_2 = \{\mathfrak{A} \in ID_n : \mathfrak{A} \models x - d_{01} \leq c_0 c_1 (-d_{01} \cdot s^0_1 c_1 x \cdot s^1_0 c_0 x)\}$.

Proof. Let $n \geq 2$. First we show that $G_n \models Ax'$. Let $\mathfrak{A} \in G_n$ with unit V, $x \in A$ and let $s \in x$. Let $i, j, k < n$, $k \neq i, j$. Let $z = s \circ [i, j]$, where $[i, j] : n \to n$ is the permutation of the set $n = \{\ell : \ell < n\}$ which interchanges i and j and leaves all the other elements fixed. Then $z \in V$ and it is easy to check that $s \in c_i c_j(\{z\})$, $z \in s^i_j c_j x$, $z \in s^j_i c_i x$, $z \in s^k_i s^i_j s^j_k c_k x$. This shows that $\mathfrak{A} \models Ax_{ij\Gamma}$ for all $i, j < n$ and for all finite $\Gamma \subseteq n$. This shows $G_n \models Ax'$. The same argument shows that G_2 satisfies the indicated equation (notice that if $s \in x - d_{01}$, then $s_0 \neq s_1$, and so z also is in $-d_{01}$).

To show the other direction of (i)-(iii), we first assume that n is finite $n \geq 2$. The proof below is an extension of the one in [6], and it will also show some similarities with the proofs in [2].

First we make an observation. Let V be a D_n-unit. Then

(*) If $s \in V$ has a repetition, then $s \circ \tau \in V$ for all $\tau : n \to n$.

Indeed, if τ is nonpermutational, then $s \circ \tau \in V$ by the definition of a D_n-unit and since n is finite. If τ is permutational, then $s \circ \tau = s \circ \tau'$ for some nonpermutational τ' because s has a repetition. (Indeed, assume $s_i = s_j$ for $i, j < n$, $i \neq j$. Let τ' be such that $\tau'(m) = \tau(m)$ if $m \neq \tau^{-1}(j)$, and $\tau'(\tau^{-1}(j)) = \tau(i)$.) This shows that (*) holds. By (*) we have that if a D_n-unit V is such that $s \circ \tau \in V$ for all repetition-free $s \in V$ and for all bijections τ, then V is a G_n-unit.

Let $\mathfrak{A} \in ID_n$, and assume that $\mathfrak{A} \models Ax$. We want to show that $\mathfrak{A}$ is isomorphic to an $\mathfrak{A}'' \in G_n$.

It is proved, implicitly, in [6] that every algebra $\mathfrak{A} \in ID_n$ is isomorphic to an $\mathfrak{A}' \in D_n$ such that the unit V of $\mathfrak{A}'$ satisfies (**) below:

(**) For all repetition-free sequences $s, z \in V$, the ranges of s and z are different, i.e. $Rng(s) \neq Rng(z)$.

Indeed, in the proof of Theorem 1 in [6], to any $\mathfrak{A} \in ID_n$ we define a set V such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}' \in D_n$ with unit V. The construction of V is given on p.674 in [6], and conditions (b),(c),(d) on the same page ensure that (**) holds for V.

(**) means, in other words, that if $s \in V$ and s is repetition-free, then no other permuted version of s is in V. By (**), the unit of $\mathfrak{A}'$ satisfies the weaker condition (***) below:

(***) For all repetition-free sequences $s \in V$, either $s \circ \tau \in V$ for all bijections $\tau : n \to n$ or else $s \circ \tau \notin V$ for all bijections $\tau : n \to n$ which are not the identity on n.

The idea of the following proof is that we can "throw in" the permuted versions of the repetition-free sequences in V such that "$\mathfrak{A}'$ will not change". Ax_{ij} will tell us "where to put the new sequence $s \circ [i, j]$". In more detail, assume that $\mathfrak{A}' \in D_n$ has unit V such that (***) holds. Let $s \in V$ be a repetition-free sequence such that "no permuted version of s is in V". Let $S = \{z_0, z_1, \ldots, z_N\}$ be a listing of all the permuted versions of s— i.e. $S = \{s \circ \tau : \tau \text{ is a bijection of } n\}$— such that

$$z_0 = s$$

$$z_i = z_j \circ [k, \ell] \quad \text{for some} \quad j < i, \; k, \ell < n$$

$$z_i \neq z_j \quad \text{if} \quad i \neq j.$$

Such a listing is possible. Then $V \cup S$ also satisfies (***). We will "represent" $\mathfrak{A}'$ on $V \cup S$, i.e. we will show that $\mathfrak{A}'$ is isomorphic to an $\mathfrak{A}''$ with unit $V \cup S$. (We will do this by putting $z_0, z_1, \ldots, z_N$ into the representation, one by one. In each step we will use an axiom $Ax_{k\ell}$ to tell us "where to put" z_i.) By an induction then $\mathfrak{A}$ is isomorphic to an $\mathfrak{A}'' \in D_n$ with unit V such that all permuted versions of all repetition-free members of V are also in V. Then V is a G_n-unit by (*) and we will be done.

We may assume that $\mathfrak{A}$ is atomic, because of the following. An equation in the language of Bo_n's is called *positive in the wider sense* if complementation $-$ occurs only in front of some constant terms or in form of Boolean meet $\cdot$. Every Boolean algebra with operators $\mathfrak{B}$ can be embedded into an atomic one $\mathfrak{B}'$ such that all the equations valid in $\mathfrak{B}$ which are positive in the wider sense continue to hold in the atomic one, $\mathfrak{B}'$, by [12, 2.15, 2.18] (see also [8, 2.7.5, 2.7.13]). All the axioms listed in this paper as well as the axioms defining ID_n in Bo_n given in [6] are positive in the wider sense. Any representation for $\mathfrak{B}'$ gives a representation for $\mathfrak{B} \subseteq \mathfrak{B}'$ with the same unit. Thus we may assume that $\mathfrak{A}$ is atomic.

A cylindric-relativised set algebra $\mathfrak{B} \in Crs_n$ is called *completely represented* if every sequence in the unit of $\mathfrak{B}$ is in some atom, i.e. if the unit of $\mathfrak{B}$ is the union – and not only the supremum – of the atoms of $\mathfrak{B}$. We also may assume that $\mathfrak{A}$ is completely represented, because of the following. The representation for $\mathfrak{A} \in ID_n$ given on p.674 in [6] is a complete one, and in the above outlined induction step we obtain a complete representation from a complete one.

We say that an atom of $\mathfrak{A}$ is *repetition-free* if it is below no d_{ij} (i.e. if it is below $\prod_{i<j<n} -d_{ij}$). Let a be an arbitrary repetition-free atom of $\mathfrak{A}$ and let $i, j < n,\ i \neq j$.

Let now $n \geq 3$ and let

$$b \leq s^i_j c_j a \cdot s^j_i c_i a \cdot \prod_{k \neq i,j} s^k_i s^i_j s^j_k c_k a$$

be an atom. Such an atom exists by $\mathfrak{A} \models Ax$. We will show that b is repetition-free.

We will work in the *atom-structure* of $\mathfrak{A}$ for a while. I.e. let At be the set of all atoms of $\mathfrak{A}$, let $E_{ij} = \{x \in At : x \leq d_{ij}\}$ and let $T_i = \{\langle x, y\rangle : x, y \in At, c_i x = c_i y\}$. Let $E_{ijk} = E_{ij} \cap E_{jk}$. We will freely use the following properties of atom-structures: For all $x, y \in At$

$x \in E_{ij},\ xT_k y$ imply $y \in E_{ij}$ if $k \neq i, j$.

$x, y \in E_{ij},\ xT_i y$ imply $x = y$.

$E_{ij} \cap E_{jk} \subseteq E_{ik} = E_{ki}$.

For any $x \in At$ and $i, j < n,\ i \neq j$ there is a unique $y \in At$ such that $xT_i y \in E_{ij}$.

It is not difficult to check that the first three statements hold in the atom-structure of any $\mathfrak{A} \in Crs_n$ while the last statement holds in the atom-structure of any $\mathfrak{A} \in D_n$.

First we show that $b \in E_{ij}$ would imply that $a \in E_{ij}$. Let $k < n$, $k \neq i, j$. Then $b \leq s_i^k s_j^i s_k^j c_k a$, and this means that there are atoms x, y, z of $\mathfrak{A}$ such that $bT_k x, x \in E_{ik}, xT_i y, y \in E_{ij}, yT_j z, z \in E_{jk}, zT_k a$, see Figure 1.

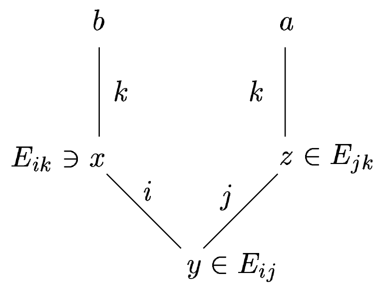

Figure 1:

Now, $b \in E_{ij}$ implies $x \in E_{ijk}$, which implies that $y = x$, $y \in E_{ijk}$ which imply that $z = y$, $z \in E_{ijk}$, which imply that $a \in E_{ij}$.

Next we show that $b \in E_{ik}$ would imply that $a \in E_{jk}$. By $b \leq s_i^j c_i a$ there is an atom x such that $bT_j x \in E_{ij}, xT_i a$, see Figure 2.

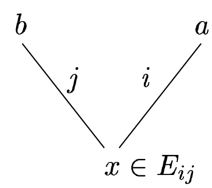

Figure 2:

Now $b \in E_{ik}$ implies $x \in E_{ijk}$ which implies that $a \in E_{jk}$.

Finally we show that $b \in E_{k\ell}$ would imply $a \in E_{k\ell}$ for all $k \neq \ell$, $j, i \neq k, \ell$. Let x be an atom as in Figure 2. Then $b \in E_{k\ell}$ implies that $x \in E_{k\ell}$ which implies that $a \in E_{k\ell}$.

Thus b is repetition-free, because a is repetition-free.

Assume now that $n \geq 2$ and $i, j \leq n,\ i \neq j$. For any repetition-free atom a choose a repetition-free atom b below $s^i_j c_j a \cdot s^j_i c_i a \cdot \prod_{k \neq i,j} s^k_i s^i_j s^j_k c_k a$ and let

$$f_{ij}(a) = b.$$

Such a b exists, because if $n \geq 3$, then we have seen this above, and if $n = 2$, then this holds by the axiom we required to hold.

Let now s be the repetition-free sequence we chose at the beginning of this proof, and recall the definition of $S = \{z_0, z_1, \ldots, z_N\}$. We will define a sequence $a_0, a_1, \ldots, a_N$ of repetition-free atoms. Let a be an atom such that $s \in a$. Such an atom exists by our assumption that $\mathfrak{A}$ is completely represented.

Let

$$a_0 \stackrel{\text{def}}{=} a.$$

Assume that a_ℓ has been defined for all $\ell < i$. Let $\ell < i$ and $m \neq j$ be such that $z_i = z_\ell \circ [m, j]$. (If there are several such ℓ, m, j, then we just select one such triple.) Now we define

$$a_i \stackrel{\text{def}}{=} f_{mj}(a_\ell).$$

We are ready to define the new representation of $\mathfrak{A}$.: For any $x \in A$ define x' as

$$x' \stackrel{\text{def}}{=} x \cup \{z_i : a_i \leq x,\ i < N\}.$$

(Intuitively, this means that "we put the sequences z_i into the atoms a_i".)

Then $V' = V \cup S$, so the unit of the new representation will be $V \cup S$ as desired. We have to show that the function $x \mapsto x'$ is an embedding of $\mathfrak{A}$ into the full (i.e. biggest) set algebra with unit $V \cup S$. Let us denote this function $x \mapsto x'$ by h.

Clearly, h is a one-to-one Boolean embedding, and $h(d_{ij}) = d_{ij}$ for all $i, j < n$. To show that h respects the cylindrifications c_k, first we prove an auxiliary statement.

Notation: If z is an n-sequence and $k, \ell < n$, then $z(k/z_\ell)$ denotes the sequence which agrees everywhere with z except on k, where it is the same as z_ℓ.

Lemma 2 *Assume that $z \in S$ and $z \in x'$, $x \in At$. Let $k, \ell < n,\ k \neq \ell$. Then for all $b \in At$ we have that*

$$(\star) \qquad z(k/z_\ell) \in b \quad \text{iff} \quad xT_k b \in E_{k\ell}.$$

Proof. By $z \in S$ there is an $m \leq N$ such that $z = z_m$. The proof will proceed by induction on m.

Assume that $m = 0$. Then $z_0 = s \in V$ and also $s \in x = a$ by $s \in x'$ and $s \in a = a_0$. Thus $(\star)$ is true because $\mathfrak{A}$ is a set algebra with D_n-unit V.

Assume that $(\star)$ is true for all $k, \ell < n$ and for all $m < p$. We will show that $(\star)$ is true for p, too. Let $z_p = z_m \circ [i, j]$ be such that $m < p$ and $a_p = f_{ij}(a_m)$. There are such m, i, j by the definition of a_p. Then $x = a_p$ by $z_p \in x'$, and also $z_m \in a_m$.

First we prove that $xT_k b \in E_{k\ell}$ imply that $z(k/z_\ell) \in b$.

<u>Case 1</u> $k \neq i, j$ and $\ell = i$ or $\ell = j$. Assume first that $\ell = j$.

By $a_p = f_{ij}(a_m)$ we have that $a_p \leq s^k_i s^i_j s^j_k c_k(a_m)$. Since in D_n the so called Merry Go Round equation $s^k_i s^i_j s^j_k c_k x = s^k_j s^j_i s^i_k c_k x$ is true, then we also have that $x = a_p \leq s^k_j s^j_i s^i_k c_k(a_m)$. Thus there are atoms d, e, b^+ such that $xT_k b^+, b^+ \in E_{kj}, b^+ T_j e, e \in E_{ij}, eT_i d, d \in E_{ki}, dT_k a_m$. By $xT_k b \in E_{kj}$ then $b^+ = b$ (by the basic properties of atom-structures). See Figure 3.

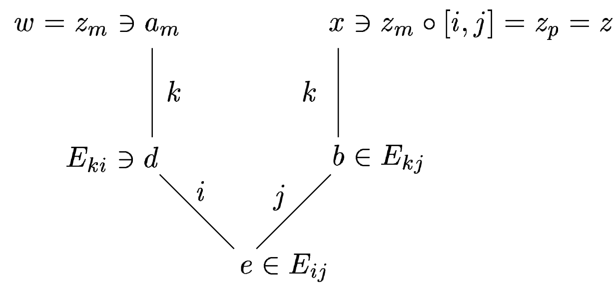

Figure 3:

Let $w = z_m$ and recall that $z = z_p$. Then $w(k/w_i) \in d$ by our induction hypothesis, and therefore $w(k/w_i)(i/w_j) \in e$ because $\mathfrak{A}$ is a set algebra. But then $w(k/w_i)(i/w_j)(j/w_i) \in b$, again because $\mathfrak{A}$ is a set algebra (notice that w_i is the k'th member of the sequence $w(k/w_i)(i/w_j)$). Finally notice that $z(k/z_j) = w(k/w_i)(i/w_j)(j/w_i)$.

The case $\ell = i$ is completely similar, except that we do not have to use the Merry Go Round equation.

<u>Case 2</u> $k \neq i, j$ and $\ell \neq i, j$.

Then $z(k/z_\ell) = z(k/z_j)(k/z_\ell)$ and therefore we will use the previous case. Let $b^+ \in E_{kj}$ be such that $xT_k b^+$. Such a b^+ exists by basic properties of atom-structures. Then by Case 1 we have that $z(k/z_j) \in b^+$, and then $z(k/z_j)(k/z_\ell) \in b$, because $\mathfrak{A}$ is a set algebra, $z(k/z_j) \in V$ and $b^+ T_k b \in E_{k\ell}$.

Case 3 $k = i$ and $\ell = j$.

By $a_p \leq f_{ij}(a_m)$ we have that $a_m \leq s^i_j c_i(a_p)$, see Figure 4.

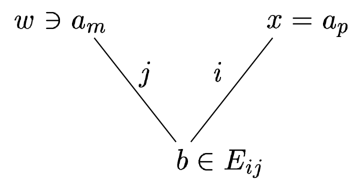

Figure 4:

Now, $w \in a_m$ implies, by the induction hypothesis, that $w(j/w_i) \in b$. But $w(j/w_i) = z(i/z_j)$.

The case $k = i,\ \ell \neq i, j$ is as above. The case $k = j$ is completely analogous.

Thus we have seen that $xT_k b \in E_{k\ell}$ imply that $z(k/z_\ell) \in b$. To see the other direction, assume that $z(k/z_\ell) \in b$. There is a b^+ such that $xT_k b^+ \in E_{k\ell}$, and we have seen that $z(k/z_\ell) \in b^+$ for this b^+. Then $b = b^+$ because distinct atoms are disjoint from each other, and so $bT_k x$ since $b^+ T_k x$. **QED(Lemma 2)**

Now we are ready to show that h is a homomorphism w.r.t. c_i. Let $j < n,\ j \neq i$. We will use that two sequences z and w differ only at i if and only if $z(i/z_j) = w(i/w_j)$.

To show that h is a homomorphism w.r.t. c_i amounts to showing (I)-(II) below for all atoms a, b and sequences $z, w \in V \cup S$:

(I) $z \in a,\ w \in b$ and $z(i/z_j) = w(i/w_j)$ imply that $aT_i b$

(II) $z \in a$ and $aT_i b$ imply that $z(i/z_j) = w(i/w_j)$ for some $w \in b$.

To prove (I)-(II), let a, b be two atoms, and let $aT_i a^+ \in E_{ij}$, $bT_i b^+ \in E_{ij}$. Then $z \in a$ implies $z(i/z_j) \in a^+$ if $z \in S$ by the previous lemma, and if $z \in V$, then this is so since $\mathfrak{A}$ is a set algebra and $z(i/z_j) \in V$. Similarly, $w \in b$ implies $w(i/w_j) \in b^+$ for all $w \in V \cup S$.

Proof of (I): Assume $z \in a,\ w \in b,\ z(i/z_j) = w(i/w_j)$. Then $z(i/z_j) \in a^+$ and $w(i/w_j) \in b^+$ by the above, hence $a^+ = b^+$, which implies that $aT_i b$ by the definition of a^+, b^+.

Proof of (II): Assume $z \in a$, aT_ib. Then $z(i/z_j) \in a^+$. Further, aT_ib implies that $a^+ = b^+$. So $z(i/z_j) \in b^+$. Now since b^+T_ib, $z(i/z_j) \in V$, and since $\mathfrak{A}$ is a set algebra, $z(i/z_j) \in b^+$ implies that $w(i/w_j) = z(i/z_j)$ for some $w \in b$.

By the above we have seen that h is an embedding of $\mathfrak{A}$ into the full set algebra with unit $V \cup S$, i.e. $\mathfrak{A}$ "can be represented on $V \cup S$." Now repeating the above procedure along a transfinite induction we get a representation of $\mathfrak{A}$ on $G(V) = \{s \circ \tau : s \in V \text{ and } \tau : n \to n\}$. Since $G(V)$ is a G_n-unit, this shows that $\mathfrak{A} \in IG_n$. By this, (i), (iii) of Theorem 1 are proved.

Let n be infinite and let Σ_n be the set of equations given in [6] which axiomatizes ID_n. We want to show that $\Sigma_n \cup Ax'$ axiomatizes the equational theory of G_n. We have already seen that $G_n \models \Sigma_n \cup Ax'$. Let now e be an arbitrary equation that holds in G_n. We want to show that e follows from $\Sigma_n \cup Ax'$. Let Γ be a finite subset of n, $|\Gamma| \geq 3$, which strictly contains all the indices occurring in e (i.e. Γ contains some other index, too). By [20, Lemma 4.13(ii)] then $G_\Gamma \models e$. Here G_Γ is the natural generalization of G_m to the case when m is any set of indices (and not necessariy an ordinal). Let $Ax_\Gamma \stackrel{\text{def}}{=} \{Ax_{ij\Gamma} : i, j \in \Gamma\}$. By Theorem 1(i) then $IG_\Gamma = \{\mathfrak{A} \in ID_\Gamma : \mathfrak{A} \models Ax_\Gamma\}$ since Γ is finite, $|\Gamma| \geq 3$. Let Σ_Γ be the set of axioms given in [6] which axiomatizes ID_Γ. Thus $\Sigma_\Gamma \cup Ax_\Gamma \models e$ by $G_\Gamma \models e$. By looking into [6], we see that $\Sigma_\Gamma \subseteq \Sigma_n$ by $\Gamma \subseteq n$. Also, $Ax_\Gamma \subseteq Ax'$. Hence $\Sigma_n \cup Ax' \models e$, this is what we wanted to show. We have seen that $\Sigma_n \cup Ax'$ axiomatizes the equational theory of G_n. Since IG_n is closed under subalgebras and direct products, the variety generated by G_n is HG_n. Thus $\mathfrak{A} \in HG_n$ iff $\mathfrak{A} \models \Sigma_n \cup Ax'$ iff ($\mathfrak{A} \in ID_n$ and $\mathfrak{A} \models Ax'$). This proves (ii) of Theorem 1. **QED(Theorem 1)**

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