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ALGEBRAIC LOGIC

We dedicate this work to J. Donald Monk *who taught us algebraic logic and more.*

CONTENTS

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INTRODUCTION

Algebraic logic can be divided into two main parts. Part I studies algebras which are relevant to logic(s), e.g. algebras which were obtained from logics (one way or another). Since Part I studies algebras, its methods are, basically, algebraic. One could say, that Part I belongs to "Algebra Country". Continuing this metaphor, Part II deals with studying and building the bridge between Algebra Country and Logic Country. Part II deals with the methodology of solving logic problems by (i) translating them to algebra (the process of algebraization), (ii) solving the algebraic problem (this really belongs to Part I), and (iii) translating the result back to logic. There is an emphasis here on step (iii), because without such a methodological emphasis one could be tempted to play the "enjoyable games" (i) and (ii), and then forget about the "boring duty" of (iii). Of course, this bridge can also be used backwards, to solve algebraic problems with logical methods. We will give some simple examples for this in the present work.

Accordingly, the present work consists of two parts, too. Parts I and II of the paper deal with the corresponding parts of algebraic logic. More specifically, Part I deals with the algebraic theory in general, and with algebras of sets of sequences, or algebras of relations, in particular. Part II deals with the methodology of algebraization of logics and logical problems, equivalence theorems between properties of logics and properties of (classes of) algebras, and in particular, discusses concrete results about logics obtained via this methodology of algebraization. Since Part II deals with general connections between logics and algebras, a general definition of what we understand by a logic or logical system is needed. Of course, such a definition has to be broad enough to be widely applicable and narrow enough to support interesting theorems. The first section of Part II is devoted to finding such a definition.

We need to make a disclaimer here. Algebraic logic, today, is an extremely broad subject. We could not cover all of it. In Part II we managed to be broader than in Part I. Even in Part II we could not come even close to discussing the important research directions, but the definitions in Part II are general enough to render the results applicable to all those logics which W. Blok and D. Pigozzi call algebraizable (cf. e.g. [BP89], [FJ94]). Most of what we say in Part II can be generalized even beyond this, e.g. to the equivalential logics of J. Czelakowski. Further possibilities of generalizing Part II beyond algebraizable logics are in recent works of Blok and Pigozzi, and others, cf. e.g. [BP86], [P91], [CzP], [ABNPS], [Cz97] .

In Part I we had to be more restrictive. We concentrated attention to those kinds of algebras which are connected to the idea of "relations" (one way or another), the idea of sets of pairs, or sets of triples, sets of sequences or something related to these. An important omission is the theory of Boolean Algebras with Operators (BAO's). BAO's are related to algebras of relations, and they provide an important unifying theory of many of the algebras we discuss here. Another important omission is Category Theoretic Logic. That branch of algebraic logic is not (at all) unrelated to what we are discussing here, but for various reasons we could not include an appropriate discussion here. In this connection more references are given in the survey [N91]. Here we mention only Makkai [Mk87], [MkR], [Z], [MkP]. We could not cover polyadic algebras, either. However, their (basic) theory is analogous to that of cylindric algebras which we do discuss in detail. There are a few exceptional points where the two theories wildly diverge, e.g. in [NS96] it was proved that the equational theory of representable polyadic algebras is highly non-computable (while that of cylindric algebras is recursively enumerable). We refer the reader to the survey paper [N91] and to [HMTII] for modern overviews of polyadic algebras. Cf. also [ST], [PS], [AGMNS]. Further important omissions are: (i) the finitization problem (cf. [N91, beginning with Remark 2], [S95], [Si93], [MNS97]); (ii) propositional modal logics of quantification, and connections with the new research direction “Logic, Language and Information” (cf. [V95], [MV], [MPM], [AvBN97], [vBtM], [vB97]); (iii) relativization as a methodology for turning negative results to positive (cf. [N96], [M93], [Ma95], [Mi95], [MV], [AvBN96]). Also there are strong connections between algebraic logic and computer science, we do not discuss these here.

On the history: The invention of Boolean algebras belongs to the “prehistory” of Part I. Algebras of sets of sequences (as in Part I) were studied by De Morgan, Peirce, and Schröder in the last century; and the modern form of their theory was created by Tarski and his school[1]. The history of Part II also goes back to Tarski and his followers, but is, in general, more recent. For more on history we refer to [AH], [ABNPS], [BP91a], [BP89], [HMTII], [Ma91], [Pr92], [TG].

[1]Relation and cylindric algebras were introduced by Tarski, polyadic algebras were introduced by Halmos, algebras of sets of finite sequences were studied by Craig; for other kinds of algebras of sets of sequences cf. e.g. [N91], [HMTII].

PART I

ALGEBRAS OF RELATIONS

GETTING ACQUAINTED WITH THE SUBJECT OF PART I.

The algebraization of classical propositional logic, yielding Boolean algebras (in short BA's), was immensely successful. What happens then if we want to extend the original algebraization yielding BA's to other, more complex logics, among others, say, to predicate logic (first–order logic)? [2]

Boolean algebras can be viewed as algebras of unary relations. Indeed, the elements of a BA are subsets of a set U, i.e. unary relations over U, and the operations are the natural operations on unary relations, e.g. intersection, complementation. The problem of extending this approach to predicate logics boils down to the problem of expanding the natural algebras of unary relations to *natural algebras of relations of higher ranks*, i.e. of relations in general. The reason for this is, roughly speaking, the fact that the basic building blocks of predicate logics are predicates, and the meanings of predicates can be relations of arbitrary ranks. [3] Indeed, already in the middle of the last century, when De Morgan wanted to generalize algebras of propositional logic in the direction of what we would call today predicate logic, he turned to algebras of binary relations. [4] That was probably the beginning of the quest for algebras of relations in general. Returning to this quest, the new algebras will, of course, have more operations than BA's, since between relations in general there are more kinds of connections than between unary relations (e.g. one relation might be the converse, sometimes called inverse, of the other). So, our algebras in most cases will be Boolean algebras with some further operations.

The *framework* for the quest for the natural algebras of relations *is universal algebra*. The reason for this is that universal algebra is the field which investigates classes of algebras in general, their interconnections, their fundamental properties etc. Therefore universal algebra can provide us for our search with a "map and a compass" to orient ourselves. There is a further good reason for using universal algebra. Namely, universal algebra is not only a unifying framework, but it also contains powerful theories. E.g. if we know in advance some general properties of the kinds of algebras we are going to investigate, then universal algebra can reward us with a powerful machinery for doing these investigations. Among the special classes of algebras concerning which universal algebra has powerful theories are the so called *discriminator varieties* and the *arithmetical varieties*. At the same

[2] The things we say here about predicate logic apply also to most logics having individual variables, hence to all quantifier logics. However, the present paper need not be "predicate logic centered" because our considerations apply also to many *propositional* logics, e.g. to Lambek Calculus, propositional dynamic logic, arrow logics, many-dimensional modal logics. C.f. e.g. [MV], [vB96], [vBtM], [Mi95], [TG].

[3] For more on this see Part II, sections 4, 7 of the present paper.

[4] De Morgan illustrated the need for expanding the algebras of unary relations (i.e. BA's) to algebras of relations in general (the topic of Part I of the present paper) by saying that the scholastics, after two millennia of Aristotelian tradition, were still unable to prove that if a horse is an animal, then a horse's tail is an animal's tail. ("v_0 is a tail of v_1" is a binary relation.)

time, algebras originating from logic turn out to fall in one of these two categories, in most cases. More concretely, more than half of these algebras are in discriminator varieties and almost all are in arithmetical ones. Certainly, all the algebras studied in the present paper are in arithmetical varieties. Therefore, awareness of these recent parts of universal algebra can be rewarding in algebraic logic. We will not assume familiarity with these theories of universal algebra, we will cite the relevant definitions and theorems when using them. [5]

Moreover, as we already said, most of our algebras will be BA's with some additional (extra-Boolean) operations. When these operations are distributive over the Boolean join, as will be the case most often, such algebras are called Boolean Algebras with Operators, in short BAO's. Many of our important classes of algebras will be discriminator varieties of BAO's. The theory of BAO's is well-developed. [6]

Let us return to our task of moving from BA's of unary relations to expanded BA's of relations in general. What are the elements of a BA? They are sets of "points". What will be the elements of the expanded new algebras? One thing about them seems to be certain, they will be sets of sequences, because relations in general are sets of sequences. These sequences may be just pairs if the relation is binary, they may be triples if the relation is ternary, or they may be longer — or even more general kinds of sequences. [7] So, one thing is clear at this point, namely that the elements of our expanded BA's of relations will be sets of sequences. Indeed, this applies to all known algebraizations of predicate logics or quantifier logics. [8]

At this point it might be useful to point out that the most obvious approach (to studying algebras of relations) based on the above observation (that the elements of the algebra are sets of sequences) leads to difficulties right at the start. [9] So, what is the most obvious approach? Consider some set U; let ${}^{<\omega}U$ denote the set of all finite sequences over U, and consider the BA $\mathcal{P}({}^{<\omega}\mathcal{U})$ (the powerset

[5] Some good introductions to universal algebra and discriminator varieties are [HMT, Chapter 0], [BS], [C65], [Gr], [MMT], [W].

[6] Distributivity of the extra-Boolean operations over join is used in the theory to build a well-working duality-theory for it (atom-structures or Kripke-frames, complex algebras). This duality theory is a quite central part of algebraic logic. Because of the limited size of the present paper, we will not deal with this here. Some references are Jónsson-Tarski [JT51], [HMT, section 2.5], Jónsson [J95], Goldblatt [G90], [G91], Venema [V96],[V97], [AGiN95], [H97], [AGoN].

[7] There is another consideration pointing in the direction of sequences. Namely, the semantics of quantifier logics is defined via satisfaction of formulas in models, which in turn is defined via evaluations of variables, and these evaluations are sequences. The meaning of a formula in a model is the *set of* those *sequences* which satisfy the formula in that model. So we arrive again at sets of sequences. For more on this see Part II, section 7 of the present paper.

[8] As mentioned earlier, this also applies to the more complex propositional logics, like e.g. many-dimensional modal logic.

[9] With further work this approach can be turned into a fruitful approach to algebraizing logic, see [N91, §7 (2–4) and the section containing Facts 2, 3 at the end of §4]; see also [HMTII, §5.6.(A survey).3, p. 265], and the references therein. The approach originates with Craig, but already the algebras in Quine [Q36] consist of sets of finite sequences.

of ${}^{<\omega}U$ conceived as a BA the standard way). Now if we are given any finitary relation, say, $R \subseteq U \times U$ over U, then $R \in \mathcal{P}({}^{<\omega}U)$. So $\mathcal{P}({}^{<\omega}U)$ contains all relations over U independently of their ranks. Therefore it might be a candidate for being the universe of an algebra of relations. Before thinking about what the new, so called extra-Boolean operations on $\mathcal{P}({}^{<\omega}U)$ should be, let us have another look at its Boolean structure: If R is a binary relation, we would like to obtain its complement $(U \times U) \smallsetminus R$ as a result of applying a Boolean operation to R. However, in our algebra $\mathcal{P}({}^{<\omega}U)$, the complement of R is not $(U \times U) \smallsetminus R$ but something infinitely bigger.

CHAPTER 1

ALGEBRAS OF BINARY RELATIONS

The above difficulty with $\mathcal{P}({}^{<\omega}U)$ motivates our concentrating first on the simplest nontrivial case, namely that of the *algebras of binary relations* (BRA's). Actually, BRA's will be strong enough to be called a truly first–order (as opposed to propositional) algebraic logic, namely the logic captured by BRA's is strong enough to serve as a vehicle for set theory and hence for ordinary metamathematics. [1]

Throughout this paper, $\mathcal{P}(U)$ denotes the powerset of U, and $\mathfrak{P}(U)$ denotes the Boolean algebra (in short BA) with universe $\mathcal{P}(U)$, for any set U. Thus $\mathcal{P}(U)$ is the set of all subsets of U, and

$$\mathfrak{P}(U) = \langle \mathcal{P}(U), \cup, - \rangle$$

where $\cup$ is the binary operation of taking union of two subsets of U, and $-$ is the unary operation of taking complement (w.r.t. U) of a subset of U. Then $\mathfrak{P}(U)$, as well as any of its subalgebras, is a natural *algebra of unary relations* on U, because a unary relation on U is just a subset of U, hence an element of $\mathcal{P}(U)$.

A binary relation is a set of pairs. Thus the usual set-theoretic (or in other words, Boolean) operations of union and complementation can be performed on binary relations. First we consider two natural operations on binary relations that use the fact that we have sets of *pairs*, namely relation-composition and relation conversion. Let R, S be binary relations. Then their composition [2] $R \circ S$ and the converse R^{-1} of R are defined as [3]

$$R \circ S = \{\langle a, b\rangle : \exists c(aRc \text{ and } cSb)\}$$

$$R^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}.$$

[1] A very interesting class of algebras of relations which is halfway between BA's and BRA's is the class RCA_2 of cylindric algebras of dimension 2. They will be discussed at the beginning of section 2.

[2] This is denoted by $R|S$ in part of the literature, e.g. in [HMTII]. The reason for this is that in a large part of the literature, $\circ$ is reserved for the case when R and S are functions and is written backwards, i.e. what we denote by $R \circ S$ is denoted by $S \circ R$.

[3] Throughout this paper we will use the convention that if R is a binary relation, then aRb means that $\langle a, b\rangle \in R$.

By a *concrete algebra of binary relations*, a cBRA, we understand an algebra whose elements are binary relations having a greatest one among them, and whose operations are the Boolean ones: union and complementation (w.r.t. this greatest relation), relation-composition and relation conversion. Thus the universe of a cBRA is closed under these operations, e.g. the union and relation composition of any two elements of the algebra are also in the universe of the algebra [4].

Formally, a cBRA is of the form

$$\mathfrak{A} = \langle A, \cup, -, \circ, ^{-1} \rangle$$

where

> A is a set of binary relations and A has a biggest element V,
>
> $\cup, -, \circ, ^{-1}$ are total operations on A, which means that $\{R \cup S, V–R, R \circ S, R^{-1}\} \subseteq A$ whenever $R, S \in A$.

An *algebra of binary relations*, a BRA, is an algebra isomorphic to a concrete algebra of binary relations. If $\mathfrak{A}$ is a BRA, then $1^{\mathfrak{A}}$ denotes the greatest element of $\mathfrak{A}$, which we shall sometimes call the *unit* of $\mathfrak{A}$.

Throughout, we use abbreviations like BRA also for denoting the corresponding class itself, e.g. BRA also denotes the class of all BRA's, and BA also denotes the class of all BA's.

The similarity type, or language, of our BRA's should contain two binary function symbols for $\cup$ and $\circ$, and two unary function symbols for $-$ and $^{-1}$. In this paper, for simplicity and suggestiveness, we use the symbols $\cup, \circ, -, ^{-1}$ for these. We hope, this will cause no confusion [5]. Typical equations holding in BRA are $(x \cup y) \circ z = (x \circ z) \cup (y \circ z)$, $(x \cup y)^{-1} = x^{-1} \cup y^{-1}$. In the literature $\vee, +$ are often used as function symbols for $\cup$, and likewise $;, ^{\smile}$ are used as function symbols for $\circ, ^{-1}$. Using these symbols, the above equations look as $(x \vee y); z = (x; z) \vee (y; z)$, $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$, or $(x + y); z = (x; z) + (y; z)$, $(x + y)^{\smile} = x^{\smile} + y^{\smile}$.

So we will use the symbol $\cup$ also in abstract Boolean algebras. Moreover, in abstract Boolean algebras we also will use $\cap$ as derived operation: $x \cap y \stackrel{\text{def}}{=} -(-x \cup -y)$. $\leq, 0, 1$ will denote the ordering $x \leq y \stackrel{def}{\iff} x \cup y = y$, smallest element and biggest element, respectively. Thus $\tau \leq \sigma$ is an equation.

[4] To understand how (and why) the theory works, it would be enough to include only "$\circ$" as "extra-Boolean" operation. Inclusion of conversion is motivated by some of the applications. Cf. the discussion of BSR below Thm.1.9 (this section).

[5] When seeing, say "$x \cup y$," the reader will have to decide whether this denotes a term of BRA's built up from the variables x, y, or whether it denotes a set (the union of the sets x and y.)

Having a fresh look at our BRA's with an abstract algebraic eye, we notice that they should be very familiar from the abstract algebraic literature. Namely, a BRA $\mathfrak{A}$ consists of two well known algebraic structures, a Boolean algebra $\langle A, \cup, -\rangle$ and an involuted semigroup $\langle A, \circ, ^{-1}\rangle$ sharing the same universe A. The two structures are connected so that they form a *normal Boolean algebra with operators*, in short a normal BAO, which means that each extra-Boolean operation is distributive over $\cup$ (additivity) and takes the value 0 whenever at least one of the arguments is 0 (normality). Also $^{-1}$ is a Boolean isomorphism and $x \mapsto 1 \circ x$, where 1 is the Boolean 1, defines a *complemented closure operation* [6] on A. The properties listed in this paragraph define a nice variety ARA containing BRA and is a reasonable starting point for an axiomatic study of the algebras of binary relations. [7]

DEFINITION 1.1 (ARA, an abstract approximation of BRA) ARA *is defined to be the class of all algebras of the similarity type of* BRA*'s which validate the following equations.*

(1) The Boolean axioms [8]

$$x \cup y = y \cup x,$$
$$x \cup (y \cup z) = (x \cup y) \cup z,$$
$$-[-(x \cup y) \cup -(x \cup -y)] = x.$$

(2) The axioms of involuted semigroups, i.e.

$$(x \circ y) \circ z = x \circ (y \circ z),$$
$$(x \circ y)^{-1} = y^{-1} \circ x^{-1},$$
$$x^{-1-1} = x.$$

(3) The axioms of normal BAO*, i.e.* [9]

[6] Closure operations are unary functions $f : U \to U$, where we have an ordering $\leq$ on U. f is called a closure operation if it is order preserving, idempotent, and increasing, i.e. if for all $u, v \in U$ we have $u \leq f(u) = ff(u)$ and $u \leq v \Rightarrow f(u) \leq f(v)$. Boolean orderings with closure operations on them are one of the central concepts of abstract algebra, for example topological spaces or subalgebras of an algebra are often represented as such. f is called a complemented closure operation if $f(-fx) = -f(x)$, i.e. the complement of a closed element is closed. For more on these see e.g. [HMT, p.38].

[7] Most of these postulates already appear in De Morgan [D1864], and since then investigations of ARA's have been carried on for almost 130 years.

[8] Problem 1.1 in [HMT, p.245], originating with H. Robbins, asks whether this is an axiom system for BA. This problem has recently been solved affirmatively (by the theorem prover program EQP developed at Argonne National Laboratory, USA). We will use this axiom system for BA in Part II, section 7.1.

[9] We are omitting some axioms that follow from the already stated ones. E.g. here we omit $x \circ (y \cup z) = (x \circ y) \cup (x \circ z)$, $x \circ 0 = 0$.

$$(x \cup y) \circ z = (x \circ z) \cup (y \circ z),$$
$$(x \cup y)^{-1} = x^{-1} \cup y^{-1},$$
$$0 \circ x = 0, \ \ 0^{-1} = 0.$$

(4) $^{-1}$ *is a Boolean isomorphism and* $x \mapsto 1 \circ x$ *is a complemented closure operation, i.e.*

$$-(x^{-1}) = (-x)^{-1},$$
$$x \leq 1 \circ x,$$
$$-(1 \circ x) = 1 \circ -(1 \circ x).$$

■

If E is a set of equations, then $\mathsf{Mod}(E)$ denotes the class of all algebras (of a given similarity type) in which E holds. A class K of algebras is called a *variety*, or an *equational class*, if $\mathsf{K} = \mathsf{Mod}(E)$ for some set E of equations. The following theorem is due to A. Tarski.

THEOREM 1.2 BRA *is an equational class, i.e. there is a set* E *of equations such that* $\mathsf{BRA} = \mathsf{Mod}(E)$.

To prove the above theorem, we will use the machinery of universal algebra. First we prove that BRA is closed under taking subalgebras and direct products. If K is a class of algebras, then $\mathbf{S}\mathsf{K}$ denotes the class of all subalgebras of elements of K, $\mathbf{P}\mathsf{K}, \mathbf{I}\mathsf{K}$, $\mathbf{H}\mathsf{K}$ and $\mathbf{Up}\mathsf{K}$ denote the classes of all algebras isomorphic to direct products, isomorphic copies, homomorphic images, and ultraproducts of elements of K respectively. [10] Thus $\mathsf{BRA} = \mathbf{I}\mathsf{cBRA}$.

LEMMA 1.3 $\mathsf{BRA} = \mathbf{SP}\{\langle \mathfrak{P}(U \times U), \circ, ^{-1}\rangle : U \textit{ is a set}\}$.

Proof. Let V be a binary relation. We say that V is an *equivalence relation* if V is symmetric and transitive, i.e. if $V^{-1} = V$ and $V \circ V \subseteq V$. The *field* of V is the smallest set U such that $V \subseteq U \times U$, i.e. $U = \{u : (\exists v)[\langle u, v\rangle \in V \text{ or } \langle v, u\rangle \in V]\}$. The following three statements $(*)$–$(***)$ will not be difficult to check:

$(*)$ If $\mathfrak{A} \in \mathsf{cBRA}$, then $1^{\mathfrak{A}}$ is an equivalence relation.

[10] Note that $\mathbf{S}$ is different from $\mathbf{P}, \mathbf{I}, \mathbf{H}$ and $\mathbf{Up}$ in that $\mathbf{S} \neq \mathbf{IS}$, while $\mathbf{P} = \mathbf{IP}$ etc. For our reasons for defining $\mathbf{S}$ this way see the remark after the definition of $\mathbf{Alg}_m$ in Part II, Def. 5.1. We will use simple facts like $\mathbf{IS} = \mathbf{SI}$, $\mathbf{IP} = \mathbf{PI}$, $\mathbf{SS} = \mathbf{S}$, etc. without mentioning them.

$(**)$ If V is an equivalence relation, then

$$\mathfrak{R}(V) \stackrel{\text{def}}{=} \langle \mathfrak{P}(V), \circ, {}^{-1} \rangle \in \mathsf{cBRA}.$$

$(***)$ Let I be a set and for all $i \in I$ let V_i be an equivalence relation. Assume that the fields of the V_i's are pairwise disjoint. Then

$$\mathfrak{R}\left(\bigcup_{i\in I} V_i\right) \cong \mathsf{P}_{i\in I}\mathfrak{R}(V_i).$$

Indeed, to see $(*)$, let $\mathfrak{A} \in \mathsf{cBRA}$ and $V \stackrel{\text{def}}{=} 1^{\mathfrak{A}}$. Then $V \in A$, hence $V \circ V, V^{-1}$ are in A as well, hence $V \circ V \subseteq V$ and $V^{-1} \subseteq V$, because V is the biggest element of A. But $V^{-1} \subseteq V$ is equivalent to $V^{-1} = V$, hence V is an equivalence relation.

To show $(**)$, one has to check that for any $R, S \subseteq V$ also $R \circ S \subseteq V$ and $R^{-1} \subseteq V$. These follow from $V \circ V \subseteq V$, $V^{-1} \subseteq V$.

To show $(***)$, we define the function $f : \mathcal{P}(\bigcup_{i\in I} V_i) \to \mathsf{P}_{i\in I}\mathcal{P}(V_i)$ by letting for all $X \subseteq \bigcup_{i\in I} V_i$,

$$f(X) \stackrel{\text{def}}{=} \langle X \cap V_i : i \in I \rangle.$$

Then it is easy to check that this f is the required isomorphism.

We are ready to prove the lemma. First we show that $\mathsf{BRA} = \mathbf{SP}\mathsf{BRA}$. By definition, cBRA is closed under taking subalgebras, so BRA is also closed under taking subalgebras (because $\mathsf{BRA} = \mathbf{I}\mathsf{cBRA}$). Let I be a set, and let $\mathfrak{A}_i \in \mathsf{cBRA}$ with unit V_i for each $i \in I$. We may assume that the V_i's have disjoint fields. Then $\mathfrak{A}_i \subseteq \mathfrak{R}(V_i)$, so $\mathsf{P}\mathfrak{A}_i \subseteq \mathsf{P}\mathfrak{R}(V_i) \cong \mathfrak{R}(\bigcup V_i) \in \mathsf{cBRA}$ by $(*)$–$(***)$. This shows that $\mathsf{P}\mathfrak{A}_i$ is isomorphic to a cBRA, i.e. BRA is closed under taking direct products.

Now let $\mathfrak{A} \in \mathsf{cBRA}$ with greatest element V. Then V is an equivalence relation, let $U_i, i \in I$ be the blocks of this equivalence relation. Then $U_i \times U_i$ are also equivalence relations with pairwise disjoint fields, and V is the union of these. Hence by $(**)$–$(***)$ we have that $\mathfrak{A} \subseteq \mathsf{P}\langle\langle \mathfrak{P}(U_i \times U_i), \circ, {}^{-1}\rangle : U_i \text{ is a block of } 1^{\mathfrak{A}}\rangle$. This completes the proof of Lemma 1.3. ■

To formulate our next lemma, we need the notions of a subdirect product and a discriminator term.

Subdirect products of algebras, and subdirectly irreducible algebras are defined in practically every textbook on universal algebra, cf. e.g. Grätzer's book [Gr], or [BS], [HMT], [MMT]. By a *subdirect product* we mean a subalgebra of a product such that the projections of the product restricted to the subalgebra remain surjective mappings. An algebra $\mathfrak{A}$ is *subdirectly irreducible* if it is not (isomorphic to) a subdirect product of algebras different from $\mathfrak{A}$. We note that the one-element algebra is not subdirectly irreducible. By Birkhoff's classical theorem, every algebra is a subdirect product of some subdirectly irreducible ones. Therefore, the subdirectly irreducible algebras are often regarded as the basic building blocks of all the other algebras. In particular, when studying an algebra $\mathfrak{A}$, it is often enough to study its subdirectly irreducible building blocks. For a class K of algebras, $\mathbf{Sir}(\mathsf{K})$ denotes the class of subdirectly irreducible members of K. For $\mathsf{K} = \mathsf{BA}$, $\mathbf{Sir}(\mathsf{BA})$ consists of the 2-element Boolean algebra only (up to isomorphisms).

We say that a class K of algebras *has a discriminator term* iff there is a term $\tau(x, y, z, u)$ in the language of K such that in every member of K we have

$$\tau(x, y, z, u) = \begin{cases} z, & \text{if } x = y, \\ u, & \text{if } x \neq y. \end{cases}$$

The term τ above is called a *discriminator term*. Sometimes instead of the four-ary τ, the ternary discriminator term $t(x, y, z) = \tau(x, y, z, x)$ is used. They are interdefinable, since $\tau(x, y, z, u) = t(t(x, y, z), t(x, y, u), u)$. Therefore, it does not matter which one is used. Moreover, in classes of algebras which have a Boolean algebra reduct, like our BRA's or ARA's, the discriminator term can be replaced with the so called *switching term*

$$c(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

By this we mean that in such a class of algebras, if $\tau(x, y, z, u)$ is a discriminator term, then $c(x) = \tau(x, 0, 0, 1)$ is a switching term, and vica versa, if $c(x)$ is a switching term, then $\tau(x, y, z, u) = [-c(x \oplus y) \cap z] \cup [c(x \oplus y) \cap u]$ is a discriminator term. Here, and later on, $\oplus$ denotes symmetric difference, i.e. $x \oplus y \stackrel{\text{def}}{=} (x \cap -y) \cup (-x \cap y)$.

LEMMA 1.4 $\mathbf{Sir}(\mathsf{BRA}) = \mathbf{IS}\{\langle \mathfrak{P}(U \times U), \circ, {}^{-1} \rangle : U$ *is a nonempty set*$\}$ *and* $\mathbf{Sir}(\mathsf{BRA})$ *has a discriminator term.*

Proof. Let $\mathsf{K} \stackrel{\text{def}}{=} \mathbf{IS}\{\langle \mathfrak{P}(U \times U), \circ, {}^{-1} \rangle : U \text{ is a nonempty set}\}$. Let $\mathfrak{A} \in \mathsf{BRA}$. Then $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathsf{P}\mathfrak{R}(U_i \times U_i)$ for some system $\langle U_i : i \in I \rangle$ of sets, by Lemma 1.3. If $U_i = \emptyset$, then $\mathfrak{R}(U_i \times U_i)$ is the one-element algebra which can be left out from any product, so we may assume that each U_i above is nonempty. But then $\mathfrak{A}$ is a subdirect product of some $\mathfrak{B}_i$, $i \in I$ where each $\mathfrak{B}_i$ is a subalgebra of $\mathfrak{R}(U_i \times U_i)$. This shows that $\mathbf{Sir}(\mathsf{BRA}) \subseteq \mathsf{K}$.

It is not difficult to check that

$$c(x) \stackrel{\text{def}}{=} 1 \circ x \circ 1$$

is a switching term on $\mathfrak{R}(U \times U)$ for all U. Hence it is a switching term on K also. Thus, K has a discriminator term.

Finally, if $\mathfrak{A}$ has a discriminator term, then $\mathfrak{A}$ has no nontrivial congruences, i.e. $\mathfrak{A}$ is simple. This is a basic fact of discriminator theory. [11] Clearly, any simple algebra is subdirectly irreducible, so $\mathsf{K} \subseteq \mathbf{Sir}(\mathsf{BRA})$. ■

We say that K is a *pseudo-axiomatizable* class if there are an expansion $\mathcal{L}$ of the language of K, a set Σ of first–order formulas in this bigger language $\mathcal{L}$ and a unary relation symbol U of $\mathcal{L}$ such that

$$\mathsf{K} = \mathbf{Rd}_U \mathsf{Mod}\Sigma,$$

where $\mathsf{Mod}\Sigma$ denotes the class of all models of Σ, and $\mathbf{Rd}_U$ denotes the operator of taking reducts to the language of K and restricting the universe to U at the same time. In more detail: Let $\mathfrak{M}$ be a model of the language $\mathcal{L}$. Then $\mathbf{Rd}\mathfrak{M}$ denotes the reduct of $\mathfrak{M}$ to the language of K, and $\mathbf{Rd}_U\mathfrak{M}$ denotes the restriction of the model $\mathbf{Rd}\mathfrak{M}$ to the interpretation $U^{\mathfrak{M}} \subseteq M$ of U in $\mathfrak{M}$. I.e. while $\mathfrak{M}$ is a model of the bigger language $\mathcal{L}$, $\mathbf{Rd}_U\mathfrak{M}$ is a model of the smaller language of K. If N is a class of models of the language of $\mathcal{L}$, then $\mathbf{Rd}_U\mathsf{N} \stackrel{\text{def}}{=} \{\mathbf{Rd}_U\mathfrak{M} : \mathfrak{M} \in \mathsf{N}\}$.

It is known that pseudo-axiomatizable classes are closed under ultraproducts, this is easy to show.

LEMMA 1.5 $\mathbf{Sir}(\mathsf{BRA})$ *is a pseudo-axiomatizable class.*

Proof. The expansion $\mathcal{L}$ of the language of BRA will be a many-sorted first-order language with three sorts: S, P and R (for set, pairs, and relations), two unary functions p_0, p_1 from P to S (first and second projections), a binary relation ε between P and R (for "is an element of"), and binary functions $\cup, \circ$ on R, unary functions $-, ^{-1}$ on R. The variables x, y, z are of sort S, the variables u, v, w are of sort P, and the variables a, b, c are of sort R. We also consider S, P, R as unary relations. [12] See Figure 1.1.

The set Σ of axioms is as follows: In the following formulas we will write ε in infix mode, like $u\varepsilon a$. Also we will write comma in place of conjunction $\wedge$. There are free variables in the elements of Σ, validity of an open formula is meant in such a way that all the free variables are universally quantified at the beginning of the formula. Σ is defined to be $\{(1a), (1b), \ldots, (4)\}$, where

The "pair-axioms" are:

[11] The reason is the following. Assume that R is a nonidentity congruence of $\mathfrak{A}$. We will show that then $R = A \times A$. Let $u, v \in A$, $u \neq v$ be such that uRv, and let $a, b \in A$ be arbitrary. Then $a = \tau(u, u, a, b) R \tau(u, v, a, b) = b$, so aRb.

[12] If one is not familiar with many-sorted models, then one can think of the above language as having S, P, R as unary relation symbols, and e.g. p_0 as a binary relation. Then to our axioms we have to add statements like $p_0(x, y) \to P(x)$, $p_0(x, y) \to S(y)$, $p_0(x, y), p_0(x, z) \to y = z$, $P(x) \to (\exists y) p_0(x, y)$. Then the fact that the variable x in the many-sorted language is of sort S while the variable u is of sort P means that one has to replace e.g. the formula $\forall x \exists u p_0(x) = u$ with $(\forall x)(S(x) \to \exists u(P(u) \wedge p_0(x, u)))$.

$$\begin{tikzcd}[column sep=large] \mathbf{R} \arrow[loop left, "{\cup, -, \circ, {}^{-1}}"] \arrow[r, dash, "\varepsilon"] & \mathbf{P} \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & \mathbf{S} \\[-2.5em] \text{(relations)} & \text{(pairs)} & \text{(base set)} \\[-2.5em] x, y, z & u, v, w & a, b, c \end{tikzcd}$$

Figure 1.1.

(1a) $(\exists u)(p_0(u) = x, p_1(u) = y)$.

(1b) $p_0(u) = p_0(v), p_1(u) = p_1(v) \ \rightarrow\ u = v$.

Extensionality of sets of pairs:

(2) $\forall u(u\varepsilon a \leftrightarrow u\varepsilon b) \rightarrow a = b$.

The definitions of the operations of cBRA:

(3a) $u\varepsilon(a \cup b) \leftrightarrow (u\varepsilon a$ or $u\varepsilon b)$.

(3b) $u\varepsilon(-a) \leftrightarrow \neg u\varepsilon a$.

(3c) $u \in (a \circ b) \leftrightarrow (\exists vw)(v\varepsilon a, w\varepsilon b, p_0(u) = p_0(v), p_1(v) = p_0(w),$
$p_1(w) = p_1(u))$.

(3d) $u \in (a^{-1}) \leftrightarrow (\exists v)(v\varepsilon a, p_0(u) = p_1(v), p_1(u) = p_0(v))$.

There are at least two elements in the relations sort:

(4) $(\exists ab) a \neq b$.

This finishes the definition of Σ. We will show that

$$\mathbf{Sir}(\mathsf{BRA}) = \mathbf{Rd}_R \mathsf{Mod}\Sigma.$$

Indeed, let $\mathfrak{A} \in \mathbf{Sir}(\mathsf{BRA})$, say $\mathfrak{A}$ is isomorphic to a subalgebra of $\langle \mathfrak{P}(U \times U), \circ, {}^{-1}\rangle$. We may assume that $\mathfrak{A} \subseteq \langle \mathfrak{P}(U \times U), \circ, {}^{-1}\rangle$. We define the three-sorted model $\mathfrak{M}$ as follows.

$$S^{\mathfrak{M}} \stackrel{\text{def}}{=} U, \quad P^{\mathfrak{M}} \stackrel{\text{def}}{=} U \times U, \quad R^{\mathfrak{M}} \stackrel{\text{def}}{=} A,$$

$$p_0^{\mathfrak{M}}(\langle u, v\rangle) = u, \quad p_1^{\mathfrak{M}}(\langle u, v\rangle) = v \text{ for all } u, v \in U,$$

$$\langle u, v\rangle \varepsilon^{\mathfrak{M}} a \text{ iff } \langle u, v\rangle \in a, \text{ for all } u, v \in U, a \in A,$$

$$a \cup^{\mathfrak{M}} b \stackrel{\text{def}}{=} a \cup b,\ -^{\mathfrak{M}} a \stackrel{\text{def}}{=} -a,\ a \circ^{\mathfrak{M}} b \stackrel{\text{def}}{=} a \circ b,\ a^{-1^{\mathfrak{M}}} b \stackrel{\text{def}}{=} a^{-1}.$$

Then it is easy to check that $\mathfrak{M} \models \Sigma$ and $\mathbf{Rd}_R\mathfrak{M} = \mathfrak{A}$. See Figure 1.2.

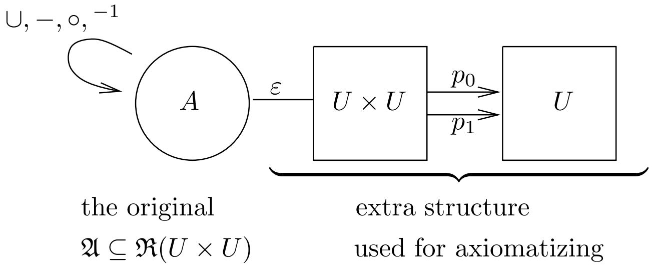

Figure 1.2.

Conversely, let $\mathfrak{M}$ be such that $\mathfrak{M} \models \Sigma$. Let $U \stackrel{\text{def}}{=} S^{\mathfrak{M}}$. Define the relation T between $U \times U$ and $P^{\mathfrak{M}}$ as follows:

$$\langle u, v\rangle T x \quad \text{iff} \quad p_0(x) = u, p_1(x) = v.$$

By (1a),(1b) in Σ then T is a bijection between $U \times U$ and $P^{\mathfrak{M}}$. Therefore we will assume that

$$U \times U = P^{\mathfrak{M}} \quad \text{and} \quad u = p_0^{\mathfrak{M}}(\langle u, v\rangle), v = p_1^{\mathfrak{M}}(\langle u, v\rangle).$$

We define now the function $Q : R^{\mathfrak{M}} \to \mathcal{P}(U \times U)$ as

$$Q(a) = \{\langle u, v\rangle : \langle u, v\rangle \varepsilon^{\mathfrak{M}} a\}.$$

See Figure 1.3. Then Q is one-to-one by (2) in Σ. Axioms (3a)–(3d) in Σ say that[13]

$$\begin{aligned} &Q(a \cup^{\mathfrak{M}} b) = Q(a) \cup Q(b), \\ &Q(-^{\mathfrak{M}} a) = (U \times U) \smallsetminus Q(a), \\ &Q(a \circ^{\mathfrak{M}} b) = Q(a) \circ Q(b), \\ &Q(a^{-1^{\mathfrak{M}}}) = (Q(a))^{-1}. \end{aligned}$$

This shows that Q is an isomorphism from $\mathbf{Rd}_R\mathfrak{M}$ into $\langle \mathfrak{P}(U \times U), \circ, ^{-1}\rangle$. Finally, (4) in Σ implies that U is nonempty, $\mathbf{Rd}_R\mathfrak{M}$ is nonempty. ■

We now are ready to apply the following theorem of universal algebra (cf. e.g. [BS, Thm.9.4 (b,c)]).

THEOREM (universal algebra) *If* $\mathbf{SUp}\mathsf{K}$ *has a discriminator term, then*

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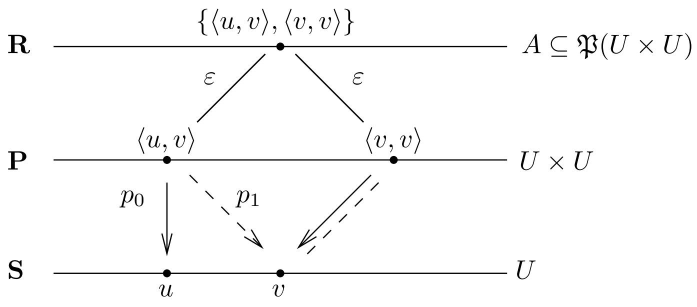
[13] $X \smallsetminus Y \stackrel{\text{def}}{=} \{x \in X : x \notin Y\}$.



The three-story structure of $\mathfrak{A}$

Figure 1.3.

$\mathbf{SPUp}\mathsf{K}$ *is an equational class.*

Indeed, let $\mathsf{K} = \mathbf{Sir}(\mathsf{BRA})$. Then $\mathsf{K} = \mathbf{S}\mathsf{K}$ by Lemma 1.4, and $\mathsf{K} = \mathbf{Up}\mathsf{K}$ by Lemma 1.5, thus $\mathsf{K} = \mathbf{SUp}\mathsf{K}$. Also, K has a discriminator term by Lemma 1.4. Thus $\mathbf{SPUp}\mathsf{K}$ is an equational class by the above theorem of universal algebra. But $\mathbf{SPUp}\mathsf{K} = \mathbf{SP}\mathsf{K} = \mathbf{SPSir}(\mathsf{BRA}) = \mathsf{BRA}$ by Lemma 1.3, and we are done with proving that BRA is an equational class.

QED(Theorem 1.2) ■

In universal algebra, an equational class K such that $\mathbf{Sir}\mathsf{K}$ has a discriminator term is called a *discriminator variety*. So we proved that BRA is a discriminator variety.

The above proof of Theorem 1.2 uses techniques that can be applied in many cases in algebraic logic. E.g. these same techniques work for cylindric and polyadic algebras. See e.g. Thms 1.10, 2.3. $\langle \mathfrak{P}(U\times U), \circ, {}^{-1}\rangle$ is called the *full* BRA over the set U. By Lemma 1.3 we could have defined BRA as

$$\begin{aligned}\mathsf{BRA} &= \mathbf{SP}\{\langle \mathfrak{P}(U\times U), \circ, {}^{-1}\rangle : U \text{ is a set}\}, \qquad \text{or as}\\ \mathsf{BRA} &= \mathbf{SP}\{\langle \mathfrak{P}(U\times U), \circ, {}^{-1}\rangle : U \text{ is a nonempty set}\}.\end{aligned}$$

Set

$$\mathsf{setBRA} \stackrel{\text{def}}{=} \mathbf{S}\{\langle \mathfrak{P}(U\times U), \circ, {}^{-1}\rangle : U \text{ is a nonempty set}\}.$$

Then $\mathsf{BRA} = \mathbf{SP}\mathsf{setBRA}$. [14] This fact, and the class setBRA will be used in Part II (section 7.4) when translating our algebraic results to logic.

[14] Because $\mathbf{SPS} = \mathbf{SP}$, this is a basic theorem in universal algebra. See e.g. [HMT, 0.3.12].

In the following, we will define our classes of algebras of relations in this style. So *when defining new kinds of algebras of relations, we will first define the simplest version* (e.g. the one with top element $U \times U \times \ldots \times U$), *and then take all subalgebras of all direct products of these*.

Let $\mathsf{K} = \mathsf{setBRA}$. Then, as we have seen, $\mathsf{BRA} = \mathbf{SP}\mathsf{K}$ is a variety because K has a discriminator term and K is pseudo-axiomatizable. [15] In almost all our cases, K, where K is the class of the corresponding set algebras, will be pseudo-axiomatizable because K is defined to be a three-story structure like BRA, only the operations on the third level will vary (and instead of $U \times U$ we may have $U \times U \times \ldots \times U$), and in most cases K will have a discriminator term. [16]

Theorem 1.2 indicates that BRA is indeed a promising start for developing a nice algebraization of stronger logics (like e.g. quantifier logics), or in the non-logical perspective, for developing an algebraic theory of relations. After Theorem 1.2, the question comes up naturally whether we can strengthen the postulates defining ARA to obtain a finite set E of equations describing the variety BRA, i.e. such that $\mathsf{BRA} = \mathsf{Mod}(E)$ would be the case. The answer is due to J. D. Monk:

THEOREM 1.6 BRA *is not finitely axiomatizable, i.e. for no finite set* Σ *of first-order formulas is* $\mathsf{BRA} = \mathsf{Mod}(\Sigma)$.

The idea of one possible proof is explained in Remark 2.9 in section 2 herein. This idea is based on the proof of Thm 2.5 which is the reason why it is postponed to that part of the paper. See Monk [M64], and also [HMTII, 5.1.57, 4.1.3], for the original proof of Theorem 1.6 (in slightly different settings).

For a class K of algebras, let $\mathbf{Eq}\mathsf{K}$ denote the set of all equations valid in K.

THEOREM 1.7 $\mathbf{Eq}(\mathsf{BRA})$ *is recursively enumerable but not decidable.*

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[15] We could have proved $\mathsf{K} = \mathbf{SUp}\mathsf{K}$ more directly, as follows. An ultraproduct of full BRA's on some sets U_i is isomorphic to the full BRA on the ultraproduct of the U_i's, namely if F is an ultrafilter on I, and $\mathfrak{F}(U)$ denotes $\langle \mathfrak{P}(U \times U), \circ, ^{-1}\rangle$, then $\mathsf{P}\mathfrak{F}(U_i)/F \cong \mathfrak{F}(\mathsf{P}U_i/F)$, and the isomorphism h is given by $a/F \mapsto \{\langle u/F, v/F\rangle : \{i \in I : \langle u_i, v_i\rangle \in a_i\} \in F\}$. The reader is invited to check that h is indeed an isomorphism. This method also is applicable in many places. We chose the method of pseudo-axiomatizability for proving that K is closed under ultraproducts, because we feel that this method reveals the real cause: our concrete algebras are usually pseudo-axiomatizable, because "concrete" very much means this, i.e. "concrete" means that there is some extra structure not coded in the operations, which means that this extra structure may disappear when taking isomorphic copies.

[16] Even if K would not have a discriminator term, then $\mathbf{SP}\mathsf{K}$ would still be a *quasi-variety*, i.e. definable by equational implications, because K will be pseudo-axiomatizable, hence $\mathsf{K} = \mathbf{Up}\mathsf{K}$, thus $\mathbf{SP}\mathsf{K} = \mathbf{SPUp}\mathsf{K}$ will hold. It is a basic theorem of universal algebra that K is a quasi-variety iff $\mathsf{K} = \mathbf{SPUp}\mathsf{K}'$ for some K'.

Proof. An equation holds in BRA iff it holds in $\mathbf{Sir}(\mathsf{BRA})$ by Lemma 1.3. Let Σ be the finite set of first-order formulas such that $\mathbf{Sir}(\mathsf{BRA}) = \mathbf{Rd}_R\mathsf{Mod}(\Sigma)$, from the proof of Lemma 1.5. Thus an equation is valid in $\mathbf{Sir}(\mathsf{BRA})$ iff it is valid in $\mathsf{Mod}(\Sigma)$ (when all the variables of the original equation are considered to be of sort R). The consequences of any finite set of first-order formulas is recursively enumerable by the completeness theorem of first-order logic. Thus $\mathbf{Eq}(\mathsf{BRA})$ is recursively enumerable (and an enumeration is given by the present proof).

The proof of undecidability of $\mathbf{Eq}(\mathsf{BRA})$ goes via interpreting the quasi-equational theory of semigroups into $\mathbf{Eq}(\mathsf{BRA})$. The proof consists of two steps:

($*$) An equational implication (i.e. a *quasi-equation*) about $\circ$ is valid in all semigroups iff it is valid in BRA.

($**$) To any equational implication q there is an equation e in the language of BRA such that BRA $\models q$ iff BRA $\models e$.

Proof of ($*$)*:* If q is true in all semigroups, then it is true in BRA because $\circ$ is associative in BRA. If q fails in a semigroup $\langle S, \cdot\rangle$, then take the Cayley-representation of this semigroup, this is an embedding of $\langle S, \cdot\rangle$ into $\langle \mathcal{P}(S' \times S'), \circ\rangle$ which is a reduct of $\mathfrak{R}(S' \times S') \in$ BRA. Thus q fails in BRA.

Proof of ($**$)*:* The reason is that BRA is a discriminator variety, and in every discriminator variety a quasi-equation q is equivalent to an equation e on the subdirectly irreducibles [17]. Now, by BRA $= \mathbf{SPSir}(\mathsf{BRA})$ we have that BRA $\models q$ iff $\mathbf{Sir}(\mathsf{BRA}) \models q$ iff(by the above) $\mathbf{Sir}(\mathsf{BRA}) \models e$ iff BRA $\models e$.

Now ($*$) and ($**$) above give an interpretation of the quasi-equations valid in all semigroups into the equations valid in BRA. Since it is known that the former is undecidable, we also have that the latter, $\mathbf{Eq}(\mathsf{BRA})$, is undecidable. ■

The above method of proof for undecidability is also widely applicable in algebraic logic. The above proof e.g. is in [CM]. For more refined uses of this technique see e.g. [Ma80], [N85a] (finite dimensional part) [KNSS], [KNSS2], [AGiN97, chapter II], [K97]. For more on *(un)decidability* in algebraic logic we refer to the just quoted works together with Jipsen [Ji92], [Ma78a], [HMTII], [N86], [N87], [N92], [MV], [Mi95], [N91].

We turn to determining the logic "captured by" BRA. We note that the connection with logic will be much more lucid in the case of cylindric (and polyadic) algebras of n-ary relations.

Let $\mathcal{L}^{\neq}_{3,2}$ denote first order logic without equality and using only three variables x, y, z, with countably many *binary* relation symbols $R_0, R_1, \ldots$ (so e.g.

[17] This is one of the basic facts of discriminator varieties. Assume that q is $\tau_1 = \sigma_1 \wedge \ldots \wedge \tau_n = \sigma_n \rightarrow \tau_0 = \sigma_0$. Then $c - (\tau_1 \oplus \sigma_1) \cap \ldots \cap c - (\tau_n \oplus \sigma_n) \leq -(\tau_0 \oplus \sigma_0)$ can be chosen for e, where c denotes the switching term and $\oplus$ denotes symmetric difference.

no ternary relation symbols), and the atomic formulas are $R_i(u,v)$ with distinct variables u, v (so atomic formulas of the form $R_i(u,u)$ are not allowed).

THEOREM 1.8 *$\mathcal{L}^{\neq}_{3,2}$ can be interpreted into* **Eq**(BRA). *I.e. there is a recursive function e mapping $\mathcal{L}^{\neq}_{3,2}$ into the set of equations on the language of* BRA *such that for every $\varphi \in \mathcal{L}^{\neq}_{3,2}$*

$$\varphi \textit{ is valid} \qquad \textit{iff} \qquad \mathsf{BRA} \models e(\varphi).$$

Theorem 1.8 will be a consequence of the following, stronger Theorem 1.9. We stated Theorem 1.8 because it states that $\mathcal{L}^{\neq}_{3,2}$ can be interpreted into **Eq**(BRA), thus **Eq**(BRA) is "at least as strong" as $\mathcal{L}^{\neq}_{3,2}$. Set Theory can be interpreted in $\mathcal{L}^{\neq}_{3,2}$, this is proved in Tarski-Givant [TG, §4.6, pp.127–134]. Thus the logic captured by BRA is strong enough to serve as a vehicle for set theory, and hence for ordinary mathematics, as we mentioned at the beginning of this chapter. [18]

We can characterize the expressive power of BRA in terms of $\mathcal{L}^{\neq}_{3,2}$. This will be stated and proved as Theorem 1.9 below. We need some preparations for stating Theorem 1.9.

In the equational language of BRA let us use the variables $v_i, i \in \omega$, where $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers. For any model $\mathfrak{M} = \langle M, R_i^{\mathfrak{M}}\rangle_{i\in\omega}$ of $\mathcal{L}^{\neq}_{3,2}$ let $k_{\mathfrak{M}}$ denote the evaluation of the variables $v_i, i \in \omega$ such that

$$k_{\mathfrak{M}}(v_i) = R_i^{\mathfrak{M}} \qquad \text{for all} \quad i \in \omega.$$

Recall that $\mathfrak{R}(M \times M) = \langle \mathfrak{P}(M \times M), \circ, ^{-1}\rangle \in \mathsf{setBRA}$.

If $\mathfrak{A}$ is an algebra, k is an evaluation of the variables, τ is a term, and e is an equation, then $\mathfrak{A} \models e[k]$ denotes that the equation e is true in the algebra $\mathfrak{A}$ under the evaluation k of the variables, and $\tau^{\mathfrak{A},k}$ denotes the element of A denoted by the term τ when the variables are evaluated according to k.

Let u, v be distinct elements of $\{x, y, z\}$. Then $\mathcal{L}_3^{uv}$ denotes the set of those elements of $\mathcal{L}^{\neq}_{3,2}$ which contain only u, v as *free* variables. If $\varphi \in \mathcal{L}_3^{xy}$ and $\mathfrak{M}$ is a model, then $\varphi^{\mathfrak{M}}$ denotes the following binary relation on M:

$$\varphi^{\mathfrak{M}} \stackrel{\text{def}}{=} \{\langle a, b\rangle \in M \times M : \mathfrak{M} \models \varphi[a, b]\}.$$

The following Theorem 1.9 says [19] that, in a way, the *expressive power of* BRA is $\mathcal{L}_3^{xy}$. We included (i) for its simple content, and (ii) states a correspondence between meanings of formulas in $\mathcal{L}_3^{xy}$ and denotation of terms in elements of setBRA. For more on the background ideas of this see Part II of the present paper.

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[18] This also gives another proof for undecidability of **Eq**(BRA), because Set Theory is undecidable.

[19] These statements and proofs are simplified versions of those in [TG]. Cf. also [HMTII, §5.3, 4.3].

THEOREM 1.9 (the expressive power of $\mathbf{Eq}(\mathsf{BRA})$)

(i) For any $\varphi \in \mathcal{L}^{\neq}_{3,2}$ *there is an equation* e *such that for all models* $\mathfrak{M}$ *of* $\mathcal{L}^{\neq}_{3,2}$

$$\mathfrak{M} \models \varphi \qquad \textit{iff} \qquad \mathfrak{R}(M \times M) \models e[k_{\mathfrak{M}}].$$

(ii) There are recursive functions $t : \mathcal{L}^{\neq}_{3,2} \to Terms$ *and* $f : Terms \to \mathcal{L}^{\neq}_{3,2}$ *such that for any* $\varphi \in \mathcal{L}^{xy}_{3}$ *and model* $\mathfrak{M}$

$$\varphi^{\mathfrak{M}} = t(\varphi)^{\langle \mathfrak{R}(M\times M), k_{\mathfrak{M}}\rangle}, \qquad \textit{and}$$

for any term τ*, set* U*, and evaluation* k*,*

$$\tau^{\langle \mathfrak{R}(U\times U), k\rangle} = f(\tau)^{\langle U, k(v_i)\rangle_{i\in\omega}}.$$

Proof. (i) follows from (ii), so it is enough to prove (ii).

The translation function $f : Terms \to \mathcal{L}^{\neq}_{3,2}$ is not hard to give. Let $u, v \in \{x, y, z\}$ be distinct, and let w be the third variable, i.e. $\{u, v, w\} = \{x, y, z\}$. We will simultaneously define the functions $f_{uv} : Terms \to \mathcal{L}^{\neq}_{3,2}$ as follows:

$$f_{uv}(v_i) \stackrel{\text{def}}{=} R_i(uv),$$

$$f_{uv}(\tau \cup \sigma) \stackrel{\text{def}}{=} f(\tau) \vee f(\sigma), \qquad f(-\tau) \stackrel{\text{def}}{=} \neg f(\tau),$$

$$f_{uv}(\tau \circ \sigma) \stackrel{\text{def}}{=} \exists w(f_{uw}(\tau) \wedge f_{wv}(\sigma)),$$

$$f_{uv}(\tau^{-1}) \stackrel{\text{def}}{=} f_{vu}(\tau).$$

For the other direction, we want to define, by simultaneous recursion, a term $\tau(\varphi, u, v)$ for all distinct variables $u, v \in \{x, y, z\}$ and $\varphi \in \mathcal{L}^{uv}_3$ such that for all models $\mathfrak{M}$ we have

$$(*) \qquad \{\langle a, b\rangle \in M \times M : \mathfrak{M} \models \varphi(u/a, v/b)\} = \tau(\varphi, u, v)^{\langle \mathfrak{R}(M\times M), k_{\mathfrak{M}}\rangle}.$$

So let $\varphi \in \mathcal{L}^{uv}_3$.

Case 1. If φ is an atomic formula, then φ is $R_i(uv)$ or $R_i(vu)$ for some $i \in \omega$ (by $\varphi \in \mathcal{L}^{uv}_3$).

$$\tau(R_i(uv), u, v) \stackrel{\text{def}}{=} v_i, \qquad \tau(R_i(vu), u, v) \stackrel{\text{def}}{=} {v_i}^{-1}.$$

Case 2. If φ is a disjunction of two formulas, say φ is $\psi \vee \eta$, then $\psi, \eta \in \mathcal{L}^{uv}_3$, and

$$\tau(\psi \vee \eta, u, v) \stackrel{\text{def}}{=} \tau(\psi, u, v) \cup \tau(\eta, u, v).$$

Case 3. If φ is a negation of another formula, then φ is $\neg\psi$ for some $\psi \in \mathcal{L}_3^{uv}$, and we define

$$\tau(\neg\psi, u, v) \stackrel{\text{def}}{=} -\tau(\psi, u, v).$$

Case 4. If φ begins with $\exists u$, then φ is $\exists u\psi$ for some $\psi \in \mathcal{L}_3^{uv}$, and then we define

$$\tau(\exists u\psi, u, v) \stackrel{\text{def}}{=} 1 \circ \tau(\psi, u, v).$$

Likewise we define

$$\tau(\exists v\psi, u, v) \stackrel{\text{def}}{=} \tau(\psi, u, v) \circ 1.$$

Case 5. Assume that φ begins with $\exists w$, i.e. φ is $\exists w\psi$. Then $\psi \in \mathcal{L}_{3,2}^{\neq}$ can be arbitrary. It is easy to prove by induction that every element of $\mathcal{L}_{3,2}^{\neq}$ is a Boolean combination of formulas in $\mathcal{L}_3^{xy}, \mathcal{L}_3^{xz}$ and $\mathcal{L}_3^{yz}$. Bring ψ into disjunctive normal form $\psi_1 \vee \ldots \vee \psi_n$ where each ψ_i is a conjunction of formulas with two free variables. Now $\exists w\psi$ is equivalent to

$$(\exists w\psi_1) \vee \ldots \vee (\exists w\psi_n),$$

so by Case 2 we may assume that ψ is of form $\psi^{uv} \wedge \psi^{uw} \wedge \psi^{vw}$ where $\psi^{uv} \in \mathcal{L}_3^{uv}$, etc. Now $\exists w\psi$ is equivalent to

$$\psi^{uv} \wedge \exists w(\psi^{uw} \wedge \psi^{vw}).$$

We now define

$$\tau(\exists w(\psi^{uw} \wedge \psi^{vw}), u, v) \stackrel{\text{def}}{=} \tau(\psi^{uw}, u, w) \circ \tau(\psi^{vw}, w, v).$$

It is not difficult to check that the so defined $\tau(\varphi, u, v)$ satisfies our requirement $(*)$. ■

One can get very far in doing algebraic logic (for quantifier or predicate logics) via BRA's. [20] As we have seen, the natural logical counterpart of BRA's is classical first-order logic restricted to three individual variables and without equality. As shown in [TG, §5.3], this system is an adequate framework for building up

[20] If we want to investigate nonclassical quantifier logics, we can replace the Boolean reduct $\mathfrak{B}$ of $\mathfrak{A} = \langle \mathfrak{B}, \circ, ^{-1} \rangle \in \mathsf{BRA}$ with the algebras (e.g. Heyting algebras) corresponding to the propositional version of the nonclassical logic in question.

set theory and hence metamathematics in it. One can illustrate most of the main results, ideas and problems of algebraic logic by using only BRA's.

We do not know how far BRA's can be simplified without losing this feature. In this connection, a natural candidate would be the class BSR of Boolean semigroups of relations defined as

$$\mathsf{BSR} = \mathbf{SP}\left\{\langle \mathfrak{P}(U \times U), \circ\rangle : U \text{ is a set}\right\}.$$

So we require only one extra-Boolean operation "$\circ$".

The question is, how far BSR could replace BRA as the simplest, "introductory" example of Tarskian algebraic logic. We conjecture that the answer will be "very far". BSR is a discriminator variety with a recursively enumerable but not decidable equational theory, and it is not finitely axiomatizable. Thus Theorems 1.2–1.7 remain true if BRA is replaced with BSR in them. [21] We conjecture that, following the lines of [TG, §5.3], set theory can be built up in BSR instead of BRA with basically the same positive properties (e.g. finitely many axioms) as the present version [TG] has [22]. It would be nice to know if this conjecture is true, and, more generally, to see a variant of algebraic logic elaborated on the basis of BSR. We do not know what natural fragment of first-order logic with three variables corresponds to BSR (if any). It certainly is difficult to simulate substitution of individual variables using only $\circ$. The converse operation, $^{-1}$, is the algebraic counterpart of substitution because, intuitively, $R(v_0, v_1)^{-1} = R(v_1, v_0)$. One can simulate quantification by $\circ$, and it is easily seen that $\circ$ is stronger than quantification but without $^{-1}$ it is not clear exactly how much stronger [23]. Curiously enough, these issues are better understood in the case of cylindric algebras to be discussed in section 2.

If we want to algebraize *first-order logic with equality*, we have to add an extra constant Id, representing equality, to the operations. RRA denotes the class of algebras embeddable into direct products of algebras of the form

$$\langle \mathfrak{P}(U \times U), \circ, ^{-1}, \mathsf{Id}\rangle$$

where $\mathsf{Id} = \mathsf{Id} \upharpoonright U = \{\langle u, u\rangle : u \in U\}$ is a constant of the expanded algebra. I.e.

$$\mathsf{RRA} = \mathbf{SP}\left\{\langle \mathfrak{P}(U \times U), \circ, ^{-1}, \mathsf{Id}\rangle : U \text{ is a set}\right\}.$$

RRA abbreviates <u>r</u>epresentable <u>r</u>elation <u>a</u>lgebras. RRA's have been investigated more thoroughly than BRA's; actually, Theorems 1.2, 1.6 above were proved first for RRA's.

[21] The proofs of Theorems 1.2, 1.7 given here go through for BSR with the obvious modifications. Nonfinite axiomatizability of BSR will follow from the later Thm.s 1.10, 1.11.

[22] Perhaps here [N85], [N86] can be useful, because an analogous task was carried through there. The last 12 lines of Jónsson [J82, p. 276], seem to be also useful here.

[23] For applications in propositional dynamic logic, BSR seems to be more relevant than BRA, because there converse (of programs or actions) is not an essential feature of the logic.

Let $\mathcal{L}^{=}_{3,2}$ denote first-order logic with three individual variables x, y, z, with equality, and with infinitely many *binary* relation symbols. (Thus the atomic formulas are $R(uv), u = v$ for any variables $u, v \in \{x, y, z\}$ and the logical connectives are $\vee, \neg, \exists x, \exists y, \exists z$.)

THEOREM 1.10 (basic properties of RRA)

(i) RRA *is a nonfinitely axiomatizable discriminator variety with a recursively enumerable but undecidable equational theory.*

(ii) The logic captured by RRA *is* $\mathcal{L}^{=}_{3,2}$*, i.e. there are recursive functions* $t : \mathcal{L}^{=}_{3,2} \to Terms$*, and* $f : Terms \to \mathcal{L}^{=}_{3,2}$ *such that the "meanings" of* φ *and* $t(\varphi)$ *as well as those of* τ *and* $f(\tau)$ *coincide, i.e. for any model* $\mathfrak{M}$*,* $\varphi \in \mathcal{L}^{=}_{3,2}$ *with free variables* x, y*, term* τ *and evaluation* k *of variables,*

$$\varphi^{\mathfrak{M}} = t(\varphi)^{\langle \Re(M\times M), k_{\mathfrak{M}}\rangle} \quad \textit{and} \quad \tau^{\langle \Re(U\times U), k\rangle} = f(\tau)^{\langle U, k(v_i)\rangle_{i\in\omega}}.$$

Proof. Obvious modifications of the proofs of Theorems 1.2, 1.7, 1.8 prove Theorem 1.10, except for nonfinite axiomatizability of RRA. For the proof of nonfinite axiomatizability of RRA see Remark 2.9. ■

The classes of algebras $\mathsf{RRA}, \mathsf{BRA}, \mathsf{BSR}$ have less operations in this order, they form a chain of subreduct classes. Note that $\mathbf{Eq}(\mathsf{K})$ denotes the set of all equations in the language of K holding in K. Thus

$$\mathbf{Eq}(\mathsf{BSR}) \subset \mathbf{Eq}(\mathsf{BRA}) \subset \mathbf{Eq}(\mathsf{RRA}).$$

The next theorem says that these classes are finitely axiomatizable over the bigger ones.

THEOREM 1.11 *Let* E_0 *denote the following set of equations:*

$$\begin{aligned}
&(x \cup y)^{-1} = x^{-1} \cup y^{-1}, \quad (x \circ y)^{-1} = y^{-1} \circ x^{-1}, \quad x^{-1\,-1} = x\\
&x \circ -(x^{-1} \circ -y) \leq y\\
&x \leq x \circ [-(y^{-1} \circ -y) \cap ((-y)^{-1} \circ y)^{-1}].
\end{aligned}$$

Then $\mathbf{Eq}(\mathsf{BSR}) \cup E_0$ *axiomatizes* BRA*, and* $\mathbf{Eq}(\mathsf{BSR}) \cup E_0 \cup \{\mathsf{Id} \circ x = x\}$ *axiomatizes* RRA.

The *proof* can be found in Andréka-Németi [AN93]. ■

Theorem 1.11 talks about interconnections between the operations of RRA. It says, in a way, that the sole cause of nonfinite axiomatizability of RRA is the operation $\circ$, it is so strong that the other operations, ${}^{-1}$ and Id, are finitely axiomatizable with its help. This is in contrast with the case of cylindric algebras of n-ary relations, where the strength of the operations are "evenly distributed".

The next figure, taken from [AN93] describes completely the interconnections between the operations $\circ, {}^{-1}, \mathsf{Id}$ (in the presence of the Boolean operations). On Figure 1.4, all classes represented by the nodes are varieties, except the ones inside a box (those are only quasi-varieties), and the classes inside a circle are not finitely axiomatizable, except BA. [24]

More on the equational theories of $\mathsf{RRA}, \mathsf{BRA}$ and BSR:

Theorem 1.2 says that there is a set E of equations which defines BRA. Let E be an arbitrary set of equations defining BRA. What do we know about E? Theorem 1.6 says that E is not finite, and Theorem 1.7 says that E can be chosen to be recursively enumerable. By using the fact that BRA is a discriminator variety and that $\mathbf{Eq}(\mathsf{BRA})$ is recursively enumerable, and by using an argument of W. Craig, one can show that E can be chosen to be decidable [25] , i.e. there is a decidable set E defining BRA. On the other hand, we know that E has to be complex in the following sense: to any number k, E must contain an equation that uses more than k variables and all of the operation symbols $\cup, -, \circ$. There is an E such that ${}^{-1}$ occurs only in finitely many members of E, by Theorem 1.11. The analogous statements are true for $\mathsf{BSR}, \mathsf{RRA}$. [26]

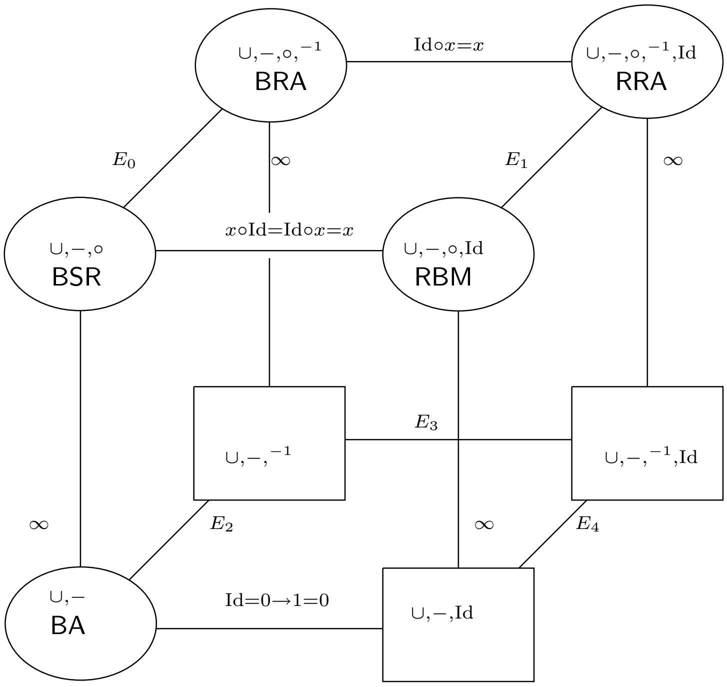
Concrete decidable sets E defining RRA are known in the literature, cf. e.g. Monk [M69]. Lyndon [Ly] outlines another recipe for obtaining a different such E. Hirsch–Hodkinson [HH] also contains such a set E. Some of these work for $\mathsf{BSR}, \mathsf{BRA}$. However, the structures of these E's are rather involved. [27] In this connection, we note that the following is still one of the most important open problems of algebraic logic:

[24] In Pratt [Pr90], the class RBM of representable Boolean monoids is obtained from our BSR's by adding Id as an extra distinguished constant. So the extra-Boolean operations of the RBM's are $\circ, \mathsf{Id}$, and thus BSR's are the Id-free subreducts of RBM's. All the results mentioned above for BSR's carry over to RBM's; e.g. RBM is a discriminator variety, hence the simple RBM's form a universally axiomatizable class, Theorems 1.2, 1.6 above apply to RBM. $\mathsf{RRA}, \mathsf{BRA}, \mathsf{BSR}, \mathsf{RBM}, \mathsf{BA}$ all occur as nodes on Figure 1.4.

[25] The idea is as follows. Let E be recursively enumerable, say $E = \{e(1), e(2), \ldots\}$ for a recursive function e. For each number n, let $\eta(n)$ denote the conjunction of n copies of $e(n)$. Since BRA is a discriminator variety, there is an equation $\varepsilon(\eta(n))$ which is equivalent to $\eta(n)$ in $\mathbf{Sir}\mathsf{BRA}$. Moreover, from $\varepsilon(\eta(n))$ we can compute back $\eta(n)$, see an earlier footnote. Then $E' \stackrel{\text{def}}{=} \{\varepsilon(\eta(1)), \varepsilon(\eta(2)), \ldots\}$ is equivalent to E and E' is decidable. The decision procedure for E' is as follows: Take any equation g. Decide whether g is $\varepsilon(f)$ for some f or no, and if yes, compute the f. If we get an f, check whether f is the conjunction of some, say n, copies of an equation h. If yes, compute $e(n)$ and check whether h is $e(n)$. If yes, g is in E', otherwise not.

[26] The need of infinitely many variables in any axiom system for RRA was proved in Jónsson [J91], the need of all the operation symbols $\cup, -, \circ$ in addition is proved in Andréka [A94]. By Theorem 1.11 then the same hold for $\mathsf{BRA}, \mathsf{BSR}$.

[27] Cf. [HMTII, pp. 112–119], for an overview.



E_0 from Thm. 1.11
$E_1 = \{x^{-1} \circ -x \leq -\mathrm{Id}, -(x^{-1}) = (-x)^{-1}\}$
$E_2 = \{x^{-1} = -x \to 1 = 0, (x \cup y)^{-1} = x^{-1} \cup y^{-1}, x^{-1-1} = x\}$
$E_3 = \{\mathrm{Id} = 0 \to 1 = 0, (x \cap \mathrm{Id})^{-1} = x \cap \mathrm{Id}\}$
$E_4 = \{(x \cup y)^{-1} = x^{-1} \cup y^{-1}, x^{-1-1} = x, (x \cap \mathrm{Id})^{-1} = x \cap \mathrm{Id}\}$

Figure 1.4.

PROBLEM 1.12 *Find simple, mathematically transparent, decidable sets E of equations axiomatizing* BSR, BRA, RRA. *(A solution for this problem has to be considerably simpler than, or at least markedly different from the E's discussed above.)*

Equational axiom systems for algebras of relations like for RRA, BRA, BSR are interesting not only because of purely aesthetical reasons, but also because such an axiom system gives an inference system for the corresponding logic. About this logical connections see e.g. Theorems 5.4, 5.5, 6.3 in Part II.

Since the classes RRA, BRA, BSR are not finitely axiomatizable, finitely axiomatizable approximations, or "computational cores" are used for them. For BRA we can take ARA as such an approximation. For RRA, the variety RA of *relation algebras*, defined by Tarski, is used in the literature as such an approximation. We get the definition of RA from the definition of ARA by replacing (4) with one stronger equation (5), and by adding the equation $\mathsf{Id} \circ x = x$.

DEFINITION 1.13 (RA, an abstract approximation of RRA) RA *is defined to be the class of all algebras of the similarity type of* RRA*'s which satisfy the equations (1)–(3) from the definition of* ARA*, together with (5), (6) below.*

$$x^{-1} \circ [-(x \circ y)] \leq -y. \tag{5}$$

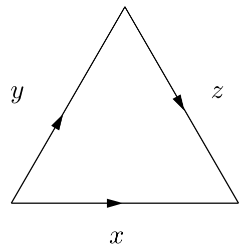
$$\mathsf{Id} \circ x = x. \tag{6}$$

■

Equation (5) is equivalent, in the presence of the other RA-axioms [28] with the following so called *triangle-rule* (5')

$$(5') \qquad x \cap (y \circ z) = 0 \quad \text{iff} \quad y \cap (x \circ z^{-1}) = 0 \quad \text{iff} \quad z \cap (y^{-1} \circ x) = 0.$$

Intuitively, (5') says that the three ways of telling that no triangle



exists, are equivalent. [29] Thus, a relation algebra is a Boolean algebra together with an involuted monoid sharing the same universe, and the interconnection between the two structures is that they form a normal BAO and the triangle rule (5') holds.

Equation (6) says that Id is the neutral element of the semigroup operation "$\circ$". We note that in algebraic logic this translates to the so called *Leibniz law* of equality in logic which says that equals cannot be distinguished [30].

[28] We note that $0 \circ x = 0, 0^{-1} = 0$ are usually omitted from the axiomatization of RA, because they follow from the rest of the axioms.

[29] In a more algebraic language, (5') says that the maps $x \mapsto a \circ x$ and $y \mapsto a^{-1} \circ y$ are *conjugates* of each other, and likewise the maps $x \mapsto x \circ a$ and $y \mapsto y \circ a^{-1}$ are conjugates. We recall from [JT51] that in a BAO the functions f, g are conjugates of each other means that $x \cap f(y) = 0$ iff $y \cap g(x) = 0$ for all x, y.

[30] For more on this see [BP89, p.10].

RA is a very strong computational core for RRA, almost all natural equations about RRA hold also in RA. [31] An RA which is not in RRA is called a *nonrepresentable* RA. Equations holding in RRA and not in RA can be obtained from each finite nonrepresentable RA by using that RA is a discriminator variety of BAO's, as follows. Let $\mathfrak{A} \in \mathsf{RA} - \mathsf{RRA}$ be finite. Then $\mathfrak{A}$ cannot be embedded in any RRA, and this can be expressed with a universal formula because $\mathfrak{A}$ is finite. Now using the switching function, this universal formula can be coded as an equation e. Then e does not hold in $\mathfrak{A} \in \mathsf{RA}$, while it holds in RRA. Many finite nonrepresentable RA's are known in the literature. The smallest such has 16 elements. [32] A source for examples of finite nonrepresentable RA's is the so called Lyndon algebras. A finite *Lyndon algebra* is a finite RA such that Id is an atom, $a^{-1} = a, a \circ a = a \cup \mathsf{Id}$, and $a \circ b = 1 - (a \cup b)$ hold for all other distinct atoms a, b. Infinitely many of the finite Lyndon algebras are nonrepresentable (and infinitely many are representable). Another way of getting finite nonrepresentable RA's is to "distorte" a representable one. There are some known methods, like *splitting* and *dilating* [33] with which we can obtain nonrepresentable RA's from representable ones. Nonrepresentable RA's are almost as important as representable ones. [34]

Some special, interesting classes of RRA's turn out to be finitely axiomatizable, below we list two such classes. These finite axiomatizations give (non-standard) finitary inference systems for $\mathcal{L}^{=}_{3,2}$, cf. Mikulás [Mi96], [Mi95]. The elegant, purely algebraic proofs for the items in the next theorem are examples for significant applications of algebra to logic, via connections between algebra and logic indicated in Part II of this paper.

THEOREM 1.14 *Let φ and ψ denote the following formulas, respectively.*

$$\exists xy(x^{-1} \circ x \leq \mathsf{Id} \ \wedge \ y^{-1} \circ y \leq \mathsf{Id} \ \wedge \ x^{-1} \circ y = 1)$$
$$\forall x \exists y(x \neq 0 \ \rightarrow \ [0 \neq y \ \wedge \ y \leq x \ \wedge \ y^{-1} \circ y \leq \mathsf{Id}]).$$

Then $\mathsf{RRA} \cap \mathsf{Mod}\Sigma = \mathsf{RA} \cap \mathsf{Mod}\Sigma$ *for* $\Sigma = \{\varphi\}$ *and* $\Sigma = \{\psi\}$. ■

For the *proofs* see Maddux [Ma78], Tarski-Givant [TG]. An RA in which φ is true is called a *quasi-projective* RA, or a QRA, and an RA in which ψ is true is called a *functionally dense* RA. [35] We can look at Theorem 1.14 in two ways: on one hand

---

[31] In other words, only complicated equations can distinguish RRA and RA.

[32] This was found by McKenzie [McK].

[33] For splitting in RA see Andréka-Maddux-Németi [AMN], for dilation in RA see Németi [N86], Németi-Simon [NSi], Simon [Si97].

[34] E.g. one proof of nonfinite axiomatizability of RRA goes by finding infinitely many nonrepresentable RA's whose ultraproduct is representable. Investigating the structure of possible axiom systems for RRA often boils down to finding suitable nonrepresentable RA's.

[35] That any QRA is representable is a theorem of Tarski, an elegant algebraic proof was given by Maddux [Ma78]. A different, illuminating proof is given in Simon [Si96]. For logical applications of this area see [TG]. The proof that every functionally dense RA is representable is in Maddux [Ma78]. See also [AGMNS].

it says that the class of quasi-projective RRA's is finitely axiomatizable (while RRA is not), and on the other hand it says that quasi-projective RA's are representable (while RA's in general are not). (And the same for functionally dense RA's.)

CHAPTER 2

ALGEBRAS OF RELATIONS IN GENERAL

By this point we might have developed some vague picture of how algebras of binary relations are introduced, investigated etc. One might even sense that they give rise to a smooth, elegant, exciting and powerful theory. However, our original intention was to develop algebras of relations in general, which should surely incorporate not only binary but also ternary, and in general n–ary relations.

Let us see how to generalize our RRA's and BRA's to relations of higher ranks. Let us first fix n to be a *finite* ordinal. As we said, we would like the new algebras to be expansions of RRA's (and BRA's). However, defining composition of n–ary relations for $n > 2$ is complicated [1] . Therefore the following sounds like a more attractive idea: We single out the simplest basic operations on n–ary relations, and hope that composition will be derivable as a term–function from these. Let us see how we could generalize our generic or full RRA's $\langle \mathfrak{P}(U \times U), \circ, ^{-1}, \mathsf{Id}\rangle$ to relations of rank n. The obvious part is that these algebras will begin with $\langle \mathfrak{P}(U \times U \times \ldots \times U), \mathsf{Id}, \ldots\rangle$, where

$$\mathsf{Id} = \{\langle u, u, \ldots, u\rangle : u \in U\}$$

is the n–ary identity relation. Again, Id is a constant, just as it was in the RRA case. Let ${}^nU$ denote $U \times U \times \ldots \times U$, e.g. ${}^3U = U \times U \times U$. The new operations (besides the Boolean ones and Id) we will need are the algebraic counterparts of quantification $\exists v_i$, for $i < n$. So, we want an operation that sends the relation defined by $R(v_0, v_1)$ to the one defined by $\exists v_0 R(v_0, v_1)$, and similarly for $\exists v_1$. For $R \subseteq U \times U$ let $Dom(R)$ and $Rng(R)$ denote the usual domain and range of R. For $n = 2$ we define

$$\mathsf{c}_0(R) = U \times Rng(R) \quad \text{and} \quad \mathsf{c}_1(R) = Dom(R) \times U.$$

Now

$$\langle \mathfrak{P}(U \times U), \mathsf{c}_0, \mathsf{c}_1, \mathsf{Id}\rangle$$

is the *full cylindric set algebra* of binary relations over U, for short the full Cs_2.

Before turning seriously to n–ary relations, we need the following:

[1] Composition for n-ary relations is studied in Marx–Németi–Sain [MNS], Marx [Ma95]. The definition is $\circ(R_1, \ldots, R_n) = \{\langle a_1, \ldots, a_n\rangle : \exists x(\langle a_1, \ldots, a_{n-1}, x\rangle \in R_1 \mathbin{\&} \langle a_1, \ldots, a_{n-2}, x, a_n\rangle \in R_2 \mathbin{\&} \ldots \mathbin{\&} \langle x, a_2, \ldots, a_n\rangle \in R_n)\}$.

CONVENTION 2.1 *Throughout we will pretend that Cartesian products and Cartesian powers are associative such that* $^nU \times {}^mU = {}^{n+m}U$, *and if e.g.* $R \subseteq {}^3U$ *then* $^2U \times R \subseteq {}^5U \supseteq R \times {}^2U$.

The full Cs_n, i.e. the *full cylindric set algebra* of n–ary relations, is the natural generalization of Cs_2 as follows. Let $R \subseteq {}^nU$. If $Rng(R) = \{\langle b_1 \dots b_{n-1}\rangle : \langle b_0 b_1 \dots b_{n-1}\rangle \in R$ for some $b_0\}$, then $\mathsf{c}_0(R) = U \times Rng(R)$ considered as a set of n-tuples. Similarly, let $Dom(R) = \{\langle b_0 \dots b_{n-2}\rangle : \langle b_0 \dots b_{n-2} b_{n-1}\rangle \in R$ for some $b_{n-1}\}$, and let $\mathsf{c}_{n-1}(R) = Dom(R) \times U$. Generalizing this to c_i with $i < n$ arbitrary, we obtain

$$\mathsf{c}_i(R) = \\ \{\langle b_0, \dots, b_{i-1}, a, b_{i+1}, \dots, b_{n-1}\rangle : \langle b_0, \dots, b_{n-1}\rangle \in R \text{ and } a \in U\}\ .$$

c_i is one of the most natural operations on relations. It simply forgets the i-th argument of the relation, or in other words, deletes the i-th column. However, since deleting the i-th column would leave us with an $(n-1)$–ary relation, $Dom(R)$ if $i = n-1$, we replace the i-th column with a dummy column i.e. in the $i = n-1$ case we represent $Dom(R)$ with the "pseudo n–ary relation" $Dom(R) \times U$. The "real rank" of an $R \subseteq {}^nU$ is always easy to recover, namely it is $\Delta(R) = \{i < n : \mathsf{c}_i(R) \neq R\}$. So c_i is the natural operation of removing i from the (real) rank of a relation.

For example, $\mathsf{c}_{\text{father}}$ when applied to the "father, mother, child" relation gives back the "mother, child" relation coded as "anybody, mother, child" (in which the anybody argument carries no information i.e. is dummy). By a *full* Cs_n we understand an algebra

$$\mathfrak{Rel}_n(U) \stackrel{\text{def}}{=} \langle \mathfrak{P}({}^nU), \mathsf{c}_0, \dots, \mathsf{c}_{n-1}, \mathsf{Id}\rangle$$

for some set U. By a Cs_n we understand a subalgebra of a full Cs_n with nonempty [2] base set U i.e.

$$\mathsf{Cs}_n \stackrel{\text{def}}{=} \mathbf{S}\{\mathfrak{Rel}_n(U) : U \text{ is a nonempty set}\}.$$

By a *representable cylindric algebra* of n–ary relations, (an RCA_n) we understand a subalgebra of a direct product of full Cs_n's (up to isomorphism), formally:

$$\mathsf{RCA}_n = \mathbf{SP}\{\langle \mathfrak{P}({}^nU), \mathsf{c}_0, \dots, \mathsf{c}_{n-1}, \mathsf{Id}\rangle : U \text{ is a set}\}.$$

Note that $\mathsf{RCA}_n = \mathbf{SP}\{\mathfrak{Rel}_n(U) : U \text{ is a set}\} = \mathbf{SP}(\text{full } \mathsf{Cs}_n) = \mathbf{SP}\mathsf{Cs}_n$. By the same argument as in the case of BRA's, every RCA_n is directly representable as an algebra of n–ary relations (with the greatest relation a disjoint union of Cartesian spaces). RCA_n is one of the "leading candidates" for being the natural algebra of n–ary relations.

[2] Excluding the empty base set here is not essential, it serves easier applicability in the second part of this paper, in section 7.

The abstract algebraic picture is simple: an RCA_n is a BA together with n closure operations and an extra constant. Accordingly, an (abstract) *cylindric algebra of dimension* n, a CA_n, is defined as a normal BAO with n self-conjugated and commuting closure operations, and with a constant satisfying two equations. In more detail:

DEFINITION 2.2 (CA_n, an abstract approximation of RCA_n) CA_n *is defined to be the class of all algebras of the similarity type of* $\mathfrak{Rel}_n(U)$ *which satisfy the following equations for all* $i, j < n$.

(1) The axioms for normal BAO*, i.e.*

the Boolean axioms,

$$\mathsf{c}_i 0 = 0, \qquad \mathsf{c}_i(x \cup y) = \mathsf{c}_i(x) \cup \mathsf{c}_i(y).$$

(2) Axioms expressing that c_i*'s are self-conjugated commuting closure operations, i.e.*

(i) $x \le \mathsf{c}_i x = \mathsf{c}_i \mathsf{c}_i x$,

(ii) $y \cap \mathsf{c}_i x = 0$ *iff* $\mathsf{c}_i y \cap x = 0$,

(iii) $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$.

Because of the above axioms, the notation $\mathsf{c}_{(\Gamma)} x = \mathsf{c}_{i_1} \ldots \mathsf{c}_{i_k} x$ *where* $\Gamma = \{i_1, \ldots, i_k\}$ *makes sense. We will use that notation from now on. We will also use the convention* [3] *that* $n = \{0, 1, \ldots, n-1\}$.

(3) The constant Id *has domain 1 and satisfies the "Leibniz-law", i.e.*

(i) $\mathsf{c}_{(n \smallsetminus \{i\})} \mathsf{Id} = 1$,

(ii) $\mathsf{c}_{(n \smallsetminus \{i,j\})} \mathsf{Id} \cap \mathsf{c}_i x = x$ *whenever* $x \le \mathsf{c}_{(n \smallsetminus \{i,j\})} \mathsf{Id}$ *and* $i \ne j$.

■

An equivalent form of saying that c_i is self-conjugated[4] is to say that the complement of a closed element is closed. Thus, in the above definition, (2)(ii) can be replaced with

$$\mathsf{c}_i - \mathsf{c}_i x = -\mathsf{c}_i x.$$

[3] For more on this see e.g. [HMT, Part I. pp. 31–32].

[4] I.e. that it is the conjugate of itself, cf. the footnote to the triangle-rule (5)' in section 1.

Note that (2)(ii) is an analogon of the triangle rule (5)' in the definition of RA. (3)(ii) expresses that the closure operator c_i is "discrete", or is the identity, when relativized to

$$\mathsf{Id}_{ij} \stackrel{\text{def}}{=} \mathsf{c}_{(n \smallsetminus \{i,j\})}\mathsf{Id},$$

$i \neq j$. Both (2)(ii) and (3)(ii) have equivalent equational forms, e.g. an equivalent form of (3)(ii) is

$$\mathsf{Id}_{ij} \cap \mathsf{c}_i(\mathsf{Id}_{ij} \cap x) = \mathsf{Id}_{ij} \cap x \quad \text{when } i \neq j$$

and an equivalent (together with the other axioms) form of (2)(ii) is

$$\mathsf{c}_i(x \cap \mathsf{c}_i y) = \mathsf{c}_i x \cap \mathsf{c}_i y.$$

Connection with geometry: The names in cylindric algebra theory come from connection with geometry. Namely, an n-ary relation is a set of n-tuples, while an n-tuple is a point of the *n-dimensional* space. E.g. $\langle a, b\rangle$ is a point in the 2-dimensional space with coordinates a and b, while $\langle a, b, c\rangle$ is a point in the 3-dimensional space with coordinates a, b and c. Thus a binary relation is a subset of the 2-dimensional space, while an n-ary relation is a subset of the n-dimensional space. Hence the name "cylindric algebra of *dimension n*".

If R is a subset of the n-dimensional space, then $\mathsf{c}_i(R)$ is the *cylinder* above R parallel to the i-th axis, and Id is the main *diagonal*. Hence the name "*cylindric* algebra". The operations c_i and Id are called "*cylindrifications*" and "*diagonals*" in CA-theory, and Id_{ij} is usually denoted by d_{ij} (for d̲iagonal). Because of these geometrical meanings, also the operations of RCA_n are easy to draw. This is illustrated on Figure 2.1, see also Figure 2.3.

How can we draw the operations of RRA? Converse is easy to draw: the converse of R is the mirror image of R w.r.t. the diagonal. However, relation composition of two relations R, S is not so easy to draw. See Figure 2.2.

Thus cylindric algebras (CA's) are simpler than relation algebras RA's in two ways: CA's have only unary operations c_i, while the central operation of RA is the binary composition operation $\circ$; and secondly, cylindrifications are easy to draw, while composition is not so easy to draw. There are furher connections with geometry, e.g. via projective planes. We do not discuss these herein, but cf. Monk [M74], [Gi97], [NS97], [NS97], [AGiN97, Chapter II].

Connection between CA's and RA's: A natural question comes up: can these "simple" CA's recapture the power of RA's? This was a requirement we expected to meet, namely we expected that the theory of n-ary relations should be an extension of that of binary relations. The answer is that RCA_n with $n \geq 3$ is strong enough to recapture RRA, while RCA_2 is not strong enough. In more detail:

Mirroring cannot be expressed by the diagonal and the cylindrifications in the plane (i.e. in 2-dimensional space), but it can be expressed if we can move out to 3-dimensional space, see Figure 2.3.

Namely, by letting

$$\mathsf{s}^i_j(x) \stackrel{\text{def}}{=} \mathsf{c}_i(\mathsf{Id}_{ij} \cap x) \qquad \text{and} \qquad P = U \times U \times \{u\}$$

for some fixed $u \in U$, we have

$$R^{-1} = P \cap \mathsf{s}^2_1\mathsf{s}^1_0\mathsf{s}^0_2\mathsf{c}_2 R$$

for $R \subseteq P$. Here, we identified the binary $R \subseteq U \times U$ with the ternary $R \times \{u\}$, and similarly for R^{-1}. Composition also can be expressed:

$$R \circ S = P \cap \mathsf{c}_2(\mathsf{s}^1_2\mathsf{c}_2 R \cap \mathsf{s}^0_2\mathsf{c}_2 S).$$

A more natural approach is based on identifying a binary relation $R \subseteq U \times U$ with the ternary relation

$$\mathbf{Dr}(R) \stackrel{\text{def}}{=} R \times U;$$

we call $\mathbf{Dr}(R)$ the <u>d</u>ummy <u>r</u>epresentation of R as a ternary relation. Then $\mathbf{Dr}$: $\mathcal{P}(U \times U) \to \mathcal{P}(U \times U \times U)$; and

$$\mathbf{Dr}(R^{-1}) = \mathsf{s}^2_1\mathsf{s}^1_0\mathsf{s}^0_2\mathbf{Dr}(R),$$

$$\mathbf{Dr}(R \circ S) = \mathsf{c}_2(\mathsf{s}^1_2\mathbf{Dr}(R) \cap \mathsf{s}^0_2\mathbf{Dr}(S)).$$

So in a sense, RRA's form a kind of a reduct of RCA_n's for $n \geq 3$. In more detail: Let $\mathfrak{A} \in \mathsf{CA}_n$, $n \geq 3$. The *relation-algebra reduct* $\mathfrak{Ra}\mathfrak{A}$ of $\mathfrak{A}$ is defined as

$$\mathfrak{Ra}\mathfrak{A} \stackrel{\text{def}}{=} \langle Ra\mathfrak{A}, \cup^{\mathfrak{A}}, -^{\mathfrak{A}}, \circ, ^{-1}, \mathsf{Id}^{\mathfrak{A}}_{01}\rangle,$$

where

$$Ra\mathfrak{A} = \{a \in A : \mathsf{c}^{\mathfrak{A}}_j a = a \text{ for all } 2 \leq j < n\}$$

and if $a, b \in Ra\mathfrak{A}$, then

$$\begin{aligned} a^{-1} &\stackrel{\text{def}}{=} \mathsf{s}^2_1\mathsf{s}^1_0\mathsf{s}^0_2 a, \\ a \circ b &\stackrel{\text{def}}{=} \mathsf{c}_2(\mathsf{s}^1_2 a \cap \mathsf{s}^0_2 b). \end{aligned}$$

Now, $\mathbf{Dr} : \langle \mathfrak{P}(U \times U), \circ, ^{-1}, \mathsf{Id}\rangle \rightarrowtail\!\!\!\!\!\rightarrow \mathfrak{Ra}\mathfrak{Rel}_n(U)$ is an isomorphism for $n \geq 3$. We define

$$\mathbf{Ra}\mathsf{CA}_n \stackrel{\text{def}}{=} \{\mathfrak{Ra}\mathfrak{A} : \mathfrak{A} \in \mathsf{CA}_n\} \quad \text{ for } 3 \leq n.$$

For $\mathsf{K} \subseteq \mathsf{CA}_n$, $\mathbf{Ra}\mathsf{K}$ is defined similarly.

Then $\mathsf{RRA} = \mathbf{SRa}\mathsf{RCA}_n$ for all $n > 2$. The classes $\mathbf{SRa}\mathsf{CA}_n$ $(n > 3)$ form a chain between RA and RRA, providing a "dimension-theory" for [5] RA. In more detail, $\mathbf{SRa}\mathsf{CA}_3 \supsetneq \mathsf{RA}$ (Monk [M61]), $\mathbf{SRa}\mathsf{CA}_4 = \mathsf{RA}$ (Maddux, [Ma78a]), $\mathsf{RA} \supset \mathbf{SRa}\mathsf{CA}_5 \supseteq \ldots \supseteq \mathsf{RRA}$ (Monk [M61]), and [6]

$$\mathsf{RRA} = \cap\{\mathbf{SRa}\mathsf{CA}_n : 3 \leq n < \omega\} = \mathbf{SRa}\mathsf{CA}_\omega.$$

Investigating the connection between RRA (RA) and RCA (CA) is an interesting subject. Some of the references are Monk [M61], Maddux [Ma78a], [Ma89], [HMTII, §5.3], Németi–Simon [NSi], [Si96], [Si97], Hirsch–Hodkinson [HH97]. Recent developments in the $\mathsf{RA}-\mathsf{CA}$ (– polyadic algebras) connection are reported in [Si97], [NSi].

Thus the answer is that RCA_n's, $n > 2$, do recapture the power of RRA's. (On the other hand, RCA_2's do not[7].)

Connection with logic: Cylindric algebras have a very close and rich connection with logic. This connection is partly described in [HMTII, §4.3], and in Examples 6, 8, 9 in section 7 herein.

Summing up this connection very briefly: RCA-theory corresponds to model theory of first-order like languages (or quantifier logics), while abstract CA-theory corresponds to their proof theory. Individual CA's correspond to theories in such logics, homomorphisms between CA's correspond to interpretations between theories, while isomorphism of CA's corresponds to definitional equivalence of models and/or theories. CA-theoretic terms and equations correspond to first-order formula schemas, an equation e is valid in RCA if its corresponding formula schema is valid, an equational derivation of e corresponds to a proof of the formula (schema) corresponding to e. More on this is written in section 7, Examples 6,8,9. Here, first-order like languages encompass finite-variable fragments of first-order language (FOL for short), usual FOL, FOL with infinitary relation symbols but with finitary logical connectives, FOL considered as a propositional multi-modal logic, FOL with several modified semantics etc.

Most of the above is discussed in [HMTII, §4.3], especially when taken together with [vB96]. Some other references illustrating the rich connection of CA's with logic are the following. In Monk [M93] the connection with FOL is treated. In [N87] and in [R] valid formula-schemas, in [N90] model theory of FOL with infinitary relation symbols, in [N96] FOL with generalized semantics, in [A], [SA] algebras of sentences, in [Se85] and Biro-Shelah [BiSh] model theoretic notions like saturated, universal, atomic models are investigated with the help of CA's, respectively. [AvBN96], [vB96], [MV] connect CA's with modal logic, [S95], [SGy]

[5] As later, in Thm. 2.14(ii), we will see, this is analogous to the chain $\mathbf{SNr}_n\mathsf{CA}_{n+k}$ $(k \geq 0)$ between CA_n and RCA_n.

[6] For the definition of CA_ω see Def. 2.12.

[7] For example, $\mathbf{Eq}\mathsf{RRA}$ is undecidable, while $\mathbf{Eq}\mathsf{RCA}_2$ is decidable, see Thm. 1.10(i), Thm. 2.3(iii).

use CA's for searching for a FOL with nicer behaviour. Further references on this line are e.g. [vB97], [V95a].

The connection of CA's with logic also sheds light on the above ways of expressing composition and converse of binary relations in CA. Namely, RCA_n is the algebraic counterpart of first-order logic with n variables (see Part II, example 6 in section 7), and in particular n-variable first-order formulas and terms in the language of RCA_n are in strong correspondence with each other (see Corollary 5.5 in Part II). The RCA-terms in the definition of an RA-reduct are just the transcripts of the 3-variable formulas defining composition and conversion of binary relations. (On the intuitive meaning of the terms s^i_j see the remark after Example 7 in section 7.)

At this point we can state the counterparts of Theorems 1.2–1.10. [8]

THEOREM 2.3 (basic properties of RCA_n) *Let n be finite.*

(i) RCA_n *is a discriminator variety, with a recursively enumerable equational theory.*

(ii) RCA_n *is not axiomatizable with a finite set of equations and its equational theory is undecidable if $n > 2$.*

(iii) RCA_2 *is axiomatizable with a finite set of equations, and its equational theory is decidable. The same is true for* RCA_1*. Any* CA_2 *satisfying for all $i, j < 2$, $i \neq j$*

$$\mathsf{c}_i x \cdot \mathsf{s}^j_i \mathsf{c}_i x \leq \mathsf{Id} \to x = \mathsf{c}_0 x \cdot \mathsf{c}_1 x, \qquad \textit{or}$$

$$\mathsf{c}_0 x \cdot \mathsf{c}_1 x - x \leq \mathsf{c}_i(\mathsf{s}^j_i \mathsf{c}_i x - \mathsf{Id})$$

is representable [9].

(iv) The logic captured by RCA_n *is first-order logic with equality restricted to n individual variables.*

Proof. The proof of (i) goes exactly as in the previous section, cf. the proofs of Thm. s 1.2, 1.7.: The subdirectly irreducible members of RCA_n are exactly the isomorphic copies of the nontrivial Cs_n's (i.e. those with nonempty base set U), and a switching term is $\mathsf{c}_{(n)}x$, i.e. in Cs_n we have[10]

$$\mathsf{c}_{(n)}x = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

[8] L. Henkin and A. Tarski proved that RCA_n is a variety, J. D. Monk [M69] proved that RCA_n is not finitely axiomatizable, L. Henkin gave a finite equational axiom system for RCA_2, and D. Scott proved that $\mathbf{Eq}(\mathsf{RCA}_2)$ is decidable.

[9] The above quasi-equation and equation then are equivalent to the so-called Henkin-equation (see [HMTII, 3.2.65]), which was further simplified in Venema [V91, §3.5.2]. On the intuitive meaning of these see the paragraph preceding Thm.7.8 in this work.

[10] Recall that $n = \{0, 1, 2, \dots, n-1\}$. Thus $n \smallsetminus 2 = \{2, 3, \dots, n-1\}$.

The proof of undecidability of BRA can be adapted here, too, by using the above outlined connection between RA and CA. Namely, the quasi-equational theory of semigroups can be interpreted in RCA_n, e.g. by using the term

$$x;y = \mathsf{c}_2(\mathsf{s}_2^1\mathsf{c}_{(n\smallsetminus 2)}x \cap \mathsf{s}_2^0\mathsf{c}_{(n\smallsetminus 2)}y).$$

The proof for nonfinite axiomatizability of $\mathsf{RCA}_n,\ n > 2$ will be discussed in Remark 2.9. The proof of the first part of (iii) can be found in [HMTII, 3.2.65, 4.2.9]. It is not hard to check that in all CA_2's, the quasi-equation and the equation in (iii) are equivalent to each other. Also, the equation in (iii) is equivalent to

$$(*) \qquad \mathsf{c}_0x \cdot \mathsf{c}_1x \leq x + \mathsf{c}_i(\mathsf{s}_i^j\mathsf{c}_ix - \mathsf{Id})$$

which is then preserved under taking perfect extensions (because negation – occurs only in front of a constant). Thus it is enough to show that any simple atomic $\mathfrak{A} \in \mathsf{CA}_2$ satisfying $(*)$ is representable. Now, $(*)$ implies that there are no defective atoms in $\mathfrak{A}$, in the sense of [HMTII, 3.2.59], and then $\mathfrak{A} \in \mathsf{RCA}_2$ by [HMTII, 3.2.59]. For (iv) see example 6 in section 7 of Part II. ■

More on the fine-structure of the equational theory of RCA_n will be said later, after Problem 2.11.

How far did we get in obtaining algebras of relations in general (binary, ternary, ..., n–ary, ...)? RCA_n is a smooth and satisfactory algebraic theory of n–ary relations. So, can our theory handle all finitary relations? The answer is both yes and no. Namely, since n is an arbitrary finite number, in a sense, we can handle all finitary relations. But, we cannot have them *all* in the same algebra or in the same variety. For any finite family of relations, we can pick n such that they are all in RCA_n. But this does not extend to infinite families of relations. To alleviate this, we could try working in the system $\langle \mathsf{RCA}_n : n \in \omega\rangle$ of varieties instead of using just one of these. To use them all together, we need a strong coordination between them. This coordination is easily derivable from the embedding function **Dr** sending R to $R \times U$ for $R \subseteq {}^nU$ defined above. Let $\mathfrak{A} \subseteq \mathfrak{Rel}_n(U) = \langle \mathfrak{P}({}^nU)\cdots\rangle$ be a Cs_n and let $\mathfrak{B}$ be the Cs_{n+1} generated by the **Dr** image of $\mathfrak{A}$, i.e. $\mathfrak{B} \subseteq \mathfrak{Rel}_{n+1}(U) = \langle \mathfrak{P}({}^{n+1}U)\cdots\rangle$ is generated by $\{\mathbf{Dr}(R) : R \in A\}$. The biggest $\mathfrak{A}$ yielding the same $\mathfrak{B}$ is called the n–ary *neat-reduct* of $\mathfrak{B}$, formally $\mathfrak{A} = \mathbf{Nr}_n(\mathfrak{B})$. Then

$$Nr_n(\mathfrak{B}) = \{b \in B : \mathsf{c}_n(b) = b\}.$$

Intuitively, $\mathbf{Nr}_n(\mathfrak{B})$ is the algebra of n–ary relations "living in" the algebra $\mathfrak{B}$ of $n+1$–ary relations. It is not hard to see that $\mathbf{Nr}_n : \mathsf{RCA}_{n+1} \longrightarrow \mathsf{RCA}_n$ is a functor, in the category theoretical sense, for every n. Now, we can use the collection of varieties RCA_n for all $n \in \omega$, synchronized via the functors $\langle \mathbf{Nr}_n : n \in \omega\rangle$, as a single mathematical entity containing all finitary relations.

Another possibility is to insist that we want all finitary relations over U represented as elements of a *single* algebra. In other words, this goal means that instead of a system of varieties we want to consider a single variety that in some sense incorporates all the original varieties taken together. Indeed, each RCA_n can be viewed as incorporating all the RCA_k's for $k \le n$, since the latter can be recovered from RCA_n by using the functors $\mathbf{Nr}_{n-1}$, $\mathbf{Nr}_{n-2}$ etc. So as n increases, RCA_n gets closer and closer to the variety we want. Indeed, we take the limit of this sequence. There are two ways of doing this, the naïve way we will follow here and the category theoretical way we only briefly mention. It is shown in the textbook Adamek–Herrlich–Strecker [AHS] that the system or "diagram"

$$\mathsf{RCA}_1 \xleftarrow{\mathbf{Nr}_1} \mathsf{RCA}_2 \xleftarrow{\mathbf{Nr}_2} \cdots \mathsf{RCA}_n \xleftarrow{\mathbf{Nr}_n} \mathsf{RCA}_{n+1} \cdots$$

is "convergent" in the category theoretic sense, i.e. that it has a limit L. Indeed, it is this class L of algebras that we will construct below in a naïve way that does not use category theoretic tools or concepts.

We first extend our Convention 2.1, stated at the beginning of the present section concerning associativity of Cartesian products and powers. *In the sequel*, ω is the smallest infinite ordinal, as well as the set of all finite numbers, and ${}^{\omega}U$ is the set of ω-sequences over U. Furthermore, ${}^{n}U \times {}^{\omega}U = {}^{\omega}U$, and if $R \subseteq {}^{n}U$ then $R \times {}^{\omega}U \subseteq {}^{\omega}U$, for $n < \omega$. We will also have to distinguish the constant Id of RCA_3 from that of RCA_4. Therefore we let

$$\mathsf{Id}^n \stackrel{\text{def}}{=} \{\langle a, \ldots, a\rangle : a \in U\}$$

denote the n–ary identity relation on U.

How do we obtain an algebra containing all finitary relations over U? If R is binary, but we want to treat it together with a 5–ary relation, then we represent R by $R \times U \times U \times U = R \times {}^{3}U$ in a Cs_5. Taking this procedure to the limit, if we want to treat R together with relations of arbitrary high ranks, then we can represent R with $R \times {}^{\omega}U$. This way we can embed all finitary relations into relations of rank ω, and relations of different ranks become "comparable" and "compatible".[11] We still haven't obtained the definition of Cs_ω's from that of Cs_n's because we do not know what to do with the constant Id. More specifically, we want to be able to use the neat reduct functor $\mathbf{Nr}_n$, as the inverse of $R \mapsto R \times {}^{\omega}U$ for $R \subseteq {}^{n}U$, in order to recover the original Cs_n's from the new Cs_ω. This means that for $\mathsf{Id}^n \subseteq {}^{n}U$ we want $\mathsf{Id}^n \times {}^{\omega}U$ to be a derived constant (distinguished element) in our algebra.

[11] In particular we avoid the problem we ran into at the end of the introduction to this part in connection with the Boolean algebra $\mathfrak{P}({}^{<\omega}U)$, because e.g. the complement of $R \times {}^{\omega}U$ is $\overline{R} \times {}^{\omega}U$ where $\overline{R}$ denotes ${}^{n}U \smallsetminus R$ if R is an n-ary relation. Instead of trying to tame the Boolean-like algebra $\{R : R \subseteq {}^{n}U \text{ for some } n\} \subseteq \mathcal{P}({}^{<\omega}U)$, we simply represent $R \subseteq {}^{n}U$ by $R \times {}^{\omega}U$ which *is* an element of $\langle \mathfrak{P}({}^{\omega}U), \mathsf{c}_0, \ldots, \mathsf{c}_n, \ldots\rangle_{n<\omega}$.

Adding $\mathsf{Id}^\omega = \{\langle a, \ldots, a, \ldots\rangle : a \in U\}$ as an extra constant does *not* ensure this any more. One of the most natural solutions is letting

$$\mathsf{Id}^\omega_{ij} = \mathsf{Id}_{ij} = \{q \in {}^\omega U : q_i = q_j\}$$

and defining a full Cs_ω as

$$\mathfrak{Rel}_\omega(U) \stackrel{\text{def}}{=} \langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathsf{Id}_{ij}\rangle_{i,j<\omega} \ ,$$

where the Id_{ij}'s are constants. The price we had to pay for replacing the finite bound n on the ranks of relations we can treat with the infinite bound ω is that we had to break up our single constant Id to infinitely many constants Id_{ij} $(i, j \in \omega)$.

RCA_ω is defined to consist of all subalgebras of direct products of full Cs_ω's (up to isomorphisms),

$$\mathsf{RCA}_\omega = \mathbf{SP}\{\langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathsf{Id}_{ij}\rangle_{i,j<\omega} : U \text{ is a set}\}.$$

Again, as it was the case with BRA's and RCA_n's, RCA_ω's are directly representable as algebras whose elements are ω–ary relations.

The elements of $\mathcal{P}({}^\omega U)$ are all the ω-ary relations over U, and not only the "representations" $R \times {}^\omega U$ of finitary relations R over U. Can we actually recover the algebras of finitary relations from the huge full Cs_ω's? Let

$$\mathsf{Rf}(U) = \{R \times {}^\omega U : R \subseteq {}^n U \text{ for some } n \in \omega\}.$$

Then $\mathsf{Rf}(U) \subseteq \mathcal{P}({}^\omega U)$; moreover it is a subalgebra of the full Cs_ω $\ \mathfrak{Rel}_\omega(U)$ with universe $\mathcal{P}({}^\omega U)$. We will denote this subalgebra by $\mathfrak{Rf}(U)$. Now, we set

$$\mathsf{Csf}_\omega \stackrel{\text{def}}{=} \mathbf{S}\{\mathfrak{Rf}(U) : U \text{ is a nonempty set}\}.$$

In a sense, Csf_ω is the narrowest reasonable class of algebras of finitary relations.[12]

If $R \in \mathsf{Rf}(U)$, then $\Delta(R) = \{i \in \omega : \mathsf{c}_i R \neq R\}$ is finite, in short R is finite-dimensional. We note that the converse is not true, there are $R \subseteq {}^\omega U$ with $\Delta(R) = \emptyset$, yet $R \notin \mathsf{Rf}(U)$. Indeed, fix $u \in U$ and set

$$R = \{s \in {}^\omega U : \{i < \omega : s_i \neq u\} \text{ is finite}\}.$$

[12]The letters Rf (and $\mathfrak{Rf}$) refer to "finitary relations". Csf_ω is the above mentioned category theoretical limit L. More precisely, for this equality to be literally true, when forming the category theoretic limit L, instead of the varieties RCA_n we have to start out from their subdirectly irreducible members, which are nothing but Cs_n's. So Csf_ω is the limit of the sequence $\mathsf{Cs}_1, \ldots, \mathsf{Cs}_n, \ldots$. The class Csf_ω and its relationship with RCA_ω was systematically investigated in Andréka [A73], [AGN73], [AGN77], [HMTAN], [HMTII]. In the first three works the class was denoted by Lv or Lr, while in the last two by $\mathsf{Cs}^{reg}_\omega \cap \mathsf{Lf}_\omega$, the latter being the standard notation today.

Then $R \notin \mathsf{Rf}(U)$ if $|U| \geq 2$, while $\mathsf{c}_i R = R$ for all $i < \omega$. A relation $R \subseteq {}^\omega U$ is called *regular* if

$$(s \in R \quad \text{iff} \quad z \in R) \qquad \text{whenever } s, z \in {}^\omega U, s \restriction \Delta(R) = z \restriction \Delta(R).$$

Then the elements of $\mathsf{Rf}(U)$ are exactly the finite-dimensional regular relations on U.

Now we turn to the connections between the classes Csf_ω, Cs_ω and RCA_ω. Intuitively, the elements of Csf_ω are algebras of finitary relations, while the elements of Cs_ω (as well as those of RCA_ω) are algebras of ω–ary relations[13].

THEOREM 2.4 (basic properties of RCA_ω)

(i) RCA_ω *is a variety with recursively enumerable and undecidable equational theory.*

(ii) $\mathbf{Eq}(\mathsf{RCA}_\omega) = \bigcup\{\mathbf{Eq}(\mathsf{RCA}_n) : n \in \omega\}$. *I.e. in the language of* RCA_n*, the same equations are true in* RCA_n *and in* RCA_ω*.*

(iii) $\mathsf{RCA}_\omega = \mathbf{SP}\mathsf{Cs}_\omega = \mathbf{SPUp}\mathsf{Csf}_\omega \neq \mathbf{SP}\mathsf{Csf}_\omega$. *I.e.* RCA_ω *is both the variety and quasi-variety generated by* Csf_ω*; the same equations and quasi-equations are true in* Csf_ω *and in* Cs_ω*, but there is an infinitary quasi-equation distinguishing* Csf_ω *and* Cs_ω*.*

Proof. (ii) follows from [HMTII, 3.1.126]. Recursive enumerability and undecidability of $\mathbf{Eq}(\mathsf{RCA}_\omega)$ follows from (ii) and Thm. 2.3 (for recursive enumerability one also has to use the proof of Thm. 2.3, namely that the recursive enumerations of $\mathbf{Eq}(\mathsf{RCA}_n)$ given there are "uniform" in n). That RCA_ω is a variety follows e.g. from [HMTII, 3.1.103], (where it is proved directly that RCA_ω is closed under taking homomorphic images). $\mathsf{RCA}_\omega = \mathbf{SPUp}\mathsf{Csf}_\omega$ follows from [HMTII, 3.2.8, 3.2.10, 2.6.52]. To show $\mathsf{RCA}_\omega \neq \mathbf{SP}\mathsf{Csf}_\omega$ consider[14] the following infinitary quasi-equation q:

$$\bigwedge\{\mathsf{c}_i x = x : i < \omega\} \wedge \mathsf{Id}_{01} \leq x \quad \rightarrow \quad x = 1.$$

Then q is valid in $\mathbf{SP}\mathsf{Csf}_\omega$ while it is not valid in RCA_ω. ■

We note that $\mathbf{SUp}\mathsf{Csf}_\omega \neq \mathsf{RCA}_\omega$. Indeed, consider the following universal formula η

$$\overline{\mathsf{Id}}_{(3)} \neq 0 \rightarrow -\mathsf{Id}_{01} \leq \mathsf{c}_2 \overline{\mathsf{Id}}_{(3)}, \qquad \text{where}$$
$$\overline{\mathsf{Id}}_{(3)} \stackrel{\text{def}}{=} -\mathsf{Id}_{01} \cap -\mathsf{Id}_{02} \cap -\mathsf{Id}_{12}.$$

[13]Thm. 2.4(i) is due to L. Henkin and A. Tarski. For the rest of the credits in connection with Thm 2.4 we refer the reader to [HMT, I, II] and [HMTAN].

[14]Another proof, exporting logical properties to algebras, can be found at the end of Example 9 in section 7.

(The intuitive content of η is that if there is a "subbase" of cardinality ≥ 3, then there are no subbases of size 2.)

RCA_ω is not a discriminator variety, e.g. because there are subdirectly irreducible but not simple RCA_ω's. But it is still an arithmetical variety of BAO's, from which many properties of RCA_ω follow by using theorems of universal algebra. It is not true that $\mathbf{Sir}(\mathsf{RCA}_\omega) = \mathbf{I}\mathsf{Cs}_\omega$, in fact no intrinsic characterization of $\mathbf{Sir}(\mathsf{RCA}_\omega)$ is known.[15] The variety RCA_ω is very well investigated, perhaps the most detailed study is in [HMTAN], [HMTII]. For more recent results see e.g. Goldblatt [G95], Monk [M93], Shelah [Sh], Serény [Se85], [Se97], Hodkinson [H97].

The theorems which say that BRA, RRA, and RCA_n are not finitely axiomatizable, carry over to RCA_ω too. However, to avoid triviality, instead of non-finite axiomatizability we have to state something stronger, because RCA_ω has infinitely many operations and finitely many axioms can speak about only finitely many operations. Taking this into account, when trying to axiomatize RCA_ω, one could still hope for a finite "schema" (in some sense) of equations treating the infinity of the RCA_ω-operations uniformly. A possible example for a finite schema is $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$ $(i, j \in \omega)$. The following theorem[16] implies that it will be hard to find such a schema, and that certain kinds of schemata are ruled out to begin with.

THEOREM 2.5 (nonfinite axiomatizability of RCA_ω) *The variety* RCA_ω *is not axiomatizable by any set* Σ *of universally quantified formulas such that* Σ *involves only finitely many variables.*

Proof. Plan: For all $m < \omega$ we will construct an algebra $\mathfrak{A}_m$ such that

a) $\mathfrak{A}_m \notin \mathsf{RCA}_\omega$

b) every m–generated subalgebra of $\mathfrak{A}_m$ is in RCA_ω.

This will prove the theorem because of the following. Assume that Σ is a set of quantifier-free formulas such that Σ involves at most m variables ($|var(\Sigma)| \leq m < \omega$) and $\mathsf{RCA}_\omega \models \Sigma$. Then Σ is valid in an algebra $\mathfrak{B}$ iff Σ is valid in every m-generated subalgebra of $\mathfrak{B}$, because $|var(\Sigma)| \leq m$ and Σ contains no quantifiers. Thus $\mathfrak{A}_m \models \Sigma$ by b) and by $\mathsf{RCA}_\omega \models \Sigma$. Then $\mathfrak{A}_m \notin \mathsf{RCA}_\omega$ shows that Σ does not axiomatize RCA_ω.

Construction of $\mathfrak{A}_m$: Let $\kappa \geq 2^m$ be finite, and let $\langle U_i : i < \omega\rangle$ be a system of pairwise disjoint sets each of cardinality κ. Let

[15] It is known that $\mathbf{Sir}(\mathsf{RCA}_\omega)$ is a proper subclass of $\mathbf{I}\mathsf{Ws}_\omega$. More on this see [HMTII, 3.1.83-3.1.88], [ANT].

[16] Monk [M69] proves that RCA_ω cannot be axiomatized with a finite number of schemas of equations like those in the definition of CA_ω. See [HMTII, 4.1.7]. Thm. 2.5, due to Andréka, is a generalization of that result and can be found in [A97] or in [M93].

$U = \bigcup\{U_i : i \in \omega\}$, let

$q \in \mathsf{P}_{i\in\omega}U_i \stackrel{\text{def}}{=} \{s \in {}^\omega U : (\forall i \in \omega)s_i \in U_i\}$ be arbitrary,

$R = \{z \in \mathsf{P}_{i\in\omega}U_i : |\{i \in \omega : z_i \neq q_i\}| < \omega\}$, and let

$\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathsf{Id}_{ij}\rangle_{i,j\in\omega}$ generated by the element R.

Then R is an atom of $\mathfrak{A}'$ because of the following. For any two sequences $s, z \in R$ there is a permutation $\sigma : U \rightarrowtail\!\!\!\!\rightarrow U$ of U taking s to z and fixing R, i.e. $s \circ \sigma = z$ and $R = \{p \circ \sigma : p \in R\}$ (the obvious choice for σ, interchanging s_i and z_i for all $i \in \omega$ and leaving everything else fixed, works). If σ is a permutation of U fixing R, then σ fixes all the elements generated by R because the operations of RCA_ω are permutation invariant. Thus if $\emptyset \neq a \in A'$ and $s \in a \cap R$ then $R \subseteq a$, showing that R is an atom of $\mathfrak{A}'$.

We now "split R into $\kappa + 1$ new atoms R_j each imitating R" obtaining a new, bigger algebra $\mathfrak{A}$ from our old $\mathfrak{A}'$. I.e. we choose a larger algebra $\mathfrak{A}$ such that $\mathfrak{A}$ satisfies the conditions below and is otherwise arbitrary.

$\mathfrak{A}' \subseteq \mathfrak{A}$, the Boolean reduct of $\mathfrak{A}$ is a Boolean algebra,

R_j are atoms of $\mathfrak{A}$ and $\mathsf{c}_i R_j = \mathsf{c}_i R$ for $j \leq \kappa, i < \omega$,

each element of $\mathfrak{A}$ is a Boolean join of an element of $\mathfrak{A}'$ and of some R_j's

c_i distributes over joins, for any $i < \omega$, i.e. $\mathfrak{A} \models \mathsf{c}_i(x \cup y) = \mathsf{c}_i x \cup \mathsf{c}_i y$.

Note that in $\mathfrak{A}$ "$\cup$" is only an abstract algebraic operation and not necessarily set theoretical union. It is easy to see that such an extension $\mathfrak{A}$ of $\mathfrak{A}'$ exists. See Figure 2.4.
By the above, we have constructed our algebra $\mathfrak{A}_m$ which in the following we will denote just by $\mathfrak{A}$. ■

CONVENTION 2.6 *In this proof we use the symbols $\cap$, $\cup$ denoting the concrete operations of our set algebras (Cs_ω's) also as the corresponding abstract algebraic operation symbols (denoting themselves in Cs_ω's). So, if x, y are variable symbols, then $x \cap y$ is a term. We hope context will help in deciding whether $x \cap y$ is meant to be a term or a concrete set. $x - y$ is the Boolean term $(x \cap -y)$ denoting the set $x \smallsetminus y$ in Cs_ω's. It is especially important to note that since, for the algebra $\mathfrak{A}$ constructed above, $\mathfrak{A} \notin \mathsf{RCA}_\omega$ was not excluded, the operations denoted by $\cup$, $\cap$, c_i etc. in $\mathfrak{A}$ are not assumed to be the real, set theoretic ones. They are just abstract operations despite of the notation "$\cup$" etc. The Boolean ordering on $\mathfrak{A}$ will be denoted by $\leq$.*

CLAIM 2.7 $\mathfrak{A} \notin \mathsf{RCA}_\omega$.

Proof. For $i, j < \omega$, $i \neq j$, let $\mathsf{s}^i_j(x) = \mathsf{c}_i(\mathsf{Id}_{ij} \cap x)$ and $\mathsf{s}^i_i(x) = x$. Let the term τ be defined by

$$\tau(x) \stackrel{\text{def}}{=} \bigcap_{i \leq \kappa} \mathsf{s}^0_i \mathsf{c}_1 \ldots \mathsf{c}_\kappa x \cap \bigcap_{i<j\leq\kappa} -\mathsf{Id}_{ij}.$$

Let ${}^\omega U^{(q)} = \{z \in {}^\omega U : \{i \in \omega : z_i \neq q_i\}$ is finite$\}$. (${}^\omega U^{(q)}$ is usually called the *weak Cartesian space determined by U and q*.) Then, for our concrete choice of R, $\mathfrak{A}' \models \tau(R) = 0$ because of the following:

$$\begin{aligned}
\mathsf{c}_1 \ldots \mathsf{c}_\kappa R &= (U_0 \times {}^\kappa U \times U_{\kappa+1} \times \ldots) \cap {}^\omega U^{(q)}, \\
\mathsf{s}^0_i \mathsf{c}_1 \ldots \mathsf{c}_\kappa R &= \left({}^i U \times U_0 \times {}^{(\kappa-i)} U \times U_{\kappa+1} \times \ldots\right) \cap {}^\omega U^{(q)}, \\
\bigcap_{i\leq\kappa} \mathsf{s}^0_i \mathsf{c}_1 \ldots \mathsf{c}_\kappa R &= \left({}^{(\kappa+1)} U_0 \times U_{\kappa+1} \times \ldots\right) \cap {}^\omega U^{(q)}.
\end{aligned}$$

Then by $|U_0| \leq \kappa$ we have that there is no repetition–free sequence in ${}^{(\kappa+1)}U_0$. Thus $\mathfrak{A}' \models \tau(R) = 0$.

Then $\mathfrak{A} \models \tau(R) = 0$ by $\mathfrak{A}' \subseteq \mathfrak{A}$ and $R \in A'$. Assume that $\mathfrak{A} \in \mathsf{RCA}_\omega$. Then there is a homomorphism $h : \mathfrak{A} \to \mathfrak{B} \in \mathsf{Cs}_\omega$ such that $h(R) \neq 0$, for some $\mathfrak{B}$. By $h(R) \neq 0$, there is $s \in h(R)$. By $R \leq \mathsf{c}_0 R_j$ we have $h(R) \subseteq \mathsf{c}_0 h(R_j)$, so there is u_j such that $\langle u_j, s_1, s_2, \ldots s_i \ldots\rangle \in h(R_j)$, for all $j \leq \kappa$. These u_j's are different from each other since the R_j's are disjoint from each other, and so the $h(R_j)$'s are disjoint from each other. Consider the sequence

$$z = \langle u_0, u_1, \ldots, u_\kappa, s_{\kappa+1}, \ldots\rangle.$$

Then $z \in \tau(h(R))$ is easily seen as follows. Obviously $z \in -\mathsf{Id}_{ij}$, if $i < j \leq \kappa$. Further
$\langle u_i, z_1, \ldots z_j \ldots\rangle \in \mathsf{Id}_{0i} \cap \mathsf{c}_1 \ldots \mathsf{c}_\kappa h(R)$, hence $z \in \mathsf{s}^0_i \mathsf{c}_1 \ldots \mathsf{c}_\kappa h(R)$ if $i \leq \kappa$. Thus $z \in \tau(h(R))$, a contradiction. ■

CLAIM 2.8 *The m–generated subalgebras of $\mathfrak{A}$ are in* RCA_ω.

Proof. Let $G \subseteq A$, $|G| \leq m$. For all $i, j \leq \kappa$ define

$$R_i \equiv R_j \quad \text{iff} \quad (\forall g \in G)[R_i \leq g \leftrightarrow R_j \leq g].$$

Then $\equiv$ is an equivalence relation on $\{R_j : j \leq \kappa\}$ which has $\leq 2^m$ blocks by $|G| \leq m$. Let p denote the number of blocks of $\equiv$, i.e. $p = |\{R_j/\equiv : j \leq \kappa\}| \leq 2^m \leq \kappa$. Define

$$B = \{a \in A : (\forall i, j \leq \kappa)([R_i \equiv R_j \text{ and } R_j \leq a] \Rightarrow R_i \leq a)\}.$$

We now show that B is closed under the operations of $\mathfrak{A}$.: Let $k < l < \omega$.

1) B clearly is closed under the Boolean operations.

2) $\mathsf{Id}_{kl} \in B$ since $(\forall j \leq \kappa) R_j \not\leq \mathsf{Id}_{kl}$.

3) Clearly, $A' \subseteq B$ (since R is an atom of $\mathfrak{A}'$), and $\mathsf{c}_k a \in A'$ for all $a \in A$. Thus $\mathsf{c}_k b \in B$ (for all $b \in A$).

Let $\mathfrak{B} \subseteq \mathfrak{A}$ be the subalgebra of $\mathfrak{A}$ with universe B. By $G \subseteq B$, it is enough to show that $\mathfrak{B} \in \mathsf{RCA}_\omega$.

We will define an embedding $h : \mathfrak{B} \rightarrowtail \langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathsf{Id}_{ij} \rangle_{i,j<\omega}$. Let $\{y_j : j < p\} = \{\sum(R_j/\equiv) : j \leq \kappa\}$. Then $\{y_j : j < p\}$ is a partition of R in $\mathfrak{B}$, i.e. they are pairwise disjoint and sum up to R, $\mathsf{c}_i y_j = \mathsf{c}_i R$ for all $j < p$ and $i < \omega$ and every element of $\mathfrak{B}$ is a join of some element of A' and of some y_j's. So, $\mathfrak{B}$ looks like the algebra on Figure 2.5.

First we define the images of the y_j's. Let $Q = \{0, 1, \ldots, \kappa - 1\}$ and let $(Q, +, 0)$ be a commutative group. For each $i < \omega$ let $f_i : U_i \rightarrowtail\!\!\!\!\rightarrow Q$ be a bijection such that $f_i(q_i) = 0$. For $j < \kappa$ define

$$R''_j = \left\{ z \in R : \sum \langle f_i(z_i) : i < \omega \rangle = j \right\},$$

where $\sum$ denotes the group theoretic sum in $(Q, +, 0)$. Then it is not difficult to check that the R''_j's are disjoint from each other and

$$\mathsf{c}_i R''_j = \mathsf{c}_i R \quad \text{for all } i < \omega$$

for the concrete set theoretic c_i's. Define for all $j < p - 1$

$$R'_j = R''_j$$

$$R'_{p-1} = \bigcup \left\{ R''_j : p - 1 \leq j < \kappa \right\}.$$

We are ready to define the embedding h of B.: We define for all $b \in B$

$$h(b) = (b - R) \cup \bigcup \{R'_j : j < p, y_j \leq b\},$$

where $b - R$ is computed in $\mathfrak{A}$, and since $(b - R) \in A' \subseteq \mathcal{P}({}^\omega\mathcal{U})$, the rest of the operations are the concrete set theoretic ones. Now it is not difficult to check that h is an embedding $h : \mathfrak{B} \rightarrowtail \langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathsf{Id}_{ij} \rangle_{i,j<\omega}$ as follows.

Clearly h preserves $\cup, -$. $h(b) = 0$ implies $b = 0$, hence h is one–one. $h(\mathsf{Id}_{kl}) = \mathsf{Id}_{kl}$. Now we check $\mathsf{c}_k h(b) = h(\mathsf{c}_k b)$.

$$\mathsf{c}_k h(b) \quad = \quad \mathsf{c}_k [(b - R) \cup \bigcup \{R'_j : y_j \leq b\}]$$

$$\begin{aligned} &= \mathsf{c}_k(b-R) \cup \bigcup\{\mathsf{c}_k R'_j : y_j \le b\} \\ &= \mathsf{c}_k(b-R) \cup \bigcup\{\mathsf{c}_k y_j : y_j \le b\} \\ &= \mathsf{c}_k[(b-R) \cup \bigcup\{y_j : y_j \le b\}] \\ &= \mathsf{c}_k b, \end{aligned}$$

where the operations in the first two lines are set theoretic while those in the last three lines are understood in the abstract algebra $\mathfrak{A}$. $h(\mathsf{c}_k b) = (\mathsf{c}_k b - R) \cup \bigcup\{R'_j : y_j \le \mathsf{c}_k b\} = \mathsf{c}_k b$, since $(\exists j) y_j \le \mathsf{c}_k b$ iff $R \le \mathsf{c}_k b$, and $R \not\le \mathsf{c}_k b$ iff $\mathsf{c}_k b = \mathsf{c}_k b - R$.
QED(Theorem 2.5) ■

Remark 2.9 below describes the modifications needed for obtaining proofs for the analogous (with Thm. 2.5) non-finitizability theorems for RCA_n $(n > 2)$ and RRA.

REMARK 2.9 Here we outline the modifications of the above proof of Theorem 2.5 yielding proofs for non-finite axiomatizability of RCA_n and RRA.

Let α be n or ω. An algebra $\mathfrak{A}$ similar to RCA_α's is said to be *representable* if $\mathfrak{A} \in \mathsf{RCA}_\alpha$. Thus representability means that $\mathfrak{A}$ is isomorphic to an algebra $\mathfrak{A}^+$ whose elements are α–ary relations and whose greatest element is a disjoint union of Cartesian spaces. $\mathfrak{A}^+$ is called a *representation* of $\mathfrak{A}$ and sometimes the isomorphism $h : \mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{A}^+$ too is called the representation of $\mathfrak{A}$. By a *homomorphic representation* we understand a homomorphism mapping $\mathfrak{A}$ into some Cs_α. This concept receives its importance from the simple but useful fact that representability of $\mathfrak{A}$ is equivalent with the existence of a set H of homomorphic representations of $\mathfrak{A}$ such that $(\forall$nonzero $x \in A)(\exists h \in H) h(x) \ne 0$.

The *intuitive idea* of the above proof of Theorem 2.5 was the following. We found two different ways of "counting" the elements of the domain $\{s_0 : s \in R\}$ of the relation R. This counting was done by looking only at the abstract, i.e. isomorphism invariant properties of $\mathfrak{A}$. The two ways of counting were: (1) Looking at the number of the disjoints elements R_j below R. This allowed us to conclude that the domain of R must be big. (2) Using the Id_{ij}'s exactly as one uses equality in first–order logic to express that a certain finite set is smaller than some κ, we concluded that the domain of R must be small. (This was done by the term $\tau(R)$ in the proof of Claim 2.7.)

We started out from an $\mathfrak{A}' \in \mathsf{RCA}_\omega$ in which the counting (2) said that "*Dom*(R)" is small. Then by splitting, we enlarged $\mathfrak{A}'$ to $\mathfrak{A}$, such that in this bigger algebra $\mathfrak{A}$ the counting (1) said that "*Dom*(R)" is big. Thus in $\mathfrak{A}$ the two countings (1) and (2) contradict each other, ensuring $\mathfrak{A} \notin \mathsf{RCA}_\omega$.

This is how we constructed one nonrepresentable algebra ($\mathfrak{A}_m$). We were able to construct an infinite sequence of such algebras in such a way that as m increases, the contradiction between (1) and (2) becomes weaker and weaker. Actually, as m approaches infinity, the contradiction between (1) and (2) vanishes. So in the ultraproduct of the $\mathfrak{A}_m$'s, (1) and (2) do not contradict each other any more, and this ultraproduct is in RCA_ω. In our construction the conflict between (1) and (2) became weaker and weaker in the sense that more and more elements had to be inspected for discovering this contradiction.[17] This finishes the intuitive idea of the proof of Theorem 2.5.

Next we would like to repeat this proof for RCA_n in place of RCA_ω, with $2 < n < \omega$. If we simply replace ω everywhere with n, the proof does not go through because the counting in (2) needs an arbitrarily large number of Id_{ij}'s and we have only $n \times n$ many.[18] So we need a new method for doing (2). This amounts to looking for an abstract algebra $\mathfrak{A}$ together with its element R and concluding that in *any* (homomorphic) representation $h : \mathfrak{A} \to \mathfrak{B} \in \mathsf{Cs}_n$ of $\mathfrak{A}$, the domain U_0 of $h(R)$ must be of smaller size than a certain κ. (The difficulty is that we have to be able to repeat this for arbitrarily large $\kappa \in \omega$.) We also need to keep in mind that we will want to have a contradiction with (1), which means that we will want to split R. In order to be able to do this, we only need that R remain an atom. There are many natural ways for ensuring (by abstract properties) smallness of a set. Perhaps the simplest way is the following. If we could "see" by looking at $\mathfrak{A}$ "abstractly" that $U_0 \times U_0$ is a union of fewer than κ functions $f_i \ \ (i < \kappa - 1)$ each of which is coded by an element of $\mathfrak{A}$, then the domain U_0 of R must be of smaller cardinality than κ in *any* representation of $\mathfrak{A}$. E.g. we can take these functions (the f_i's) to be powers of a single suitable permutation f of U_0; say let $U_0 = \kappa$, and let f be the usual successor modulo κ. Let $F \stackrel{\text{def}}{=} f \times {}^{n-2}U$. Then $F \subseteq {}^nU$. We include into our algebra $\mathfrak{A}'$, besides R, also F as a new generator element. It can be checked that R remains an atom (because no subset of U_0 became "definable"). Now, similarly to the way we used the equation $\tau(R) = 0$ in the proof of Claim 2.7, by studying the abstract properties of F and R in the new $\mathfrak{A}'$ we can conclude that in any homomorphic representation h of $\mathfrak{A}'$, the Cartesian square of the domain of $h(R)$ is contained in the union of fewer than κ powers of a function coded by $h(F)$. But then this domain must be of cardinality $\leq \kappa$. (Exactly what we proved in Claim 2.7 of the old proof. So we can prove our new Claim 2.7.) After this modification, the whole proof goes through by replacing all occurrences of ω with n.[19] This completes the outline of the proof that RCA_n cannot be axiomatized with quantifier free formulas using finitely many variables, if $n > 2$.

Let us turn to the RRA case, i.e. to Theorems 1.6, 1.10. The idea is basically the

[17]This allowed us to avoid ultraproducts in the final argument. We find it more natural to explain the intuitive idea in terms of ultraproducts, which incidentally happens to be the way the original proof of nonfinitizability went.

[18]We can construct the algebras $\mathfrak{A}_m$ as in the proof of Thm. 2.5 for $m < log_2(n)$. But for the contradiction to vanish, we need $\mathfrak{A}_m$ for arbitrarily large m.

[19]We need $n \geq 3$ to be able to see abstractly that the f_i's are functions. This proof is worked out in detail in [A97, Thm.1].

same as in the above outlined RCA_n case. Exactly as in the RCA_n case, here too we use two counting principles (1), (2) and construct algebras $\mathfrak{A}_m$ in which (1) and (2) contradict each other. Again we want a controllable contradiction such that as m approaches infinity the contradiction vanishes. Here we have to take a less obvious principle for counting in (2), because in RRA, functions interfere with splitting[20] elements $R = U_0 \times U_1$. E.g. we can use colorings of the full graph $U_0 \times U_0$ with finitely many colors without monochromatic triangles, and then apply Ramsey's theorem. This means that we arrange $U_0 \times U_0$ to be a disjoint union of symmetric relations $G_0, \dots, G_r$ such that $G_r = \mathsf{Id} \upharpoonright U_0$ (symmetric means $G_i = G_i^{-1}$) and $(G_i \circ G_i) \cap G_i = 0$. To ensure splittability of R we also arrange that $G_i \circ G_j \supseteq G_k$ whenever $|\{i,j,k\}| > 1$, $i,j,k < r$. We let our $\mathfrak{A}'$ be generated in this case by $\{G_0, \dots, G_r, R\}$. All these properties of the G_i's were abstract, "equational" ones.[21] This ensures that in every representation of $\mathfrak{A}'$, the domain U_0 of R must be finite (by Ramsey's theorem). We split R into ω many R_i's obtaining $\mathfrak{A}$ from $\mathfrak{A}'$ as we did in the RCA_ω, RCA_n cases before. The rest of the proof goes through as before with replacing ω (or n) everywhere by 2, except for the following change. In the RRA case we have to look at the ultraproduct of the $\mathfrak{A}_m$'s and observe that it is representable (since the contradiction between (1) and (2) disappeared as both counting gives us continuum many elements). Therefore this proof gives only non-finite axiomatizability of RRA (i.e. Monk's theorem) without proving (Jónsson's result saying) that infinitely many variables are needed. For the latter, one has to fine-tune the construction some more.[22] ■

In section 1, Theorem 1.6 leads to Problem 1.12 in a natural way. Exactly the same way our present Theorem 2.5 leads to the following important open problem.

PROBLEM 2.10 *Find simple, mathematically transparent, decidable sets E of equations axiomatizing* RCA_ω. *The* RCA_n, $2 < n < \omega$ *version of this problem is open and interesting, too.*

The RCA_n version is strongly related to Problem 1.12 in section 1. On the other hand, the present, RCA_ω version has a logical counterpart, cf. e.g. [HMTII, Prob.4.16, p.180]. This is one of the central problems of Algebraic Logic, cf. [HMTII, Prob.4.1], Henkin–Monk [HM74, Prob.5], etc. For strongly related results (or for partial solutions) see [HMTII, pp.112–119], [V91], [V95], [Si91], [Si93], [HH].

[20] Splitting in RA is defined, and the conditions for splittability are described, in [AMdN].

[21] In fact, it is an open problem in RA-theory (see e.g. [AMN, section on open problems], whether there are such concrete relations $G_0, \dots, G_r$ on some set U_0 or not. What we should do here is that we state these properties abstractly on some abstract relations $G_0, \dots, G_r$. The only difference from the previous proof will then be that we do not know whether $\mathfrak{A}' \in \mathsf{RRA}$. But this does not matter, what we need is that $\mathfrak{A}_m \notin \mathsf{RRA}$ and $\mathsf{P}\mathfrak{A}_m/F \in \mathsf{RRA}$.

[22] This is done in [AN90], where we use projective geometries for the purposes of counting in (2).

PROBLEM 2.11 *Is there a finite schema axiomatizable quasi-variety* K *such that* $\mathbf{Eq}(\mathsf{K}) = \mathbf{Eq}(\mathsf{RCA}_\omega)$*, i.e. the variety generated by* K *is* RCA_ω*? The same for* RCA_n *for* $n < \omega$*. I.e. is there a finitely axiomatizable quasi-variety* $\mathsf{K} \subseteq \mathsf{RCA}_n$ *such that* $\mathsf{RCA}_n = \mathbf{H}\mathsf{K}$*?*

This problem is related to the existence of weakly sound Hilbert-style inference systems for first-order logic, see Part II, Thm.6.5 and Open Problem 7.2.

On the structure of the equational axiomatizations of RCA_n**,** RCA_ω**:**

Let E be an arbitrary set of equations axiomatizing RCA_n. As in the RRA-case, E must be infinite, but it can be chosen to be decidable. Unlike the RRA-case, here every operation symbol has to occur infinitely many times in E (in the RRA-case, only the Booleans and $\circ$ had to occur infinitely many times). A similar statement is true for RCA_ω in place of RCA_n. For more on this see Figure 3.1 and [A97], [A94]. Concrete decidable sets E are known, see e.g. [HMTII, pp.112-119], cf. also [V91], [V95], [Si91], [Si93], [HH97]. However, it would be important to find choices of E with more perspicuous structures, see Problem 2.10.

Let us turn to the relationship between RCA_ω and its abstract approximation CA_ω. These investigations yield information on proof theoretical properties of first-order logic and of some related logics. See Examples 6, 8, 9 in section 7, especially theorems 7.4 - 7.7.

DEFINITION 2.12 (CA_ω, an abstract approximation of RCA_ω) *A* CA_ω *is a normal* BAO *of the same similarity type as* RCA_ω *in which the* c_i*'s are self-conjugated commuting closure operations, and in which the constants* Id_{ij} *satisfy the following equations:*

(3)' For all $i, j, k < \omega$

(i) $\mathsf{c}_i\mathsf{Id}_{ij} = 1$, $\mathsf{c}_k\mathsf{Id}_{ij} = \mathsf{Id}_{ij}$ *if* $k \neq i, j$, $\mathsf{Id}_{ii} = 1$, $\mathsf{Id}_{ij} = \mathsf{Id}_{ji}$, *and* $\mathsf{Id}_{ij} \cap \mathsf{Id}_{jk} \leq \mathsf{Id}_{ik}$.

(ii) $\mathsf{Id}_{ij} \cap \mathsf{c}_i x = x$ *whenever* $x \leq \mathsf{Id}_{ij}$ *and* $i \neq j$.

■

To treat $\mathsf{RCA}_n, \mathsf{CA}_n$ and $\mathsf{RCA}_\omega, \mathsf{CA}_\omega$ in a unified manner, we replace ω in the definitions of RCA_ω and CA_ω with an arbitrary but fixed ordinal α, obtaining $\mathsf{RCA}_\alpha, \mathsf{CA}_\alpha$ (here $\alpha = n$ and $\alpha = \omega$ are of course permitted).[23]

[23]This generalization will also be useful in algebraizing various quantifier logics different from classical first–order logic.

If $\alpha = n < \omega$, then the newly defined RCA_n and CA_n are only definitionally equivalent with the previously defined ones, because in Def. 2.2 we had only one constant Id in place of the present $n \times n$–many constants Id_{ij}. This definitional equivalence is given by

$$\mathsf{Id} = \bigcap_{i,j<n} \mathsf{Id}_{ij} \quad \text{and} \quad \mathsf{Id}_{ij} = \mathsf{c}_{(n \smallsetminus \{i,j\})}\mathsf{Id}.$$

Since definitional equivalence is a very close connection between classes of algebras, we did not give new names for RCA_n and CA_n.

DEFINITION 2.13 *(Locally finite, dimension-complemented* CA*'s, and neat-reducts) Let* α, β *be any ordinals.*

(i) Let $\mathfrak{A} \in \mathsf{CA}_\alpha$*,* $x \in A$*. Then* $\Delta(x) \stackrel{\text{def}}{=} \{i \in \alpha : \mathsf{c}_i x \neq x\}$.

$$\mathsf{Lf}_\alpha \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{CA}_\alpha : (\forall x \in A)(\Delta(x) \textit{ is finite})\}.$$

$$\mathsf{Dc}_\alpha \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{CA}_\alpha : (\forall x \in A)(\alpha \smallsetminus \Delta(x) \textit{ is infinite})\}.$$

(ii) Assume $\alpha \leq \beta$ *and* $\mathfrak{A} \in \mathsf{CA}_\beta$*. Then*

$$Nr_\alpha\mathfrak{A} \stackrel{\text{def}}{=} \{x \in A : \Delta(x) \subseteq \alpha\}, \quad \mathfrak{Nr}_\alpha\mathfrak{A} \stackrel{\text{def}}{=} \langle Nr_\alpha\mathfrak{A}, \mathsf{c}_i^{\mathfrak{A}}, \mathsf{Id}_{ij}^{\mathfrak{A}}\rangle_{i,j<\alpha}.$$

In the above, $\mathsf{c}_i^{\mathfrak{A}}, \mathsf{Id}_{ij}^{\mathfrak{A}}$ *denote the corresponding operations of* $\mathfrak{A}$*. It can be checked that* $\mathfrak{Nr}_\alpha\mathfrak{A} \in \mathsf{CA}_\alpha$*.*

$$\mathbf{Nr}_\alpha\mathsf{CA}_\beta \stackrel{\text{def}}{=} \{\mathfrak{Nr}_\alpha\mathfrak{A} : \mathfrak{A} \in \mathsf{CA}_\beta\}.$$

■

The elements of Lf_α and Dc_α are called *locally finite* and *dimension–complemented* CA_α's respectively. $\mathfrak{Nr}_\alpha\mathfrak{A}$ is called the α*-neat reduct* of $\mathfrak{A}$. We note that

$$\mathsf{Dc}_\alpha = \{\mathfrak{A} \in \mathsf{CA}_\alpha : (\forall x \in A)(\alpha \smallsetminus \Delta(x) \neq \emptyset)\}.$$

THEOREM 2.14 (Relationships [24] between $\mathsf{Lf}_a, \mathsf{Dc}_\alpha, \mathbf{Nr}_\alpha\mathsf{CA}_\beta$ and RCA_α.)

[24] The classes $\mathbf{SNr}_\alpha\mathsf{CA}_\beta$ were introduced by Henkin, and Monk [M61] proved that they are varieties. Thm. 2.14(i) is due to J. D. Monk. The equalities in (ii) are due to L. Henkin [H55], while the inequality in (ii) was proved by J. D. Monk [M69].

(i) $\mathsf{RCA}_\omega = \mathbf{SUpLf}_\omega \neq \mathbf{SPLf}_\omega$ *and* $\mathsf{Dc}_\omega \subseteq \mathsf{RCA}_\omega$. *I.e. there is no universal formula distinguishing* RCA_ω *from* Lf_ω*, and every* Dc_ω *is representable. The same hold for all* $\alpha \geq \omega$ *in place of* ω.

(ii) $\mathsf{RCA}_\alpha = \mathbf{SNr}_\alpha \mathsf{CA}_{\alpha+\omega} = \bigcap\{\mathbf{SNr}_\alpha \mathsf{CA}_{\alpha+m} : m \in \omega\} \neq \mathbf{SNr}_a \mathsf{CA}_{\alpha+m}$ *for all* α *and for all finite* m. $\mathbf{SNr}_\alpha \mathsf{CA}_{\alpha+m}$ *is a variety for all* α *and* m.

Proof. The positive statements follow from [HMTII, 3.2.10, 2.6.32(ii), 2.6.50, 2.6.52 3.2.11]. The negative statements are also proved in [HMTII] taken together with [HMTAN]. ■

The above theorem gives information on the proof theory of first order logic (FOL) and on its n-variable fragment $\mathbf{L}_n$. Intuitively, it says (in several different forms) that the important feature of FOL is not that each formula involves only finitely many variables, but that given any formula, there are infinitely many variables it does not involve [25].

Based on the above theorem, an inference system is given both for FOL and $\mathbf{L}_n$ which uses the finite-schema axiomatization of CA_α together with a supply of variables which do not occur in our original formulas.[26] I.e. these variables can occur in a proof, but not in the final formula we want to prove.

[25] The earlier mentioned theorem saying that quasi-projective RA's are representable, also speaks about this phenomenon: the projections are used for coding together the already involved variables, so that we get one more "unused" variable. This idea comes through clearly in [Si96]. The same idea is used for proving finite schema axiomatizability (i.e. completeness of the corresponding logic) in [S95], [SGy]. The same idea is used for obtaining an unorthodox completeness theorem in [Si91].

[26] Cf. e.g. [HMTII, p.157], [N96], [AGN77, Thm. 3.15] and [Si91]. Cf. also section 7 herein.

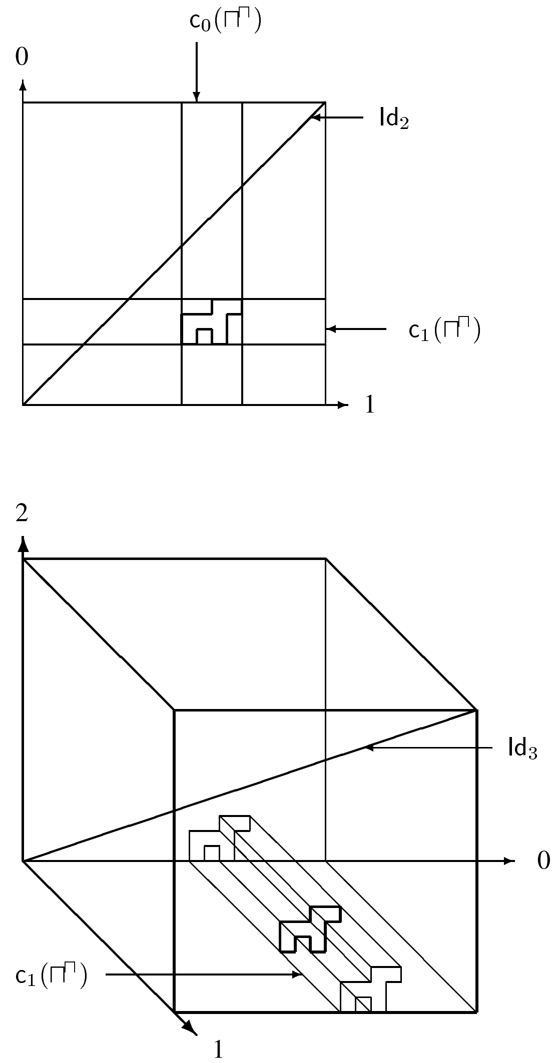
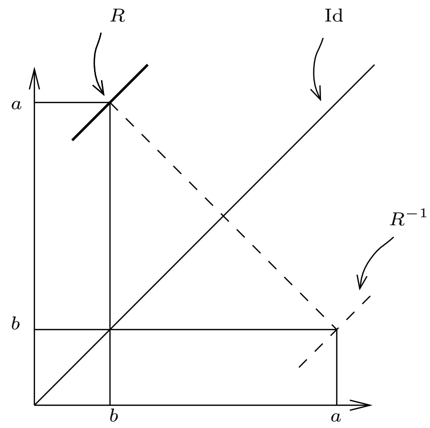
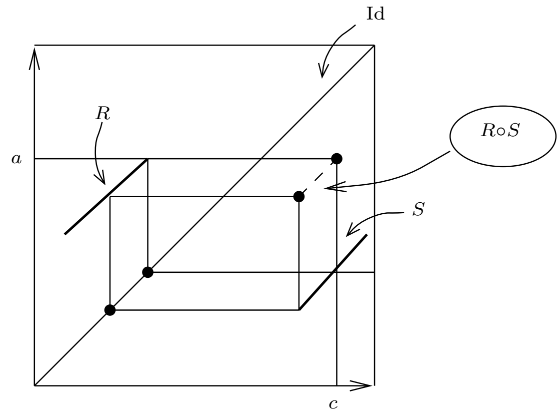

$c_0(\sqcap\sqcap)$
0
Id_2
$c_1(\sqcap\sqcap)$
1
2
Id_3
0
$c_1(\sqcap\sqcap)$
1

Figure 2.1.

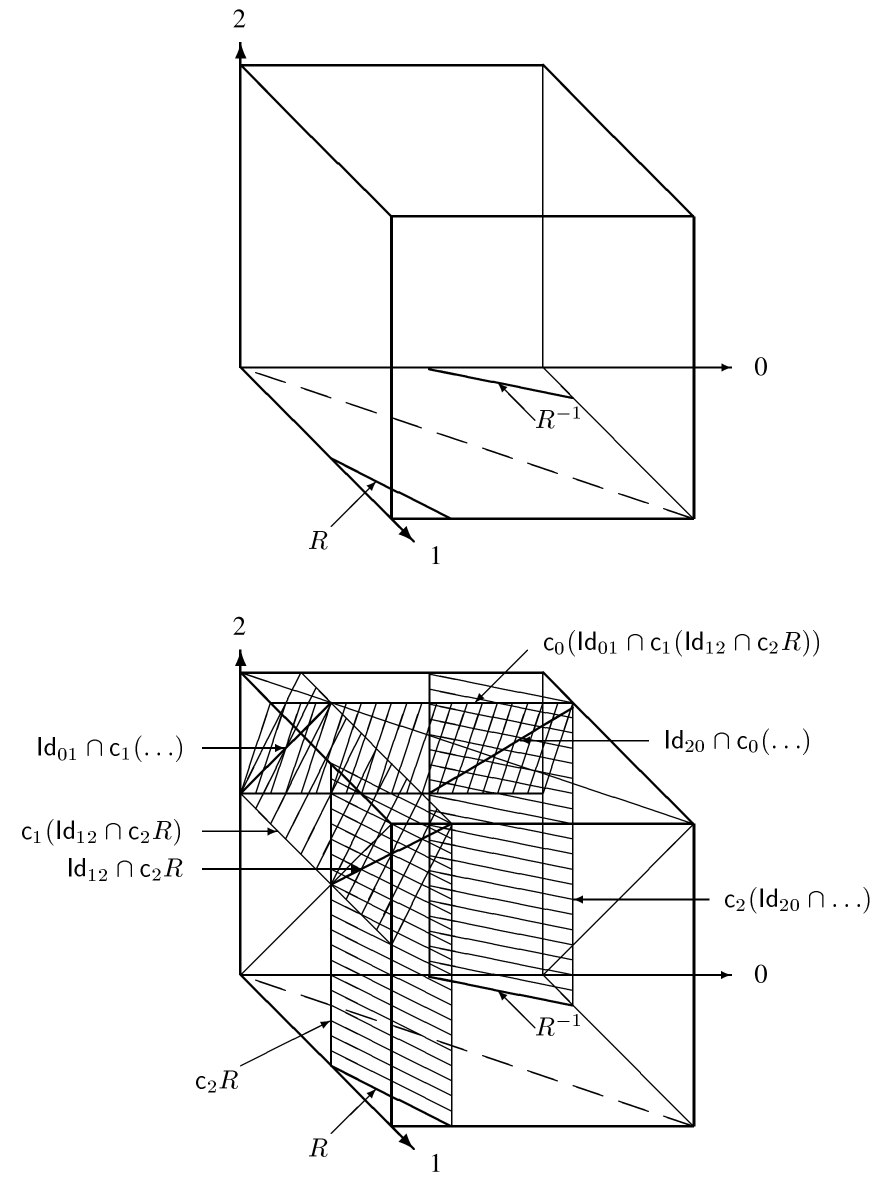


converse is mirroring



composition is not so easy to draw

Figure 2.2.



Mirroring in 3-dimensional space
can be expressed by $\mathsf{c}_i,\ \mathsf{Id}_{ij},\ \cap$.

Figure 2.3.

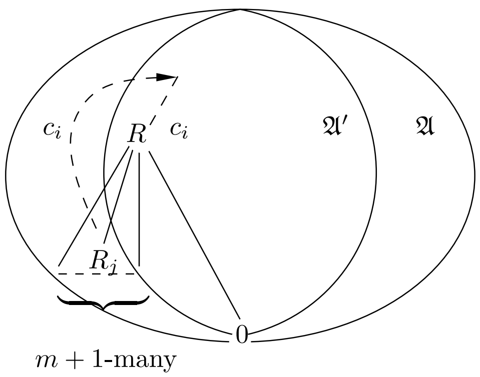


Figure 2.4.

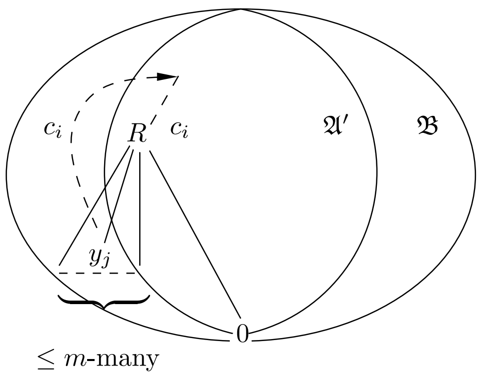


Figure 2.5.

CHAPTER 3

ALGEBRAS FOR LOGICS WITHOUT IDENTITY

We start from RCA_α, and would like to consider its "natural" Id-free reducts. If we simply omit Id (or Id_{ij}), then we lose not only identity (or equality), but also our ability to "algebraize" substitution of individual variables like $\varphi \mapsto \varphi[v_i/v_j]$ in the logic to be algebraized. Therefore, before dropping the Id_{ij}'s, we first add our term functions $\mathsf{s}^i_j(x) = \mathsf{c}_i(\mathsf{Id}_{ij} \cap x)$ for $i \neq j$, $\mathsf{s}^i_i(x) = x$. Now,

$$\mathsf{RSC}_\alpha \stackrel{\text{def}}{=} \mathbf{SP}\{\langle \mathfrak{P}({}^\alpha U), \mathsf{c}_i, \mathsf{s}^i_j\rangle_{i,j<\alpha} : U \text{ is a set}\}.$$

RSC_α's are called *representable substitution-cylindric* algebras, cf. [N91], [AGMNS]. They are the simplest kind in the family of polyadic-style algebras. The theory of RSC_α is analogous with that of RCA_α, in particular, if $\alpha > 2$, then RSC_α is not finitely axiomatizable, cf. [ST].

Let $\mathbf{Rd}_{sc}$ be the operator which to any cylindric-type algebra $\mathfrak{A} = \langle A, \cup, -, \mathsf{c}_i, \mathsf{Id}_{ij}\rangle_{i,j<\alpha}$ associates the RSC_α-type algebra

$$\mathbf{Rd}_{sc}(\mathfrak{A}) \stackrel{\text{def}}{=} \langle A, \cup, -, \mathsf{c}_i, \mathsf{s}^i_j\rangle_{i,j<\alpha}$$

where s^i_j is the derived operation of $\mathfrak{A}$ defined above. Now, $\mathsf{RSC}_\alpha = \mathbf{SRd}_{sc}\mathsf{RCA}_\alpha$. The finitely axiomatizable approximation SC_α of RCA_α is defined analogously,

$$\mathsf{SC}_\alpha \stackrel{\text{def}}{=} \mathbf{SRd}_{sc}\mathsf{CA}_\alpha.$$

SC_α's are called *substitution-cylindric* algebras.

THEOREM 3.1 SC_α *is a finite schema axiomatizable variety containing* RSC_α. ■

For the simple set of axioms, and for information on the proof we refer the reader to [AGMNS]. Cf. also [N91, §8].

The theory of the pair $\mathsf{SC}_\alpha, \mathsf{RSC}_\alpha$ is almost completely analogous with that of $\mathsf{CA}_\alpha, \mathsf{RCA}_\alpha$. The connection between SC_α-theory and first order logic without equality is analogous with the connection between CA_α-theory and logic with equality. In particular, the logic counterpart of the algebraic operation s^i_j is the "substitution-modality" $[v_i/v_j]$, cf. Part II, section 7. The logics $\mathbf{L}_n^s$ and $\mathbf{L}_n^{s=}$ are introduced in section 7, example 7. Let $\mathbf{L}_n^z$ and $\mathbf{L}_n^{z=}$ denote the fragments of these logics obtained by dropping the connective $[v_i, v_j]$. Then the $\mathsf{CA}_n, \mathsf{RCA}_n$ pair is strongly connected to $\mathbf{L}_n^{z=}$ while the $\mathsf{SC}_n, \mathsf{RSC}_n$ pair is completely analogously connected to $\mathbf{L}_n^z$. More on the connection between SC_α and logic can be found in [SN], [MV], [V95a], [vB96], [N91].

The classes $\mathsf{CA}_\alpha, \mathsf{RCA}_\alpha$ and $\mathsf{SC}_\alpha, \mathsf{RSC}_\alpha$ introduced so far constitute the hearts of the following two "worlds": the algebraic counterpart of logics with equality (the "cylindric world"), and the algebraic counterpart of logics without equality (the "polyadic world"). In both worlds one can introduce natural extra operations like e.g. cardinality quantifiers, generalized cylindrifications, but what determines the most basic theorems (remaining true for the expanded algebras) remains the CA_α-structure or the SC_α-structure. Therefore it seems reasonable to pay somewhat more attention (say, as a default) to CA_α and SC_α than to their versions enriched with extra operators. [1]

SC_α's with extra operators (like e.g. $[v_i, v_j]$) are discussed in the literature under the names *quasi-polyadic* algebras (QPA_α's) and *polyadic* algebras respectively.[2] The most important extra operator p_{ij} is a substitution operator like s_j^i (p_{ij} corresponds to the logical connective $[v_i, v_j]$ in section 7, Example 7). Let $X \subseteq {}^\alpha U$. Then

$$\mathsf{p}_{01}(X) \stackrel{\text{def}}{=} \{\langle q_1, q_0, q_2, \ldots\rangle : q \in X\}.$$

I.e., p_{01} interchanges q_0 and q_1 in a sequence q. For $i, j < \alpha$, p_{ij} is defined completely analogously. Now, RQPA_α's are defined to be RSC_α's enriched with the p_{ij}'s $(i, j < \alpha)$:

$$\mathsf{RQPA}_\alpha \stackrel{\text{def}}{=} \mathbf{SP}\{\langle \mathfrak{P}({}^\alpha U), \mathsf{c}_i, \mathsf{s}_j^i, \mathsf{p}_{ij}\rangle_{i,j<\alpha} : U \text{ is a set}\}.$$

The abstract class QPA_α approximating RQPA_α is defined by finitely many axiom-schemes analogously to the definition of CA_α or SC_α, cf. [N91], [ST], [AGMNS].

Polyadic algebras (RPA_α and PA_α) are obtained from RSC_α and SC_α by adding infinitary substitutions and infinitary cylindrifications denoted as s_τ, $\mathsf{c}_{(\Gamma)}$, for $\tau : \alpha \to \alpha$ and $\Gamma \subseteq \alpha$. For the theory of these algebras we refer to [Ha62], [HMTII], [N91]. Cf. also [NS96].

We note that the theory of QPA's seems to be very strongly analogous with that of SC_α's. (However, in certain studies, e.g. when investigating the connection between RA's and CA's, CA's enriched with the p_{ij}'s play a very illuminating role

[1]Of course, no such rule is valid in general. Actually, pushing the above considerations further, SC_α's seem to be at the heart of CA_α-theory, too, therefore they could be considered as the core of (or basis for) the algebraizations of quantifier logics in general. This unified perspective for algebraic logic has not been elaborated yet.

[2]PA's, QPA's, and their versions with equality like QPEA's, originate with P. Halmos (cf. [Ha62]). SC's originate with C. Pinter, cf. e.g. [AGMNS] or [N91].

(cf. [NSi]). The latter algebras are called QPA's with equality, or QPEA's. Cf. e.g. [HMTII] for their theory.)

For lack of space, we do not discuss further the theory of SC_α's with extra operators (like QPA's, PA's, etc).

The next figure, taken from [A97] describes the interconnections between the operations $\mathsf{c}_i, \mathsf{Id}_{ij}, \mathsf{s}^i_j$ and p_{ij} (in the presence of the Boolean operations). (On the figure, $\mathsf{Id}_{ij}, \mathsf{s}^i_j$ are denoted as $\mathsf{d}_{ij}, \mathsf{s}_{ij}$, respectively.) On Figure 3.1, nodes represent classes of algebras of relations where the units are Cartesian spaces ${}^nU$ ($3 \leq n < \omega$), and the operations are those along the path leading to the node. A broken edge between two nodes means that the second class is finitely axiomatizable over the first one, a bold edge means non-finite axiomatizability over, and a normal line means that it is unknown (to the authors) whether finite or non-finite axiomatizability holds.

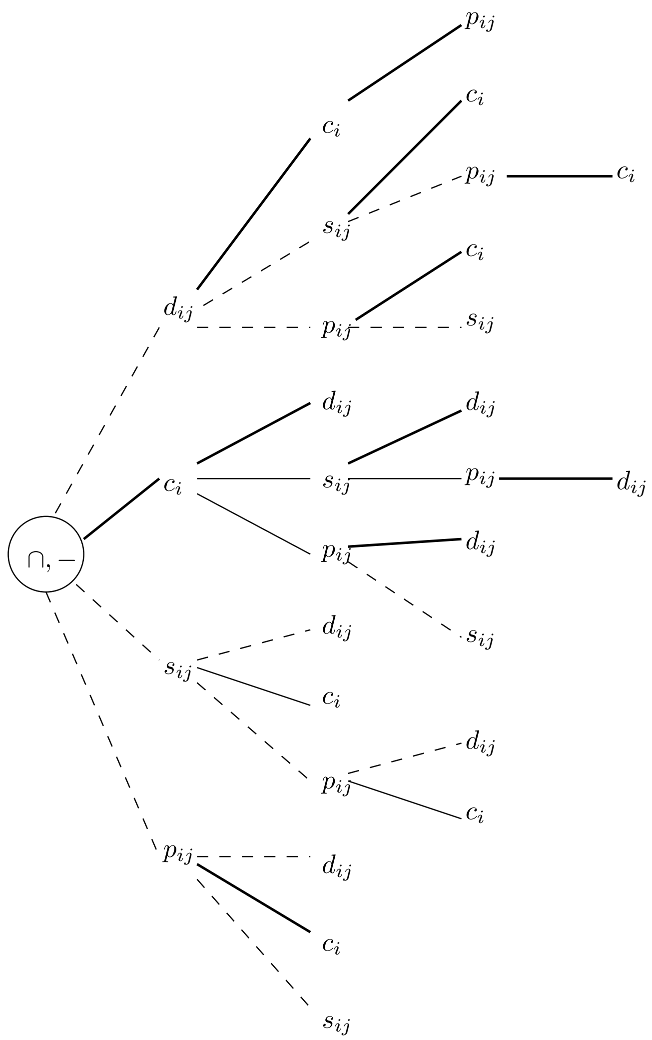


Figure 3.1.

PART II

BRIDGE BETWEEN LOGIC AND ALGEBRA: ABSTRACT ALGEBRAIC LOGIC

INTRODUCTION TO PART II

Let us start by putting the subject matter of the present part (i.e. Part II of the present paper) into perspective.

The idea of solving problems in logic by first translating them to algebra, then using the well developed methodology of algebra for solving them, and then translating the solution back to logic, goes back to Leibnitz and Pascal. Papers on the history of Logic (e.g. Anellis–Houser [AH], Maddux [Ma91]) point out that this method was fruitfully applied in the 19$^{\text{th}}$ century not only to propositional logics but also to quantifier logics (De Morgan, Peirce, etc. applied it to quantifier logics too). The number of applications grew ever since. (Though some of these remained unnoticed, e.g. the celebrated Kripke–Lemmon completeness theorem for modal logic w.r.t. Kripke models was first proved by Jónsson and Tarski in 1948 using algebraic logic.)

For brevity, we will refer to the above method or procedure as "applying Algebraic Logic (AL) to Logic". This expression might be somewhat misleading since AL itself happens to be a part of logic, and we do not intend to deny this. We will use the expression all the same, and hope, the reader will not misunderstand our intention.

In items (i) and (ii) below we describe two of the main motivations for applying AL to Logic.

(i) This is the more obvious one: When working with a relatively new kind of problem, it often proved to be useful to "transform" the problem into a well understood and streamlined area of mathematics, solve the problem there and translate the result back. Examples include the method of Laplace Transform in solving differential equations (a central tool in Electrical Engineering).[3]

In the present part we define the algebraic counterpart $\mathbf{Alg}(\mathcal{L})$ of a logic $\mathcal{L}$ together with the algebraic counterpart $\mathbf{Alg}_m(\mathcal{L})$ of the semantical-model theoretical ingredients of $\mathcal{L}$. Then we prove equivalence theorems, which to essential logical properties of $\mathcal{L}$ associate natural and well investigated properties of $\mathbf{Alg}(\mathcal{L})$ such that if we want to decide whether $\mathcal{L}$ has a certain property, we will know what to ask from our algebraician colleague about $\mathbf{Alg}(\mathcal{L})$. The same devices are suitable for finding out what one has to change in $\mathcal{L}$ if we want to have a variant of $\mathcal{L}$ having a desirable property (which $\mathcal{L}$ lacks). To illustrate these applications we include several examples (which deal with various concrete logics) in section 7. For all this, first we have to define what we understand by a logic $\mathcal{L}$ in general (because

[3] At this point we should dispell a misunderstanding: In certain circles of logicians there seems to be a belief that AL applies only to syntactical problems of logic and that semantical and model-theoretic problems are not treated by AL or at least not in their original model theoretic form. Nothing can be as far from the truth as this belief, as e.g. looking into the present part (i.e. Part II) should reveal. A variant of this belief is that the main bulk of AL is about offerring a cheap pseudo semantics to Logics as a substitute for intuitive, model theoretic semantics. Again, this is very far from being true.

otherwise it is impossible to define e.g. the function **Alg** associating a class $\mathbf{Alg}(\mathcal{L})$ of algebras to each logic $\mathcal{L}$).

(ii) With the rapidly growing variety of applications of logic (in diverse areas like computer science, linguistics, AI, law, physics, etc.) there is a growing number of new logics to be investigated. In this situation AL offers us a tool for economy and a tool for unification in various ways. One of these is that $\mathbf{Alg}(\mathcal{L})$ is always a class of algebras, therefore we can apply the same machinery, namely universal algebra, to study all the new logics. In other words, we bring all the various logics to a kind of "normal form" where they can be studied, compared, and even combined by uniform methods. Moreover, for most choices of $\mathcal{L}$, $\mathbf{Alg}(\mathcal{L})$ tends to appear in the same "area" of universal algebra, hence specialized powerful methods lend themselves to studying $\mathcal{L}$. There is a fairly well understood "map" available for the landscape of universal algebra. By using our algebraization process and equivalence theorems, we can project this "map" back to the (far less understood) landscape of possible logics.

In section 7, we will illustrate the above outlined "application of AL to logic" by using the AL-results of Part I, as follows. In Part I, we studied various distinguished classes of algebras, like e.g. RCA_n. Here, after studying the bridge ($\mathcal{L} \mapsto \mathbf{Alg}(\mathcal{L})$ etc.) between the world of logics and that of algebras, we look up those distinguished logics to which the distinguished algebras of Part I belong. E.g. we will find a certain logic $\mathbf{L}_n$ for which $\mathsf{RCA}_n = \mathbf{Alg}(\mathbf{L}_n)$. Then we will use results in Part I about RCA_n to establish properties of $\mathbf{L}_n$. For this, we will use the "equivalence theorems" established in section 6 (of Part II). Besides RCA_n and $\mathbf{L}_n$, a similar procedure will be applied to other distinguished classes of algebras (from Part I) and other distinguished logics.

The approach reported here is part of a broader, joint approach with W. Blok and D. Pigozzi outlined in [ABNPS]. The present part contains only a somewhat specialized version of that general approach, in order to suit the special needs of the present work. Besides [ABNPS], we refer to [BP89], [BP91], [Cz97], [P91], [FJ94], [FJ97], [PP], [CzP], as well as [ANSK], [HMTII, sections 5.6, 4.3], [AKNS], [NA], [M96], [H96], [Mi95] for the more general approach. The semantic aspect of this approach goes back to e.g. [AN75], [AGN77], [AS].

CHAPTER 4

GENERAL FRAMEWORK FOR STUDYING LOGICS

DEFINITION 4.1 (logic) *By a* logic $\mathcal{L}$ *we mean an ordered quadruple*

$$\mathcal{L} \stackrel{\text{def}}{=} \langle F,\ \vdash,\ M,\ \models\rangle,$$

where (i)–(iv) below hold.

(i) F *(called the set of* formulas *of* $\mathcal{L}$*) is a set of finite*[1] *sequences (called words) over some set* X *(called the alphabet of* $\mathcal{L}$*).*

(ii) $\vdash$ *(called the* provability relation *of* $\mathcal{L}$*) is a relation between sets of formulas and formulas, that is,* $\vdash\ \subseteq\ \mathcal{P}(F) \times F$*. Following tradition*[2]*, instead of "*$\langle \Sigma, \varphi\rangle \in \vdash$*" we write "*$\Sigma \vdash \varphi$*".*

(iii) M *is a class*[3] *(called the class of* models *of* $\mathcal{L}$*).*

(iv) $\models$ *(called the* validity relation*) is a relation between* M *and* F *that is,* $\models\ \subseteq M \times F$*. Instead of "*$\langle \mathfrak{M}, \varphi\rangle \in \models$*" we write "*$\mathfrak{M} \models \varphi$*".*

If $\mathcal{L}$ *is a logic, then by* $F_{\mathcal{L}}$*,* $\vdash_{\mathcal{L}}$*,* $M_{\mathcal{L}}$*,* $\models_{\mathcal{L}}$ *we denote its corresponding parts.*

■

Intuitively, F is the collection of "texts" or "sentences" or "formulas" that can be "said" in the language $\mathcal{L}$. For $\Gamma \subseteq F$ and $\varphi \in F$, the intuitive meaning of $\Gamma \vdash \varphi$ is that φ is provable (or derivable) from Γ with the syntactic inference system (or

[1] With this we exclude infinitary languages like $\mathcal{L}_{\kappa,\lambda}, \mathcal{L}^n_{\infty,\omega}$. This exclusion is not necessary, all the methods go through with some modifications. Actually, occasionally we will look into properties of the finite variable fragment $\mathcal{L}^n_{\infty,\omega}$ of infinitary logic, because it naturally admits applications of our methods and plays an essential role in finite model-theory and in theoretical computer science.

[2] This tradition is used for all binary relations: if R is a binary relation, then instead of $\langle a, b\rangle \in R$ we sometimes write $a\ R\ b$.

[3] Although it is not automatically permitted in the "most official" version of set theory (ZF), we *may* assume that for any four classes $M_1, \ldots, M_4$ the tuple $\langle M_1, \ldots, M_4\rangle$ exists and is again a class. This does not lead to set theoretical paradoxes. What one should avoid is assuming that the collection of *all* classes would (exist and) form a class again (and variants of this). For more on this cf. [HMT, p.34, first 10 lines], [HMT, p.25], [AHS, §2, pp.5-8], or almost any work on abstract model theory.

deductive mechanism) of $\mathcal{L}$. In all important cases, $\vdash$ is subject to certain (well-known) conditions like $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$ imply $\Gamma \vdash \psi$ for any $\Gamma \subseteq F$ and $\varphi, \psi \in F$. The class M of models is understood in the spirit of model theory: The models $\mathfrak{M} \in M$ of $\mathcal{L}$ are thought of as "possible environments" or "possible interpretations" or "possible worlds", cf. Allén [Al]. Here a possible world is *not* the same as the technical devices called possible worlds in a Kripke model. The validity relation tells us which texts are "true" in which possible environments (or worlds or models) under what conditions. Usually F and $\vdash$ are defined by what are called grammars in mathematical linguistics. $\langle F, \vdash\rangle$ together with the grammar defining them is called the *syntactical part* of $\mathcal{L}$, while $\langle M_{\mathcal{L}}, \models\rangle$ is the *semantical part or model theoretical part* of $\mathcal{L}$ [4].

As a binary relation between M and F, $\models$ induces a Galois-connection between M and F, and in particular, it defines two closure operators, one on M and one on F. Next, we collect some of the relevant definitions.

DEFINITION 4.2 (theory, models, consequences)

(i) Let $K \subseteq M$ and $\Sigma \subseteq F$. Then

$$K \models_{\mathcal{L}} \Sigma \quad \textit{iff} \quad (\forall \mathfrak{M} \in K)(\forall \varphi \in F)\mathfrak{M} \models_{\mathcal{L}} \varphi.$$

We will write $K \models \varphi$ in place of $K \models \{\varphi\}$ and similarly when $K = \{\mathfrak{M}\}$.
$\mathsf{Th}_{\mathcal{L}}(K) \overset{\text{def}}{=} \{\varphi \in F : K \models \varphi\}$*, the theory of K, and*
$\mathsf{Mod}_{\mathcal{L}}(\Sigma) \overset{\text{def}}{=} \{\mathfrak{M} \in M : \mathfrak{M} \models \Sigma\}$*, the class of models of Σ.*

(ii) Semantical consequence, valid formulas: Let $\Sigma \cup \{\varphi\} \subseteq F$. Then

$$\Sigma \overset{\bullet}{\models}_{\mathcal{L}} \varphi \quad \textit{iff} \quad \mathsf{Mod}(\Sigma) \models_{\mathcal{L}} \varphi,$$

φ is a semantical consequence of Σ.
$\models_{\mathcal{L}} \varphi$ iff $M \models_{\mathcal{L}} \varphi$. In this case we say that φ is a valid formula of $\mathcal{L}$.

[4]Cf. Sections 14, 15 of Gabbay [Gab3] for more intuitive motivation on how and why these parts are highlighted in a logic.

At this point a natural objection suggests itself: Why is $M_{\mathcal{L}}$ an arbitrary class? Why did we not assume (like in Barwise-Feferman [BF]) that $M_{\mathcal{L}}$ is a class of first order structures or of algebraic systems? The answer is (i)-(iii) below. (i) In institutions theory they do the same what we do and for the same reasons. Cf. the subsection "Connections with the literature" at the very end of the present section. (ii) We are developing a general theory, and we do not know in advance what kinds of structures will be the models of our $\mathcal{L}$. E.g. they may be classical first order models, they may or may not have a topological structure too, they may be propositional Kripke-models, they may have infinitary relations on them (cf. [HMT, §4.3]), they may be models of intensional logic in the sense of Montague etc. Therefore, at the very beginning, we do not want to commit ourselves on exactly what kinds of mathematical objects will the elements of $M_{\mathcal{L}}$ be. (iii) All the same, during the development of our theory, we will impose *some* structure on $M_{\mathcal{L}}$ (but only gradually). This structure-imposing process is carried even further in [ABNPS], cf. e.g. "concrete semantical systems" therein.

(iii) Axiomatizable classes of models:
$\mathsf{Mod}_{\mathcal{L}}\mathsf{Th}_{\mathcal{L}}(K)$ *is called the axiomatizable hull of* K. K *is axiomatizable iff* $K = \mathsf{Mod}_{\mathcal{L}}(\Sigma)$ *for some* $\Sigma \subseteq F$. *In this case we also say that* Σ *describes or defines* K.

(iv) Provability, or derivability:
$\vdash_{\mathcal{L}} \varphi$ *iff* $\emptyset \vdash_{\mathcal{L}} \varphi$, *in this case we say that* φ *is provable or derivable in* $\mathcal{L}$.
If $\Sigma \vdash_{\mathcal{L}} \varphi$, *then we say that* φ *is provable from* Σ *(in* $\mathcal{L}$*).*

If there is no danger of confusion, we will omit the subscript $\mathcal{L}$ *from* $\stackrel{\bullet}{\models}_{\mathcal{L}}$, $\mathsf{Th}_{\mathcal{L}}$, $\mathsf{Mod}_{\mathcal{L}}$ *etc.* ■

REMARK ThMod *and* ModTh *are the two closure operators induced by* $\models$. *The semantical consequence relation* $\stackrel{\bullet}{\models}$ *is a binary relation between* $\mathcal{P}(F)$ *and* F, *just like* $\vdash$ *is. To treat* $\vdash$ *and* $\stackrel{\bullet}{\models}$ *uniformly, in some places a logical system is defined to be* $\langle F, \models\!\equiv \rangle$ *where* $\models\!\equiv\; \subseteq \mathcal{P}(F) \times F$. *E.g. Blok-Pigozzi [BP89] uses this definition. In this notion,* $\models\!\equiv$ *can mean either the derivability relation or the semantical consequence relation* $\stackrel{\bullet}{\models}$. *Connections between our conception of a logic and the literature will be discussed at the end of this section.* ■

The definition of a logic in Def. 4.1 is very broad. Actually, it is too broad for proving interesting theorems about logics. Now we will define a subclass of logics which we will call *algebraizable semantical logics*. The notion of an algebraizable logic is broad enough to cover a very large part of the logics investigated in the literature [5]. On the other hand, the class of algebraizable logics is narrow enough for proving interesting theorems about such logics, that is, we will be able to establish typical logical facts that hold for most logics studied in the literature.

Below, in Definitions 4.3–4.10, we collect some common features of logics.

We will discuss the usual extra assumptions one usually makes about a logic $\mathcal{L}$ in the following order. First we discuss (assumptions on) the distinguished parts of $\mathcal{L}$ beginning with $F_{\mathcal{L}}$ and ending with $\models_{\mathcal{L}}$. Then in Def. 4.10, we will discuss (assumptions on) how these parts are put together. Often, what we call "extra assumptions" here will also imply "extra structure".

The set F of formulas is usually defined by fixing a set Cn of logical connectives and a set P of atomic formulas:

[5]Moreover, in Remark 4.11 we will indicate how to extend the methods of the present work from algebraizable logics to a broader class called protoalgebraic semantical logics. The latter is really broad enough to cover most logics in the literature.

DEFINITION 4.3 ($\mathcal{L}$ has connectives) *(i) Assume that two sets, P and Cn are given, such that every element of Cn has a finite rank. Then $F(P, Cn)$ denotes the smallest set H satisfying (1),(2) below:*

(1) $P \subseteq H$, and

(2) for every $\varphi_1, \ldots, \varphi_k \in H$ and $c \in Cn$ of rank k, $c(\varphi_1, \ldots, \varphi_k) \in H$.

Note that $F(P, Cn)$ is the universe of the word-algebra of type Cn generated by P.

(ii) We say that $F_{\mathcal{L}}$ is given by $\langle P, Cn\rangle$ if $F_{\mathcal{L}} = F(P, Cn)$. In this case we say that P is the set of atomic formulas *or* atomic propositions *of $\mathcal{L}$, and Cn is the set of* logical connectives *of $\mathcal{L}$. P is also called the vocabulary of $\mathcal{L}$* [6]. *The word-algebra generated by P and using the logical connectives of Cn as algebraic operations is denoted by $\mathfrak{F}$, and is called the* formula algebra *of $\mathcal{L}$. Note that $\mathfrak{F} = \langle F, c^{\mathfrak{F}}\rangle_{c\in Cn}$ where $c^{\mathfrak{F}}(\varphi_1, \ldots, \varphi_k) \stackrel{\text{def}}{=} c(\varphi_1, \ldots, \varphi_k) \in F$ for all $\varphi_1, \ldots, \varphi_k \in F$ and k-ary connective $c \in Cn$.*

(iii) We say that $\mathcal{L}$ has connectives *if $F_{\mathcal{L}}$ is given by $\langle P, Cn\rangle$ for some P, Cn. In this case usually we assume that $\langle P, Cn\rangle$ is given together with $\mathcal{L}$.* ■

Next we turn to *inference systems* $\vdash_{\mathcal{L}}$. Inference systems (usually denoted as $\vdash$) are syntactical devices serving to recapture (or at least to approximate) the semantical consequence relation of the logic $\mathcal{L}$. The idea is the following. Suppose $\Sigma \stackrel{\bullet}{\models} \varphi$. This means that, in the logic $\mathcal{L}$, the assumptions collected in Σ semantically imply the conclusion φ. (In any possible world or model $\mathfrak{M}$ of $\mathcal{L}$ whenever Σ is valid in $\mathfrak{M}$, then also φ is valid in $\mathfrak{M}$.) Then we would like to be able to reproduce this relationship between Σ and φ by purely syntactical, "finitistic" means. That is, by applying some formal rules of inference (and some axioms of the logic $\mathcal{L}$) we would like to be able to derive φ from Σ by using "paper and pencil" only. In particular, such a derivation will always be a finite string of symbols. If we can do this, that will be denoted by $\Sigma \vdash \varphi$.

Inference systems are usually given by *axioms* and *inference* rules. These axioms and rules use formula-schemes in place of concrete formulas. A formula-scheme is just like a formula, the only difference is that it is built up from formula-variables (i.e. metavariables ranging over formulas) in place of atomic formulas.

DEFINITION 4.4 (formula-scheme, Hilbert-style inference system) *(i) Assume that $F_{\mathcal{L}}$ is given by $\langle P, Cn\rangle$. We will call ϕ_i, $i < \omega$* formula-variables*, and FV will*

[6] Sometimes, informally, P is also called the set of "propositional variables". Here one should emphasize that the connotations of this name are misleading.

denote the set of all formula-variables, i.e. $FV = \{\phi_i : i < \omega\}$. *The elements of* $F(FV, Cn)$ *are called* formula-schemes*, and* Fs *will denote the set of formula-schemes of* $\mathcal{L}$. *I.e.* $Fs \stackrel{\text{def}}{=} F(FV, Cn)$. *An* instance *of a formula-scheme is obtained by substituting formulas for the formula variables in it. A formula-scheme is called* valid *if all of its instances are valid.*

(ii) An inference-rule for $\mathcal{L}$ *is a pair* $\big\langle\langle\Phi_1, \ldots, \Phi_k\rangle, \Phi_0\big\rangle$, *where every* Φ_i $(i \leqslant k)$ *is a formula scheme of* $\mathcal{L}$. *This inference rule will be denoted by*

$$\frac{\Phi_1, \ldots, \Phi_k}{\Phi_0}\,.$$

An instance of an inference rule *is obtained by substituting formulas for the formula variables in the formula schemes occurring in the rule. An inference rule* $\langle\langle\Phi_1, \ldots, \Phi_k\rangle, \Phi_0\rangle$ *is called* valid *if* $\{\varphi_1, \ldots, \varphi_k\} \stackrel{\bullet}{\models} \varphi_0$ *for all instances* $\langle\langle\varphi_1, \ldots, \varphi_k\rangle, \varphi_0\rangle$ *of it. Valid inference rules are also called* admissible rules *or strongly sound rules in the literature.*

(iii) A Hilbert-style inference system *(or* calculus*) for* $\mathcal{L}$ *is a pair* $\langle Ax, Ru\rangle$ *where* Ax *is a finite set of formula-schemes and* Ru *is a finite set of inference rules for* $\mathcal{L}$.

(iv) A Hilbert-style inference system $I = \langle Ax, Ru\rangle$ *defines a provability or derivability relation* $\vdash$ *as follows. Assume* $\Sigma \cup \{\varphi\} \subseteq F$. *We say that* φ *is* $\vdash$-derivable *(or* $\vdash$-provable*) from* Σ *iff there is a finite sequence* $\langle\varphi_1, \ldots, \varphi_n\rangle$ *of formulas (an* $\vdash$-proof of φ from Σ*) such that* φ_n *is* φ *and for every* $1 \leqslant i \leqslant n$

- $\varphi_i \in \Sigma$ *or*
- φ_i *is an instance of an axiom scheme (an* axiom *for short) of* I *or*
- *there are* $j_1, \ldots, j_k < i$, *and there is an inference rule of* I *such that* $\frac{\varphi_{j_1}, \ldots, \varphi_{j_k}}{\varphi_i}$ *is an instance of this rule.*

We write $\Sigma \vdash \varphi$ *if* φ *is* $\vdash$*-provable from* Σ. *Now* $\vdash = \{\langle\Sigma, \varphi\rangle : \Sigma \vdash \varphi\}$. *We say that* $\vdash$ is given by $\langle Ax, Ru\rangle$. *Throughout, we identify* I *with* $\vdash$, *e.g. we say that* φ *is an axiom of* $\vdash$. ■

Next, we turn to the semantics $M_{\mathcal{L}}$, $\models_{\mathcal{L}}$ of $\mathcal{L}$. Usually, validity of formulas in models, i.e. $\models_{\mathcal{L}}$ is defined indirectly by first defining something more basic, namely the *meanings* or denotations of formulas (and of other kinds of syntactic entities belonging to the language) in models. The idea is that the meaning of some syntactic entity (like a noun-phrase, or a sentence) need not always be a truth value. Therefore, first we define a so-called meaning function which to each syntactic entity φ and each model $\mathfrak{M}$ associates some semantic entity $mng(\varphi, \mathfrak{M})$ called the meaning of φ in $\mathfrak{M}$. After knowing what the syntactic entities mean in the models, one may be able to derive information about which sentences are true or valid in which models.

DEFINITION 4.5 (meaning function, $\mathcal{L}$ is compositional) *(i) Let mng be any function mapping $F \times M$ into a class; and let us call $mng(\varphi, \mathfrak{M})$* the meaning of φ in $\mathfrak{M}$. *For a fixed $\mathfrak{M} \in M$, the function $mng_{\mathfrak{M}}$ mapping F to the set of meanings* [7] *is defined by letting for all $\varphi \in F$*

$$mng_{\mathfrak{M}}(\varphi) \stackrel{\text{def}}{=} mng(\varphi, \mathfrak{M}).$$

We say that mng is a meaning-function *for $\mathcal{L}$, if validity of a formula depends only on its meaning, i.e. if $(*)$ below holds:*

$$(*) \qquad mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi) \implies [\mathfrak{M} \models \varphi \ \textit{ iff } \ \mathfrak{M} \models \psi].$$

(ii) Assume that $\mathcal{L}$ has connectives. We say that the meaning-function mng is compositional *if the meanings of formulas are built up from the meanings of their subformulas, i.e. if the condition below is satisfied for all $\varphi_i, \psi_i \in F$, $1 \leq i \leq k$ and k-ary connective $c \in Cn$:*

$$\bigwedge_{i=1}^{k} mng_{\mathfrak{M}}(\varphi_i) = mng_{\mathfrak{M}}(\psi_i) \implies$$
$$mng_{\mathfrak{M}}(c(\varphi_1, \ldots, \varphi_k)) = mng_{\mathfrak{M}}(c(\psi_1, \ldots, \psi_k)).$$

This condition says exactly that $mng_{\mathfrak{M}}$ is a homomorphism on the formula algebra. We say that $\mathcal{L}$ is compositional *if it has connectives and a compositional meaning-function (w.r.t. the connectives of $\mathcal{L}$). This property is traditionally called Frege's principle of compositionality.*

(iii) From now on, by a logic $\mathcal{L}$ we understand a logic with a meaning-function, i.e. $\mathcal{L} = \langle F \vdash, M, mng, \models \rangle$, where mng is a meaning-function for the rest of $\mathcal{L}$. ■

On the ingredients of a logic: Let $\mathcal{L} = \langle \ldots mng, \models \rangle$ be a logic. Then, using the terminology of Frege, Carnap, Montague as in [vBtM], mng represents the *intensional* (or denotational) aspects of semantics, while $\models$ represents the *extensional*

[7] We use the word "meaning" in the sense Frege used "intension" or "sense". It is important to emphasize that "meaning" is much more general than "extension" or truthvalue (though for some logics the two may coincide). Further denotation (e.g. in Partee [Pa]) can be identified with what we call "meaning". As an example, let $\mathcal{L}$ be Richard Montague's intensional logic. Then $mng(\varphi, \mathfrak{M})$ is exactly what Montague calls the intension of φ in $\mathfrak{M}$. See also [Pa, pp. 1–18] on meaning(s). It seems that we are using the word "meaning" in the same sense as Janssen [Ja, pp. 419–470] does. What we call the meaning function $mng_{\mathfrak{M}}$ is called a meaning assignment in [Ja, p. 423].

(or truth-value oriented) aspect of semantics, in $\mathcal{L}$. Therefore, those approaches to general logics in which mng is repressed, seem to be extensionally oriented (cf. e.g. institutions theory e.g. in Gabbay [Ga94, p. 359]), while the ones emphasizing mng (e.g. Andréka-Sain [AS], [HMTII], Epstein [E], Sain [S79]) seem to be intensions-oriented (or sense or denotations oriented).

In many logics we have a derived connective $\leftrightarrow$ and a formula denoted as $True$ which establish a strong connection between mng and $\models$, namely

(i) $mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi)$ iff $\mathfrak{M} \models \varphi \leftrightarrow \psi$, and

(ii) $\mathfrak{M} \models \varphi$ iff $mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(True)$.

In these logics there is a strong connection between $\mathsf{Th}(\mathfrak{M}) \subseteq F$ and the kernel of the meaning function $ker(mng_{\mathfrak{M}}) \subseteq F \times F$, namely the kernel of $mng_{\mathfrak{M}}$ and $\mathsf{Th}(\mathfrak{M})$ are recoverable from each other.[8] We will say that $\mathcal{L}$ has the *filter-property* iff there are derived connectives that generalize the above situation.

DEFINITION 4.6 ($\mathcal{L}$ has the filter-property) *(i) A k-ary* derived connective *is a formula-scheme $\Delta \in Fs$ using the formulavariables $\phi_0, \ldots, \phi_{k-1}$ only. If $\varphi_0, \ldots, \varphi_{k-1}$ are formulas, then $\Delta(\varphi_0, \ldots, \varphi_{k-1})$ denotes the instance of Δ when we replace $\phi_0, \ldots, \phi_{k-1}$ by $\varphi_0, \ldots, \varphi_{k-1}$ respectively.*[9]

(ii) We say that $\mathcal{L}$ has the filter-property *iff there are derived connectives $\varepsilon_0, \ldots, \varepsilon_{m-1}$ and $\delta_0, \ldots, \delta_{m-1}$ (unary) and $\Delta_0, \ldots, \Delta_{n-1}$ (binary) $(m, n \in \omega)$ of $\mathcal{L}$ with the following properties: For all $\varphi, \psi \in F$ and for all $\mathfrak{M} \in M$*

(1) $mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi) \iff (\forall i < n)[\mathfrak{M} \models \varphi \Delta_i \psi]$.

(2) $\mathfrak{M} \models \varphi \iff (\forall j < m)[mng_{\mathfrak{M}}(\varepsilon_j(\varphi)) = mng_{\mathfrak{M}}(\delta_j(\varphi))]$.

■

In the case of classical logic, we can choose the above derived connectives such that Δ is "$\leftrightarrow$", $\varepsilon(\varphi)$ is φ, and $\delta(\varphi)$ is "True".

Finally, most logics have some substitution-invariance properties. In this respect we usually expect a logic to be *substitutional* in the sense of Def. 4.7 below. If it is not, then we rather treat it as a "theory" of a substitutional logic. For such examples see section 7 (Examples) or [AKNS].

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[8] If f is a function, then $ker(f) = \{\langle a, b\rangle : f(a) = f(b)\}$.

[9] The algebraic counterpart of "derived connective" is "term function". If Δ is binary, then we will write $\varphi\Delta\psi$ in place of $\Delta(\varphi, \psi)$.

DEFINITION 4.7 ($\mathcal{L}$ has the substitution property) *(i) By a* substitution s *we understand a function* $s : P \to F$ *(we will "substitute"* $p \in P$ *with* $s(p) \in F$*). If* $\varphi \in F$*, then* $\varphi(\overline{p}/s(\overline{p}))$ *denotes the formula we obtain from* φ *after simultaneously substituting* $s(p)$ *for every occurrence of* p*, for all* $p \in P$ *in* φ*. In other words,* $\varphi(\overline{p}/s(\overline{p})) \stackrel{\text{def}}{=} \hat{s}(\varphi)$*, where* $\hat{s}$ *is the (unique) extension of* $s : P \to F$ *to a homomorphism* $\hat{s} : \mathfrak{F} \to \mathfrak{F}$*.* [10]

(ii) $\mathcal{L}$ *has the* (syntactic) substitution property *(or* $\mathcal{L}$ *is* substitutional*) iff for any formula* $\varphi \in F$ *and substitution* $s : P \to F$

$$\models \varphi \quad \textit{implies} \quad \models \varphi(\overline{p}/s(\overline{p})).$$

This means that a formula of $\mathcal{L}$ *is valid iff the corresponding formula scheme of* $\mathcal{L}$ *is valid (where we get the corresponding formula scheme by substituting atomic formulas* $p_i \in P$ *with formula variables* $\phi_i \in FV$*).*

(iii) $\mathcal{L}$ *has the* semantical substitution property *iff for any model* $\mathfrak{M} \in M$ *and substitution* $s : P \to F$ *there is another model* $\mathfrak{N} \in M$ *such that*

$$mng_{\mathfrak{N}}(p) = mng_{\mathfrak{M}}(s(p)) \quad \textit{for all } p \in P.$$

Intuitively, the model $\mathfrak{N}$ *is the substituted version of* $\mathfrak{M}$ *along* s*.* ■

The semantical substitution property says that the atomic formulas can have the meanings of any other formulas. (This statement will be made precise in Proposition 5.2.) Examples where we have and do not have this property are given in section 7. We state the next proposition without its simple proof.

PROPOSITION 4.8 *If a logic* $\mathcal{L}$ *has the filter-property and the semantical substitution property, then it has the syntactic substitution property, too.* ■

Thus, a *"fully-fledged"* logic $\mathcal{L} = \langle F, \vdash, M, mng, \models\rangle$ sometimes is given as $\mathcal{L} = \langle\langle P, Cn\rangle, \langle Ax, Ru\rangle, M, mng, \models\rangle$. Often, not all parts of a logic are given. Sometimes we have only $\langle F, \vdash\rangle$ and we are searching for a "semantics" $\langle M, mng, \models\rangle$ for it such that e.g. $\langle F, \vdash, M, mng, \models\rangle$ is complete.[11] Or, even more often, we have $\langle F, M, mng, \models\rangle$ and we are searching for a provability relation $\vdash$ such that $\langle F, \vdash, M, mng, \models\rangle$ would be complete. Sometimes $\langle F, \vdash\rangle$ is called a "*deductive logic" (or syntactic one), while* $\langle F, M, mng, \models\rangle$ *is called a* "semantic logic" (cf. e.g. [ABNPS]). (Though, one should keep in mind that $\langle F, \stackrel{\bullet}{\models}\rangle$ is a "deductive logic" in this sense.) From now on we will often omit some parts of a logic. Most

[10] Such a unique extension exists because $\mathfrak{F}$ is the word-algebra generated by P, i.e., in algebraic terms, it is freely generated by P.

[11] Completeness of a logic will be defined in Def. 6.1 in section 6.

often we will deal with $\langle F, M, mng, \models\rangle$ and we will say that $\mathcal{L} = \langle F, M, mng, \models\rangle$ is a logic or more carefully, a semantical logic. Most of the notions are meaningful for it, e.g. that $\mathcal{L}$ is compositional etc.

DEFINITION 4.9 (Algebraizable semantical logic, structural logic) *Let* $\mathcal{L} = \langle F, M, mng, \models\rangle$ *be a logic in the above sense.*

- *We say that* $\mathcal{L}$ *is* structural *if* $\mathcal{L}$ *is compositional and has the semantical substitution property.*
- *We say that* $\mathcal{L}$ *is an* algebraizable [12] *semantical logic if* $\mathcal{L}$ *is structural and has the filter-property.*

■

In most cases, the set P of atomic formulas is a parameter in the definition of the logic $\mathcal{L}$. Namely, P is a fixed but *arbitrary* set. So in a sense, $\mathcal{L}$ is a function of P, and we could write $\mathcal{L}^P$ (instead of $\mathcal{L}$) to make this explicit. Most often the choice of P has only limited influence on the behaviour of $\mathcal{L}$. However, we will have to remember that P is a freely chosen parameter because in certain investigations, the choice of P does influence the behaviour of $\mathcal{L}^P$.

DEFINITION 4.10 (General logic, algebraizable general logic)

(i) A general logic *is a function (or indexed family)*

$$\mathbf{L} \overset{\text{def}}{=} \langle \mathcal{L}^P \ : \ P \textit{ is a set}\rangle,$$

where for each set P, $\mathcal{L}^P = \langle F^P, M^P, mng^P, \models^P\rangle$ *is a logic in the above sense.*

(ii) We say that $\mathbf{L}$ has connectives *iff there is a set* Cn *of connectives such that for every set* P, Cn *is the set of connectives of* $\mathcal{L}^P$ *in the sense of Def. 4.3 and* P *is the set of atomic formulas* [13] *of* $\mathcal{L}^P$, *i.e.* $F^P = F(P, Cn)$ *for all* P. *Sometimes* P *is called the vocabulary of* $\mathcal{L}^P$.

(iii) $\mathbf{L}$ *is* compositional *if it has connectives and* $\mathcal{L}^P$ *is compositional for all* P.

(iv) $\mathbf{L}$ has the filter-property *iff there are derived connectives* $\varepsilon_0, \ldots, \varepsilon_{m-1}$, $\delta_0, \ldots, \delta_{m-1}$, $\Delta_0, \ldots, \Delta_{n-1}$ *(common for all possible choices of* P*) such that* $\mathcal{L}^P$ *has the filter-property with these, for all* P.

[12] The definition of algebraizability originates with Blok and Pigozzi, cf. e.g. [BP89].

[13] We are making simplifications now. It is not necessary to have $\mathcal{L}^P$ in $\mathbf{L}$ for all sets P. What is needed for our investigations is that for all cardinality κ there be a P such that the set of atomic formulas of $\mathcal{L}^P$ has cardinality at least κ.

(v) $\mathbf{L}$ *has the* substitution property *iff for all* P, Q*,* $s : P \to F^Q$*, and* $\varphi \in F^P$*,*

$$\models^P \varphi \quad \textit{implies} \quad \models^Q \varphi(\overline{p}/s(\overline{p})).$$

(vi) $\mathbf{L}$ *has the* semantical substitution property *iff for all sets* P, Q*,* $s : P \to F^Q$ *and* $\mathfrak{M} \in M^Q$ *there is* $\mathfrak{N} \in M^P$ *such that* $mng^Q_{\mathfrak{M}} \circ \hat{s} = mng^P_{\mathfrak{N}}$*.*

(vii) $\mathbf{L}$ *is an* algebraizable general logic *iff* $\mathbf{L}$ *is compositional, has the filter-property, and has both substitution properties,* $\mathbf{L}$ *is* structural *if it is compositional and has the semantical substitution property.*

(viii) The notions of a formula-scheme, valid formula-scheme, valid rule, and Hilbert-style inference system for a general logic are the obvious generalizations of their versions given for (non-general) logics $\mathcal{L}$*, cf. e.g. Def. 4.4.*

(ix) By a fully fledged general logic *we understand a function*

$$\mathbf{L} = \langle \mathcal{L}^P : P \quad \textit{is a set} \quad \rangle$$

such that for each set P*,* $\mathcal{L}^P = \langle F^P, \vdash^P, M^P, mng^P, \models^P \rangle$ *is a fully fledged logic i.e.* $\langle F, \vdash \rangle$ *is a deductive logic,* $\langle F, M, mng, \models \rangle$ *is a semantical logic, and* $\vdash^P \subseteq \stackrel{\bullet}{\models}^P$*. Items (ii)–(vii) above extend to the fully fledged case the natural way.* ■

REMARK 4.11 *We note that* $\mathbf{L}$ *is an algebraizable general logic iff* $\mathcal{L}^P$ *is an algebraizable semantical logic for all* P*, the connectives and the derived connectives for the filter-property are the same for all* P*, and the condition below holds for all* $P \subseteq Q$*:*

$$\left\{ mng^P_{\mathfrak{M}} \ : \ \mathfrak{M} \in M^P \right\} = \left\{ (mng^Q_{\mathfrak{M}}) \restriction F^P \ : \ \mathfrak{M} \in M^Q \right\}.$$

Intuitively, this condition says that $\mathcal{L}^P$ *is the natural restriction of* $\mathcal{L}^Q$*.* ■

REMARK 4.12 *The theories of semantical logics* $\mathcal{L}_s = \langle F, M, mng, \models \rangle$ *and deductive logics* $\mathcal{L}_d = \langle F, \vdash, \rangle$ *are best when developed in a parallel fashion, cf. e.g. [ABNPS]. Throughout this remark we assume that* $\mathcal{L}_s$ *is structural (cf. Def.4.9).*

We called $\mathcal{L}_s$ *algebraizable iff it has the filter property. An analogous definition for algebraizability of* $\mathcal{L}_d$ *was given in the papers of Blok and Pigozzi, cf. [BP89]. There are weaker properties of* $\mathcal{L}_d$ *studied in the Blok and Pigozzi papers which properties already enable one to apply (at least part of) the methodology of algebraic logic to the logics in question.*

Logics with these properties are called "protoalgebraic", "equivalential", and "weakly algebraizable".[14] *(There are other such properties in the literature, but the weakest one facilitating application of our methodology seems to be being protoalgebraic.) As we implied, the properties of being protoalgebraic and equivalential naturally extend to semantical logics.*

We call a semantical logic $\mathcal{L}_s$ protoalgebraic *iff there is a set* $\Delta(\varphi,\psi) = \{\Delta_i(\varphi,\psi) : i \in I\}$ *of derived connectives such that*

$$(*) \qquad \models \Delta(\varphi,\varphi) \quad \textit{and} \quad \mathfrak{M} \models \Delta(\varphi,\psi) \Rightarrow [\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M} \models \psi],$$

for all $\mathfrak{M} \in M$, *and* $\varphi, \psi \in F$.

$\mathcal{L}_s$ *is called* equivalential *iff there is* Δ *as in* $(*)$ *above, but such that Condition (1) in Def.4.6 holds for this* Δ *and* $\mathcal{L}_s$.

$\mathcal{L}_s$ *is called* weakly algebraizable *iff it is protoalgebraic and there are sets* $\varepsilon(x) = \{\varepsilon_i(x) : i \in I\}$ *and* $\delta(x) = \{\delta_i(x) : i \in I\}$ *as in Def.4.6(2).*

Clearly, if $\mathcal{L}_s$ *is both equivalential and weakly algebraizable, then* $\mathcal{L}_s$ *is infinitely algebraizable, where "infinitely" means that* $\Delta, \varepsilon, \delta$ *may be infinite sets of derived connectives. (If they are finite, then* $\mathcal{L}_s$ *is algebraizable.) To keep the rest of this discussion short, we concentrate on protoalgebraic and equivalential (but all what we say can be extended to weakly algebraizable, too).*

We note that if $\mathcal{L}_s$ *is protoalgebraic (or equivalential), then so is its deductive counterpart* $\langle F, \ \stackrel{\bullet}{\models}\ \rangle$ *in the sense of e.g. [CzP]. Moreover,* $\mathcal{L}_d$ *is protoalgebraic/equivalential in the sense of [CzP] iff there is a semantical logic* $\mathcal{L}_s$ *such that* $\mathcal{L}_d = \langle F, \stackrel{\bullet}{\models}_{\mathcal{L}_s}\rangle$ *and* $\mathcal{L}_s$ *is protoalgebraic/equivalential, respectively. For the method of proving such equivalence theorems we refer to Font-Jansana [FJ94] and [ABNPS]. We also note that*

$$\textit{protoalgebraic} \supsetneq \textit{equivalential} \supsetneq \textit{algebraizable}$$

for semantical logics (the same applies to deductive ones, too). The machinery (algebraization process, equivalence theorems etc) developed in the present work does extend to protoalgebraic and equivalential semantical logics (from algebraizable ones). Cf. e.g. Hoogland [H96], Hoogland-Madarász [HoM] for part of this extension. For brevity, in this work we present the above mentioned machinery for the case of algebraizable logics only, at the same time inviting the interested reader to extend this machinery to the protoalgebraic and/or equivalential cases, too. ■

Connections with the literature: What we call a fully fledged logic $\mathcal{L} = \langle F, \vdash, \ldots, \models\rangle$ was called[15] a "formalism" in Tarski-Givant [TG, p.16 (section 1.6)], an "axiomatic system with semantics $\langle A, \vdash, \mathsf{Mod}, \models\rangle$" in Aczel [Ac, p.265] (here

[14] Cf. Czelakowski-Pigozzi [CzP].

[15] A difference is that the intensional part mng is suppressed in most of the quoted works but it is not suppressed in e.g. Epstein [E].

A coincides with our F), and a "logic $\langle \text{Sign}, sen, Mod, \vdash, \models\rangle$" in Marti-Oliet–Meseguer [MOM]. More precisely, the latter corresponds to our fully fledged general logics (cf. the subitem below).

What we call a semantic logic was called a "semantical system" in [ABNPS], a "semantical system $\langle A, \mathsf{Mod}, \models\rangle$" in Aczel [Ac, p.265]; and our general logic $\mathbf{L} = \langle \mathcal{L}^P : \ldots\rangle$ corresponds to an institution $\langle \text{Sign}, sen, \mathsf{Mod}, \models\rangle$ in Marti-Oliet–Meseguer [MOM, p.358] (the latter will be elaborated below).

Connection with institutions: Institutions theory (e.g. [MOM]) emphasizes the category theoretic aspects of a general logic $\mathbf{L} = \langle \mathcal{L}^P : \ldots\rangle$ (which are downplayed here), and suppresses the intensional aspects represented by mng. To see that such a general logic is a category whose objects are the logics $\mathcal{L}^P$, let $\text{Sign} = \{P : P \text{ occurs as a set of atomic formulas in } \mathbf{L}\}$ be fixed[16]. (We know that according to our conventions, Sign $=$ "all sets", but let us abstract away from this, and just assume that Sign is a fixed proper class.) Let $P, P_1 \in$ Sign. Next we define what a logic morphism $h : \mathcal{L}^P \to \mathcal{L}^{P_1}$ is. A *logic morphism* is a pair $h = \langle f, \mu\rangle$ such that $f : \mathfrak{F}^P \to \mathfrak{F}^{P_1}$ is a homomorphism of the formula-algebras and $\mu : M^{P_1} \to M^P$ "makes everything commute", e.g. $\mathfrak{M} \models f(\varphi) \iff \mu(\mathfrak{M}) \models \varphi$.[17] Now, if $\langle f, \mu\rangle$ is such a logic morphism, then $(f \restriction P)$ is called a *signature morphism*. Let **Sign** be the category of these signature-morphisms (as arrows, and Sign itself is the class of objects).

Let **Log** be the category of logics $\mathcal{L}^P$ and logic morphisms $h = \langle f, \mu\rangle$ occurring in $\mathbf{L}$. Then there is a functor $\mathbf{F} : \mathbf{Sign} \to \mathbf{Log}$ sending P to $\mathcal{L}^P$. This $\mathbf{F}$ is almost the institution we are looking for. There is one ingredient missing, though. Namely, M^P is not only an arbitrary class, but is a category in all the applications we know of. This category character of M^P is de-emphasized in the present paper, but it does show up in the theory later, cf. e.g. the category of concrete semantical systems in [ABNPS]. So, let us assume that each M^P is a category. Then our functor $\mathbf{F}$ above induces two functors $fml : \mathbf{Sign} \to \mathbf{Fmla}$ and $\mathsf{Mod} : \mathbf{Sign} \to \mathbf{Cat}^{op}$, where **Fmla** is the category of formula-algebras (of the form $\mathfrak{F}^P$), and for each P,

$$\mathbf{F}(P) = \langle fml(P), \mathsf{Mod}(P), \models^P\rangle, \quad \text{while}$$

$$\mathbf{F}(\sigma) = \langle fml(\sigma), \mathsf{Mod}(\sigma)\rangle$$

for the morphisms σ of **Sign**. Now the tuple

$$I(\mathbf{L}) = \langle \mathbf{Sign}, fml, \mathsf{Mod}, \models\rangle$$

is called an *institution*, where

$$\models \ = \ \langle \models^P : P \in \text{Sign}\rangle.$$

[16] What we call here a set P of atomic formulas is called a signature in institutions theory.

[17] If meaning functions are also present, then μ should induce a function μ^+ on the meanings such that $mng_{\mu(\mathfrak{M})}(\varphi) = \mu^+(mng_{\mathfrak{M}}(f(\varphi)))$, cf. [ABNPS].

At this point one can see that to a general logic $\mathbf{L}$ there is an equivalent institution $I(\mathbf{L})$, and conversely to an institution $\mathbf{I}$ there is a general logic $L(\mathbf{I})$ such that the two can be recovered from each other. (Here we assumed that in $L(\mathbf{I})$ the models still form a category.) Therefore, institutions and general logics can be studied interchangeably, depending on the kind of mathematical tools (universal algebra or categories) one wants to use.[18]

In institutions theory, our "fml" is denoted by "sen", exactly because there mng is suppressed and therefore meanings are replaced by truthvalues. So, when the theory is applied to e.g. first-order logic, then attention has to be restricted to sentences (=closed formulas) because meanings of open formulas are more complex objects than just truthvalues.

We do not treat here the different notions of equivalence of logics, morphisms acting between logics, concrete semantical logics, cf. e.g. [ABNPS]. Also, we do not treat here interpretability between logics, and combining logics, [Gab], [Gab2], [JKE], [ABNPS]. These are important and very interesting subjects.

For the rest of this work, one of the most important definitions of section 4 is that of an algebraizable general logic. It is summarized in Remark4.11.

[18]To make this statement hundred percent true, one should include the intensional aspect mng into institutions and make M^P into a category in general logics. We do not see why one would not do these amendments.

CHAPTER 5

THE PROCESS OF ALGEBRAIZATION

The algebraic counterpart of classical sentential logic $\mathcal{L}_S$ is the variety BA of Boolean algebras. Why is this so important? The answer lies in the general experience that sometimes it is easier to solve a problem concerning $\mathcal{L}_S$ by translating it to BA, solving the algebraic problem, and then translating the result back to $\mathcal{L}_S$ (than solving it directly in $\mathcal{L}_S$).

In this section we extend applicability of BA to $\mathcal{L}_S$ to applicability of algebra in general to logics in general. We will introduce a standard translation method from logic to algebra, which to each logic $\mathcal{L}$ associates a class $\mathbf{Alg}(\mathcal{L})$ of algebras. (Of course, $\mathbf{Alg}(\mathcal{L}_S)$ will be BA.) Further, this translation method will tell us how to find the algebraic question corresponding to a logical question. If the logical question is about $\mathcal{L}$, then its algebraic equivalent will be about $\mathbf{Alg}(\mathcal{L})$. For example, if we want to decide whether $\mathcal{L}$ has the property called Craig's interpolation property, then it is sufficient to decide whether $\mathbf{Alg}(\mathcal{L})$ has the so called amalgamation property (for which there are powerful methods in the literature of algebra). If the logical question concerns connections between several logics, say between $\mathcal{L}_1$ and $\mathcal{L}_2$, then the algebraic question will be about connections between $\mathbf{Alg}(\mathcal{L}_1)$ and $\mathbf{Alg}(\mathcal{L}_2)$. (The latter are quite often simpler, hence easier to investigate.) This "bridge" also enables us to solve algebraic problems by logical methods (for an example see section 7).

DEFINITION 5.1 (meaning algebra, $\mathbf{Alg}_m$, $\mathbf{Alg}$) *Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a compositional logic with $F \neq \emptyset$.*

(i) First we turn every model into an algebra. Compositionality of $mng_{\mathfrak{M}}$ means that we can define an algebra of type Cn on the set $\{mng_{\mathfrak{M}}(\varphi) : \varphi \in F\}$ of meanings. This algebra is $mng_{\mathfrak{M}}(\mathfrak{F})$, it will be called the meaning algebra of $\mathfrak{M}$ *and it will be denoted by $\mathfrak{Mng}(\mathfrak{M})$. In more detail, to any logical connective c of arity k we can define a k-ary function $c^{\mathfrak{M}}$ on the meanings in $\mathfrak{M}$ by setting for all formulas $\varphi_1, \dots, \varphi_k$*

$$c^{\mathfrak{M}}(mng_{\mathfrak{M}}(\varphi_1), \dots, mng_{\mathfrak{M}}(\varphi_k)) \stackrel{\text{def}}{=} mng_{\mathfrak{M}}(c(\varphi_1, \dots, \varphi_k)).$$

(We could say that $c^{\mathfrak{M}}$ is the meaning of the logical connective c.) Then $\mathfrak{Mng}(\mathfrak{M}) \stackrel{\text{def}}{=} \langle \{mng_{\mathfrak{M}}(\varphi) : \varphi \in F\}, c^{\mathfrak{M}}\rangle_{c \in Cn}$.

(ii) $\mathbf{Alg}_m(\mathcal{L})$ *denotes the class of all meaning-algebras of* $\mathcal{L}$*, i.e.*

$$\mathbf{Alg}_m(\mathcal{L}) \stackrel{\text{def}}{=} \{mng_{\mathfrak{M}}(\mathfrak{F}) : \mathfrak{M} \in M_{\mathcal{L}}\} = \{\mathfrak{Mng}(\mathfrak{M}) : \mathfrak{M} \in M_{\mathcal{L}}\}.$$

(iii) Let $K \subseteq M_{\mathcal{L}}$*. Then for every* $\varphi, \psi \in F$

$$\varphi \sim_K \psi \stackrel{def}{\iff} (\forall \mathfrak{M} \in K)\, mng_{\mathfrak{M}}(\varphi) = mng_{\mathfrak{M}}(\psi).$$

Then $\sim_K$ *is an equivalence relation, which is a congruence on* $\mathfrak{F}$ *by compositionality of* $\mathcal{L}$*.* $\mathfrak{F}/{\sim_K}$ *denotes the factor-algebra of* $\mathfrak{F}$*, factorized by* $\sim_K$*. It is called the Lindenbaum-Tarski algebra of* K*. Now,*

$$\mathbf{Alg}(\mathcal{L}) \stackrel{\text{def}}{=} \mathbf{I}\,\{\mathfrak{F}/{\sim_K} :\ K \subseteq M_{\mathcal{L}}\}\,.$$

Thus, $\mathbf{Alg}(\mathcal{L})$ *is the class of isomorphic copies of the Lindenbaum-Tarski algebras of* $\mathcal{L}$*.*

(iv) Let $\mathbf{L} = \langle \mathcal{L}^P : P \textit{ is a set}\rangle$ *be a general logic. Then*

$$\mathbf{Alg}_m(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup\{\mathbf{Alg}_m(\mathcal{L}^P) : P \textit{ is a set}, F^P \neq \emptyset\},$$

and

$$\mathbf{Alg}(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup\{\mathbf{Alg}(\mathcal{L}^P) : P \textit{ is a set}, F^P \neq \emptyset\}.$$

■

REMARK *In the definition of* $\mathbf{Alg}_m(\mathcal{L})$ *above, it is important that* $\mathbf{Alg}_m(\mathcal{L})$ *is not an abstract class in the sense that it is not closed under isomorphisms. The reason for defining* $\mathbf{Alg}_m(\mathcal{L})$ *in such a way is that since* $\mathbf{Alg}_m(\mathcal{L})$ *is the class of algebraic counterparts of the* models *of* $\mathcal{L}$*, we need these algebras as concrete algebras and replacing them with their isomorphic copies would lead to loss of information (about semantic-model theoretic matters). See e.g. the algebraic characterization of the* weak Beth definability property*, Theorem 6.12 in the next section.* ■

For a logic $\mathcal{L}$, let $Mng_{\mathcal{L}} \stackrel{\text{def}}{=} \{mng_{\mathfrak{M}} : \mathfrak{M} \in M_{\mathcal{L}}\}$. That is, $Mng_{\mathcal{L}}$ is the class of "meaning-homomorphisms" of the logic $\mathcal{L}$ (or equivalently, the unary meaning-functions induced by the models of $\mathcal{L}$). If $\mathfrak{A}$ is an algebra and K is a class of algebras, then $Hom(\mathfrak{A}, \mathsf{K})$ denotes the class of all homomorphisms $h : \mathfrak{A} \to \mathfrak{B}$ where $\mathfrak{B} \in \mathsf{K}$.

PROPOSITION 5.2 (characterization of structural logics) *Let $\mathcal{L}$ and $\mathbf{L}$ be a compositional logic and a compositional general logic, respectively. Then (i)–(ii) below hold.*

(i) *$\mathcal{L}$ has the semantic substitution property iff*
$Mng_{\mathcal{L}} = Hom(\mathfrak{F}, \mathbf{Alg}_m(\mathcal{L}))$, *iff*
$Mng_{\mathcal{L}} = Hom(\mathfrak{F}, \mathsf{K})$ *for some class* K *of algebras.*

(ii) *$\mathbf{L}$ has the semantic substitution property iff*
$Mng^P = Hom(\mathfrak{F}^P, \mathbf{Alg}_m(\mathbf{L}))$, *for all P, iff*
$Mng^P = Hom(\mathfrak{F}^P, \mathsf{K})$, *for all P, for some* K.

Proof. To prove the first equivalence in (i), assume that $\mathcal{L}$ has the semantic substitution property, and $h : \mathfrak{F} \to mng_{\mathfrak{M}}(\mathfrak{F})$ for some $\mathfrak{M} \in M_{\mathcal{L}}$. We want to show that $h = mng_{\mathfrak{N}}$ for some $\mathfrak{N} \in M_{\mathcal{L}}$. For each $p \in P$ let $\varphi_p \in F$ be such that $h(p) = mng_{\mathfrak{M}}(\varphi_p)$, and let $s : P \to F$ be defined such that $s(p) = \varphi_p$ for all $p \in P$. By the semantical substitution property of $\mathcal{L}$, there is $\mathfrak{N} \in M_{\mathcal{L}}$ such that for all φ, $mng_{\mathfrak{N}}(\varphi) = mng_{\mathfrak{M}}(\hat{s}\varphi)$. Then for all $p \in P$, $mng_{\mathfrak{N}}(p) = mng_{\mathfrak{M}}(s(p)) = h(p)$, i.e. $mng_{\mathfrak{N}}$ and h agree on P. Since $\mathfrak{F}$ is generated by P, this implies that $h = mng_{\mathfrak{N}}$. The other direction of the first part of (i) is trivial: Let $\mathfrak{M} \in M_{\mathcal{L}}$ and $s : P \to F$. We have to show that $mng_{\mathfrak{M}} \circ \hat{s} \in Mng$, which is true by $mng_{\mathfrak{M}} \circ \hat{s} : \mathfrak{F} \to mng_{\mathfrak{M}}(\mathfrak{F}) \in \mathbf{Alg}_m(\mathcal{L})$. To prove the equivalence of the second and third statements in (i), assume that $Mng_{\mathcal{L}} = Hom(\mathfrak{F}, \mathsf{K})$. We want to show that $Mng_{\mathcal{L}} = Hom(\mathfrak{F}, \mathbf{Alg}_m(\mathcal{L}))$. Notice first that $Mng_{\mathcal{L}} = Hom(\mathfrak{F}, \mathsf{K})$ implies that $\mathbf{Alg}_m(\mathcal{L}) \subseteq \mathsf{SK}$. So let $h : \mathfrak{F} \to \mathfrak{A}, \mathfrak{A} \in \mathbf{Alg}_m(\mathcal{L})$. Then $h : \mathfrak{F} \to \mathfrak{B}$ for some $\mathfrak{B} \in \mathsf{K}$, by $\mathbf{Alg}_m(\mathcal{L}) \subseteq \mathsf{SK}$. Thus $h \in Mng_{\mathcal{L}}$ by $Mng_{\mathcal{L}} = Hom(\mathfrak{F}, \mathsf{K})$.

The proof of (ii) is completely analogous, we omit it. ■

THEOREM 5.3 (connection between $\mathbf{Alg}_m$ and $\mathbf{Alg}$)

(i) *Let $\mathcal{L}$ be a compositional logic. Then*

$$\mathbf{SPAlg}(\mathcal{L}) = \mathbf{SPAlg}_m(\mathcal{L}).$$

(ii) *Let $\mathbf{L}$ be a structural general logic. Then*

$$\mathbf{Alg}(\mathbf{L}) = \mathbf{SPAlg}_m(\mathbf{L}).$$

Proof. Proof of (i): First we show $\mathbf{Alg}_m(\mathcal{L}) \subseteq \mathbf{IAlg}(\mathcal{L})$. Let $\mathfrak{A} \in \mathbf{Alg}_m(\mathcal{L})$, say $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$. Set $K = \{\mathfrak{M}\}$. Then $\sim_K = ker(mng_{\mathfrak{M}})$, so $\mathfrak{F}/\sim_K$ is isomorphic to $\mathfrak{A}$. To show the other direction, let $\mathfrak{A} \in \mathbf{Alg}(\mathcal{L})$, say $\mathfrak{A} = \mathfrak{F}/\sim_K$ for some $K \subseteq M_{\mathcal{L}}$. Then there is a subset $K' \subseteq K$ such that $\mathfrak{F}/\sim_K = \mathfrak{F}/\sim_{K'}$ (this holds because F is a set). Thus we may assume that $K \subseteq M_{\mathcal{L}}$ is a set. We define for all $\varphi \in F$

$$h\,(\varphi/{\sim_K}) \stackrel{\text{def}}{=} \langle mng_{\mathfrak{M}}(\varphi) \ : \ \mathfrak{M} \in K\rangle.$$

This is a sound definition by the definition of $\sim_K$. It is not difficult to check that h is one-to-one and a homomorphism, so $h : \mathfrak{A} \rightarrowtail \mathsf{P}\langle mng_{\mathfrak{M}}(\mathfrak{F}) \ : \ \mathfrak{M} \in K\rangle$, showing that $\mathfrak{A} \in \mathbf{SPAlg}_m(\mathcal{L})$. Since $\mathbf{SP}$ is a closure operator, we are done with proving (i).

Proof of (ii): First we note that, by (i), $\mathbf{Alg}(\mathcal{L}^P) \subseteq \mathbf{SPAlg}_m(\mathcal{L}^P)$ for any set P, thus $\mathbf{Alg}(\mathbf{L}) \subseteq \mathbf{SPAlg}_m(\mathbf{L})$ holds.

We are going to prove $\mathbf{SPAlg}_m(\mathbf{L}) \subseteq \mathbf{Alg}(\mathbf{L})$. Let $\mathfrak{A} \in \mathbf{SPAlg}_m(\mathcal{L})$, say $\mathfrak{A} \subseteq \mathsf{P}_{i\in I}\mathfrak{A}_i$ for a set I and algebras $\mathfrak{A}_i \in \mathbf{Alg}_m(\mathbf{L})$. Let $h : \mathfrak{F}^A \twoheadrightarrow \mathfrak{A}$ be any onto homomorphism (e.g. we can take for h the homomorphic extension of the identity mapping $h' : A \to A$). For each $i \in I$ let π_i denote the projection function onto $\mathfrak{A}_i$, and let $h_i \stackrel{\text{def}}{=} \pi_i \circ h$. Then $h_i : \mathfrak{F}^A \to \mathfrak{A}_i \in \mathbf{Alg}_m(\mathbf{L})$. By Proposition 5.2 (ii) then $h_i = mng_{\mathfrak{M}_i}$ for some $\mathfrak{M}_i \in M^A$. Let $K = \{\mathfrak{M}_i : i \in I\}$. Then it is easy to check that $h(\varphi) = h(\psi)$ iff $\varphi \sim_K \psi$ for all $\varphi, \psi \in F^A$. Thus $\mathfrak{A}$ is isomorphic to $\mathfrak{F}^A/{\sim_K} \in \mathbf{Alg}(\mathbf{L})$, and we are done. ∎

We note that we also proved that for structural logics $\mathcal{L}$,

$$\mathbf{Alg}(\mathcal{L}) = \mathbf{SPAlg}_m(\mathcal{L}) \cap \{\mathfrak{A} : |A| \leq |F_{\mathcal{L}}|\}.$$

Now we turn to proving that the equations valid in $\mathbf{Alg}(\mathcal{L})$ correspond to the valid formula-schemes of $\mathcal{L}$, and the quasi-equations valid in $\mathbf{Alg}(\mathcal{L})$ correspond to the valid rules of $\mathcal{L}$. Here we will use the filter-property. If $\mathcal{L}$ is algebraizable, then the equational and quasi-equational theories of $\mathbf{Alg}(\mathcal{L})$ recapture the validities and the semantical consequence relation $\stackrel{\bullet}{\models}$ of $\mathcal{L}$, respectively. Thus, when a logic $\mathcal{L}$ is given, it is interesting to investigate the equational and quasi-equational theories of $\mathbf{Alg}(\mathcal{L})$. Note that by Theorem 5.3 above, $\mathbf{Alg}(\mathcal{L})$ and $\mathbf{Alg}_m(\mathcal{L})$ have the same equational and quasi-equational theories.

First we note that formulas and formula-schemes are terms in the language of $\mathbf{Alg}(\mathcal{L})$. Hence if $\varphi, \psi \in Fs$, then $\varphi = \psi$ is an equation in the language of $\mathbf{Alg}(\mathcal{L})$ where we consider the formula-variables ϕ_i as algebraic variables (ranging over the elements of the algebras). Similarly, if $\varphi, \psi \in F$, then $\varphi = \psi$ is also an equation in the language of $\mathbf{Alg}(\mathcal{L})$, where we consider the elements of P as algebraic variables.

THEOREM 5.4 (valid rules of $\mathcal{L}$ and quasi-equations of $\mathbf{Alg}(\mathcal{L})$) *Let $\mathcal{L}$ be a compositional logic with filter-property. Let $\varepsilon, \delta, \Delta, m, n$ be as in the definition of filter-property. Then (i)–(ii) below hold.*

(i) A rule $\langle\langle\varphi_0,\ldots,\varphi_{k-1}\rangle,\varphi_k\rangle$ of $\mathcal{L}$ is valid (or admissible) iff

$$\mathbf{Alg}(\mathcal{L}) \models \bigwedge_{\substack{\ell<k\\ i<m}} \varepsilon_i(\varphi_\ell) = \delta_i(\varphi_\ell) \to \varepsilon_j(\varphi_k) = \delta_j(\varphi_k) \quad \textit{for each } j < n.$$

(ii) A quasi-equation $\varphi_0 = \psi_0 \wedge \ldots \wedge \varphi_{k-1} = \psi_{k-1} \to \varphi_k = \psi_k$ (with variables from FV) is valid in $\mathbf{Alg}(\mathcal{L})$ iff

the rules

$$\frac{\varphi_0\Delta_0\psi_0,\ldots,\varphi_0\Delta_{n-1}\psi_0,\ \ldots,\ \varphi_{k-1}\Delta_0\psi_{k-1},\ldots,\varphi_{k-1}\Delta_{n-1}\psi_{k-1}}{\varphi_k\Delta_j\psi_k}$$

are valid in $\mathcal{L}$ for all $j < n$.

Proof. Assume that ρ is a valid rule of $\mathcal{L}$ of the form $\langle\langle\varphi_0,\ldots,\varphi_{k-1}\rangle,\varphi_k\rangle$. Let qeq_j denote the quasi-equation associated to it in (i). We want to show that qeq_j is valid in $\mathbf{Alg}(\mathcal{L})$. By Theorem 5.3 it is enough to prove that it is valid in $\mathbf{Alg}_m(\mathcal{L})$. Let $\mathfrak{A} \in \mathbf{Alg}_m(\mathcal{L})$, and let $h : FV \to A$ be an evaluation of the variables in qeq_j such that the hypothesis part of qeq_j is true in $\mathfrak{A}$ under the evaluation h, i.e. assume that

$$(\star) \qquad \mathfrak{A} \models \bigwedge_{\substack{\ell<k\\ i<m}} \varepsilon_i(\varphi_\ell) = \delta_i(\varphi_\ell)[h].$$

We want to show

$$\mathfrak{A} \models \varepsilon_j(\varphi_k) = \delta_j(\varphi_k)[h].$$

By $\mathfrak{A} \in \mathbf{Alg}_m(\mathcal{L})$, there is $\mathfrak{M} \in M_\mathcal{L}$ such that $\mathfrak{A} = mng_\mathfrak{M}(\mathfrak{F})$. For any $\phi_i \in FV$ take $\psi_i \in F$ such that $h(\phi_i) = mng_\mathfrak{M}(\psi_i)$ and let $\langle\langle\varphi'_0,\ldots,\varphi'_{k-1}\rangle,\varphi'_k\rangle$ be the instance of our rule ρ by replacing each ϕ_i with ψ_i. Then for each $i < m$ and $\ell \le k$ we have that $\hat{h}(\varepsilon_i(\varphi_\ell)) = mng_\mathfrak{M}(\varepsilon_i(\varphi'_\ell))$ and the same for δ, i.e. $\hat{h}(\delta_i(\varphi_\ell)) = mng_\mathfrak{M}(\delta_i(\varphi'_\ell))$. (Here $\hat{h}$ denotes the homomorphic extension of h to Fs.) Then by the filter-property of $\mathcal{L}$, and by our assumption $(\star)$, we have $\mathfrak{M} \models_\mathcal{L} \varphi'_\ell$ for all $\ell < k$. Since $\langle\langle\varphi_0,\ldots,\varphi_{k-1}\rangle,\varphi_k\rangle$ is a valid rule, and $\langle\langle\varphi'_0,\ldots,\varphi'_{k-1}\rangle,\varphi'_k\rangle$ is an instance of it, this implies $\mathfrak{M} \models_\mathcal{L} \varphi'_{k-1}$. Then by the filter-property again, $mng_\mathfrak{M}(\varepsilon_j(\varphi'_k)) = mng_\mathfrak{M}(\delta_j(\varphi'_k))$, i.e. $\hat{h}(\varphi_k)) = \hat{h}(\delta_j(\varphi_k))$ and we are done.

Conversely, assume that the quasi-equation is valid in $\mathbf{Alg}(\mathcal{L})$, and we want to show that the rule is valid. Let $\langle\langle\varphi'_0,\ldots,\varphi'_{k-1}\rangle,\varphi'_k\rangle$ be an instance of the rule that we got by substituting ψ_i to the formulavariables ϕ_i, for all $i < \omega$. Assume $\mathfrak{M} \in M_\mathcal{L}$ and $\mathfrak{M} \models_\mathcal{L} \{\varphi'_0,\ldots,\varphi'_{k-1}\}$. We want to show $\mathfrak{M} \models_\mathcal{L} \varphi'_k$. By the filter-property we have $mng_\mathfrak{M}(\varepsilon_i(\varphi'_\ell)) = mng_\mathfrak{M}(\delta_i(\varphi'_\ell))$. Let $h : Fs \to F$ be a homomorphism such that $h(\phi_i) = \psi_i$ for all $i < \omega$. Then $h(\varepsilon_i(\varphi_\ell)) =$

$mng_{\mathfrak{M}}(\varepsilon_i(\varphi'_\ell))$ and the same for δ_i, thus $(\star)$ above holds with $\mathfrak{A} = mng_{\mathfrak{M}}(\mathfrak{F})$. Thus $\mathfrak{A} \models \varepsilon_j(\varphi_k) = \delta_j(\varphi_k)[h]$, since the quasi-equation is valid in $\mathfrak{A} \in \mathbf{Alg}(\mathcal{L})$, i.e. $mng_{\mathfrak{M}}(\varepsilon_j(\varphi'_k)) = mng_{\mathfrak{M}}(\delta_j(\varphi'_k))$, for all $j < m$. By the filter-property then $\mathfrak{M} \models_{\mathcal{L}} \varphi'_k$ as was to be shown.

We omit the proof of (ii). It is analogous to the above proof of (i). ■

COROLLARY 5.5 *(*Valid formula-schemes, validities, and $\mathbf{Eq}(\mathbf{Alg}(\mathcal{L}))$)

(i) Let $\mathcal{L}$ be a compositional logic with filter-property. Let $\varepsilon, \delta, \Delta, m, n$ be as in the definition of the filter-property. Then for every formula-scheme φ of $\mathcal{L}$

φ *is a valid formula-scheme of* $\mathcal{L}$ *iff*

$\mathbf{Alg}(\mathcal{L}) \models \varepsilon_j(\varphi) = \delta_j(\varphi)$ *for all* $j < m$.

(ii) Assume further that $\mathcal{L}$ is algebraizable. Then for any formulas $\varphi, \varphi_0, \ldots, \varphi_k$ of $\mathcal{L}$,

$\models_{\mathcal{L}} \varphi$ *iff*

$\mathbf{Alg}(\mathcal{L}) \models \varepsilon_j(\varphi) = \delta_j(\varphi)$ *for each* $j < m$.

$\{\varphi_0, \ldots, \varphi_{k-1}\} \overset{\bullet}{\models}_{\mathcal{L}} \varphi_k$ *iff*

$\mathbf{Alg}(\mathcal{L}) \models \bigwedge_{\substack{\ell < k \\ i < m}} \varepsilon_i(\varphi_\ell) = \delta_i(\varphi_\ell) \to \varepsilon_j(\varphi_k) = \delta_j(\varphi_k)$ *for each* $j < m$.

(iii) The set of valid formula-shemes of $\mathcal{L}$ is decidable (recursively enumerable) iff $\mathbf{Eq}(\mathbf{Alg}(\mathcal{L}))$ *is decidable (recursively enumerable). The set of valid (admissible) rules of $\mathcal{L}$ is decidable (recursively enumerable) iff the quasi-equational theory of* $\mathbf{Alg}(\mathcal{L})$ *is decidable (recursively enumerable).*

(iv) Statements (i) and (ii) above hold for general logics $\mathbf{L}$ *in place of* $\mathcal{L}$. ■

We say that the *validity problem* of the logic $\mathcal{L}$ is *decidable* iff the set of valid formulas of $\mathcal{L}$ is decidable. If $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set} \rangle$ is a general logic, then the validity problem of $\mathbf{L}$ is decidable iff it is decidable for $\mathcal{L}^P$, for all P.

COROLLARY 5.6 *Let $\mathcal{L}$ be an algebraizable logic with $|P| \geq \omega$ or an algebraizable general logic. Then the validity problem of $\mathcal{L}$ is decidable iff* $\mathbf{Eq}(\mathbf{Alg}(\mathcal{L}))$ *is decidable.* ■

CHAPTER 6

EQUIVALENCE THEOREMS

In this part we give algebraic characterizations of some logical properties. In the next section we will apply these theorems to some well known logics. Instead of giving the proofs of the theorems in this section, we will refer to where they can be found. First we characterize completeness and compactness properties.

DEFINITION 6.1 (complete, sound inference systems) *Let $\mathcal{L} = \langle F, M, mng, \models\rangle$ be a logic, and let* $\vdash \subseteq \mathcal{P}(F) \times F$.

- $\vdash$ *is* weakly complete *for $\mathcal{L}$ iff each valid formula is derivable, i.e. iff*
$$(\forall \varphi \in F)\,(\models \varphi \implies \vdash \varphi)\,.$$
- $\vdash$ *is* finitely complete *for $\mathcal{L}$ iff consequences of finite sets are derivable, i.e. iff* [1]
$$(\forall \Sigma \subseteq_\omega F)(\forall \varphi \in F)\left(\Sigma \overset{\bullet}{\models} \varphi \implies \Sigma \vdash \varphi\right).$$
- $\vdash$ *is* strongly complete *for $\mathcal{L}$ iff any semantical consequence is derivable (not only consequences of finite sets), i.e. iff*
$$(\forall \Sigma \subseteq F)(\forall \varphi \in F)\left(\Sigma \overset{\bullet}{\models} \varphi \implies \Sigma \vdash \varphi\right).$$
- $\vdash$ *is* weakly sound *for $\mathcal{L}$ iff derivable formulas are valid, i.e. iff*
$$(\forall \varphi \in F)\,(\vdash \varphi \implies \models \varphi)\,.$$
- $\vdash$ *is* strongly sound *for $\mathcal{L}$ iff derivable consequences are valid consequences, i.e. iff*
$$(\forall \Sigma \subseteq F)(\forall \varphi \in F)\left(\Sigma \vdash \varphi \implies \Sigma \overset{\bullet}{\models} \varphi\right).$$

[1] $X \subseteq_\omega Y$ denotes that X is a finite subset of Y.

- $\vdash$ *is* strongly complete and sound *for* $\mathcal{L}$ *iff the derivability and semantic consequence relations coincide, i.e. iff*

$$(\forall \Sigma \subseteq F)(\forall \varphi \in F)\left(\Sigma \vdash \varphi \text{ iff } \Sigma \stackrel{\bullet}{\models} \varphi\right).$$

Let $\mathcal{I} = \langle Ax, Ru \rangle$ be a Hilbert-style inference system. We say that $\mathcal{I}$ is (weakly, finitely, strongly) complete for $\mathcal{L}$ if the derivability relation $\vdash$ given by $\langle Ax, Ru \rangle$ is such for $\mathcal{L}$. We say that $\mathcal{I}$ is (weakly, finitely, strongly) complete for a general logic $\mathbf{L} = \langle \mathcal{L}^P : P$ is a set$\rangle$ if $\mathcal{I}$ is such for all $\mathcal{L}^P$. We use an analogous terminology for the soundness properties. ■

The next theorem is a characterization of existence of strongly complete and sound Hilbert-style inference systems for general logics. It is proved in [AKNS, Thm.3.2.21]. A class K is a *quasi-variety* iff it can be axiomatized with a set of *quasi-equations*, i.e. equational implications. See a footnote in section 1.

THEOREM 6.2 *Assume that $\mathbf{L}$ is an algebraizable general logic. Then there is a strongly complete and sound Hilbert-style inference system for $\mathbf{L}$ iff $\mathbf{Alg}(\mathbf{L})$ is a finitely axiomatizable quasi-variety.* ■

In Theorem 6.3 below we give a sufficient and necessary condition for an algebraizable semantic logic to have a finitely complete Hilbert-style inference system. Its proof can be found in [AKNS, Thm.3.2.3].

THEOREM 6.3 *Assume that $\mathcal{L}$ is an algebraizable semantic logic and $Cn(\mathcal{L})$ is finite*[2]*. Then there is a finitely complete and strongly sound Hilbert-style inference system for $\mathcal{L}$ iff $\mathbf{Alg}(\mathcal{L})$ generates a finitely axiomatizable quasi-variety.* ■

The following theorem is a characterization of existence of weakly complete and strongly sound Hilbert-style inference systems. It is [AKNS, Thm.3.2.4].

THEOREM 6.4 *Assume that $\mathcal{L}$ is an algebraizable semantic logic and $Cn(\mathcal{L})$ is finite. Then there is a weakly complete and strongly sound Hilbert-style inference system for $\mathcal{L}$ iff there is a finitely axiomatizable quasi-variety K such that $\mathbf{Alg}_m(\mathcal{L}) \subseteq \mathsf{K} \subseteq \mathbf{HSPAlg}(\mathcal{L})$. The same is true for algebraizable general logics.* ■

[2]One can eliminate the assumption of $Cn(\mathcal{L})$ being finite. Then the finitary character of a Hilbert-style inference system has to be ensured in a more subtle way. Also, "finitely axiomatizable quasi-variety" must be replaced by "finite-schema axiomatizable quasi-variety" in the second clause, cf. e.g. Monk [M69], or Németi [N91].

The following theorem, due to J. Madarász, is a characterization of existence of weakly complete and weakly sound Hilbert-style inference systems. Such an inference-system is sometimes called a Gabbay-style inference system. Its rules are not necessarily valid, but the formula-schemes they derive (from the empty set $\Sigma = \emptyset$ of premises) should be valid. [3]

THEOREM 6.5 *Assume that* $\mathbf{L}$ *is an algebraizable general logic and* $Cn(\mathbf{L})$ *is finite. Assume that* $\mathbf{HAlg}(\mathbf{L}) \models \overline{\varepsilon}(x\overline{\Delta}y) = \overline{\delta}(x\overline{\Delta}y) \;\to\; x = y$. *Then there is a weakly complete and weakly sound Hilbert-style inference system for* $\mathbf{L}$ *iff there is a finitely axiomatizable quasi-variety* K *such that* $\mathbf{H}\mathsf{K} = \mathbf{HAlg}(\mathbf{L})$ [4]. ■

DEFINITION 6.6 (compactness)

- $\mathcal{L}$ *is* satisfiability compact *iff a set* Γ *has a model whenever all of its finite subsets* Σ *have models, i.e.*

$$(\forall\Sigma \subseteq_\omega \Gamma)\mathsf{Mod}(\Sigma) \neq \emptyset \implies \mathsf{Mod}(\Gamma) \neq \emptyset.$$

- $\mathcal{L}$ *is* consequence compact *if a consequence of* Γ *is always a consequence of a finite subset* Σ *already, i.e. iff*

$$\Gamma \overset{\bullet}{\models} \varphi \implies (\exists\Sigma \subseteq_\omega \Gamma)\Sigma \overset{\bullet}{\models} \varphi.$$

A general logic $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set}\rangle$ *is (satisfiability, consequence) compact if* $\mathcal{L}^P$ *is such for all* P. ■

We note that in general, consequence compactness and satisfiability compactness are independent properties, i.e. neither of them implies the other. However, for algebraizable general logics, consequence compactness implies satisfiability

[3] Cf. e.g. [Mi95], [MV], [Si91]. We note that many Gabbay-style rules are even more "liberal" than not being strongly sound in that in addition their form does not satisfy Def. 4.4 (ii). (It is not known yet which of these two liberties is responsible for their behaviour. Our feeling is that non-strongly soundness is the more essential.) These extremely liberal (Gabbay-style) inference systems correspond to classes $\mathsf{K} \subseteq \mathbf{Alg}(\mathbf{L})$ such that K is finitely axiomatized by $\forall\exists$-formulas (of a certain form) and $\mathbf{HS}\mathsf{K} = \mathbf{HAlg}(\mathbf{L})$. An example for such K is the class of rectangularly dense cylindric algebras, the representation theorem of which ([AGMNS]) was used in [Mi95] for obtaining a weakly-sound completeness theorem for $\mathbf{L}_n$ (which will be defined in section 7). Cf. also the open problem below [Mi95, Thm.1.3.11, p.27].

[4] Here, $x\overline{\Delta}y = \{x\Delta_i y : i < n\}$ and for a set H of formulas, $\overline{\varepsilon}(H) = \overline{\delta}(H)$ denotes $\{\varepsilon_i(\varphi) = \delta_i(\varphi) : i < m, \varphi \in H\}$. We do not know whether the condition "$\mathbf{HAlg}(L) \models \ldots$" is needed for this theorem. It is not needed for direction "$\Longrightarrow$". In the other direction, we can always obtain a weakly complete and weakly sound "$\vdash$", but this $\vdash$ may not be completely Hilbert-style (this $\vdash$ is more into the "Gabbay-Venema-Simon-Mikulás" direction).

compactness. Also, if we have some kind of negation, then the two notions of compactness coincide.[5] In all our examples in section 7, the two notions of compactness coincide.

Our next theorem characterizes consequence compactness of algebraizable general logics. For a proof see [AKNS, Thm.3.2.20], or [ANSK, Cor.3.10].

THEOREM 6.7 (characterization of compactness) *Assume that* $\mathbf{L}$ *is an algebraizable general logic. Then* $\mathbf{L}$ *is consequence compact iff* $\mathbf{Alg}(\mathbf{L})$ *is closed under taking ultraproducts, i.e. iff* $\mathbf{Alg}(\mathbf{L}) = \mathbf{UpAlg}(\mathbf{L})$. ■

Now we turn to characterization of some definability properties. Beth's definability properties of logics were defined e.g. in Barwise-Feferman [BF]. Here we give the definitions in the framework of the present paper.

DEFINITION 6.8 (implicit definition, explicit definition) *Let* $\langle \mathcal{L}^P : P$ *is a set*$\rangle$ *be a general logic. Let* $P \subsetneqq Q$ *be sets with* $F^P \neq \emptyset$*, and let* $R \stackrel{\text{def}}{=} Q \smallsetminus P$.

- *A set* $\Sigma \subseteq F^Q$ *of formulas defines* R *implicitly in* Q *iff a* P*-model can be extended to a* Q*-model of* Σ *at most one way, i.e. iff*

  $(\forall \mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}^Q(\Sigma))[mng^Q_{\mathfrak{M}} \upharpoonright F^P = mng^Q_{\mathfrak{N}} \upharpoonright F^P \longrightarrow mng^Q_{\mathfrak{M}} = mng^Q_{\mathfrak{N}}]$.

- Σ *defines* R *implicitly in* Q *in the strong sense iff, in addition, any* P*-model that in principle can, indeed can be extended to a* Q*-model of* Σ*, i.e. iff*

  Σ *defines* R *implicitly in* Q *and*

  $(\forall \mathfrak{M} \in \mathsf{Mod}^P([\mathsf{Th}^Q \mathsf{Mod}^Q \Sigma] \cap F^P))(\exists \mathfrak{N} \in \mathsf{Mod}^Q(\Sigma))$

  $mng^Q_{\mathfrak{N}} \upharpoonright F^P = mng^P_{\mathfrak{M}}$.

- Σ *defines* R *explicitly in* Q *iff any element of* R *has an "explicit definition" that works in all models of* Σ*, i.e. iff*

  $(\forall r \in R)(\exists \varphi_r \in F^P)(\forall \mathfrak{M} \in \mathsf{Mod}^Q(\Sigma)) mng^Q_{\mathfrak{M}}(r) = mng^Q_{\mathfrak{M}}(\varphi_r)$.

[5] More on this can be found in [AKNS],[ANSK].

- Σ *defines* R *local-explicitly in* Q *iff the above definition can vary from model to model, i.e. iff*

  $(\forall\mathfrak{M} \in \mathsf{Mod}^Q(\Sigma))(\forall r \in R)(\exists\varphi_r \in F^P)mng^Q_{\mathfrak{M}}(r) = mng^Q_{\mathfrak{M}}(\varphi_r).$

■

DEFINITION 6.9 (Beth definability properties) *Let* $\mathbf{L}$ *be a general logic.*

- $\mathbf{L}$ *has the* (strong) Beth definability *property iff for all* P, Q, R *and* Σ *as in Def. 6.8,*

  (Σ *defines* R *implicitly in* $Q \Longrightarrow \Sigma$ *defines* R *explicitly in* Q*).*

- $\mathbf{L}$ *has the* local Beth definability *property iff for all* P, Q, R *and* Σ *as in Def. 6.8,*

  (Σ *defines* R *implicitly in* $Q \Longrightarrow \Sigma$ *defines* R *local-explicitly in* Q*).*

- $\mathbf{L}$ *has the* weak Beth definability *property iff for all* P, Q, R *and* Σ *as in Def. 6.8,*

  (Σ *defines* R *implicitly in* Q *in the strong sense* $\Longrightarrow \Sigma$ *defines* R *explicitly in* Q*).*

■

DEFINITION 6.10 (patchwork property of models) *Let* $\mathbf{L}$ *be a general logic.* $\mathbf{L}$ *has the patchwork property of models iff*
for all sets P, Q*, and models* $\mathfrak{M} \in M^P$, $\mathfrak{N} \in M^Q$,

$F^{P\cap Q} \neq \emptyset$ *and* $mng^P_{\mathfrak{M}} \restriction (P \cap Q) = mng^Q_{\mathfrak{N}} \restriction (P \cap Q) \Longrightarrow$

$(\exists\mathfrak{P} \in M^{P\cup Q})(mng^{P\cup Q}_{\mathfrak{P}} \restriction F^P = mng^P_{\mathfrak{M}}$ *and* $mng^{P\cup Q}_{\mathfrak{P}} \restriction F^Q = mng^Q_{\mathfrak{N}}$*).*

■

Recall that if K is a class of algebras, then by a morphism of K we understand a triple $\langle \mathfrak{A}, h, \mathfrak{B} \rangle$, where $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ and $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism. A morphism $\langle \mathfrak{A}, h, \mathfrak{B} \rangle$ is an *epimorphism of* K iff for every $\mathfrak{C} \in \mathsf{K}$ and every pair $f, k : \mathfrak{B} \to \mathfrak{C}$ of homomorphisms we have $f \circ h = k \circ h$ implies $f = k$. Typical examples of epimorphisms are the surjections. But for certain choices of K there are epimorphisms of K which are not surjective. Such is the case, e.g., when K is the class of distributive lattices.

Let $\mathsf{K}_0 \subseteq \mathsf{K}$ be two classes of algebras. Let $\langle \mathfrak{A}, h, \mathfrak{B} \rangle$ be a morphism of K. h is said to be K_0-*extensible* iff for every algebra $\mathfrak{C} \in \mathsf{K}_0$ and every homomorphism $f : \mathfrak{A} \to \mathfrak{C}$ there exists some $\mathfrak{N} \in \mathsf{K}_0$ and $g : \mathfrak{B} \to \mathfrak{N}$ such that $\mathfrak{C} \subseteq \mathfrak{N}$ and $g \circ h = f$. It is important to emphasize here that $\mathfrak{C}$ is a concrete subalgebra of $\mathfrak{N}$ and *not* only is embeddable into $\mathfrak{N}$.

THEOREM 6.11 (characterization of Beth properties) [6] *Let* $\mathbf{L}$ *be an algebraizable general logic which has the patchwork property of models.*

(i) $\mathbf{L}$ *has the Beth definability property iff all the epimorphisms of* $\mathbf{Alg}(\mathbf{L})$ *are surjective.*

(ii) $\mathbf{L}$ *has the local Beth definability property iff all the epimorphisms of* $\mathbf{Alg}_m(\mathbf{L})$ *are surjective.*

(iii) $\mathbf{L}$ *has the weak Beth definability property iff every* $\mathbf{Alg}_m(\mathbf{L})$*-extensible epimorphism of* $\mathbf{Alg}_m(\mathbf{L})$ *is surjective.*

■

In the formulation of Thm. 6.11 (ii),(iii) above, it was important that $\mathbf{Alg}_m(\mathbf{L})$ is not an abstract class in the sense that it is not closed under isomorphisms, since the definition of K-extensibility strongly differentiates isomorphic algebras.

If K is a class of algebras, then $\mathbf{max}\mathsf{K}$ denotes the class of all $\subseteq$-maximal elements of K:

$$\mathbf{max}\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{K} : (\forall \mathfrak{B} \in \mathsf{K})(\mathfrak{A} \subseteq \mathfrak{B} \Longrightarrow \mathfrak{A} = \mathfrak{B})\}.$$

We note that e.g. $\mathbf{max}\mathsf{Cs}_n$ is the class of all full Cs_n's.

[6] The proof of (i) is in Németi [N82] and in Hoogland [H96]. A less general version of (i) is proved in [HMTII, Thm.5.6.10]. Part (ii) is due to J. Madarász. An early version of (iii) is in Sain [S90], and the full version is proved in [H96]. The finite Beth property is obtained from the Beth property by restricting R to be finite. The emphasis in [N82] and [HMTII] was on the finite Beth property. E. Hoogland and J. Madarász [HoM] extended the characterization of Thm.6.11(i) to the broader (than algebraizable) class of equivalential logics.

We will use the notions of "reflective subcategory", and "limits of diagrams of algebras" as in Mac Lane [MacL]. We will not recall these. Throughout, by a reflective subcategory we understand a full and isomorphism closed one.

The weak Beth property was introduced in Friedman [Fr] (cf. references of [BF]) and has been investigated since then, cf. e.g. [BF] pp.73–76, 689–716.

THEOREM 6.12 (characterization of weak Beth property) [7] *Let* $\mathbf{L}$ *be an algebraizable general logic which has the patchwork property of models. Assume that every element of* $\mathbf{Alg}_m(\mathbf{L})$ *can be extended to a maximal element of* $\mathbf{Alg}_m(\mathbf{L})$*, i.e. that* $\mathbf{Alg}_m(\mathbf{L}) \subseteq \mathbf{SmaxAlg}_m(\mathbf{L})$*. Then conditions (i)–(iii) below are equivalent.*

(i) $\mathbf{L}$ *has the weak Beth definability property.*

(ii) $\mathbf{Alg}(\mathbf{L})$ *is the smallest full reflective subcategory of* $\mathbf{Alg}(\mathbf{L})$ *containing* $\mathbf{maxAlg}_m(\mathbf{L})$.

(iii) $\mathbf{maxAlg}_m(\mathbf{L})$ *generates* $\mathbf{Alg}(\mathbf{L})$ *by taking limits of diagrams of algebras. I.e. there is no limit-closed proper subclass separating these two classes of algebras.*

■

Now we turn to characterizing Craig's interpolation properties.

DEFINITION 6.13 (interpolation porperties) *Let* $\mathcal{L} = \langle F, M, mng, \models\rangle$ *be a logic with connectives. For each formula* $\varphi \in F$ *let* $voc(\varphi)$ *denote the set of atomic formulas occurring in* φ*. Let* $\to$ *be a binary connective of* $\mathcal{L}$*.*

- $\mathcal{L}$ *has the* $\models$*-interpolation property iff*

  for all $\varphi, \psi \in F$ *such that* $\varphi \models \psi$ *there is* $\chi \in F$ *such that* $voc(\varphi) \subseteq voc(\varphi) \cap voc(\psi)$ *and* $\varphi \models \chi \models \psi$.

- $\mathcal{L}$ *has the* $\to$*-interpolation property iff*

  for all $\varphi, \psi \in F$ *such that* $\models \varphi \to \psi$ *there is* $\chi \in F$ *such that* $voc(\chi) \subseteq voc(\varphi) \cap voc(\psi)$ *and* $\models \varphi \to \chi$ *and* $\models \chi \to \psi$.

■

[7] This is due to I. Sain, J. Madarász, and I. Németi. For the origins of this characterization of weak Beth property see [S90, p.223 and on].

Next we recall from the literature the amalgamation and superamalgamation properties of classes of algebras. Let K be a class of algebras. We say that K *has the amalgamation property* iff any two algebras in K can be jointly embedded into a third element of K such that we can require some parts of them to go to the same place. I.e. for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbf{I}\mathsf{K}$ with $\mathfrak{B} \supseteq \mathfrak{A} \subseteq \mathfrak{C}$, there are $\mathfrak{N} \in \mathsf{K}$ and injective homomorphisms (embeddings) $f : \mathfrak{B} \rightarrowtail \mathfrak{N}$, $h : \mathfrak{C} \rightarrowtail \mathfrak{N}$ such that $f \upharpoonright A = h \upharpoonright A$.

By a *partially ordered algebra* we mean a structure $\langle \mathfrak{A}, \leq \rangle$ where $\mathfrak{A}$ is an algebra and $\leq$ is a partial ordering on the universe A of $\mathfrak{A}$. A class K of partially ordered algebras has the *superamalgamation property* iff any two algebras as above can be embedded in a third one such that only the necessary coincidences and ordering would hold, i.e. if for any $\mathfrak{A}_i \in \mathsf{K}$, $i \leq 2$ and for any embeddings $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ there exist an $\mathfrak{A} \in \mathsf{K}$ and embeddings $m_1 : \mathfrak{A}_1 \to \mathfrak{A}$ and $m_2 : \mathfrak{A}_2 \to \mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$ and for $\{j, k\} = \{1, 2\}$,

$$(\forall x \in A_j)(\forall y \in A_k)[m_j(x) \leq m_k(y) \Longrightarrow (\exists z \in A_0)(x \leq i_j(z) \text{ and } i_k(z) \leq y)].$$

DEFINITION 6.14 *Let $\mathcal{L}$ be a compositional logic. We say that $\mathcal{L}$ has a* deduction theorem *iff there is a binary derived connective ∇ such that for all $\Sigma \subseteq F$, $\varphi, \psi \in F$*

$$\Sigma \cup \{\varphi\} \stackrel{\bullet}{\models} \psi \quad \textit{iff} \quad \Sigma \stackrel{\bullet}{\models} \varphi \nabla \psi.$$

Such a ∇ is called a deduction term. ∎

THEOREM 6.15 (characterization of interpolation properties) [8]
Let $\mathcal{L}$ be an algebraizable semantic logic.

(i) Assume that $\mathcal{L}$ is consequence compact and usual conjunction $\wedge$ is in $Cn(\mathcal{L})$. Assume that $\mathcal{L}$ has a deduction theorem. Then $\mathcal{L}$ has the $\models$-interpolation property iff $\mathbf{Alg}(\mathcal{L})$ *has the amalgamation property.*

(ii) Assume that $\mathbf{Alg}(\mathcal{L})$ *consists of normal* $\mathsf{BAO}'s$. *Assume that* $\mathbf{Alg}(\mathcal{L})$ *is algebraized via the usual Boolean biconditional $\leftrightarrow$, i.e. in the filter-property Δ is $\leftrightarrow$. Let $\to$ denote the usual Boolean implication term. Then $\mathcal{L}$ has the $\to$-interpolation property iff* $\mathbf{HSPAlg}(\mathcal{L})$ *has the superamalgamation property.* ∎

[8] The proof of (i) is in Czelakowski [Cz, Thm.3]. The proof of (ii) is in Madarász [M96], [M97].

The above is only a sample of the equivalence theorems in algebraic logic. Other kinds of investigations are connecting deduction property of a logic $\mathcal{L}$ with $\mathbf{Alg}(\mathcal{L})$ having equationally definable principal congruences (EDPC) [BP89a]-[BP89c], [BP97]; theorems connecting e.g. atomicity of the formula-algebra of $\mathcal{L}$ with Gödel's incompleteness property of $\mathcal{L}$ ([N85], [N86]), theorems connecting logical meanings to neat-reducts of formula-algebras ([A], [SA]), etc.

CHAPTER 7

EXAMPLES AND APPLICATIONS

In this section we give some applications for the previous theorems. Most of the logics we use are well-known, but we recall their definitions for illustrating how they are special cases of the concept defined herein, and also for fixing our notation. More and also different kinds of examples are given in [AKNS], [NA], [ANSK].

1. Classical sentential logic $\mathbf{L}_S = \langle \mathcal{L}_S^P : P \text{ is a set}\rangle$.

Below, we often will omit the index S.

The set of logical connectives is $Cn = \{\wedge, \neg\}$, $\wedge$ is binary, $\neg$ is unary. Let P be any set. Thus the set of formulas of $\mathcal{L}_S^P$ is $F_S^P = F(P, Cn)$.

A model of sentential logic $\mathcal{L}_S^P$ is a function assigning 0 (false) or 1 (true) to each atomic proposition $p \in P$. Thus the class M_S^P of models of $\mathcal{L}_S^P$ is ${}^P2$, the set of all functions mapping P to 2. (Recall that $2 = \{0, 1\}$.)

We can extend any model $\mathfrak{M} : P \to 2$ to the set F_S^P of all formulas: for all $\varphi, \psi \in F_S^P$ we let

$$\mathfrak{M}(\neg\varphi) = \begin{cases} 1 & \text{if } \mathfrak{M}(\varphi) = 0 \\ 0 & \text{if } \mathfrak{M}(\varphi) = 1, \end{cases}$$

$$\mathfrak{M}(\varphi \wedge \psi) = \begin{cases} 1 & \text{if } \mathfrak{M}(\varphi) = \mathfrak{M}(\psi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, the meaning of φ in $\mathfrak{M}$ is $\mathfrak{M}(\varphi)$, i.e. $mng^P(\varphi, \mathfrak{M}) = \mathfrak{M}(\varphi)$, and φ is valid in $\mathfrak{M}$, $\mathfrak{M} \models_S \varphi$, iff $\mathfrak{M}(\varphi) = 1$.

We let $\mathcal{L}_S^P \overset{\text{def}}{=} \langle F_S^P, M_S^P, mng^P, \models_S\rangle$ and $\mathbf{L}_S \overset{\text{def}}{=} \langle \mathcal{L}_S^P : P \text{ is a set}\rangle$.

By this, we have defined $\mathbf{L}_S$. We are going to show that $\mathbf{L}_S$ is an algebraizable general logic with $\mathbf{Alg}(\mathbf{L}_S) = \mathsf{BA}$.

That $\mathbf{L}_S$ is compositional comes immediately from the definition. Let $\mathbf{2}$ denote the 2-element Boolean algebra with universe 2. Then $\mathfrak{Mng}(\mathfrak{M}) = \mathbf{2}$ for all $\mathfrak{M} \in$

M^P, $P \neq 0$. Thus $\mathbf{Alg}_m(\mathbf{L}_S) = \{\mathbf{2}\}$, and it can be seen from the definition that any homomorphism $h : \mathfrak{F}^P \to \mathbf{2}$ is a meaning-function of some model (namely, that of $h \restriction P$), thus $Mng^P = Hom(\mathfrak{F}^P, \{\mathbf{2}\})$. Thus by Proposition 5.2, $\mathbf{L}_S$ has the semantical substitution property, so $\mathbf{L}_S$ is structural. Then $\mathbf{Alg}(\mathbf{L}_S) = \mathbf{SP}\{\mathbf{2}\} = \mathsf{BA}$ by Theorem 5.3 (and also $\mathbf{Alg}(\mathcal{L}_S^P) =$ "BA's of cardinality $\leq max(\omega, |P|)$"). It can be seen that $\mathbf{L}_S$ has the filter-property with $m = n = 1$, $\Delta_0(\varphi, \psi) \stackrel{\text{def}}{=} (\varphi \leftrightarrow \psi) \stackrel{\text{def}}{=} \neg(\neg\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\psi)$, $\varepsilon_0(\varphi) \stackrel{\text{def}}{=} \varphi$ and $\delta_0(\varphi) \stackrel{\text{def}}{=} TRUE \stackrel{\text{def}}{=} \neg(\neg p \wedge p)$ for a fixed $p \in P$.

Thus $\mathbf{L}_S$ is an algebraizable general logic with $\mathbf{Alg}(\mathbf{L}_S) = \mathsf{BA}$, $\mathbf{Alg}_m(\mathbf{L}_S) = \{\mathbf{2}\}$.

By Theorem 6.7, $\mathbf{L}_S$ is compact, because $\mathsf{BA} = \mathbf{Up}\mathsf{BA}$. By Theorem 6.2, $\mathbf{L}_S$ has a strongly complete and sound Hilbert-style inference system, because BA is a finitely axiomatizable quasi-variety. Moreover, the proof of Thm. 6.2 (in [AKNS]) constructs such a Hilbert-style inference system (given any axiomatization of BA). We give here the inference system we get from the proof.

In the next inference system, we will use φ, ψ, χ and δ as formula-variables, and we will use $\leftrightarrow$ and $TRUE$ as derived connectives.

The axioms are:

$$\varphi \wedge \psi \ \leftrightarrow \ \psi \wedge \varphi,$$

$$\varphi \wedge (\psi \wedge \chi) \ \leftrightarrow \ (\varphi \wedge \psi) \wedge \chi,$$

$$\varphi \ \leftrightarrow \ \neg(\neg(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\psi)).$$

The rules are as follows:

$$\frac{}{\varphi \leftrightarrow \varphi}, \qquad \frac{\varphi \leftrightarrow \psi}{\psi \leftrightarrow \varphi}, \qquad \frac{\varphi \leftrightarrow \psi,\ \psi \leftrightarrow \chi}{\varphi \leftrightarrow \chi},$$

$$\frac{\varphi \leftrightarrow \psi}{\neg\varphi \leftrightarrow \neg\psi}, \qquad \frac{\varphi \leftrightarrow \psi,\ \chi \leftrightarrow \delta}{(\varphi \wedge \chi) \leftrightarrow (\psi \wedge \delta)},$$

$$\frac{\varphi \leftrightarrow TRUE}{\varphi}, \qquad \frac{\varphi}{\varphi \leftrightarrow TRUE}.$$

It is easy to check that $\mathbf{L}_S$ has the patchwork property. $\mathbf{L}_S$ has the Beth property by Theorem 6.11, because epimorphisms are surjective in BA. Thus $\mathbf{L}_S$ has the weak Beth property, so by Theorem6.12, **2** generates BA by limits, since $\mathbf{maxAlg}_m(\mathbf{L}_S) = \mathbf{Alg}_m(\mathbf{L}_S) = \{\mathbf{2}\}$.

A deduction term for $\mathbf{L}_S$ is $\phi_0 \rightarrow \phi_1 \stackrel{\text{def}}{=} \neg(\phi_0 \wedge \neg\phi_1)$. Since BA has superamalgamation, Theorem 6.15 implies that $\mathbf{L}_S$ has the interpolation properties.

The validity problem of $\mathbf{L}_S$ is decidable, the set of admissible rules of $\mathbf{L}_S$ is decidable, the set of valid formula-schemes of $\mathbf{L}_S$ is decidable by Theorem 5.4 and Corollary 5.5, because the quasi-equational theory of BA is decidable.

2. Sentential logic in a slightly different form, $\mathbf{L}_S{}'$.

The set of connectives, thus the set of formulas are just like in the previous case. The models are different. Let P be a set.

$$M'_S \stackrel{\text{def}}{=} \{\langle W, v\rangle : W \text{ is a non-empty set and } v : P \rightarrow \mathcal{P}(W)\}.$$

Thus a model $\mathfrak{M} = \langle W, v\rangle$ is a non-empty set together with an assignment assigning a subset of W to each $p \in P$. Let $\mathfrak{M} = \langle W, v\rangle$ be a model. We call W the set of possible situations (or states, or worlds) of $\mathfrak{M}$. For any formula φ, we define $\mathfrak{M}, w \Vdash \varphi$, which we read as "$\varphi$ is true in $\mathfrak{M}$ at w", as follows:

$$\mathfrak{M}, w \Vdash p \quad \text{iff} \quad w \in v(p), \quad \text{for } p \in P.$$

$$\mathfrak{M}, w \Vdash \neg\varphi \quad \text{iff} \quad \mathfrak{M}, w \nVdash \varphi,$$

$$\mathfrak{M}, w \Vdash \varphi \wedge \psi \quad \text{iff} \quad (\mathfrak{M}, w \Vdash \varphi \;\text{ and }\; \mathfrak{M}, w \Vdash \psi).$$

We say that φ is valid in $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash \varphi$ for all $w \in W$.

The above amounts to saying that the meaning-function $mng_{\mathfrak{M}}$ is the homomorphic extension of v into the algebra $\langle \mathcal{P}(W), \cap, \smallsetminus\rangle$, i.e. $mng_{\mathfrak{M}} : \mathfrak{F}^P \rightarrow \langle \mathcal{P}(W), \cap, \smallsetminus\rangle$, and

$$\mathfrak{M}, w \Vdash \varphi \quad \text{iff} \quad w \in mng_{\mathfrak{M}}(\varphi),$$

$$\mathfrak{M} \models \varphi \quad \text{iff} \quad W = mng_{\mathfrak{M}}(\varphi).$$

Now, $\mathbf{L}_S{}'$ is defined. It is compositional, $\mathbf{Alg}_m(\mathbf{L}_S{}') = \mathsf{setBA} \stackrel{\text{def}}{=} \mathbf{S}\{\langle \mathcal{P}(W), \cap, \smallsetminus\rangle : W \text{ is a non-empty set}\} =$"the class of all non-trivial set Boolean algebras", and $Mng^P(\mathbf{L}_S{}') = Hom(\mathfrak{F}^P, \mathsf{setBA})$.

Thus $\mathbf{L}_S{}'$ is algebraizable, $\mathbf{Alg}(\mathbf{L}_S{}') = \mathsf{BA}$, $\mathbf{Alg}_m(\mathbf{L}'_S) = \mathsf{setBA}$. By Theorem 5.4, it has the same semantical consequence relation $\models^{\bullet}$, same admissible rules, same valid formula-schemes and same valid formulas as $\mathbf{L}_S$.

3. Modal logic S5.

The set of connectives is $\{\wedge, \neg, \Diamond\}$, $\wedge$ binary, $\neg, \Diamond$ unary. The class of models is the same as for $\mathbf{L}_S{}'$. The "meaning of $\Diamond$" is as follows:

$$\mathfrak{M}, w \Vdash \Diamond\varphi \quad \text{iff} \quad (\mathfrak{M}, w' \Vdash \varphi \text{ for some } w' \in W).$$

This is the same as saying that

$$mng_{\mathfrak{M}}(\Diamond\varphi) = C_0^W(mng_{\mathfrak{M}}(\varphi)), \quad \text{where for any set } X \subseteq W$$

$$C_0^W(X) = \begin{cases} W & \text{if } X \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The rest of the definition of **S5** goes the same way as in the case of $\mathbf{L}_S{}'$ above.

It can be checked that **S5** has the filter property with the same terms as sentential logic $\mathbf{L}_S$.

Thus **S5** is an algebraizable general logic with $\mathbf{Alg}_m(\mathbf{S5}) = \mathsf{Cs}_1 = \mathbf{S}\{\langle \mathcal{P}(W), \cap, \smallsetminus, C_0^W \rangle : W \text{ is a nonempty set}\}$, $\mathbf{Alg}(\mathbf{S5}) = \mathbf{SP}\mathsf{Cs}_1 = \mathsf{RCA}_1$.

A deduction term for **S5** is $\neg\Diamond\neg\phi_0 \to \neg\Diamond\neg\phi_1$.

Therefore **S5** is decidable, compact, has a strongly complete and sound Hilbert-style inference system, and has the Beth and interpolation properties by Theorems 5.4, 6.7, 6.2, 6.11, 6.15, because RCA_1 is a decidable, finitely axiomatizable variety having the superamalgamation property, see Thm.2.3(ii) .

4. Arrow logic $\mathbf{L}_{REL}$.

The field of Arrow Logics grew out of application areas in Logic, Language and Computation, and plays an important role there, cf. e.g. van Benthem [vB96], [vB91a], and the proceedings of the Arrow Logic day at the conference "Logic at Work" (Dec.1992, Amsterdam). These arrow logics go back to the investigations in Tarski-Givant [TG]. Tarski defined in 1951 basically $\mathcal{L}_{REL}$ to give the first example of an undecidable propositional logic.

The set of connectives of $\mathbf{L}_{REL}$ is $\{\wedge, \neg, \circ, ^{-1}\}$, $\wedge, \circ$ binary, $\neg, ^{-1}$ unary. The models are as in **S5**, except that we require that the elements of W be all pairs over some set U, i.e.

$$M^P_{REL} \stackrel{\text{def}}{=} \{\langle W, v\rangle : W = U \times U \text{ for some } U \text{ and } v : P \to \mathcal{P}(W)\}.$$

The definition of $\mathfrak{M}, w \Vdash \varphi$ is as in the previous case, and we define the meanings of $\circ$ and $^{-1}$ as

$$\mathfrak{M}, \langle u, z\rangle \Vdash \varphi \circ \psi \quad \text{iff} \quad (\mathfrak{M}, \langle u, x\rangle \Vdash \varphi \text{ and } \mathfrak{M}, \langle x, z\rangle \Vdash \psi \text{ for some } x),$$

$$\mathfrak{M}, \langle u, z\rangle \Vdash \varphi^{-1} \quad \text{iff} \quad \mathfrak{M}, \langle z, u\rangle \Vdash \varphi.$$

This amounts to saying that the meaning of $\circ$ is relation composition, and the meaning of $^{-1}$ is relation conversion, i.e.

$$mng_{\mathfrak{M}}(\varphi \circ \psi) = mng_{\mathfrak{M}}(\varphi) \circ mng_{\mathfrak{M}}(\psi),$$

$$mng_{\mathfrak{M}}(\varphi^{-1}) = (mng_{\mathfrak{M}}(\varphi))^{-1}.$$

Otherwise everything is the same as before, e.g. $\mathfrak{M} \models \varphi$ iff $mng_{\mathfrak{M}}(\varphi) = W$.

Now, $\mathbf{L}_{REL}$ is an algebraizable general logic with $\mathbf{Alg}(\mathbf{L}_{REL}) = \mathsf{BRA}$, $\mathbf{Alg}_m(\mathbf{L}_{REL}) = \mathsf{setBRA}$.

Thus, by our equivalence theorems in section 6 and by our algebraic theorems on BRA in section 1, we obtain that $\mathbf{L}_{REL}$ is undecidable, compact, has no finitely complete and strongly sound Hilbert-style inference system. Since in BRA epimorphisms are not surjective [1], $\mathbf{L}_{REL}$ does not have the Beth property.

A deduction term for $\mathbf{L}_{REL}$ is $TRUE \circ \phi_0 \circ TRUE \to TRUE \circ \phi_1 \circ TRUE$. Since BRA does not have the amalgamation property, by Theorem 6.15 $\mathbf{L}_{REL}$ does not have the interpolation property.

We can add "equality" to $\mathbf{L}_{REL}$, obtaining $\mathbf{L}^{=}_{REL}$ as follows. We add Id to the set of connectives as a zero-ary connective, and we define its meaning as for any model $\mathfrak{M} = \langle U \times U, v\rangle$

$$mng_{\mathfrak{M}}(\mathsf{Id}) = \{\langle u, u\rangle : u \in U\}.$$

[1] See the methods in Sain [S90].

Then $\mathbf{L}^{=}_{REL}$ is an algebraizable general logic with $\mathbf{Alg}(\mathbf{L}^{=}_{REL}) = \mathsf{RRA}$, $\mathbf{Alg}_m(\mathbf{L}^{=}_{REL}) = \mathsf{setRRA}$. Just as in the previous cases we get properties of $\mathbf{L}^{=}_{REL}$ by using the theorems about RRA stated in section 1, and the equivalence theorems stated in section 6.

5. First-order logic with n variables, with substituted atomic formulas, $\mathbf{L}_n{}'$.

Let $n \in \omega$, let $V_n \stackrel{\text{def}}{=} \{v_i : i < n\}$, our set of variables. Let $\mathcal{R}$ be any set (our relation symbols). The set P of atomic formulas of the logic $\mathcal{L}'_n \stackrel{\text{def}}{=} {\mathcal{L}'_n}^{\mathcal{R}}$ is

$$P \stackrel{\text{def}}{=} \{R(x_0, \ldots, x_{n-1}) : R \in \mathcal{R}, x_0, \ldots, x_{n-1} \in V_n\}.$$

The set of connectives is $\{\wedge, \neg, v_i = v_j, \exists v_i : i, j < n\}$, $\wedge$ binary, $\neg, \exists v_i$ unary, and $v_i = v_j$ zero-ary.[2] This defines the set ${F'_n}^{\mathcal{R}}$ of formulas of ${\mathcal{L}'_n}^{\mathcal{R}}$. The class of models is the usual one,

$$M_n^{\mathcal{R}} \stackrel{\text{def}}{=} \{\langle M, R^{\mathfrak{M}}\rangle_{R\in\mathcal{R}} : M \text{ is a nonempty set and}$$
$$R^{\mathfrak{M}} \subseteq {}^nM \text{ for all } R \in \mathcal{R}\}.$$

Let $\mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle_{R\in\mathcal{R}}$ be a model. Then we define $\Vdash$ as before: Let $\varphi \in F$ and $h \in {}^nM$. We call h an evaluation of the variables. Because of the tradition, we will write $\mathfrak{M} \models \varphi[h]$ in place of $\mathfrak{M}, h \Vdash \varphi$.

$\mathfrak{M} \models R(v_{i_0} \ldots v_{i_{n-1}})[h]$ iff $\langle h(i_0), \ldots, h(i_{n-1})\rangle \in R^{\mathfrak{M}}$,

$\mathfrak{M} \models v_i = v_j[h]$ iff $h_i = h_j$,

$\mathfrak{M} \models \exists v_i\varphi[h]$ iff ($\mathfrak{M} \models \varphi[h']$ for some $h' \in {}^nM$ such that h and h' differ at most at i.)

$\mathfrak{M} \models (\varphi \wedge \psi)[h]$ and $\mathfrak{M} \models \neg\varphi[h]$ are as before.

$\mathfrak{M} \models_n \varphi$ iff ($\mathfrak{M} \models \varphi[h]$ for all $h \in {}^nM$).

We define[3]

$$mng_{\mathfrak{M}}(\varphi) \stackrel{\text{def}}{=} \{h \in {}^nM : \mathfrak{M} \models \varphi[h]\}.$$

We define ${\mathcal{L}'_n}^{\mathcal{R}} \stackrel{\text{def}}{=} \langle {F'_n}^{\mathcal{R}}, M_n^{\mathcal{R}}, mng, \models_n\rangle$, and $\mathbf{L}_n{}' \stackrel{\text{def}}{=} \langle {\mathcal{L}'_n}^{\mathcal{R}} : \mathcal{R} \text{ is a set}\rangle$.

Now, $\mathbf{L}_n{}'$ is compositional, and has the filter property. But it is not structural (and then it is not semantically structural either) as the following example shows. Let

$$p \stackrel{\text{def}}{=} R(v_0, v_1) \quad \text{and} \quad q \stackrel{\text{def}}{=} R(v_0, v_0).$$

[2] Notice that $v_i = v_j$ is not an atomic formula but rather a zero-ary logical connective.

[3] In the literature, $mng_{\mathfrak{M}}(\varphi)$ is called the relation defined by φ in the model $\mathfrak{M}$. Thus $\mathfrak{Mng}(\mathfrak{M})$ is the algebra of n-variable definable relations in $\mathfrak{M}$.

Then it can be checked that

$$\models_n q \;\leftrightarrow\; \exists v_1(v_1 = v_0 \wedge p), \qquad \text{but}$$
$$\not\models_n p \;\leftrightarrow\; \exists v_1(v_1 = v_0 \wedge p).$$

Another example is the following. Let

$$p \stackrel{\text{def}}{=} R(v_0, v_1) \quad \text{and} \quad q \stackrel{\text{def}}{=} R(v_1, v_0).$$

Then it can be checked that

$$\models_n \exists v_1 p \;\leftrightarrow\; \exists v_1(v_1 = v_0 \wedge \exists v_0 q), \qquad \text{but}$$
$$\not\models_n \exists v_1 q \;\leftrightarrow\; \exists v_1(v_1 = v_0 \wedge \exists v_0 q).$$

The logic $\mathcal{L}'_n$ is not structural because the meanings of the atomic formulas are not independent of each other: as soon as we know the meaning of $R(v_0 \dots v_{n-1})$, this will determine the meanings of $R(x_0 \dots x_{n-1})$ where $x_0, \dots, x_{n-1}$ are arbitrary variables. It would be natural to treat only $R(v_0 \dots v_{n-1})$ as an atomic formula. Then we would like to obtain the substituted atomic formulas $R(x_0 \dots x_{n-1})$ as "complex", built-up formulas. We will achieve this in two different ways. In the first case we will use Tarski's observation that substitution can be expressed with quantifiers and equality[4] and in the second case we will introduce substitutions explicitly as logical connectives [5].

6. First-order logic with n variables, structural version [6] $\mathbf{L}_n$, for $n < \omega$ and for n any ordinal.

This is exactly like the previous example, except that we keep as atomic formulas only $R(v_0, \dots, v_{n-1}), R \in \mathcal{R}$. Since the order of the variables is fixed in our atomic formulas, we will simply write R in place of $R(v_0, \dots, v_{n-1})$. The set of connectives, the class of models, and the meaning function are exactly as before, the only difference is that now the set of atomic formulas is $\mathcal{R}$ itself. When n is an infinite ordinal, everything is analogous (then R stands for $R(v_0 \dots v_i \dots)_{i<n}$). Notation: $\mathbf{L}_n = \langle \mathcal{L}_n^{\mathcal{R}} : \mathcal{R} \text{ is a set} \rangle,\ \mathcal{L}_n^{\mathcal{R}} = \langle F_n^{\mathcal{R}}, \dots \rangle$.

Then, $\mathbf{L}_n$ is compositional, and $Mng_{\mathcal{L}_n^{\mathcal{R}}} = Hom(\mathfrak{F}^{\mathcal{R}}, \mathsf{Cs}_n)$, so $\mathbf{L}_n$ has the semantic substitution property by Proposition 5.2. $\mathbf{L}_n$ has the filter property (with

[4] C.f. Tarski [T51], [T65].

[5] For more detail see [BP89, Appendix C], and [HMTII, §4.3].

[6] This is called a *full restricted* first-order language in [HMTII, §4.3]. "Restricted" refers to the fact that we keep atomic formulas only with a fixed sequence of variables, and "full" refers to the fact that the arity (or rank) of each relation symbol is n. This logic is investigated in [BP89, Appendix C], too. Most of what we say in this example, can be generalized to the infinitary version $\mathbf{L}^n_{\infty\omega}$ of $\mathbf{L}_n$ studied in finite model theory, see Example 11 herein.

$\phi_0, TRUE, \phi_0 \leftrightarrow \phi_1$ as $\varepsilon, \delta, \Delta$), so $\mathbf{L}_n$ is an algebraizable general logic. It is easy to check that $\mathbf{Alg}_m(\mathcal{L}_n^{\mathcal{R}})$ is the class of $|\mathcal{R}|$-generated Cs_n's, so $\mathbf{Alg}_m(\mathbf{L}_n) = \mathsf{Cs}_n$. Thus $\mathbf{Alg}(\mathbf{L}_n) = \mathsf{RCA}_n$ by Thm.5.3(ii), since $\mathsf{RCA}_n = \mathbf{SP}\mathsf{Cs}_n$. Using $\mathbf{Alg}_m(\mathbf{L}_n) = \mathsf{Cs}_n$ and $\mathbf{Alg}(\mathbf{L}) = \mathsf{RCA}_n$, we begin to apply the theorems in sections 2,6 to $\mathbf{L}_n$.

$\mathbf{L}_n$ is compact for all n by Thm.6.7, because $\mathsf{RCA}_n = \mathbf{Up}\mathsf{RCA}_n$ by Thm.2.3. For finite n, a deduction term for $\mathbf{L}_n$ is $\neg\exists v_0 \ldots \exists v_{n-1}\neg\phi_0 \to \neg\exists v_0 \ldots \exists v_{n-1}\neg\phi_1$.

THEOREM 7.1 *Let $n > 2$. There is no weakly complete and strongly sound Hilbert-style inference system for $\mathbf{L}_n$. As a contrast, there are strongly complete and sound Hilbert-style inference systems for $\mathbf{L}_2$, $\mathbf{L}_1$, $\mathbf{L}_0$.*

Proof. For $n > 2$, this follows from Thm.6.4, because RCA_n is a nonfinitely axiomatizable variety (by Thms.2.3, 2.4, 2.5). For $n \leq 2$, this follows from Thm.6.2, because RCA_n, $n \leq 2$ is a finitely axiomatizable quasi-variety (by Thm.2.3). ■

Soon we will give a strongly sound and complete inference system $\models_{2+}$ for $\mathbf{L}_2$.

The above negative result can be meaningfully generalized to most known variants [7] of $\mathbf{L}_n$, $\mathbf{L}_n$ without equality, and the infinitary version $\mathbf{L}^n_{\infty}\omega$ of $\mathbf{L}_n$ studied e.g. in finite model theory (e.g. [EF], [O]). See Example 11 herein.

The proof of Thm. 7.1 above is a typical example of applying algebraic logic to logic. There are analogous theorems (using the same "general methodology"). An example is provided by the positive results giving completeness theorems for relativized versions of $\mathbf{L}_n$ cf. e.g. [AvBN97] or [N96]. Different kinds of positive results relevant to Theorem 7.1 above are in [S95], [SGy].

OPEN PROBLEM 7.2 *Is there a weakly complete and weakly sound Hilbert-style inference system for $\mathbf{L}_n$, $n > 2$?*

By Thm.6.5, Open Problem 7.2 above is equivalent to Problem 2.11 (i.e. whether $\mathsf{RCA}_n = \mathbf{H}\mathsf{K}$ for some finitely axiomatizable quasi-variety K), because $\mathsf{RCA}_n \models (x \leftrightarrow y) = 1 \to x = y$, where $x \leftrightarrow y = -(x \oplus y)$. Actually, in the present case a positive answer would imply the existence of a strongly complete and weakly sound "$\vdash$" for $\mathbf{L}_n$, because $\mathbf{Alg}(\mathbf{L}_n)$ is a variety.

[7] e.g. to $\mathbf{L}_n{}'$ of Example 5.

Next we turn to investigating inference systems suggested by the connections between RCA_n, CA_n, and $\mathbf{SNr}_n\mathsf{CA}_m$, for $m \geq n$ (see Thm.2.14). We will define two provability relations, $\vdash_n$ and $\vdash_{n,m}$ for $\mathbf{L}_n$. (Of these, $\vdash_n$ is given by a Hilbert-style inference system, while $\vdash_{n,m}$ is not.)

In the following we will heavily use that $F_n^{\mathcal{R}} \subseteq F_{n+m}^{\mathcal{R}}$ (which is so because the atomic formulas of $\mathcal{L}_n^{\mathcal{R}}$ and $\mathcal{L}_{n+m}^{\mathcal{R}}$ are identified).

DEFINITION 7.3 (provability relations $\vdash_n$ and $\vdash_{n,m}$ for $\mathbf{L}_n$) [8]

(i) First we define $\vdash_n$ which will be given by the Hilbert-style inference system $\langle Ax_n, Ru_n\rangle$. In the formula-shemes below we will use φ, ψ as formula-variables (instead of ϕ_0, ϕ_1), and $\forall v_i$, $\to$ are derived connectives:

$$\begin{aligned} \forall v_i\varphi &\stackrel{\text{def}}{=} \neg\exists v_i\neg\varphi \\ \varphi \to \psi &\stackrel{\text{def}}{=} \neg(\neg\varphi \wedge \psi). \end{aligned}$$

Recall that $\exists v_i$, $v_i = v_j$ are logical connectives for $i, j < n$.

Ax_n consists of the following formula-shemes of $\mathbf{L}_n$: For all $i, j, k < n$

χ, $\quad$ *χ a propositional tautology [9], i.e. a valid formula-sheme of $\mathbf{L}_0$*

$\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi)$

$\forall v_i\varphi \to \varphi$

$\forall v_i\forall v_j\varphi \to \forall v_j\forall v_i\varphi$

$\forall v_i\varphi \to \forall v_i\forall v_i\varphi$

$\exists v_i\varphi \to \forall v_i\exists v_i\varphi$

$v_i = v_i$

$v_i = v_j \to (v_i = v_k \to v_j = v_k)$

$\exists v_i(v_i = v_j)$

$v_i = v_j \to \forall v_k(v_i = v_j)$ $\quad$ *if $k \neq i, j$.*

$v_i = v_j \to (\varphi \to \forall v_i(v_i = v_j \to \varphi)$, $\quad$ *if $i \neq j$.*

[8] $\vdash_{n,m}$, in a slightly different form, is defined in [HMTII, p. 157], and also in [N86], [N92], [N95], [BP89, Appendix C], [CzP]. In [BP89], $\vdash_n$ is denoted by $\vdash_{PR_n}$. One could get the definition of $\vdash_n$ by mechanically translating the CA_n-axioms, as we did with the BA-axioms in Example 1, and then polishing the so obtained axioms and rules.

[9] To keep Ax_n finite for $n < \omega$, we replace the infinitely many shemes here with Ax_0, where $\langle Ax_0, \{(MP)\}\rangle$ is a strongly complete and sound Hilbert-style inference system for $\mathbf{L}_0$. Such systems are known, cf. e.g. [ABNPS].

Ru_n consists of the rules Modus Ponens (MP) *and* Generalization $(G)_i$ *for $i < n$, where (MP) and $(G)_i$ are, respectively:*

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad \textit{and} \qquad \frac{\varphi}{\forall v_i \varphi}.$$

Now, $\vdash_n$ is the derivability relation given by $\langle Ax_n, Ru_n\rangle$ (for $\mathbf{L}_n$).

(ii) To define $\vdash_{n,m}$, let $n \leq m$. Let $Ax^n_m \subseteq F^{\mathcal{R}}_m$ consist of the following formulas:

$$(R) \qquad R \to \forall v_i R \qquad \textit{if} \quad n \leq i < m, \quad \textit{and} \quad R \in \mathcal{R}.$$

Now the inference system $\vdash_{n,m} \subseteq \mathcal{P}(F^{\mathcal{R}}_n) \times F^{\mathcal{R}}_n$ is defined to be "$(Ax^n_m \vdash_m)$ restricted to $\mathbf{L}_n$", i.e.

$$\vdash_{n,m} \stackrel{\text{def}}{=} \{\langle \Sigma, \varphi\rangle : \Sigma \cup Ax^n_m \vdash_m \varphi, \quad \Sigma \cup \{\varphi\} \subseteq F^{\mathcal{R}}_n\}.$$

Thus, in an $\vdash_{n,m}$-proof, in addition to the instances of Ax_m, we also can use $R \to \forall v_i R$, for $n \leq i < m$, $R \in \mathcal{R}$. If $\vdash_{n,m} \varphi$, then we say that "φ is provable with m variables"*, or* "φ is m-variable provable". ■

It is not hard to check that both $\vdash_n$ and $\vdash_{n,m}$ are strongly sound for $\mathbf{L}_n$, and $\vdash_n$ is given by a Hilbert-style inference system (if n is finite).

Therefore, from Thm.7.1 we can conclude that there are infinitely many valid $\mathbf{L}_n$-formulas which are not $\vdash_n$-provable, i.e. $\vdash_n$ is incomplete for $\mathbf{L}_n$ (in a rather strong way). On the other hand, we will see below that $\vdash_{n,\omega}$ is strongly complete for $\mathbf{L}_n$.

We are going to prove that the inference system $\vdash_n$ is the logical equivalent of the algebraic axiom system defining the variety CA_n.

Any formula $\varphi \in \mathcal{L}^{\mathcal{R}}_n$ can be identified with a term in the algebraic language of CA_n such that the elements of $\mathcal{R}$ are considered as (algebraic) variables, assuming that we identify the operations of CA_n with the connectives of $\mathbf{L}_n$. Hence $\varphi = 1$ is an equation in the language of CA_n (for $\varphi \in \mathcal{L}^{\mathcal{R}}_n$). We write $\varphi \in \mathcal{L}^{\mathcal{R}}_n$ for $\varphi \in F^{\mathcal{R}}_n$. Also, $\varphi \in \mathbf{L}_n$ means ($\exists$ set $\mathcal{R}$) $\varphi \in \mathcal{L}^{\mathcal{R}}_n$.

THEOREM 7.4 *Let $\varphi \in \mathcal{L}^{\mathcal{R}}_n$, n any ordinal, $\mathcal{R}$ any set. Then (i)–(iii) below hold for all $n \leq m$.*

(i) $\vdash_n \varphi$ *iff* $\mathsf{CA}_n \models \varphi = 1$.

(ii) $\vdash_{n,m} \varphi$ *iff* $\mathbf{SNr}_n\mathsf{CA}_m \models \varphi = 1$.

(iii) $\models_n \varphi$ *iff* $\mathsf{RCA}_n \models \varphi = 1$.

Proof. (i)–(iii) are proved in [HMTII] as Cor. 4.3.26, Thm. 4.3.25, and Thm. 4.3.17, respectively. See also 4.3.57, 4.3.59 therein. To check that $\vdash_r$ of [HMTII] is the same as our $\vdash_{n,m}$, it suffices to check that in the proof of 4.3.22, only the axioms of our Ax_n are used. ■

Now we are ready to state the logical corollaries of Thm.2.14(ii).

COROLLARY 7.5

(i) $\vdash_{n,n+\omega}$ *is strongly complete for* $\mathbf{L}_n$*, for all* n*.*

(ii) $\vdash_{n,n+m}$ *is not even weakly complete for* $\mathbf{L}_n$*, if* $m < \omega$*.*

(iii) Let $\varphi \in \mathbf{L}_n$*. Then* $\models_n \varphi$ *iff* $\vdash_{n,n+m} \varphi$ *for some* $m < \omega$*.*

(iv) For all $n, m < \omega$ *there are valid* n*-variable formulas which cannot be proved with* m *variables. For each valid* n*-variable formula* φ *there is an* $m < \omega$ *such that* φ *is* m*-variable provable.*

Proof. (i): Weak completeness of $\vdash_{n,n+\omega}$ follows immediately from Thm. 7.4, Thm. 2.14 ($\mathsf{RCA}_n = \mathbf{SNr}_n\mathsf{CA}_{n+\omega}$). For $n < \omega$, then strong completeness follows, because $\mathbf{L}_n$ is compact and has a deduction theorem. For $n \geq \omega$, $\mathbf{L}_n$ is still compact, but it does not have a deduction theorem. However, $\vdash_{n,n+\omega}$ is strongly complete for $\mathbf{L}_n$ by [HMTII, 4.3.23(ii)].

(ii)–(iv) follow from Thm. 7.4 and Thm. 2.14. E.g., assume $\varphi \in \mathbf{L}_n$. Then $\models_n \varphi$ iff $\mathsf{RCA}_n \models \varphi = 1$ iff (by $\mathsf{RCA}_n = \mathbf{SNr}_n\mathsf{CA}_{n+\omega}$) $\mathbf{SNr}_n\mathsf{CA}_{n+\omega} \models \varphi = 1$ iff $\models_{n,n+\omega} \varphi$ iff (by the definition of $\vdash_{n,m}$) $\vdash_{n,n+m} \varphi$ for some $m < \omega$. ■

We note that Corollary 7.5 speaks also about usual first-order logic, because an n-variable formula is valid in $\mathbf{L}_n$ iff it is valid in usual first-order logic (and every first order formula φ has a normal form φ' which is in $\mathbf{L}_n$, for some $n \in \omega$).

To investigate further the provability relations $\vdash_n, \vdash_{n,m}$, now we compare their "deductive powers". Results in cylindric algebra theory yield the following.

THEOREM 7.6 (the deductive powers of $\vdash_n, \vdash_{n,m}$)

(i) For any $1 < n < \omega$*,* $\vdash_n \neq (\vdash_{n+1} \restriction \mathbf{L}_n)$*, i.e. there is an* n*-variable formula* φ *such that*

$$\not\vdash_n \varphi \quad \text{but} \quad \vdash_{n+1} \varphi.$$

(ii) If $n \geq \omega$*, then* $\vdash_{n,m} \neq \vdash_{n,m+1}$ *for all* $m < \omega$*, i.e. there is a* $\varphi \in \mathbf{L}_\omega$ *such that*

$$\not\vdash_{\omega,m} \varphi \quad \text{and} \quad \vdash_{\omega,m+1} \varphi.$$

(iii) $\vdash_2 \neq \vdash_{2,3} = \vdash_{2,m}$ *for all* $m \geq 3$.

(iv) If $n \leq 1$ *or* $n \geq \omega$, *then for any* n*-variable* φ, $\vdash_n \varphi$ *iff* $\vdash_m \varphi$, *for all* $m \geq n$. *If* $n \leq 1$, *then* $\vdash_n = \vdash_{n,m}$ *for all* $m \geq n$.

Proof. (i) follows from Thm 7.4 and [HMT, 2.6.14], as follows. Let $2 \leq n < \omega$. Then there is an equation e in the language of CA_n such that $\mathsf{CA}_n \not\models e$ and $\mathsf{CA}_{n+1} \models e$, by [HMT, 2.6.14]. We may assume that e is of form $\varphi = 1$, and then Thm. 7.4(i) finishes the proof.

Similarly, (iv) follows from Thm. 7.4 and [HMT, 2.6.8, 2.6.9], and from $\mathsf{CA}_n = \mathsf{RCA}_n = \mathbf{SNr}_n\mathsf{CA}_{n+m}$ if $n \leq 1$.

(ii) follows from (Thm. 7.4(ii) and) $\mathbf{SNr}_n\mathsf{CA}_{n+m} \neq \mathbf{SNr}_{n+m+1}$ for all $n \geq \omega$, $m < \omega$ which is an unpublished result of Don Pigozzi.

(iii) follows from $\mathsf{CA}_2 \neq \mathbf{SNr}_2\mathsf{CA}_3 = \mathsf{RCA}_2 = \mathbf{SNr}_2\mathsf{CA}_\omega$, see [HMT, 2.6.42, 3.2.65] ■

OPEN PROBLEM 7.7 *Is it true that* $\vdash_{n,m} \neq \vdash_{n,m+1}$ *whenever* $3 \leq n \leq m < \omega$*? I.e. is there a* 3*-variable formula* φ *for all* $3 \leq m < \omega$ *such that* φ *is not provable with* m *variables but it is provable with* $m+1$ *variables?* ■

The above is Problem 2.12 in [HMT]. Results in this direction are proved e.g. in Maddux [Ma83], [Ma91a], Andréka [A97]. The present status of this problem is summarized in [A97, Remark 3].

REMARK $\vdash_{n,m}$ *is a (structural) derivability relation for* $\mathbf{L}_n$ *in the sense of [BP89], i.e. for any* $\varphi \in \mathcal{L}_n^{\mathcal{R}}$ *and* $s : \mathcal{R} \to \mathcal{L}_n^{\mathcal{R}}$, *if* $\vdash_{n,m} \varphi$ *then* $\vdash_{n,m} \hat{s}(\varphi)$. *This follows from Thm. 7.4(ii). Thm. 7.4(ii) also implies that* $\vdash_{n,n+1}$ *for* $n < \omega$ *can be given by some Hilbert-style inference system* $\langle Ax'_n, Ru'_n \rangle$*; while* $\vdash_{n,n+m}$ *with* $m \geq 2$ *cannot be given with such. The latter is so because* $\mathbf{SNr}_n\mathsf{CA}_{n+1}$ *is a finitely axiomatizable variety while* $\mathbf{SNr}_n\mathsf{CA}_{n+m}$, $m \geq 2$ *is not finitely axiomatizable (see [A97, Thm. 2.3]), and then one can use the presently discussed "methodogy of algebraization", cf. Thm.6.3, to infer the above information.* ■

$\vdash_{n,n+\omega}$ is strongly complete for $\mathbf{L}_n$, but the $\vdash_{n,n+\omega}$-proofs use formulas that are not in $\mathbf{L}_n$. Different kinds of complete inference systems for $\mathbf{L}_n$, where the proofs use only $\mathbf{L}_n$-formulas, are in [Si91], [V91], [MV, §5.5.1], [Mi95], [Mi96]. A common feature of the latter inference systems is that they are not strongly sound. (This is natural to expect because by Thm. 7.1 there cannot exist strongly sound and complete Hilbert-style inference systems for $\mathbf{L}_n$ if $n > 2$.)

Now we turn to $\mathbf{L}_2$. If $n \leq 1$, then $\vdash_n$ is strongly complete for $\mathbf{L}_n$ by Thm. 7.4, because then $\mathsf{CA}_n = \mathsf{RCA}_n$. $\vdash_2$ is not complete for $\mathbf{L}_2$ (but $\vdash_{2,3}$ is). We will show that if we add a rule or axiom expressing

$$(*) \qquad |\mathit{Dom}(R)| \leq 1 \Longrightarrow R = \mathit{Dom}(R) \times \mathit{Rng}(R),$$

(and the same for $\mathit{Rng}(R)$), then we get a strongly complete Hilbert-style inference system for $\mathbf{L}_2$. Namely, consider the following formula-shema and rule for $i \neq j$, $i, j < 2$:

$$(SA) \qquad \exists v_0\varphi \wedge \exists v_1\varphi \wedge \neg\varphi \to \exists v_i(\exists v_j(v_0 = v_1 \wedge \exists v_i\varphi) \wedge v_0 \neq v_1)$$

$$(SR) \qquad \frac{\exists v_i\varphi \wedge \exists v_j(v_0 = v_1 \wedge \exists v_i\varphi) \to v_0 = v_1}{\varphi \leftrightarrow (\exists v_0\varphi \wedge \exists v_1\varphi)}.$$

THEOREM 7.8 *Both* $\langle Ax_2 \cup (SA), Ru_2\rangle$ *and* $\langle Ax_2, Ru_2 \cup (SR)\rangle$ *are strongly complete for* $\mathbf{L}_2$.

Proof. This follows from Thm. 7.4 and Thm. 2.3(iii). ■

Now we turn to checking what Theorems 6.11 and 6.15 say about definability and interpolation properties of $\mathbf{L}_n$. $\mathbf{L}_n$ has the patchwork property of models.

So $\mathbf{L}_n$ for $n \geq 2$ does not have the local Beth definability property by Thm.6.11, because epimorphisms are not surjective in Cs_n (see [KMPT]+[M97a, T.7.4.(i)], for $2 \leq n < \omega$ see [ACN], for $n \geq \omega$ see [N88], [S90, Thm. 10]).

It is proved in [KSS] that $\mathbf{L}_3$ does not have the weak Beth property, and we conjecture that this extends to $3 < n < \omega$, while $\mathbf{L}_2$ has the weak Beth property. It is proved in [KSS] that for $n < \omega$, $\mathbf{L}_n$ has the weak local Beth property. By Thm.6.15, $\mathbf{L}_n$ does not have the interpolation property for all $n > 1$, since RCA_n does not have the amalgamation property (see [KMPT], this is a result of Comer [Co69]).

Further definability and interpolation results for $\mathbf{L}_n$, $\vdash_n$ (both $n < \omega$ and $n \geq \omega$) are in Madarász [M97b]. That paper is devoted to solving problems from Pigozzi [P72].

Summing up:

$\mathbf{L}_0$ is equivalent to sentential logic $\mathbf{L}_S{}'$.

$\mathbf{L}_1$ is equivalent to **S5**.

$\mathbf{L}_2$: Our characterization theorems in section 6 and the corresponding algebraic theorems in section 2 give the following properties for $\mathbf{L}_2$: $\mathbf{L}_2$ is decidable, it has a strongly complete and sound Hilbert-style inference system, which can be obtained from the equational axiomatization of RCA_2. $\mathbf{L}_2$ has the finite model property. The algebraic version of this is stated in [HMTII, 3.2.66]. It does not have the Beth (definability) property , and it does not have the interpolation property. We conjecture that $\mathbf{L}_2$ has the weak Beth property.

$\mathbf{L}_n$ for $3 \leq n < \omega$: The characterization theorems and the corresponding algebraic theorems give the following properties of $\mathbf{L}_n$: $\mathbf{L}_n$ is undecidable, $\mathbf{L}_n$ does not have a strongly sound and complete Hilbert-style inference system. It is open whether it has a weakly sound and weakly complete Hilbert-style inference system, cf. Problem 2.11 and Theorem 5.4. $\mathbf{L}_n$ has neither the Beth property nor the interpolation property.

$\mathbf{L}_\omega$: This is called "Finitary logic of infinitary relations". Model theoretic results (using AL) are in Németi [N90].

7. First-order logic with n variables with substitutions, with and without equality, $\mathbf{L}_n{}^{s=}$, $\mathbf{L}_n{}^{s}$, ($n \leq \omega$).

First we define $\mathbf{L}_n^{s=}$. The set of connectives is $\{\wedge, \neg, \exists v_i, [v_i/v_j], [v_i, v_j], v_i = v_j : i, j < n\}$, $\wedge$ binary, $v_i = v_j$ zero-ary, and the rest unary. Everything is as in the previous example, we only have to give the meanings of the logical connectives $[v_i/v_j], [v_i, v_j]$. Let $\mathfrak{M}$ be a model, and recall [10] the operations $[i/j], [i, j]$ mapping n to n. Now

$$mng_{\mathfrak{M}}([v_i/v_j]\varphi) = \{h \in {}^nM : h \circ [i/j] \in mng_{\mathfrak{M}}(\varphi)\},$$

$$mng_{\mathfrak{M}}([v_i, v_j]\varphi) = \{h \in {}^nM : h \circ [i, j] \in mng_{\mathfrak{M}}(\varphi)\}.$$

By this, we have defined $\mathbf{L}_n^{s=}$. It is not hard to check that $\mathbf{L}_n^{s=}$ is an algebraizable general logic.

The theory of quasi-polyadic algebras QPA's is analogous with that of cylindric algebras. Exactly as cylindric algebras are the algebraic counterparts of quantifier logics with equality, QPA's are the algebraic counterparts of quantifier logics *without* equality, cf. section 3 herein. RQPA_n and QPs_n denote the classes of representable QPA's of dimension n and quasi-polyadic set algebras (of dimension n) respectively as introduced e.g. in [N91] and in section 3 herein. Analogously, QPEA_n and QPse_n denote the same classes but with equality.

Now, $\mathbf{Alg}_m(\mathbf{L}_n^{s=}) = \mathsf{QPse}_n$, $\mathbf{Alg}(\mathbf{L}_n^{s=}) = \mathsf{RQPEA}_n$.

[10] $[i/j]$ sends i to j and leaves everything else fixed, and $[i, j]$ interchanges i and j and leaves everything else fixed.

If we omit equality from the set of connectives, then we get the equality-free version $\mathbf{L}_n^s$ of the logic. This is also an algebraizable general logic with $\mathbf{Alg}_m(\mathbf{L}_n^s) = \mathsf{QPs}_n$, $\mathbf{Alg}(\mathbf{L}_n^s) = \mathsf{RQPA}_n$.

We turn to showing how to retrieve substituted atomic formulas $R(v_{i_0} \dots v_{i_{n-1}})$ in $\mathbf{L}_n$, $\mathbf{L}_n^s$. Here we assume $n < \omega$.

First we treat the case $\mathbf{L}_n^s$. Since a finite mapping can always be written as a product of $[i,j]$'s and $[i/j]$'s, we obtain that for any sequence $x_0, \dots, x_{n-1}$ of variables there is a sequence $[i_1, j_1], \dots, [i_\ell/j_\ell]$ of "substitutions" such that for all models $\mathfrak{M}$ and relation symbols R,

$$mng_{\mathfrak{M}}(R(x_0 \dots x_{n-1})) = mng_{\mathfrak{M}}([v_{i_1}, v_{j_1}] \dots [v_{i_\ell}, v_{j_\ell}]R).$$

(Here the first meaning-function is taken from $\mathcal{L}_n'$, while the second one from $\mathcal{L}_n^s$.) This shows that in $\mathbf{L}_n^s$ we do have our substituted atomic formulas back as "complex" formulas. (On the other hand, the expressive power of $\mathbf{L}_n^s$ is not bigger than that of $\mathbf{L}_n'$, because of the following. It can be proved with a simple induction that the meaning of the formula $[v_i, v_j]\varphi$ is the same as that of the formula we get from φ by interchanging v_i and v_j in it everywhere (in the connectives $\exists v_i$ also), and the meaning of the formula $[v_i/v_j]\varphi$ coincides with that of the formula we get from φ by replacing v_i everywhere it it with v_j. Here φ is a formula of $\mathcal{L}_n'$.)

Now we show how to get substituted atomic formulas back in $\mathbf{L}_n$ by using Tarski's observation that substitution can be expressed with quantifiers and equality. By the above, it is enough to express the meaning of the formulas $[v_i/v_j]\varphi$ and $[v_i, v_j]\varphi$, for $i \neq j$. So let $\mathfrak{M}$ be a model and h an evaluation of the variables in M. Then it can be checked that

$$\mathfrak{M} \models_n ([v_i/v_j]\varphi \leftrightarrow \exists v_i(v_i = v_j \land \varphi))[h],$$

and if $k \neq i, j$, $\mathfrak{M} \models_n \varphi \leftrightarrow \exists v_k \varphi$, then

$$\mathfrak{M} \models_n ([v_i, v_j]\varphi \leftrightarrow [v_i/v_k][v_k/v_j][v_j/v_i]\varphi)[h].$$

Thus to express $[v_i, v_j]$ we need one extra free variable. We can get this e.g. by treating ${\mathcal{L}_n'}^{\mathcal{R}}$ as the following theory of $\mathcal{L}_{n+1}^{\mathcal{R}}$:

$$\{(\exists v_n R) \leftrightarrow R : R \in \mathcal{R}\}$$

and then treat the atomic formula $R(v_0, \dots, v_{n-1})$ of $\mathcal{L}_n'$ as the atomic formula R of $\mathcal{L}_{n+1}$. For more on this see [HMTII, §4.3], [BP89].

8. First-order logic, ranked [11] version, $\mathbf{L}_{FOL}^{ranked}$.

The set of connectives is $Cn = \{\wedge, \neg, \exists v_i, v_i = v_j : i, j < \omega\}$, $\wedge$ binary, $\neg, \exists v_i$ unary, and $v_i = v_j$ zero-ary. (This is the same as that of $\mathbf{L}_\omega$.)

Let $\mathcal{R}$ be a set (the set of relation-symbols), and let $\rho : \mathcal{R} \to \omega$ be a function (the *rank-function*, $\rho(R)$ is the *rank* of R). First we define the logic $\mathcal{L}_{FOL}^{\rho}$.

Our atomic formulas will be $R(v_0, \ldots, v_{\rho(R)-1})$ for $R \in \mathcal{R}$. We do not include $R(v_{i_0}, \ldots, v_{i_{\rho R-1}})$ into the set of atomic formulas for the same reason as in our previous examples: because they would immediately make our logic unstructural. However, these substituted atomic formulas will be present in our logic as (complex) formulas, because they can be expressed by quantifiers and equality (see our previous remark on this). Since the sequence $(v_0, \ldots, v_{\rho(R)-1})$ of variables is determined by ρ, we will just write R in place of $R(v_0, \ldots, v_{\rho(R)-1})$. (This will be convenient also when we will compare our present logic with $\mathbf{L}_\omega$.) Thus the set of atomic formulas is $\mathcal{R}$. Then the formula-algebra $\mathfrak{F}^\rho$ of $\mathcal{L}_{FOL}^{\rho}$ has universe $F(\mathcal{R}, Cn)$.

The models are $\mathfrak{M} = \langle M, R^{\mathfrak{M}} \rangle_{R \in \mathcal{R}}$ where $R^{\mathfrak{M}}$ is a $\rho(R)$-ary relation on M for all $R \in \mathcal{R}$. I.e.

$$M^\rho = \{\langle M, R^{\mathfrak{M}} \rangle_{R \in \mathcal{R}} : R \subseteq {}^{(\rho R)}M \text{ for all } R \in \mathcal{R}\}.$$

Validity and the meaning function are practically the same as those of $\mathbf{L}_\omega$, therefore we only give here the concise algebraic definition: Let $\mathfrak{M}$ be a model.

$mng_{\mathfrak{M}}(R) = \{h \in {}^\omega M : h \restriction \rho(R) \in R^{\mathfrak{M}}\}$, and

$mng_{\mathfrak{M}} : \mathfrak{F}^\rho \to \langle \mathfrak{P}({}^\omega M), C_i^M, D_{ij}^M \rangle_{i,j<\omega}$ is a homomorphism.

$\mathfrak{M} \models \varphi \quad \text{iff} \quad mng_{\mathfrak{M}}(\varphi) = {}^\omega M.$

Now

$$\mathbf{L}_{FOL}^{ranked} = \langle \mathcal{L}_{FOL}^{\rho} : \rho \text{ is a function into } \omega \rangle.$$

Let $\mathbf{L} = \mathbf{L}_{FOL}^{ranked}$. Then $\mathbf{L}$ is compositional, and has the filter-property. Also we have that

$$\mathbf{Alg}_m(\mathbf{L}) = \mathsf{Csf}_\omega \qquad \text{and} \qquad \mathbf{Alg}(\mathbf{L}) = \mathsf{Lf}_\omega.$$

The second statement is proved in [HMTII, 4.3.28(iii)].

But $\mathbf{L}$ is not substitutional, even $\mathcal{L}_{FOL}^{\rho}$ is not substitutional if $\rho \neq \emptyset$. An example is: Let R be an n-ary relation symbol in ρ and let φ denote the formula $v_0 = v_1 \wedge \ldots \wedge v_0 = v_n$. Then

$$\models R \to \forall v_n R \qquad \text{while} \qquad \not\models \varphi \to \forall v_n \varphi.$$

[11] These are called ordinary languages in [HMTII, §4.3].

It is easy to see that $\mathbf{L}$ is compact. Since $\mathbf{Up}\mathsf{Lf}_\omega \neq \mathsf{Lf}_\omega$, this logic shows that the condition of structurality in Theorem 6.7 is necessary.

We can extend our inference system $\vdash_\omega$ of the non-ranked logic $\mathbf{L}_\omega$ to get a complete one for $\mathbf{L}^{ranked}_{FOL}$, as follows.

For any rank-function $\rho : \mathcal{R} \to \omega$, let Ax^ρ denote the set of the following formulas:

$$(R') \qquad R \to \forall v_i R, \quad \text{if} \quad \rho(R) \leq i < \omega, \quad \text{and} \quad R \in \mathcal{R}.$$

(Ax^ρ is a straightforward modification of Ax^m_n.) Then $\vdash^\rho$ is defined as

$$\vdash^\rho \stackrel{\text{def}}{=} \{\langle \Sigma, \varphi\rangle : \Sigma \cup Ax^\rho \vdash_\omega \varphi,\ \Sigma \cup \{\varphi\} \subseteq \mathcal{L}^\rho_{FOL}\}.$$

Now, $\vdash^\rho$ provides a complete inference system for the ranked version of first-order logic $\mathbf{L}^{ranked}_{FOL}$. I.e.:

THEOREM 7.9 (Gödel's completeness theorem) *For every formula φ of $\mathcal{L}^\rho_{FOL}$ we have*

$$\models \varphi \quad \textit{iff} \quad \vdash^\rho \varphi.$$

Proof. This is a corollary of Thm. 7.4(ii) and $\mathsf{Lf}_\omega \subseteq \mathsf{RCA}_\omega = \mathbf{HSP}\mathsf{Csf}_\omega$ (Thm's 2.14(i), 2.4(iii)), as follows. Let $\equiv \stackrel{\text{def}}{=} \{(\varphi,\psi) : \vdash^\rho \varphi \leftrightarrow \psi\}$. Then $\mathfrak{F}^\rho/\equiv \in \mathsf{Lf}_\omega$ by Thm. 7.4(i) and *Rng*$(\rho) \subseteq \omega$. Assume $\not\vdash^\rho \varphi$. Then $\mathfrak{F}^\rho/\equiv \not\models \varphi = 1$, hence $\mathsf{Csf}_\omega \not\models \varphi = 1$ by $\mathsf{Lf}_\omega \subseteq \mathbf{HSP}\mathsf{Csf}_\omega$, i.e. $\not\models \varphi$. ■

At this point we should emphasize that there are valid schemes [12] Φ of $\mathcal{L}^\rho_{FOL}$ such that although $\models \Phi$, we have $\not\vdash^\rho \Phi$. This is so because there is no difference between the schema languages of $\mathcal{L}^\rho_{FOL}$ and $\mathbf{L}_\omega$, and also the valid schemes of $\mathbf{L}^{ranked}_{FOL}$ and $\mathbf{L}_\omega$ coincide by Cor.5.5, because $\mathbf{Eq}(\mathsf{Lf}_\omega) = \mathbf{Eq}(\mathsf{RCA}_\omega)$, and the $\vdash^\rho$-provable and $\vdash_\omega$-provable schemas coincide [13].

How it is possible that there is an $\vdash^\rho$-unprovable valid formula-scheme Φ? This means that though each instance of Φ in $\mathcal{L}^\rho_{FOL}$ is $\vdash^\rho$-provable (because of Thm. 7.9), these $\vdash^\rho$-proofs vary form instance to instance. We cannot give a "uniform" $\vdash^\rho$–proof for these instances, in spite of there being a uniform "cause" Φ of their validity.

[12] If Φ is a formula schema of $\mathcal{L}$ (cf. Def.4.4), then by $\mathcal{L}$-derivability $\vdash_\mathcal{L}$ of Φ we understand the natural extension of Def.4.4 to a mixed language consisting of both $\mathcal{L}$-formulas and schemes. I.e. in a derivation $\langle \Phi_1, \ldots, \Phi_n\rangle$ of Φ, Φ_i is built up from atomic formulas $p_j \in P$ of $\mathcal{L}$ and formula-variables $\phi_j \in FV$ (using the connectives Cn of $\mathcal{L}$).

[13] This not quite trivial, but can be proved with CA-theoretic methods, e.g. one can use [HMT, 2.5.26].

Theorem 2.14(i) stating $\mathsf{Dc}_\omega \subseteq \mathsf{RCA}_\omega$ can be used to overcome schema-incompleteness of $\vdash^\rho$. Using this theorem, one can obtain enriched inference systems $\vdash^{\rho+}$ by adding brand new variables w_i $(i < \omega)$ to the language and new axioms postulating the effects of the fact that w_i does not occur in the *old* formulas. Roughly, these axioms say that

$$\begin{array}{ll} R \to \forall v_i R & \text{if } \rho(R) \leq i < \omega \quad \text{and} \\ \phi_j \to \forall w_i \phi_j & \text{if } i < \omega,\ \phi_j \text{ is a formula-variable.} \end{array}$$

These inference systems are strongly complete for the formula-schemas of $\mathbf{L}_{FOL}^{ranked}$. These completeness theorems (based on the Dc-representation result) are proved in [AGN77], [HMTII, §4.3].

The reason for $\mathbf{L}_{FOL}^{ranked}$ not being substitutional is that the atomic formulas cannot take the meanings of any formula, because an atomic formula has a fixed finite rank, while formulas can have meanings of arbitrarily large finite ranks. This will be repared in our next example.

9. First-order logic, rank-free [14] (or type-less) version, $\mathbf{L}_{FOL}$.

The set of connectives are as in the previous case. Let $\mathcal{R}$ be a set (of relation symbols). Then the set of atomic formulas of $\mathcal{L}_{FOL}^{\mathcal{R}}$ is $\mathcal{R}$, as before.

The models will be different (as the information $\rho : \mathcal{R} \to \omega$ is missing): We only know that R denotes a finitary relation, we do not know what its arity is. The actual arity will be given by the model. I.e., the models are $\mathfrak{M} = \langle M, R^{\mathfrak{M}} \rangle_{R \in \mathcal{R}}$ where $R^{\mathfrak{M}}$ is an arbitrary finitary relation on M for all $R \in \mathcal{R}$,

$$M^{\mathcal{R}} = \{\langle M, R^{\mathfrak{M}} \rangle_{R \in \mathcal{R}} : (\forall R \in \mathcal{R})(\exists n \in \omega) R \subseteq {}^n M\}.$$

Validity and the meaning function are the same as in the previous case, the only difference is that

$$mng_{\mathfrak{M}}(R) = \{h \in {}^\omega M : h \restriction n \in R^{\mathfrak{M}} \text{ for some } n\}.$$

Let $\mathbf{L}_{FOL}$ denote the system of these logics. Now this general logic is structural, since

$$Mng^{\mathcal{R}} = Hom(\mathfrak{F}^{\mathcal{R}}, \mathsf{Csf}_\omega).$$

Thus $\mathbf{L}_{FOL}$ is an algebraizable general logic with

$$\mathbf{Alg}_m(\mathbf{L}_{FOL}) = \mathsf{Csf}_\omega \quad \text{and} \quad \mathbf{Alg}_m(\mathbf{L}_{FOL}) = \mathbf{SP}\mathsf{Csf}_\omega.$$

[14] Rank-free first-order logic was introduced in Henkin-Tarski [HT], and elaborated in more detail in [A73], [AGN77, sec. IV]. See also [HMTII, section 4.3]. A nice proof system for this logic is given in Simon [Si91].

Thus Theorem 6.7 says that $\mathbf{L}_{FOL}$ is not compact because $\mathbf{SP}\mathsf{Csf}_\omega \neq \mathbf{UpSP}\mathsf{Csf}_\omega$ by Thm. 2.4(iii). Or vice versa, one can prove the algebraic theorem $\mathbf{SP}\mathsf{Csf}_\omega \neq \mathbf{UpSP}\mathsf{Csf}_\omega$ by showing that $\mathbf{L}_{FOL}$ is not compact, as follows: $\mathbf{L}_{FOL}$ is not compact because the set $\Sigma = \{\neg(R \leftrightarrow \exists v_i R) : i < \omega\}$ of formulas, where R is any relation symbol, is not satisfiable while all of its finite subsets are. Thus Theorem 6.7 says that $\mathbf{SP}\mathsf{Csf}_\omega \neq \mathbf{UpSP}\mathsf{Csf}_\omega$ because $\mathbf{Alg}(\mathbf{L}_{FOL}) = \mathbf{SP}\mathsf{Csf}_\omega$.

Thm.6.4 admits a generalization to logics like $\mathbf{L}_{FOL}$ above. Then we obtain the following corollary of this generalized result and of Thm.2.5 (saying that RCA_n is not finite schema axiomatizable).

COROLLARY 7.10 *Assume that $\langle Ax, Ru\rangle$ defines a strongly sound and weakly complete inference system $\vdash$ for $\mathbf{L}_{FOL}$. Then $\langle Ax, Ru\rangle$ must involve an infinite set of formula-variables. I.e. $\mathbf{L}_{FOL}$ is not finite-schema axiomatizable. The same applies for $\mathbf{L}_\omega$ in place of $\mathbf{L}_{FOL}$.* ■

Improved versions of this negative result are in [A97] where it is proved that $\langle Ax, Ru\rangle$ has to be extremely complex, too, besides involving infinitely many formula-variables. Positive results kind of side-stepping Corollary 7.10 above are in [S95], [S97], [SGy]. These present expansions of $\mathbf{L}_{FOL}$ with further logical connectives, such that the new $\mathbf{L}^+_{FOL}$ becomes finite schema axiomatizable.

At this point the reader might have the impression that Corollary 7.10 seems to contradict Gödels completeness theorem. However, Gödels theorem holds for the ranked version $\mathbf{L}^{ranked}_{FOL}$ of first order logic but not for $\mathbf{L}_{FOL}$. The essential difference between these two logics is that $\mathbf{L}^{ranked}_{FOL}$ is *not* structural (substitutional). No structural version of first order logic is known for which Gödel's completeness theorem would hold. More precisely, the only such versions are the logics presented in [SGy] etc. cited above. The presently discussed issue is highly relevant to the propositional modal versions of first order logic, cf. e.g. van Benthem [vB97], [vB96], [vBtM], [V95a], [MV].

Now we briefly compare our three versions of FOL: non-ranked $\mathbf{L}_\omega$, the ranked version $\mathbf{L}^{ranked}_{FOL}$, and the rank-free one, $\mathbf{L}_{FOL}$. The same formula-schemes are valid in them, and they have the same admissible rules by Theorem 5.4, because the same quasi-equations are true in their algebraized forms by $\mathbf{SPUp}\mathsf{Csf}_\omega = \mathsf{RCA}_\omega = \mathbf{Alg}(\mathbf{L}_\omega)$. Also, this set of admissible formula-schemes is recursively enumerable, and the validity problem in these logics is not decidable, by Theorem 2.4, Theorem 5.4, Corollary 5.6.

As a contrast, here we will give a logic which has a decidable validity problem and at the same time the set of valid formula-schemes is not even recursively enumerable.

10. Equality logic, monadic logic.

First we treat equality logic $\mathcal{L}^e$. This is the same as first-order logic with ω variables and with no atomic formulas, i.e. $\mathcal{L}^e \stackrel{\text{def}}{=} \mathcal{L}^{\emptyset}_{\omega}$. Therefore, this is not a general logic. Notice that the set of formulas is non-empty because $v_i = v_j$ is a zero-ary logical connective. A model is just a set M and the meaning-algebra $\mathfrak{Mng}(M)$ of this model is the subalgebra of $\langle \mathfrak{P}({}^{\omega}M), C_i, D_{ij}\rangle_{i,j<\omega}$ generated by $\{D_{ij} : i, j < \omega\}$. These are called the *minimal cylindric set algebras*, and their class is denoted by setMn_{ω}, while $\mathsf{Mn}_{\omega} \stackrel{\text{def}}{=} \mathbf{I}\mathsf{setMn}_{\omega}$.

$\mathcal{L}^e$ is an algebraizable semantic logic with $\mathbf{Alg}_m(\mathcal{L}^e) = \mathsf{setMn}_{\omega}$ and $\mathbf{Alg}(\mathcal{L}^e) = \mathbf{SP}\mathsf{Mn}_{\omega}$. ($\mathcal{L}^e$ is substitutional because its set of atomic formulas is empty.)

It is well known that the validity problem of $\mathcal{L}^e$ is decidable, it has the finite model property, and it admits an elimination-of-quantifiers theorem. (See e.g. [M64a].)

However, the set of valid formula-schemes of $\mathcal{L}^e$ is not even recursively enumerable. This is so by Corollary 5.5, because $\mathbf{Eq}(\mathsf{Mn}_{\omega})$ is not recursively enumerable.[15]

More generally, consider now ranked first-order logics $\mathcal{L}^{\rho}_{\omega}$. Ranked first-order logic $\mathcal{L}^{\rho}_{n}$ with n variables, $n < \omega$ can be defined analogously for $\rho : \mathcal{R} \to n$. Let $n \leq \omega$. If every relation symbol is unary, i.e. if $Rng\rho \subseteq \{1\}$, then $\mathcal{L}^{\rho}_{n}$ is called a *monadic* logic. Let $\mathcal{L}^{m\mathcal{R}}_{n}$ denote monadic logic with relation symbols $\mathcal{R}$, i.e. $\mathcal{L}^{m\mathcal{R}}_{n} \stackrel{\text{def}}{=} \mathcal{L}^{\rho}_{n}$ where $\rho = \mathcal{R} \times \{1\}$.

$\mathcal{L}^{m\mathcal{R}}_{n}$ is a compositional logic with filter-property. It is not substitutional.

It is known that the validity problem of monadic logics is also decidable, they have the finite model property, and they admit elimination of quantifiers. (See also [M64a].)

Let $n > 2$. If n is infinite, then the valid schemes of $\mathcal{L}^{m\mathcal{R}}_{n}$ are not recursively enumerable. If ρ is not monadic, then the valid schemes of $\mathcal{L}^{\rho}_{n}$ are recursively enumerable (and the validities become undecidable). (If $n \leq 2$, then the set of valid schemes of $\mathcal{L}^{\rho}_{n}$ is decidable.) These are proved in [N87] by showing that the equational theories of the corresponding classes of algebras are not recursively enumerable (and using Theorem 5.4). The logical implications and the reasons for this behaviour are also explained carefully in [N87].

11. Infinitary version $\mathbf{L}^{n}_{\infty\omega}$ of the finite variable fragments $\mathbf{L}_n$.

Let κ be an infinite cardinal. $\mathbf{L}^{n}_{\kappa\omega}$ is obtained from $\mathbf{L}_n$ by adding κ-ary conjunction to the logical connectives. More formally, let F_n be the set of formulas of $\mathcal{L}^{\mathcal{R}}_{n} = \langle F_n, \ldots\rangle$. Let F^{n}_{κ} be the smallest set satisfying (i)–(iii) below.

[15] This was proved by M. Rubin, and independently by I. Németi, see [N87].

(i) $F^n_\kappa \supseteq F_n$,

(ii) F^n_κ is closed under the connectives of $\mathcal{L}_n$,

(iii) $|H| < \kappa \;\Rightarrow (\wedge H) \in F^n_\kappa$, for any $H \subseteq F^n_\kappa$.

The models of $\mathcal{L}^{n\mathcal{R}}_{\kappa\omega}$ are the same as those of $\mathcal{L}^{\mathcal{R}}_n$, and mng_κ, F_κ are the obvious generalizations of the definition given for $\mathcal{L}^{\mathcal{R}}_n$. Then

$$\mathcal{L}^{n\mathcal{R}}_{\kappa\omega} \stackrel{\text{def}}{=} \langle F^n_\kappa, M_n, mng_\kappa, F_\kappa\rangle \quad \text{and} \quad \mathbf{L}^n_{\kappa\omega} \stackrel{\text{def}}{=} \langle \mathcal{L}^{n\mathcal{R}}_{\kappa\omega} \;:\; \mathcal{R} \text{ is a set}\rangle\,.$$

$\mathbf{L}^n_{\infty\omega}$ is obtained from $\mathbf{L}^n_{\kappa\omega}$ by removing all conditions of the form "$\ldots < \kappa$". That is, $F^n_{\infty\omega} := \bigcup\{F^n_{\kappa\omega} \;:\; \kappa \text{ is a cardinal}\}$, etc. $\mathbf{L}^n_{\infty\omega}$ and $\mathbf{L}^n_{\kappa\omega}$ (with $\kappa > \omega$) are *not* logics in the sense of Def .4.1 because they involve infinitely long strings of symbols. All the same, $\mathbf{L}^n_{\infty\omega}$ is an interesting mathematical structure whose study is motivated by studying logics. Most properties of logics make sense for the "pseudo-logic" $\mathbf{L}^n_{\infty\omega}$, too. Studying mathematical structures like $\mathbf{L}^n_{\infty\omega}$, $\mathbf{L}^n_{\kappa\omega}$ seems to be useful for obtaining a better understanding of logics (in the sense of Def. 4.1).

Most of the results obtained for $\mathbf{L}_n$ via the methods of algebraic logic can be pushed through for $\mathbf{L}^n_{\infty\omega}$ by the same kinds of algebraic methods. In particular, by stretching the algebraic methods which lead to Theorem 7.1 , one can obtain the following. The notions of formula schema, inference system, axiom schema, rule schema can be generalized to $\mathbf{L}^n_{\kappa\omega}$ the natural way. Herein we do not go into the details of this.

COROLLARY 7.11 *Assume $\vdash$ is a strongly sound and weakly complete provability relation for $\mathbf{L}^n_{\infty\omega}$ or for $\mathbf{L}^n_{\kappa\omega}$ ($\kappa \geq \omega$). Then $\vdash$ is not definable by a Hilbert-style inference system. Moreover, any schema $\langle Ax, Ru\rangle$ axiomatizing $\vdash$ must involve infinitely many formula variables (cf. Def.4.4 for $\langle Ax, Ru\rangle$ axiomatizing $\vdash$.)*

The next table summarizes the algebraic counterparts of some of the distinguished logics.

Logic $\mathcal{L}$	$\mathbf{Alg}(\mathcal{L})$	$\mathbf{Alg}_m(\mathcal{L})$	substitutional, compact
$\mathbf{L}_s$ sentential logic	BA	$\{\mathbf{2}\}$	$+ \quad +$
S5 modal logic	RCA_1	Cs_1	$+ \quad +$
$\mathbf{L}_{REL}$ arrow logic	BRA	setBRA	$+ \quad +$
$\mathbf{L}'_n$ n-var. FOL with substituted atomic fmlas	$\mathbf{Rd}_{ca}$ RPEA_n	$\mathbf{Rd}_{ca}$ Pse_n	$- \quad +$
$\mathbf{L}_n$ structural n-var. FOL	RCA_n	Cs_n	$+ \quad +$
$\mathbf{L}_\omega$ finitary FOL of ω-ary rel.'s	RCA_ω	Cs_ω	$+ \quad +$
$\mathbf{L}_{FOL}^{ranked}$ ranked FOL	Lf_ω	Csf_ω	$- \quad +$
$\mathbf{L}_{FOL}$ rank-free FOL	$\mathbf{SP}\mathsf{Csf}_\omega$	Csf_ω	$+ \quad -$

Table 7.1.

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