SIMPLE PROOF FOR DECIDABILITY OF THE UNIVERSAL THEORY OF CYLINDRIC SET ALGEBRAS OF DIMENSION 2

Hajnal Andréka and István Németi

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We present here a proof of decidability of the universal formulas valid in Cs_2, the class of cylindric set algebras of dimenson 2 (this is Thm.4.2.8 of [1]). This proof is analogous to the proof of decidability of the universal theory of CA_2 in [1, 2.5.4], due to Henkin, and, in our opinion, is quite elementary (compared to the proof in [1, 4.2.8]).

Let $\mathfrak{A} \in Cs_2$ and $X \subseteq A$ be arbitrary. We will define a $\mathfrak{B} \in Cs_2$ with finite base and a function $f : X \to B$ such that f is a weak embedding between the partial structures $\mathfrak{A} \upharpoonright X$ and $\mathfrak{B} \upharpoonright f(X)$ (see Claim 2) [1] . This will show that any universal formula that fails in Cs_2, fails also in a Cs_2 with finite base, and thus gives a recursive enumeration of the universal formulas not valid in Cs_2. Since the universal formulas valid in Cs_2 are recursively enumerable [2], too, this will prove that the set of universal formulas valid in Cs_2 is decidable [3] . (Recall that if $\mathfrak{A} \in Cs_2$ then $A \subseteq Sb(U \times U)$ for some set U, and this set U is called the base, or base set, of $\mathfrak{A}$.)

For any set H, SbH denotes the powerset of H, and by a partition of a set H we mean $\mathcal{H} \subseteq SbH$ such that $\mathcal{H}$ consists of pairwise disjoint nonempty sets such that $\bigcup \mathcal{H} = H$.

Since any finite subset of A generates only a finite subset by the Boolean operations of $\mathfrak{A}$, we may assume that X is closed under the Boolean operations of $\mathfrak{A}$ and $d_{01} \in X$. Let $At = \{R_1, ..., R_k\}$ be the set of atoms of X, let U denote the base set of $\mathfrak{A}$, and let $\mathcal{P}$ denote the Boolean subalgebra of SbU generated by $\{DoR, RgR : R \in At\}$, and let $\mathcal{R}$ denote the Boolean subalgebra of SbU generated by $\{DoS, RgS : S \in \{R \cap (P \times Q) : R \in At, P, Q \in \mathcal{P}\}\}$. Then $\mathcal{P} \subseteq \mathcal{R}$ because

[1]We cannot do better, i.e. there are an $\mathfrak{A} \in Cs_2$ and a finite $X \subseteq A$ such that X generates an infinite subset of A.

[2]This can be seen e.g. by "translating" Cs_2 into a finitely axiomatizable class of many-sorted structures.

[3]Another way of concluding decidability is the following: An upper bound for the size of the base set of $\mathfrak{B}$ can be effectively computed from the size of X (by tracing sizes in the proof given hereby). This gives us an effectively computed upper bound $\beta(\varphi)$ for any universal formula φ such that φ fails in Cs_2 iff φ fails in some Cs_2 with base $\leq \beta(\varphi)$.

$R\cap(DoR\times RgR) = R$ for any R. For any finite Boolean algebra $\mathfrak{B}$, let $At\mathfrak{B}$ denote the set of atoms of $\mathfrak{B}$. Then $At\mathcal{P}$, $At\mathcal{R}$ are partitions of U, and $At\mathcal{R}$ is a refinement of the partition $At\mathcal{P}$.

For any $H \in At\mathcal{R}$ let H' be a set such that $|H'| = 1$ if $H \in \mathcal{P}$ with $|H| = 1$, otherwise let $|H'| = k$, the number of atoms of X, and $H'\cap Q' = \emptyset$ if $H\cap Q = \emptyset$. (Note that if $H \in \mathcal{R}, |H| = 1$ but $H \notin \mathcal{P}$ then $|H'| = k$.) Let $U' = \bigcup\{H' : H \in At\mathcal{R}\}$. Then U' is finite and $\{H' : H \in At\mathcal{R}\}$ is a partition of U'. (See Figure 1.)

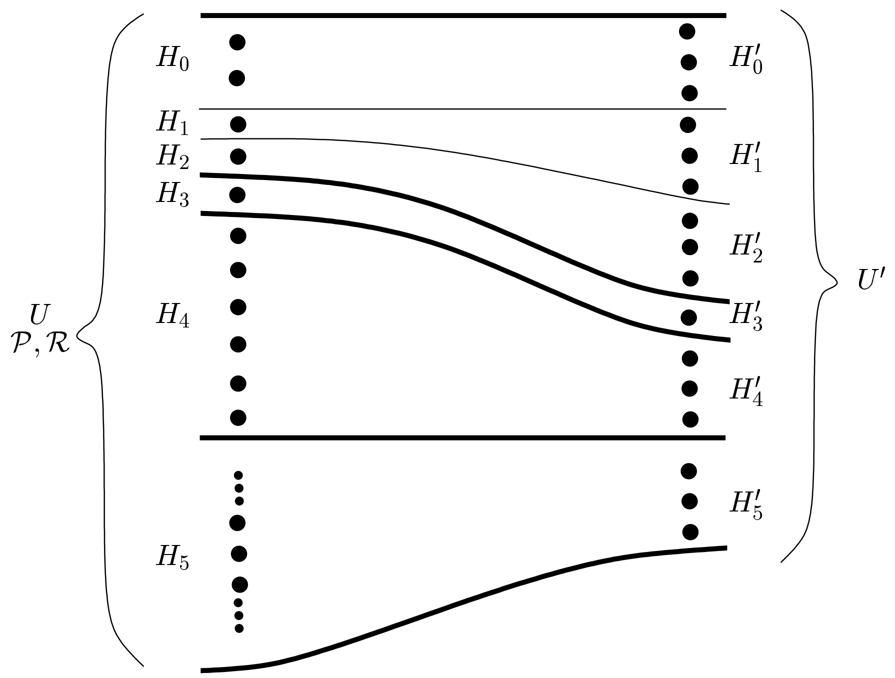

Figure 1: Example: $At\mathcal{R} = \{H_0, \ldots, H_5\}$, $At\mathcal{P} = \{(H_0 \cup H_1 \cup H_2), H_3, H_4, H_5\}$, $k = 3$. Note that $|H'_1| = 3$, $|H'_3| = 1$, $|H_5| = \infty$, $|H'_5| = 3$.

Let $g : \mathcal{R} \to \mathcal{R}' \subseteq SbU'$ be a Boolean isomorphism such that $g(H) = H'$ for any $H \in At\mathcal{R}$. For any $H \in \mathcal{R}$, we will write H' for $g(H)$. Next we define a function on $\{R\cap(P\times Q) : R \in At,\ P, Q \in At\mathcal{P}\}$. We will denote this function by f. Let $d'_{01} = \{(u, u) : u \in U'\}$. We note that $d_{01} = \{(u, u) : u \in U\}$.

Let $P, Q \in At\mathcal{P}$ and assume first that $|P| = 1$ or $|Q| = 1$.

Then if $\rho = R \cap (P \times Q)$ for $R \in At$, then $\rho = Do\rho \times Rg\rho$ and $Do\rho, Rg\rho \in \mathcal{R}$. Define

$$f(\rho) = (Do\rho)' \times (Rg\rho)'.$$

Let $P, Q \in At\mathcal{P}$ and assume next $|P| > 1, |Q| > 1$.

Then $|P'| \geq k,\ |Q'| \geq k$, and $\mathcal{H} = \{R \cap (P \times Q) : R \in At\} \smallsetminus \{\emptyset\}$ is a partition of $P \times Q$ (since At is a partition of $U \times U$). We want to have a partition $\mathcal{H}'$ of $P' \times Q'$ for which (a)-(c) below hold.

(a) $|\mathcal{H}'| = |\mathcal{H}|$,

(b) $|\mathcal{H}' \cap Sb(d'_{01})| = |\mathcal{H} \cap Sb(d_{01})|$, and $(\forall S \in \mathcal{H}')[S \subseteq d'_{01}$ or $S \cap d'_{01} = \emptyset]$,

(c) $(\forall S \in \mathcal{H}')[DoS = P',\ RgS = Q']$. [4]

To show that such partition $\mathcal{H}'$ exists, assume first that $P \neq Q$. Then $P \cap Q = \emptyset$, and thus $\mathcal{H} \cap Sbd_{01} = \emptyset$, $(P' \times Q') \cap d'_{01} = \emptyset$, so condition (b) will hold automatically. Let $|\mathcal{H}| = m$ and let $P_0, ..., P_{m-1}, Q_0, ..., Q_{m-1}$ be partitions of P', Q' respectively. Such partitions exist by $m \leq k,\ |P'| \geq k,\ |Q'| \geq k$. For any $j < m$ let

$$H_j = \bigcup \{P_i \times Q_{i+j (mod\ m)} : i < m\},$$

and let $\mathcal{H}' = \{H_j : j < m\}$. Then it can be checked that $\mathcal{H}'$ is a partition of $P' \times Q'$ which satisfies (a),(c).

Assume next $P = Q$. First we will show that $Id_P = \{(u,u) : u \in P\} \in \mathcal{H}$. Indeed, let $R \in At,\ R \subseteq d_{01}$ and assume that $R \cap (P \times P) \neq \emptyset$. By $R \in At$ we have $DoR \in \mathcal{P}$, thus $P \subseteq DoR$ by $P \in At\mathcal{P},\ R \cap (P \times P) \neq \emptyset$. But then $R \cap (P \times P) = Id_P$ by $R \subseteq d_{01}$.

By $Id_P \in \mathcal{H}$ we have that $\mathcal{H} \cap Sbd_{01} = \{Id_P\}$, since $\mathcal{H}$ is a partition of $P \times P$. Thus (b) will be taken care of if we set $Id_{P'} = \{(u,u) : u \in P'\} \in \mathcal{H}'$.

To construct $\mathcal{H}'$ in the case $P = Q$, let $|\mathcal{H}| = m$, let $P_0, ..., P_{m-1}$ be a partition of P', and let

$$H_j = \bigcup \{P_i \times P_{i+j (mod\ m)} : i < m\}$$

as in the previous case. Then $Id_{P'} \subseteq H_0$. Set

[4] We note that $\{R_1, ..., R_k\}$ can be an arbitrary partition of $U \times U$, with the only restriction that $R_i \subseteq d_{01}$ or $R_i \cap d_{01} = \emptyset$ for all $1 \leq i \leq k$. Then, $\mathcal{H}$ can be an arbitrary partition of $P \times Q$ with the only restriction that $\{(u,u) : u \in P\} \in \mathcal{H}$ if $P = Q$ (see an argument later in the proof). In particular, DoS, RgS can be "arbitrary" for $S \in \mathcal{H}$. We want property (c) of $\mathcal{H}'$ to ensure Claim 1.(iii), which we will use to show that f is a partial embedding w.r.t. c_0, c_1.

$H_0' = Id_{P'}$, $\quad H_1' = H_1 \cup \{H_0 \setminus Id_{P'}\}$ $\qquad$ and

$\mathcal{H}' = \{H_0', H_1'\} \cup \{H_j : 2 \leq j < m\}$.

(We note that $m \geq 2$ by $|P| > 1$, $Id_P \in \mathcal{H}$, and $\bigcup \mathcal{H} = P \times P$.)

Let now $f : \mathcal{H} \cup \{\emptyset\} \to \mathcal{H}' \cup \{\emptyset\}$ be a bijection such that $f(\emptyset) = \emptyset$ and for all $S \in \mathcal{H}$ we have that $S \subseteq d_{01}$ iff $f(S) \subseteq d_{01}'$.

Let $f : X \to Sb(U' \times U')$ be defined by

$$f(x) = \sum\{f(R \cap (P \times Q)) : R \in At,\ P, Q \in \mathcal{P},\ R \cap (P \times Q) \subseteq x\}.$$

Claim 1

(i) $\{f(R) : R \in At\}$ is a partition of $U' \times U'$.

(ii) $\{f(R) : R \in At, R \subseteq d_{01}\}$ is a partition of d_{01}'.

(iii) $Do(f(R)) = (DoR)'$ and $Rg(f(R)) = (RgR)'$ for all $R \in At$.

Proof of Claim 1 First we show that $\{f(R) : R \in At\}$ is a partition of $U' \times U'$. Assume $R, S \in At$, $R \neq S$. Then $R \cap S = \emptyset$. Let $P, Q \in At\mathcal{P}$. If $|P| > 1$, $|Q| > 1$, then $f(R \cap (P \times Q)) \cap f(S \cap (P \times Q)) = \emptyset$ by our construction. Assume $|P| = 1$. Then $Rg(R \cap (P \times Q)) \cap Rg(S \cap (P \times Q)) = \emptyset$, by $R \cap S = \emptyset$, thus $f(R \cap (P \times Q)) \cap f(S \cap (P \times Q)) = \emptyset$ by our construction. The case $|Q| = 1$ is similar. The above together with $f(R \cap (P \times Q)), f(S \cap (P \times Q)) \subseteq P \times Q$ show that $f(R) \cap f(S) = \emptyset$.

Next we show that $\bigcup\{f(R) : R \in At\} = \bigcup\{f(R \cap (P \times Q)) : R \in At, P, Q \in At\mathcal{P}\} = U' \times U'$. Let $P, Q \in At\mathcal{P}$. If $|P| > 1, |Q| > 1$ then $\bigcup\{f(R \cap (P \times Q)) : R \in At\} = P' \times Q'$ by our construction. Assume $|P| = 1$. Then by $P \times Q = \bigcup\{R \cap (P \times Q) : R \in At\}$ we have $Q = \bigcup\{Rg(R \cap (P \times Q)) : R \in At\}$, hence $Q' = \bigcup\{f(Rg(R \cap (P \times Q)) : R \in At\}$, thus $P' \times Q' = \bigcup\{f(R \cap (P \times Q)) : R \in At\}$ by our construction. The case $|Q| = 1$ is analogous. Let $R \in At$. Clearly, $f(R) \neq \emptyset$, since by $R \neq \emptyset$ we have $R \cap (P \times Q) \neq \emptyset$ for some $P, Q \in At\mathcal{P}$, and then $\emptyset \neq f(R \cap (P \times Q)) \subseteq f(R)$. We have seen that $\{f(R) : R \in At\}$ is a partition of $U' \times U'$.

Let $R \in At$. We want to show $Dof(R) = (DoR)'$. Let $P \in At\mathcal{P}$. By $DoR \in \mathcal{P}$ we have either $P \subseteq DoR$ or $P \cap DoR = \emptyset$. By our construction we have that

$Dof(R \cap (P \times Q)) \subseteq P'$ for all $P, Q \in At\mathcal{P}$. Thus if $P \cap DoR = \emptyset$, then $f(R \cap (P \times Q)) = \emptyset$ for all $Q \in At\mathcal{P}$, and thus $P' \cap Dof(R) = \emptyset$.

Assume now $P \subseteq DoR$. We want to show $P' \subseteq Dof(R)$. By $P \in \mathcal{P} \subseteq \mathcal{R}$ we have that $P = \bigcup\{T : T \in At\mathcal{R}, T \subseteq P\}$. So let $T \in At\mathcal{R}, T \subseteq P$. Let $u \in T \cap DoR$. Then $(u, v) \in R$ for some v. Let $Q \in At\mathcal{P}$ be such that $v \in Q$. If $|Q| = 1$ or $|P| = 1$ then $T = Do(R \cap (P \times Q))$ and $T' = Dof(R \cap (T \times Q)) \subseteq Dof(R)$ by our construction. Otherwise, if $|P| > 1, |Q| > 1$ then $T' \subseteq P' = Dof(R \cap (P \times Q)) \subseteq Dof(R)$.

By the above we have seen $Dof(R) = (DoR)'$ for all $R \in At$. The proof for $Rgf(R) = (RgR)'$ is completely analogous, we omit it. **QED(Claim 1)**

Claim 2

(i) f is a Boolean isomorphism on X.

(ii) $f(d_{01}) = d'_{01}$.

(iii) If $x, c_i x \in X$ then $f(c_i)x = c_i f(x)$, for $i \in \{0, 1\}$.

Proof of Claim 2 (i) follows from Claim 1(i). (ii) follows from Claim 1(i),(ii). (iii) follows from Claim 1(iii) as follows: By $Dox = \bigcup\{DoR : R \subseteq x, R \in At\}$ we have $Dof(x) = (Dox)'$ for all $x \in X$. Let $i = 1$. Then $c_1 x = Dox \times U$ and $c_1 f(x) = (Dox)' \times U'$. Now $c_1 x = \bigcup\{R : R \in At, DoR \subseteq Dox\}$, hence $f(c_1 x) = \sum\{f(R) : DoR \subseteq Dox\} = \sum\{f(R) : Do(f(R)) \subseteq (Dox)'\} = (Dox)' \times U'$. The case $i = 0$ is analogous. **QED(Claim 2)**

References

[1] L. Henkin, J. D. Monk, and A. Tarski. *Cylindric Algebras Parts I and II.* North–Holland, Amsterdam, 1971 and 1985.