Functionally dense relation algebras

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Abstract. We give a new proof of a theorem due to Maddux and Tarski that every functionally dense relation algebra is representable. Our proof is very close in spirit to the original proof of the theorem of Jónsson and Tarski that atomic relation algebras with functional atoms are representable. We prove that a simple, functionally dense relation algebra is either atomic or atomless, and that every functionally dense relation algebra is essentially isomorphic to a direct product $\mathtt{B} \times \mathtt{C}$, where $\mathtt{B}$ is a direct product of simple, functionally dense relation algebras each of which is either atomic or atomless, and $\mathtt{C}$ is a functionally dense relation algebra that is atomless and has no simple factors at all. We give several new structural descriptions of all atomic relation algebras with functional atoms. For example, each such algebra is essentially isomorphic to an algebra of matrices with entries from the complex algebra of some group. Finally, we construct examples of functionally dense relation algebras that are atomless and simple, and examples of functionally dense relation algebras that are atomless and have no simple factors at all.

Jónsson and Tarski [8] proved that every (abstract) atomic relation algebra with functional atoms is representable as an algebra of binary relations. They drew the corollary that every relation algebra in which the unit is a sum of finitely many functional elements is representable as an algebra of binary relations. Tarski posed the question whether every functionally dense relation algebra—that is to say, every relation algebra in which the unit is a sum of some set (finite or infinite) of functional elements —is representable. A positive answer to this question was given by Maddux and Tarski—see the abstract [10] and the paper [9]. Jónsson and Tarski also gave a description of all complete and atomic relation algebras with functional atoms, using complex algebras of axiomatically defined generalized Brandt groupoids. In particular, they proved that a simple relation algebra is complete and atomic with functional atoms if and only if it is isomorphic to the complex algebra of a Brandt groupoid.

In this paper we give a new proof of Maddux and Tarski's representation theorem, one that in our opinion is very close in spirit to the original proof of Jónsson and Tarski's representation theorem and also to the standard Cayley representation theorem for groups. We then show that a functionally dense relation algebra that is simple is either atomic or atomless. Consequently, the completion of every functionally dense relation algebra can be decomposed into a direct product $\mathfrak{B} \times \mathfrak{C}$ with the following properties: $\mathfrak{B}$ is a direct product of simple, functionally dense relation algebras each of which is either atomic or atomless; and $\mathfrak{C}$ is either trivial (that is to say, it has just one element) or else it is a non-trivial functionally dense relation algebra that is atomless and has no simple factors because its Boolean algebra of ideal elements is atomless. We give several different structural descriptions of the simple relation algebras

that are atomic with functional atoms. We believe that these descriptions are more visual and more familiar than the Jónsson-Tarski description using axiomatically defined Brandt groupoids. We also construct a class of examples of simple, atomless, functionally dense relation algebras. The problem whether every simple, atomless, functionally dense relation algebras is essentially isomorphic to one of these examples is left open. Finally, we close with a class of examples of functionally dense relation algebras for which the Boolean algebras of ideal elements are also atomless.

1. Preliminaries

A *relation algebra* is an algebra

$$\mathfrak{A} = (A\,,\,+\,,\,-\,,\,;\,,\,{}^{\smile}\,,\,1')$$

in which $+$ and $;$ are binary operations on the universe A, while $-$ and ${}^{\smile}$ are unary operations on A, and 1' is a distinguished constant of A, such that $(A\,,\,+\,,\,-)$ is a Boolean algebra, and the following additional postulates (due to Tarski) are satisfied for all elements r, s, and t in A.

(i) (Associative law for Relative Multiplication) $r\,;(s\,;t) = (r\,;s)\,;t$.
(ii) (Identity Law for Relative Multiplication) $r\,;1' = r$.
(iii) (First Involution Law) $r^{\smile\smile} = r$.
(iv) (Second Involution Law) $(r\,;s)^{\smile} = s^{\smile}\,;r^{\smile}$.
(v) (Distributive Law for Relative Multiplication) $(r+s)\,;t = r\,;t + s\,;t$.
(vi) (Distributive Law for Converse) $(r+s)^{\smile} = r^{\smile} + s^{\smile}$.
(vii) (Tarski's Law) $r^{\smile}\,;-(r\,;s) + -s = -s$.

The set A is called the *universe* of $\mathfrak{A}$. The (Boolean) operations $+$ and $-$ are called *addition* and *complement* (or *complementation*) respectively. The (Peircean) operations $;$ and ${}^{\smile}$ are called *relative multiplication* and *converse* (or *conversion*) respectively. The distinguished constant 1' is called the *identity element*. The conventions regarding the order in which operations are to be performed when parentheses are omitted are as follows: unary operations take precedence over binary operations, and among binary operations, multiplications take precedence over additions. For example, in fully parenthesized form, axioms (iv) and (vii) would be written as

$$(r\,;s)^{\smile} = (s^{\smile})\,;(r^{\smile}) \qquad \text{and} \qquad ((r^{\smile})\,;(-(r\,;s))) + (-s) = -s$$

respectively. The associative law for relative multiplication permits us to write expressions such as $r\,;s\,;t$ without any parentheses. We shall often employ this law without any explicit reference to it.

The classic example of a relation algebra is the *full set relation algebra* $\mathfrak{Re}(E)$ of all subrelations of an equivalence relation E on a set U. The operations of $\mathfrak{Re}(E)$ are the set-theoretic ones of union, complement with respect

to E, relational composition, and converse that are respectively defined by

$$\begin{aligned} R \cup S &= \{(\alpha,\beta) : (\alpha,\beta) \in R \text{ or } (\alpha,\beta) \in S\}, \\ {\sim}R &= \{(\alpha,\beta) : (\alpha,\beta) \in E \text{ and } (\alpha,\beta) \notin R\}, \\ R|S &= \{(\alpha,\beta) : (\alpha,\gamma) \in R \text{ and } (\gamma,\beta) \in S \text{ for some } \gamma \in U\}, \\ R^{-1} &= \{(\alpha,\beta) : (\beta,\alpha) \in R\}. \end{aligned}$$

The distinguished constant is the identity relation id_U on the set U. If E is the universal relation $U \times U$, then $\mathfrak{Re}(E)$ is called the *full set relation algebra on* U and is denoted by $\mathfrak{Re}(U)$, so that in this case $\mathfrak{Re}(E) = \mathfrak{Re}(U)$.

Another important example of a relation algebra is the complex algebra of a group $(G\,,\ \circ\,,\ ^{-1}\,,\ \iota)$, where ι is the identity element of the group. The elements in this complex algebra are the subsets (or *complexes*) of the set G, and the operations of the algebra are the set-theoretic ones of union, complement with respect to G, group complex multiplication, and group complex inverse, the latter two being respectively defined by

$$H \circ K = \{h \circ k : h \in H \text{ and } k \in K\} \qquad \text{and} \qquad H^{-1} = \{h^{-1} : h \in H\}.$$

The identity element of the algebra is the singleton $\{\iota\}$. We write $\mathfrak{Cm}(G)$ for this complex algebra. In practice, we identify the elements in G with the atoms in $\mathfrak{Cm}(G)$, that is to say, we identify group elements h with their singletons $\{h\}$, and speak about h as if it were the element $\{h\}$ in $\mathfrak{Cm}(G)$.

Since a relation algebra $\mathfrak{A}$ is a Boolean algebra with additional operations, the usual Boolean notions make sense in the context of relation algebras. For example, the Boolean product of two elements r and s in $\mathfrak{A}$ is defined by

$$r \cdot s = -(-r + -s),$$

and a partial ordering can be defined on $\mathfrak{A}$ by

$$r \leq s \qquad \text{if and only if} \qquad r + s = s.$$

The Boolean zero and unit elements are denoted by 0 and 1 respectively. Two elements in $\mathfrak{A}$ are said to be *disjoint* if their Boolean product is 0. An *atom* in $\mathfrak{A}$ is defined to be a minimal non-zero element (in the sense of the partial ordering just defined), and a relation algebra is *atomic* if every non-zero element is above an atom. The *supremum* (or *Boolean sum*) and *infimum* (or *Boolean product*) of a set X of elements in $\mathfrak{A}$ are, respectively, the least upper bound and the greatest lower bound of X, and if they exist, then they are respectively denoted by $\sum X$ and $\prod X$. Of course, suprema and infima of arbitrary sets of elements need not exist, but if they always do exist, then the relation algebra $\mathfrak{A}$ is said to be *complete*. A *complete subalgebra* of a complete relation algebra $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ such that the supremum (in $\mathfrak{A}$) of every subset of $\mathfrak{B}$ belongs to $\mathfrak{B}$. A *regular subalgebra* of a (not necessarily complete) relation algebra $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ such that, for every subset X of $\mathfrak{B}$, if

X has a supremum r in $\mathfrak{B}$, then X has a supremum in $\mathfrak{A}$ as well, and that supremum is r.

The operations of converse and relative multiplication in a relation algebra are *completely distributive* over addition in the sense that if the suprema of sets of elements X and Y exist in the algebra, then the supremum of the set $\{r^\smile : r \in X\}$ exists and

$$\sum\{r^\smile : r \in X\} = (\sum X)^\smile,$$

and the supremum of the set $\{r\,;s : r \in X \text{ and } s \in Y\}$ exists and

$$\sum\{r\,;s : r \in X \text{ and } s \in Y\} = (\sum X)\,;(\sum Y).$$

There are a number of important laws in the theory of relation algebras that we shall need. Proofs of most of them can be found in [1], [2], or [8]. Among these laws, the following equivalences play a fundamental role in the development of the entire arithmetic of relation algebras.

Lemma 1.1. *The following three equations are equivalent*:

$$(r\,;s)\cdot t = 0, \qquad (r^\smile\,;t)\cdot s = 0, \qquad (t\,;s^\smile)\cdot r = 0.$$

The next lemma summarizes some of the general relation algebraic laws that we shall need. The second and sixth laws are called the *monotony laws* for converse and relative multiplication respectively. The seventh and eighth laws are the *left-hand identity law* and the *left-hand distributive law* for relative multiplication over addition.

Lemma 1.2. (i) $1'^\smile = 1'$, $\quad 0^\smile = 0$, $\quad$ *and* $\quad 1^\smile = 1$.
(ii) $r \leq s$ *if and only if* $r^\smile \leq s^\smile$.
(iii) $r\,;0 = 0$ *and* $0\,;r = 0$.
(iv) $r \leq r\,;1$ *and* $r \leq 1\,;r$.
(v) $1\,;1 = 1$.
(vi) *If* $r \leq t$ *and* $s \leq u$, *then* $r\,;s \leq t\,;u$.
(vii) $1'\,;r = r$.
(viii) $t\,;(r+s) = t\,;r + t\,;s$.
(ix) $(r\,;s)\cdot t \leq r\,;[s\cdot(r^\smile\,;t)]$.

Several special kinds of elements will play an important role in our discussion. An *equivalence element* in a relation algebra $\mathfrak{A}$ is an element e satisfying the inequalities $e^\smile \leq e$ and $e\,;e \leq e$. In set relation algebras, a relation satisfies the first, respectively the second, of these inequalities just in case it is symmetric, respectively transitive. If r and s are arbitrary elements in $\mathfrak{A}$ that are below an equivalence element e, then

$$r + s \leq e, \qquad e\cdot -r \leq e, \qquad r\,;s \leq e, \qquad r^\smile \leq e, \qquad e\cdot 1' \leq e.$$

The set of all elements below e is therefore a Boolean algebra under the operation of Boolean addition and complement relative to e. The preceding inequalities also show that this set is closed under relative multiplication and

converse, and contains the relativized identity element $e \cdot 1'$ (where 1' is the identity element in $\mathfrak{A}$). The corresponding algebra is called the *relativization of* $\mathfrak{A}$ *to* e. It is a relation algebra and, in fact, it is almost a subalgebra of $\mathfrak{A}$ (see [8]). The only differences between a relativization and a subalgebra of $\mathfrak{A}$ are that the unit is e and not the unit 1 of $\mathfrak{A}$, the identity element is $e \cdot 1'$ and not the identity element 1' of $\mathfrak{A}$, and complements are formed with respect to e and not with respect to 1. Here is a concrete example: if E is an equivalence relation on a set U, then $\mathfrak{Re}(E)$ is just the relativization of $\mathfrak{Re}(U)$ to E.

The relativization of a relation algebra $\mathfrak{A}$ to an equivalence element e may inherit a number of properties from $\mathfrak{A}$. For instance, if $\mathfrak{A}$ is complete, then so is the relativization. An element r in $\mathfrak{A}$ is an atom in the relativization if and only if r is an atom in $\mathfrak{A}$ and $r \leq e$. In particular, if $\mathfrak{A}$ is atomic, then the relativization is also atomic.

A *subidentity* in a relation algebra is an element $x \leq 1'$. A *right-ideal element* is an element r with the property that $r = r\,;1$. Right-ideal elements r are characterized by the fact that $r = s;1$ for some element s (and in this case s is said to *generate* the right-ideal element r); in fact, s may always be assumed to be a subidentity element. Analogously, r is a *left-ideal element* if $r = 1\,;r$, or equivalently, if there is an element s such that $r = 1;s$ (and as before, s may always be assumed to be a subidentity element). The names come from the fact that these elements play a special role in determining what are sometimes called (relation algebraic) right and left ideals. In a full set relation algebra on a set U, right-ideal elements, respectively left-ideal elements, are relations R that can be written in the form $R = X \times U$, respectively $R = U \times X$, for some subset X of U. These relations may be thought of as vertical, respectively horizontal, strips in the Cartesian plane determined by $U \times U$.

We shall need the following laws concerning right-ideal elements.

Lemma 1.3. *Let r and s be right-ideal elements, say $r = x\,;1$ and $s = y\,;1$ for some subidentity elements x and y.*

(i) *$-r$ is a right-ideal element, and in fact, $-r = -(x\,;1) = (1' - x)\,;1$.*

(ii) *If the supremum of an arbitrary set of right-ideal elements exists, then that supremum is a right-ideal element. In particular, $r + s$ is a right-ideal element, and in fact, $r + s = x\,;1 + y\,;1 = (x + y)\,;1$.*

(iii) *$r \cdot (s\,;t) = (r \cdot s)\,;t$ for all elements s and t.*

The set of all right-ideal elements forms a regular Boolean subalgebra B of the Boolean part of a relation algebra $\mathfrak{A}$, by parts (i) and (ii) of the preceding lemma. It is called the *Boolean algebra of right-ideal elements* in $\mathfrak{A}$. The law in (iii) is called the *strong modular law for right-ideal elements*, and it actually characterizes those elements r that are right-ideal elements: r is a right-ideal element in $\mathfrak{A}$ if and only if r satisfies the equation in (iii) for all s and t in $\mathfrak{A}$.

A *rectangle* in a relation algebra is an element of the form $x\,;1\,;y$, where x and y are subidentity elements. The elements x and y are called the *sides* of the rectangle. Since $x\,;1\,;y = (x\,;1) \cdot (1\,;y)$, a rectangle is just an element

that is the Boolean product of a right-ideal element with a left-ideal element. Consequently, in a full set relation algebra on a set U, the rectangles are sets of the form $X \times Y$, where X and Y are subsets of U. Here are the laws about rectangles that we shall need.

Lemma 1.4. *Let x, y, u, v be subidentity elements in a relation algebra $\mathfrak{A}$.*

(i) $(x\,;1\,;y)\cdot(u\,;1\,;v) = (x\cdot u)\,;1\,;(y\cdot v)$.

(ii) $(x\,;1\,;y)^{\smile} = y\,;1\,;x$.

(iii) $x\,;1\,;y \neq 0$ *whenever $\mathfrak{A}$ is simple and the sides x, y are non-zero.*

(iv) $(x\,;1\,;y)\,;(u\,;1\,;v) \leq x\,;1\,;v$, *and equality holds whenever $\mathfrak{A}$ is simple and $y\cdot u$ is non-zero.*

(v) $x\,;1\,;y = u\,;1\,;v$ *if and only if $x = u$ and $y = v$, whenever $\mathfrak{A}$ is simple and the sides of the rectangles are non-zero.*

A *square* in a relation algebra is a rectangle for which the two sides are equal. It is easy to check, using Lemma 1.4(ii),(iv), that a square is always an equivalence element. Consequently, it makes sense to speak of the relativization of a relation algebra to a square.

The *domain* and *range* of an element r in a relation algebra are defined to be the subidentity elements

$$\text{domain } r = (r\,;1)\cdot 1' \qquad \text{and} \qquad \text{range } r = (1\,;r)\cdot 1'$$

respectively. In a set relation algebra, the domain (range) of a relation R is the set of pairs (α,α) such that α is in the domain (range) of R in the standard sense of this word. The domain and range of a rectangle $x\,;1\,;y$ are, respectively, the sides x and y. The domain of a right-ideal element $r = x\,;1$ is x, and the range of a left-ideal element $r = 1\,;y$ is y. If x and y are respectively the domain and range of an element r, then

$$x\,;r = r\,;y = r.$$

Thus, the domain and range of an element r act respectively as left-hand and right-hand identity elements for r.

A *function*, or a *functional element*, in a relation algebra is an element f with the property that $f^{\smile}\,;f \leq 1'$. A *bijection*, or a *bijective element*, is a function f with the property that the converse $f^{\smile}$ is also a function. In a set relation algebra, functional and bijective elements are respectively just functions and one-to-one functions in the set-theoretic sense of the word. Here are the laws about functions that we need. The law in (iii) says that a function is left-distributive over Boolean multiplication. It follows that a bijection is both left- and right-distributive over Boolean multiplication.

Lemma 1.5. (i) *If f is a function with domain x and range y, and if g is a function with domain y and range z, then $f\,;g$ is a function with domain x and range z.*

(ii) *If f is a function, and $g \leq f$, then g is a function and $(g\,;1)\cdot f = g$.*

(iii) *If f is a function, then $f\,;(r\cdot s) = (f\,;r)\cdot(f\,;s)$ for all elements r and s.*

(iv) *The supremum of a set of functions generating mutually disjoint right-ideal elements, if it exists, is a function. In particular, if f and g are functions such that $(f\,;1)\cdot(g\,;1)=0$, then $f+g$ is a function.*

The next lemma concerns atoms.

Lemma 1.6. (i) *If r is an atom, then so are the domain and range of r.*
(ii) *If r is an atom, then $r\,;1$ is an atom in the Boolean algebra of right-ideal elements. In particular, if r is an atom and $0<s\leq r\,;1$, then $s\,;1=r\,;1$.*
(iii) *If r is an atom in the Boolean algebra of right-ideal elements, and if f is a function such that $r\cdot f\neq 0$, then $r\cdot f$ is an atom. In particular, if r is an atom (in the relation algebra) and if f is a function satisfying $0<f\leq r\,;1$, then f is an atom.*
(iv) *A function is an atom if and only if its domain is an atom.*
(v) *If r is an atom and f a function whose domain includes the range of r, then $r\,;f$ is an atom.*
(vi) *If r is an atom, then so is $r^{\smile}$.*

An element that is both a right-ideal element and a left-ideal element is called an *ideal element*. Ideal elements r are characterized by the fact that $r=1\,;s\,;1$ for some element s (and in this case s is said to *generate* the ideal element r); as with right-ideal elements, s may always be taken to be a subidentity element. In the full relation algebra on an equivalence relation E, the ideal elements are just the relations R that can be written in the form

$$R=\bigcup\{V\times V : V\in X\}$$

for some set X of equivalence classes of E. In particular, if V is an equivalence class of E, then $V\times V$ is an ideal element.

It follows from Lemma 1.3(i),(ii) and the analogue of this lemma for left-ideal elements, together with Lemma 1.2(iii),(v), that the set of ideal elements in a relation algebra $\mathfrak{A}$ contains 0 and 1 and is closed under the Boolean operations of $\mathfrak{A}$. Consequently, this set is a Boolean algebra under the Boolean operations of $\mathfrak{A}$, and in fact it is a regular subalgebra of the Boolean part of $\mathfrak{A}$ in the strong sense that the supremum of a set of ideal elements exists in the Boolean algebra of ideal elements if and only if it exists in $\mathfrak{A}$, and when these suprema do exist, they are equal (see [1] and [8]).

There is a very close connection between ideal elements and ideals in relation algebras. Let $\mathfrak{A}$ be a relation algebra and B the Boolean algebra of ideal elements in $\mathfrak{A}$. If M is a relation algebraic ideal in $\mathfrak{A}$, then the set $K_M=B\cap M$ is a Boolean ideal in B, and if K is a Boolean ideal in B, then the set

$$M_K=\{r\in A:\ 1\,;r\,;1\in K\}$$

is a relation algebraic ideal in $\mathfrak{A}$. The function that maps each relation algebraic ideal M in $\mathfrak{A}$ to the Boolean ideal K_M in B is a lattice isomorphism from the lattice of relation algebraic ideals in $\mathfrak{A}$ to the lattice of Boolean ideals in B, and the inverse of this isomorphism maps each Boolean ideal K in B to

the relation algebraic ideal M_K in $\mathfrak{A}$ (see [1] and [8]). In particular, M is a principal ideal in $\mathfrak{A}$ just in case K_M is a principal ideal in B, or what amounts to the same thing, just in case M is the set of all elements in $\mathfrak{A}$ that are below some given ideal element.

Ideal elements are always equivalence elements, so it makes sense to speak of the relativization of a relation algebra to an ideal element (see [8]). In contrast to arbitrary equivalence elements, however, ideal elements determine ideals (and therefore congruence relations) on relation algebras. Consequently, systems of ideal elements can be used to obtain direct decompositions of relation algebras. If a system $(r_i : i \in I)$ of ideal elements in a relation algebra $\mathfrak{A}$ forms a *partition of unity* in the sense that the elements are mutually disjoint and sum to 1, then $\mathfrak{A}$ is isomorphic to the direct product of the system of relation algebras $(\mathfrak{A}_i : i \in I)$, where $\mathfrak{A}_i$ is the relativization of $\mathfrak{A}$ to the ideal element r_i. We shall refer to this result as the Jónsson-Tarski decomposition theorem (see [8]). In particular, if r is an ideal element in $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to the direct product of the relation algebras $\mathfrak{B}$ and $\mathfrak{C}$ that are the relativizations of $\mathfrak{A}$ to r and $-r$ respectively.

A relation algebra $\mathfrak{A}$ is said to be *simple* if it is *non-trivial* (that is to say, it has at least two elements), and if it has only two ideals: the trivial ideal $\{0\}$ and the improper ideal A. The class of simple relation algebras admits a very simple arithmetic characterization: a relation algebra $\mathfrak{A}$ is simple if and only if it has precisely two ideal elements, namely 0 and 1. Equivalently, $\mathfrak{A}$ is simple if and only if $0 \neq 1$ and for all elements r in $\mathfrak{A}$, if $r \neq 0$, then $1\,;r\,;1 = 1$ (see [8]). It follows from this characterization that every subalgebra of a simple relation algebra is simple (see [8]). It also follows that if r is an atom in the Boolean algebra of ideal elements in $\mathfrak{A}$—an *ideal element atom*, so to say —then the relativization of $\mathfrak{A}$ to r is a simple relation algebra.

We shall need the following consequence (due to Givant) of the Jónsson-Tarski decomposition theorem.

Theorem 1.7. *Every complete relation algebra $\mathfrak{A}$ is isomorphic to the direct product of two complete relation algebras $\mathfrak{B}$ and $\mathfrak{C}$ with the following properties*: *$\mathfrak{B}$ is the direct product of relativizations of $\mathfrak{A}$ to ideal element atoms — in particular, each of these relativizations is simple — and $\mathfrak{C}$ is a relativization of $\mathfrak{A}$ that has an atomless Boolean algebra of ideal elements.*

Proof. Let $(r_i : i \in I)$ be an enumeration of the distinct ideal element atoms in $\mathfrak{A}$. The sum $r = \sum_i r_i$ of these elements exists in $\mathfrak{A}$, because $\mathfrak{A}$ is assumed to be complete; and this sum is an ideal element, by Lemma 1.3(ii) and its analogue for left-ideal elements. Let $\mathfrak{B}$ and $\mathfrak{C}$ be the relativizations of $\mathfrak{A}$ to r and $-r$ respectively. Then $\mathfrak{A}$ is isomorphic to the direct product of $\mathfrak{B}$ and $\mathfrak{C}$, by the Jónsson-Tarski decomposition theorem. Each of the elements r_i remains an ideal element in $\mathfrak{B}$, and the set of these ideal elements is a partition of unity in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is isomorphic to the direct product of the system $(\mathfrak{B}_i : i \in I)$, where $\mathfrak{B}_i$ is the relativization of $\mathfrak{B}$ to the element

r_i, again by the Jónsson-Tarski decomposition theorem. The relativization $\mathfrak{B}_i$ coincides with the relativization of $\mathfrak{A}$ to r_i, so $\mathfrak{B}$ is isomorphic to a direct product of relativizations of $\mathfrak{A}$ to ideal element atoms—in particular, each of these relativizations is simple. There can be no ideal element atoms below $-r$, because every ideal element atom is, by definition, below r. Consequently, the Boolean algebra of ideal elements in $\mathfrak{C}$ is atomless. □

The fact that the Boolean algebra of ideal elements in the relation algebra $\mathfrak{C}$ from the preceding theorem is atomless is equivalent to saying $\mathfrak{C}$ has no simple factors. Consequently, $\mathfrak{C}$ cannot be further decomposed into a direct product containing at least one simple factor.

A relation algebra $\mathfrak{A}$ is said to be *integral* if it is non-trivial and if the relative product of two non-zero elements is always non-zero. It turns out that a relation algebra is integral if and only if 1' is an atom (see [8]). This characterization immediately implies, for example, that the complex algebra of a group is always integral, and a full set relation algebra on a set of cardinality at least two is never integral. It also follows from this characterization (and Lemma 1.6(ii) and its analogue for left-ideal elements) that in an integral relation algebra, the unit is an atom in the Boolean algebra of ideal elements, and therefore an integral relation algebra must be simple.

Every relation algebra $\mathfrak{A}$ has two important extensions, both of which will be needed in this paper. The first is the *canonical extension*. It is a complete and atomic relation algebra $\mathfrak{B}$ that includes $\mathfrak{A}$ as a subalgebra and has the following two additional properties: first, for any two atoms a and b in $\mathfrak{B}$, there is an element r in $\mathfrak{A}$ that is above a and disjoint from b (this is called the *atom separation property*); and second, if 1 is the supremum in $\mathfrak{B}$ of some subset X of elements from $\mathfrak{A}$, then 1 is already the supremum of some finite subset of X (this is called the *compactness property*). The canonical extension of $\mathfrak{A}$ exists and is unique up to isomorphisms that are the identity function on $\mathfrak{A}$ (see [7]). If $\mathfrak{A}$ is simple, then the canonical extension of $\mathfrak{A}$ is simple.

The second extension of $\mathfrak{A}$ is the *completion* of $\mathfrak{A}$. It is a complete relation algebra $\mathfrak{B}$ that includes $\mathfrak{A}$ as a regular, dense subalgebra. To say that $\mathfrak{A}$ is *dense* in $\mathfrak{B}$ means that every non-zero element in $\mathfrak{B}$ is above a non-zero element from $\mathfrak{A}$, or what amounts to the same thing, every element r in $\mathfrak{B}$ is the supremum of the set of elements in $\mathfrak{A}$ that are below r. The completion of $\mathfrak{A}$ also exists and is unique up to isomorphisms that are the identity function on $\mathfrak{A}$ (see [11]). If $\mathfrak{A}$ is simple, then the completion of $\mathfrak{A}$ is also simple. The completion $\mathfrak{B}$ is in general not atomic; in fact, $\mathfrak{B}$ is atomic if and only if $\mathfrak{A}$ is atomic, since an element r in $\mathfrak{B}$ is an atom in $\mathfrak{B}$ if and only if r is already an atom in $\mathfrak{A}$.

A *representation* of a relation algebra $\mathfrak{A}$ is a function φ that maps $\mathfrak{A}$ into a full set relation algebra $\mathfrak{Re}(E)$ for some equivalence relation E. The subalgebra of $\mathfrak{Re}(E)$ that is the image of $\mathfrak{A}$ under φ is also referred to as a representation of $\mathfrak{A}$. A representation (function) φ of $\mathfrak{A}$ is said to be *complete* if it preserves all

existing suprema as unions. This means that if an element r is the supremum in $\mathfrak{A}$ of a set of elements X, then the relation $\varphi(r)$ in $\mathfrak{Re}(E)$ is the union of the set of relations $\{\varphi(s) : s \in X\}$. A relation algebra is said to be *representable* or *completely representable* according to whether it has some representation or some complete representation.

To give a concrete example of a relation algebraic representation, consider the complex algebra $\mathfrak{Cm}(G)$ of a group $(G\,,\ \circ\,,\ ^{-1}\,,\ \iota)$. The *Cayley representation* of an element f in G is the permutation

$$R_f = \{(g, g \circ f) : g \in G\}$$

of G, and the Cayley representation of a subset H of G is the union of the Cayley representations of the elements in H:

$$\begin{aligned} R_H = \bigcup\{R_f : f \in H\} &= \{(g, g \circ f) : g \in G \text{ and } f \in H\} \\ &= \{(g, h) : g, h \in G \text{ and } h \leq g \circ H\}, \end{aligned}$$

where $g \circ H = \{g \circ f : f \in H\}$. The validity of the following equations for all subsets H, K, and L of G is easy to check:

$$\begin{aligned} R_H \cup R_K &= R_L, \qquad \text{where} \qquad L = H \cup K, \\ {\sim}R_H &= R_L, \qquad \text{where} \qquad L = {\sim}H, \\ R_H | R_K &= R_L, \qquad \text{where} \qquad L = H \circ K, \\ R_H^{-1} &= R_L, \qquad \text{where} \qquad L = H^{-1}, \end{aligned}$$

and $id_G = R_\iota$. (The operations on the left are performed in $\mathfrak{Re}(G)$, while those on the right are performed in $\mathfrak{Cm}(G)$.) The equations make clear that the set of all relations of the form R_H for subsets H of G is a subuniverse of $\mathfrak{Re}(G)$, and in fact it is a complete subuniverse. The corresponding complete subalgebra is denoted by $\mathfrak{Ca}(G)$. The equations also imply that the function φ from $\mathfrak{Cm}(G)$ to $\mathfrak{Ca}(G)$ defined by

$$\varphi(H) = R_H = \{(g, h) : g, h \in G \text{ and } h \leq g \circ H\}$$

is an isomorphism and in fact a complete representation of $\mathfrak{Cm}(G)$. It is called the *Cayley representation* of $\mathfrak{Cm}(G)$, and $\mathfrak{Ca}(G)$ is also referred to as the Cayley representation of $\mathfrak{Cm}(G)$.

An atomic relation algebra $\mathfrak{A}$ need not be complete, since there may be many infinite subsets of the universe for which the supremum does not exist. This may pose a problem when trying to represent $\mathfrak{A}$ as an isomorphic copy of an algebra in some given concrete class of set relation algebras, and not just as a subalgebra of some member of the class; for the intended representing algebras in the class may all be complete, as in the case of the Cayley representations of group complex algebras. Incompleteness is a relatively minor defect, one that is correctable by first "filling in" all of the missing infinite sums, without otherwise modifying the essential structure of $\mathfrak{A}$. The technical way of doing this is to pass to the completion of $\mathfrak{A}$. The atoms in the completion are the

same as the atoms of $\mathfrak{A}$, and the operations of the completion when restricted to elements of $\mathfrak{A}$, agree with the operations of $\mathfrak{A}$; this is even true in a limited sense for the infinitary partial operations of forming suprema and infima: if the supremum or infimum of an infinite subset of $\mathfrak{A}$ exists in $\mathfrak{A}$, then the subset has the same supremum or infimum in the completion. We shall say that two relation algebras are *essentially isomorphic* if their completions are isomorphic. It should be pointed out that it is not always possible to establish such an essential isomorphism when dealing with representations; for there exist atomic relation algebras $\mathfrak{A}$ with the property that $\mathfrak{A}$ is representable but the completion of $\mathfrak{A}$ is not, as was shown by Hodkinson [5].

We close this section with a theorem that is really a special case of a more general result about atomic Boolean algebras with operators. It gives sufficient conditions for a set W of mutually disjoint non-zero elements in a relation algebra $\mathfrak{A}$ to generate an atomic subalgebra of $\mathfrak{A}$ in which the elements in W are the atoms.

Theorem 1.8. *Suppose $\mathfrak{A}$ is a relation algebra, and W a set of non-zero elements in $\mathfrak{A}$ with the following properties.*

(i) *The elements in W are mutually disjoint and sum (in $\mathfrak{A}$) to 1.*
(ii) *The identity element 1' is the sum (in $\mathfrak{A}$) of the elements in W that are below 1'.*
(iii) *If p is in W, then so is $p^\smile$.*
(iv) *If p and q are in W, then $p\,;q$ is the sum (in $\mathfrak{A}$) of the elements in W that are below $p\,;q$.*

The set of sums $\sum X$ such that X is a subset W and $\sum X$ exists in $\mathfrak{A}$ is then the universe of an atomic subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, and the atoms of $\mathfrak{B}$ are just the elements in W. If $\mathfrak{A}$ is complete, then $\mathfrak{B}$ is a complete subalgebra of $\mathfrak{A}$.

2. The representation theorem for atomic relation algebras with functional atoms

We begin by presenting a slightly stronger version of Jónsson and Tarski's representation theorem for atomic relation algebras with functional atoms, and a slightly different version of their proof. The representation that we use is close in spirit to the Cayley representation of the complex algebra of a group. We focus on the parts of the proof that differ from the proof we shall give of Maddux and Tarski's representation theorem for functionally dense relation algebras. The following preliminary observation may be helpful in orienting the reader: if every atom in a relation algebra is a function, then the converse of every atom is also a function (since the converse of an atom is an atom—see Lemma 1.6(vi)), and therefore every atom is actually a bijection.

Theorem 2.1. *Every atomic relation algebra with functional atoms is completely representable.*

Proof. Let $\mathfrak{A}$ be an atomic relation algebra with functional atoms, and U the set of atoms in $\mathfrak{A}$. Define a function φ from the universe of $\mathfrak{A}$ into $\mathfrak{Re}(U)$ by

$$\varphi(r) = \{(a, b) : a, b \in U \text{ and } b \leq a\,;\, r\}.$$

The proofs that φ maps 0, 1', and 1 to the empty relation, the identity relation, and an equivalence relation E on U respectively, and that φ preserves Boolean sums as unions, Boolean products as intersections, complements as set-theoretic complements relative to E, and converses as relational converses are all identical to the arguments that we shall give in the proof of Theorem 3.3 below. For that reason, we omit those arguments here—the interested reader is referred to the relevant sections of that proof — and focus on showing that φ preserves all existing suprema as unions, preserves relative multiplication as relational composition, and is one-to-one.

To see that φ preserves all existing suprema as unions, suppose r is the supremum of a set X of elements in $\mathfrak{A}$. The complete distributivity of relative multiplication over addition implies that

$$a\,;\, r = \sum(a\,;\, X) = \sum\{a\,;\, s : s \in X\}.$$

For atoms a and b, we therefore have $b \leq a\,;\, r$ if and only if $b \leq a\,;\, s$ for some s in X, so

$$\begin{aligned}
(a, b) \in \varphi(r) \qquad & \text{if and only if} \qquad b \leq a\,;\, r, \\
& \text{if and only if} \qquad b \leq a\,;\, s \text{ for some } s \text{ in } X, \\
& \text{if and only if} \qquad (a, b) \in \varphi(s) \text{ for some } s \text{ in } X, \\
& \text{if and only if} \qquad (a, b) \in \textstyle\bigcup_{s \in X} \varphi(s),
\end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of the union of a system of elements. Thus, $\varphi(r) = \bigcup_{s \in X} \varphi(s)$.

For the proof that φ preserves relative multiplication, let a and b be atoms, and r and s arbitrary elements, in $\mathfrak{A}$. Take X to be the set of atoms below $a\,;\, r$, and observe that $a\,;\, r = \sum X$, since $\mathfrak{A}$ is assumed to be atomic. The complete distributivity of relative multiplication over addition implies that

$$a\,;\, r\,;\, s = (\textstyle\sum X)\,;\, s = \sum\{c\,;\, s : c \in X\} = \sum\{c\,;\, s : c \in U \text{ and } c \leq a\,;\, r\}.$$

Therefore,

$$b \leq a\,;\, r\,;\, s \qquad \text{if and only if} \qquad c \leq a\,;\, r \text{ and } b \leq c\,;\, s$$

for some c in U. Consequently,

$$\begin{aligned}
(a, b) \in \varphi(r\,;\, s) \qquad & \text{if and only if} \qquad b \leq a\,;\, r\,;\, s, \\
& \text{if and only if} \qquad c \leq a\,;\, r \text{ and } b \leq c\,;\, s \text{ for some } c \text{ in } U, \\
& \text{if and only if} \qquad (a, c) \in \varphi(r) \text{ and } (c, b) \in \varphi(s) \\
& \qquad\qquad\qquad\qquad\qquad \text{for some } c \text{ in } U, \\
& \text{if and only if} \qquad (a, b) \in \varphi(r)|\varphi(s),
\end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of relational composition. Thus, $\varphi(r\,;s) = \varphi(r)|\varphi(s)$ for all elements r and s in $\mathfrak{A}$.

To check that φ is one-to-one, consider elements r and s in $\mathfrak{A}$. Take X to be the set of atoms below 1', and observe that 1' is the supremum of X, by the assumption that $\mathfrak{A}$ is atomic. Consequently,

$$r = 1'\,;r = (\textstyle\sum X)\,;r = \sum\{a\,;r : a \in X\} = \sum\{a\,;r : a \in U \text{ and } a \leq 1'\}.$$

For each atom b, we therefore have $b \leq r$ if and only if $b \leq a\,;r$ for some atom $a \leq 1'$. A similar argument leads to the conclusion that $b \leq s$ if and only if $b \leq a;s$ for some atom $a \leq 1'$. Suppose now that $\varphi(r) = \varphi(s)$. This assumption and the definition of φ imply that for any two atoms a and b, we have $b \leq a\,;r$ if and only if $b \leq a\,;s$. Combine these observations to arrive at the conclusion that an atom is below r if and only if it is below s. The elements r and s are the sums of the atoms they dominate, by the assumed atomicity of $\mathfrak{A}$, so the last conclusion implies that $r = s$. □

The definition of the representation in the preceding proof may be viewed as a natural extension to atomic relation algebras with functional atoms of the definition of the Cayley representation of the complex algebra of a group.

Jónsson-Tarski [8] drew the following conclusion from their version of Theorem 2.1.

Corollary 2.2. *If the unit of a relation algebra $\mathfrak{A}$ is the sum of a finite set of functional elements, then $\mathfrak{A}$ is representable.*

Proof. Let $\mathfrak{B}$ be the canonical extension of $\mathfrak{A}$. Finite sums are preserved under the passage to canonical extensions, so the unit of $\mathfrak{B}$ must also be the sum of a finite set of functions. The algebra $\mathfrak{B}$ is atomic, and each atom in $\mathfrak{B}$, being below the unit and therefore below a sum of functions, must be below some function. Elements below functions are functions, by Lemma 1.5(ii), so $\mathfrak{B}$ is an atomic relation algebra with functional atoms. Consequently, $\mathfrak{B}$ is completely representable, by Theorem 2.1. It follows that $\mathfrak{A}$ is also representable. □

3. Functionally dense relation algebras

A relation algebra is said to be *functionally dense* if below every non-zero element there is a non-zero function. The following five conditions on a relation algebra $\mathfrak{A}$ are easily seen to be equivalent: (1) $\mathfrak{A}$ is functionally dense; (2) the unit in $\mathfrak{A}$ is the supremum of the set of all functions; (3) the unit in $\mathfrak{A}$ is the supremum of some set of functions; (4) every element in $\mathfrak{A}$ is the supremum of the set of all functions that are below it; (5) every element in $\mathfrak{A}$ is the supremum of some set of functions. The goal of this section is to give a new proof of Maddux and Tarski's theorem that every functionally dense relation algebra is representable (see [9]). Their theorem improves Theorem 2.1 by removing the requirement that the relation algebra be atomic; and it improves Corollary 2.2

by removing the requirement that the set of functions of which the unit is the supremum be finite. In contrast to the representation in Theorem 2.1, however, the representation that is obtained in the Maddux-Tarski theorem is not necessarily complete.

The underlying idea of our proof is to assume first that $\mathfrak{A}$ is a functionally dense relation algebra that is complete, and to pass to the canonical extension of $\mathfrak{A}$. This extension is in general not functionally dense, but it has enough functional atoms to build a representation of $\mathfrak{A}$ in the style of Theorem 2.1. The general case when $\mathfrak{A}$ is not complete is handled by passing to the completion of $\mathfrak{A}$, which continues to be functionally dense.

Lemma 3.1. *If a functionally dense relation algebra is complete, then below every non-zero element r there is a (non-zero) function f such that $f;1 = r;1$*[1].

Proof. Let $\mathfrak{A}$ be a functionally dense relation algebra that is complete, and consider a non-zero element r in $\mathfrak{A}$. By means of a potentially transfinite induction on ordinal numbers i, we construct a system of non-zero functions f_i that are all below r and that generate mutually disjoint right-ideal elements. For the base case $i = 0$, use the functional density of $\mathfrak{A}$ to choose a non-zero function f_0 that is below r.

Assume now that f_i has been chosen for all ordinals i less than a given ordinal j. The sum

$$g_j = \textstyle\sum_{i<j} f_i \tag{1}$$

certainly exists, because $\mathfrak{A}$ is assumed to be complete. Furthermore, g_j is a function, by Lemma 1.5(iv) and the induction hypothesis that the functions f_i generate mutually disjoint right-ideal elements; and g_j is below r because each of the functions f_i is below r, by the induction hypothesis. If the generated right-ideal elements $g_j\,;1$ and $r\,;1$ are equal, then the function $f = g_j$ has the requisite properties.

If the generated right-ideal elements are not equal, then

$$0 < g_j\,;1 < r\,;1$$

(since $0 < g_j \leq r$), and therefore

$$-(g_j\,;1) \cdot (r\,;1) \neq 0.$$

The complement $-(g_j\,;1)$ is a right-ideal element, by Lemma 1.3(i). Apply the strong modular law for right-ideal elements (Lemma 1.3(iii), with r, s, and t replaced by $-(g_j : 1)$, r, and 1 respectively) to the preceding inequality to obtain

$$[-(g_j\,;1) \cdot r]\,;1 \neq 0.$$

---

[1]Maddux has recently communicated the following information to us. Tarski formulated and proved the result given in this lemma in 1976. The result is stated without proof in the abstract [10] and it is also mentioned (but not proved) in [9].

It follows that $-(g_j;1)\cdot r \neq 0$, by Lemma 1.2(iii). Invoke the functional density of $\mathfrak{A}$ to get a non-zero function f_j that is below $-(g_j\,;1)\cdot r$. In particular, f_j is below r. Because f_j is also below $-(g_j\,;1)$, the monotony law for relative multiplication and the fact that $-(g_j\,;1)$ is a right-ideal element imply that

$$f_j\,;1 \leq [-(g_j\,;1)]\,;1 = -(g_j\,;1).$$

Thus, the right-ideal elements $g_j\,;1$ and $f_j\,;1$ are disjoint. Since

$$g_j\,;1 = (\textstyle\sum_{i<j} f_i)\,;1 = \sum_{i<j}(f_i\,;1),$$

it follows that $f_j\,;1$ is disjoint from $f_i\,;1$ for each $i<j$. This completes the induction step of the construction.

This process must stop at some ordinal number, because it is impossible to construct more non-zero, mutually disjoint functions in $\mathfrak{A}$ than there are elements in $\mathfrak{A}$. If the process stops at the ordinal j, then the sum g_j in (1) is a function below r that generates the same right-ideal element as r. (If it did not generate the same right-ideal element, then the inductive construction would continue for at least one more step.) The function $f = g_j$ has the requisite properties. □

We continue with the assumption that $\mathfrak{A}$ is a complete, functionally dense relation algebra. Let $\mathfrak{B}$ be the canonical extension of $\mathfrak{A}$, and let U be the set of atoms in $\mathfrak{B}$ that are below functions in $\mathfrak{A}$. In other words, an element a belongs to U if and only if a is an atom in $\mathfrak{B}$ and $a \leq f$ for some function f in $\mathfrak{A}$. The set U will be the base set of the representation of $\mathfrak{A}$. The next lemma gives a key property that will be needed to construct the representation.

Lemma 3.2. *Suppose $(a_i : i \in I)$ is a finite, non-empty system of elements in U, and $(r_i : i \in I)$ a corresponding system of elements in $\mathfrak{A}$. If the Boolean product $\prod_i (a_i\,;r_i)$ is not zero, then this product must be above an atom in U.*

Proof. Write

$$t = \textstyle\prod_i (a_i\,;r_i), \tag{1}$$

and assume $t \neq 0$. It is to be shown that t is above some atom in U. Apply Lemma 1.2(iv), the definition of t, and the monotony and associative laws for relative multiplication to obtain

$$0 < t \leq t\,;1 \leq (a_i\,;r_i)\,;1 = a_i\,;(r_i\,;1) \leq a_i\,;1 \tag{2}$$

for each index i. The element a_i is an atom in $\mathfrak{B}$, so the generated right-ideal element $a_i\,;1$ is an atom in the Boolean algebra of right-ideal elements in $\mathfrak{B}$, and

$$t\,;1 = a_i\,;1, \tag{3}$$

by (2) and Lemma 1.6(ii). It follows that $t;1$ is an atom in the Boolean algebra of right-ideal elements in $\mathfrak{B}$.

The assumption that a_i belongs to U, and the definition of U, imply that a_i is below some function f_i in $\mathfrak{A}$. Apply Lemma 1.5(ii) to obtain

$$(a_i\,;1)\cdot f_i = a_i. \tag{4}$$

Use the strong modular law for right-ideal elements (Lemma 1.3(iii), with r, s, and t replaced by $t\,;1$, f_i, and r_i respectively), (3), and (4) to see that

$$(t\,;1)\cdot(f_i\,;r_i) = [(t\,;1)\cdot f_i]\,;r_i = [(a_i\,;1)\cdot f_i]\,;r_i = a_i\,;r_i. \tag{5}$$

Conclude that

$$(t\,;1)\cdot\textstyle\prod_i(f_i\,;r_i) = \prod_i[(t\,;1)\cdot(f_i\,;r_i)] = \prod_i(a_i\,;r_i) = t > 0, \tag{6}$$

by Boolean algebra, (5), and (1).

As a Boolean product of relative products of finitely many elements in $\mathfrak{A}$, the element $\prod_i(f_i\,;r_i)$ must also belong to $\mathfrak{A}$. Use (6), Lemma 3.1, and the assumed completeness of $\mathfrak{A}$ to get a non-zero function g in $\mathfrak{A}$ such that

$$g \leq \textstyle\prod_i(f_i\,;r_i) \qquad \text{and} \qquad g\,;1 = (\prod_i(f_i\,;r_i))\,;1. \tag{7}$$

The short calculation given below shows that

$$(t\,;1)\cdot(g\,;1) \neq 0. \tag{8}$$

Indeed,

$$(t\,;1)\cdot(g\,;1) = (t\,;1)\cdot[(\textstyle\prod_i(f_i\,;r_i))\,;1] = [(t\,;1)\cdot\prod_i(f_i\,;r_i)]\,;1 = t\,;1 \neq 0,$$

by the second part of (7), the strong modular law for right-ideal elements (with r, s, and t replaced by $t\,;1$, $\prod_i(f_i\,;r_i)$, and 1 respectively), (6), and (2). Use (8) and Lemmas 1.1 and 1.2(i) to arrive at

$$(t\,;1\,;1)\cdot g \neq 0. \tag{9}$$

The element

$$c = (t\,;1\,;1)\cdot g = (t\,;1)\cdot g$$

is non-zero, by (9). Since g is a function and $t\,;1$ is an atom in the Boolean algebra of right-ideal elements in $\mathfrak{B}$, it follows from Lemma 1.6(iii) that c is in fact an atom in $\mathfrak{B}$. Also, c is below the function g, which is in $\mathfrak{A}$, so c must belong to the set U, by the definition of U. Finally,

$$c = (t\,;1)\cdot g \leq (t\,;1)\cdot\textstyle\prod_i(f_i\,;r_i) = t,$$

by the definition of c, the first part of (7), the monotony law for relative multiplication, and (6). Thus, c is an atom in U that is below t, as desired. □

Here is Maddux and Tarski's representation theorem for functionally dense relation algebras.

Theorem 3.3. *Every functionally dense relation algebra is representable.*

Proof. Let $\mathfrak{A}$ be a functionally dense relation algebra, and assume first that $\mathfrak{A}$ is complete. Take $\mathfrak{B}$ to be the canonical extension of $\mathfrak{A}$, and U the set of atoms in $\mathfrak{B}$ that are below functions in $\mathfrak{A}$. Define a function φ from the universe of $\mathfrak{A}$ into $\mathfrak{Re}(U)$ by

$$\varphi(r) = \{(a, b) : a, b \in U \text{ and } b \leq a\,;\, r\}$$

for r in $\mathfrak{A}$.[2] Clearly,

$$\varphi(0) = \varnothing \qquad \text{and} \qquad \varphi(1') = id_U.$$

Indeed, $a\,;\,0 = 0$, by Lemma 1.2(iii), so for no atoms a and b can we have $b \leq a\,;\,0$. Also, $a\,;\,1' = a$, by the identity law for relative multiplication, so for atoms a and b we have $b \leq a\,;\,1'$ if and only if $a = b$.

To prove that φ preserves the operations of $\mathfrak{A}$, fix atoms a and b in U, and elements r and s in $\mathfrak{A}$. Consider first the operation of Boolean addition. The left-hand distributive law for relative multiplication over addition implies that

$$a\,;\,(r + s) = a\,;\,r + a\,;\,s.$$

Since b is an atom, we have

$$b \leq a\,;\,(r + s) \qquad \text{if and only if} \qquad b \leq a\,;\,r \;\text{ or }\; b \leq a\,;\,s.$$

It follows that

$$\begin{aligned}
(a, b) \in \varphi(r + s) \qquad & \text{if and only if} \qquad b \leq a\,;\,(r + s), \\
& \text{if and only if} \qquad b \leq a\,;\,r \;\text{ or }\; b \leq a\,;\,s, \\
& \text{if and only if} \qquad (a, b) \in \varphi(r) \;\text{ or }\; (a, b) \in \varphi(s), \\
& \text{if and only if} \qquad (a, b) \in \varphi(r) \cup \varphi(s),
\end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of union. Thus, $\varphi(r + s) = \varphi(r) \cup \varphi(s)$.

Turn now to the operation of Boolean multiplication. Every element in U is below a function in $\mathfrak{A}$, by the definition of U, and is therefore a function, by Lemma 1.5(ii). The distributive law for functions (Lemma 1.5(iii)) implies

$$a\,;\,(r \cdot s) = (a\,;\,r) \cdot (a\,;\,s),$$

and therefore

$$\begin{aligned}
b \leq a\,;\,(r \cdot s) \qquad & \text{if and only if} \qquad b \leq (a\,;\,r) \cdot (a\,;\,s), \\
& \text{if and only if} \qquad b \leq a\,;\,r \text{ and } b \leq a\,;\,s.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(a, b) \in \varphi(r \cdot s) \qquad & \text{if and only if} \qquad b \leq a\,;\,(r \cdot s), \\
& \text{if and only if} \qquad b \leq a\,;\,r \text{ and } b \leq a\,;\,s, \\
& \text{if and only if} \qquad (a, b) \in \varphi(r) \text{ and } (a, b) \in \varphi(s),
\end{aligned}$$

[2] Maddux has called our attention to the fact that the function φ appears in the abstract [10], but it does not appear in the paper [9].

$$\text{if and only if} \qquad (a, b) \in \varphi(r) \cap \varphi(s),$$

by the definition of φ, the preceding remarks, and the definition of intersection. Thus, $\varphi(r \cdot s) = \varphi(r) \cap \varphi(s)$.

As regards the preservation of converse, observe that

$$a \leq b\,;r \qquad \text{if and only if} \qquad b \leq a\,;r^{\smile},$$

by (the contrapositive of) Lemma 1.1 (with b, r, and a in place of r, s, and t respectively) and the assumption that a and b are atoms. Consequently,

$$\begin{aligned}(a, b) \in \varphi(r^{\smile}) \qquad &\text{if and only if} \qquad b \leq a\,;r^{\smile},\\ &\text{if and only if} \qquad a \leq b\,;r,\\ &\text{if and only if} \qquad (b, a) \in \varphi(r),\\ &\text{if and only if} \qquad (a, b) \in \varphi(r)^{-1},\end{aligned}$$

by the definition of φ, the preceding remarks, and the definition of the converse of a relation. Thus, $\varphi(r^{\smile}) = \varphi(r)^{-1}$.

As is to be expected, the proof that φ preserves relative multiplication is different than in the proof of Theorem 2.1. Suppose first that a pair (a, b) belongs to the relational composition $\varphi(r)|\varphi(s)$. There must then be an atom c in U such that

$$(a, c) \in \varphi(r) \qquad \text{and} \qquad (c, b) \in \varphi(s),$$

by the definition of relational composition. It follows from the definition of φ that $c \leq a\,;r$ and $b \leq c\,;s$. Consequently,

$$b \leq c\,;s \leq a\,;r\,;s,$$

by the monotony law for relative multiplication, so the pair (a, b) belongs to $\varphi(r\,;s)$. This argument shows that $\varphi(r)|\varphi(s)$ is included in $\varphi(r\,;s)$.

To establish the reverse inclusion, suppose that (a, b) belongs to $\varphi(r\,;s)$. The definition of φ implies that b is below $a\,;r\,;s$, and therefore

$$(a\,;r\,;s) \cdot b \neq 0.$$

Apply Lemma 1.1 (with r and t replaced by $a\,;r$ and b) to obtain

$$(a\,;r) \cdot (b\,;s^{\smile}) \neq 0.$$

Invoke Lemma 3.2 (with I a two-element index set) to get an atom c in U satisfying

$$c \leq (a\,;r) \cdot (b\,;s^{\smile}).$$

Since c is below $a\,;r$, the pair (a, c) is in $\varphi(r)$, by the definition of φ. Since c is also below $b\,;s^{\smile}$, the pair (b, c) is in $\varphi(s^{\smile})$. Consequently, the pair (c, b) is in $\varphi(s)$, because—as we have already seen—φ preserves converse. The pair (a, b) is therefore in the relational composition $\varphi(r)|\varphi(s)$, by the definition of this composition. Conclusion: $\varphi(r\,;s) = \varphi(r)|\varphi(s)$.

The fact that φ preserves the identity element and the operations of addition, converse, and relative multiplication implies that the relation $E = \varphi(1)$ is an equivalence relation on the base set U. In more detail,

$$id_U = \varphi(1') \leq \varphi(1) = E,$$
$$E^{-1} = \varphi(1)^{-1} = \varphi(1^\smile) = \varphi(1) = E,$$
$$E|E = \varphi(1)|\varphi(1) = \varphi(1\,;1) = \varphi(1) = E.$$

A mapping on a relation algebra that preserves addition, Boolean multiplication, zero, and the unit must also preserve complement in the sense that $\varphi(-r) = {\sim}\varphi(r)$, where the complement on the right is formed in $\mathfrak{Re}(E)$. The argument so far therefore proves that φ is a homomorphism from $\mathfrak{A}$ into $\mathfrak{Re}(E)$.

It remains to check that φ is one-to-one. Suppose $r \neq 0$. The element $1\,;r^\smile$ is then certainly not zero, by Lemma 1.2(i),(iv), and the second involution law, so there must be a non-zero function f in $\mathfrak{A}$ that is below $1\,;r^\smile$, by the assumed functional density of $\mathfrak{A}$. In particular, $f \cdot (1\,;r^\smile) \neq 0$. Apply Lemma 1.1 to obtain that $(f\,;r)\cdot 1 \neq 0$. Thus, $f\,;r$ is not zero.

Let X be the set of atoms in the canonical extension $\mathfrak{B}$ that are below f. Every atom in X is below a function in $\mathfrak{A}$, namely f, so every atom in X belongs to the set U, by the definition of U. Since $\mathfrak{B}$ is atomic, f must be the supremum of the set X of atoms that it dominates, so

$$0 < f\,;r = (\textstyle\sum X)\,;r = \sum\{a\,;r : a \in X\},$$

by the complete distributivity of relative multiplication over addition. It follows that $a\,;r \neq 0$ for some atom a below f. Apply Lemma 3.2 (with I a one-element index set) to obtain an atom b in U that is below $a\,;r$. The pair (a,b) belongs to $\varphi(r)$, by the definition of φ. Conclusion: the image under φ of every non-zero element in $\mathfrak{A}$ is a non-empty relation on the set U, so the kernel of φ consists of just zero. Thus, φ is one-to-one and therefore an embedding of $\mathfrak{A}$ into $\mathfrak{Re}(E)$. In other words, φ is a representation of $\mathfrak{A}$.

Consider now the case when $\mathfrak{A}$ is an arbitrary functionally dense relation algebra. The completion of $\mathfrak{A}$ is also functionally dense. Indeed, each non-zero element in the completion is above a non-zero element in $\mathfrak{A}$, by the definition of the completion, and each non-zero element in $\mathfrak{A}$ is above a non-zero function, by the assumption that $\mathfrak{A}$ is functional dense. Consequently, each non-zero element in the completion is above a non-zero function. It follows that the completion is representable via some function φ, by the first part of the proof. The restriction of φ to $\mathfrak{A}$ is a representation of $\mathfrak{A}$. □

The reader may wonder why the argument in the proof of Theorem 2.1, showing that the representation defined in that proof is complete, cannot also be applied in the proof of Theorem 3.3 to show that a complete, functionally dense relation algebra is completely representable. The proof of Theorem 3.3 uses the canonical extension of $\mathfrak{A}$, even when $\mathfrak{A}$ is complete; and suprema of infinite subsets of $\mathfrak{A}$ are in general different in $\mathfrak{A}$ than they are in the canonical

extension. In fact, if $\mathfrak{A}$ did possess a complete representation, then $\mathfrak{A}$ would have to be atomic, by a theorem of Hirsch and Hodkinson [3], and would therefore fall under the purview of Theorem 2.1. In this connection, the following observation may be of some interest. The canonical extension of a representable relation algebra is always completely representable. (This result—due to Hirsch and Hodkinson [4], and independently to the authors—improves an earlier result of Monk, according to which the canonical extension of a representable relation algebra is representable.) Consequently, the canonical extension of a functionally dense relation algebra *is* completely representable.

4. A decomposition theorem for functionally dense relation algebras

The representation in Theorem 3.3 raises the problem of giving a complete structural description of all functionally dense relation algebras. Although this problem is not yet settled, some interesting observations can be made. As we shall see in a moment, simple, functionally dense relation algebras are not always atomic. However, the presence of a single atom in such a relation algebra does imply atomicity.

Theorem 4.1. *A functionally dense relation algebra that is simple must be either atomic or atomless.*

Proof. Let $\mathfrak{A}$ be a functionally dense relation algebra that is simple. Observe, first of all, that if the left side of a rectangle in $\mathfrak{A}$ is an atom, then the rectangle must be the sum of a set of functional atoms in $\mathfrak{A}$. Indeed, consider a subidentity atom x in $\mathfrak{A}$, and let y be an arbitrary subidentity element. The rectangle $x\,;1\,;y$ is the sum of a (possibly empty) set X of non-zero functions, by the assumed functional density of $\mathfrak{A}$. Each function f in X is below $x\,;1$, by the monotony law for relative multiplication, and $x\,;1$ is an atom in the Boolean algebra of right-ideal elements, by Lemma 1.6(ii), so f is an atom in $\mathfrak{A}$, by Lemma 1.6(iii). It follows in particular that every rectangle with atomic sides in $\mathfrak{A}$ is the sum of a set of functional atoms.

Assume now that the algebra $\mathfrak{A}$ is not atomless, say r is an atom in $\mathfrak{A}$. The domain of r—call it x—is a subidentity atom, by Lemma 1.6(i). The rectangle $x\,;1\,;1'$ is therefore the sum of a set X of functional atoms, by the observation of the previous paragraph. The range of every atom is again an atom, by Lemma 1.6(i), so $(1\,;f)\cdot 1'$ is an atom for every f in X. The assumed simplicity of $\mathfrak{A}$ implies that $1\,;x\,;1=1$. Use also the identity law for relative multiplication and the complete distributivity of Boolean and relative multiplication over addition to arrive at

$$\begin{aligned}1' = (1\,;1')\cdot 1' = [(1\,;x\,;1)\,;1']\cdot 1' = [1\,;(x\,;1\,;1')]\cdot 1'\\ = (1\,;\textstyle\sum X)\cdot 1' = \sum\{(1\,;f)\cdot 1' : f\in X\}.\end{aligned}$$

This calculation shows that $1'$ is the sum of the set W of all subidentity atoms.

It now follows easily that the unit in $\mathfrak{A}$ is the sum of the set of all rectangles with atomic sides, since

$$1 = 1' ; 1 ; 1' = (\sum W) ; 1 ; (\sum W) = \sum\{y ; 1 ; z : y, z \in W\}.$$

We saw in the first paragraph that every rectangle with atomic sides is the sum of a set of functional atoms. Combine these two observations to conclude that the unit is the sum of the set of all functional atoms. Consequently, $\mathfrak{A}$ is atomic. □

Here is the direct decomposition theorem for functionally dense relation algebras.

Theorem 4.2. *Every functionally dense relation algebra is essentially isomorphic to a direct product of simple, functionally dense relation algebras—each of which is either atomic or atomless—and a single functionally dense relation algebra that is atomless and has an atomless Boolean algebra of ideal elements.*

Proof. Given a functionally dense relation algebra, pass to its completion $\mathfrak{A}$. It is easily seen that $\mathfrak{A}$ is also functionally dense. According to Theorem 1.7, the algebra $\mathfrak{A}$ is isomorphic to the direct product of two (possibly trivial) complete relation algebras $\mathfrak{B}$ and $\mathfrak{C}$ with the property that $\mathfrak{B}$ is the direct product of (complete) simple relation algebras, and $\mathfrak{C}$ is atomless and has an atomless Boolean algebra of ideal elements. Moreover, the algebras $\mathfrak{B}$ and $\mathfrak{C}$, and also the simple algebras in the direct decomposition of $\mathfrak{B}$, are all relativizations of $\mathfrak{A}$. It is easy to check that a relativization of a functionally dense relation algebra remains functionally dense. Consequently, the factor $\mathfrak{B}$ is the direct product of complete, simple, functionally dense relation algebras, each of which is either atomic or atomless, by Theorem 4.1; and the factor $\mathfrak{C}$ is a complete, atomless, functionally dense relation algebra with an atomless Boolean algebra of ideal elements. □

It may happen that one or more of the factors in the decomposition of the preceding theorem is trivial. For example, if $\mathfrak{A}$ is atomic, then it has no non-trivial atomless factors, so the atomless factor $\mathfrak{C}$ in the proof of the preceding theorem is trivial, and the system of simple factors in the direct decomposition of the factor $\mathfrak{B}$ contains no atomless algebras at all. On the other hand, if $\mathfrak{A}$ itself has an atomless Boolean algebra of ideal elements, then the factor $\mathfrak{B}$ in the proof of the preceding theorem is trivial.

An atomic relation algebra is functionally dense if and only if every atom is a function. Indeed, an atomic relation algebra with functional atoms is obviously functionally dense, since every element is above an atom and therefore above a non-zero function. On the other hand, in an atomic, functionally dense relation algebra, every atom must be above a non-zero function and therefore every atom must be a function.

Corollary 4.3. *Every atomic relation algebra with functional atoms is essentially isomorphic to a direct product of simple, atomic relation algebras with functional atoms.*

5. A description of atomic relation algebras with functional atoms

Jónsson and Tarski [8] gave a description of all atomic relation algebras with functional atoms in terms of (axiomatically defined) complex algebras of generalized Brandt groupoids. In its application to simple relation algebras, their theorem says that a simple relation algebra is complete and atomic with functional atoms if and only it is isomorphic to the complex algebra of an Brandt groupoid, and every (not necessarily complete) simple relation algebra that is atomic with functional atoms is embeddable into the complex algebra of a Brandt groupoid. (Observe that, by Corollary 4.3, the problem of describing all atomic relation algebras with functional atoms reduces to the problem of describing all such algebras that are simple.) As a consequence of their description, Jónsson and Tarski concluded that a relation algebra is representable if and only if it is embeddable into the complex algebra of a generalized Brandt groupoid, and a simple relation algebra is representable if and only if it is embeddable into the complex algebra of a Brandt groupoid.

In this section we shall give a somewhat different description than the one in [8]—a description that (in our opinion) is more visual and also more familiar. We begin with a lemma. A relation algebra is said to be *bijectively dense* if below every non-zero element there is a non-zero bijection.

Lemma 5.1. *A functionally dense relation algebra is bijectively dense. In particular, every atom is a bijection.*

Proof. Let $\mathfrak{A}$ be a functionally dense relation algebra, and r a non-zero element in $\mathfrak{A}$. The converse $r^\smile$ is also non-zero, by Lemma 1.2(i) and the first involution law, so there is a non-zero function f below $r^\smile$, by the functional density of $\mathfrak{A}$. The converse $f^\smile$ is non-zero, so there is a non-zero function g below $f^\smile$, again by the functional density of $\mathfrak{A}$. It is easy to check that g is a bijection below r. Indeed, $g^\smile$ must be below $f^{\smile\smile}$, by the monotony law for converse, and $f^{\smile\smile} = f$, by the first involution law, so $g^\smile$ is below f and is therefore a function, by Lemma 1.5(ii). Since g is a function, by assumption, it follows that g is a bijection. Also, $f^\smile$ is below $r^{\smile\smile}$, and $r^{\smile\smile} = r$, so $g \leq f^\smile \leq r$. □

We shall also need the following theorem, due to Jónsson-Tarski [8].

Theorem 5.2. *An integral relation algebra is complete and atomic with functional atoms if and only if it is isomorphic to the complex algebra of a group.*

Proof. Obviously, the complex algebra of a group is an integral relation algebra that is complete and atomic with functional atoms (see the remarks following

Theorem 1.7). Consider now an arbitrary integral relation algebra $\mathfrak{A}$ that is complete and atomic with functional atoms. The set G of atoms in $\mathfrak{A}$ consists of bijections, by Lemma 5.1, and the domain and range of each of these bijections is the identity element 1', by Lemma 1.6(i). The elements in G therefore form a group under the restricted operations of relative multiplication and converse from $\mathfrak{A}$, by Lemma 1.5(i) and the definition of a bijection; and the identity element of the group coincides with the identity element of $\mathfrak{A}$. The function ϑ that maps f to $\{f\}$ for each f in G is clearly a bijection from the set of of atoms in $\mathfrak{A}$ to the set of atoms in $\mathfrak{Cm}(G)$. Moreover, this function maps the identity element 1' in $\mathfrak{A}$ to the identity element $\{1'\}$ in $\mathfrak{Cm}(G)$, and it preserves the operations of converse and relative multiplication on atoms. For example,

$$\vartheta(f\,;g) = \{f\,;g\} = \{f\}\,;\{g\} = \vartheta(f)\,;\vartheta(g),$$

by the definitions of ϑ and of relative multiplication in $\mathfrak{Cm}(G)$. The natural extension of ϑ to $\mathfrak{A}$, the function φ that is defined for every element r in $\mathfrak{A}$ by

$$\varphi(r) = \{f \in G : f \leq r\},$$

is easily seen to be a well-defined isomorphism from $\mathfrak{A}$ to $\mathfrak{Cm}(G)$. □

Corollary 5.3. *An integral relation algebra is atomic with functional atoms if and only if it is essentially isomorphic to the complex algebra of a group.*

In order to describe all *simple* (not necessarily integral) atomic relation algebras with functional atoms, it is necessary to construct a broader class of examples than group complex algebras. Fix a complete relation algebra $\mathfrak{C}$ and a cardinal number $\kappa > 0$ (which, for notational convenience, we assume to be the set of all smaller ordinals, as is standard in set theory). A κ-by-κ *matrix* with entries from $\mathfrak{C}$ is a function r from the Cartesian product $\kappa \times \kappa$ into the universe of $\mathfrak{C}$. We write $r = [r_{ij}]$ for this matrix. Let B be the set of all such matrices, and define Boolean operations $+$ and $-$, and Peircean operations $;$ and $^\smile$, on B by

$$r + s = t, \qquad \text{where} \qquad t_{ij} = r_{ij} + s_{ij}, \tag{1}$$

$$-r = t, \qquad \text{where} \qquad t_{ij} = -r_{ij}, \tag{2}$$

$$r\,;s = t, \qquad \text{where} \qquad t_{ij} = \textstyle\sum_{k<\kappa} r_{ik}\,;s_{kj}, \tag{3}$$

$$r^\smile = t, \qquad \text{where} \qquad t_{ij} = r^\smile_{ji}. \tag{4}$$

Here, the operation symbols on the left denote the operations being defined on B, whereas the operation symbols on the right denote the operations of the relation algebra $\mathfrak{C}$. (The assumed completeness of $\mathfrak{C}$ is needed in order to ensure that the sum on the right side of (3) exists.) For the most part, the operations being defined are just standard operations on matrices from linear algebra, except that the operations on the entries of the matrices are performed in the relation algebra $\mathfrak{C}$ instead of in a field. Thus, the operation of addition is the standard operation of forming the (componentwise) sum of two matrices, except that the sums of entries are formed as Boolean sums in $\mathfrak{C}$;

the operation of forming the complement is the standard operation of forming the (componentwise) negative of a matrix, except that the negatives of entries are formed as complements in $\mathfrak{C}$; and the operation of relative multiplication is the standard operation of multiplying two matrices, except that the sums and products of the entries are formed as Boolean sums and relative products in $\mathfrak{C}$. The converse of a matrix is obtained by first forming the (componentwise) converse of each entry of the matrix and then forming the transpose of the result. The identity matrix 1' in B is just the matrix with the identity element of $\mathfrak{C}$ down the diagonal and the zero element of $\mathfrak{C}$ elsewhere. The algebra

$$\mathfrak{B} = (B\,,\,+\,,\,-\,,\,;\,,\,{}^{\smile}\,,\,1')$$

is easily seen to be a relation algebra (see [6] and [12]). Indeed, the validity of most of the relation algebraic axioms in $\mathfrak{B}$ is an immediate consequence of the validity of these laws in the algebra $\mathfrak{C}$, and the standard laws governing matrix multiplication and transposition. We shall call $\mathfrak{B}$ the κth *matrix relation algebra over* $\mathfrak{C}$, and we shall refer to $\mathfrak{C}$ as the *base* of $\mathfrak{B}$.

When $\mathfrak{C}$ is the complex algebra of a group G, every matrix algebra over $\mathfrak{C}$ is an example of a simple relation algebra that is complete and atomic with functional atoms (see below). The atoms are the matrices that have exactly one non-zero entry, and that entry is some atom in $\mathfrak{C}$ (that is to say, it is some element of the group G).

To obtain a different perspective on matrix relation algebras, suppose $\mathfrak{B}$ is the kth matrix relation algebra over a base $\mathfrak{C}$ that is a complete set relation algebra on some base set G; in other words, $\mathfrak{C}$ is assumed to be a complete subalgebra of $\mathfrak{Re}(G)$. The intention is that $\mathfrak{C}$ is the Cayley representation of the complex algebra of some group G, but this intention does not play a role in the discussion. The immediate goal is to construct a complete representation of $\mathfrak{B}$ by associating with each matrix $[R_{ij}]$ in $\mathfrak{B}$ a binary relation that is the disjoint union of copies of the relations R_{ij} which form the entries of the given matrix. Here are the details.

Let $(G_i : i < \kappa)$ be a system of pairwise disjoint copies of the set G, selected so that $G_0 = G$. Take F_{00} to be the identity function on G, and for each index i with $0 < i < \kappa$, take F_{0i} be an arbitrary bijection from G to G_i. For each element g in G, write g_i for the image of g (in G_i) under F_{0i}. Put $F_{ij} = F_{0i}^{-1}|F_{0j}$, and observe that F_{ij} is a bijection from G_i to G_j (and hence a subrelation of $G_i \times G_j$); in fact,

$$F_{ij} = \{(g_i, g_j) : g \in G\}.$$

The following properties of these bijections are readily verified:

$$F_{ii} = id_{G_i}, \quad F_{ij}^{-1} = F_{ji}, \quad F_{ik}|F_{kj} = F_{ij}, \quad \text{and} \qquad F_{ik}|F_{\ell j} = \varnothing \tag{5}$$

when $k \neq \ell$.

Write $U = \bigcup_{i<\kappa} G_i$ and observe that the rectangles $G_i \times G_j$ for $i, j < \kappa$ form a partition of $U \times U$ in the sense that they are non-empty, mutually

disjoint, and have $U \times U$ as their union (see Figure 1). The isomorphic copy

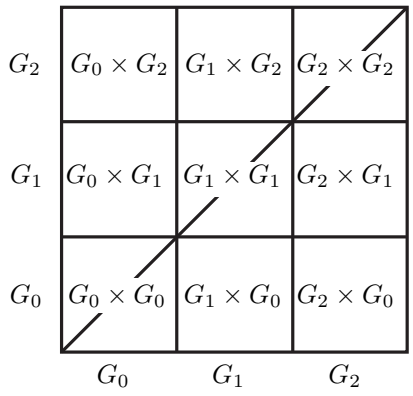

FIGURE 1. Rectangles partitioning the unit set $U \times U$.

of $\mathfrak{B}$ that we are going to define is a subalgebra of $\mathfrak{Re}(U)$. If R_{ij} is an element in $\mathfrak{C}$ (and thus a subrelation of $G \times G$), then write

$$R^*_{ij} = F_{i0}|R_{ij}|F_{0j} = \{(f_i, g_j) : (f, g) \in R_{ij}\} \tag{6}$$

and observe that R^*_{ij} is the copy of R_{ij} that is a subrelation of $G_i \times G_j$. Using (5), it is not difficult to verify that the following equations are valid:

$$R^*_{ij} \cup S^*_{ij} = T^*_{ij}, \qquad \text{where} \qquad T_{ij} = R_{ij} \cup S_{ij}, \tag{7}$$

$$G_i \times G_j \sim R^*_{ij} = T^*_{ij}, \qquad \text{where} \qquad T_{ij} = {\sim}R_{ij}, \tag{8}$$

$$R^*_{ik}|S^*_{kj} = T^*_{ij}, \qquad \text{where} \qquad T_{ij} = R_{ik}|S_{kj}, \tag{9}$$

$$(R^*_{ji})^{-1} = T^*_{ij}, \qquad \text{where} \qquad T_{ij} = R^{-1}_{ji}, \tag{10}$$

$$R^*_{ik}|S^*_{\ell j} = \varnothing \qquad \text{when} \qquad k \neq \ell. \tag{11}$$

The operations on the left are performed in $\mathfrak{Re}(U)$, whereas the operations on the right are performed in $\mathfrak{C}$, since the elements on the right come from $\mathfrak{C}$. As an example, here is the verification of (9):

$$\begin{aligned} R^*_{ik}|S^*_{kj} &= (F_{i0}|R_{ik}|F_{0k})|(F_{k0}|S_{kj}|F_{0j}) = F_{i0}|R_{ik}|(F_{0k}|F_{k0})|S_{kj}|F_{0j} \\ &= F_{i0}|R_{ik}|F_{00}|S_{kj}|F_{0j} = F_{i0}|R_{ik}|id_G|S_{kj}|F_{0j} \\ &= F_{i0}|R_{ik}|S_{kj}|F_{0j} = F_{i0}|T_{ij}|F_{0k} = T^*_{ij}. \end{aligned}$$

Notice that this derivation depends only on the definition in (6) and the properties of the bijections that are stated in (5).

With each matrix $R = [R_{ij}]$ in $\mathfrak{B}$, we correlate a relation R^* on the set U that is defined by

$$R^* = \bigcup\{R^*_{ij} : i, j < \kappa\}. \tag{12}$$

Keep in mind that R_{ij} is a relation in $\mathfrak{C}$, and R^*_{ij} is the copy of this relation that is a subrelation of the rectangle $G_i \times G_j$. Thus, for each i and j we make

a copy beneath $G_i \times G_j$ of the ijth entry of the matrix R, and we take R^* to be the (disjoint) union of these copies of the entries in R (see Figure 2). Since the rectangles $G_i \times G_j$ are mutually disjoint, the function that maps

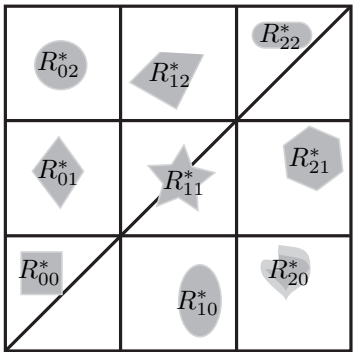

FIGURE 2. A typical relation R^*.

each matrix R to the relation R^* is an injection from $\mathfrak{B}$ into $\mathfrak{Re}(U)$. With the help of equations (7)–(11), we see that the following equations are valid for all matrices $R = [R_{ij}]$, $S = [S_{ij}]$, and $T = [T_{ij}]$ in $\mathfrak{B}$:

$$R^* \cup S^* = T^*, \qquad \text{where} \qquad T_{ij} = R_{ij} \cup S_{ij}, \tag{13}$$

$$\sim R^* = T^*, \qquad \text{where} \qquad T_{ij} = \sim R_{ij}, \tag{14}$$

$$R^* | S^* = T^*, \qquad \text{where} \qquad T_{ij} = \textstyle\bigcup_k (R_{ik} | S_{kj}), \tag{15}$$

$$(R^*)^{-1} = T^*, \qquad \text{where} \qquad T_{ij} = R_{ji}^{-1}, \tag{16}$$

$$id_U = T^*, \qquad \text{where} \qquad T_{ij} = id_{G_i} \text{ or } T_{ij} = \varnothing, \tag{17}$$

$$\text{according as } i = j \text{ or } i \neq j.$$

For example, here is the verification of (15), where the matrix T is as specified:

$$\begin{aligned} R^* | S^* &= (\textstyle\bigcup_{ij} R^*_{ij}) | (\bigcup_{ij} S^*_{ij}) = \bigcup_{ijk\ell} (R^*_{ik} | S^*_{\ell j}) \\ &= \textstyle\bigcup_{ijk} (R^*_{ik} | S^*_{kj}) = \bigcup_{ij} \bigcup_k (R^*_{ik} | S^*_{kj}) = \bigcup_{ij} T^*_{ij} = T^*. \end{aligned}$$

The first and last equalities hold by (12), the second by the complete distributivity of relational composition over unions, the third by (11), and the fourth by the general associative law for unions. For the fifth equality, observe that

$$T^*_{ij} = (\textstyle\bigcup_k (R_{ik} | S_{kj}))^* = \bigcup_k (R_{ik} | S_{kj})^* = \bigcup_k (R^*_{ik} | S^*_{kj}),$$

by the definition of T_{ij} in (15) and the equations in (7) (generalized to arbitrary unions) and (9).

A comparison of (1)–(4) with (13)–(16), and of the definition of the identity matrix in $\mathfrak{B}$ with (17), shows that the function mapping R to R^* for each matrix R in $\mathfrak{B}$ preserves the operations of $\mathfrak{B}$ and also preserves arbitrary sums in $\mathfrak{B}$ as unions, so this mapping is a complete embedding of $\mathfrak{B}$ into $\mathfrak{Re}(U)$. We shall call the image of this embedding — that is to say, the complete

subalgebra of $\mathfrak{Re}(U)$ consisting of all relations of the form R^* for R in $\mathfrak{B}$ — the κth *semipower* of $\mathfrak{C}$, because in some sense it is just κ^2 disjoint copies of the algebra $\mathfrak{C}$, all amalgamated together. The argument just presented shows that the kth matrix relation algebra over $\mathfrak{C}$ is isomorphic to the κth semipower of $\mathfrak{C}$. Observe that this semipower is obviously simple, since it is a complete subalgebra of the simple set relation algebra $\mathfrak{Re}(U)$; and it is obviously atomic whenever $\mathfrak{C}$ is atomic, its atoms being the copies beneath $G_i \times G_j$, for each pair $i, j < \kappa$, of the atoms in $\mathfrak{C}$.

In particular, if $\mathfrak{C}$ is the Cayley representation of a group G, so that $\mathfrak{C} = \mathfrak{Ca}(G)$, then the kth semipower of $\mathfrak{C}$ is a simple, complete and atomic relation algebra with functional atoms. The complex algebra $\mathfrak{Cm}(G)$ is canonically isomorphic to $\mathfrak{Ca}(G)$ via the Cayley representation, and therefore the kth matrix relation algebra over $\mathfrak{Cm}(G)$ is canonically isomorphic to the kth matrix relation algebra over $\mathfrak{Ca}(G)$. The latter is isomorphic to the kth semipower of $\mathfrak{Ca}(G)$, by the remarks of the preceding paragraph. We summarize what has been accomplished so far.

Theorem 5.4. *For each group G and each cardinal number $\kappa > 0$, the* κth *matrix relation algebra over $\mathfrak{Cm}(G)$ is a simple, complete and atomic relation algebra with functional atoms. It is isomorphic to the* kth *semipower of $\mathfrak{Ca}(G)$, and is therefore completely representable.*

The next theorem and its corollary say that, up to essential isomorphism, the algebras of Theorem 5.4 are the only simple relation algebras that are atomic with functional atoms.

Theorem 5.5. *A simple relation algebra is complete and atomic with functional atoms if and only if it is isomorphic to the* κth *matrix relation algebra over $\mathfrak{Cm}(G)$—or equivalently, to the* κth *semipower of $\mathfrak{Ca}(G)$—for some cardinal $\kappa > 0$ and some group G.*

Proof. The implication from right to left in the statement of the theorem has already been proved in the discussion leading up to Theorem 5.4. To establish the implication in the reverse direction, fix a simple relation algebra $\mathfrak{A}$ that is complete and atomic with functional atoms. The idea of the proof is to imitate in an abstract setting the proof given above that the κth matrix relation algebra over $\mathfrak{Cm}(G)$ is isomorphic to the κth semipower of $\mathfrak{Ca}(G)$.

The assumption that $\mathfrak{A}$ is atomic implies that the identity element 1' in $\mathfrak{A}$ is the sum of a non-empty set of subidentity atoms. Let κ be the cardinality of this set of atoms, and let $(1'_i : i < \kappa)$ be an enumeration of the distinct subidentity atoms in $\mathfrak{A}$. For each pair $i, j < \kappa$, define the *local unit* 1_{ij} to be the rectangle

$$1_{ij} = 1'_i ; 1 ; 1'_j.$$

These local units are non-zero, by Lemma 1.4(iii) and the assumed simplicity of $\mathfrak{A}$; they are mutually disjoint, by Lemmas 1.4(i) and 1.2(iii); and they sum

to 1, since

$$1 = 1' ; 1 ; 1' = (\textstyle\sum_i 1'_i) ; 1 ; (\sum_i 1'_i) = \sum_{ij}(1'_i ; 1 ; 1'_j) = \sum_{ij} 1_{ij}, \tag{1}$$

by the identity laws for relative multiplication, the complete distributivity of relative multiplication over addition, and the definition of the local units. In other words, the local units form a partition of unity in $\mathfrak{A}$. They are the analogues in $\mathfrak{A}$ of the rectangles $G_i \times G_j$ in the construction of the κth semipower of $\mathfrak{Ca}(G)$. The following identities concerning the local units are not difficult to derive using the laws in Lemma 1.4 that govern rectangles:

$$1' \cdot 1_{ii} = 1'_i, \qquad 1_{ij}^{\smile} = 1_{ji}, \qquad 1_{ik} ; 1_{kj} = 1_{ij}, \qquad 1_{ik} ; 1_{\ell j} = 0 \tag{2}$$

for $k \neq \ell$, and if r is a non-zero element below 1_{ij}, then

$$\text{domain } r = 1'_i, \qquad \text{range } r = 1'_j, \qquad \text{and} \qquad 1'_i ; r = r ; 1'_j = r. \tag{3}$$

Put $f_{00} = 1'_0$, and for $0 < i < \kappa$ choose an atom f_{0i} below the local unit 1_{0i}. Each atom f_{0i} must be a bijection, by Lemma 5.1; and the domain and range of this bijection must be $1'_i$ and $1'_j$ respectively, by (3). Write $f_{ij} = f_{0i}^{\smile} ; f_{0j}$ and observe that f_{ij} is a (bijective) atom below 1_{ij}, by Lemma 1.6(v),(vi), the monotony law for relative multiplication, and (2). The atoms f_{ij} are the analogues in $\mathfrak{A}$ of the functions F_{ij} defined in the construction of the κth semipower of $\mathfrak{Ca}(G)$. The following properties of these atoms are readily verified using (2), (3), and Lemma 1.5:

$$f_{ii} = 1'_i, \qquad f_{ij}^{\smile} = f_{ji}, \qquad f_{ik} ; f_{kj} = f_{ij}, \qquad \text{and} \qquad f_{ik} ; f_{\ell j} = 0 \tag{4}$$

when $k \neq \ell$.

The local unit 1_{00} is a non-zero square, so the relativization of $\mathfrak{A}$ to 1_{00} is a relation algebra in which the identity element $1'_0$ is an atom. Consequently, this relativization—call it $\mathfrak{C}$ — is an integral relation algebra, by the remarks following Theorem 1.7. The algebra $\mathfrak{C}$ inherits from $\mathfrak{A}$ the properties of being complete and atomic with functional atoms. Apply Theorem 5.2 (and its proof) to conclude that the set of atoms in $\mathfrak{C}$ is a group G under the (restricted) operations of relative multiplication and converse in $\mathfrak{A}$, with $1'_0$ as the group identity element, and that $\mathfrak{C}$ is isomorphic to the complex algebra $\mathfrak{Cm}(G)$. For each element r_{ij} in $\mathfrak{C}$, put

$$r^*_{ij} = f_{i0} ; r_{ij} ; f_{0j} \tag{5}$$

and observe that r^*_{ij} is below 1_{ij}, by the monotony laws and (2):

$$r^*_{ij} = f_{i0} ; r_{ij} ; f_{0j} \leq 1_{i0} ; 1_{00} ; 1_{0j} = 1_{ij}.$$

The intuition behind this definition is that r^*_{ij} is a copy of r_{ij} below 1_{ij}. Using (5), it is not difficult to verify that the following equations are valid in $\mathfrak{A}$:

$$r^*_{ij} + s^*_{ij} = t^*_{ij}, \qquad \text{where} \qquad t_{ij} = r_{ij} + s_{ij}, \tag{6}$$

$$1_{ij} \cdot -r^*_{ij} = t^*_{ij}, \qquad \text{where} \qquad t_{ij} = -r_{ij}, \tag{7}$$

$$r^*_{ik} ; s^*_{kj} = t^*_{ij}, \qquad \text{where} \qquad t_{ij} = r_{ik} ; s_{kj}, \tag{8}$$

$$(r^*_{ji})^\smile = t^*_{ij}, \qquad \text{where} \qquad t_{ij} = r^\smile_{ji}, \tag{9}$$

$$r^*_{ik} \mathbin{;} s^*_{\ell j} = 0 \qquad \text{when} \qquad k \neq \ell. \tag{10}$$

The operations on the left are performed in $\mathfrak{A}$, whereas the operations on the right are performed in the relativization $\mathfrak{C}$, since the elements on the right come from $\mathfrak{C}$. As an example, here is the verification of (8):

$$\begin{aligned} r^*_{ik} \mathbin{;} s^*_{kj} &= (f_{i0} \mathbin{;} r_{ik} \mathbin{;} f_{0k}) \mathbin{;} (f_{k0} \mathbin{;} s_{kj} \mathbin{;} f_{0j}) = f_{i0} \mathbin{;} r_{ik} \mathbin{;} (f_{0k} \mathbin{;} f_{k0}) \mathbin{;} s_{kj} \mathbin{;} f_{0j} \\ &= f_{i0} \mathbin{;} r_{ik} \mathbin{;} f_{00} \mathbin{;} s_{kj} \mathbin{;} f_{0j} = f_{i0} \mathbin{;} r_{ik} \mathbin{;} 1'_0 \mathbin{;} s_{kj} \mathbin{;} f_{0j} \\ &= f_{i0} \mathbin{;} r_{ik} \mathbin{;} s_{kj} \mathbin{;} f_{0j} = f_{i0} \mathbin{;} t_{ij} \mathbin{;} f_{0k} = t^*_{ij}, \end{aligned}$$

by the definitions of r^*_{ik}, s^*_{kj}, and t^*_{ij} (see (5)), the associative law for relative multiplication, (4), (3), and the definition of t_{ij} in (8). We shall also need the equivalence

$$r^*_{ij} = s^*_{ij} \qquad \text{if and only if} \qquad r_{ij} = s_{ij}. \tag{11}$$

The implication from right to left is obvious. For the reverse implication, observe that

$$\begin{aligned} r_{ij} = 1'_0 \mathbin{;} r_{ij} \mathbin{;} 1'_0 &= f_{00} \mathbin{;} r_{ij} \mathbin{;} f_{00} = (f_{0i} \mathbin{;} f_{i0}) \mathbin{;} r_{ij} \mathbin{;} (f_{0j} \mathbin{;} f_{j0}) \\ &= f_{0i} \mathbin{;} (f_{i0} \mathbin{;} r_{ij} \mathbin{;} f_{0j}) \mathbin{;} f_{j0} = f_{0i} \mathbin{;} r^*_{ij} \mathbin{;} f_{j0}, \end{aligned}$$

by (3)–(5). Consequently, if $r^*_{ij} = s^*_{ij}$, then

$$r_{ij} = f_{0i} \mathbin{;} r^*_{ij} \mathbin{;} f_{j0} = f_{0i} \mathbin{;} s^*_{ij} \mathbin{;} f_{j0} = s_{ij}.$$

Let $\mathfrak{B}$ be the κth matrix relation algebra over $\mathfrak{C}$. With each matrix

$$r = [r_{ij}] = (r_{ij} : i, j < \kappa)$$

in $\mathfrak{B}$, correlate an element r^* in $\mathfrak{A}$ that is defined by

$$r^* = \sum\{r^*_{ij} : i, j < \kappa\}. \tag{12}$$

Keep in mind that r_{ij} is an element in the relativization $\mathfrak{C}$, and r^*_{ij} is an element in $\mathfrak{A}$. The assumption that $\mathfrak{A}$ is complete is needed in order to ensure that the sum r^* exists in the case when κ is infinite. The intuition behind the definition in (12) is that for each entry r_{ij} in the matrix r, we make a copy of r_{ij} that is below 1_{ij} — this is the element r^*_{ij}—and we take r^* to be the disjoint sum of these copies. Thus, r^* is a reasonable representative in $\mathfrak{A}$ of the matrix r.

Indeed, the function φ that maps each matrix $r = [r_{ij}]$ in $\mathfrak{B}$ to its representative r^* in $\mathfrak{A}$ is an injection from $\mathfrak{B}$ into $\mathfrak{A}$, by (11). To see that φ is also a surjection, consider an arbitrary element t in $\mathfrak{A}$. For each pair of indices $i, j < \kappa$, the element $t \cdot 1_{ij}$ is the portion of t that is below 1_{ij}. Put

$$r_{ij} = f_{0i} \mathbin{;} (t \cdot 1_{ij}) \mathbin{;} f_{j0}, \tag{13}$$

and observe that r_{ij} is below 1_{00}, by (2) and the monotony law for relative multiplication:

$$r_{ij} = f_{0i} \,;\, (t \cdot 1_{ij}) \,;\, f_{j0} \leq 1_{0i} \,;\, 1_{ij} \,;\, 1_{j0} = 1_{00}.$$

Consequently, r_{ij} belongs to the relativization $\mathfrak{C}$. Furthermore,

$$\begin{aligned} r_{ij}^* = f_{i0} \,;\, r_{ij} \,;\, f_{0j} &= f_{i0} \,;\, (f_{0i} \,;\, (t \cdot 1_{ij}) \,;\, f_{j0}) \,;\, f_{0j} \\ &= (f_{i0} \,;\, f_{0i}) \,;\, (t \cdot 1_{ij}) \,;\, (f_{j0} \,;\, f_{0j}) = f_{ii} \,;\, (t \cdot 1_{ij}) \,;\, f_{jj} \\ &= 1_i' \,;\, (t \cdot 1_{ij}) \,;\, 1_j' = t \cdot 1_{ij}, \end{aligned} \tag{14}$$

by (3)–(5) and (13). The matrix $r = [r_{ij}]$ belongs to $\mathfrak{B}$, by the definition of $\mathfrak{B}$, and

$$r^* = \textstyle\sum_{ij} r_{ij}^* = \sum_{ij} (t \cdot 1_{ij}) = t \cdot (\sum_{ij} 1_{ij}) = t \cdot 1 = t,$$

by (12), (14), the complete distributivity of Boolean multiplication over addition, and (1). Therefore, φ maps the matrix r to the element t, so every element in $\mathfrak{A}$ is the image under φ of some element in $\mathfrak{B}$.

Using equations (6)–(10), it is not too difficult to check that the following equations are valid in $\mathfrak{A}$ for all matrices $r = [r_{ij}]$, $s = [s_{ij}]$, and $t = [t_{ij}]$ in $\mathfrak{B}$:

$$r^* + s^* = t^*, \qquad \text{where} \qquad t_{ij} = r_{ij} + s_{ij}, \tag{15}$$

$$-r^* = t^*, \qquad \text{where} \qquad t_{ij} = -r_{ij}, \tag{16}$$

$$r^* \,;\, s^* = t^*, \qquad \text{where} \qquad t_{ij} = \textstyle\sum_k (r_{ik} \,;\, s_{kj}), \tag{17}$$

$$(r^*)^\smile = t^*, \qquad \text{where} \qquad t_{ij} = r_{ji}^\smile, \tag{18}$$

$$1' = t^*, \qquad \text{where} \qquad t_{ij} = 1_0' \text{ or } t_{ij} = 0, \tag{19}$$

$$\text{according as } i = j \text{ or } i \neq j.$$

For example, here is the verification of (17), where the matrix t is as specified:

$$\begin{aligned} r^* \,;\, s^* = (\textstyle\sum_{ij} r_{ij}^*) \,;\, (\sum_{ij} s_{ij}^*) &= \textstyle\sum_{ijk\ell} (r_{ik}^* \,;\, s_{\ell j}^*) \\ &= \textstyle\sum_{ijk} (r_{ik}^* \,;\, s_{kj}^*) = \sum_{ij} \sum_k (r_{ik}^* \,;\, s_{kj}^*) = \sum_{ij} t_{ij}^* = t^*. \end{aligned}$$

The first and last equalities hold by (12), the second by the complete distributivity of relative multiplication over Boolean addition, the third by (10), and the fourth by the general associative law for Boolean addition. For the fifth equality, observe that

$$t_{ij}^* = (\textstyle\sum_k (r_{ik} \,;\, s_{kj}))^* = \sum_k (r_{ik} \,;\, s_{kj})^* = \sum_k (r_{ik}^* \,;\, s_{kj}^*),$$

by the definition of t_{ij} in (17) and the equations in (6) (generalized to arbitrary Boolean sums) and (8).

A comparison of equations (1)–(4) in the definition of the matrix relation algebra $\mathfrak{B}$ with equations (15)–(18) above, and of the definition of the identity matrix in $\mathfrak{B}$ with equation (19) above, makes clear that the function φ preserves the operations and the distinguished element of $\mathfrak{B}$ and is therefore an isomorphism from $\mathfrak{B}$ onto $\mathfrak{A}$. It has already been proved in Theorem 5.4 that $\mathfrak{B}$ is isomorphic to the κth semipower of $\mathfrak{Ca}(G)$. Since $\mathfrak{B}$ is isomorphic

to $\mathfrak{A}$, by the proof given above, it may be concluded that $\mathfrak{A}$ is isomorphic to the κth semipower of $\mathfrak{Ca}(G)$. □

Corollary 5.6. *A simple relation algebra is atomic with functional atoms if and only if it is essentially isomorphic to a matrix relation algebra over the complex algebra of some group G—or equivalently, to a semipower of $\mathfrak{Ca}(G)$.*

It follows from Corollary 5.6 (without the help of Theorem 2.1) that every atomic relation algebra with functional atoms is completely representable. The proof of Theorem 5.5 therefore provides an alternative proof of Theorem 2.1, one that is longer but that also gives much more information, namely a clear description of what the representing algebras are.

6. An alternative description of atomic relation algebras with functional atoms

The goal of this section is to give an alternative description of the simple relation algebras that are atomic with functional atoms. This alternative description will lead (in the next section) to the construction of a natural class of simple relation algebras that are functionally dense and atomless.

Consider a group $(G\,,\ \circ\,,\ ^{-1}\,,\ \iota)$ and a subgroup H. The *right cosets* of H are the sets of the form

$$H \circ h = \{g \circ h : g \in H\},$$

and they form a partition of G. The *index* of H is the number of right cosets of H in G. Associated with each element f in G is its Cayley representation

$$R_f = \{(g, g \circ f) : g \in G\},$$

which is a permutation of the set G. The Cayley representations of elements in G form a partition of $G \times G$. Each Cayley representation maps each right coset of H bijectively to another right coset of H. More precisely, R_f maps the coset $H \circ h$ bijectively to the coset $H \circ h \circ f$, since

$$R_f(g \circ h) = g \circ h \circ f$$

for every element g in H. The various restrictions of R_f to right cosets of H are therefore bijections between these cosets. If K is such a coset, write $R_f{\restriction}K$ for the restriction of R_f to K.

Lemma 6.1. *Suppose G is a group and H a subgroup of G. The set of all restrictions of the Cayley representations R_f (for f in G) to right cosets of H is the set of atoms of a (simple) subalgebra of $\mathfrak{Re}(G)$ that is complete and atomic with functional atoms.*

Proof. Let W be the set of restrictions of the Cayley representations of elements in G to right cosets of H. We proceed to verify that W satisfies conditions (i)–(iv) of Theorem 1.8. Cayley representations of distinct elements in G are disjoint, as are distinct right cosets of H. Consequently, if f and g

are distinct elements in G, or if K and L distinct right cosets of H, then the restrictions $R_f{\restriction}K$ and $R_g{\restriction}L$ are disjoint. The union of the right cosets of H is G, so the union of the various restrictions to right cosets of a specific Cayley representation R_f is just R_f:

$$\begin{aligned} R_f = R_f{\restriction}G = R_f{\restriction}\bigcup\{K : K \text{ is a right coset of } H\} \\ = \bigcup\{R_f{\restriction}K : K \text{ is a right coset of } H\}. \end{aligned} \tag{1}$$

The unit $G \times G$ of the full set relation algebra $\mathfrak{Re}(G)$ is the union of the Cayley representations R_f for f in G, so the unit is also the union of the relations in W, by (1). Conclusion: the relations in W form a partition of $G \times G$. This verifies condition (i) of Theorem 1.8.

To verify condition (ii), take the element f in (1) to be the identity element ι of the group G. Since R_ι coincides with the identity relation id_G in $\mathfrak{Re}(G)$, it follows from (1) that id_G is the union of the relations in W that are below id_G, and in fact

$$R_\iota = \bigcup\{R_\iota{\restriction}K : K \text{ is a right coset of } H\}. \tag{2}$$

As regards condition (iii), if a permutation R_f maps a right coset K to a right coset L, then the converse of R_f, which is the permutation $R_{f^{-1}}$, maps the right coset L to the right coset K. The converse of the restriction $R_f{\restriction}K$ is therefore the restriction $R_{f^{-1}}{\restriction}L$, so the converse of every relation in W is another relation in W. In fact, since R_f maps a right coset K to the right coset $K \circ f$, the preceding observation may be summarized by writing

$$(R_f{\restriction}K)^{-1} = R_{f^{-1}}{\restriction}L, \qquad \text{where} \qquad L = K \circ f. \tag{3}$$

To verify condition (iv), consider restrictions $R_f{\restriction}K$ and $R_g{\restriction}L$ in W. If the image of K under R_f is L—that is to say, if $L = K \circ f$—then the relational composition of R_f and R_g, which is the permutation $R_{f \circ g}$, must map K to the right coset of H that is the image of L under R_g. On the other hand, if the image of K under R_f is not L, then this image coset must be disjoint from L, and therefore the composition of the two restrictions is the empty relation. Summarizing,

$$(R_f{\restriction}K)|(R_g{\restriction}L) = \begin{cases} R_{f \circ g}{\restriction}K & \text{if} \quad L = K \circ f, \\ \varnothing & \text{otherwise.} \end{cases} \tag{4}$$

In either case, the composition of the two restrictions is a union of relations in W, so condition (iv) holds.

Apply Theorem 1.8 to conclude that the set of all unions of subsets of W is a complete subuniverse of $\mathfrak{Re}(G)$, and the atoms of this subuniverse are just the relations in W. Each of these relations is a bijection, so the corresponding complete subalgebra of $\mathfrak{Re}(G)$ is atomic with functional atoms. The subalgebra is simple because $\mathfrak{Re}(G)$ is simple, and subalgebras of simple relation algebras are simple (see the remarks preceding Theorem 1.7). □

Denote the subalgebra of the previous lemma by $\mathfrak{F}(G, H)$. When H is the improper subgroup $H = G$, this subalgebra is just the Cayley representation $\mathfrak{Ca}(G)$ of the group complex algebra $\mathfrak{Cm}(G)$. At the other extreme, when H is the trivial subgroup $H = \{\iota\}$, the subalgebra coincides with the full set relation algebra $\mathfrak{Re}(G)$. It follows from Theorem 5.5 and Lemma 6.1 that $\mathfrak{F}(G, H)$ is isomorphic to a matrix relation algebra over the complex algebra of some group. The next theorem says that the matrix relation algebra is in fact the κth matrix relation algebra over the complex algebra $\mathfrak{Cm}(H)$, where κ is the index of H in G.

Theorem 6.2. *If G is a group and H a subgroup of G with index κ, then $\mathfrak{F}(G, H)$ is isomorphic to the κ*th *matrix relation algebra over $\mathfrak{Cm}(H)$—or equivalently, to the κ*th *semipower of $\mathfrak{Ca}(H)$.*

Proof. We continue with the notation introduced before Lemma 6.1. Write $\mathfrak{A}$ for the relation algebra $\mathfrak{F}(G, H)$. By Lemma 6.1, $\mathfrak{A}$ is a simple subalgebra of $\mathfrak{Re}(G)$ that is complete and atomic with functional atoms. Thus, $\mathfrak{A}$ satisfies the conditions of Theorem 5.5. Apply the theorem to conclude that for some cardinal $\kappa^* > 0$ and some group G^*, the algebra $\mathfrak{A}$ is isomorphic to the κ^*th matrix relation algebra over the group complex algebra G^*. The proof of Theorem 5.5 makes clear that κ^* is the number of subidentity atoms in $\mathfrak{A}$, and G^* is the group of atoms in $\mathfrak{A}$ that are below the local unit $1_{00} = 1_0' ; 1 ; 1_0'$.

The subgroup H is assumed to have κ right cosets in G, say

$$(H_i : i < \kappa) \tag{1}$$

is an enumeration (without repetitions) of these cosets, with $H_0 = H$. The atoms in $\mathfrak{A}$ are the restrictions to these cosets of the Cayley representations R_f of the elements f in G, since these restrictions are the elements in the set W from the proof of Lemma 6.1. Thus, the atoms in $\mathfrak{A}$ are the relations $R_f \restriction H_i$ for f in G and $i < \kappa$.

The subidentity atoms are the restrictions to right cosets of the identity relation R_ι (where ι is the identity element of the group G), by item (2) in the proof of Lemma 6.1, so the distinct subidentity atoms in $\mathfrak{A}$ are the restrictions $R_\iota \restriction H_i$ for $i < \kappa$. There are κ such restrictions, one for each $i < \kappa$, so the number of subidentity atoms in $\mathfrak{A}$ equals the index of H in G, that is to say, $\kappa^* = \kappa$.

Turn now to the determination in $\mathfrak{A}$ of the local unit 1_{00} and the group G^*. The subidentity atom $1_0'$ from the proof of Theorem 5.5 is the relation $R_\iota \restriction H$ in $\mathfrak{A}$, by the observations of the preceding paragraph, the choice of the atom $1_0'$ as the first element in the enumeration of the subidentity atoms (see the proof of Theorem 5.5), and the assumption that the coset H_0 is just H. The unit of $\mathfrak{A}$ is the universal relation $G \times G$, since $\mathfrak{A}$ is a subalgebra of $\mathfrak{Re}(G)$. Consequently, the local unit 1_{00} in $\mathfrak{A}$ is the relation

$$(R_\iota \restriction H) | (G \times G) | (R_\iota \restriction H), \tag{2}$$

by the definition of 1_{00}. The relation $G \times G$ is the union of the atomic relations $R_f \restriction H_i$ for f in G and $i < \kappa$, by the observations made after (1), so the local unit in (2) is the union of the compositions

$$(R_\iota \restriction H) | (R_f \restriction H_i) | (R_\iota \restriction H) \tag{3}$$

for f in G and $i < \kappa$, by the distributivity of relational composition over arbitrary unions. If $i \neq 0$—that is to say, if the coset H_i is different from H—then the composition $(R_\iota \restriction H) | (R_f \restriction H_i)$ is empty, by item (4) from the proof of Lemma 6.1, since the domain H_i of the second relation is disjoint from the range H of the first relation. Assuming $i = 0$ and hence $H_i = H$, a similar argument shows that if f is not in H, then the composition $(R_f \restriction H_i) | (R_\iota \restriction H)$ is empty, because the range $H \circ f$ of the first relation is disjoint from the domain H of the second relation. It follows that the only non-empty compositions in (3) are those for which $i = 0$ and f is in H; and for these compositions we have

$$\begin{aligned}(R_\iota \restriction H) | (R_f \restriction H_i) | (R_\iota \restriction H) &= (R_\iota \restriction H) | (R_f \restriction H) | (R_\iota \restriction H) \\ &= (R_\iota \circ f \circ \iota) \restriction H = R_f \restriction H,\end{aligned}$$

by item (4) from the proof of Lemma 6.1 and the fact that $\iota \circ f \circ \iota = f$ in G. Combine these observations to conclude that the local unit in (2) is the union of the atomic relations $R_f \restriction H$ for f in H. Thus, the group G^* is the set of these atomic relations under the operations (from $\mathfrak{A}$) of relational composition and converse, and with the identity element $R_\iota \restriction H$.

It is easy to check, using items (3) and (4) from the proof of Lemma 6.1 that the function mapping each element f in H to the relation $R_f \restriction H$ in G^* is a group isomorphism from H to G^*. In fact, this isomorphism is just the Cayley representation of the group H. The group isomorphism lifts to a relation algebraic isomorphism from the κth matrix relation algebra over $\mathfrak{Cm}(H)$ to the κth matrix relation algebra over $\mathfrak{Cm}(G^*)$. As we have already noted, $\mathfrak{A}$ is isomorphic to the latter, so $\mathfrak{A}$ must be isomorphic to the former as well. □

It is now easy to see that the relation algebras $\mathfrak{F}(G, H)$ suffice to represent all simple relation algebras that are complete and atomic with functional atoms.

Theorem 6.3. *A simple relation algebra is complete and atomic with functional atoms if and only if it is isomorphic to a relation algebra of the form $\mathfrak{F}(G, H)$ for some group G and some subgroup H of G.*

Proof. It was already shown in Lemma 6.1 that the relation algebras $\mathfrak{F}(G, H)$ are simple, complete, and atomic with functional atoms. To prove the reverse direction of the theorem, consider any simple relation algebra $\mathfrak{A}$ that is complete and atomic with functional atoms. By Theorem 5.5, $\mathfrak{A}$ is isomorphic to an algebra $\mathfrak{B}$ that is the κth matrix relation algebra over a group complex algebra $\mathfrak{Cm}(H)$, for some cardinal $\kappa > 0$ and some group H. Let G be any group that contains H as a subgroup with index κ. The algebra $\mathfrak{F}(G, H)$ is

isomorphic to $\mathfrak{B}$, by Theorem 6.2. Consequently, $\mathfrak{A}$ is isomorphic to $\mathfrak{F}(G, H)$, as was to be shown.

It remains to check that there actually is a group G containing H as a subgroup with index κ. Let K be any group of cardinality κ. For example K could be the cyclic group of order κ when κ is a finite or countably infinite cardinal, and K could be the weak κth direct power of the group of integers when κ is uncountable. Let e be the (group) identity element in K. Form the direct product $H \times K$, and observe that $H \times \{e\}$ is a subgroup of $H \times K$ with index κ. Indeed, the distinct cosets of $H \times \{e\}$ in $H \times K$ are just the sets $H \times \{k\}$ for k in K. Since the given group H is isomorphic to the subgroup $H \times \{e\}$ via the function that maps each element h in H to the pair (h, e), the Exchange Principle from general algebra may be applied to obtain a group G that is isomorphic to $H \times K$ and contains H as a subgroup of index κ. □

Corollary 6.4. *A simple relation algebra is atomic with functional atoms if and only if it is essentially isomorphic to $\mathfrak{F}(G, H)$ for some group G and some subgroup H of G.*

The preceding construction of the algebras $\mathfrak{F}(G, H)$ gives some insight into how simple, non-integral relation algebras that are atomic with functional atoms arise. One starts with the complex algebra of a group G, which is an integral relation algebra. As was mentioned before, $\mathfrak{Cm}(G)$ is isomorphic to its Cayley representation $\mathfrak{Ca}(G)$, which turns out to coincide with $\mathfrak{F}(G, G)$. One then splits each atom—and in particular each subidentity atom—in $\mathfrak{Ca}(G)$ into several pieces by introducing a proper subgroup H to form $\mathfrak{F}(G, H)$. The exact number of pieces into which each atom is split depends on the index of H in G. In the extreme case when H is the trivial subgroup of G, one ends up with the full set relation algebra $\mathfrak{Re}(G)$, but in general one gets a simple, non-integral, atomic relation algebra with functional atoms that lies between $\mathfrak{Ca}(G)$ and $\mathfrak{Re}(G)$.

7. Atomless functionally dense relation algebras

The problem of giving a set-theoretical description of all simple, functionally dense relation algebras that are atomless is still open. We content ourselves with constructing an interesting class of examples, and then looking at a concrete instance of the construction. We return to the relation algebras $\mathfrak{F}(G, H)$ constructed in the previous section. The algebra $\mathfrak{F}(G, H)$ is atomic, by Lemma 6.1. In order to construct an atomless algebra, it is necessary to establish some connection between $\mathfrak{F}(G, H)$ and $\mathfrak{F}(G, K)$ for distinct subgroups H and K of G.

Lemma 7.1. *If H is a subgroup of G, and K a proper subgroup of H, then $\mathfrak{F}(G, H)$ is a proper subalgebra of $\mathfrak{F}(G, K)$, and each atom in $\mathfrak{F}(G, H)$ is split in $\mathfrak{F}(G, K)$ into the union of as many atoms as there are cosets of K in H.*

Proof. Suppose K has index κ in H, that is to say, suppose there are κ right cosets of K in H (where κ is a cardinal number). For every right coset H' of H in G, there are then κ distinct right cosets of K in G that are included in H', and H' is the union of these right cosets. Consequently,

$$R_f{\restriction}H' = \bigcup\{R_f{\restriction}K' : K' \text{ is a right coset of } K \text{ and } K' \subseteq H'\}.$$

It follows that each atom in $\mathfrak{F}(G, H)$ is the union of κ atoms in $\mathfrak{F}(G, K)$. Since $\mathfrak{F}(G, K)$ is closed under arbitrary unions, each of the atoms—and therefore each of the elements — in $\mathfrak{F}(G, H)$ belongs to $\mathfrak{F}(G, K)$. Both algebras are complete subalgebras of $\mathfrak{Re}(G)$, so the operations in $\mathfrak{F}(G, H)$ and in $\mathfrak{F}(G, K)$ are restrictions of the corresponding operations of $\mathfrak{Re}(G)$. Conclusion: $\mathfrak{F}(G, H)$ is a complete subalgebra of $\mathfrak{F}(G, K)$. It is a proper subalgebra because none of the atoms in $\mathfrak{F}(G, K)$ belongs to $\mathfrak{F}(G, H)$. □

Call an (infinite) system $(H_i : i \in I)$ of subgroups of G *strictly downward directed* if for any pair of indices i and j, there is an index k such that H_k is a proper subgroup of H_i and H_j. Taking $i = j$ shows, in particular, that every subgroup in such a system has a proper subgroup in the system.

Theorem 7.2. *If $(H_i : i \in I)$ is a strictly downward directed system of subgroups of G, then $(\mathfrak{F}(G, H_i) : i \in I)$ is a (strictly upward) directed system of complete subalgebras of $\mathfrak{Re}(G)$, and the union of this system is a (simple) subalgebra of $\mathfrak{Re}(G)$ that is functionally dense and atomless.*

Proof. Lemmas 6.1 and 7.1, together with the hypothesis that $(H_i : i \in I)$ is a strictly downward directed system of subgroups of G, imply that

$$(\mathfrak{F}(G, H_i) : i \in I) \tag{1}$$

is a system of complete subalgebras of $\mathfrak{Re}(G)$ that is strictly (upward) directed in the sense that for every pair of indices i and j, there is an index k such that $\mathfrak{F}(G, H_i)$ and $\mathfrak{F}(G, H_j)$ are proper subalgebras of $\mathfrak{F}(G, H_k)$. The union of the system in (1)—call it $\mathfrak{A}$—is therefore a subalgebra of $\mathfrak{Re}(G)$ (since the union of a directed system of subalgebras is always a subalgebra). In particular, $\mathfrak{A}$ must be simple, since $\mathfrak{Re}(G)$ is simple.

To see that $\mathfrak{A}$ is atomless, consider an arbitrary non-empty relation R in $\mathfrak{A}$. Since $\mathfrak{A}$ is the union of the system in (1), the relation R must belong to $\mathfrak{F}(G, H_i)$ for some index i, and therefore R must be above an atom S in the atomic algebra $\mathfrak{F}(G, H_i)$. Let k be an index in I such that H_k is a proper subgroup of H_i, say with κ cosets in H_i ($\kappa \geq 2$). The relations R and S belong to $\mathfrak{F}(G, H_k)$, and S is split into the union of κ atoms in $\mathfrak{F}(G, H_k)$, by Lemma 7.1. Consequently, R cannot be an atom in $\mathfrak{F}(G, H_k)$, and therefore R cannot be an atom in $\mathfrak{A}$.

Finally, $\mathfrak{A}$ is functionally dense because the union of any directed system of functionally dense relation algebras is functionally dense. In more detail, a non-zero relation R in $\mathfrak{A}$ belongs to $\mathfrak{F}(G, H_i)$ for some index i, and is therefore

above a non-zero function S in $\mathfrak{F}(G, H_i)$, by the functional density of $\mathfrak{F}(G, H_i)$. The relation S belongs to $\mathfrak{A}$, so R is above the non-zero function S in $\mathfrak{A}$. $\square$

As a concrete example of the construction in Theorem 7.2, take G to be the group $\mathbb{Z}$ of integers under addition, and let $H_0 = \mathbb{Z}$. In this case $\mathfrak{F}(\mathbb{Z}, H_0)$ is just the Cayley representation of the group complex algebra $\mathfrak{Cm}(\mathbb{Z})$. The atoms of $\mathfrak{F}(\mathbb{Z}, H_0)$ are the Cayley representations of the integers k, that is to say, they are the functions R_k defined by

$$R_k(m) = m + k$$

for every integer m (see Figure 3). The operations of relational converse and composition on atoms in $\mathfrak{F}(\mathbb{Z}, H_0)$ are determined by the formulas

$$R_k^{-1} = R_{-k} \qquad \text{and} \qquad R_k | R_\ell = R_{k+\ell}$$

for integers k and ℓ.

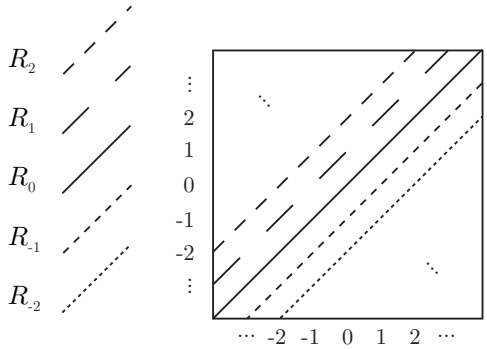

FIGURE 3. $\mathfrak{F}(\mathbb{Z}, H_0)$

Let H_1 be the subgroup of $\mathbb{Z}$ consisting of the even integers. The cosets of H_1 are the sets

$$K_i = \{m \in \mathbb{Z} : m \equiv i \mod 2\}$$

for $i = 0, 1$, that is to say, they are the sets of even and odd integers. The atoms of the algebra $\mathfrak{F}(\mathbb{Z}, H_1)$ are the restrictions $R_k{\restriction}K_0$ and $R_k{\restriction}K_1$ for integers k. For $i = 0, 1$, the function R_k maps the coset K_i bijectively to the coset K_i when k is even, and it maps K_i bijectively to K_{1-i} when k is odd (see Figure 4). Each atom R_k in $\mathfrak{F}(\mathbb{Z}, H_0)$ is split into the union of the two atoms $R_k{\restriction}K_0$ and $R_k{\restriction}K_1$ in $\mathfrak{F}(\mathbb{Z}, H_1)$. The converses of atoms in $\mathfrak{F}(\mathbb{Z}, H_1)$ are determined by the formulas

$$(R_k{\restriction}K_i)^{-1} = R_{-k}{\restriction}K_i \qquad \text{and} \qquad (R_k{\restriction}K_i)^{-1} = R_{-k}{\restriction}K_{1-i},$$

according to whether k is even or odd. For example,

$$(R_4{\restriction}K_1)^{-1} = R_{-4}{\restriction}K_1 \qquad \text{and} \qquad (R_3{\restriction}K_1)^{-1} = R_{-3}{\restriction}K_0.$$

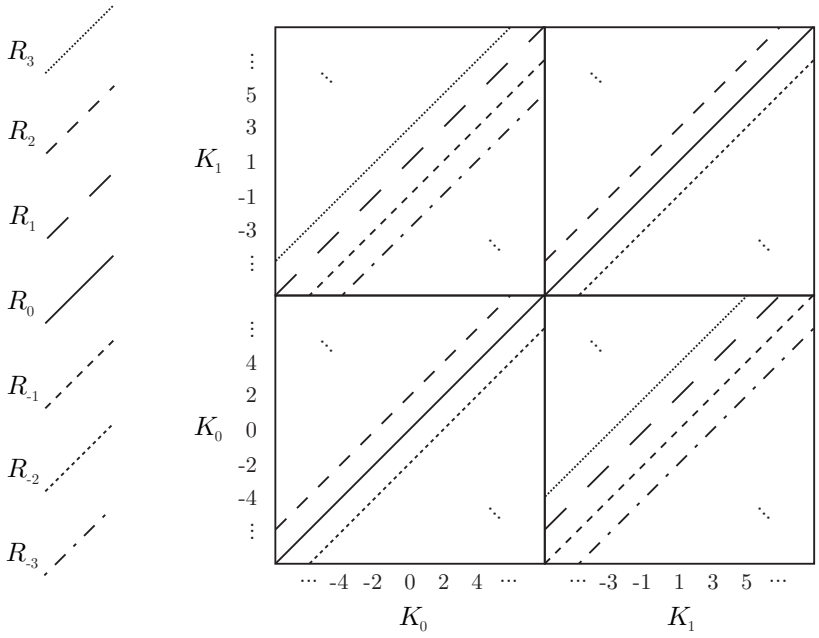

FIGURE 4. $\mathfrak{F}(\mathbb{Z}, H_1)$

The compositions of atoms in $\mathfrak{F}(\mathbb{Z}, H_1)$ are determined by the formulas

$$(R_k{\restriction}K_i)|(R_\ell{\restriction}K_i) = R_{k+\ell}{\restriction}K_i \quad \text{and} \quad (R_k{\restriction}K_i)|(R_\ell{\restriction}K_{1-i}) = \varnothing$$

when k is even, and

$$(R_k{\restriction}K_i)|(R_\ell{\restriction}K_i) = \varnothing \quad \text{and} \quad (R_k{\restriction}K_i)|(R_\ell{\restriction}K_{1-i}) = R_{k+\ell}{\restriction}K_i$$

when k is odd. For example,

$$(R_5{\restriction}K_0)|(R_4{\restriction}K_0) = \varnothing \quad \text{and} \quad (R_5{\restriction}K_1)|(R_4{\restriction}K_0) = R_9{\restriction}K_1.$$

Next, take H_2 to be the subgroup of $\mathbb{Z}$ consisting of the integers that are divisible by 4. The cosets of H_2 are the sets

$$K_{ij} = \{m \in \mathbb{Z} : m \equiv ij \mod 4\},$$

where each of i and j is either 0 or 1, and ij is the binary notation for an integer between 0 and 3. For instance, if $i = j = 1$, then ij is the binary notation for the integer 3, and

$$K_{11} = \{m \in \mathbb{Z} : m \equiv 3 \mod 4\}.$$

The atoms of the relation algebra $\mathfrak{F}(\mathbb{Z}, H_2)$ are the restrictions $R_k{\restriction}K_{ij}$. If ij, $i'j'$, and $i''j''$ are respectively the binary notations for the remainders of integers k, ℓ, and m upon division by 4, and if $ij + i'j' = i''j''$ in binary arithmetic modulo 4 (so that $k + \ell \equiv m \mod 4$), then the function R_k maps

the coset $K_{i'j'}$ bijectively to the coset $K_{i''j''}$. For example, when k is congruent to 3 mod 4, the function R_k bijectively maps

$$K_{00} \text{ to } K_{11}, \qquad K_{01} \text{ to } K_{00}, \qquad K_{10} \text{ to } K_{01}, \qquad K_{11} \text{ to } K_{10},$$

because 3 in binary notation is 11, and

$$00 + 11 = 11, \qquad 01 + 11 = 00, \qquad 10 + 11 = 01, \qquad 11 + 11 = 10$$

in binary arithmetic modulo 4. The atom $R_k{\restriction}K_i$ in $\mathfrak{F}(\mathbb{Z}, H_1)$ is split into the union of the two atoms $R_k{\restriction}K_{0i}$ and $R_k{\restriction}K_{1i}$ in $\mathfrak{F}(\mathbb{Z}, H_2)$. For example, the set K_1 of odd integers is the disjoint union of the set K_{01} of integers congruent to 1 mod 4 and the set K_{11} of integers congruent to 3 mod 4, so the atom $R_7{\restriction}K_1$ in $\mathfrak{F}(\mathbb{Z}, H_1)$ is split into the union of the two atoms $R_7{\restriction}K_{01}$ and $R_7{\restriction}K_{11}$ in $\mathfrak{F}(\mathbb{Z}, H_2)$ (see Figure 5). The formulas for computing converses and compositions of atoms in $\mathfrak{F}(\mathbb{Z}, H_2)$ are quite similar to the formulas for $\mathfrak{F}(\mathbb{Z}, H_1)$ but are notationally more complicated to express in a general way. A few examples should suffice to illustrate the main ideas. The function R_7 maps K_{10} to K_{01}, and the function R_6 maps K_{10} to K_{00}, so

$$(R_7{\restriction}K_{10})^{-1} = R_{-7}{\restriction}K_{01} \qquad \text{and} \qquad (R_6{\restriction}K_{10})^{-1} = R_{-6}{\restriction}K_{00}.$$

Also,

$$(R_7{\restriction}K_{10})|(R_6{\restriction}K_{01}) = R_{13}{\restriction}K_{10} \qquad \text{and} \qquad (R_7{\restriction}K_{10})|(R_6{\restriction}K_{ij}) = \varnothing$$

if $i \neq 0$ or $j \neq 1$.

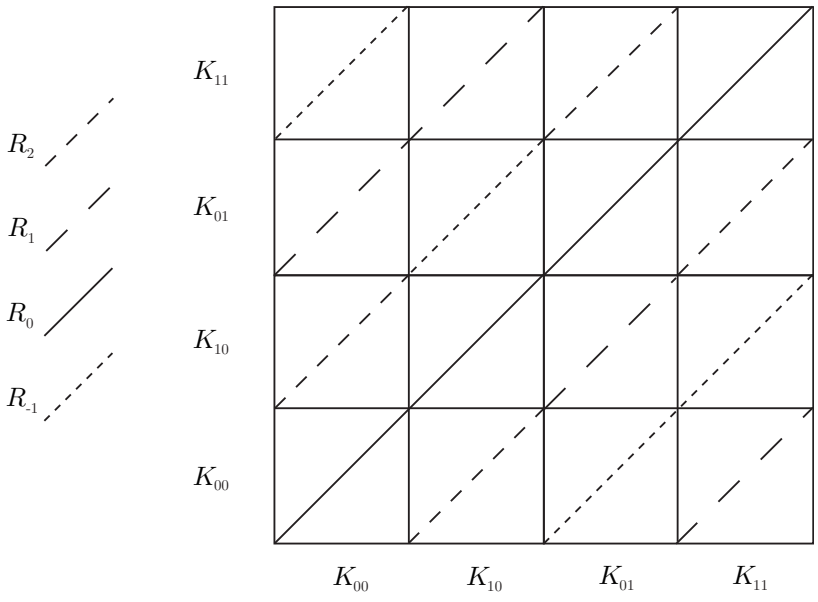

FIGURE 5. $\mathfrak{F}(\mathbb{Z}, H_2)$

In general, take H_n to be the subgroup of $\mathbb{Z}$ consisting of the integers that are divisible by 2^n. The cosets of H_n are the sets

$$K_a = \{m \in \mathbb{Z} : m \equiv a \mod 2^n\},$$

where a is a string of n zeros and ones that is the binary notation for an integer between 0 and $2^n - 1$. For instance, 101 is the binary notation for 5, so K_{101} is the coset of the subgroup of H_3 consisting of the integers that are congruent to 5 mod 8. (In this example, $n = 3$ and therefore $2^n = 8$.) If an integer k is congruent to 6 mod 8, then the function R_k maps the coset K_{101} bijectively to the coset K_{011}. The atom $R_k{\restriction}K_a$ in $\mathfrak{F}(\mathbb{Z}, H_n)$ is split into the union of the two atoms $R_k{\restriction}K_{0a}$ and $R_k{\restriction}K_{1a}$ in $\mathfrak{F}(\mathbb{Z}, H_{n+1})$. For instance, the atom $R_k{\restriction}K_{101}$ in $\mathfrak{F}(\mathbb{Z}, H_3)$ is split into the union of the two atoms $R_k{\restriction}K_{0101}$ and $R_k{\restriction}K_{1101}$ in $\mathfrak{F}(\mathbb{Z}, H_4)$.

The sequence $H_0, H_1, H_2, \ldots$ forms a strictly descending chain of subgroups of $\mathbb{Z}$, so the sequence

$$\mathfrak{F}(\mathbb{Z}, H_0), \quad \mathfrak{F}(\mathbb{Z}, H_1), \quad \mathfrak{F}(\mathbb{Z}, H_2), \ldots$$

forms a strictly ascending chain of complete subalgebras of $\mathfrak{Re}(\mathbb{Z})$. The union of this ascending chain is an example of a simple, atomless, functionally dense subalgebra of $\mathfrak{Re}(\mathbb{Z})$.

8. Functionally dense relation algebras with atomless Boolean algebras of ideal elements

In the previous section, we looked at some examples of functionally dense relation algebras that are atomless. The examples given are all simple algebras, and consequently their Boolean algebras of ideal elements are all just the two-element Boolean algebra. We know from Theorem 4.2, however, that an arbitrary functionally dense relation algebra may have a single atomless factor that is functionally dense and has an atomless Boolean algebra of ideal elements. The problem of describing all such functionally dense relation algebras with atomless Boolean algebras of ideal elements is also open. Again, we content ourselves with presenting a class of examples and then looking at a concrete example of the construction.

Start with an arbitrary relation algebra $\mathfrak{A}$ and an arbitrary infinite set I. For concreteness, we shall take I to be the set of natural numbers, but only because this choice simplifies the notation somewhat. Form the Ith direct power $\mathfrak{A}^I$. We define a strictly increasing sequence $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \ldots$ of subalgebras of $\mathfrak{A}^I$ with the property that every element in $\mathfrak{B}_n$ is split in $\mathfrak{B}_{n+1}$ into a sum of at least two disjoint non-zero elements.

Fix a natural number n. For each element r in the finite power $\mathfrak{A}^{2^n}$, define an element $\hat{r}$ in the infinite power $\mathfrak{A}^I$ by

$$\hat{r}(i) = r_j, \qquad \text{where} \qquad 0 \leq j < 2^n \quad \text{and} \quad i \equiv j \mod 2^n.$$

For example, if $n = 0$, then r is a function with domain $\{0\}$, and $r_0 = p$ is an element in $\mathfrak{A}$, so that $\hat{r} = (p, p, p, p, \dots)$. If $n = 1$, then r is a function with domain $\{0, 1\}$, and $r_0 = p$ and $r_1 = q$ are elements in $\mathfrak{A}$, so in this case $\hat{r} = (p, q, p, q, \dots)$. In general, $\hat{r}$ is the element in $\mathfrak{A}^I$ obtained by concatenating r with itself infinitely many times.

The set B_n of elements of the form $\hat{r}$ for r in $\mathfrak{A}^{2^n}$ is a subuniverse of $\mathfrak{A}^I$. Indeed, B_n contains the identity element

$$\hat{1}' = (1', 1', 1', 1' \dots)$$

of $\mathfrak{A}^I$, and B_n is closed under the operations of $\mathfrak{A}^I$ because these operations are defined coordinatewise:

$$\begin{aligned} \hat{r} + \hat{s} &= \hat{t}, \qquad \text{where} \qquad t = r + s, \\ -\hat{r} &= \hat{t}, \qquad \text{where} \qquad t = -r, \\ \hat{r} \,;\, \hat{s} &= \hat{t}, \qquad \text{where} \qquad t = r \,;\, s, \\ \hat{r}^{\smile} &= \hat{t}, \qquad \text{where} \qquad t = r^{\smile}. \end{aligned}$$

(The operations on the left are performed in $\mathfrak{A}^I$, and the ones on the right are performed in $\mathfrak{A}^{2^n}$.) The subalgebra of $\mathfrak{A}^I$ with universe B_n—call it $\mathfrak{B}_n$— is isomorphic to $\mathfrak{A}^{2^n}$ via the function that maps $\hat{r}$ to r for each element r in $\mathfrak{A}^{2^n}$. Consequently, $\mathfrak{B}_n$ inherits all of the properties of $\mathfrak{A}^{2^n}$. For example, if r is an ideal element in $\mathfrak{A}^{2^n}$, then $\hat{r}$ is an ideal element in $\mathfrak{B}_n$; and if $\mathfrak{A}$ possesses some density property such as functional density, then so does $\mathfrak{A}^{2^n}$ and therefore so does $\mathfrak{B}_n$.

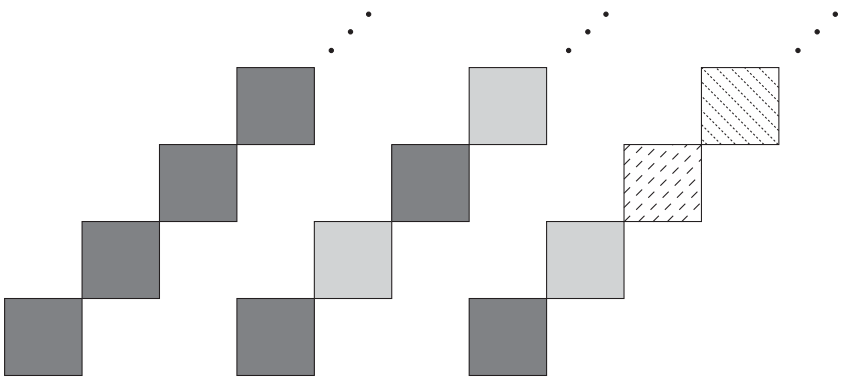

FIGURE 6. Schematic of the algebras $\mathfrak{B}_0$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$.

Each element in $\mathfrak{B}_n$ clearly belongs to $\mathfrak{B}_{n+1}$. Indeed, if r is an element in $\mathfrak{A}^{2^n}$, and if s is the element in $\mathfrak{A}^{2^{n+1}}$ defined by

$$s_i = r_j, \qquad \text{where} \qquad 0 \leq j < 2^n \quad \text{and} \quad i \equiv j \mod 2^n$$

(so that s is essentially the element obtained by concatenating r with itself), then the element $\hat{r}$ in $\mathfrak{B}_n$ coincides with the element $\hat{s}$ in $\mathfrak{B}_{n+1}$. The operations

of $\mathfrak{B}_n$ and $\mathfrak{B}_{n+1}$ are, by definition, restrictions of the corresponding operations of $\mathfrak{A}^I$. Consequently, $\mathfrak{B}_n$ must be a subalgebra of $\mathfrak{B}_{n+1}$ (see Figure 6). Each non-zero element $\hat{r}$ in $\mathfrak{B}_n$ is split in $\mathfrak{B}_{n+1}$ into the sum of the two disjoint non-zero elements. Indeed, if s is the element in $\mathfrak{A}^{2^{n+1}}$ that was just defined (so that $\hat{r} = \hat{s}$), and if t and u are the elements in $\mathfrak{A}^{2^{n+1}}$ defined by

$$t_i = \begin{cases} s_i & \text{if} \quad 0 \leq i < 2^n, \\ 0 & \text{if} \quad 2^n \leq i < 2^{n+1}, \end{cases} \quad \text{and} \quad u_i = \begin{cases} 0 & \text{if} \quad 0 \leq i < 2^n, \\ s_i & \text{if} \quad 2^n \leq i < 2^{n+1}, \end{cases}$$

then t and u are non-zero elements in $\mathfrak{A}^{2^{n+1}}$ that are disjoint and have s as their sum. It follows that $\hat{t}$ and $\hat{u}$ are non-zero elements in $\mathfrak{B}_{n+1}$ that are disjoint and have $\hat{s}$ as their sum. Since $\hat{s}$ coincides with $\hat{r}$, we can write $\hat{r}$ as the disjoint sum of the non-zero elements $\hat{t}$ and $\hat{u}$. Notice, in particular, that if r is a non-zero ideal element in $\mathfrak{A}^{2^n}$, then t and u are ideal elements in $\mathfrak{A}^{2^{n+1}}$ and therefore the non-zero ideal element $\hat{r}$ in $\mathfrak{B}_n$ is split in $\mathfrak{B}_{n+1}$ into the sum of the two disjoint non-zero ideal elements $\hat{t}$ and $\hat{u}$.

The sequence $\mathfrak{B}_0$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, ... forms a strictly ascending chain of subalgebras of $\mathfrak{A}^I$ with the property that each non-zero element in $\mathfrak{B}_n$ is split into a sum of two disjoint non-zero elements in $\mathfrak{B}_{n+1}$, and each non-zero ideal element in $\mathfrak{B}_n$ is split into a sum of two disjoint non-zero ideal elements in $\mathfrak{B}_{n+1}$. Furthermore, if $\mathfrak{A}$ possesses some density property such as functional density, then so does each algebra $\mathfrak{B}_n$ in the chain, and the union of the chain inherits this density property.

Theorem 8.1. *If $\mathfrak{A}$ is a non-trivial relation algebra, and I the set of natural numbers, then the union of the chain $\mathfrak{B}_0$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, ... of subalgebras of $\mathfrak{A}^I$ constructed above is an atomless subalgebra of $\mathfrak{A}^I$ with no ideal element atoms. If $\mathfrak{A}$ is functionally dense, then so is the union of the chain.*

To obtain a concrete example of a functionally dense relation algebra that has no ideal element atoms, take $\mathfrak{A}$ to be any simple, functionally dense relation algebra (atomic or atomless), or any non-empty product of such algebras. For instance, $\mathfrak{A}$ could be the complex algebra of some fixed group.

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