Free Boolean algebras with closure operators and a conjecture of Henkin, Monk, and Tarski*

by

J. X. Madarász and I. Németi

Abstract

We characterize the finite-dimensional elements of a free cylindric algebra. This solves Problem 2.10 in [Henkin, Monk, Tarski: Cylindric Algebras, North-Holland, 1971 and 1985]. We generalize the characterization to quasi-varieties of Boolean algebras with operators in place of cylindric algebras.

Free algebras play an important role in universal algebra, see e.g. Andréka-Jónsson-Németi [2]. Free algebras are even more important in algebraic logic, because they give information on proof-theoretic properties of a logic. Cf. e.g. [7, §5.6], [4], [12], [8].

Cylindric algebras are special Boolean algebras with closure operators. Tarski proved that any element of a free cylindric algebra behaves the same way for all but finitely many of these operators: x is either closed for all but finitely many or only finitely many operators. See Theorem 2.6.23 in [6]. It remained open which elements of a free algebra are closed to all but finitely many, and which are closed only to finitely many. [6, 2.6.24] contains a conjecture for a characterization, but Henkin, Monk, and Tarski write there that they were unable to verify this conjecture. They also formulate the conjecture as Problem 2.10 in [6].

In this paper we solve Problem 2.10 of [6] by showing that their conjecture was right. We prove more: we give information about which element is closed under which operators, and also we generalize the result to quasi-varieties of Boolean algebras with operators.

*Research supported by the Hungarian National Foundation for scientific research grants No T30314 and T23234.

Boolean algebras with operators (Bo_α's) were introduced in Jónsson-Tarski [9] and they have been well investigated ever since. See e.g. the papers in [3] and in [13], or [1], [11], [10]. Bo_α is often called also BAO in the literature. An α-dimensional *Boolean algebra with operators* is an algebra $\mathfrak{A} = \langle A, +, -, c_i, d_{ij}\rangle_{i,j\in\alpha}$ in which each $c_i : A \to A$ is a closure operator in the usual sense, and the d_{ij}'s are constants. I.e. c_i is additive and $x \le c_i x = c_i c_i x$ holds in $\mathfrak{A}$. We say that $\mathfrak{A}$ has *complemented closure operators* if the complement of a c_i-closed element is c_i-closed again, that is if $\mathfrak{A} \models \{c_i - c_i x = -c_i x : i \in \alpha\}$. An operator c_i is *normal* if $c_i 0 = 0$, and a Bo_α is normal if all of its operators are normal. α-dimensional *cylindric algebras*, CA_α's, are Bo_α's with commuting complemented closure operators in which three additional equations are postulated for the constants d_{ij}.

Some terminology: Let $\mathfrak{A} = \langle A, +, -, c_i, d_{ij}\rangle_{i,j\in\alpha}$ be a Bo_α and let $a \in A$. We say that a is *sensitive to an operator* c_i if $a \ne c_i a$ in $\mathfrak{A}$. We say that a is *finite (cofinite) dimensional* if a is sensitive to finitely many (co-finitely many) of the operators c_i $(i \in \alpha)$. We say that a is *zero-dimensional* if a is not sensitive to any of the operators, i.e. if $a = c_i a$ for all $i \in \alpha$. We say that a is a *constant* element in $\mathfrak{A}$ if a is generated by the constants $0, 1, d_{ij}$ $(i, j \in \alpha)$, or in other words, if a is an element of the smallest subalgebra of $\mathfrak{A}$. If a generator set G of $\mathfrak{A}$ is fixed, then by a *generating term* of a we understand a term $\tau(x_1, \ldots, x_n)$ in the language of Bo_α such that $a = \tau(g_1, \ldots, g_n)$ in $\mathfrak{A}$ for some $g_1, \ldots, g_n \in G$.

Theorem 1 (Solution of Problem 2.10 in [6]) *Let α be an infinite ordinal and β be a nonzero cardinal.*

(i) The set of all finite-dimensional elements of the β-generated free algebra $\mathfrak{Fr}_\beta CA_\alpha$ is the subuniverse generated by $\{g \cdot -c_0(-d_{01}) : g \in G\}$, where G is the set of free generators of $\mathfrak{Fr}_\beta CA_\alpha$.

(ii) If an element is not finite-dimensional in $\mathfrak{Fr}_\beta CA_\alpha$, then it is sensitive to all operators which do not occur in all of its generating terms.

We will prove Theorem 1 as a corollary of the following theorem about all quasi-varieties of normal Bo_α's.

Theorem 2 *Assume $\alpha \ge \omega$. Let K be any quasi-variety of normal Bo_α's in which the equations (1) – (4) below are valid for all $i, j, k \in \alpha$.*

(1) $c_i(c_i x \cdot y) = c_i c_i x \cdot c_i y$ *(weak distributivity)*

(2) $c_k d_{ij} = d_{ij}$ *if* $k \notin \{i, j\}$

(3) $c_i d_{ij} = 1$

(4) $c_i - d_{ij} = 1$.

Let $\beta \neq 0$. Then (i) and (ii) below hold.

(i) All non-constant elements of the β-generated K-free algebra $\mathfrak{Fr}_\beta K$ are cofinite-dimensional.

(ii) Moreover, a non-constant element a of the free algebra can be insensitive to an operator c_i only if c_i occurs in all generating terms of a from the free generators G of $\mathfrak{Fr}_\beta K$. In more detail: Assume that $a = \tau(\bar{g})$ for some sequence $\bar{g}$ of free generators and c_i does not occur in the term τ. Then $a \neq c_i a$ in $\mathfrak{Fr}_\beta K$.

In proving Theorem 2, we will use the following two lemmas. Lemma 3 below is a version of the known theorem in cyindric algebra theory that relativization with zero-dimensional elements is a homomorphism. We have to prove this because we want to use the statement of this known theorem under much weaker conditions.

Lemma 3 *Assume that Γ is a set and $\mathfrak{A}$ is a Bo_Γ enriched perhaps with additional constants, and assume that the equations* (1) *(weak distributivity) hold in $\mathfrak{A}$. Let b be a zero-dimensional element of $\mathfrak{A}$, $a_1, \ldots, a_n$ arbitrary elements of $\mathfrak{A}$, and let $\tau(x_1, \ldots, x_n)$ be a term in the language of $\mathfrak{A}$. Then*

$$b \cdot \tau^{\mathfrak{A}}(a_1, \ldots, a_n) = b \cdot \tau^{\mathfrak{A}}(a_1 \cdot b, \ldots, a_n \cdot b) .$$

Proof. We proceed by induction on τ. The only non-trivial cases are when τ is $c_i\sigma$ or $-\sigma$. In the first case we use weak distributivity, and the proof of the second case is much the same as in the cylindric algebra case. We denote $\sigma^{\mathfrak{A}}(a_1, \ldots, a_n)$ and $\sigma^{\mathfrak{A}}(a_1 \cdot b, \ldots, a_n \cdot b)$ by $\sigma(\bar{a})$ and $\sigma(\bar{a} \cdot b)$ respectively. Proof of homomorphism with respect to c_i: $b \cdot c_i\sigma(\bar{a}) = c_i(b \cdot \sigma(\bar{a})) = c_i(b \cdot \sigma(\bar{a} \cdot b)) = b \cdot c_i\sigma(\bar{a} \cdot b)$, using (1), $b = c_i b$ and the inductive hypothesis on σ. Proof of homomorphism with respect to complementation $-$: $b \cdot -\sigma(\bar{a}) = b \cdot -(b \cdot \sigma(\bar{a})) = b \cdot -(b \cdot \sigma(\bar{a} \cdot b)) = b \cdot -\sigma(\bar{a} \cdot b)$. ∎

Lemma 4 *Assume that K satisfies the hypotheses of Theorem 2. Then the following is valid in K:*

$$c_k - d_{ij} = -d_{ij} \quad \text{if} \quad k \notin \{i, j\}.$$

Proof. It is enough to prove that $c_k(-d_{ij}) + d_{ij} = 1$ and $c_k(-d_{ij}) \cdot d_{ij} = 0$. We will use (1) – (4), additivity and normality of the operators c_i, and of course Boolean algebra. Proof of the first equation: $c_k(-d_{ij}) + d_{ij} = c_k(-d_{ij}) + c_k(d_{ij}) = c_k(-d_{ij} + d_{ij}) = c_k 1 = c_k(d_{ki} + -d_{ki}) = c_k d_{ki} + c_k - d_{ki} = 1$. Proof of the second equation: $c_k(-d_{ij}) \cdot d_{ij} = c_k(-d_{ij}) \cdot c_k c_k d_{ij} = c_k(-d_{ij} \cdot c_k d_{ij}) = c_k(-d_{ij} \cdot d_{ij}) = c_k 0 = 0$. ∎

We are ready to prove Theorem 2.

Proof of Theorem 2: It is enough to prove the second statement of Theorem 2, because it is a stronger version of the first one. Let α, β, K be as in the hypotheses of the theorem. Set $\mathfrak{F} \stackrel{\text{def}}{=} \mathfrak{Fr}_\beta K$, let a be a non-constant element of $\mathfrak{F}$, let $\tau(x_1, \ldots, x_n)$ be a term in the language of Bo_α and $g_1, \ldots, g_n$ be free generators in $\mathfrak{F}$ such that $a = \tau(g_1, \ldots, g_n)$. Let Γ be the set of all indices of operators (i.e. c_i's) occurring in τ. We will show that $a \neq c_i a$ in $\mathfrak{F}$ for all $i \in \alpha \smallsetminus \Gamma$.

Let $i \in \alpha \smallsetminus \Gamma$. If $K \not\models c_i\tau(\bar{x}) = \tau(\bar{x})$, then $c_i\tau(\bar{g}) \neq \tau(\bar{g})$ because the g_k's are free generators in $\mathfrak{F} = \mathfrak{Fr}_\beta K$, i.e. $a \neq c_i a$ and we are done.

Assume therefore that $K \models c_i\tau(\bar{x}) = \tau(\bar{x})$. We will derive a contradiction. Let $j \in \alpha \smallsetminus \Gamma, j \neq i$. Such a j exists because α is infinite and Γ is finite. Now

$$(\star) \qquad K \models \tau(\bar{x}) = c_i(\tau(\bar{x}) \cdot d_{ij}) = c_i(\tau(\bar{x}) \cdot -d_{ij})$$

by (1), (3) and (4). For every $1 \leq k \leq n$ define

$$e_k \stackrel{\text{def}}{=} g_k \cdot d_{ij} + d_{01} \cdot -d_{ij} \,.$$

Let $\mathfrak{N}$ denote the Γ-reduct of $\mathfrak{F}$ together with all the constants, i.e. the universe of $\mathfrak{N}$ is the same as that of $\mathfrak{F}$ and the operations of $\mathfrak{N}$ are those of $\mathfrak{F}$ except that we omit those operations c_k where $k \notin \Gamma$. Then $\mathfrak{N}$ is a Bo_Γ with additional constants in which (1) holds. Also, d_{ij} is zero-dimensional in $\mathfrak{N}$ by (2) and $i, j \notin \Gamma$. Similarly, $-d_{ij}$ is zero-dimensional in $\mathfrak{N}$ by Lemma 4, and the term τ is in the language of $\mathfrak{N}$. Thus we can apply Lemma 3.

By applying $(\star)$ and Lemma 3 twice, and using the definition of $\bar{e}$ we get $\tau(\bar{e}) = c_i(\tau(\bar{e}) \cdot d_{ij}) = c_i(\tau(\bar{e} \cdot d_{ij}) \cdot d_{ij}) = c_i(\tau(\bar{g} \cdot d_{ij}) \cdot d_{ij}) = c_i(\tau(\bar{g}) \cdot d_{ij}) = \tau(\bar{g})$. Completely analogously we obtain $\tau(\bar{e}) = \tau(\bar{d}_{01})$. Thus $\tau(\bar{g}) = \tau(\bar{d}_{01})$. But this is a contradiction because $a = \tau(\bar{g})$ is a non-constant element by our assumption and since $\tau(\bar{d}_{01})$ is a constant element. ∎

Corollary 5 *Let K be a variety of Boolean algebras with complemented closure operators. Then* (2)-(4) *of Theorem 2 imply the conclusion of Theorem 2.*

Proof. A Bo_α with complemented closure operators is always normal and weak distributivity (1) holds in it. This is easy to check. ∎

We are ready to prove Theorem 1 now. We note that neither (i) nor (ii) of Theorem 2 is true for $K = CA_\alpha$. Indeed, if g is a free generator, then $a = g \cdot -c_0(-d_{01})$ is a zero-dimensional element in $\mathfrak{Fr}_\beta CA_\alpha$, yet a is not a constant in $\mathfrak{Fr}_\beta CA_\alpha$. Also, $a = c_2 a$ because it is zero-dimensional, yet 2 does not occur in the generating term $\tau = x \cdot -c_0(-d_{01})$ of a.

Proof of Theorem 1. First we prove (ii). Let M denote the subuniverse of $\mathfrak{Fr}_\beta CA_\alpha$ generated by $\{g \cdot -c_0(-d_{01}) : g \in G\}$. Assume that $a \in Fr_\beta CA_\alpha \setminus M$, $a = \tau(g_1, \ldots, g_n)$, $g_1, \ldots, g_n \in G$ and c_j does not occur in τ. We want to show that $a \neq c_j a$ in $\mathfrak{Fr}_\beta CA_\alpha$.

Let K denote the variety of those CA_α's in which the equation $c_0 - d_{01} = 1$ holds. Then K satisfies the hypotheses of Theorem 2. (We note that CA_α does not satisfy those hypotheses, because (4) fails for CA_α.) Assume that the set of free generators of $\mathfrak{Fr}_\beta K$ is also G, and let $h : \mathfrak{Fr}_\beta CA_\alpha \to \mathfrak{Fr}_\beta K$ be a homomorphism which is the identity on G (i.e. $h(g) = g$ for all $g \in G$). Such a homomorphism exists because $\mathfrak{Fr}_\beta K \in CA_\alpha$. We are going to show that $h(a)$ is non-constant in $\mathfrak{Fr}_\beta K$.

Let $\delta \stackrel{\text{def}}{=} c_0(-d_{01})$. Then it is a cylindric algebraic theorem that δ is a zero-dimensional element. Cf. [6, 1.6.9(i)]. Let $e(x) \stackrel{\text{def}}{=} x \cdot \delta$ for any $x \in Fr_\beta CA_\alpha$. Then by Lemma 3, e is a homomorphism on $\mathfrak{Fr}_\beta CA_\alpha$ in the sense of [6, Def.0.2.1, p.67], i.e. there is a unique algebra $\mathfrak{A}$ such that e is a surjective homomorphism from $\mathfrak{Fr}_\beta CA_\alpha$ onto $\mathfrak{A}$. Cf. [6, Thm.0.2.4]. Then $\mathfrak{A} \in CA_\alpha$ because CA_α is a variety. Then $\mathfrak{A} \in K$ by $\mathfrak{A} \models \delta = 1$. We have $A \subseteq Fr_\beta CA_\alpha$ because $A = \{a \cdot \delta : a \in Fr_\beta CA_\alpha\}$. Let h' denote the restriction of h to A. We are going to show that h' is an isomorphism between $\mathfrak{A}$ and $\mathfrak{Fr}_\beta K$. It is easy to check that $h' : \mathfrak{A} \to \mathfrak{Fr}_\beta K$ is a homomorphism because $h(\delta) = 1$. It remains to show that h' is one-to-one on A. Let $f : \mathfrak{Fr}_\beta K \to \mathfrak{A}$ be a homomorphism such that $f(g) = g \cdot \delta$ for all $g \in G$. Such a homomorphism exists, because $\mathfrak{A} \in K$ and $g \cdot \delta \in A$ for all $g \in G$. Thus the two homomorphisms $f \circ h'$ and $h' \circ f$ are homomorphisms on $\mathfrak{A}$ and $\mathfrak{Fr}_\beta K$ respectively, such that they are identity on the generating sets $\{g \cdot \delta : g \in G\}$ and G respectively. Thus both $f \circ h'$ and $h' \circ f$ are identity on the corresponding algebras, showing that h' and f both are isomorphisms.

Now we are ready to show that $h(a)$ is non-constant in $\mathfrak{Fr}_\beta K$. Assume the contrary, i.e. that $h(a) = \sigma$ in $\mathfrak{Fr}_\beta K$ for some constant term σ. Then $h(\sigma \cdot \delta) = h(\sigma^{\mathfrak{A}}) = h'(\sigma^{\mathfrak{A}}) = \sigma$ because h' is a homomorphism, and then $a \cdot \delta = \sigma \cdot \delta$ because h is one-to-one on A (since h' is an isomorphism) and $a \cdot \delta \in A$. Since δ is zero-dimensional, so is $-\delta$, and thus $a \cdot -\delta = \tau(\bar{g} \cdot -\delta)$ by Lemma 3. So $(a \cdot -\delta) \in M$ by the definition of M. This is a contradiction, because $a \notin M$ but we have seen that $a \cdot \delta = \sigma \cdot \delta$ is a constant, hence in M, and M is closed under addition. We have shown that $h(a)$ is not a constant in $\mathfrak{Fr}_\beta K$.

Now we can apply Theorem 2(ii), which yields that $h(a) \neq c_j h(a)$ in $\mathfrak{Fr}_\beta K$ because τ is also a generating term for $h(a)$ in $\mathfrak{Fr}_\beta K$. But then $a \neq c_j a$ in $\mathfrak{Fr}_\beta CA_\alpha$ because h is a homomorphism. This completes the proof of (ii).

To prove (i), it is enough now to show that each element of M is finite-dimensional. This follows from the cylindric algebraic theorem that $\eta \stackrel{\text{def}}{=} -c_0(-d_{01})$ is a hereditarily zero-dimensional element, which means that not only η, but all elements smaller than η are zero-dimensional. See [6, 1.6.20]. Thus $g \cdot \eta$ is zero-dimensional for each $g \in G$, and then one can apply the easy cylindric algebraic theorem that finite-dimensional elements generate finite-dimensional ones (cf. [6, 2.1.5(i)]). ∎

References

[1] H. Andréka, S. Givant, and I. Németi. Perfect extensions and derived algebras. *J. of Symbolic Logic*, 60(3):775–796, 1995.

[2] H. Andréka, B. Jónsson, and I. Németi. Free algebras in discriminator varieties. *Algebra Universalis*, 28:401–447, 1991. Abstracted in [5], 1–14.

[3] H. Andréka, J. D. Monk, and I. Németi (editors). *Algebraic Logic* (Proc. Conf. Budapest 1988). Colloq. Math. Soc. J. Bolyai Vol **54**, North–Holland, Amsterdam, 1991.

[4] H. Andréka, I. Németi, and I. Sain. Algebraic logic. In *Handbook of Philosophical Logic Vol. II, second edition.* D. Gabbay and F. Guenthner (eds), D. Reidel Publ. Co., 2000. To appear.

[5] C. H. Bergman, R. D. Maddux, and D. L. Pigozzi, editors. *Algebraic logic and universal algebra in computer science. Lecture Notes in Computer Science*, volume 425. Springer–Verlag, Berlin, 1990. xi+292 pp.

[6] L. Henkin, J. D. Monk, and A. Tarski. *Cylindric Algebras Part I.* North–Holland, Amsterdam, 1971 and 1985.

[7] L. Henkin, J. D. Monk, and A. Tarski. *Cylindric Algebras Part II.* North–Holland, Amsterdam, 1985.

[8] L. Henkin, J. D. Monk, A. Tarski, H. Andréka, and I. Németi. *Cylindric Set Algebras*. Springer–Verlag, Berlin, 1981. Lecture Notes in Mathematics Vol. 884, v+323 pp.

[9] B. Jónsson and A. Tarski. Boolean algebras with operators, Part I. *Amer. J. Math.*, 73:891–939, 1951.

[10] J. X. Madarász. Interpolation in algebraizable logics. Semantics for non-normal multi-modal logic. *Journal of Applied Non-Classical Logics*, 8(1-2):67–105, 1998.

[11] J. X. Madarász. Interpolation and amalgamation; pushing the limits (Parts I,II). *Studia Logica*, 60,61(3,1):311–345, 1998, 1999.

[12] I. Németi. *Free Algebras and Decidability in Algebraic Logic.* Dissertation for D.Sc. with Hung. Academy of Sciences, Budapest, 1986. In Hungarian.

[13] I. Németi and I. Sain (editors). *Special Issue on Algebraic Logic.* Logic Journal of IGPL, Vol. 8, No 4, 2000.

Alfréd Rényi Mathematical Institute
Budapest, Pf. 127
H–1364, Hungary
e-mail: madarasz nemeti@math-inst.hu