Logical analysis of relativity theories[1]

Andréka, H. Madarász, J. X. and Németi, I.
andreka@renyi.hu, madarasz@renyi.hu, nemeti@renyi.hu
Alfréd Rényi Institute of Mathematics, Budapest, H-1364 Hungary, Pf. 127

1 Introduction

In this paper we try to give a small sample illustrating the approach of Andréka et al. [2],[5],[21]–[24] to a logical analysis of relativity theory conducted purely in first-order logic, FOL. We stick with FOL for methodological reasons. Here we first concentrate on special relativity, but in [22],[2],[5],[24] steps are made in the direction of generalizing this FOL-approach towards general relativity. We discuss that direction in the second half of this introduction. In [5] we build up variants of relativity theory as "competing" axiom systems formalized in FOL. The reason for having several versions for the theory, i.e. several axiom systems, is that this way we can study the *consequences* of the various axioms, enabling us to find out which axiom is responsible for what interesting or "exotic" prediction of relativity theory. Among others, this enables us to refine the *conceptual analysis* of relativity in Friedman [12] and Rindler [28], or compare the Reichenbach-Grünbaum-Salmon approach to relativity (cf. [31] or [12]) with the standard one. Later we will refer to the just indicated "several competing axiom systems" feature of our theory as flexibility feature or *modularity feature*.

One of our FOL axiom systems will be called *Specrel*. We will see that *Specrel* is a faithful, streamlined FOL axiomatization of (the kinematics of)

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special relativity theory. *Specrel* is intended to consist of simple, intuitively convincing, logically transparent, natural axioms. Besides *Specrel* we study its fragments, its generalizations towards general relativity and other versions of relativity.

As explained in [5, §1], the present approach is (in some sense) more ambitious (as a relativity theory) than e.g. a formalization of, say, Minkowskian geometry in first-order logic would be, in various respects: (i) One respect is that *if* we identified Minkowskian geometry with special relativity, then this would yield an *uninterpreted* (in the physical sense) version of relativity, while the first-order theory which we develop in [5] contains "its own interpretation", too. (ii) It is not clear to us how the conceptual analysis[2] suggested e.g. in [12] could be squeezed into Minkowskian geometry. (iii) Our formalized relativity theory is undecidable, while the FOL-theory of Minkowskian geometry in [13] is decidable, pointing in the direction that in our theory one can talk about things which do not appear in pure Minkowskian geometry. Someone may argue that Minkowskian geometry is the heart of special relativity theory; but it is *only* the heart, and we would like to formalize the full theory and not only its heart. (iv) The observational/theoretical duality outlined in [12] motivates us to keep our vocabulary and axioms on the modular, observational side (while Minkowskian geometry remains more on the "monolithic", "theoretical" side).[3] (v) Besides building up observational relativity in FOL, we also formalize the "monolithic", theoretically oriented geometric theory of space-time in FOL in e.g. [21],[23]. Then we prove that these two FOL theories are FOL-definitionally equivalent. So the user of our FOL theory can switch between the observational and theoretical versions whenever he so wishes. (vi) We also work on generalizing gradually our FOL theory of special relativity in the direction of general relativity in e.g. [2],[22],[24]. For this gradual generalization we will rely on the modularity feature of our observational theory mentioned way above. This feature is tied to the theory's having many small building blocks each of which carries some intuitive and natural meaning and which blocks can be removed or added one-by-one like in a lego toy world yielding new, meaningful and coherent versions of the theory. Besides generalization towards general relativity, this

[2]Which axiom is responsible for what prediction, which axiom is intuitively more natural than the other, etc.

[3]We use the observational/theoretical distinction in relativity in the sense of [12] going back to Reichenbach (1920). Sometimes it is useful to think of this as bottom-up/top-down distinction.

modularity feature is used in answering the so-called *why-type questions* and for conceptual analysis. This degree of modularity does not seem to be easily available if one starts out with an axiomatization of Minkowskian geometry or some other "top-down" approach.

After having formalized relativity theory in first-order logic, one can use the well developed machinery of FOL for studying properties of the theory, e.g. *Specrel* (e.g. the number of non-elementarily equivalent models, or its relationships with Gödel's incompleteness theorems, independence issues, etc). The reasons why we find it important to stick with FOL as a framework throughout the logical analysis of relativity can be found e.g. in van Benthem [8] when read together with Sain [29]. These reasons are further explained in [5, Appendix], Väänänen [32], Woleński [34]. It is explained e.g. in Feferman [10] and in Ferreirós [11] why and how we can stay in the framework of FOL throughout all our developments, if we want to.

As already indicated, the present work intends to give samples from a broader ongoing project represented by e.g. [5],[2],[21],[22],[24],[1]. A general plan for this broader project goes as follows: First we build up (the kinematics of) special relativity theory in FOL obtaining the finitely axiomatized FOL theory *Specrel*. *Specrel* was mentioned already at the beginning of this introduction. We put emphasis on making the axioms of *Specrel* streamlined, transparent, and intuitively convincing. First, as usual, we establish adequateness of *Specrel* for special relativity (completeness theorem, independence of the axioms, etc).[4] Then we elaborate a conceptual analysis of special relativity, its variants, and its generalizations. This analysis is based on the FOL axiom system *Specrel*, on variants and fragments of *Specrel* and their generalizations. Among others, we analyse *Specrel* both from the logical point of view (model theory, proof theory, "reverse mathematics" etc) and from the physico-philosophical relativity theoretic point of view. Much of this is done in [5],[21],[2],[24]. As a natural continuation of all this, we also experiment with generalizing *Specrel* in the direction of general relativity.

The first two steps in this generalization are (I) and (II) below. (I) We extend *Specrel* to accomodate accelerated observers which, via Einstein's equivalence principle, enables us to study some features of gravity. E.g. the Twin Paradox and the Tower Paradox (gravity slows time down) become provable in the accelerated observers version *Acc*(*Specrel*) of *Specrel*, cf.

[4] In some sense, we consider this as "Step Zero".

e.g. [2]. (II) As a second step in this direction, we make *Acc*(*Specrel*) *local* where "local" is understood in the sense of general relativity. We do this via the so-called method of localization which can be applied basically to any version of *Specrel* and *Acc*(*Specrel*). The localized versions of these theories are built up also in FOL (we make special efforts to ensure this) for methodological reasons mentioned earlier. Since localization turns out to be such a general procedure, we can denote the so obtained theories as *Loc*(*Specrel*), *Loc*(*Acc*(*Specrel*)) etc. So, *Loc*(−) can be regarded as some kind of a general "operator" applicable to theories (which are variants of *Specrel*).

It is explained in the classic textbook [25, pp.163-5] on general relativity that by suitably combining accelerated observers and localization one can safely move towards general relativity by starting out from special relativity, cf. e.g. Box 6.1 on p.164 therein. This motivates our study of the FOL theory *Loc*(*Acc*(*Specrel*)) and its variants. The investigation of *Loc*(*Acc*(*Specrel*)) is analogous with that of *Specrel*, i.e. after introducing the theory and proving theorems about its basic properties comes a fine-scale conceptual analysis both from the logic point of view and from the relativity theoretic point of view. The operator *Loc*(−) and *Loc*(*Specrel*) in particular are discussed in the present volume in [22]. The ideas in [22] are easily combined with those in [2] on *Acc*(*Specrel*) in order to obtain a comprehensive understanding of *Loc*(*Acc*(*Specrel*)). More on *Loc*(−) and *Loc*(*Specrel*) is in [22],[24], while more on *Acc*(*Specrel*) is found in [2], and the works quoted therein.

The research project reported herein is part of a much broader background literature of logic-based approaches to space-time. E.g., axiomatizations of special relativity are abundant in the literature. To mention some: axiomatizations of special relativity have been studied in works of Robb, Reichenbach, Carathéodory, Alexandrov and his school, Suppes and his school, Szekeres, Ax, Friedman, Mundy, Goldblatt, Schutz, Walker. This is only a small sample. There are more works listed in the bibliographies of [2],[5],[21]. Latzer [19], Buseman [9] represent moves towards general relativity in a spirit similar to that of our [22],[2],[24].

2 The frame language

We introduce the first-order logic language, which we will use for formalizing (first special) relativity, with an eye open for the subsequent generalization of the theory. We want to talk about *motion of bodies*.[5] What is motion? It is changing location in time. Therefore we will talk about *bodies*, *time*, *space*, and about a *location-function* which tells us which body is where at a given time. We want to talk about *relativity* theories; therefore these location functions will depend on *observers*; different observers may see the same motion differently. (The location function determined by an observer m will be called the world-view function w_m of observer m.) We will treat observers as special bodies whose motion will be represented exactly the same way as that of the rest of the bodies. These observers are often called, in the literature, *reference frames*.[6]

It will be convenient for us to be flexible about the dimension of space: we will not only treat 3-dimensional space, but 1 or 2, or higher-dimensional spaces as well. We will treat time as a special dimension of *space-time*. n will denote the dimension of our space-time.[7] Thus, usually $n = 4$ (3 space-dimensions and 1 time-dimension), but we will consider also $n = 2, 3$ or $n > 4$. Our bodies will be idealized, pointlike.

The vocabulary of our language is the following: unary relations

B (bodies)

Obs (observers)

Ph (photons)

Q (quantities used for giving location and "measuring time");

an $n + 2$-ary relation, the *location-* or *world-view* relation

W (world-view relation, $W(m, b, t, s_1, \ldots, s_{n-1})$ intends to mean that according to observer (or reference-frame) m, the body b is present at time t and location $\langle s_1, \ldots, s_{n-1}\rangle$);

for dealing with quantities, we will have two binary functions, and a binary relation:

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[5]In this paper we concentrate only on kinematics; the same kind of investigations can be carried out concerning mass, forces, energy etc. However, if a theorem can be proved without referring to these extra notions, we consider that a virtue.

[6]This difference is only a matter of linguistic convention.

[7]Recent generalizations of general relativity in the literature (e.g. M-theory) indicate that it might be useful to leave n as a variable.

$+, \quad \cdot, \quad \leq$.

In our theories, we will always assume the following:

- observers and photons are bodies,
- $W(m, b, t, s_1, \dots, s_{n-1})$ implies that m is an observer, b is a body, and $t, s_1, \dots, s_{n-1}$ are quantities,
- $\langle Q, +, \cdot, \leq \rangle$ is a Euclidean[8] linearly ordered field.

We found that the simplest way of treating these assumptions is to use a 2-sorted first-order language, where

B, Q are sorts or universes,

Obs, Ph are unary relations of sort B,

W is an $n+2$-ary relation of sort $B \times B \times Q \times Q \times \dots \times Q$,

$+, \cdot$ and $\leq$ are operations and relation of sort Q.

Let

$$\mathsf{M} = \langle B^{\mathsf{M}}, Obs^{\mathsf{M}}, Ph^{\mathsf{M}}; Q^{\mathsf{M}}, +^{\mathsf{M}}, \cdot^{\mathsf{M}}, \leq^{\mathsf{M}}; W^{\mathsf{M}} \rangle$$

be a model of our two-sorted language. This means that B^{M} and Q^{M} are sets, they are called the *universes of sort* B *and* Q respectively, $Obs^{\mathsf{M}}, Ph^{\mathsf{M}} \subseteq B^{\mathsf{M}}$ etc. We will omit the superscripts $^{\mathsf{M}}$. We call M a *frame-model* if $\langle Q, +, \cdot, \leq \rangle$ is a Euclidean linearly ordered field and $W \subseteq Obs \times B \times Q \times \dots \times Q$. $\models$ denotes the usual semantical consequence relation *induced by frame-models*, i.e. $Th \models \varphi$ means that for every *frame-model* M, if $\mathsf{M} \models Th$, then $\mathsf{M} \models \varphi$.

Next we introduce some terminology in connection with arbitrary frame-models $\mathsf{M} = \langle B, Obs, Ph; Q, +, \cdot, \leq; W \rangle$.

The essence, the "heart" of a frame-model is the world-view relation W. Since $W \subseteq Obs \times B \times {}^nQ$, for every observer $m \in Obs$ it induces a function $w_m : {}^nQ \to \{X : X \subseteq B\}$ as follows: for every $p \in {}^nQ$

$$w_m(p) := \{b \in B \ : \ W(m, b, p)\}.$$

Thus $w_m(p)$ is the set of bodies present at space-time location p for m. We

[8] An ordered field is called *Euclidean* if every positive element has a square root in it, i.e. if $(\forall x > 0)(\exists y) x = y \cdot y$ is valid in it.

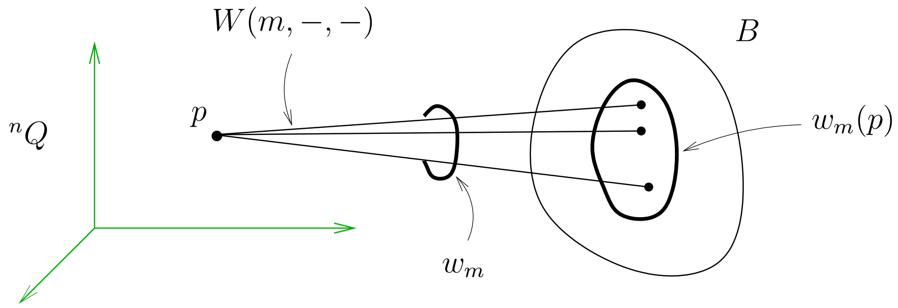

Figure 1: The world-view function w_m.

call a set of bodies an *event*, and w_m is called the *world-view function* of m, which to each space-time location p tells us what event observer m observes or "sees happening" at location p. "Seeing" has nothing to do with photons here, it really means "coordinatizing" only.

The *trace* or *life-line* of a body b according to an observer m is the set of space-time locations where m sees b, i.e.

$$tr_m(b) := \{p \in {}^nQ : W(m,b,p)\}.$$

The world-view function w_m can be recovered from the family of traces of all bodies (from $\langle tr_m(b) : b \in B\rangle$), and the world-view-relation W can be recovered from all the world-view functions (from $\langle w_m : m \in Obs\rangle$). Thus we can "represent" the function w_m by the *world-view* of m, which is just the indexed family $\langle tr_m(b) : b \in B\rangle$, and which, in turn, we represent by drawing the traces of bodies that we are interested in. See Figure 2.

Since $\mathsf{Q} = \langle Q, +, \cdot\rangle$ is a field, we can define n-dimensional straight lines as follows (these will be the life-lines of "inertial bodies"). We will use the vector-space structure of ${}^n\mathsf{Q}$, i.e. if $p, q \in {}^nQ$ and $\lambda \in Q$ then $p+q, p-q, \lambda p \in {}^nQ$ and $\overline{0}$ denotes the *origin*, i.e. $\overline{0} = \langle 0, \ldots, 0\rangle$, where 0 is the zero-element of the field. Let $\ell \subseteq {}^nQ$. We say that ℓ is a *straight line* iff there are $p, \alpha \in {}^nQ$ such that $\alpha \neq \overline{0}$ and

$$\ell = \{p + r \cdot \alpha \ : \ r \in Q\}.$$

Lines denotes the set of all straight lines (of dimension n). $\bar{t}$ denotes the *time axis*,

$$\bar{t} := \{\langle r, 0, \ldots, 0\rangle \ : \ r \in Q\}.$$

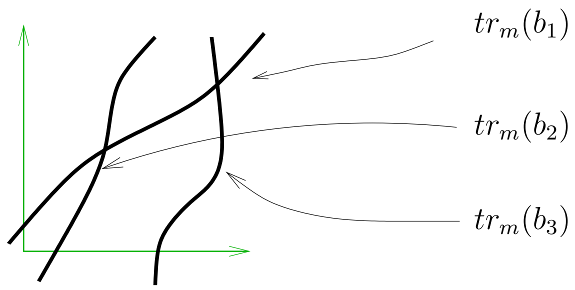

Figure 2: World-view of m.

$\bar{t}$ is a straight line. If $\ell \in \mathit{Lines}$, then $ang(\ell)$, defined below, represents the *angle*[9] between ℓ and $\bar{t}$ (where $\alpha = \langle \alpha_0, \dots, \alpha_{n-1} \rangle$ is associated to ℓ as before):

$$ang(\ell) := \frac{\alpha_1^2 + \dots + \alpha_{n-1}^2}{\alpha_0^2} \quad \text{if } \alpha_0 \neq 0, \text{ and}$$

$$ang(\ell) := \infty \quad \text{if } \alpha_0 = 0. \text{ Here } \infty \text{ is any element not in } Q.$$

$ang(\ell) = 1$ means intuitively that the angle between ℓ and $\bar{t}$ is 45^o. (See Figure 3.) Assume that $tr_m(k) = \ell$ is a straight line. Then $ang(\ell)$ represents the *velocity*[10] of k as seen by m:

$$v_m(k) := ang(tr_m(k)), \quad \text{if } tr_m(k) \in \mathit{Lines}.$$

E.g. $v_m(k) = 0$ means that $tr_m(k)$ is parallel with $\bar{t}$, i.e. k's location does not change with time, i.e. k is *at rest* w.r.t. m. The bigger $v_m(k)$ is, the bigger distance k travels in a unit time (as seen by m).

3 Basic axioms of special relativity

As already indicated, a *plurality* of "competing" axiom systems (or "relativity theories") is an essential feature of a logical analysis of relativity as developed in e.g. [5],[21],[2]. In this section we recall one of these axiom systems

[9] Actually, $ang(\ell)$ is the square of the tangent of the angle between ℓ and $\bar{t}$.

[10] Instead of "velocity", the precise expression would be "speed", since $v_m(k)$ is a scalar and not a vector.

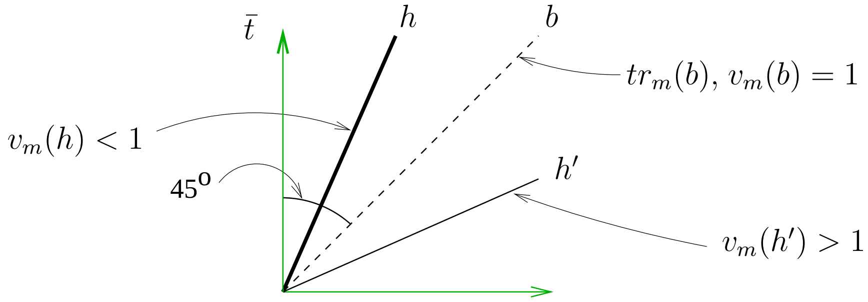

Figure 3: Velocities.

and will call it $Specrel_0$. It consists of five axioms. In the following axioms, m, k stand for arbitrary observers, h for an arbitrary body, ℓ for an arbitrary straight line (i.e. element of *Lines*), and *ph* for an arbitrary photon. We use the standard custom in logic that free variables should be understood as universally quantified, e.g. the axiom $tr_m(m) = \bar{t}$ means $(\forall m \in Obs) tr_m(m) = \bar{t}$.

Our first axiom says that the traces of observers and photons, as seen by any observer, are straight lines:

AxLine $tr_m(h) \in Lines \quad$ for $h \in Obs \cup Ph$.

Since translating our intuitive statements to first-order formulas in the language of our frame-models (M's) will be straightforward, we will not give these translations,[11] we will only give the intuitive forms.

The second axiom says that any observer sees himself at rest in the origin:

AxSelf $tr_m(m) = \bar{t}$.

The third axiom says that we have the tools for thought-experiments: on any appropriate straight line we can assume there is a potential observer; and the same for photons:[12]

[11] They can be found in [2],[21].

[12] This axiom can be "tamed" by using modal logic, such that space-time does not get crowded with k's and ph's, cf. [5].

AxPot $ang(\ell) < 1 \quad \Rightarrow \quad (\exists k \in Obs)\ell = tr_m(k)$, and
$ang(\ell) = 1 \quad \Rightarrow \quad (\exists ph \in Ph)\ell = tr_m(ph)$.

The fourth axiom says that all observers "see" the same events (possibly at different space-time locations):[13]

AxEvents $Rng(w_m) = Rng(w_k)$.

The last axiom says that the velocity of a photon is 1, for each observer:

AxPh $v_m(ph) = 1 \quad$ (and $\quad tr_m(ph) \in Lines$).

Our choice for a "first possible" axiom system for special relativity is:

$$Specrel_0 := \{\textbf{AxLine, AxSelf, AxPot, AxEvents, AxPh}\}.$$

When we want to indicate explicitly the number of dimensions, we will write $Specrel_0(n)$ in place of $Specrel_0$. We note that **AxPh** together with the photon part of **AxPot** is the relativistic part of $Specrel_0$. (The rest are true in Newtonian Mechanics.)

Let $n > 2$. In this paper we show that $Specrel_0(n)$ is consistent, it is not independent, and it forbids faster than light observers but permits faster than light bodies.[14] We show that $Specrel_0$ generates an *undecidable* first-order theory but we can strengthen it so that it becomes decidable (moreover categorical); and also we can strengthen it so that it becomes hereditarily undecidable, further both of Gödel's incompleteness properties hold for this strengthened version. We will see that both kinds of extension of $Specrel_0$ are natural.

[13]This will have to be considerably weakened, when preparing for a generalization of our axiom systems like $Specrel_0$ towards general relativity, cf. [5],[2],[22]. For a function f, its *range* is $Rng(f) := \{y : \exists x(f(x) = y)\}$.

[14]The point in proving things like $Specrel_0 \models$ *no FTL observer* is in the small number of axioms and concepts needed. Actually in [22] we show that a much weaker version of $Specrel_0$ is enough for proving this conclusion. A more refined version of the theorem says that FTL observers "lose most of their meter rods", cf. [5].

4 Traveling with light, traveling faster than light

As a warm-up, we begin with a simple statement about our axiom system $Specrel_0$. When Einstein was a child, he once dreamed that he traveled together with a photon, and then he tried to imagine how the world could look like when one sees it while traveling with a photon. Our first proposition says that in models of $Specrel_0$, you can't see the world while traveling with a photon. (By "seeing" we mean "coordinatizing".)

Proposition 1 *$Specrel_0 \models tr_m(k) \neq tr_m(ph)$ for any $m, k \in Obs$ and $ph \in Ph$.*

Proof. Assume that $tr_m(k) = tr_m(ph)$ for some $m, k \in Obs, ph \in Ph$ in a model of $Specrel_0$. Then $tr_k(k) = \bar{t}$ and $v_k(ph) = 1$ by **AxSelf, AxPh**. Thus $tr_k(k) \neq tr_k(ph)$. Then k sees an event in which k is present but ph is not present (namely such is $w_k(p)$ for any $p \in tr_k(k) \setminus tr_k(ph)$). However, m does not see such an event by $tr_m(k) = tr_m(ph)$. This contradicts **AxEvents**, proving the proposition. See Figure 4. **QED**

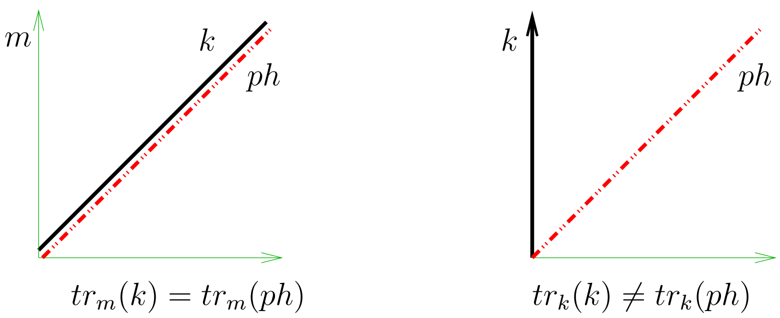

Figure 4: An observer cannot travel together with a photon.

Theorem 1 *Let $n > 2$.*

(i) $Specrel_0(n)$ is not independent, namely

$$\{\textbf{AxSelf, AxPot, AxEvents, AxPh}\} \models \textbf{AxLine}.$$

(ii) $Specrel_0(2)$ is independent, i.e. for any $\mathbf{Ax} \in Specrel_0(2)$ we have

$$Specrel_0(2) \setminus \{\mathbf{Ax}\} \quad \not\models \quad \mathbf{Ax}.$$

Proof. For brevity, we will write $Specrel_0 - \mathbf{Ax}$ for $Specrel_0 \setminus \{\mathbf{Ax}\}$. It is not difficult to check that $Specrel_0 - \mathbf{Ax} \not\models \mathbf{Ax}$ for any $\mathbf{Ax} \in Specrel_0$, if $\mathbf{Ax} \neq \mathbf{AxLine}$. So we have to show that

$Specrel_0(n) - \mathbf{AxLine} \models \mathbf{AxLine}$ and

$Specrel_0(2) - \mathbf{AxLine} \not\models \mathbf{AxLine}$.

Assume that M is a model of $Specrel_0(n) - \mathbf{AxLine}$. Let $m, k \in Obs^{\mathsf{M}}$ and define

$$f_{mk} := \{\langle p, q\rangle \in {}^nQ \times {}^nQ \ : \ w_m(p) = w_k(q)\}.$$

Thus f_{mk} is a binary relation on space-time locations; two space-time locations are related when m and k see the same "events" at those points. We now show that

(*) f_{mk} is a bijective mapping of ${}^nQ$ onto ${}^nQ$, in any model of $\{$**AxPot, AxEvents**$\}$.

Let $p, q \in {}^nQ$ be distinct. Then there is a straight line ℓ with $ang(\ell) < 1$ separating them, i.e. $p \in \ell$ and $q \notin \ell$. By **AxPot**, ℓ is the trace of some observer h. Then $h \in w_m(p), h \notin w_m(q)$, showing that w_m is injective for any observer m. By **AxEvents** we have that both the domain and the range of f_{mk} is ${}^nQ$ (since $f_{mk}^{-1} = f_{km}$). These facts imply (*).

f_{mk} is called the *world-view transformation* between m and k: its intuitive meaning is that m thinks that k is "crazy" to the extent that his seeing is distorted by this function f_{mk} (whatever event m sees at space-time location p, k sees it at location $f_{mk}(p)$).

Now, **AxPh, AxPot** require that f_{mk} preserve *light-lines* (i.e. straight lines with angle 1). By a slight generalization of the celebrated Alexandrov-Zeeman theorem (that we will recall in a moment) then f_{mk} has to preserve all straight lines, in other words, it is a *collineation*. Then $tr_k(m) = f_{mk}(tr_m(m)) = f_{mk}(\bar{t})$ is a straight line by **AxSelf**. Thus **AxLine** holds.

To show $Specrel_0(2) - \mathbf{AxLine} \not\models \mathbf{AxLine}$ we construct a bijection $f : {}^2R \to {}^2R$, where R is the set of reals, which preserves light-lines, but which takes $\bar{t}$ onto a curve which is not a straight line. Here is the idea of the construction (see Figure 5):

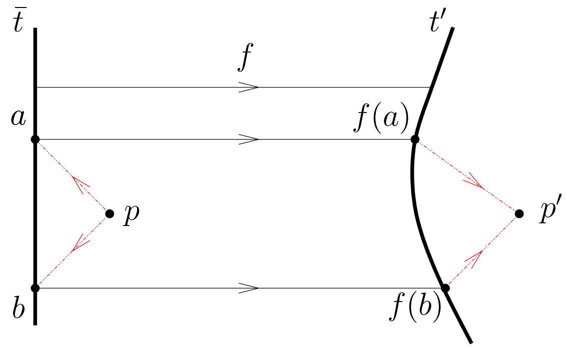

Figure 5: Illustration for the proof of Thm.1(ii). f preserves all light-lines but not all straight lines. Thus $\bar{t}$ cannot be defined from light-lines in ${}^2R$.

Let t' be a "slightly bent" version of $\bar{t}$, and let f be any bijection between $\bar{t}$ and t'. We extend f to any point p not on $\bar{t}$ as follows: Let a and b be the two points where the two light-lines through p intersect $\bar{t}$, and let $f(p)$ be the intersection point of the two corresponding light-lines going through $f(a)$ and $f(b)$. With some care this extension of f will be a bijection, and it preserves all light-lines by its construction. Now it is not difficult to construct a model of $Specrel_0(2)$ − **AxLine** where this f is one of the world-view transformations; and so in this model **AxLine** does not hold.

We now briefly recall the *Alexandrov-Zeeman theorem*. This theorem states that a permutation of ${}^4R$ which preserves light-lines is a collineation of a special form (namely a so-called Lorentz-transformation up to a dilation, a translation, and a field-automorphism-induced transformation[15]). An illuminating logical proof can be found in Appendix B of Goldblatt [13]. That proof can be generalized to any Euclidean field Q and $n > 2$ in place of R and 4. About the Alexandrov-Zeeman theorem see also [22] in this volume. We sketch the proof for $n = 3$. Let ℓ be any light-line. Let P be the set of those points through which no light-line intersecting ℓ goes through. Then it is not difficult to see that P is just the plane tangent to any light-cone[16] containing ℓ, see Figure 6. Now we can obtain all straight lines ℓ with $ang(\ell) > 1$ by intersecting such tangent planes; then we can define all planes using these

[15] This latter will matter when R will be replaced with Q.
[16] A *light-cone* is the union of all light-lines going through a given point.

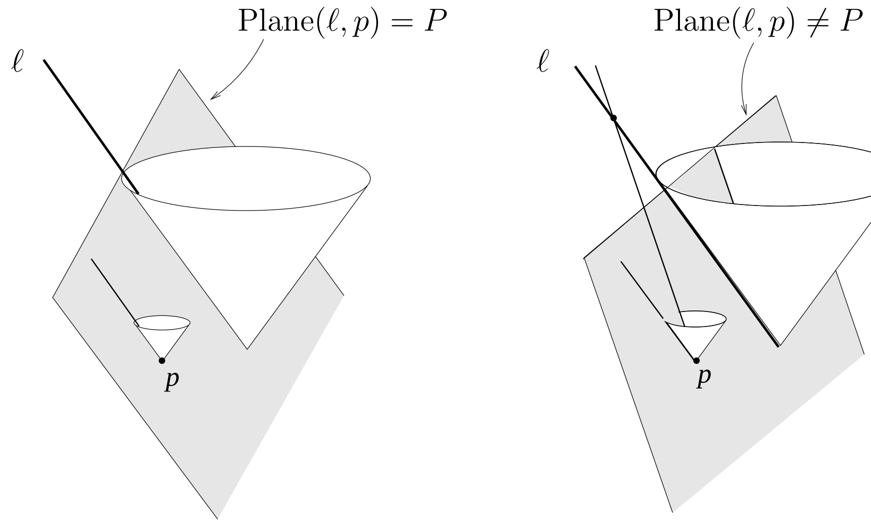

Figure 6: Illustration for the proof of the Alexandrov-Zeeman theorem. Definition of tangent planes: P is the set of points p through which no light-line intersecting ℓ goes. All straight lines can be defined from light-lines in ${}^3R$.

newly obtained straight lines, and then we can obtain all the straight lines by intersecting again these new planes. Hence, any light-line preserving permutation is a collineation. We omit the proof of the rest, but for an idea of the proof see the proof of Thm.3 herein. **QED**

Let M be a frame-model, and k be an observer in it. We say that k is a *faster than light* (FTL) observer, if $v_m(k) > 1$ for some observer m. Below, *no FTL observer* abbreviates the sentence $(\forall m, k \in Obs) v_m(k) < 1$, i.e. that there is no FTL observer in the model.[17]

Theorem 2 *Let $n > 2$.*

(i) $Specrel_0(2) \not\models$ no FTL observer.

(ii) $Specrel_0(n) \models$ no FTL observer.

[17] There are well known common sense arguments, going back to Einstein, against FTL (cf. e.g. [27, p.11]). These involve "causality" among other undefined concepts. As e.g. Gödel pointed out, these arguments are *not* proofs in the logical sense. Our present Theorem 2 is of an essentially different character from this point of view (contrast e.g. (i) with (ii)).

Proof. Since we want to stay visual, we give a proof for $n = 3$. We give a proof that is centered around the notion of Minkowski-orthogonality. Let ℓ, k be two straight lines. We say that ℓ is *Minkowski-orthogonal* (or shortly, M-orthogonal) to k if ℓ is orthogonal in the usual Euclidean sense to the reflection k' of k to the xy-plane. We say that ℓ is Minkowski-orthogonal to the plane P if it is Minkowski-orthogonal to at least two distinct straight lines lying in P, see Figure 7.

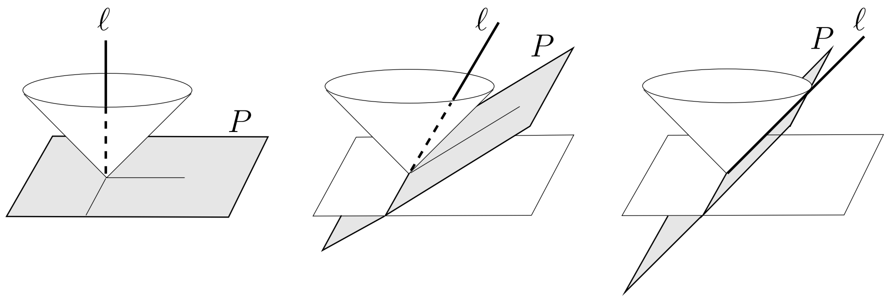

Figure 7: ℓ is Minkowski-orthogonal to P.

Minkowski-orthogonality is exhaustively investigated, e.g. fully axiomatized, in Goldblatt [13]. We will use here the following corollary of the generalized Alexandrov-Zeeman theorem:

(1) If a bijection of ${}^nQ$ preserves light-lines, then it preserves Minkowski-orthogonality.

We call a plane *space-like* if it contains no light-lines, and we call a straight line *time-like* if it is Minkowski-orthogonal to a space-like plane. It is not difficult to check (see Figure 8) that

(2) ℓ is time-like iff $ang(\ell) < 1$.

Clearly $\bar{t}$ is time-like, since it is M-orthogonal to the xy-plane which contains no light-line. Now we have seen in the proof of Theorem 1 that $f := f_{km}$ is a bijective collineation that preserves light-lines. Thus f takes the xy-plane to a space-like plane to which $f[\bar{t}]$ is M-orthogonal by (1), thus

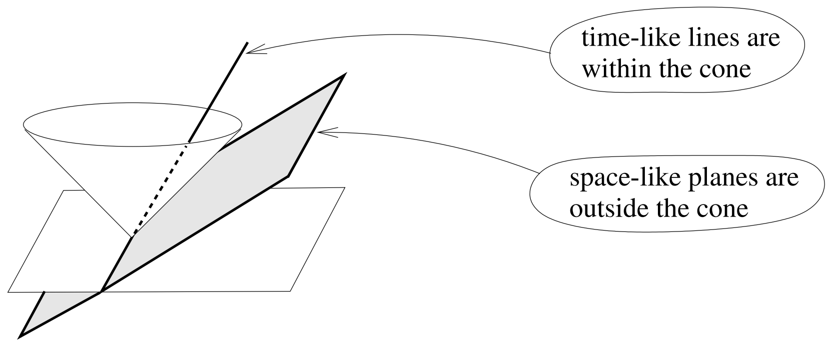

Figure 8: Time-like lines and space-like planes.

$f[\bar{t}]$ is time-like. By (2) then $ang(f[\bar{t}]) < 1$. But $f[\bar{t}] = f_{km}[tr_k(k)] = tr_m(k)$, thus $v_m(k) < 1$ in M.

To show $Specrel_0(2) \not\models$ *no FTL observer*, we have to give a model of $Specrel_0(2)$ in which there are FTL-observers. Such models are given in [5], in section 2.4. **QED**

On pushing the limits of Theorem 2: The Alexandrov-Zeeman theorem is not true for functions f not defined everywhere in ${}^4R$. Therefore the above simple proof does not generalize to the local version $Loc(Specrel_0)$ of $Specrel_0$. In [22, Thm.3], "*no FTL observer*" is proved from a very weak axiom system, where the world-view transformations are only partial functions and where **AxPh** is substantially weakened. Theorems proving "*no FTL observer*" from weak axiom systems are also in [5] and in [21]. In the process of finding the "limits" of the "no FTL theorems", we also gave some intuitively appealing axiom systems (such is e.g. *Relphax* in [5, §3]) which do have models with faster than light observers. More about these FTL investigations is in e.g. [2]. We also note that in [22] a new Alexandrov-Zeeman style theorem is proved for the local version $Loc(Specrel_0)$ of $Specrel_0$.

Now we are going to introduce seven extra natural axioms that will make $Specrel_0$ categorical over any field. The theory $Specrel_0$ extended with these seven axioms (and with any decidable theory of fields) is decidable. We will see that if we leave out any one of six of these axioms, the theory will become undecidable, and such that it can be extended to a hereditarily undecidable theory where both Gödel's incompleteness theorems hold.

5 A principle of relativity

The *world-view transformation* f_{mk} between two observers m, k is defined as

$$f_{mk} := \{\langle p, q\rangle : w_m(p) = w_k(q) \text{ and } w_k(q) \neq \emptyset\} .$$

We already used f_{mk} in the proof of Theorem 1. From our previous axioms it follows that f_{mk} is a transformation of ${}^nQ$ (and not only an arbitrary binary relation) if m, k are observers.[18] Therefore we will use f_{mk} as a function. Then $f_{mk}(p)$ is the "place" where k sees the same event that m sees at p, i.e.

$$w_m(p) = w_k(f_{mk}(p)) .$$

When $p = \langle p_0, \dots, p_{n-1}\rangle \in {}^nQ$, we will denote p_0 by p_t in order to emphasize that p_t is the "*time component*" of p. Let $p, q \in {}^nQ$. Then $p_t - q_t$ is the time passed between the events $w_m(p)$ and $w_m(q)$ as seen by m and $f_{mk}(p)_t - f_{mk}(q)_t$ is the time passed between the same two events as seen by k. Hence $\|(f_{mk}(p)_t - f_{mk}(q)_t)/(p_t - q_t)\|$ is the rate with which k's clock runs slow or fast as seen by m. Here, $\|a\|$ denotes the *absolute value* of a when $a \in Q$, i.e. $\|a\| \in \{a, -a\}$ and $\|a\| \geq 0$.

AxSym All observers see each other's clocks run slow to the same extent,

$$\|f_{mk}(p)_t - f_{mk}(q)_t\| = \|f_{km}(p)_t - f_{km}(q)_t\|, \text{ when } m, k \in Obs \text{ and } p, q \in \bar{t}.$$

AxSym states only that any two observers "see" each other's clocks "change" the same way. In principle, this allows the clocks run fast, be right, or run slow. In the Newtonian world **AxSym** is true because there each observer sees that the other's clocks are right. In models of $Specrel_0$, **AxSym** can be true only in the way that any observer sees that the clocks of any other observer not at rest wr.r.t. it *run slow.* Figure 12 in the proof of Thm.3 shows how it is possible in models of $Specrel_0$ that *both* observers "see" the clock of the other run slow.

On the choice of our symmetry axiom **AxSym**.: Under mild extra assumptions, $Specrel_0$ implies that **AxSym** is equivalent with an instance of

[18] This is a typical example of a property of special relativity which is relaxed in the process of localization (towards general relativity) in [22]. Namely, the axioms of the local theories in [22] will not imply that the function f_{mk} is everywhere defined in ${}^nQ$. This is an essential generalization towards general relativity.

Einstein's *special principle of relativity* SPR as it was formalized in [21, pp.87-89]. The principle SPR goes back to Galileo, intuitively it says that the "laws of nature" are the same for all inertial observers. A careful logic based analysis of SPR and its role in relativity is in [21, pp.84-91]. See also Friedman [12, p.153]. We note that in models of $Specrel_0$, **AxSym** is equivalent to the potential axiom requiring that, in space, in the direction orthogonal (in the Euclidean sense) to the direction of the movement there is no relativistic distortion, i.e. there is no length-contraction. Other equivalent formalizations of **AxSym** can be found in [5, §3.7].

6 Axioms making $Specrel_0$ categorical

Here we introduce six more axioms that will make $Specrel_0$ categorical (over any given field). As in section 3, in the following m, k stand for observers, ℓ for a straight line, ph_i for photons; and free variables in the axioms should be understood as universally quantified.

The first two axioms deal with the direction of flow of time. We define for any two observers m, k

$$m \uparrow k \quad \text{iff} \quad (f_{km}(1_t) - f_{km}(\bar{0}))_t > 0.$$

Intuitively this means that time flows in the same direction for m and k, see Figure 9.

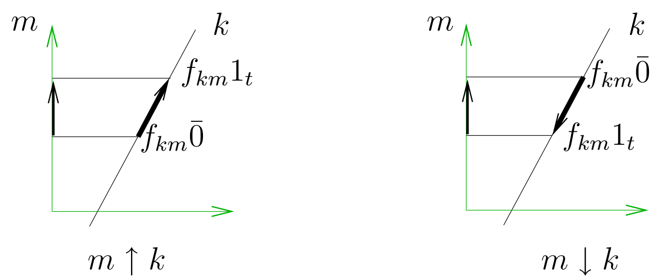

Figure 9: $m \uparrow k$ means that time flows in the same direction for m and k.

Our first axiom is a stronger version of part of **AxPot**, it says that every appropriate straight line is the life-line of an observer whose time flows "forwards".

AxPot$^+$ $ang(\ell) < 1 \quad \Rightarrow \quad (\exists k \in Obs)[\ell = tr_m(k)$ and $m \uparrow k]$.

The next axiom says that time flows in the same direction for any observers at rest in the origin.

Ax$\uparrow$ $tr_m(k) = \bar{t} \quad \Rightarrow \quad m \uparrow k$.

The next axiom says that every observer can "re-coordinatize" his world-view with a so-called Galilean transformation. To formalize the next axiom, first we single out special transformations, that we will call Galilean transformations. A mapping $f : {}^nQ \to {}^nQ$ is called a *Galilean transformation* if it preserves Euclidean distance and $f(1_t) - f(\bar{0}) = 1_t$ where $1_t = \langle 1, 0, 0, \ldots \rangle$ and 1 denotes the unit element of the field Q. In other words, a Galilean transformation is a congruence transformation which is the identity map on $\bar{t}$, composed with a translation. See Figure 10. It is known that a Galilean transformation is a linear transformation composed with a translation, hence the next axiom is a first-order logic one.

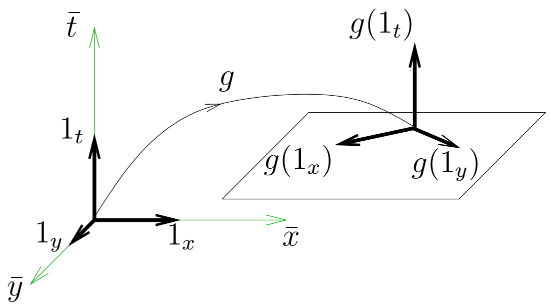

Figure 10: A Galilean transformation takes the unit vectors into pairwise orthogonal vectors of length 1, and does not change the direction of the time-unit vector.

AxGal $G(\bar{0}) \in \bar{t} \;\Rightarrow\; (\exists k \in Obs) f_{mk} = G,$ for every Galilean transformation G.

The next two axioms say, intuitively, that of each kind of observers and photons we have only one copy (or in other words, according to Leibniz's principle, if we cannot distinguish two observers or photons via some observable properties, then we treat them as equal).[19] In other words, these are so-called *extensionality axioms*. Id denotes the identity mapping.

AxExt$_1$ $f_{mk} = Id \quad \Rightarrow \quad m = k.$

AxExt$_2$ $tr_m(ph_1) = tr_m(ph_2) \quad \Rightarrow \quad ph_1 = ph_2.$

The last axiom says that every body is an observer or photon.

AxNobody $B = Obs \cup Ph.$

$Compl := \{$**AxSym, AxPot**$^+$**, AxGal, AxExt**$_1$**, AxExt**$_2$**, AxNobody**$\}$,

$Specrel := Specrel_0 \cup \{$**AxSym**$\}$,

$Specrel^+ := Specrel \cup Compl \cup \{\mathrm{Ax}\!\uparrow\}.$

In the terminology of e.g. Malament and Hogarth, $Specrel_0, Specrel$ and $Specrel^+$ correspond to *causal space-time* (or metric-free space-time), *space-time*, and *time-oriented space-time* respectively, cf. e.g. Hogarth [18]. $Specrel_0$ is also strongly connected to the "conformal structure of space-time". When we write "causal space-time", we have in mind the *symmetrized* version of the strict "causality relation" $\ll$. (Sometimes "metric-free space-time", "space-time", "time-oriented space-time" are used.)[20]

We did not include **Ax**$\uparrow$ into $Compl$ because, as we will see, its effects are different from those of the the elements of $Compl$.[21]

Theorem 3 *Let*[22] $n > 2$ *and let* $\mathsf{Q} = \langle Q, +, \cdot, \leq \rangle$ *be any Euclidean field.*

(i) There are exactly two models of $Specrel \cup Compl$ *with field-reduct* Q*, up to isomorphism.*

---

[19]We could have named these axioms after Occam, too.

[20]The terminology varies with different authors, but what we wanted to point out is that the levels of abstraction corresponding to $Specrel_0, Specrel$ and $Specrel^+$ seem to be generally distinguished levels of abstraction in the literature of relativity.

[21]Intuitively, **Ax**$\uparrow$ excludes only one model of two choices, while the rest exclude an infinite number of possibilities, cf. Thm.s 3-5. Sci.Am.(Sept. 2002, special issue, pp.30-31) discusses the justification of assuming **Ax**$\uparrow$ which turns out to be not as straightforward as one might think at first sight.

[22]We exclude the case $n = 2$ for simplicity only.

(ii) There is a unique model of $Specrel^+$ with field-reduct Q, up to isomorphism.

On the proof. We illustrate that in any model of *Specrel*, all the world-view transformations are so-called Poincaré-transformations (i.e. Lorentz-transformations composed with translations), and this is the most important part of the proof of Theorem 3.

Let m, k be observers in a model of *Specrel*, we will investigate the world-view transformation $f := f_{mk}$. We have already seen that $f : {}^nQ \to {}^nQ$ is a *bijection*. It is a *collineation* by the Alexandrov-Zeeman theorem in case $n > 2$, and by [22, Thm.2] in case $n = 2$. By **AxPh**, f takes light-lines onto light-lines, and this implies that f takes the unit vectors into vectors of the *same length* and *Minkowski-orthogonal* to each other. Figure 11 illustrates the idea of the proof of this part.

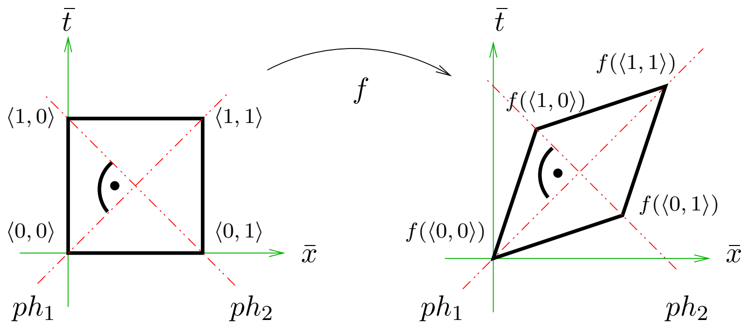

Figure 11: World-view transformations in models of $Specrel_0$ take the unit vectors to vectors Minkowski-orthogonal to each other and of the same length.

Finally, **AxSym** implies that the length of the unit vectors is fixed, as follows. We write out this part of the proof in more detail, because e.g. it shows how it is possible that both observers see each other's clocks run slow.

Let $1_t = \langle 1, 0, 0, \ldots \rangle$, and let us see where $e := f_{km}(1_t)$ is on $tr_m(k)$. Let a, b and a' be as in Figure 12; i.e. they are the points on $tr_m(k)$ and on $\bar{t}$ such that the straight line connecting 1_t and a is parallel with $\bar{x}$, and the straight lines connecting 1_t and b and connecting a and a' are parallel with $f_{km}[\bar{x}]$. See Figure 12. If $e = a$, then m sees that k's clock shows 1 just when his clock shows 1, because 1_t and a are simultaneous for m. But k will see that m's clock shows $a' < 1$ when his clock shows 1, because for k, $e = a$ and a'

are simultaneous. So k will think that m's clocks run slow, but m will think that k's clocks are right. Analogously, m thinks that k's clocks are right (run slow or fast, respectively) iff $e = b$ ($> b$ or $< b$ respectively). And, k thinks that m's clocks are right (run slow or fast, respectively) iff $e = a$ ($< a$ or $> a$ respectively). Thus both think that the other's clocks run slow iff $b < e < a$. The rate of "slowness" is the same for them at a unique point in between a and b, because the change of rate is a continuous and strictly monotonic function (of the "number" $\|e\|$). Now, *Minkowski-distance* is defined so that the Minkowski-distance is 1 between $\bar{0}$ and this unique point (where the rates of "running slow" are the same for m and k). Figure 13 shows the points whose Minkowski-distance from $\bar{0}$ is 1, i.e. it shows Minkowski-circle with radius 1 and center $\bar{0}$.

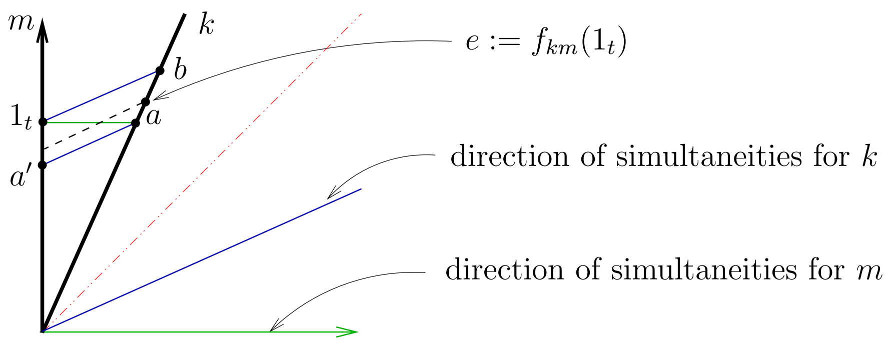

Figure 12: Both m and k think that the other's clocks run slow iff $f_{mk}(1_t)$ is in between a and b. The rates of "running slow" will be equal at a unique point.

It is known that any collineation is an affine transformation composed with a field-automorphism-induced transformation. Using that the above line of thought is valid for any $p \in \bar{t}$ in place of 1_t, one can show that the world-view transformations are actually *affine* transformations. Summing up: in models of *Specrel*, the world-view transformations take the unit vectors into pairwise Minkowski-orthogonal vectors of Minkowski-length 1. These kinds of affine transformations are called in the literature *Poincaré-transformations*. **QED**

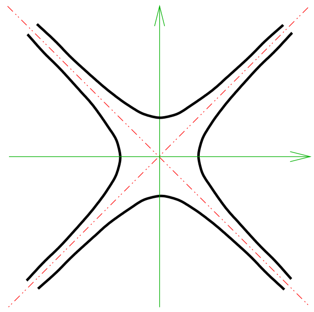

Figure 13: Minkowski-distance 1.

7 Decidability and Gödel incompleteness

We now turn to decidability questions. We start this by recalling the definition of real-closed fields and by recalling some facts from the literature.

An ordered field F is real-closed if it is Euclidean (i.e. every positive element has a square root), and if every polynomial of odd degree has zero as a value. This last requirement can be expressed with the infinite set $\{\phi_{2n+1} : n \in \omega\}$ of first-order formulas, where for every $n \in \omega$, ϕ_n denotes the following sentence

$$\forall x_0 \ldots \forall x_n \exists y[x_n \neq 0 \rightarrow (x_0 + x_1 \cdot y + \ldots + x_n \cdot y^n = 0)].$$

By a *theory* we will understand an arbitrary set of first-order formulas (i.e. we will not assume that it is closed under semantical consequence). We call a theory *Th decidable* (or *undecidable* respectively) if the set of all first-order semantical consequences of *Th* is decidable (or undecidable respectively). We call *Th complete* if it implies either ϕ or $\neg\phi$ for each first-order formula ϕ (of its language). Propositions 2,3 below are known in the literature. Prop.2 is a corollary of Tarski's famous elimination of quantifiers for real-closed fields.

Proposition 2 *The theory of real-closed fields is decidable and complete.*

Proposition 3 *The theories of ordered fields and Euclidean fields are undecidable.* [23]

[23]Note that if a finitely (or more generally, recursively) axiomatizable theory is undecidable, then it is not complete.

Conjecture 1 *Any finitely axiomatizable consistent theory of ordered fields is undecidable.*

Corollary 1 *$Specrel_0$, Specrel and $Specrel^+$ are undecidable.*

Proof. This is a corollary of Prop.3, and the theorem that for any Euclidean field F there is a model of $Specrel^+$ with F as the field reduct (Theorem 3).: Let ϕ be any field-theoretic first-order formula written by using variables of our quantity sort. Then ϕ is valid in a frame-model M with field reduct F iff ϕ is valid in F. Thus ϕ is valid in the class of Euclidean fields iff ϕ is true in all models of $Specrel^+$. Since the first-order theory of the Euclidean fields is undecidable by Prop.3, the first-order consequences of $Specrel^+$ is undecidable, too. Since this is a finite theory, then any subset of it is undecidable, too. **QED**

The above suggests that if we want to obtain interesting decision-theoretic results, we have to concentrate on real-closed fields; or at least include a decidable theory of field-axioms into our theories. Let Φ denote the theory of real-closed fields.

Theorem 4 *Let $n > 2$.*

(i) $Specrel_0 \cup Compl \cup \Phi$ *is decidable.*

(ii) $Specrel_0 \cup Compl \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$ *is decidable and complete.*

(iii) $Specrel_0 \cup (Compl \setminus \{\mathbf{Ax}\}) \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$ *is undecidable, for any axiom* $\mathbf{Ax} \in Compl$.

Proof. We show that (i) and (ii) are corollaries of Theorem 3, we sketch the proof of (ii). Let M and M' be models of $Specrel_0 \cup Compl \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$. We cannot apply Theorem 3 yet, because the field-reducts F and F' of M and M' respectively may not be the same. But they are elementarily equivalent, because Φ is complete, so by the Keisler-Shelah isomorphic ultrapowers theorem they have isomorphic ultrapowers, say F_1 and F'_1. Let M_1 and M'_1 be the ultrapowers of M and M' respectively, taken by the same ultrafilter. Then the field-reducts of these are F_1 and F'_1 respectively. Now we can apply Theorem 3 to M_1 and M'_1 because F_1 and F'_1 are isomorphic, getting that M_1 and M'_1 are isomorphic, so elementarily equivalent. But then M and M' are elementarily equivalent, too, since the former two models are ultrapowers of these. This finishes the proof of (ii). (iii) is a corollary of the next theorem; we included it here because it nicely contrasts (i) and (ii). **QED**

We now turn to the analog of Gödel's first incompleteness theorem.

Theorem 5 *Let $n > 0$ and let* **Ax** *be any member of Compl. There is a formula ν (in our frame-language) such that*

(i) ν is consistent with $Specrel_0 \cup (Compl \setminus \{\mathbf{Ax}\}) \cup \{\mathbf{Ax}\uparrow\} \cup \Phi$

and for any theory Th consistent with ν

(ii) Th is hereditarily undecidable in the sense that no consistent extension of Th is decidable.

(iii) The conclusion of Gödel's first incompleteness theorem applies to the theory Th, i.e. no consistent recursively enumerable extension of Th is complete; moreover there is an algorithm that to each consistent, recursively enumerable extension Th' of Th gives us a formula ϕ such that $Th' \not\models \phi$ and $Th' \not\models \neg\phi$.

Proof. The idea of the proof is to show that absence of any member of *Compl* allows us to interpret Robinson's Arithmetic into our theory. We sketch this for the case **Ax** = **AxNobody**. We will see that in this case ν will be quite natural: it will state the existence of a periodically moving body. Consider the following formulas (with free variables m, b and t):

$$I(t) \;:=\; I(m,b,t) \;:=\; W(m,b,t,\overline{0}), \quad \text{and}$$

$$\nu \;:=\; I(0) \;\wedge\; (\forall t,s)$$

$$([t<1 \;\wedge\; t\neq 0] \to \neg I(t) \quad \wedge$$

$$t \geq 0 \to [I(t) \leftrightarrow I(t+1)] \quad \wedge$$

$$[I(t)\wedge I(s)] \to [I(t+s)\wedge I(t\cdot s)]).$$

Add, for a moment, m and b as constants to our language. Then t remains the only free variable of I which then specifies a subset of the field-reduct in any frame-model: the set of time-points where the observer m sees the body b at the origin. Now the formula ν requires that this subset behaves like the set of integers: it is a discrete periodic subset containing $0, 1$ and closed under $+, \cdot$. Since the field-reduct of a frame-model is a field, then Robinson's arithmetic will be true in the field-reduct restricted to the subset defined by I. In other words, I is an *interpretation of Robinson's Arithmetic in* $Th \cup \{\nu\}$, whenever ν is consistent with Th. For definition of Robinson's Arithmetic and (semantical) interpretation see e.g. Monk [26, Def.14.17, Def.11.43]. Thus, Robinson's Arithmetic can be interpreted in $Th \cup \{\nu\}$. Then $Th \cup \{\nu\}$ is inseparable (which is a strong version of undecidability) by Thm.16.1 and Prop.15.6 in [26]; and thus (ii) and (iii) of our Theorem hold by Monk [26, Thm.s 15.9 and 15.8]. Finally, if we omit the constants m, b, then semantical consequence does not change, so (ii) and (iii) will hold for the original language (set of formulas not containing the constants m or b), too (in (iii) a further little argument is needed).

To show (i), we have to construct a model of $Specrel_0 \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi \cup \{\nu\} \cup (Compl \setminus \{\mathbf{AxNobody}\})$. This is not difficult as ν basically states the existence of a periodically moving body; see Figure 14.

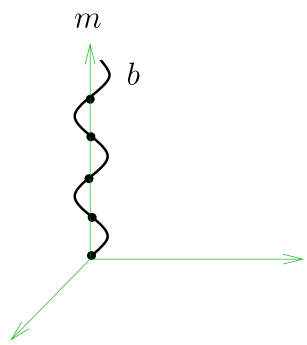

Figure 14: b is a periodically moving body in m's world-view.

Take a "standard" model with minimum set of observers and photons; and add one periodically moving body. We omit the details of the definition of this model.

The proofs for the other cases are analogous; we only give different interpretations of Robinson's arithmetic. This means that we give a different formula I, but ν will be the same (speaking about I), and then we only have to show that $Th \cup \{\nu\}$ is consistent, where Th is the theory in (i). To give a flavor, we give this new interpretation I for the case when $\mathbf{Ax} = \mathbf{AxPot}^+$.

$$I(m,t) := (\forall \ell)[ang(\ell) = \tfrac{1}{t} \quad \Rightarrow \quad (\exists k)(tr_m(k) = \ell \wedge m \uparrow k)] \text{ or } t = 0, 1.$$

This finishes the proofidea of Theorem 5. **QED**

A theorem analogous to Theorem 5 but concerning *Gödel's second incompleteness theorem* can also be stated and proved with analogous methods. For details see [4].

For current research directions in logic started by Gödel's incompleteness theorems we refer to Hájek and Pudlák [14], Willard [33], the latter in the present volume. The connections between the "observations oriented" and the "theoretically oriented" approaches to relativity were studied in [21] where the logical theories of definability and identifiability are used and further elaborated in the spirit of works of Hodges (cf. [17]) and Hintikka [15].[24] Actually, these logic based relativistic investigations induced new research in definability and identifiability theory. In later work continuing [2],[22] we plan to look into the logical structure of general relativistic space-times permitting *closed time-like loops* (which can be regarded as causing a kind of self-reference[25]). In Scheffler [30, p.179], and in Lewis [20, pp.67-80, pp.212-3] it is pointed out that these causal loops do not imply logical contradictions or even logical paradoxes. They simply have more complex logical structures than "linear causation". We plan to extend the mathematical logic based approach to further analyzing these and related possibilities thoroughly and carefully.

Acknowledgements. We would like to express our deep gratitude to the organizers and to the participants of the FOL75 conference for useful discussions, suggestions, encouragement and for a very fruitful, creative, and supportive atmosphere.

[24]In [21, Thm.2.2.23, p.168] we use and extend Tarski's elimination of quantifiers for real-closed fields [16], for analyzing our theories of relativity.

[25]like the one in Gödel's incompleteness proof, Tarski's proof of undefinability of truth, or Barwise and Etchemendy's book on the "Liar"

8 Bibliography

[1] Andai, A. Andréka, H. Madarász, J. X. and Németi, I. Logical approach to relativity theory. Mathematical Physics Meeting, Sept. 2001. (In Hungarian.) www.kfki.hu/˜elftterm/matfiz01.html

[2] Andréka, H. Madarász, J. X. and Németi, I. Logical axiomatizations of space-time. Samples from the literature. In: Non-Euclidean Geometries (János Bolyai Memorial Volume). Kluwer Academic Publishers, to appear.

[3] Andréka, H. Madarász, J. X. and Németi, I. Defining new universes in many-sorted logic. Preprint, Rényi Institute of Mathematics, Budapest, 2001.

[4] Andréka, H. Madarász, J. X. and Németi, I. Decidability, undecidability, and Gödel incompleteness in relativity theories. Manuscript, Budapest, 1999.

[5] Andréka, H. Madarász, J. X. Németi, I. Sain, I. and Sági, G. Analyzing the logical structure of relativity theory via model theoretic logic. Lecture Notes for A'dam course (spring 1998). Updated version July 5, 2002, 1312pp. http://www.math-inst.hu/pub/algebraic-logic/Contents.html.

[6] van Benthem, J. The logic of time. Reidel, Dordrecht, 1983.

[7] van Benthem, J. Semantics of time. In: Time, Mind, and Behaviour (eds: Michon, J. A. and Jackson, J. L.), Springer-Verlag Berlin 1985. pp.266-278.

[8] van Benthem, J. Content versus wrapping: an essay in semantic complexity. In: Arrow Logic and Multimodal Logic (eds: Marx, M. Pólos, L. and Masuch, M.) CSLI Publications, Stanford, 1996. pp.203-219.

[9] Buseman, H. Time-like spaces. Dissertationes Math. (Rozprawy Math.), 53, 1967.

[10] Feferman, S. Theories of finite type related to mathematical practice. In J. Barwise, editor, Handbook of Mathematical Logic, North-Holland, 1977. pp.913-971.

[11] Ferreirós, J. The road to modern logic - an interpretation. The Bulletin of Symbolic Logic 7,4 (2001), 441-484.

[12] Friedman, M. Foundations of space-time theories. Relativistic physics and philosophy of science. Princeton University Press, 1983.

[13] Goldblatt, R. Orthogonality and space-time geometry. (With 126 illustrations.) Springer-Verlag, 1987.

[14] Hájek, P. and Pudlák, P. Mathematics of first order arithmetic. Springer-Verlag, Berlin, 1993.

[15] Hintikka, J. Towards a general theory of identifiability. In: Definitions and definability: philosophical perspectives. Kluwer Academic Publishers, 1991. pp.161-183.

[16] Hodges, W. Model theory. Cambridge University Press, 1993.

[17] Hodges, W. Hodkinson, I. and Macpherson, H. Omega-categoricity, relative categoricity and coordinatisation. Annals of Pure and Applied Logic 46,4 (1990), 169-199.

[18] Hogarth, M. Conventionality of simultaneity: Malament's result revisited. Preprint, Cambridge University, 2003.

[19] Latzer, R. W. Nondirected light signals and the structure of time. Synthese 24 (1972), 236-280.

[20] Lewis, D. The paradoxes of time travel. In: David Lewis Philosophical Papers Vol. II, Oxford University Press, 1986. pp.67-80.

[21] Madarász, J. X. Logic and Relativity (in the light of definability theory). PhD Thesis, Eötvös Loránd University, Budapest, 2002. http://www.math-inst.hu/pub/algebraic-logic/Contents.html.

[22] Madarász, J. X. Németi, I. and Tőke, Cs. On generalizing the logic-approach to space-time towards general relativity: first steps. This volume.

[23] Madarász, J. X. and Tőke, Cs. Duality between observational and theoretical concepts in relativity theory analyzed by FOL and definability theory. In: Proceedings of the Kalmár Workshop on Logic and Computer Science, July 2003. pp.133-142.

[24] Madarász, J. X. and Tőke, Cs. Partial relativity theories. Research report, Alfréd Rényi Institute of Mathematics, Budapest, July 2003.

[25] Misner, C. W. Thorne, K. S. and Wheeler, J. A. Gravitation. W. H. Freeman, San Francisco, 1973.

[26] Monk, J. D. Mathematical Logic. Springer-Verlag, Berlin, 1976.

[27] Rakić, N. Common sense time and special relativity. Dissertation, University of Amsterdam, 1997. ILLC Dissertation series 97-02.

[28] Rindler, W. Essential relativity. Special, general and cosmological. Springer-Verlag, Berlin, 1969, 1977.

[29] Sain, I. There are general rules for specifying semantics: observations on abstract model theory. CL&CL 13 (1990), 195-250.

[30] Scheffler, U. Ereignis und Zeit. Ontologische Grundlagen der Kausalrelationen. Logos Verlag Berlin, 2001.

[31] Szabó, L. E. The problem of open future, determinism in the light of relativity and quantum theory. Manuscript, Dept. of Theoretical Physics, Eötvös Loránd University, Budapest 1999. (Also: Typotex Kiadó, Budapest, 2002. 256pp. (In Hungarian.))

[32] Väänänen, J. Second-order logic and foundations of mathematics. The Bulletin of Symbolic Logic 7,4 (2001), 504-520.

[33] Willard, D. A version of the second incompleteness theorem for axiom systems that recognize addition but not multiplication as total functions. This volume.

[34] Woleński, J. First-order logic: (philosophical) pro and contra. This volume.