Chapter 1

THE LOGIC OF SPACE-TIME

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1. Introduction

Our goal is to make General Relativity accessible and transparent for any reader with logical background. You do not have to "believe" anything. The emphasis is on the logic-based approach to relativity theory. The purpose is giving insights as opposed to mere recipes for calculations. Therefore proofs will be visual geometric ones, efforts will be made to replace computational proofs with suggestive drawings (but providing complete proof).

Relativity theory comes in (at least) 2 versions, special relativity (Einstein 1905) and general relativity (Einstein, Hilbert 1915). They differ in scope, the scope of general relativity is broader. Special relativity is a theory of motion and light propagation in vacuum far away from any gravitational object. I.e. special relativity does not deal with gravity. Also, special relativity is a "prelude" for general relativity, it provides a foundation or framework or starting point (in some sense) for the general theory. General relativity unifies special relativity and the theory of gravitation. In some sense, general relativity is an "extension" of special relativity putting also gravity into the picture. General relativity can be used as a foundation for cosmology, e.g. it is a suitable framework for discussing the (evolution, properties, etc. of the) whole universe (expanding or otherwise). Special relativity, on the other hand, is not rich enough for this purpose.

Special relativity shows us that there is no such thing as space in itself, instead, a unified space-time exists. This inseparability of space and time becomes more dramatic in general relativity. Namely, general relativity shows us that gravity is nothing but the curvature of space-time. It is extremely difficult, if not impossible, to explain gravity without invoking the curvature (i.e. geometry) of space-time. But the crucial point is that curvature of space is not enough (by far), it is space-time whose curvature explains gravity.[1]

From a different angle: general relativity is a "geometrization" of much of what we know about the world surrounding us. E.g. it provides a full geometrization of our understanding of gravity and related phenomena like motion and light signals.

2. Axiomatization of special relativistic space-time

In our approach, axiomatization is not the end of the story but, instead, it is the beginning. Namely: axiomatizations of relativity are not ends in themselves (goals), instead, they are only tools. Our goals are to obtain simple, transparent, easy-to-communicate insights into the nature of relativity, to get a deeper understanding of relativity, to simplify it, to provide a foundation for it. Another aim is to make relativity theory accessible for many people (as fully as possible). Further, we intend to analyze the logical structure of the theory: which assumptions are responsible for which predictions; what happens if we weaken/fine-tune the assumptions, what we could have done differently. We seek insights, a deeper understanding. We could call this approach "reverse relativity" in analogy with "reverse mathematics".

In this section we build up special relativity theory as a theory of first-order logic, using only a handful of simple, streamlined axioms.

2.1 Motivation for special relativistic kinematics in place of Newtonian kinematics

Why should we replace Newtonian Kinematics with such an exotic or counter-common-sense theory as special relativity? The Newtonian theory proved very successful for 200 years. By today, the Newtonian picture of motion became the same as the common-sense picture of motion. Hence the question is why we have to throw away our common-sense understanding of motion.[2]

There are several independent reasons for this any single one of which would be sufficient for justifying this move/step (actually, this paradigm-shift). We will mention a few of these reasons, but for simplicity of presentation, we will base this work on a fixed one of these reasons, namely on the outcome of the Michelson-Morley (MM for short) experiment. We will call this outcome of the MM experiment the Light Axiom.

There are deeper, more philosophical reasons for replacing the Newtonian world-view with relativity theory, which might convince readers who are not experimentally minded, i.e. who are not easily convinced by mere facts about how results of certain experiments turned out. These philosophical reasons are intimately intertwined with issues which were significantly present through the last 2500 years of the history of our culture. We briefly sketch this direction in subsection 7.1.

We now turn to the Light Axiom which will play a central role in this work. The first test of the Light Axiom was the MM experiment in 1887 and it has been tested extremely many times and in many radically different ways ever since. As a consequence, the Light Axiom has been generally accepted. An informal, intuitive formulation of the axiom is the following. (Later we will present this axiom in more formal, more precise terms, too, see AxPh in subsection 2.3.)

Light Axiom:
The speed of light is finite and is direction independent, in the world-view of any inertial observer.

In other words, the Light Axiom means the following. Imagine a (huge) spaceship drifting through outer space in inertial motion. (Inertial here means that the rockets of the spaceship are switched off, and that the spaceship is not spinning.) Assume a scientist in this inertial spaceship is making experiments. The claim is that if the scientist measures the speed of light, then he will find that this speed is the same in all directions and that it is finite. It is essential here that this is claimed to hold for all possible inertial spaceships irrespective of their velocities relative to the Earth or the Sun or the center of our galaxy or whatever reference system would be chosen.[3] The point is that no matter which inertial spaceship we choose, the speed of light in that spaceship is independent of the direction in which it was measured, i.e. it is “isotropic”.

In the technical language what we called “inertial spaceship” above is called an inertial reference frame, and the scientist in the spaceship making the experiments is called an “observer”. Later “ observer ” and reference frame tend be identified.

Let us notice that the Light Axiom is surprising, it is in sharp contrast with common-sense. Namely, common-sense says that if we send out a light signal from Earth, and a spaceship is racing with this light signal moving with almost the speed of the signal in the same direction as the signal does, then the velocity of the signal relative to the spaceship should be very small. Hence, one would think that the astronaut in the spaceship will observe the motion of the light signal as very slow.

With the same kind of reasoning, the astronaut should observe light signals moving in the opposite direction as moving very fast. But the Light Axiom states that light moves with the same speed in all directions for the astronaut in the spaceship, too. Hence, the Light Axiom flies in the face of common-sense. This gives us a hint/promise that very interesting, surprising things might be in the making.

In fact, if we add the Light Axiom to Newtonian Kinematics, then we obtain a logical contradiction. I.e. (Newtonian Kinematics + Light Axiom) is an inconsistent theory in the usual sense of logic as we will outline soon (cf. Prop.2.1). Seeing this contradiction, Einstein did the natural thing. He weakened Newtonian Kinematics (NK for short) to a weaker theory NK^- such that NK^- became consistent with the Light Axiom. Then the theory (NK^- + Light Axiom) became known as Special Relativistic Kinematics (SRK for short). We will study this theory under the name $\mathbf{Specrel}_0$ to be introduced in a logical language soon. We represent the above outlined process by the following diagram:

(NK + Light Axiom) **leads to Contradiction** (!)

$\Downarrow$

NK **gets replaced with the weaker** NK^-

$\Downarrow$

(NK^- + Light Axiom) **receives the name** Special Relativity (SRK).

SRK **is consistent** (this will be proved).

To see the above process more clearly, let us invoke a possible axiomatization of NK, still on the intuitive level.

Preparation for NK: If we want to represent motion of "particles" or "bodies" or "mass-points", it is natural to use a 4-dimensional Cartesian coordinate system $R \times R \times R \times R$ (where R is the set of real numbers), with one time dimension t and three space dimensions x,y,z. A three-dimensional part of this looks like in Figure 1.1. Representing the motion of a body, say $b1$, in a 4-coordinate system can be done by specifying a function f which to each time instance $t \in R$ tells us the space coordinates x, y, z where the body $b1$ is found at time t. Hence a function f : Time $\rightarrow$ Space specifies motion of a particle in this sense. The function f representing motion of $b1$ is called the *life-line* or world-line of $b1$. Figure 1.1 represents motion of bodies, in this spirit. Besides the coordinate axes, we represented the life-lines of inertial bodies $b1, b2$ and $b3$, in Figure 1.1. The straight line labeled by $b1$ is the life-line of $b1$. The slope of the life-line of $b1$ is greater than that of $b2$ which means

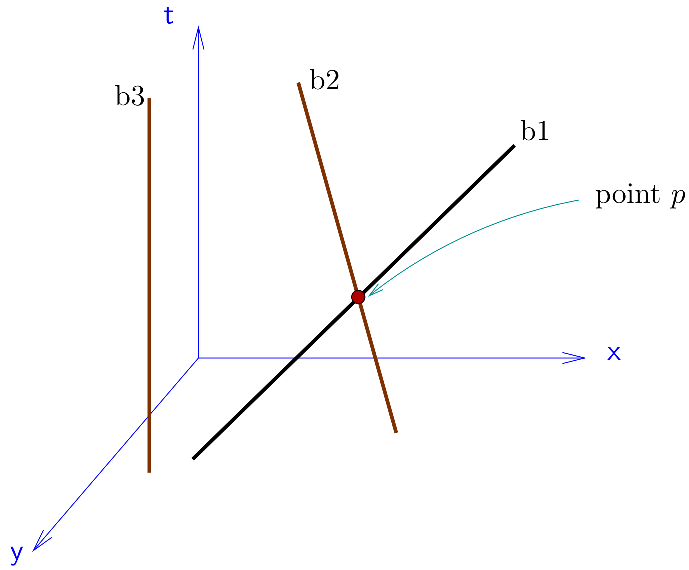

Figure 1.1. A spacetime diagram. Life-lines of bodies $b1, b2, b3$ represent motion. (Coordinate z is not indicated in the figure.) $b3$ is motionless and $b1$ moves faster than $b2$.

that $b1$ moves faster than $b2$ does. The life-line of the third body $b3$ is parallel with the time axis, this means that $b3$ is motionless. Bodies $b1$ and $b2$ meet at spacetime point $p = \langle t, x, y, z \rangle$. Such a meeting (of two or more bodies) is called an *event*. We will extensively refer to such 4-dimensional coordinate systems and such life-lines of bodies and events.

The axioms of[4] NK are summarized as (i)-(v) below.

(i) Each observer "lives" in a 4-coordinate system as described above. The observer in its own coordinate system is motionless in the origin, i.e. its life-line is the time-axis.

(ii) Inertial motion is straight: Let $o1$ and $o2$ be arbitrary inertial observers. Then in $o1$'s 4-coordinate system the lifeline of $o2$ is a straight line. I.e. in an inertial observer's world-view or 4-coordinate system all life-lines of inertial observers appear as straight lines.

As we said, an observer in his metaphorical "spaceship" is inertial if his rockets are turned off and the spaceship is not spinning. In special relativity, we discuss only inertial motion, hence in our axiomatization the adjective "inertial" could be omitted. (Of course, then we need a general claim that only inertial things/objects will be studied.)

(iii) Motion is permitted: In the world-view or 4-coordinate system of any inertial observer it is possible to move through any point p in any direction with any finite speed.

(iv) Any two observers "observe" the same events. I.e. if according to $o1$ bodies $b1$ and $b2$ have met, then the same is true in the 4-coordinate system of any $o2$. We postulate the same for triple meetings e.g. of $b1, b2, b3$.

(v) Absolute time: Any two observers agree about the amount of time elapsed between two events. (Hence temporal relationships are absolute.)

So, now, NK is defined as the theory axiomatized by $\{(\mathsf{i}) - (\mathsf{v})\}$ above.

It is easy to see that (NK + Light Axiom) is inconsistent. Einstein's idea was to check which ones of (i)-(v) are responsible for contradicting the Light Axiom and to throw away or weaken the "guilty" axioms of NK. We will see that (v) is guilty and that part of (iii) is suspicious. Hence we throw away (v) and weaken (iii) to a safer form (iii$^-$) where (iii$^-$) is the following.

(iii$^-$) Slower than light motion is possible: in the world-view of any inertial observer, through any point in any direction it is possible to move with any speed slower than that of light (here, light-speed is understood as measured at that place and in that direction where we want to move).

In the formal part we will carefully study whether all of these modifications are really needed and to what extent. We define NK$^-$ as the remaining theory:

$$\mathrm{NK}^- := \{(\mathsf{i}), (\mathsf{ii}), (\mathsf{iii}^-), (\mathsf{iv})\}$$

and Special Relativistic Kinematic is defined as

$$\mathrm{SRK} := (\mathrm{NK}^- + \mathsf{Light\ Axiom}).$$

The formalized version of this SRK will appear later as the theory **Specrel**$_0$. We will prove that **Specrel**$_0$ is consistent (i.e. contradiction-free) and will study its properties. Therefore SRK is also consistent, since, as we said, **Specrel**$_0$ is a formalized version of SRK. Actually, the whole process of arriving from NK and the Light Axiom (or some alternative for the latter) to SRK will be subjected to logic-based conceptual analysis in subsection 2.5.

Before turning to formalizing (and studying) Special Relativity SRK in logic, let us prove (informally only) on the present level of precision why absolute time (i.e. axiom (v)) is excluded by the Light Axiom, or more precisely, it is excluded if we want to keep a fragment of our intuitive picture of the world, i.e. if we want to keep (i), (ii), (iv) of NK. We will prove:

$$(\mathrm{NK}^- + \textsf{Light Axiom}) \vdash \text{Negation of (v)},$$

where we use turnstile " $\vdash$ " as the symbol of logical provability or derivability. I.e. $A \vdash B$ means that from statement A one can prove, rigorously, statement B.

Actually, we will prove something stronger and stranger from the Light Axiom (and a fragment of NK^-). We will prove that the time elapsed between two events may be different for different observers even in the special case when this elapsed time is zero for one of the observers. I.e. the very question whether two events happened at the same time or not will depend on the observer: two events A and B may happen at the same time for me, while event A happened much later than event B for the Martian in his spaceship. We will refer to this phenomenon by saying that " simultaneity is not absolute ". Moreover, we will see later that the temporal order of some events may be switched: event A may precede event B for me, while for the Martian in his spaceship, event B precedes event A.

We say that events e and $e1$ are simultaneous for observer O if in O's coordinate system the two events $e, e1$ happen at the same time.

Proposition 2.1. (Simultaneity is not absolute) *Assume* SRK. *Moving clocks get out of synchronism, i.e.: Assume that a spaceship S is in uniform motion relative to another one, say E, and assume that two events $e, e1$ happen simultaneously at the rear and at the nose of the spaceship according to the spaceship. Then e and $e1$ take place at different times in E's coordinate system.*

I.e., the time elapsed between e and $e1$ is zero as "seen" from the spaceship, but the time elapsed between e and $e1$ is nonzero as "seen" from E. See Figure 1.3.

Intuitive proof. Assume that we are in spaceship E, and let us call E "Earth". Assume that spaceship S – let us call it "Spaceship" – moves away from us in a uniform motion with, say, with 0.9 light-speed. The captain of Spaceship positions his brothers called Rear, Middle, and Nose at the rear, middle and nose of the spaceship, respectively, and asks Rear and Nose to switch on their flashlights towards Middle exactly at the same time. Then the light signals (photons[5]) Ph1 and Ph2 from the two flashlights arrive to Middle at the same time, because Middle is exactly in the middle of the spaceship, and because the speed of Ph1 sent by Rear is the same as the speed of Ph2 sent by Nose (by the Light Axiom). See Figure 1.2.

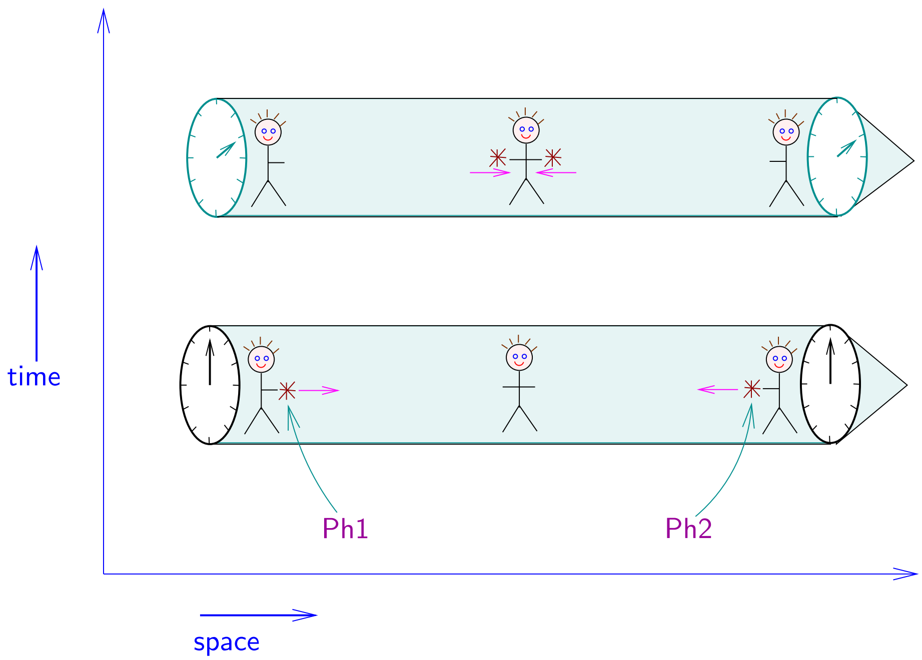

Figure 1.2. Seen from the spaceship, the two light-signals (i.e. photons) Ph1 and Ph2 are sent out at the same time, and meet in the middle. This is indicated by the clocks at the rear and at the nose of the spaceship. Notice that time in this figure is running upwards! I.e., this figure is similar to drawings in cartoons in that a sequence of scenes is represented in it. However, here the temporal order of the scenes is switched: the scene at the bottom took place earliest. The reason for this convention is our seeking compatibility with the usual spacetime diagrams like Figure 1.1.

How do we see all this from the Earth? We see that Rear and Nose send light signals (or photons) Ph1 and Ph2 towards Middle, and we also see that Ph1, Ph2 arrive to Middle at the same time (because this is a 3-meeting of bodies/entities and axiom (iv) in NK^-). However, by the Light Axiom, the speeds of Ph1 and Ph2 are the same for us on the Earth, too. Since Spaceship moves away from us (with 0.9 light-speed), we see Ph1 crawl very slowly along the hull of Spaceship because the ship is "running away" from us. On the other hand, the other photon, Ph2, flashes along the hull of the spaceship towards us with enormous relative speed (relative to the hull of the spaceship). Because of this difference of their speeds relative to Spaceship, according to Earth, Ph1 and Ph2 either meet close to the rear of the spaceship, or if they meet in the middle, then Nose had to switch on his flashlight much later than Rear did. See Figure 1.3.

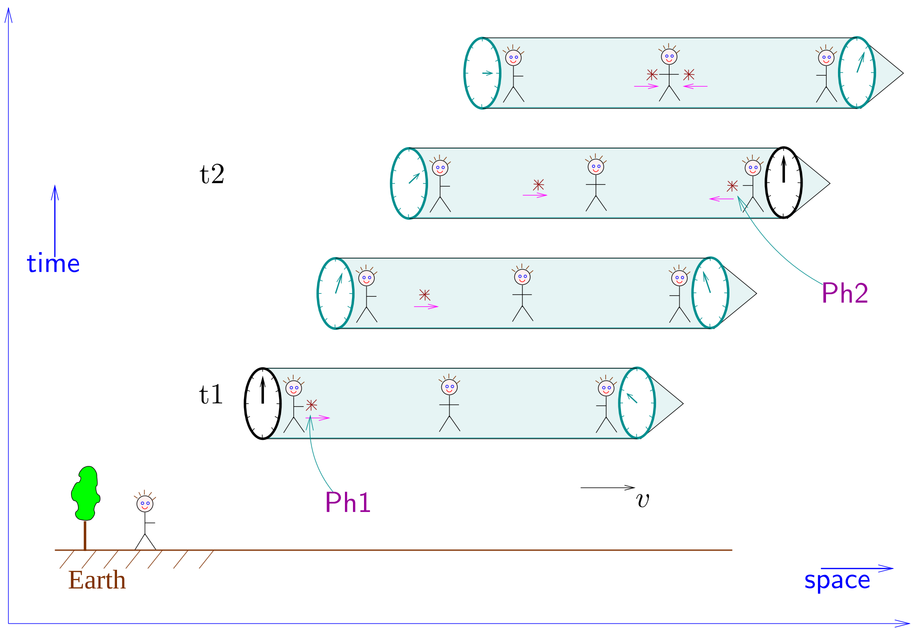

Figure 1.3. Seen from the Earth, the photon Ph2 had to be sent out later in order that it arrive in the middle at the same time as Ph1 does. But seen from the spaceship, they were sent out at the same time. Hence the clocks at the nose and the rear are out of synchronism.

So far we proved that at least one of two things cannot be absolute. These are (a) being in the middle of the spaceship, and (b) simultaneity. Here, (a) means that Spaceship observes Middle in the middle of the ship, while Earth observes that Middle is not in the middle of the ship; and (b) means that emissions of photons Ph1 and Ph2 are simultaneous for Spaceship but not for Earth.

The first possibility is that Middle stands closer to the rear of Spaceship as seen from the Earth, i.e. that he is not in the middle of ship according to Earth observers, while he is in the middle according to ship observers. Here is a thought-experiment which shows that this is not possible. Let us ask the captain to give mirrors to Rear and Nose, and order Middle to send photons Ph3, Ph4 at the same time to these two mirrors. Since Middle is exactly in the middle of the ship, the bouncedback photons arrive to him at the same time, by the Light Axiom. By (iv) in NK^{-}, we on the Earth also see that the two photons Ph3, Ph4 meet again at Middle after bouncing back, so they traveled their round-trips in the same amount of time. We will show that, as seen from the Earth, the time needed for the round-trip is proportional to the covered distance: if, say, Nose is twice as far from Middle as Rear is, then the time needed for Ph4 for the round-trip Middle-Nose-Middle is twice as much as the time needed for Ph3 for the round-trip Middle-Rear-Middle, even in a fast-moving spaceship. Thus, since from the Earth we see that the round-trip took the same time for Ph3 and for Ph4, we have to infer that Middle is really in the middle of the ship. See Figure 1.4.

We now prove that the time needed for the round-trip is proportional to the covered distance. Indeed, assume that the distance Middle-Nose is twice as much as the distance Middle-Rear. We will show that the round-trip Middle-Nose-Middle takes twice as much time for a photon Ph4 as the round-trip Middle-Rear-Middle for a photon Ph3. Let us watch from the Earth how the two photons Ph3 and Ph4 move relative to the spaceship (as in Figure 1.4, but now Middle standing closer to Rear). We will see that Ph3 covers the segment Middle-Rear fast, traveling towards us, and then covers the segment Rear-Middle slowly, moving away from us. The same way, Ph4 covers the segment Middle-Nose slowly, moving away from us, while Ph4 covers the segment Nose-Middle fast, moving towards us. The relative speed of Ph4 in the "towards-us" segment Nose-Middle is the same as the relative speed of Ph3 in the "towards-us" segment Middle-Rear; hence this part of the trip takes twice as much time for Ph4 as for Ph3 because we assumed that the distance Nose-Middle is twice as much as the distance Middle-Rear. The situation is completely analogous for the "away-from-us" segments, so the trip Middle-Nose takes twice as much time for Ph4 as the trip Rear-Middle for Ph3. Summarizing the segments, the round-trip takes twice as much time for Ph4 as for Ph3.

As we said earlier, we observe from the Earth that Ph3, Ph4 and Middle meet in a single event. Therefore, since we observe that Ph3 arrives to Middle exactly when Ph4 arrives to Middle after their round-trips, we have to infer, on the Earth, that Middle really stands exactly in the middle of Spaceship. Remains the only possibility that Nose sent out his photon Ph2, which we see as fast-moving along the hull of the space ship, much later than Rear sent Ph1 which we see as slowly-moving along the hull of the spaceship. Thus, as seen from the Earth, the clocks at the nose and at the rear of the spaceship show different times (at the same Earth-moment). This is what we referred to as the clocks of the spaceship get out of synchronism.

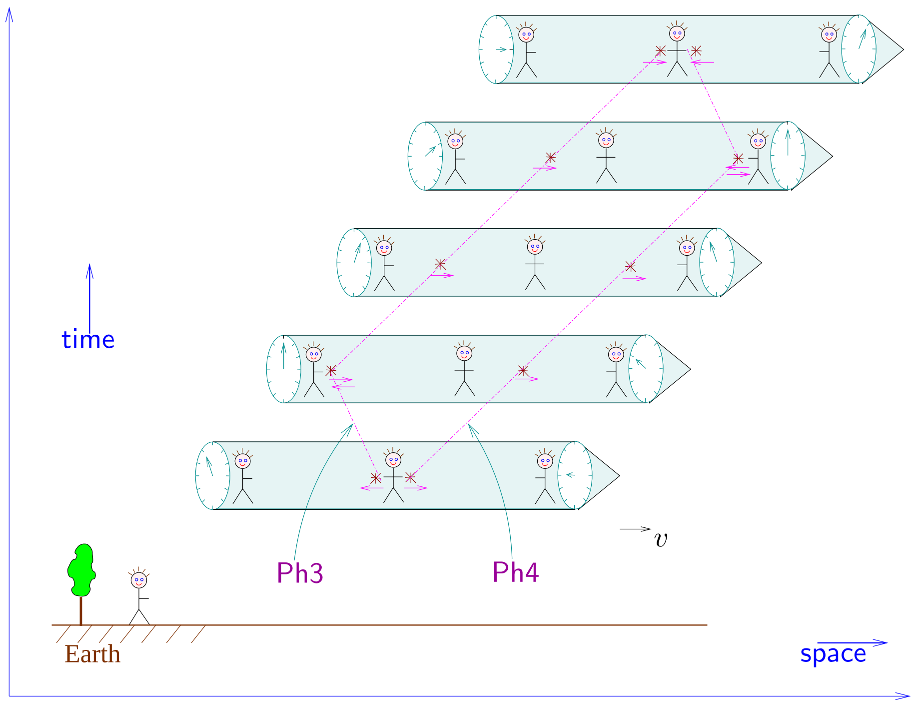

Figure 1.4. The round-trip for Ph3 takes the same time as for Ph4, seen both from the spaceship and from the Earth. Hence Earth infers that Middle is indeed in the middle of the ship.

Summing up: Let e and $e1$ be the events when Rear sends his photon Ph1 towards Middle, and when Nose sends his photon Ph2 towards Middle, respectively. Then these two events took place at the same time as seen from Spaceship, while as seen from the Earth, $e1$ took place later than e did. This finishes the proof of Proposition 2.1. □

Let us notice that Proposition 2.1 above is a far reaching claim. It implies that one of the most basic words of natural language refers to an illusion only and carries no real meaning. The word in question is the word "now". (If challenging "now" sounds too daring, then let us talk about the word "when". We mean this in the context of "A happened when B did".)

In order to be able to carry out the proof of Proposition 2.1 and other similar chains of thought in the "safe", precise setting of mathematical logic, in the next subsection we introduce a first-order logic language in which we formalize our axioms, statements, and proofs.

2.2 Language

Motivation for language. We want to talk about space-time as relativity theory conceives it. We will talk about space-time as experienced through motion. Though we discuss here kinematics (theory of motion) only, one can derive (logically) dynamic predictions of relativity, too (i.e. phenomena involving forces, energy) using the same approach/axioms. We will use first-order logic, and the most important decision is to choose the language (vocabulary), i.e. what objects and what relations between them will belong to the language. We will specify our first-order language by specifying its models.

We want to axiomatize motion. What moves? Bodies. Hence our model has a universe B for bodies. What does it mean to move? To move means changing location in time. We will have co-ordinate systems, or reference frames in other words, for marking locations and time, and we will use quantities in setting up these co-ordinate systems. So our model has another universe F for quantities. We will think of quantities as real numbers, so F together with the operations $+, \cdot, <$ will form a linearly ordered *quadratic* field, i.e. a field in which every positive member has a square root.[6] We will think of co-ordinate systems as belonging to special bodies called observers. Hence Obs is a unary relation on B. We will have another kind of special bodies, photons, too. Hence Ph is another unary relation on B. The heart of our model is the world-view-relation W. This is a 6-ary relation connecting bodies and quantities. We think of $W(o, b, t, x, y, z)$ as the statement that the body b is in location xyz at time t in observer o's co-ordinate system. We will simply pronounce this as

o sees the body b at $txyz$

though this has no connection with optical seeing, instead, it is an act of co-ordinatizing only. With this intuition in mind we now fix the language of our theories of special relativity.

The language. Let us fix a natural number $n > 1$. n will be the number of space-time dimensions. In most works $n = 4$, i.e. one has 3 space-dimensions and one time-dimension. Recent generalizations of general relativity in the literature indicate that it might be useful to leave n as a variable (e.g. string-theory uses 11 dimensions).

We declare two sorts of objects. One sort is for "quantities", it will be denoted by F (for field). (This is the same as "real numbers" in other treatments.) We have binary operations $+, \cdot$ and a binary relation $<$ of sort "quantities".

The other sort, B, is for entities which do the "moving". We will call these "*bodies*". (We call the moving entities "bodies" whatever they may be, but in reality they can be e.g. co-ordinate systems or electromagnetic waves.) We have two kinds of special bodies, observers and photons. Thus Obs and Ph are unary predicate symbols of sort B.

We have a relation which connects these two sorts, the $n+2$-ary relation W which is of sort $B \times B \times F \times \cdots \times F$. The sentence "observer o observes body b at space-time location $p_1, ..., p_n$" is denoted as $W(o, b, p_1, ..., p_n)$, or as $W(o, b, p)$ in short.

Summing up, a model $\mathfrak{M}$ of our language is of form

$$\langle F, +, \cdot, <;\ B, Obs, Ph;\ W\rangle$$

where $\langle F, +, \cdot, <\rangle$ is a structure similar to ordered fields, Obs, Ph are subsets of B, and $W \subseteq B \times B \times F \times \cdots \times F$.