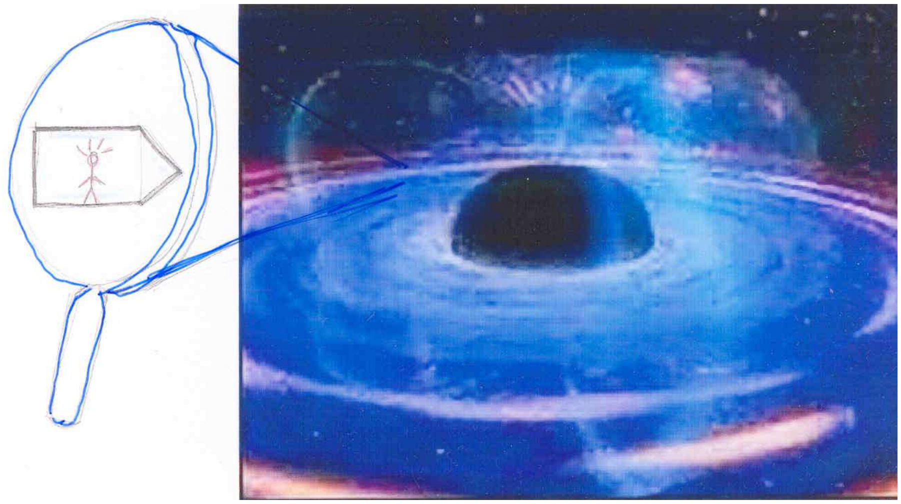

Figure 1.13. Einstein's locally special relativity principle: where-ever we drop a small enough spaceship, for a short enough time it will experience special relativity.

4.2 Space-time in general relativity

Next we implement Einstein's locally special relativity principle (formulated in subsection 4.1 above) for formalizing general relativity.

In this section, for simplicity, we will use $\mathbb{R}$ in place of an arbitrary linearly ordered quadratic field F, and also we will use $n = 4$. For a logical analysis of what we "sweep under the rug" by the $\mathsf{F} = \mathbb{R}$ assumption we refer to Madarász-Németi-Székely 2005. Also, a careful logical analysis of the Twin Paradox (acceleration causes time run slow) is elaborated there purely in FOL without the $\mathsf{F} = \mathbb{R}$ assumption. Further, the method there can be applied to studying the effects of gravity, e.g. on clocks, in particular to proving the Tower Paradox purely in FOL. The tower paradox means roughly that gravity causes slow time. E.g. clocks in the basement of a tower and the top floor of the tower run differently, the one in the basement being slower. This leads to the prediction that clocks on the event horizon of a black hole stand still. The techniques used there also indicate how the $\mathsf{F} = \mathbb{R}$ assumption can be eliminated from the material that comes below, and how what we do below can be nailed down in pure FOL. Having said this, in order for a better communication of the essential ideas, below we do make simplifying assumptions like $\mathsf{F} = \mathbb{R}$. We claim that they can and will be eliminated (on the long run) along the ideas in Madarász-Németi-Székely 2005.

For general relativity, we will use *global coordinate frames* GFR's. A global coordinate frame is based on an open subset of $\mathbb{R}^4$. So a global coordinate system looks like a special relativity frame, we even call one of the coordinates time, the others space etc. The difference is that in a global frame the coordinates do not carry any physical or intuitive meaning. They serve only as a matter of convention in gluing the local special relativity frames (LFR's) together. For simplicity, at the beginning we will pretend that the coordinate system of our global frame is the whole of $\mathbb{R}^4$. Later we will refine this to saying that the global frame is an open subset of $\mathbb{R}^4$. (And even later we will generalize this further, to be a manifold.) Since the differences are extremely minor and secondary (from the relativistic point of view), let us first pretend that the global frame is $\mathbb{R}^4$.

Imagine a general relativistic coordinate system, a GFR, representing the whole universe, with a black hole in the middle etc. So we are looking at the bare coordinate grid of $\mathbb{R}^4$ intending to represent the whole of space-time.

What is the first thing you (as author) would want to specify for your readers about the points of this grid $\mathbb{R}^4$? Well, it is how the local tiny little inertial special relativistic spaceships are associated to the points p of $\mathbb{R}^4$. (Recall that these tiny inertial spaceships come from Einstein's locally special relativity principle formulated in subsection 4.1.) More precisely, to every point p of $\mathbb{R}^4$ you would want to specify how the local special relativity space-time at point p is squeezed, distorted, rotated etc when it is squeezed into the local neighborhood of p . Rindler's book formulates this by saying that to each p of $\mathbb{R}^4$ we have to specify an LFR (Local Frame of Reference = local special relativity frame) sitting in there at p. The point is in specifying how the clocks of the LFR slow down or speed up at p, and which axis of the local LFR points in what direction and is distorted (shortened/lengthened) in what degree.

An important aspect of the present section is the following. We use that special relativistic space-time and Minkowski geometry have been defined (cf. subsection 2.7). We will use the local special relativity frames (the LFR's) for importing the notions of special relativity to our general relativistic space-time model.

Specifying the local frames LFR. A *local frame* m at p will be a bijective mapping $m : \mathbb{R}^4 \to \mathbb{R}^4$ such that $m(0) = p$. We will think of the first $\mathbb{R}^4$ as the coordinate system of special relativity or of the

Minkowski space and of the second $\mathbb{R}^4$ as the global frame upon which we want to build our general relativistic space-time model. Since we want to use our local frames only to specify how the tiny clocks slow down in the limit (roughly, in an "infinitesimally small" neighborhood of p) etc, we will choose these m's to be affine mappings. In the figure, m is denoted as $L(p)$, for reasons to be discussed below.

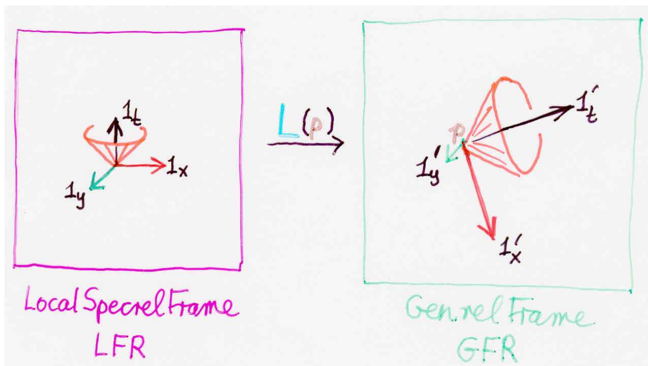

Figure 1.14. The local special relativity frame at p.

To specify the local frame at p, let $G(p) = \langle G_t(p), ..., G_z(p)\rangle$ be a 4-tuple of vectors (i.e. elements of $\mathbb{R}^4$) associated to point p in $\mathbb{R}^4$. Recall that our global frame coordinate system is (also) $\mathbb{R}^4$. Now, the sole purpose of the datum $G(p)$ is to specify a affine transformation $L(p)$ from the special relativistic space-time to our global frame coordinate system $\mathbb{R}^4$. Formally, this amounts to saying that $L(p) : \mathbb{R}^4 \to \mathbb{R}^4$ is the affine transformation mapping the origin $\langle 0, 0, 0, 0\rangle$ to p, $1_t = \langle 1, 0, 0, 0\rangle$ to $G_t(p) + p$, $1_x = \langle 0, 1, 0, 0\rangle$ to $G_x(p) + p$, $1_y = \langle 0, 0, 1, 0\rangle$ to $G_y(p) + p$, and $1_z = \langle 0, 0, 0, 1\rangle$ to $G_z(p) + p$. See Figures 1.14,1.15.

So, the key device in building our general relativity space-time is associating to each point p in $\mathbb{R}^4$ of our global coordinate grid an affine transformation $L(p)$ mapping the Minkowski space to the global frame $\mathbb{R}^4$. We will use the inverse $L(p)^{-1}$ of this affine transformation $L(p)$ to translate our general relativistic problems to special relativity, i.e. to Minkowski space, and we will use $L(p)$ to bring back the answers special relativity gives us.

A word of caution. We will use this translation $L(p)$ induced by $G(p)$ only in small enough neighborhoods of p (i.e. we will use it in the limit, more and more accurately as we close on p). Restricting attention to small neighborhoods of p is what is meant by saying that general

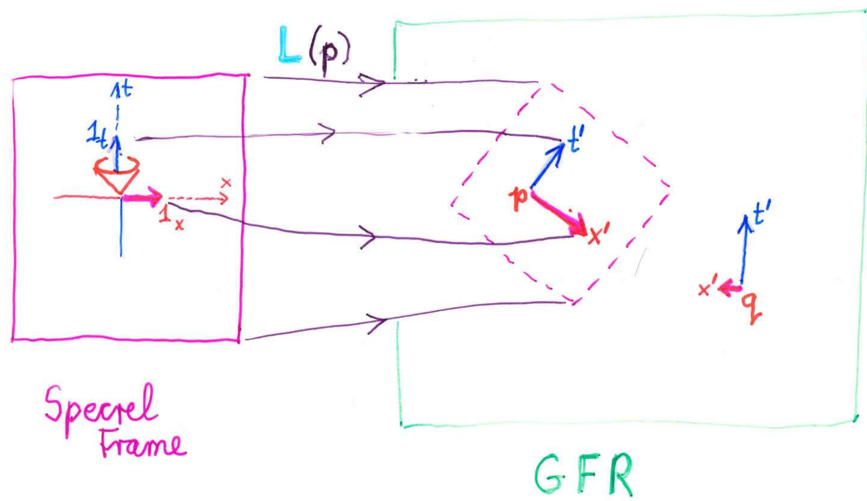

Figure 1.15. The local special relativity frame at p.

relativity can be *locally* reduced to special relativity, but only locally. If we want to solve a problem at a point q farther away from p , then we will have to use the mapping $L(q)$ associated to q in place of using $L(p)$.

Definition of a general relativity space-time model GRM. So we can specify how special relativity (*Specrel* in the following) sits in our global coordinate system at each point by associating four vectors $G_t(p)$, $G_x(p)$, $G_y(p)$, $G_z(p)$ to each point p of $\mathbb{R}^4$. The vectors $G_t(p)$ etc. are simply elements of $\mathbb{R}^4$ (i.e. ordinary vectors over $\mathbb{R}$). This amounts to specifying four vector fields $G_t : \mathbb{R}^4 \to \mathbb{R}^4$ etc. So, a general relativity space-time model is a 4-tuple $\langle G_t, ..., G_z \rangle$ of vector fields over $\mathbb{R}^4$. What are the conditions that this 4-tuple should satisfy (in order to qualify as a general relativistic space-time model)? They are (i)-(ii) below.

(i) Each $G_i : \mathbb{R}^4 \to \mathbb{R}^4$ is continuous, differentiable, and infinitely many times such, i.e. it is what is called a smooth function in analysis. $(i \in \{t, x, y, z\})$

(ii) The vectors $G_t(p), ..., G_z(p)$ at each point p should be linearly independent in the usual sense. (This means only that the affine mapping they specify is a bijection.)

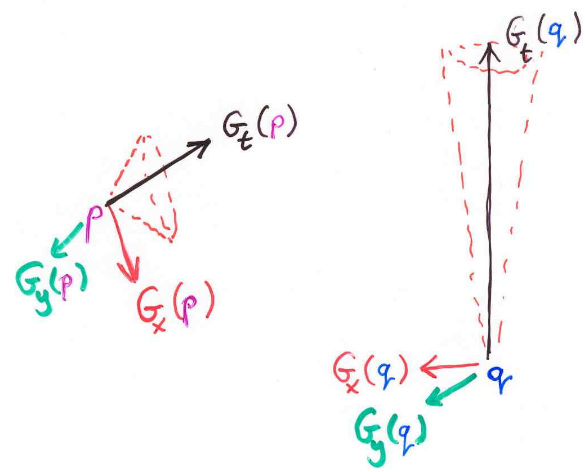

Figure 1.16. A general relativity space-time model is a 4-tuple of vector-fields.

Now, how do we use a general relativity space-time model, i.e. such a 4-tuple of vector fields, for representing some aspects of reality? Clearly, for fixed p of $\mathbb{R}^4$, the vector tetrad $G_t(p), ..., G_z(p)$ wants to represent the local frame LFR sitting at point p . More precisely, it wants to represent how the special relativistic frame is situated at point p. I.e. it represents how the local frame is distorted when it is glued into the holistic picture of the whole global frame. The important point is how the individual local frames are distorted, rotated etc w.r.t. *each other*, the big global frame grid is only a theoretical, conventional device to serve as a common denominator in arranging the little local frames relative to each other.

Very-very roughly, the information content of a general relativistic space-time model $\mathfrak{M} = \langle G_t, ..., G_z \rangle$ can be visualized as follows. The vector tetrad $G_t(p), ..., G_z(p)$ at point p tells us how the measuring instruments (clocks, meter-rods) of the tiny little inertial observer we imagine as being dropped at p go crazy (go wrong) from the point of view of the big, global general relativistic coordinate grid $\mathbb{R}^4$ we are using in $\mathfrak{M}$. This information is very subjective, since as we said, the big global coordinate grid carries no physical meaning, it is subjective, i.e. it is very tentative or conventional. But some objective content can be extracted from this subjective information, e.g. we can ask ourselves how the instruments at point p go crazy from the point of view of some local observer with some fixed life-line crossing some point q in $\mathbb{R}^4$. (We plan to return to clarifying this more, at some later time.)

OK, OK, the reader may say "so much for philosophy but let us be more concrete". What can be described in terms of the 4-tuple $\langle G_t, ..., G_z \rangle$ of vector fields? Well, we can describe the (potential) life-lines of inertial bodies and the life-lines of photons. Other things like gravity etc. come later and are built on top of these two basic concepts. We emphasize that knowing what the potential life-lines of inertial bodies and photons are tells us everything important about a general relativistic space-time model. (If we want to use the rest of "fancy words" like gravity, curvature, they can be defined explicitly from knowing the above mentioned life-lines.)

For simplicity, we will talk about life-lines of inertial observers instead of those of inertial bodies. (This is justified because in general relativity, it is assumed that, with the exception of photons, the life-lines of inertial bodies are also life-lines of observers.)

Defining the life-lines of inertial observers and photons in a general relativistic space-time model. The life-lines of inertial observers will be described mathematically as time-like geodesic curves.

By a *curve* f we understand a smooth mapping $f : D \to \mathbb{R}^4$, where D is an open interval of $\mathbb{R}$. By a point of the curve we mean a point in its range. (Here we think of $\mathbb{R}^4$ as a global gen. rel. frame grid, which we could indicate informally by saying that a curve is an "$f : D \to (GFR - grid)$" or "$f : D \to$ GenRel".)

Intuitively, the curve f is called *time-like* at point p iff the local frame at p "sees" an observer co-moving with the curve at p. In more detail, the curve f is called time-like at p iff the speed of f as seen by the local frame at p is smaller than 1. This means that the tangent of the curve $L(p)^{-1} \circ f$ has slope smaller than 1, at the origin. (Here, the time axis has slope 0, while a photon life-line has slope 1.) The curve f is called time-like iff the curve f is time-like at each of its points p . (Note: this is independent of how the curve f is parameterized.)

Note that talking about the tangent of $L(p)^{-1} \circ f : D \to \mathbb{R}^4$ involves nothing "fancy", since (at this step) we are in a special relativity space-time model and we are using its Euclidean geometry over $\mathbb{R}^4$ and we are looking at a smooth curve in it. We are using the intuitive scheme $L(p)$: *Specrel* $\to$ *Genrel* and $f : D \to$ *Genrel*. Hence $L(p)^{-1} \circ f : D \to$ *Specrel*. Further, the slope of a line ℓ is the (tangent of the) angle between the time axis and ℓ.

We think of a time-like curve as a curve that in principle can be the life-line of a (moving) observer.

OK. When is a time-like curve $f : D \to \mathbb{R}^4$ called a time-like geodesic? First we have to check whether the curve f represents (or measures) relativistic time correctly. Here, the parametrization will be important.

Definition 4.1. *We say that f* represents time correctly *if the following statement holds. For every $t \in D$, and for every positive ε there is a positive δ such that for all $s \in [t-\delta, t+\delta]$ it holds that $|s-t|$ agrees with what $d(f(s), f(t))$ is as measured by the local frame determined by $G_t(f(t)), ..., G_z(f(t))$ up to an error bound by $\varepsilon \cdot |t-s|$. (We note that here "$d(f(s), f(t))$ as measured by the local frame..." coincides with the Minkowski distance between $L(p)^{-1}(f(s))$ and $L(p)^{-1}(f(t))$ understood in the special relativity model.) See Figure 1.17.*

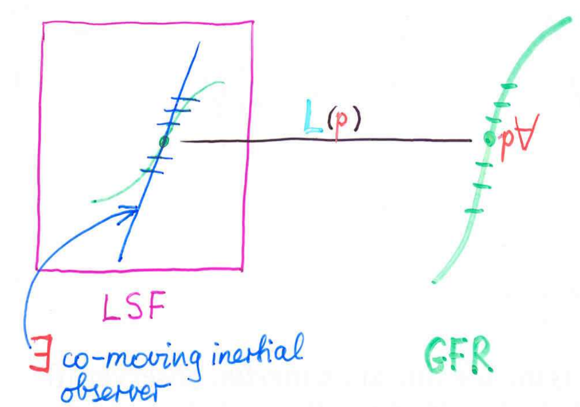

Figure 1.17. A time-faithful curve.

We call a curve *time-faithful* iff it is time-like and represents relativistic time correctly.

Intuitively, a time-like curve is time-faithful iff at each point p of the curve, the local frame at p "sees" an observer co-moving with the curve such that the parametrization of the curve agrees with how time passes for this co-moving observer.

We imagine that a time-faithful curve f is the life-line of an observer b such that the parameter t measures proper time of b; or in other words, t shows the time on the wrist-watch of b. We imagine the motion of b such that $f(t)$ in $\mathbb{R}^4$ is the location of observer b at his wrist-watch time t. The condition in Definition 4.1 serves to ensure that wrist-watch-time

t of observer b (whose motion is represented by the curve f) agrees with the relativistic time interval measured by the relativistic metric d at the local frame which is situated at the location $f(t)$. More precisely, small time intervals on the wrist-watch of b agree with the relativistic time interval measured by the d's of the local special relativity frames.

Yet in other words, we think that a time-like curve can be the life-line of a spaceship which uses fuel (i.e. uses its ship-drive) for accelerating and decelerating. The curve is time-faithful if the parametrization of the curve agrees with "inner time" of the spaceship.

Definition 4.2. *(*time-like geodesic*) By a* time-like geodesic *we understand a time-faithful curve $f : D \to \mathbb{R}^4$ satisfying the following. For any t in D there is a neighborhood S of $f(t)$ (understood in $\mathbb{R}^4$) such that inside S, f is a "straightest possible" curve in the following sense: For any two points p, q of S connected by f, the distance of p and q as measured by f is maximal among the distances measured by time-faithful curves inside S.*

Formally, this maximality condition is expressed by the following. Assume that h is a time-faithful curve. Assume that $p = f(s) = h(s')$, $q = f(r) = h(r')$ and $h(t')$ is in S for all t' which are between s' and r'. Then $|s - r|$ is greater or equal to $|s' - r'|$. See Figure 1.18.

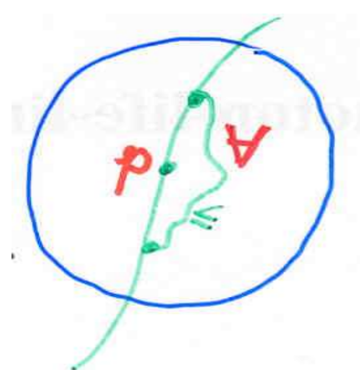

Figure 1.18. A time-like geodesic.

If f is a time-like geodesic, then we imagine that an inertial observer b can move along it. By an inertial observer b we imagine a spaceship with ship-drive switched off, i.e. a spaceship which does not use fuel for accelerating (i.e. for influencing its motion).

The above definition of time-like geodesics is the natural reformulation of the Euclidean notion of geodesics (straight curves in a possibly curved surface of Euclidean 3-space) with “minimal” replaced by “maximal”. If one thinks about this maximality condition, one will find that it is strongly connected to the Twin Paradox of special relativity. Indeed, this maximality condition expresses the property that the local frames through which f passes (as a curve) will think that f is the life-line of an inertial observer.

By the above definition of time-like geodesics, we can express what life-lines of inertial observers are in general relativity space-time models $\langle Gt, ..., Gz\rangle$. The definition of photon-geodesics is analogous, and goes as follows.

The curve f is called *photon-like* at p iff the speed of f as seen by the local frame at p equals 1. In more detail, this means that the tangent of the curve $L(p)^{-1} \circ f$ at the origin has slope 1. The curve f is called photon-like iff the curve f is photon-like at each of its points p . (Note: this is independent of how the curve f is parameterized.)

We imagine that a photon-like curve can be the life-line of a photon perhaps directed (diverted) by suitable mirrors.

A *photon-like geodesic* is a photon-like curve f with the property that each point in the curve has a neighborhood in which through any two points of f , f is the unique photon-like curve. (In more detail, let F denote the range of f. Then any point in F has a neighborhood S such that whenever f' is a photon-like curve connecting two points of $F \cap S$ and such that F'=*the range of* f' is inside S, we have that $F' \subseteq F$.)

Maximal photon-like geodesics are called life-lines of photons. (The meaning of maximal here will be explained soon. Intuitively, it means “non-extendible.) Similarly, maximal time-like geodesics are called life-lines of inertial observers. Let us notice at this point that a general relativistic space-time model $\langle G_t, ..., G_z\rangle$ determines “inertial motion” and also determines how photons move.

On how we use the adjective maximal: Since the domain D of a curve f may be a subset of the domain of the curve g, a function f is smaller than g iff f is a restriction of g.

Definition of isomorphisms between general relativistic space-time models. We said that the big global frame grid carries no physical meaning, and only time-like and photon-like geodesics carry physical meanings, everything else (e.g. gravity) can be defined from these geodesics. We give "meaning" to this statement (or claim) in the form of defining what isomorphisms of general relativistic space-time models are.

Let $G = \langle G_t, G_x, G_y, G_z \rangle$ and $G' = \langle G'_t, G'_x, G'_y, G'_z \rangle$ be two general relativistic space-time models. An *isomorphism* between these two GRT models is a bijection $Iso : R^4 \to R^4$ such that (i)-(iii) below hold.

(i) Both *Iso* and the inverse of *Iso* are smooth.

(ii) *Iso* preserves time-like geodesics. In more detail, for any curve $f : D \to \mathbb{R}^4$, f is a time-like geodesic in G iff $f \circ Iso$ is a time-like geodesic in G'.

(iii) *Iso* preserves photon-like geodesics (in the above sense).

We note that we could omit (iii), because one can prove that it follows from (i)-(ii) above.

By the above, we have the main building blocks of General Relativity and we can start working. So, let us make first sure that we have a common understanding of what models $\langle G_t, ..., G_z \rangle$ of General Relativity are and how time-like geodesics and photon-like geodesics move in them. It is a worthwhile project to clarify what these are, because there is nothing else that would be basic in General Relativity. Hence, after having digested these things we can start discussing the Expanding Universe, black holes or whatever one likes.

Remark: A relatively important generalization: Let D be an open subset of $\mathbb{R}^4$. Assume that the vector-fields $G_t, ..., G_z$ are defined only over D and they satisfy the required conditions (e.g. continuity) only on D. Then already we call the 4-tuple $\langle G_t, ..., G_z \rangle$ of vector-fields a general relativity space-time model. The pragmatic use of this comes from situations where we have a nice and intuitive definition of the fields $Gt, ..., Gz$ but they satisfy the continuity conditions on $\mathbb{R}^4$ only with finitely many exceptions. Then we can throw away those finitely many points from $\mathbb{R}^4$, i.e. from our coordinate domain, and still call our $\langle G_t, ..., G_z \rangle$ a general relativity space-time model. For explicitness we sometimes write $\mathfrak{M} = \langle G_t, ..., G_z, D \rangle$ with $D = \mathsf{Dom}(G_t) = ... = \mathsf{Dom}(G_z)$ for a general relativity space-time model. So in this more flexible notation, a general relativity space-time model is a 5-tuple (but consists only of 4 vector fields all the same).