# Finite Schematizable Algebraic Logic *

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**Abstract**

In this work, we attempt to alleviate three (more or less) equivalent negative results. These are (i) non-axiomatizability (by any finite schema) of the valid formula schemas of first order logic, (ii) non-axiomatizability (by finite schema) of any propositional logic equivalent with classical first order logic (i.e., modal logic of quantification and substitution), and (iii) non-axiomatizability (by finite schema) of the class of representable cylindric algebras (i.e., of the algebraic counterpart of first order logic). Here we present two finite schema axiomatizable classes of algebras that contain, as a reduct, the class of representable quasi-polyadic algebras and the class of representable cylindric algebras, respectively. We establish positive results in the direction of finitary algebraization of first order logic without equality as well as that with equality. Finally, we will indicate how these constructions can be applied to turn negative results (i), (ii) above to positive ones.

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# 1 Introduction

Since it was introduced, first order logic became the most frequently used and studied logical system. Its logical connectives and basic concepts are motivated by intuitive "mathematical" reasoning. It also plays a fundamental role in logical (formal) investigations of not necessarily mathematical reasoning, cf. e.g. [GG85], [vB91].

First order logic is only one of the logical systems investigated in modern logic. For many logical systems, or logics for short, the methodology of algebraic logic proved to be very useful. This methodology is summarized e.g. in [ANSK] and in [NA93]. Cf. also Blok–Pigozzi [BP89]. The key step in this methodology is that, to any logic $L$, we associate a class $\mathrm{Alg}(L)$ of algebras, called the algebraic counterpart of the logic $L$. In many cases, investigating $\mathrm{Alg}(L)$ is easier than investigating $L$, e.g. because, in the study of $\mathrm{Alg}(L)$, we can use the well developed tools of algebra. A number of problems concerning certain logics $L$ were often solved by translating the problem to $\mathrm{Alg}(L)$, solving the algebraic problem by the methodology of algebra, and then translating the algebraic result back to the logical context in which $L$ was originally investigated. Another equally important value of algebraization is obtaining insights into the essential nature of a logic $L$, via abstraction provided by algebraization.

The algebraic counterpart of classical propositional logic is the class of Boolean algebras. Boolean algebras proved immensely useful in studying classical propositional logic, and in understanding its connections with other logical systems. Similarly, for a great number of other logics, studying their algebraic counterparts has proven very useful. One example is propositional modal logic, the algebraic counterpart of which is Boolean algebras with operators. Similar positive examples are most of the non-classical propositional logics, e.g. intuitionistic propositional calculus whose algebraic counterpart is the variety of Heyting Algebras. Cf. e.g. [W88].

In the case of first order logic, when trying to apply the methodology of algebraic logic in the above outlined fashion, we have to face the following problem. As we said, the algebraic counterpart of classical propositional logic is the class of Boolean algebras. This class turns out to be a finitely axiomatizable equational class. The same is true for the other successful examples we referred to. In each case, $\mathrm{Alg}(L)$ is axiomatizable by a finite set of equations or quasi-equations[1]. As a contrast, in the case of first order logic, all the algebraic counterparts found before 1987 are very far from being finitely axiomatizable. Namely, J. D. Monk proved in [M70] that the most thoroughly investigated algebraic counterpart of first order logic with equality, the class $\mathrm{RCA}_\omega$ of representable cylindric algebras is not axiomatizable by a finite schema of equations[2]. A similar result was obtained in Sain–Thompson [SaT91] about the class $\mathrm{RQPA}_\omega$ of representable quasi-polyadic algebras (the algebraic counterpart of first order logic without equality)[3]. Monk's theorem was improved by Andréka in [A91]. She proved that any axiomatization of $\mathrm{RCA}_\omega$, besides being infinite, has to be extremely complex.

These results motivate the problem of finding an algebraization of first order logic where the class of algebras associated to this logic is axiomatizable by a preferably nice (simple, transparent, mathematically elegant) schema of equations or quasi-equations. There is a harder version of this problem asking for a strictly finite axiomatization. However, it is understood that a finite schema solution could already be very useful if the schema of axioms is simple enough. This problem is known as the *finitization problem* in algebraic logic. Actually, as it turns out, it is not a single problem, but rather a family of problems or, perhaps, even a research direction of algebraic logic, cf. e.g. Németi [N93], Simon [Si93], Sain [Sa92]. Here we will simply refer to it as the finitization problem. The problem as we formulated above is not concrete enough to be regarded as a tangible mathematical problem. To illustrate this, we note that Simon [Si93] is devoted to showing that many seemingly relevant results are, though mathematically interesting, not even partial solutions of the finitization problem. Therefore, below we recall a sufficiently concrete formulation of the finitization problem from the literature.

The problem of finding a "finitizable" algebraization of first order logic was raised in Monk [M70, p.20] and in Henkin–Monk [HM74].[4] The formulation in the latter goes as follows:

> *"Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based equational class."*

An equational class is called *finitely based* if it is axiomatizable by finitely many equations. In [HM74] an algebra is called representable if it is isomorphic to a

[1] Quasi-equations are equational implications.

[2] Monk obtained similar negative results for $\mathrm{RCA}_3$ already in 1964.

[3] The Sain–Thompson result was preceded by a relevant result of James Johnson [Jo69] who proved that $\mathrm{RQPA}_n$ $(2 < n < \omega)$ is not finitely axiomatizable.

[4] This problem has a long history and many people worked on it, e.g. A. Tarski did so already in the 1940's; but the explicit formulation we would like to use is in the quoted papers of L. Henkin and J. D. Monk. For completeness, we also refer to Henkin–Tarski (1961) [HT61] to the open problem formulated below Theorem 1.14 therein (lines 21–29 on page 96). There they ask for a finite schema axiomatization of $Eq(\mathrm{RCA}_\omega)$. This turns out possible (but only) if we add to $\mathrm{RCA}_\omega$ operations like $\mathsf{s}_{\mathrm{suc}}, \mathsf{s}_{\mathrm{pred}}$ defined herein.

subdirect product of *set algebras*. Therefore the key step is finding an appropriate notion of set algebra. In devising such definitions, usually, an ordinal $\alpha$ is fixed in advance, and then the elements of a set algebra are (certain) $\alpha$-ary relations over some set called the base of the algebra. One piece of the connection with logic is that the base of the algebra corresponds to the universe of that model of first order logic from which it is obtained (see e.g. [HMTII, sections 4.3, 5.6] for details on this connection). Following Jónsson and Tarski, we require that the operations of a set algebra should be invariant under any permutation of the base set[5]. An operation with this property is called *logical.* (This property is called "permutation invariant" in Jónsson [Jó86].) In this work we concentrate on that version of the above quoted problem of Henkin and Monk in which *all operations of the set algebras are required to be logical.*[6] This condition admits the following equivalent formulation. The class of representable algebras can be obtained in the form $\mathrm{Alg}(L)$ for some logic $L$ such that, in the model theory of $L$, isomorphic models satisfy the same formulas — the latter condition is one of the basic axioms of abstract model theory.

A solution to the truly finite axiomatization version of the finitization problem was given in Sain [Sa87a] part of which is revised as [Sa87] and abstracted in [Sa92] and [Sa94]. These papers give a complete solution to the case of first order logic without equality, while the solution there to logic with equality is slightly weaker. Much of the effort in the just quoted papers goes into ensuring that the number of axioms is strictly finite (as opposed to a finite schema). Therefore the idea comes up that if we require only a finite schema axiomatization (as opposed to "truly finite") then, perhaps, we could achieve substantial improvements in other directions such as simplifying the axioms. Indeed, after [Sa87a] was presented at the Algebraic Logic conference in Asilomar, 1987, both Leon Henkin and J. Donald Monk suggested looking for a solution which, instead of a truly finite set of axioms, provides only a finite schema of axioms, but in such a way that this schema would be simple and easy to work with. This is one of the things we intend to do in the present paper. [7]

The question of comparing our notion of a schema with the notion used in Keisler's completeness theorem and in the Daigneault–Monk representation theorem for polyadic algebras (without equality) comes up at this point (see [HMTII] 5.4). We will make such a comparison using results of [NS95] and [Sá95] in Subsection 2.6 at the end of this section.

[5] Tarski mentions this requirement already in his 1941 paper on the calculus of relations. See also next footnote.

[6] This condition goes back to Lindenbaum–Tarski [LT36], and was extensively discussed in Tarski [Ta87] where it was suggested that the condition is essential for investigations of truly logical nature. Further considerations for necessity of requiring all the operations to be logical (i.e. permutation invariant) are in [Sh91] and [vB96] 3.3.2 (pp.67–68). It was proved e.g. in Biró [Bi89] that permutation invariance is a necessary condition in the finitization problem. Cf. also Németi [N91].

[7] A related open problem is to simplify the axiomatization given in the original papers (e.g. in [Sa87]).

A further aim of this paper is to improve the positive results to a tighter or more orthodox class of set algebras. This tighter class is denoted below as $\mathrm{Cdf}_G$.

To *first order logic without equality*, we will associate a fairly simple and powerful class $\mathrm{Cdf}_G$ of set algebras. A $\mathrm{Cdf}_G$ algebra $\mathcal{A}$ is a subalgebra of the Boolean algebra of subsets of the Cartesian space ${}^{\omega}U$ expanded with some extra operations corresponding to quantification and substitution[8]. We will axiomatize $\mathrm{Cdf}_G$ by a relatively simple finite schema of quasi-equations (cf. Theorem 2.5). A modified version of this class will receive an equational axiomatization. It remains an open problem to simplify these axiomatizations. To the same logic we will also associate a slightly bigger class $\mathrm{Gwdf}_G$ of set algebras. We will be able to prove more results about this second class (and also we will be able to simplify the axioms for the latter). $\mathrm{Cdf}_G$ will meet all the requirements of the (schema version of the) finitization problem quoted way above. E.g., its operations will be logical and natural set theoretic operations on relations. The saem applies to $\mathrm{Gwdf}_G$.

To *first order logic with equality*, we will associate a class $\mathrm{Cs}_{\mathrm{suc,pred}}$ of set algebras, which has the following positive properties. The equational class $V$ generated by $\mathrm{Cs}_{\mathrm{suc,pred}}$ is axiomatizable by a finite schema of equations, which is even simpler than the one we give to logic without equality. Further, $\mathrm{Cs}_{\mathrm{suc,pred}}$ is an expansion of cylindric set algebras with some extra operations. These extra operations are again logical and simple set theoretic ones, just as in the previous case. Now, our finite schema axiomatizable $V$ has the property that if we omit the extra operations from its members then we get exactly the class $\mathrm{RCA}_\omega$ of representable cylindric algebras. Let us see what this means in more detail.

To the natural algebraic counterpart $\mathrm{RCA}_\omega$ of first order logic with equality, we add simple logical operations in such a way that, in the language of the so expanded class, we can write up a finite schema of axioms which completely describes the original operations of $\mathrm{RCA}_\omega$, but not necessarily the new operations themselves. More concretely, any model $\mathcal{A}$ of our finite schema is representable as a true set algebra in the following sense. $\mathcal{A}$ is isomorphic to some $\mathcal{B}$, where the elements of $\mathcal{B}$ are real relations, and all the old cylindric algebraic operations are the usual set theoretic ones on $\mathcal{B}$, but perhaps the new operations of $\mathcal{B}$ remain “abstract”.

At this point it might be of interest to note that, in the book [TaG87] (on page 62), the finitization problem was formulated in this weaker sense which we just described in connection with logic with equality. That is, Tarski and Givant ask for expanding the existing algebras with new logical operations in a way that there would be a finite set of axioms in the expanded language which

[8] These algebras are often called “square algebras”.

describes the old operations completely, but not necessarily the new ones. For a comparison of the two versions of the finitization problem see the series of remarks in [N93] beginning with Remark 2.

Summing up, we can give a solution to the more "ambitious" formulation of the schema version of the finitization problem for logic without equality, while for logic with equality we could solve only the above outlined Tarski-Givant style formulation of the finitization problem.

When defining the finite schema axiomatizable (finitizable for short) algebraic counterparts $\mathrm{Gwdf}_G, \mathrm{Cs}_{\mathrm{suc,pred}}$ of first order logic, one has a certain freedom of choice concerning exactly how these algebras will look like. We will explore this freedom of choice in a metatheory, asking ourselves the question which of these choices of $\mathrm{Gwdf}_G, \mathrm{Cs}_{\mathrm{suc,pred}}$ are workable, and which are already non-finitizable.

We would like to note that our approach to the case with equality is strongly related to William Craig's pioneering work summarized, e.g., in his book [Cr74]. Actually, our work can be viewed as a continuation of the approach initiated by Craig.

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The formulation of the results mentioned above together with the basic definitions can be found in Section 2. In detail, Subsection 2.1 contains preliminary material for the rest of the paper, while Subsections 2.2 and 2.3 are devoted to the finitization of first order logic without equality and with equality, respectively. Subsection 2.5 concerns applications to 'pure' logic. The proofs are given in Section 3. At the end of the paper there are further historical remarks.

Any proof which would rely on references not available to the reader (or would seem incomplete for any other reason) is available with all such missing data added from the authors.

# 2 Results

## 2.1 Preliminaries

Throughout this paper, $\omega$ denotes the set of all natural numbers. If $A$ and $B$ are arbitrary sets then ${}^AB$ denotes the set of all functions from $A$ into $B$. Thus, for any set $U$, ${}^\omega U$ denotes the set of all $\omega$–sequences over $U$. We often call the elements of ${}^\omega\omega$ *transformations* of $\omega$. A function $f \in {}^\omega\omega$ is called a *finite transformation* of $\omega$ if $f(i) = i$ for all but finitely many $i \in \omega$. Examples of finite transformations of $\omega$ are *transpositions* and *replacements*, which are defined as follows. For each $i, j \in \omega$, the transposition $[i, j]$ interchanges $i$ and $j$ and leaves all the other elements of $\omega$ fixed; the replacement $[i/j]$ maps $i$ to

$j$ and leaves all the other elements of $\omega$ fixed. It is easy to see that the set of all finite transformations of $\omega$ forms a semigroup with function composition $\circ$ as the semigroup operation. Clearly, this semigroup is generated by the set of all replacements and transpositions.

Examples of transformations of $\omega$ that are not finite are the successor suc and the predecessor pred defined as follows. For every $i \in \omega$, $\mathrm{suc}(i) = i + 1$ and $\mathrm{pred}(i) = \begin{cases} i-1 & \text{if } i \neq 0 \\ i & \text{if } i = 0 \end{cases}$.

If $H$ is a set then $H^*$ denotes the set of sequences of elements of $H$. If $T$ is a subset of ${}^{\omega}\omega$ then $[T]$ denotes the subsemigroup of $\langle {}^{\omega}\omega, \circ\rangle$ generated by $T$.

For any class $K$ of similar algebras, $\mathbf{I}K$, $\mathbf{H}K$, $\mathbf{S}K$, $\mathbf{P}K$ and $\mathbf{Up}K$ denote, respectively, the classes of isomorphic copies, homomorphic images, subalgebras, isomorphic copies of direct products and isomorphic copies of ultraproducts of members of $K$. $\mathbf{Sir}K$ denotes the class of subdirectly irreducible members of $K$.

Let $t$ and $t'$ be similarity types with $t' \subseteq t$. For any class $K$ of algebras of similarity type $t$, $\mathbf{Rd}_{t'}K$ denotes the class of algebras of similarity type $t'$ defined as follows. $\mathcal{A} \in \mathbf{Rd}_{t'}K$ if and only if $\mathcal{A}$ is the $t'$–reduct of some algebra in $K$. Informally speaking, we obtain the $t'$–reduct of an algebra by deleting those operations of it that are not in $t'$.

For any set $\Sigma$ of first order formulas of similarity type $t$, $Mod_t(\Sigma)$ denotes the collection of those models (of type $t$) in which $\Sigma$ is valid. When $t$ is known from context, we will often write $Mod(\Sigma)$ instead of $Mod_t(\Sigma)$.

For a class of algebras $K$, $Eq(K)$ denotes the set of equations valid in $K$ while $Qeq(K)$ is the set of valid quasi-equations of $K$.

If $f$ is a function and $U$ is a subset of its domain then $f{\restriction}_U$ denotes the restriction of $f$ to $U$.

For the pair of sets $U, V$, $U \subseteq_\omega V$ expresses the relation that $U$ is a *finite* subset of $V$.

As we outlined in the Introduction, we will introduce several (new) kinds of set algebras. These algebras will be defined to be Boolean algebras of relations augmented with some extra operations. The greatest element of a Boolean algebra (with perhaps some extra operations) is called its *unit*. Therefore, when speaking about a class $K$ of special Boolean algebras, then it is meaningful to talk about the class of unit elements of members of $K$. These will be called $K$–units.

Of these classes of set algebras, our first theorem (Theorem 2.5) will involve only the most important one called $\mathrm{Cdf}_G$. Actually, one of the main results of this paper is Theorem 2.5 (i), (ii). To understand that result it is enough to read only a small fraction of the definitions below. Namely, after reading Definition 2.1 (i), the $\mathrm{Cdf}_H$ part of Definition 2.2 and Definition 2.4 the reader can go directly to read Theorem 2.5.

`df-unit`

**Definition 2.1 ([HMTII] Def.3.1.2)**
Let $U$ be an arbitrary set.

(i). The set ${}^{\omega}U$ is called the *Cartesian space with base $U$ (and dimension $\omega$)*.

(ii). By a *generalized Cartesian space* we mean a disjoint union of Cartesian spaces. That is, a set is called a generalized Cartesian space (of dimension $\omega$) if it has the form $\bigcup_{i\in I} {}^{\omega}U_i$, where $I$ is a set, and for all distinct $i, j \in I$, $U_i \cap U_j = \emptyset$.

(iii). For every $p \in {}^{\omega}U$, we set

$$ {}^{\omega}U^{(p)} = \{q \in {}^{\omega}U \ : \ \{i \in \omega \ : \ q(i) \neq p(i)\} \text{ is finite}\}, $$

and we call ${}^{\omega}U^{(p)}$ the *weak Cartesian space (of dimension $\omega$) determined by $p$*. $U$ is called the *base* of ${}^{\omega}U^{(p)}$.

(iv). By a *generalized weak Cartesian space* or $\mathrm{Gws}_{\omega}$*–unit* we mean a disjoint union of weak Cartesian spaces. That is, a set is called a generalized weak Cartesian space of dimension $\omega$ if it has the form $\bigcup_{i\in I} {}^{\omega}U_i^{(p_i)}$ for some sets $I, U_i$ and functions $p_i \in {}^{\omega}U_i$ $(i \in I)$ such that for all distinct $i, j \in I$, ${}^{\omega}U_i^{(p_i)} \cap {}^{\omega}U_j^{(p_j)} = \emptyset$.

(v). We say that a $\mathrm{Gws}_{\omega}$–unit $\bigcup_{i\in I} {}^{\omega}U_i^{(p_i)}$ is *normal* if for all distinct $i, j \in I$, $U_i$ and $U_j$ are either disjoint or equal.

(vi). Finally, we say that a $\mathrm{Gws}_{\omega}$–unit $\bigcup_{i\in I} {}^{\omega}U_i^{(p_i)}$ is *compressed* if $U_i = U_j$ for all $i, j \in I$.

$\square$

## 2.2 Finitizing algebras of first order logic without equality

d-igwsg

**Definition 2.2** Suppose that $H$ is a fixed subset of ${}^{\omega}\omega$. An algebra is called a *full generalized weak $H$–cylindric set algebra with unit $V$* if it has the form

$$ \langle \mathcal{P}(V), \cup, -, \mathbf{s}_{\tau}, \mathbf{c}_k \rangle_{\tau\in H, k\in\omega} \ , $$

where

- $V$ is a $\mathrm{Gws}_{\omega}$–unit,
- for every $\tau \in H$ and $x \subseteq V$, $\mathbf{s}_{\tau}(x) = \{q \in V \ : \ q \circ \tau \in x\}$,

- $\mathbf{s}_\tau(V) = V$ whenever $\tau \in H$,
- $\mathbf{c}_k$ is the $k$-th cylindrification, that is, for any $x \subseteq V$,

$$\mathbf{c}_k(x) = \{q \in V \ : \ (\exists q' \in x)(\forall i \in \omega) \ i \neq k \Rightarrow q(i) = q'(i)\}.$$

If $\mathcal{A}$ is a full generalized weak $H$–cylindric set algebra with unit $V$, then $\mathcal{A}$ will often be denoted by $\underline{\mathcal{P}}(V)$, if $H$ is known from context.

By a *generalized weak $H$–cylindric set algebra* (or briefly, by a $\mathrm{Gwdf}_H$) we mean a subalgebra of a full generalized weak $H$–cylindric set algebra. The class of all generalized weak $H$–cylindric set algebras is denoted by $\mathrm{Gwdf}_H$[9]. The similarity type of $\mathrm{Gwdf}_H$ is denoted by $t_H$.

The classes of all algebras of $\mathrm{Gwdf}_H$, whose unit elements are a Cartesian space, a generalized Cartesian space, a normal $\mathrm{Gws}_\omega$–unit or a compressed $\mathrm{Gws}_\omega$–unit, are denoted by $\mathrm{Cdf}_H$, $\mathrm{Gdf}_H$, $\mathrm{Gwdf}_H^{\mathrm{nm}}$ or $\mathrm{Gwdf}_H^{\mathrm{cm}}$ respectively.

$\square$

d-111

**Definition 2.3** Let $H \subseteq {}^\omega\omega$, $V$ be a generalized Cartesian space[10], $x \subseteq V$. We define the operator $\mathbf{c}_{(\omega)} : \mathcal{P}(V) \to \mathcal{P}(V)$ as

$$\mathbf{c}_{(\omega)}x \stackrel{\mathrm{def}}{=} \{s \in V : (\exists q \in \mathbf{c}_0 x) \ s(0) = q(0)\}.$$

$\mathcal{A} = \langle \mathcal{B}, \mathbf{c}_{(\omega)} \rangle$ is called $\mathrm{Gdf}_H^{\mathrm{ec}}$ iff $\mathcal{B}$ is a $\mathrm{Gdf}_H$ with unit $V$. An $\mathcal{A} \in \mathrm{Gdf}_H^{\mathrm{ec}}$ is called $\mathrm{Cdf}_H^{\mathrm{ec}}$ if its unit element is a Cartesian space. $\square$

To help the intuition we point out that $\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}} = \mathbf{SP}\mathrm{Cdf}_H^{\mathrm{ec}}$ while in $\mathrm{Cdf}_H^{\mathrm{ec}}$ $\mathbf{c}_{(\omega)}x = 1$ iff $x > 0$ and $\mathbf{c}_{(\omega)}0 = 0$.

Adding the new operator $\mathbf{c}_{(\omega)}$ to $\mathrm{Gdf}_H$ amounts to adding the formation of *existential closure* $\varphi \to \exists\overline{x}\varphi$ to the logic being algebraized (where $\overline{x}$ contains all the variables occuring in $\varphi$). The abbreviation 'ec' in $\mathrm{Gdf}_H^{\mathrm{ec}}$ (and in $\mathrm{Cdf}_H^{\mathrm{ec}}$) refers to this fact.

d-g

**Definition 2.4** Throughout this paper, $G$ denotes the following set.

$$G \stackrel{\mathrm{def}}{=} \{[i/j] \ : \ i, j \in \omega\} \cup \{\mathrm{suc}, \mathrm{pred}\}.$$

$\square$

The following theorem summarizes the main results of this section.

csicsa1

[9]Gwdf refers to the name 'generalized weak diagonal free cylindric set algebra' used in [HMTI] to denote the $H$-free reduct of our class.

[10]The definition makes perfect sense for the case when $V$ is only a generalized weak Cartesian space (or even any subset of ${}^\omega U$ for some set $U$). For brevity, we will not discuss here the more general cases.

**Theorem 2.5** *(i).* $\mathbf{I}\mathrm{Cdf}_G$ *is a finite schema axiomatizable quasi-variety. That is, there is a finite schema* $\mathrm{Qx}_G$[11] *of quasi-equations such that*

$$\mathbf{I}\mathrm{Cdf}_G = Mod(\mathrm{Qx}_G)\,.$$

*(ii).* $\mathbf{I}\mathrm{Gdf}_G^{\mathrm{ec}}$ *is a finite schema axiomatizable discriminator variety.*

*(iii).* $\mathbf{H}\mathrm{Cdf}_G$ *is a variety axiomatized by a finite set* $\mathrm{Ax}_G$[12] *of equation schemas. Thus,* $\mathrm{Ax}_G$ *axiomatizes* $Eq(\mathrm{Cdf}_G)$ *i.e.*

$$\mathrm{Cdf}_G \models e \quad \textit{iff} \quad \mathrm{Ax}_G \vdash e,$$

*for any equation* $e$ *in the language of* $\mathrm{Cdf}_G$.

The statements of this theorem are restated in Theorems 2.7, 2.15, 2.16 below in a more detailed form. In the following part of this section we present the finite sets of equations and quasi-equations that axiomatize $\mathbf{H}\mathrm{Cdf}_G$, $\mathbf{I}\mathrm{Cdf}_G$ and $\mathbf{I}\mathrm{Gdf}_G^{\mathrm{ec}}$. Further, the related weak classes are examined and a detailed characterization is given. (Theorem 2.7) Finally, we show that many of the results remain valid if we replace the set $G$ by an arbitrary set $H$ of transformations provided that $H$ satisfies certain properties. (Theorems 2.11, 2.19, 2.20)

d-ax

**Definition 2.6** We define the finite set $\mathrm{Ax}_G$ of axiom schemas to be the union of sets (1)–(6) below.

(1). A finite set of equations axiomatizing Boolean algebras.

(2). For every $\tau \in G$, the following axioms stating that $\mathbf{s}_\tau$ is a Boolean homomorphism:
$\mathbf{s}_\tau(x \vee y) \;=\; \mathbf{s}_\tau(x) \vee \mathbf{s}_\tau(y), \qquad \mathbf{s}_\tau(-x) \;=\; -\mathbf{s}_\tau(x).$

(3). The axioms governing $\mathbf{s}_{[i/j]}$'s: for all distinct $i, j, k \in \omega$,
$\mathbf{s}_{[j/i]}\mathbf{s}_{[i/j]}x \;=\; \mathbf{s}_{[j/i]}x, \qquad \mathbf{s}_{[j/k]}\mathbf{s}_{[i/j]}x \;=\; \mathbf{s}_{[j/k]}\mathbf{s}_{[i/k]}x.$

(4). The axioms describing the connections between $\mathbf{s}_{[i/j]}$'s, $\mathbf{s}_{\mathrm{suc}}$ and $\mathbf{s}_{\mathrm{pred}}$:
$$\begin{aligned} \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}x &= x,\\ \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{\mathrm{pred}}x &= \mathbf{s}_{[0/1]}x,\\ \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{[i/j]}x &= \mathbf{s}_{[\mathrm{suc}(i)/\mathrm{suc}(j)]}\mathbf{s}_{\mathrm{suc}}x \quad \text{for all distinct } i, j \in \omega. \end{aligned}$$

(5). Two axiom schemas about $\mathbf{c}_i$ and the Boolean operations: for all $i \in \omega$,
$x \leq \mathbf{c}_i x, \quad \mathbf{c}_i(x \vee y) = \mathbf{c}_i x \vee \mathbf{c}_i y.$

(6). The axiom schemas describing the connection between $\mathbf{s}_{[i/j]}$'s and $\mathbf{c}_k$'s: for all distinct $i, j, k \in \omega$,
$$\begin{aligned} \mathbf{s}_{[i/j]}\mathbf{c}_i x &= \mathbf{c}_i x,\\ \mathbf{c}_i\mathbf{s}_{[i/j]}x &= \mathbf{s}_{[i/j]}x,\\ \mathbf{c}_k\mathbf{s}_{[i/j]}x &= \mathbf{s}_{[i/j]}\mathbf{c}_k x. \end{aligned}$$

[11] $\mathrm{Qx}_G$ is presented in Definition 2.14 below.
[12] $\mathrm{Ax}_G$ is presented in Definition 2.6 below.

th-main

**Theorem 2.7 (characterization of $\mathbf{I}\mathrm{Gwdf}_G$ and related classes)** *Let $G$ and $\mathrm{Ax}_G$ be as in Definitions 2.4 and 2.6. Then statements (1)–(4) below hold.*

*(1).* $\mathbf{I}\mathrm{Gwdf}_G$ *is axiomatizable by the finite set* $\mathrm{Ax}_G$ *of equation schemas i.e.* $\mathbf{I}\mathrm{Gwdf}_G = Mod(\mathrm{Ax}_G)$.

*(2).* $\mathbf{I}\mathrm{Gwdf}_G$ *is a variety.*

*(3).* $\mathbf{I}\mathrm{Gwdf}_G = \mathbf{I}\mathrm{Gwdf}_G^{\mathrm{nm}} = \mathbf{I}\mathrm{Gwdf}_G^{\mathrm{cm}} = \mathbf{H}\mathrm{Cdf}_G$.

*(4). The set $Eq(\mathrm{Cdf}_G)$ of equations is axiomatizable by the finite set $\mathrm{Ax}_G$ of equation schemas i.e.*

$$\mathrm{Cdf}_G \models e \quad \textit{iff} \quad \mathrm{Ax}_G \vdash e,$$

*for any equation $e$ in the language of* $\mathrm{Cdf}_G$.

**Proof** (2) is a corollary of (1) while (1) is proved in Subsections 3.1 – 3.3.

To prove (3) we note that statement (3) is a special case of Theorem 2.11 (ii), (iii). To see that the later is applicable, we need to show that the successor, the predecessor and all the substitutions are bounded transformations of $\omega$ and they have right quasi-inverses in $[G]$. (Consult Definition 2.9 for the notion of bounded, quasi-bounded and quasi-invertible transformation.) It easy to verify that the given transformations have the required properties using facts as $\mathrm{suc} \circ \mathrm{pred}(i) = \mathrm{pred} \circ \mathrm{suc}(i) = i$ whenever $i > 0$.

Finally, (4) follows from (3) and (1). ∎

Later, in Remark 2.22, we show that for a slightly modified version of $\mathbf{I}\mathrm{Gwdf}_G$ (where only the 0-th cylindrification is a basic operation, the other cylindrifications are terms of the reduced language) similar results can be achieved.

In the proof of the previous theorem we referred to a metatheorem (2.11). There we show that if $H$ satisfies certain requirements then the classes $\mathbf{I}\mathrm{Gwdf}_H$ and $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}}$ actually coincide with $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$.

nt-approx

**Notation 2.8** We will often deal with pairs of sequences that agree on all but finitely many coordinates. If $q_1$ and $q_2$ are sequences then

$$q_1 \approx q_2 \text{ iff } |\{i \in \omega \ : \ q_1(i) \neq q_2(i)\}| < \omega.$$

Clearly, $\approx$ is an equivalence relation.

d-bounded

**Definition 2.9** Let $f, g \in {}^{\omega}\omega$ be arbitrary.

- $f$ is said to be *bounded* if it does not send infinitely many elements of $\omega$ into the same place, that is, if $|f^{-1}(i)| < \omega$ for all $i \in \omega$.

- $f$ is said to be *quasi-bounded* if there is a $k \in \omega$ for which $|f^{-1}(f(k))| < \omega$.

- $g$ is called the *right (left) quasi-inverse* of $f$ if $f \circ g \approx Id$ ($g \circ f \approx Id$), where $Id$ is the identity transformation of $\omega$. $f$ is said to be *right (left) quasi-invertible* in $H \subseteq {}^{\omega}\omega$ if it has a left (right) quasi-inverse in $H$.

$\square$

d-rich

**Definition 2.10 (rich semigroup)**
Suppose that $i, j, k \in \omega$, $A \subseteq \omega$, $f, g \in {}^{\omega}\omega$. Then $f[i/j]$ and $f[A/g]$ are transformation of $\omega$ defined as

$$f[i/j](k) = \begin{cases} j & \text{if } k = i \\ f(k) & \text{otherwise} \end{cases} \qquad f[A/g](k) = \begin{cases} g(k) & \text{if } k \in A \\ f(k) & \text{otherwise.} \end{cases}$$

(Note that $f[i/j]$ and $f \circ [i/j]$ are different transformations.)[13]
A semigroup $Q \subseteq {}^{\omega}\omega$ is called *rich* if it satisfies the following two conditions:

- $(\forall i, j \in \omega)(\forall f \in Q)\ f[i/j] \in Q$,

- $(\exists \hat{s}, \hat{p} \in Q)(\forall f \in Q)$
  $\hat{p} \circ \hat{s} = Id \wedge Rng(\hat{s}) \neq \omega \wedge (\hat{s} \circ f \circ \hat{p})[(\omega - Rng(\hat{s}))/Id] \in Q$.

$\square$

th-nc

**Theorem 2.11** *Let $H$ be an arbitrary set of transformations.*

*(i).* $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}} = \mathbf{HSP}\mathrm{Cdf}_H$ *is a variety. Thus* $Eq(\mathrm{Cdf}_H) = Eq(\mathrm{Gwdf}_H^{\mathrm{nm}})$.

*(ii). If for all $\sigma \in H$, $\sigma$ is bounded then*

$$\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}} = \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}} = \mathbf{H}\mathrm{Cdf}_H.$$

*(iii). If for all $\sigma \in H$, $\sigma$ is quasi-bounded and right quasi-invertible in $[H]$, then*

$$\mathbf{I}\mathrm{Gwdf}_H = \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}.$$

*(iv). Assume that $[H]$ is a finite schema presented*[14] *rich semigroup. Then,* $\mathbf{I}\mathrm{Gwdf}^{\mathrm{nm}}$ *is axiomatizable by a finite schema $\Sigma$ of equations.*

(i) and (iv) are quoted from [Sa92] (Theorem 2.1 (2)), the proof of (ii) and (iii) can be found in Subsection 3.3.

The example below shows that the requirement given in Theorem 2.11 (iii) cannot be omitted. In some cases $Eq(\mathrm{Gwdf}_H) \neq Eq(\mathrm{Gwdf}_H^{\mathrm{nm}})$.

ex0

**Example 2.12** *Let* doub $\in {}^{\omega}\omega$ *be defined by* $\mathrm{doub}(i) = 2i$ *for every* $i \in \omega$, *let* $H = \{\mathrm{doub}\}$ *and let* $e$ *be the equation* $\mathbf{c}_0\mathbf{s}_{\mathrm{doub}}x = \mathbf{s}_{\mathrm{doub}}\mathbf{c}_0 x$.

*Then*

$$\mathrm{Gwdf}_H^{\mathrm{nm}} \models e \quad \textit{but} \quad \mathrm{Gwdf}_H \not\models e.$$

**Proof** Let $V = {}^{\omega}2^{(\langle 0,1,0,1,\ldots\rangle)} \cup {}^{\omega}3^{(\langle 0,0,0,0,\ldots\rangle)}$. Then $\underline{\mathcal{P}}(V) \in \mathrm{Gwdf}_H \setminus \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$ since $e$ fails in $\underline{\mathcal{P}}(V)$ (choose $x = \{\langle 2,0,0,0,\ldots\rangle\}$) but $e$ is satisfied in every algebra of $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$. ∎

We turn our attention to the study of classes of algebras with square (Cartesian space) unit elements. We prove that the class $\mathbf{I}\mathrm{Cdf}_G$ is a finite schema axiomatizable quasi-variety and is strictly smaller than $\mathbf{I}\mathrm{Gwdf}_G$.

ex1

**Example 2.13** *Let* $q$ *be the quasi-equation* $\mathbf{s}_{\mathrm{pred}}x = -x \Rightarrow 0 = 1$. *Then*[15]

$$\mathrm{Gdf}_G \models q \quad \textit{but} \quad \mathbf{I}\mathrm{Gwdf}_G \not\models q.$$

**Proof** In a $G$–cylindric generalized set algebra we can pick a constant sequence $s$ in the universe. For any element $x$ of the algebra, the chosen $s$ is in $x$ if and only if it is in $\mathbf{s}_{\mathrm{pred}}x$. It implies that $q$ is in the symmetric difference of $-x$ and $\mathbf{s}_{\mathrm{pred}}x$ so $\mathbf{s}_{\mathrm{pred}}x \neq -x$.

On the other hand, if we put $V = {}^{\omega}2^{(\langle 1,0,1,0,\ldots\rangle)} \cup {}^{\omega}2^{(\langle 0,1,0,1,\ldots\rangle)}$ then $e$ fails in $\underline{\mathcal{P}}(V)$. (Put $x$ to be ${}^{\omega}2^{(\langle 0,1,0,1,\ldots\rangle)}$ and for that $x$, $\mathbf{s}_{\mathrm{pred}}x = -x$.) ∎

d-qx1

**Definition 2.14** (i). Let $n \in \omega$. We define $\mathbf{c}_{(n)}x \stackrel{\mathrm{def}}{=} \mathbf{c}_0\mathbf{c}_1 \ldots \mathbf{c}_{n-1}x$. Further, $\mathbf{s}_{n\text{-pred}}x \stackrel{\mathrm{def}}{=} \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[0/1]}\mathbf{s}_{[1/2]} \ldots \mathbf{s}_{[n-1/n]}x$. Hence, $\mathbf{s}_{n\text{-pred}}$ is a term function[16].

---

[13] $f[i/j]$ can be distinguished from $f[A/g]$ by noticing that always $|g| \geq \omega > i$.

[14] We use the notion of a finite schema presented semigroup in the usual sense. For completeness we will recall this concept in Definition 3.7 in the 'Proofs' section.

[15] For completeness we note connections of Example 2.13 and the theory of polyadic algebras ($\mathrm{PA}_\omega$'s) and PA's with equality ($\mathrm{PEA}_\omega$'s) without recalling these two classes. We use the notation of [HMTII]. Although $\mathrm{Ax}_G \not\models q$, we have $\mathrm{PA}_\omega \models q$ (e.g. by the Daigneault–Monk theorem). On the other hand, there are equations $e$ such that $\mathrm{PEA}_\omega \not\models e$ but $\mathrm{RPEA}_\omega \models e$, cf. Németi-Sági [NS95].

[16] In set algebras, the term function $\mathbf{s}_{n\text{-pred}}$ coincides with $\mathbf{s}_\tau$ for $\tau = \langle 0, 1, \ldots, n-1, n-1, n, n+1, \ldots\rangle$, that is,

$$\tau(i) = (\mathrm{pred}\lceil_{\omega - n}) \cup (Id\lceil_n) = \begin{cases} i & \text{if } i < n \\ \mathrm{pred}(i) & \text{otherwise.} \end{cases}$$

(ii). We define $\mathrm{Qx}_G$ to be the following schema of quasi-equation.

$$\bigcup_{i\leq m} \mathbf{c}_{(n_i)} x_i = 1 \wedge \bigwedge_{i\leq m} \mathbf{s}_{n_i\text{-pred}} x_i \leq -x_i \Rightarrow 0 = 1,$$

for $n_0, n_1, \ldots, n_m, m \in \omega$.

□ cj1

**Theorem 2.15**

$$\mathbf{I}\mathrm{Cdf}_G = Mod(\mathrm{Ax}_G \cup \mathrm{Qx}_G).$$

**Proof** [17] The general Theorem 2.19 should be applied to $H = G$. Qx can be simplified to $\mathrm{Qx}_G$ and Theorem 2.7 implies that $Eq(\mathbf{I}\mathrm{Gwdf}_G)$ follows from $\mathrm{Ax}_G$. It is left to the reader to show the first implication. ∎

We conjecture that $\mathrm{Qx}_G$ can be *considerably* simplified. Cf. e.g. the definition of $\mathrm{Ax}_2$ and proof of Proposition 1 in Andréka [A91] part II.

t-nemeti

**Theorem 2.16** $\mathbf{I}\mathrm{Gdf}_G^{\mathrm{ec}}$ *is a finite schema axiomatizable discriminator variety. Furthermore,* $\mathbf{I}\mathrm{Gdf}_G^{\mathrm{ec}} = Mod(\Sigma)$ *where* $\Sigma$ *is* $\mathrm{Ax}_G$ *together with the following set of axiom schemas:*

*(i). axioms stating that* $\mathbf{c}_{(\omega)}$ *is a complemented closure operator, i.e.* $x \leq \mathbf{c}_{(\omega)} x$, $\mathbf{c}_{(\omega)}(x \cup y) = \mathbf{c}_{(\omega)} x \cup \mathbf{c}_{(\omega)} y$, $\mathbf{c}_{(\omega)} \mathbf{c}_{(\omega)} x = \mathbf{c}_{(\omega)} x$, $\mathbf{c}_{(\omega)} - \mathbf{c}_{(\omega)} x = -\mathbf{c}_{(\omega)} x$,

*(ii). axiom schemas stating that* $\mathbf{c}_{(\omega)}$ *majorizes all the other operators, i.e.* $\mathbf{c}_{(\omega)} x \geq f(x)$ *for all* $f \in \{\mathbf{c}_i, \mathbf{s}_\tau : i \in \omega, \tau \in G\}$,

*(iii). axioms corresponding to* $\mathrm{Qx}_G$*: for all* $n_0, n_1, \ldots, n_m, m \in \omega$,
$-\mathbf{c}_{(\omega)} - (\bigcup_{i\leq m} \mathbf{c}_{(n_i)} x_i) \leq \bigcup_{i\leq m} \mathbf{c}_{(\omega)}(x_i \cap \mathbf{s}_{n_i\text{-pred}}(x_i))$.

$\mathbf{I}\mathrm{Cdf}_G^{\mathrm{ec}}$ *is the class of subdirectly irreducible members of* $\mathbf{I}\mathrm{Gdf}_G^{\mathrm{ec}}$.

Theorem 2.16 is a special case of the general Theorem 2.20.

ques1

**Open question 2.17** Find simpler axiom schemas aximatizing $\mathbf{I}\mathrm{Cdf}_G$ and $\mathbf{I}\mathrm{Gdf}_G^{ec}$, respectively.

Below we show that $\mathbf{I}\mathrm{Cdf}_H$ is a quasi-variety, provided that $H$ satisfies some properties. Furthermore, we give a set Qx of quasi-equations that axiomatize $\mathbf{I}\mathrm{Cdf}_H$ over $\mathbf{I}\mathrm{Gwdf}_H$. Results 2.19 and 2.20 were obtained jointly with István Németi.

d-qx2

**Definition 2.18** (i). Let $n \in \omega$, $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\} \subseteq \omega$. We recall from [HMTI] that $\mathbf{c}_{(\Gamma)} x \stackrel{\mathrm{def}}{=} \mathbf{c}_{\gamma_0} \ldots \mathbf{c}_{\gamma_{n-1}} x$.

[17] In [Sa87] a simple proof is given to the fact that $\mathbf{I}\mathrm{Cdf}_G$ is closed under **SPUp**, i.e. it forms a quasi-variety. Finite schema axiomatizability is not proved there.

(ii). We define Qx to be

$$\bigcup_{i\leq m} \mathbf{c}_{(\Gamma_i)} x_i = 1 \wedge \bigwedge_{i\leq m} \bigcap_{\tau\in H_i} \mathbf{s}_\tau(x) = 0 \Rightarrow 0 = 1,$$

for $m \in \omega, \Gamma_0, \ldots, \Gamma_{n-1} \subseteq_\omega \omega$, $H_i \subseteq_\omega [H]$ such that $(\tau \in H_i \Rightarrow \tau\lceil_{\Gamma_i} = Id\lceil_{\Gamma_i})$.

$\square$

**Theorem 2.19** *Let $H \subseteq {}^\omega\omega$ such that $[H]$ contains all finite transformations and every element of $H$ is bounded, left and right quasi-invertible in $H$. Then* t-csax

*(1).* $\mathbf{I}\mathrm{Cdf}_H$ *is a quasi-variety,*

*(2).* $\mathbf{I}\mathrm{Cdf}_H$ *is axiomatized by* Qx *over* $\mathrm{Gwdf}_H$*, that is,* $\mathrm{Qx} \cup Eq(\mathrm{Gwdf}_H)$ *axiomatizes* $\mathbf{I}\mathrm{Cdf}_H$.

(1) is a consequence of (2) while (2) is proved in Subsection 3.4.

t-222

**Theorem 2.20** *Assume that $H \subseteq {}^\omega\omega$ is such that $\mathbf{I}\mathrm{Cdf}_H$ is a quasi-variety. Then $\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}}$ is a discriminator variety whose subdirectly irreducible members are exactly the elements of $\mathbf{I}\mathrm{Cdf}_H^{\mathrm{ec}}$.*

*If all the assumptions of Theorem 2.19 hold for $H$ then $\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}}$ is axiomatized over $\mathbf{I}\mathrm{Gwdf}_H$ by the axioms (i) and (ii) of Theorem 2.16 together with*

*(iii').* *axioms corresponding to* Qx*: for all $m \in \omega, \Gamma_0, \ldots, \Gamma_{n-1} \subseteq_\omega \omega$, $H_i \subseteq_\omega [H]$ such that $(\tau \in H_i \Rightarrow \tau\lceil_{\Gamma_i} = Id\lceil_{\Gamma_i})$ we state*

$$-\mathbf{c}_{(\omega)} - (\bigcup_{i\leq m} \mathbf{c}_{(\Gamma_i)} x_i) \leq \bigcup_{i\leq m} \mathbf{c}_{(\omega)} (\bigcap_{\tau\in H_i} \mathbf{s}_\tau x).$$

**Proof** Let $\Sigma$ be the set equations valid in $\mathrm{Gdf}_H^{\mathrm{ec}}$, that is, $\Sigma = Eq(\mathrm{Gdf}_H^{\mathrm{ec}})$. We want to prove $\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}} = Mod(\Sigma)$. $\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}} \subseteq Mod(\Sigma)$ is obvious.

For proving the other direction, it is enough to show that

$$\mathbf{Sir}(Mod(\Sigma)) \subseteq \mathbf{I}\mathrm{Cdf}_H^{\mathrm{ec}} \subseteq \mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}}.$$

Therefore, assume $\mathcal{A} \in \mathbf{Sir}(Mod(\Sigma))$. $\mathcal{A} \models x \neq 0 \to \mathbf{c}_{(\omega)} x = 1$ can be verified as follows. Suppose for contradiction that $\mathbf{c}_{(\omega)} x \neq 1$ for some element $x$ of $\mathcal{A}$. Since axioms (i) and (ii) certainly hold in $\mathrm{Gdf}_H^{\mathrm{ec}}$, they are in $\Sigma$. But using axioms (i) and (ii) we can prove that the relativization with $x$ and that with $-x$ decomposes $\mathcal{A}$ to the subdirect product of the two images.

Let $\mathcal{A} = \langle \mathcal{B}, \mathbf{c}_{(\omega)} \rangle$. If $\bigwedge_{i<n} \tau_i = \sigma_i \to \tau = \sigma$ is a quasi-equation valid in $\mathrm{Cdf}_H$ then its equational translation $\mathbf{c}_{(\omega)}(\tau \oplus \sigma) \leq \bigcup_{i<n} \mathbf{c}_{(\omega)}(\tau_i \oplus \sigma_i)$ ($\oplus$ is the symmetric difference operator derivable from the Boolean connectives) holds in $\mathrm{Cdf}_H^{\mathrm{ec}}$ and

also in $\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}} = \mathbf{SP}\mathrm{Cdf}_H^{\mathrm{ec}}$ so it is in $\Sigma$. Further, if the equational translation of a quasi-equation is valid in $\mathcal{A}$ then the original quasi-equation is satisfied in $\mathcal{B}$. This proves that $Qeq(\mathrm{Cdf}_H)$ holds in $\mathcal{B}$ so $\mathcal{B} \in \mathbf{I}\mathrm{Cdf}_H$ using the assumption that $\mathbf{I}\mathrm{Cdf}_H$ is a quasi-variety. This immediately yields $\mathcal{A} \in \mathbf{I}\mathrm{Cdf}_H^{\mathrm{ec}}$. Since every member of $\mathbf{I}\mathrm{Cdf}_U^{\mathrm{ec}}$ is subdirectly irreducible we get $\mathbf{Sir}(\mathbf{I}\mathrm{Gdf}_H^{\mathrm{ec}}) = \mathbf{I}\mathrm{Cdf}_H^{\mathrm{ec}}$

The second part of the theorem follows from the first and from Theorem 2.19. ∎

ques

**Open question 2.21** The authors would like to know if the quasi-variety $\mathrm{Gdf}_{G^\times}$ in Sain [Sa87], [Sa93] (cf. e.g. Section 4 in [Sa93]) admits a finite schema axiomatization (perhaps by the techniques of 2.15 or 2.16 herein).

rm-allci

**Remark 2.22** In fixing the basic operations, one of the motivations could have been that none of the operations shall be definable by any term of the others. One can easily notice that in $\mathrm{Gwdf}_G$ this is not the case. For any $i \in \omega, i > 0$, $\mathbf{c}_i$ can be defined by a term of $\mathbf{c}_0$ and the other operations as follows.

$$\mathbf{c}_i x = \mathbf{s}_{[i,0]}\mathbf{c}_0\mathbf{s}_{[i,0]}x,$$

where $\mathbf{s}_{[i,j]}$ abbreviates the term $\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[0/\mathrm{suc}(j)]}\mathbf{s}_{[\mathrm{suc}(j)/\mathrm{suc}(i)]}\mathbf{s}_{[\mathrm{suc}(i)/0]}\mathbf{s}_{\mathrm{suc}}$.

If we had chosen the smaller set $\{\cup, -, \mathbf{s}_\tau, \mathbf{c}_0\}_{\tau \in H} \stackrel{\mathrm{def}}{=} t_H^* \subseteq t_H$ to be the set of basic operations then the class $\mathrm{Gwdf}_H^*$ of representable algebras would be defined as

$$\mathrm{Gwdf}_H^* = \mathbf{S}\left\{\langle \mathcal{P}(V), \cup, -, \mathbf{s}_\tau, \mathbf{c}_0\rangle_{\tau\in H} \ : \begin{array}{l} V \text{ satisfies the conditions} \\ \text{listed in Definition 2.2} \end{array}\right\}.$$

With this modified class in place of $\mathrm{Gwdf}_G$, Theorem 2.7 remains true provided that the set $\mathrm{Ax}_G$ is replaced with the following bigger set.

For all distinct $i, j, k, l \in \omega$ we state:

('1). Same as (1).

('2). Same as (2).

('3). (3) together with the following 3 schemas:

$$\begin{array}{rclrcl} \mathbf{s}_{[i/k]}\mathbf{s}_{[i/j]}x &=& \mathbf{s}_{[i/j]}x, & \mathbf{s}_{[k/j]}\mathbf{s}_{[i/j]}x &=& \mathbf{s}_{[i/j]}\mathbf{s}_{[k/j]}x, \\ &&& \mathbf{s}_{[k/l]}\mathbf{s}_{[i/j]}x &=& \mathbf{s}_{[i/j]}\mathbf{s}_{[k/l]}x. \end{array}$$

('4). Same as (4).

('5). (5) restricted to $i = 0$, that is, $x \leq \mathbf{c}_0 x$, $\quad \mathbf{c}_0(x \vee y) = \mathbf{c}_0 x \vee \mathbf{c}_0 y$.

('6).

$$\begin{array}{rcl} \mathbf{s}_{[0/1]}\mathbf{c}_0 x &=& \mathbf{c}_0 x, \\ \mathbf{c}_0\mathbf{s}_{[0/1]}x &=& \mathbf{s}_{[0/1]}x, \\ \mathbf{c}_0\mathbf{s}_{[i/j]}x &=& \mathbf{s}_{[i/j]}\mathbf{c}_0 x. \end{array}$$

## 2.3 Finitizing algebras of first order logic with equality

As we said in the Introduction, the present subsection is strongly related to Craig [Cr74], and can be viewed as a continuation of the approach initiated therein. Indeed, it is very likely that Theorem 2.27 below could be proved from Craig's results. However, we show a generalization of these results and in later work we intend to generalize it in various directions, a part of which was indicated in [Sa87] and [Sa92].

df-gwsp

**Definition 2.23** Suppose that $H$ is a fixed subset of ${}^{\omega}\omega$. An algebra is called a *generalized weak $H$–cylindric set algebra with equality* (or briefly, a $\mathrm{Gws}_H$) if it is of the form

$$\langle A, \cup, -, \mathbf{s}_\tau, \mathbf{c}_i, \mathbf{d}_{ij}\rangle_{\tau\in H, i,j\in\omega}$$

where $\langle A, \cup, -, \mathbf{s}_\tau, \mathbf{c}_i, \rangle_{\tau\in H, i\in\omega} \in \mathrm{Gwdf}_H$ and the constants $\mathbf{d}_{ij}$ are the set theoretic diagonal elements (defined e.g. in [HMTII] Def. 3.1.1: $\mathbf{d}_{ij} = \{q \in 1 : q(i) = q(j)\}$ — where 1 is the unit element of the algebra).

The class of all generalized weak $H$–cylindric set algebras with equality is denoted by $\mathrm{Gws}_H$. The similarity type of $\mathrm{Gws}_H$ is denoted by $t^{=}_{H}$. ◻

Similarly to the case where equality was not present, we define subclasses of $\mathrm{Gws}_H$ on the bases of differences between the unit elements of the algebras.

df-norm2

**Definition 2.24** Let $H$ be a subset of ${}^{\omega}\omega$. The class of all algebras of $\mathrm{Gws}_H$, whose unit elements are Cartesian spaces, (generalized Cartesian spaces) is denoted by $\mathrm{Cs}_H$ ($\mathrm{Gs}_H$ respectively). ◻

The class $\mathrm{Gs}_\emptyset$ is the class of representable cylindric algebras. In the literature (e.g. in [HMTII]) it is often denoted by $\mathrm{Gs}_\omega$ or $\mathrm{RCA}_\omega$. Further parallel notations are $\mathrm{Gws}_\emptyset = \mathrm{Gws}_\omega$, $\mathrm{Cs}_\emptyset = \mathrm{Cs}_\omega$. Since the $t^{=}_{\emptyset}$–reduct of a $\mathrm{Gws}_H$ is a cylindric algebra, we introduce the notation $\mathbf{Rd}_{\mathrm{CA}}$ to be $\mathbf{Rd}_{t^{=}_{\emptyset}}$.

csicsa2

**Theorem 2.25** *(i).* $\mathbf{HSP}\mathrm{Cs}_{\mathrm{suc,\ pred}}$ *is a variety axiomatized by a finite set* $\mathrm{Ax}^{=}$ *of equation schemas. Thus,* $\mathrm{Ax}^{=}$ *axiomatizes* $Eq(\mathrm{Cs}_{\mathrm{suc,\ pred}})$ *i.e.*

$$\mathrm{Cs}_{\mathrm{suc,\ pred}} \models e \quad \textit{iff} \quad \mathrm{Ax}^{=} \vdash e,$$

*for any equation $e$ in the language of* $\mathrm{Cs}_{\mathrm{suc,\ pred}}$.

*(ii).* $\mathbf{S}\,\mathbf{Rd}_{\mathrm{CA}}\mathbf{HSP}\mathrm{Cs}_{\mathrm{suc,\ pred}} = \mathrm{RCA}_\omega$.

Theorem 2.25 above is a corollary of Theorem 2.27. In the latter we give a more detailed description of the classes $\mathrm{Cs}_{\mathrm{suc,\ pred}}$ and $\mathrm{Gws}_{\mathrm{suc,\ pred}}$ i.e. we give a set of axioms (Definition 2.26) that axiomatizes them.

Results 2.15 and 2.16 have the corollary that

*$Qeq(\mathrm{Cdf}_G)$ and $Eq(\mathrm{Cdf}^{\mathrm{ec}}_G)$ are finite schema axiomatizable.* (*)

In view of the results above, it seems natural to expand some generalization of (*) for the case with equality. This was proved impossible in Craig [Cr74] Chapter 11. It is proved there (using a result of McKenzie) that neither $Qeq(\mathrm{Cs}_{\mathrm{suc,\ pred}})$ nor $Eq(\mathrm{Gs}^{\mathrm{ec}}_{\mathrm{suc,\ pred}})$ is recursively enumerable.

Moreover, the following seems to be an even sharper contrast with 2.15 and 2.16. Let $K$ be the class of algebras that are expansions of elements of $\mathrm{Cs}_{\mathrm{suc,\ pred}}$ with the operation $\mathbf{c}_{(\omega\setminus 2)}$ defined as follows:

$$\mathbf{c}_{(\omega\setminus 2)}x \stackrel{\mathrm{def}}{=} \{s \in V \ : \ (\exists s' \in x)\ s(0) = s'(0), s(1) = s'(1)\}.$$

It is proved in [NS95] that $Eq(K)$ is $\Pi^1_1$ complete (i.e. highly non-computational).

d-ax+

**Definition 2.26** We define the finite set $\mathrm{Ax}^=$ of axiom schemas to be the union of sets (1)–(3) below.

(1). The set of axiom schemas characterizing cylindric algebras with equality (this set is quoted from [HMTI] Def. 1.1.1 p.162): for all $i, j, k \in \omega$,
a finite set of equations axiomatizing Boolean algebras

| | |
|---|---|
| $\mathbf{c}_i 0 = 0$, | $x \leq \mathbf{c}_i x$, |
| $\mathbf{c}_i(x \wedge \mathbf{c}_i y) = \mathbf{c}_i x \wedge \mathbf{c}_i y$, | $\mathbf{c}_i \mathbf{c}_j x = \mathbf{c}_j \mathbf{c}_i x$, |
| $\mathbf{d}_{ii} = 1$, | |
| $\mathbf{d}_{ij} = \mathbf{c}_k(\mathbf{d}_{ik} \wedge \mathbf{d}_{kj})$ | whenever $k \notin \{i, j\}$, |
| $\mathbf{c}_i(\mathbf{d}_{ij} \wedge x) \wedge \mathbf{c}_i(\mathbf{d}_{ij} \wedge -x) = 0$ | whenever $i \neq j$. |

(2). The axioms stating that $\mathbf{s}_{\mathrm{suc}}$ and $\mathbf{s}_{\mathrm{pred}}$ are Boolean homomorphisms:

$$\begin{array}{rclcrcl} \mathbf{s}_{\mathrm{suc}}(x \vee y) &=& \mathbf{s}_{\mathrm{suc}}(x) \vee \mathbf{s}_{\mathrm{suc}}(y), && \mathbf{s}_{\mathrm{suc}}(-x) &=& -\mathbf{s}_{\mathrm{suc}}(x), \\ \mathbf{s}_{\mathrm{pred}}(x \vee y) &=& \mathbf{s}_{\mathrm{pred}}(x) \vee \mathbf{s}_{\mathrm{pred}}(y), && \mathbf{s}_{\mathrm{pred}}(-x) &=& -\mathbf{s}_{\mathrm{pred}}(x). \end{array}$$

(3). The axioms describing the connection between $\mathbf{s}_{\mathrm{suc}}$, $\mathbf{s}_{\mathrm{pred}}$ and the other cylindric operations:

$$\begin{array}{rcll} \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}x &=& x, & \\ \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{\mathrm{pred}}x &=& \mathbf{c}_0(\mathbf{d}_{01} \wedge x), & \mathbf{s}_{\mathrm{suc}}(x - \mathbf{c}_0(-\mathbf{d}_{01})) = x - \mathbf{c}_0(-\mathbf{d}_{01}). \\ \mathbf{s}_{\mathrm{suc}}\mathbf{c}_i x &=& \mathbf{c}_{\mathrm{suc}(i)}\mathbf{s}_{\mathrm{suc}}x, & \\ \mathbf{s}_{\mathrm{suc}}\mathbf{d}_{ij} &=& \mathbf{d}_{\mathrm{suc}(i)\mathrm{suc}(j)}. & \end{array}$$

□

th-rca

**Theorem 2.27** *(1).* $\mathrm{RCA}_\omega = \mathbf{SRd}_{\mathrm{CA}} Mod(\mathrm{Ax}^=$ *without (4)) that is, the equations valid in* $\mathrm{RCA}_\omega$ *are exactly those consequences of* $\mathrm{Ax}^=$ *without (4) that do not use the new operations* $\mathbf{s}_{\mathrm{suc}}, \mathbf{s}_{\mathrm{pred}}$.

*(2).* $\mathbf{H}\mathrm{Gws}_{\mathrm{suc,\ pred}} = Mod(\mathrm{Ax}^=)$.

*(3).* $\mathbf{HSP}\mathrm{Cs}_{\mathrm{suc,\ pred}} = Mod(\mathrm{Ax}^=)$.

*(4). The set* $Eq(\mathrm{Cs}_{\mathrm{suc,\ pred}})$ *of equations valid in* $\mathrm{Cs}_{\mathrm{suc,\ pred}}$ *is axiomatized by the finite set* $\mathrm{Ax}^=$ *of equations. I.e.*

$$\mathrm{Cs}_{\mathrm{suc,\ pred}} \models e \quad \textit{iff} \quad \mathrm{Ax}^= \vdash e,$$

*for any equation $e$ in the language of* $\mathrm{Cs}_{\mathrm{suc,\ pred}}$.

**Proof** We apply the general theorem 2.30 to the special case $H = \{\mathrm{suc}, \mathrm{pred}\}$. First we note that $\overline{[\mathrm{suc}, \mathrm{pred}]} = G$. (See Definition 2.29 below.) We have already seen that $G$ is rich, the cardinality of $G$ is $\omega$, every element of $G$ is bounded, and right quasi-invertible in $G$. (The right quasi-inverse of $[i/j]$ is $Id$, while that of suc is pred and vica versa.)

(1) We apply Theorem 2.30 (1). Since all the equations valid in $\mathbf{I}\mathrm{Gwdf}_G$ — the equations of H2 — are consequences of $\mathrm{Ax}_G$ (Theorem 2.7 (1)), we need to show that the set $\mathrm{Ax}^{=}$ without (4) implies H1 and $\mathrm{Ax}_G$. But most of the axioms of H1 and $\mathrm{Ax}_G$ is actually listed in $\mathrm{Ax}^{=}$ while the rest can be easily derived.

(2), (3) and (4) are corollaries of Theorem 2.30 (3). (We need only that H3 is also implied by $\mathrm{Ax}^{=}$.) ∎

Clearly, $Mod(\mathrm{Ax}^{=}) \supset \mathrm{Gws}_{\mathrm{suc,\ pred}}$. We have seen in Theorem 2.27 (2) that equation (4) is valid in $\mathbf{H}\mathrm{Gws}_{\mathrm{suc,\ pred}}$. In the following example we show that it is not a consequence of $\mathrm{Ax}^{=}$ giving that the variety generated by $\mathrm{Gws}_{\mathrm{suc,\ pred}}$, that is $\mathbf{H}\mathrm{Gws}_{\mathrm{suc,\ pred}}$, is a strictly smaller class than $Mod(\mathrm{Ax}^{=})$.

ex2

**Example 2.28** *Let $\mathcal{A}$ be the four element Boolean algebra augmented with the unary operations $\mathbf{c}_i$ $(i \in \omega)$, $\mathbf{s}_{\mathrm{suc}}, \mathbf{s}_{\mathrm{pred}}$ and the constants $\mathbf{d}_{ij}$ $(i, j \in \omega)$. $\mathbf{s}_{\mathrm{suc}}$ and $\mathbf{s}_{\mathrm{pred}}$ interchanges the atoms of $\mathcal{A}$ and fixes 1 and 0 while $\mathbf{c}_i$ $(i \in \omega)$ are the identity functions. $\mathbf{d}_{ij} = 1$ for all $(i, j \in \omega)$. Equation (4) of $\mathrm{Ax}^{=}$ fails in $\mathcal{A}$ if we set $x$ to be an atom of $\mathcal{A}$.*

Theorem 2.30 below states that we can obtain positive results not only for the specific choice $H = \{\mathrm{suc}, \mathrm{pred}\}$ but for an arbitrary $H \subseteq {}^{\omega}\omega$ provided that $H$ satisfies certain conditions. In that sense Theorem 2.30 is a metatheorem and Theorem 2.27 is its special case. In Definition 2.29 below we define those axioms that are used in the metatheorem 2.30.

d-H13

**Definition 2.29** For any set $H \subseteq {}^{\omega}\omega$, we define $\bar{H} = H \cup \{[i/j] \ : \ i, j \in \omega\}$. Below we define the sets H1, H2, H3 of equations.

(H1). The set of all equations valid in $\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$, that is $Eq(\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}})$. Since $t_{\bar{H}} \not\subseteq t_H^{=}$, we need to define, how to interpret an equation of $\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$ in $\mathrm{Gws}_H$. If $[i/j] \notin H$ then the symbol $\mathbf{s}_{[i/j]}x$ denotes the term $\mathbf{c}_i(\mathbf{d}_{ij} \cap x)$ in the language of $t_H^{=}$.

(H2).
$$\begin{array}{lll} \mathbf{s}_{[i/j]}x = \mathbf{c}_i(\mathbf{d}_{ij} \cap x) & \text{when} & [i/j] \in H, \\ \mathbf{s}_{\sigma}\mathbf{d}_{ij} = \mathbf{d}_{\sigma(i)\sigma(j)} & \text{when} & \sigma \in H, \\ \mathbf{c}_k\mathbf{d}_{ij} = \mathbf{d}_{ij} & \text{when} & i, j, k \in \omega, i \neq k \neq j. \end{array}$$

(H3).
$$\begin{array}{lll} \mathbf{s}_{\sigma}(x - \mathbf{c}_0(-\mathbf{d}_{01})) = x - \mathbf{c}_0(-\mathbf{d}_{01}) & \text{when} & \sigma \in H, \\ \mathbf{c}_i(-\mathbf{c}_0(-\mathbf{d}_{01})) = -\mathbf{c}_0(-\mathbf{d}_{01}) & \text{when} & i \in \omega, \\ \mathbf{c}_i\mathbf{c}_0(-\mathbf{d}_{01}) = \mathbf{c}_0(-\mathbf{d}_{01}) & \text{when} & i \in \omega. \end{array}$$

□ t-gen=

**Theorem 2.30 (general)**
*Let $H$ be a set of transformations of $\omega$. Then (1)–(4) below hold.*

*(1).* *If $[\bar{H}]$ is rich, then* $\mathrm{RCA}_\omega = \mathbf{S}\,\mathbf{Rd}_{\mathrm{CA}} Mod(H1, H2)$.

*(2).* *If the conditions (a)–(e) are satisfied then* $\mathrm{Eq}(\mathrm{Cs}_H)$ *is finitely axiomatizable i.e. there is a finite schema $\Sigma$ of equations such that*

$$\mathrm{Cs}_H \models e \quad \textit{iff} \quad \Sigma \vdash e,$$

*for any equation $e$ in the language of* $\mathrm{Cs}_H$.

*(a)* $[\bar{H}]$ *is rich,*

*(b)* $|H| \leq \omega$,

*(c) for any $\sigma \in H$, $\sigma$ is bounded,*

*(d) for any $\sigma \in H$, $\sigma$ is right quasi-invertible in $[\bar{H}]$,*

*(e) $[\bar{H}]$ is a finitely presented semigroup (see Definition 3.7).*

*(3).* *If the conditions (a)–(c) above and (f) below are satisfied then* $\mathbf{H}\mathrm{Gws}_H = Mod(H1, H2, H3)$ *and therefore,*

$$\mathrm{Gws}_H \models e \quad \textit{iff} \quad (H1, H2, H3) \vdash e,$$

*for any equation $e$ in the language of* $\mathrm{Gws}_H$.

*(f) for any $\nu, \mu \in [\bar{H}]$ there are $\nu^*, \mu^* \in [\bar{H}]$ with $\nu \circ \nu^* \approx \mu \circ \mu^*$.*

*(4).* *If the conditions (a)–(d) are satisfied then* $\mathbf{HSP}\mathrm{Cs}_H = Mod(H1, H2, H3)$ *and therefore,*

$$\mathrm{Cs}_H \models e \quad \textit{iff} \quad (H1, H2, H3) \vdash e,$$

*for any equation $e$ in the language of* $\mathrm{Cs}_H$. *Furthermore, if conditions (a)–(d) are satisfied then* $\mathbf{H}\mathrm{Gws}_H = \mathbf{H}\mathrm{Gs}_H = \mathbf{HSP}\mathrm{Cs}_H$.

(1) is proved in Subsection 3.5. (2) follows from the observation that H2, H3 are already finite schemas of equations, and that by Theorem 2.11 (iv) above there is a finite schema $\Sigma_0$ such that $\Sigma_0 \vdash H1$. Now, $\Sigma = \Sigma_0 \cup (H1, H2)$. (3) and (4) is proved in Subsection 3.7.

## 2.4 Open Qusetions

The authors would like to collect those questions that where raised in the sections listing the results of the paper.

Find simpler axiom schemas aximatizing$\mathbf{I}\mathrm{Cdf}_G$ and $\mathbf{I}\mathrm{Gdf}_G^{ec}$, respectively. (Cf. Theorem's 2.15, 2.16 and the definition of $\mathrm{Qx}_G$).

Is the quasi-variety $\mathrm{Gdf}_{G^\times}$ in Sain [Sa87], [Sa93] (cf. e.g. Section 4 in [Sa93]) admits a finite schema axiomatization (perhaps by the techniques of 2.15 or 2.16 herein)?

## 2.5 Application to logic

Consider the following three types of algebraic results:

(i). $\mathbf{I}\mathrm{Gwdf}_H$ is axiomatizable by the finite schema $\mathrm{Ax}_H$ of equations (if $H$ satisfies certain conditions).

(ii). $\mathbf{I}\mathrm{Gdf}_H$ is closed under ultraproducts (under some conditions in $H$).

(iii). $Eq(\mathrm{Cdf}_H)$ or $Eq(\mathrm{Cs}_H)$ is axiomatizable by a finite set of equation schemas.

In the present work we prove results of type (i)–(iii). A detailed elaboration of some of the logical applications of results of type (i)–(iii) is given in Sain [Sa93] Section 4. Though therein the explanation is given for a particular choice (denoted by $G^+$) of $H$, the same argument applies to any other choice of $H$ (satisfying the conditions of our theorems). The reader interested in applications to logic and in relevance to the original logic oriented formulation of the finitization problem is referred to Sain [Sa93] Section 4 and Németi [N91].

We plan to elaborate these considerations together with new developments in this direction in a sequel to the present work.

## 2.6 Comparisons with the polyadic notion of a schema

First half of the material studied in this paper is concerned with algebras of logic *without* equality. This half seems to be related to the axiomatization of representable polyadic algebras of infinite dimension ($\mathrm{RPA}_\omega$). The set of axiom schemas approximating $\mathrm{RPA}_\omega$ in [HMTII] p225 is denoted by $P_0$–$P_{11}$.

A difference between the polyadic approach and ours is that $\mathrm{RPA}_\omega$ has uncountably many operation symbols, hence the computability or recursive enumerability aspects of, say, a completeness theorem (or equivalently of an axiomatizability theorem) are not automatic consequences of the existence of a finite schema axiomatization provided by the theorem in question. As a contrast, in our approach the basic operation symbols ($\mathbf{s}_{\mathrm{suc}}, \mathbf{s}_{\mathrm{pred}}, \mathbf{s}_{[i/j]}, \mathbf{c}_i \;\; i, j \in \omega$) are effectively enumerated and hence the finite schemas (e.g. $\mathrm{Ax}_G$ in Definition 2.6) of axioms yield a recursive enumeration of their consequences. To illustrate the computational differences between the two notions of schema we recall the following result of Gábor Sági.

There are three primitive recursive functions $\tau_1, \tau_2, \tau_3 \in {}^\omega\omega$ such that the set $E$ of equations of the language of $\mathrm{RPA}_\omega$ that involve only $\{\mathbf{s}_{\tau_i} : i = 1, 2, 3\}$ as operations and are consequences of $P_0$–$P_{11}$ is not recursively enumerable. This seems to mean that there are effective and decidable sublanguages $L$ of the language of $\mathrm{RPA}_\omega$ such that $E_L = \{e \in L : P_0\text{–}P_{11} \vdash e\}$ is not recursively enumerable. Cf. Sági [Sá95]. Since $L$ is decidable (as a subset of the full language of $\mathrm{RPA}_\omega$), this is in contrast with the expected behaviour of finite schema axiomatizations. This observation can be interpreted as saying that the finite schemas $P_0$–$P_{11}$ are less informative from the *effective* or *constructive* point of view as one would normally expect. This seems to imply that for *any* reasonable generalization (from $\omega$ to $L_0$) of the notion of computability the set $Eq(Mod(P_0\text{–}P_{11}))$ will turn non-recursively enumerable.

As a contrast, $E_L$ becomes recursively enumerable if we replace $P_0$–$P_{11}$ with any (e.g. $\mathrm{Ax}_G$) of *our* axiom schemas and $L$ with any decidable (or even recursively enumerable) subset of our equational language.

The other half of the present paper, that deals with algebras associated to logics *with* equality, does not seem to have a polyadic algebraic ($\mathrm{RPA}_\omega$–theoretic) counterpart in the literature yet. Such a counterpart would deal with $\mathbf{SUp}\mathrm{RPEA}_\omega$ or $Eq(\mathrm{RPEA}_\omega)$. (It has been known for long that $\mathbf{SUp}\mathrm{RPEA}_\omega \neq \mathrm{RPEA}_\omega$ hence the subject of study is *not* $\mathrm{RPEA}_\omega$ but rather $\mathbf{SUp}\mathrm{RPEA}_\omega$.)

In [NS95] it is proved about the polyadic notion (or Halmos' notion) of a schema that $Eq(\mathrm{RPEA}_\alpha)$ is not axiomatizable by any set of polyadic schemas if $\alpha \geq \omega$. Therefore the schema $P_0$–$E_3$ used by Halmos to axiomatize $\mathrm{PEA}_\alpha$ and quoted in [HMTII, pp.225-6] does not describe $Eq(\mathrm{RPEA}_\alpha)$, moreover, one cannot add similar schemas to $P_0$–$E_3$ such that the result would axiomatize $Eq(\mathrm{RPEA}_\alpha)$.

# 3 Proofs

## 3.1 An axiomatization of $\mathbf{I}\mathrm{Gwdf}_G$

To prove Theorem 2.7 (1), we use the method and some theorems described in [Sa92].

cl-rich

**Claim 3.1** *If the subsemigroup $Q$ of $\langle {}^\omega\omega, \circ\rangle$ is rich then all the finite transformations belong to $Q$.*

**Proof** $Id \in Q$ since $Id = \hat{p} \circ \hat{s}$. For all $i, j \in \omega$, $[i/j] \in Q$ since $[i/j] = Id[i/j]$ and $[i, j] \in Q$ since $[i, j] = [i/j][j/i]$. The semigroup of finite transformations is generated by the set of transpositions and replacements. ∎

cl-g-reach

**Claim 3.2** *The semigroup* $[G]$ *is rich.*

**Proof** All the transpositions belong to $[G]$, since

$$[i,j] = \text{pred} \circ [0/j] \circ [j/i] \circ [i/0] \circ \text{suc}.$$

This implies that every finite transformation of $\omega$ belongs to $[G]$.

A transformation $f$ of $\omega$ is called *$n$-shifted* if $f(x) = x + n$ for all but finitely many members of $\omega$ ($n$ is a given integer). Clearly, $\text{suc}^n$ is $n$-shifted, $\text{pred}^n$ is $(-n)$-shifted and $Id$ is 0-shifted i.e. it is a finite transformation. If $f$ is $n$-shifted, $g$ is $(-n)$-shifted then both $f \circ \text{pred}^n$ and $\text{suc}^n \circ g$ are finite transformations. (Here $n$ is a natural number.) This implies that $f$ is equal to the composition of a finite transformation (namely $f \circ \text{pred}^n$) and $\text{suc}^n$, $g$ is equal to the composition of $\text{pred}^n$ and a finite transformation (namely $\text{suc}^n \circ g$). So every $n$-shifted transformation is in $[G]$.

An $f \in {}^\omega\omega$ is called *shifted* if it is $n$-shifted for some integer $n$. All shifted transformations belong to $[G]$ and every member of $[G]$ is shifted. (The generators of $[G]$ are shifted, composition of shifted transformations is shifted.)

The first condition of Definition 2.10 is satisfied since if $f$ is shifted then so is $f[i/j]$ and that implies that it belongs to $[G]$. The elements $\hat{s}$ and $\hat{p}$ of $[G]$ required in the second condition can be suc and pred. $\omega - Rng(\text{suc}) = \{0\}$. The previous argument shows that $(\text{suc} \circ f \circ \text{pred})[\{0\}/Id]$ belongs to $[G]$. ∎

In the definition of $\text{Gwdf}_H$, $\tau \in H$ plays a double role. Namely, $\mathbf{s}_\tau$ is a symbol hence $\tau$ itself is also used as a symbol. On the other hand, $\tau$ as an element of ${}^\omega\omega$ is a concrete function i.e. it occurs on the semantic side of $\text{Gwdf}_H$. In the proofs below, it will be most useful to distinguish $\tau$ as a *symbol* from $\tau$ as a *function*. (The reason for this will be clear e.g. in Definition 3.3 below.) Therefore, from that point on until the end of Subsection 3.2 we use temporarily $\underline{\tau}$ when we want to refer to $\tau$ as a concrete *transformation* of $\omega$, and we use $\tau$ to refer to the *symbol* or *name* of the transformation. For example, $\text{suc}, \text{pred}, [i/j], [i,j]$ are symbols while $\underline{\text{suc}}, \underline{\text{pred}}, \underline{[i/j]}, \underline{[i,j]}$ are transformations of $\omega$ ($i, j \in \omega$).

Furthermore, if $\sigma$ is a finite sequence (or word) over $H$, that is $\langle \tau_1, \ldots, \tau_n \rangle = \sigma \in H^*$, then $\underline{\sigma}$ denotes the transformation $\underline{\tau}_1 \circ \ldots \circ \underline{\tau}_n$. If $H$ is a set of symbols then $\underline{H} = \{\underline{\tau} : \tau \in H\}$. It follows that $\underline{H^*} = [\underline{H}]$.

d-*17

**Definition 3.3 (Definition 2.2.2 of [Sa93])**
Let $H$ be a set of names of transformations of $\omega$ with the property that $\mathbf{c}_i$ and $\mathbf{s}_{[i/j]}$ are term definable in $\mathbf{I}\text{Gwdf}^*_H$ for every $i, j \in \omega$. Below we define postulates (*1)-(*7). The postulates are stated for all $\sigma, \nu \in H^*$ and for all $i, j \in \omega$. The symbols $\mathbf{c}_i$ and $\mathbf{s}_{[i/j]}$ abbreviate the terms that define $\mathbf{c}_i$ and $\mathbf{s}_{[i/j]}$ respectively. For any word $\sigma \in H^*$, $\sigma = \langle \tau_1 \ldots \tau_n \rangle$, $\mathbf{s}_\sigma(x)$ abbreviates the term $\mathbf{s}_{\tau_1} \ldots \mathbf{s}_{\tau_n}(x)$.

(*1). The finite set of equations axiomatizing Boolean algebras. The axioms stating that $\mathbf{c}_i$ is an additive complemented closure operation and $\mathbf{c}_i$, $\mathbf{c}_j$ commute: $x \leq \mathbf{c}_i x$, $\mathbf{c}_i x = \mathbf{c}_i \mathbf{c}_i x$, $\mathbf{c}_i(x \vee y) = \mathbf{c}_i x \vee \mathbf{c}_i y$, $\mathbf{c}_i(-\mathbf{c}_i x) = -\mathbf{c}_i x$, $\mathbf{c}_i \mathbf{c}_j x = \mathbf{c}_j \mathbf{c}_i x$.

(*2). The axioms stating that $\mathbf{s}_\sigma$ is a Boolean homomorphism: $\mathbf{s}_\sigma(x \vee y) = \mathbf{s}_\sigma(x) \vee \mathbf{s}_\sigma(y)$, $\quad \mathbf{s}_\sigma(-x) = -\mathbf{s}_\sigma(x)$.

(*3). $\mathbf{s}_\sigma(x) = \mathbf{s}_\nu(x)$ if $\underline{\sigma} = \underline{\nu}$.

(*4). $\mathbf{s}_\sigma(x) = x$ if $\underline{\sigma} = Id$.

(*5). $\mathbf{s}_\sigma \mathbf{c}_i x = \mathbf{s}_\nu \mathbf{c}_i x$ if $j \neq i \Rightarrow \underline{\nu}(j) = \underline{\sigma}(j)$.

(*6). $\mathbf{s}_\sigma \mathbf{c}_i x = \mathbf{c}_j \mathbf{s}_\sigma x$ if $\underline{\sigma}^{-1}(j) = \{i\}$.

(*7). $\mathbf{c}_i \mathbf{s}_{[i/j]} x = \mathbf{s}_{[i/j]} x$ if $i \neq j$.

$\square$

th-*17

**Theorem 3.4 (Theorem 2.2.4 of [Sa93])**
*Let $H$ be a set as in the previous definition. Assume that $[\underline{H}]$ is rich. Then*

$$Mod_{t_H^*}((*1)\text{-}(*7)) = \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}*}.$$

cr-*171

**Corollary 3.5** *Let $H$ be as in the previous theorem. Also assume that the set theoretic operations $\mathbf{c}_i$ $(i > 0)$ are term definable in $\mathrm{Gwdf}_H^*$. Then (*1)-(*7) together with the definition of $\mathbf{c}_i$ $(i > 0)$ is a complete axiomatization of $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$.*

cr-*17

**Corollary 3.6** *(*1)-(*7) together with the definition of $\mathbf{c}_i$ $(i > 0)$ is a complete axiomatization of $\mathbf{I}\mathrm{Gwdf}_G^{\mathrm{nm}}$. Furthermore, that set is a complete axiomatization of $\mathbf{I}\mathrm{Gwdf}_G$ since the two class coincide as it will be proved in Subsection 3.3.*

## 3.2 Theorem 2.7 (1): finite schema axiomatizing $\mathbf{I}\mathrm{Gwdf}_G$

The axiom system defined in Definition 3.3 is not a schema axiomatization since the variable $\sigma$ ranges over $G^*$ and not $G$. In this subsection we will correct the axiom system.

In the definition below we recall a standard concept of universal algebra. (See e.g. [HMTII].)

d-fin-pr

**Definition 3.7** An algebra $\mathcal{A}$ is said to be *presented* by a set $\Sigma$ of defining equation in a variety $V$ if all of the following conditions hold.

- $\mathcal{A}$ is generated by some set $H \subseteq A$.

- We denote the expansion of the algebra $\mathcal{A}$ with constants corresponding to $H$ by $\mathcal{A}'$. $\Sigma$ is a set of equations of the language of $\mathcal{A}'$ that contain no variables.

- For every pair of terms $\tau$ and $\nu$ of the language of $\mathcal{A}'$ that contain no variables
$$\mathcal{A}' \models \tau = \nu \;\Leftrightarrow \Sigma \cup Eq(V) \models \tau = \nu.$$

$\square$

Since the equational logic has a strongly complete and sound inference system, in the case when $V$ is the variety SG of all semigroups we can simplify the last condition as

- $(\forall \tau, \nu \in H^*)\ \underline{\tau} = \underline{\nu} \;\Leftrightarrow\; \Sigma \cup Eq(\mathrm{SG}) \vdash \tau = \nu.$

th-g-pr

**Theorem 3.8** $[\underline{G}]$ *is presented by the following set of equations in the variety of all semigroups:*

*(3.1).* $\mathrm{pred}\,\mathrm{suc} = \lambda$ *where* $\lambda$ *is the empty word,*

*(3.2).* $\mathrm{suc}\,\mathrm{pred} = [0/1]$,

*(3.3).* $\mathrm{suc}[i/j] = [i+1/j+1]\mathrm{suc}$ *for every distinct* $i, j \in \omega$,

*(3.4).* $[i/j]\mathrm{pred} = \mathrm{pred}[i+1/j+1]$ *for every distinct* $i, j \in \omega$ *with* $i \neq 0$,

*(3.5).* $[0/j]\mathrm{pred} = \mathrm{pred}[0/j+1][1/j+1]$ *for every* $j \in \omega$.

*(3.6). Thompson's axioms from [To93] that present the semigroup generated by the replacements only. All* $i, j, k, l \in \omega$ *are distinct.*

*(B1)* $[i/k][i/j] = [i/j]$,
*(B2)* $[j/i][i/j] = [j/i]$,
*(B3)* $[j/k][i/j] = [j/k][i/k]$,
*(B4)* $[k/j][i/j] = [i/j][k/j]$,
*(B5)* $[k/l][i/j] = [i/j][k/l]$,
*(B6)* $[i/j][j/k][k/i][i/j] = [i/k][k/j][j/i]$,
*(B7)* $[i/j][j/k][k/l][l/i] = [i/k][k/l][l/j][j/i][i/l]$.

**Proof** We will use Thompsons result saying that the semigroup generated by the replacements is presented by a set of axioms. The set listed in [To93] contains three additional axioms but they are derivable from the others:
$[i/j][i/j] = [i/j][i/i+j+1][i/j] = [i/i+j+1][i/j] = [i/j]$
$[i/j][k/j][i/j] = [i/j][i/j][k/j] = [i/j][k/j]$ and
$[i/j][k/l][i/j] = [i/j][i/j][k/l] = [i/j][k/l]$.

In this proof we will write $\vdash$ instead of (3.1)-(3.6)$\cup Eq$(SG) $\vdash$.

We say that $\tau \in G^*$ is in *normal form* if $\tau = \text{pred}^n \tau' \text{suc}^m$ for some $\tau' \in \{[i/j]\}^*_{i,j\in\omega}$ and $n, m \in \omega$. Every word can be reduced to a normal formed word using only (3.1)-(3.5). This implies that

$$(\forall \tau \in G^*)(\exists \tau' \in G^*) \quad \vdash \tau = \tau' \text{ and } \tau' \text{ is in normal form}$$

Let $\tau$ and $\nu$ be arbitrary words from $G^*$, assume that $\underline{\tau} = \underline{\nu}$. There are $n, m, l, k \in \omega$ and $\tau', \nu' \in \{[i/j]\}^*_{i,j\in\omega}$ for that $\vdash \tau = \text{pred}^n \tau' \text{suc}^m$, $\vdash \nu = \text{pred}^l \nu' \text{suc}^k$.

Since $\underline{\tau}$ is $m-n$-shifted, $\underline{\nu}$ is $k-l$-shifted we can conclude that $m-n = k-l$ or $m = n+k-l$.

Without loss of generality we may assume that $l \leq n$. $\vdash \text{pred}^l \nu' \text{suc}^k = \text{pred}^l \text{pred}^{n-l} \text{suc}^{n-l} \nu' \text{suc}^k = \text{pred}^n \nu'' \text{suc}^{n-l} \text{suc}^k = \text{pred}^n \nu'' \text{suc}^m$ can be derived using (3.1), (3.3) and the previous observation. $\vdash \text{suc}^n \tau \text{pred}^m = [0/1]^n \tau' [0/1]^m$ and similarly $\vdash \text{suc}^n \nu \text{pred}^m = [0/1]^n \nu'' [0/1]^m$ using (3.2).

$\underline{\tau} = \underline{\nu}$ implies $\underline{\text{suc}}^n \circ \underline{\tau} \circ \underline{\text{pred}}^m = \underline{\text{suc}}^n \circ \underline{\nu} \circ \underline{\text{pred}}^m = \underline{[1/0]}^n \circ \underline{\tau'} \circ \underline{[1/0]}^m = \underline{[1/0]}^n \circ \underline{\nu''} \circ \underline{[1/0]}^m$.

¿From the main result of Thompson [To93] we know that $\underline{[1/0]}^n \circ \underline{\tau'} \circ \underline{[1/0]}^m = \underline{[1/0]}^n \circ \underline{\nu''} \circ \underline{[1/0]}^m$ imply $\vdash [0/1]^n \tau' [0/1]^m = [0/1]^n \nu'' [0/1]^m$. Collecting all the information we have: $\vdash \text{suc}^n \tau \text{pred}^m = \text{suc}^n \nu \text{pred}^m$. But this implies $\vdash \text{pred}^n \text{suc}^n \tau \text{pred}^m \text{suc}^m = \text{pred}^n \text{suc}^n \nu \text{pred}^m \text{suc}^m$.

Using (3.1) again we can conclude $\vdash \tau = \nu$ and this completes the proof. ∎

The set of defining equations listed in 3.8 is not independent. Below we show that equations (3.4), (3.5), (B6), (B7) are implied by the others.

**Claim 3.9** *(3.1),(3.2),(3.3),(B1),(B3),(B4),(B5)$\vdash$(B6)* cl-b7

**Proof** First we prove that the listed B-axioms imply

$$[i/j][j/k][k/i][i/j][a/b] = [i/k][k/j][j/i][a/b]$$

where all $i, j, k, a, b$ are distinct:

$$\begin{aligned}
[i/j][j/k][k/i][i/j][a/b] &\overset{B1}{=} [i/j][j/k][k/i][i/j][a/j][a/b] \\
&\overset{B4}{=} [i/j][j/k][k/i][a/j][i/j][a/b] \\
&\overset{B1}{=} [i/j][j/k][k/i][a/i][a/j][i/j][a/b] \\
&\overset{B3,B4}{=} [i/j][j/k][a/i][k/a][a/j][i/j][a/b] \\
&\overset{B5,B3,B4,B1}{=} [a/j][i/j][j/k][i/j][k/a][a/b] \\
&\overset{B3}{=} [a/j][i/j][j/k][i/k][k/a][a/b]
\end{aligned}$$

$$
\begin{aligned}
&\stackrel{B4,B1,B3}{=} [a/j][i/k][j/i][k/a][a/b]\\
&\stackrel{B5}{=} [i/k][a/j][k/a][j/i][a/b]\\
&\stackrel{B3,B4,B5}{=} [i/k][k/j][a/j][a/b][j/i]\\
&\stackrel{B1,B5}{=} [i/k][k/j][j/i][a/b].
\end{aligned}
$$

Now we can compute:
$[i/j][j/k][k/i][i/j] \stackrel{3.1}{=} \mathrm{pred}\,\mathrm{suc}[i/j][j/k][k/i][i/j] \stackrel{3.3,3.1,3.2}{=}$
$\mathrm{pred}[i+1/j+1][j+1/k+1][k+1/i+1][i+1/j+1][0/1]\mathrm{suc} =$
$\mathrm{pred}[i+1/k+1][k+1/j+1][j+1/i+1][0/1]\mathrm{suc} \stackrel{3.3,3.1,3.2}{=}$
$[i/k][k/j][j/i]$. ∎

cl-b8

**Claim 3.10** *(3.1),(3.2),(3.3),(B1),(B3),(B4),(B5)⊢(B7)*

**Proof** First we prove that the listed B-axioms imply

$$[i/k][k/l][l/j][j/i][i/l][a/b] = [i/j][j/k][k/l][l/i][a/b]$$

where all $i, j, k, l, a, b$ are distinct:

$$
\begin{aligned}
[i/k][k/l][l/j][j/i][i/l][a/b] &\stackrel{B1,B4}{=} [i/k][k/l][l/j][j/i][a/l][i/l][a/b]\\
&\stackrel{B1,B4,B3}{=} [i/k][k/l][l/j][a/i][j/a][a/l][i/l][a/b]\\
&\stackrel{B4,B5,B3}{=} [a/k][i/k][k/l][l/j][i/l][j/a][a/b]\\
&\stackrel{B4,B3}{=} [a/k][i/k][k/l][i/j][l/i][j/a][a/b]\\
&\stackrel{B5,B1}{=} [a/k][i/j][k/l][l/i][j/a][a/b]\\
&\stackrel{B5}{=} [i/j][a/k][j/a][k/l][l/i][a/b]\\
&\stackrel{B3,B4,B5,B1}{=} [i/j][j/k][a/k][a/j][k/l][l/i][a/b]\\
&\stackrel{B1,B5}{=} [i/j][j/k][k/l][l/i][a/b].
\end{aligned}
$$

Now we can compute:
$[i/k][k/l][l/j][j/i][i/l] \stackrel{3.1}{=} \mathrm{pred}\,\mathrm{suc}[i/k][k/l][l/j][j/i][i/l] \stackrel{3.3,3.1,3.2}{=}$
$\mathrm{pred}[i+1/k+1][k+1/l+1][l+1/j+1][j+1/i+1][i+1/l+1][0/1]\mathrm{suc} =$
$\mathrm{pred}[i+1/j+1][j+1/k+1][k+1/l+1][l+1/i+1][0/1]\mathrm{suc} \stackrel{3.3,3.1,3.2}{=}$
$[i/j][j/k][k/l][l/i]$. ∎

cl-35

**Claim 3.11** *(3.1),(3.2),(3.3),(B3),(B4)⊢(3.5)*

**Proof** $[0/j]\mathrm{pred} \stackrel{3.1}{=} \mathrm{pred}\,\mathrm{suc}[0/j]\mathrm{pred} \stackrel{3.3}{=} \mathrm{pred}[1/j+1]\mathrm{suc}\,\mathrm{pred} \stackrel{3.2}{=}$ $\mathrm{pred}[1/j+1][0/1] \stackrel{B3,B4}{=} \mathrm{pred}[0/j+1][1/j+1]$ ∎

**Claim 3.12** *(3.1),(3.2),(3.3),(B4),(B5)⊢(3.4)* cl-34

**Proof** $[i/j]\mathrm{pred} \stackrel{3.1}{=} \mathrm{pred}\,\mathrm{suc}[i/j]\mathrm{pred} \stackrel{3.3}{=} \mathrm{pred}[i+1/j+1]\mathrm{suc}\,\mathrm{pred} \stackrel{3.2}{=}$ $\mathrm{pred}[i+1/j+1][0/1] \stackrel{B4 or B5}{=} \mathrm{pred}[0/1][i+1/j+1] \stackrel{3.2}{=}$ $\mathrm{pred}\,\mathrm{suc}\,\mathrm{pred}[i+1/j+1] \stackrel{3.1}{=} \mathrm{pred}[i+1/j+1]$ ∎

We quote an important result of Pinter.

**Claim 3.13 (Lemma 2.2 of [P73])** cl-p
*For every distinct $i,j,k,l \in \omega$ we have*

$$\begin{array}{lll} \mathrm{Ax}_G & \vdash & \mathbf{s}_{[i/k]}\mathbf{s}_{[i/j]} = \mathbf{s}_{[i/j]}, \\ \mathrm{Ax}_G & \vdash & \mathbf{s}_{[k/j]}\mathbf{s}_{[i/j]} = \mathbf{s}_{[i/j]}\mathbf{s}_{[k/j]}, \\ \mathrm{Ax}_G & \vdash & \mathbf{s}_{[k/l]}\mathbf{s}_{[i/j]} = \mathbf{s}_{[i/j]}\mathbf{s}_{[k/l]}. \end{array}$$

We will need the following technical observation that originates from Sándor Csizmazia:

**Lemma 3.14** $\mathrm{Ax}_G$ *together with the equations (*3), (*4), (*5), (*1) without* $\mathbf{c}_i\mathbf{c}_j x = \mathbf{c}_j\mathbf{c}_i x$ *implies that for any* $\tau \in \{[i/j] \mid i,j \geq 1\} \cup \{[1/0]\mathrm{suc}, \mathrm{pred}[1/2]\}$ lm-uj7

$$\mathbf{s}_\tau\mathbf{c}_0 x = \mathbf{c}_0\mathbf{s}_\tau x$$

**Proof** There is nothing to prove in $\mathbf{s}_{[i/j]}\mathbf{c}_0 x = \mathbf{c}_0\mathbf{s}_{[i/j]}x$ $(i,j \geq 1)$ since it is included in $\mathrm{Ax}_G$. It was formulated in the lemma only for further applications.

To prove the other two statements we will need the following claim quoted from Pinter [P73]:

**Claim 3.15** $\mathbf{c}_i x = \min \overbrace{\{y \mid y = \mathbf{s}_{[i/j]}y, y \geq x\}}^{B}$ cl-ci

**Proof** $\mathbf{c}_i x \in B$ since $\mathbf{s}_{[i/j]}\mathbf{c}_i x = \mathbf{c}_i x$ and $\mathbf{c}_i x \geq x$ that is (5) and (6) of $\mathrm{Ax}_G$. For any $y \in B$ $y \geq \mathbf{c}_i x$ since $\mathbf{c}_i x \stackrel{*1}{\leq} \mathbf{c}_i y = \mathbf{c}_i\mathbf{s}_{[i/j]}y \stackrel{*7}{=} \mathbf{s}_{[i/j]}y = y$. ∎

Further observations are needed.

**Claim 3.16** $\mathbf{c}_i x = \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x$ cl-defci

**Proof** Using the previous claim, we have to prove that $\mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x \in B \stackrel{\mathrm{def}}{=} \{y \mid y = \mathbf{s}_{[i/j]}y, y \geq x\}$ and $y \in B \Rightarrow y \geq \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x$.
$\mathbf{s}_{[i/j]}\mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x \stackrel{*3}{=} \mathbf{s}_{[0,i]}\mathbf{s}_{[0/j]}\mathbf{c}_0\mathbf{s}_{[0,i]}x \stackrel{6}{=} \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x$.

(5) implies $\mathbf{c}_0\mathbf{s}_{[0,i]}x \geq \mathbf{s}_{[0,i]}x$, using (2) we have $\mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x \geq \mathbf{s}_{[0,i]}\mathbf{s}_{[0,i]}x \stackrel{*4}{=} x$.
$y \in B \;\Rightarrow\; y \geq x \stackrel{*1,2}{\Rightarrow} \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x \leq \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}y \stackrel{y\in B}{=} \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}\mathbf{s}_{[i/j]}y \stackrel{*3}{=}$
$\mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0/j]}\mathbf{s}_{[0,i]}y \stackrel{6}{=} \mathbf{s}_{[0,i]}\mathbf{s}_{[0/j]}\mathbf{s}_{[0,i]}y \stackrel{*3}{=} \mathbf{s}_{[0/j]}y \stackrel{y\in B}{=} y$. ∎

cl-c1s01

**Claim 3.17** $\mathbf{c}_0\mathbf{s}_{[1/0]}x = \mathbf{c}_1\mathbf{s}_{[0/1]}x$

**Proof** $\mathbf{c}_0\mathbf{s}_{[1/0]}x \stackrel{*3}{=} \mathbf{c}_0\mathbf{s}_{[1/2]}\mathbf{s}_{[1/0]}x \stackrel{6}{=} \mathbf{s}_{[1/2]}\mathbf{c}_0\mathbf{s}_{[1/0]}x \stackrel{*3}{=} \mathbf{s}_{[1/0]}\mathbf{s}_{[1/2]}\mathbf{c}_0\mathbf{s}_{[1/0]}x \stackrel{6}{=}$
$\mathbf{s}_{[1/0]}\mathbf{c}_0\mathbf{s}_{[1/2]}\mathbf{s}_{[1/0]}x \stackrel{*3}{=} \mathbf{s}_{[1/0]}\mathbf{c}_0\mathbf{s}_{[1/0]}x \stackrel{*3}{=} \mathbf{s}_{[0,1]}\mathbf{s}_{[0/1]}\mathbf{c}_0\mathbf{s}_{[0,1]}\mathbf{s}_{[0/1]}x \stackrel{6}{=}$
$\mathbf{s}_{[0,1]}\mathbf{c}_0\mathbf{s}_{[0,1]}\mathbf{s}_{[0/1]}x \stackrel{Cl.3.16}{=} \mathbf{c}_1\mathbf{s}_{[0/1]}x$ ∎

cl-cisuc

**Claim 3.18** $\mathbf{c}_{i+1}\mathbf{s}_{\mathrm{suc}}x = \mathbf{s}_{\mathrm{suc}}\mathbf{c}_i x$

**Proof** Let
$$\begin{aligned} B_1 &= \{y \mid \mathbf{s}_{[i+1/i+2]}y = y, y \geq \mathbf{s}_{\mathrm{suc}}x\},\\ B_{12} &= \{y \mid \mathbf{s}_{[i+1/i+2]}y = \mathbf{s}_{[0/1]}y = y, y \geq \mathbf{s}_{\mathrm{suc}}x\},\\ B_{21} &= \{\mathbf{s}_{\mathrm{suc}}y \mid \mathbf{s}_{[i/i+1]}y = y, y \geq x\},\\ B_2 &= \{y \mid \mathbf{s}_{[i/i+1]}y = y, y \geq x\}.\end{aligned}$$
The following statements prove this claim:
$$\begin{array}{lrcl} \text{I} & \mathbf{c}_{i+1}\mathbf{s}_{\mathrm{suc}}x &=& \min B_1,\\ \text{II} & \min B_1 &=& \min B_{12},\\ \text{III} & B_{12} &=& B_{21},\\ \text{IV} & \min B_{21} &=& \mathbf{s}_{\mathrm{suc}}\min B_2,\\ \text{V} & \min B_2 &=& \mathbf{s}_{\mathrm{suc}}\mathbf{c}_i x.\end{array}$$

I, V follow from Claim 3.15, IV from the observation that $y \in B_2 \leftrightarrow \mathbf{s}_{\mathrm{suc}}y \in B_{21}$.

Similarly to the proof of the previous claim, in II we need to show that $B_{12} \subseteq B_1$ (trivial) and $(\forall y \in B_1)\; -\mathbf{c}_0(-y) \in B_{12} \wedge -\mathbf{c}_0(-y) \leq y$.
So suppose $y \in B_1$. Then $\mathbf{s}_{[i+1/i+2]}(-\mathbf{c}_0(-y)) \stackrel{Ax}{=} -\mathbf{c}_0(-\mathbf{s}_{[i+1/i+2]}y) = -\mathbf{c}_0(-y)$,
$\mathbf{s}_{[0/1]}(-\mathbf{c}_0(-y)) = -\mathbf{s}_{[0/1]}\mathbf{c}_0(-y) = -\mathbf{c}_0(-y)$ and
$y \geq \mathbf{s}_{\mathrm{suc}}x \Rightarrow -\mathbf{c}_0(-y) \geq -\mathbf{c}_0(-\mathbf{s}_{\mathrm{suc}}x) = -\mathbf{c}_0\mathbf{s}_{\mathrm{suc}}(-x) \stackrel{*}{=} -\mathbf{s}_{\mathrm{suc}}(-x) = \mathbf{s}_{\mathrm{suc}}x$.
In * we used that $\mathbf{s}_{\mathrm{suc}}x = \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}x = \mathbf{s}_{[0/1]}\mathbf{s}_{\mathrm{suc}}x = \mathbf{c}_0\mathbf{s}_{[0/1]}\mathbf{s}_{\mathrm{suc}}x = \mathbf{c}_0\mathbf{s}_{\mathrm{suc}}x$.

Turning to III, suppose that $y \in B_{12}$. Then $y = \mathbf{s}_{[0/1]}y = \mathbf{s}_{\mathrm{suc}}(\mathbf{s}_{\mathrm{pred}}y)$.
$\mathbf{s}_{[i/i+1]}(\mathbf{s}_{\mathrm{pred}}y) = \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}\mathbf{s}_{[i/i+1]}(\mathbf{s}_{\mathrm{pred}}y) = \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[i+1/i+2]}\mathbf{s}_{\mathrm{suc}}\mathbf{s}_{\mathrm{pred}}y =$
$\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[i+1/i+2]}\mathbf{s}_{[0/1]}y = \mathbf{s}_{\mathrm{pred}}y$,
$y \geq \mathbf{s}_{\mathrm{suc}}x \Rightarrow \mathbf{s}_{\mathrm{pred}}y \geq \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}x = x$.

On the other hand if we suppose that $\mathbf{s}_{\mathrm{suc}}y \in B_{21}$ then
$\mathbf{s}_{[i+1/i+2]}\mathbf{s}_{\mathrm{suc}}y = \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{[i/i+1]}y = \mathbf{s}_{\mathrm{suc}}y$,
$\mathbf{s}_{[0/1]}\mathbf{s}_{\mathrm{suc}}y = \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{[\mathbf{s}_{\mathrm{suc}}/y]} = \mathbf{s}_{\mathrm{suc}}y$ and
$y \geq x \Rightarrow \mathbf{s}_{\mathrm{suc}}y \geq \mathbf{s}_{\mathrm{suc}}x$. ∎

Now we can prove the two statements formulated in the lemma:
$\mathbf{c}_0\mathbf{s}_{[1/0]}\mathbf{s}_{\mathrm{suc}}x \stackrel{*}{=} \mathbf{s}_{\mathrm{suc}}\mathbf{c}_0 x \stackrel{6}{=} \mathbf{s}_{\mathrm{suc}}\mathbf{s}_{[0/1]}\mathbf{c}_0 x \stackrel{4}{=} \mathbf{s}_{[1/2]}\mathbf{s}_{\mathrm{suc}}\mathbf{c}_0 x \stackrel{3}{=}$

$\overset{3}{=} \mathbf{s}_{[1/0]}\mathbf{s}_{[1/2]}\mathbf{s}_{\mathrm{suc}}\mathbf{c}_0 x = \mathbf{s}_{[1/0]}\mathbf{s}_{\mathrm{suc}}\mathbf{c}_0 x$,
$\mathbf{c}_0\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[1/2]}x \overset{4}{=} \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}\mathbf{c}_0\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[1/2]}x \overset{*}{=} \mathbf{s}_{\mathrm{pred}}\mathbf{c}_0\mathbf{s}_{[0/1]}\mathbf{s}_{\mathrm{suc}}\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[1/2]}x =$
$= \mathbf{s}_{\mathrm{pred}}\mathbf{c}_0\mathbf{s}_{[1/2]}x = \mathbf{s}_{\mathrm{pred}}\mathbf{s}_{[1/2]}\mathbf{c}_0 x$.
In * we used $\mathbf{c}_0\mathbf{s}_{[1/0]}\mathbf{s}_{\mathrm{suc}}x \overset{Cl.3.17}{=} \mathbf{c}_1\mathbf{s}_{[0/1]}\mathbf{s}_{\mathrm{suc}}x \overset{Ax}{=} \mathbf{c}_1\mathbf{s}_{\mathrm{suc}}\mathbf{s}_{\mathrm{pred}}\mathbf{s}_{\mathrm{suc}}x \overset{Ax}{=}$
$\overset{Ax}{=} \mathbf{c}_1\mathbf{s}_{\mathrm{suc}}x \overset{Cl.3.18}{=} \mathbf{s}_{\mathrm{suc}}\mathbf{c}_0 x$. ∎

**Proof of Theorem 2.7 (1)** Using the result of the previous subsection stated in Corollary 3.6 we need to prove only that $\mathrm{Ax}_G \vdash$(*1)–(*7) and $\mathrm{Ax}_G \vdash \mathbf{c}_i x = \mathbf{s}_{[0,i]}\mathbf{c}_0\mathbf{s}_{[0,i]}x$.

(*2) is a trivial consequence of (2).

(3) and (4) together imply (*3) and (*4) since if $\mathcal{A}$ is an algebra of $\mathbf{I}\mathrm{Gwdf}_G$ and $\mathcal{A}^*$ is the semigroup of all polynomials of $\mathcal{A}$, expanded with the constant symbols $[i/j]$, suc, pred evaluated into the polynomials $\mathbf{s}_{[i/j]}, \mathbf{s}_{\mathrm{suc}}, \mathbf{s}_{\mathrm{pred}}$ then $\mathcal{A}^*$ is a model of (3.1)–(3.6) ((3) and (4) is the relevant part of the set of equations (3.1)–(3.7) — see Claim 3.9–3.13 — written in the language of $\mathcal{A}^*$) and so $\underline{\tau} = \underline{\nu}$ forces $\mathcal{A}^* \models \tau = \nu$ and that is $\mathbf{s}_\tau x = \mathbf{s}_\nu x$.

(*7) is included in (6).

To prove (*5), suppose $\tau$ and $\nu$ are in $G^*$ and $\underline{\tau}$ and $\underline{\nu}$ agree everywhere but in $i$. Then $\underline{\tau} \circ \underline{[i/i+1]}$ and $\underline{\nu} \circ \underline{[i/i+1]}$ are equal everywhere.

$$\begin{aligned} \mathbf{s}_\tau\mathbf{c}_i x &\overset{6}{=} \mathbf{s}_\tau\mathbf{s}_{[i/i+1]}\mathbf{c}_i x = \mathbf{s}_{\tau[i/i+1]}\mathbf{c}_i x = \\ &= \mathbf{s}_{\nu[i/i+1]}\mathbf{c}_i x = \mathbf{s}_\nu\mathbf{s}_{[i/i+1]}\mathbf{c}_i x \overset{6}{=} \mathbf{s}_\nu\mathbf{c}_i x \end{aligned}$$

In (*1) $x \leq \mathbf{c}_i x$, $\mathbf{c}_i(x \vee y) = \mathbf{c}_i x \vee \mathbf{c}_i y$ follow from (2) and (5). $\mathbf{c}_i\mathbf{c}_i x = \mathbf{c}_i x$ and $\mathbf{c}_i(-\mathbf{c}_i x) = -\mathbf{c}_i x$ is a consequence of (2) and (6): $\mathbf{c}_i\mathbf{c}_i x = \mathbf{c}_i\mathbf{s}_{[i/i+1]}\mathbf{c}_i x = \mathbf{s}_{[i/i+1]}\mathbf{c}_i x = \mathbf{c}_i x$ and $\mathbf{c}_i(-\mathbf{c}_i x) = \mathbf{c}_i(-\mathbf{s}_{[i/i+1]}\mathbf{c}_i x) = \mathbf{c}_i\mathbf{s}_{[i/i+1]}(-\mathbf{c}_i x) =$
$= \mathbf{s}_{[i/i+1]}(-\mathbf{c}_i x) = -\mathbf{s}_{[i/i+1]}\mathbf{c}_i x = -\mathbf{c}_i x$

To prove (*6) we will need the following observation: *If* $T_0 = \{f \in [\underline{G}] \mid f^{-1}(0) = \{0\}\}$ *and* $G_0 = \{[i/j] \mid i,j \geq 1\} \cup \{[1/0]\mathrm{suc}, \mathrm{pred}[1/2]\}$ *then* $T_0$ *is generated by the meanings of the elements of* $G_0$. To verify this statement define the function $'$ mapping $[\underline{G}]$ into itself

$$f' = (\underline{\mathrm{suc}} \circ f \circ \underline{\mathrm{pred}})[0/0]$$

(Since $[\underline{G}]$ is rich $f' \in [\underline{G}]$.) It is easy to check that $'$ is a homomorphism ($(f \circ g)' = f' \circ g'$), furthermore, the $'$ image of $[\underline{G}]$ is $T_0$ (if $f \in [\underline{G}]$ then $f'^{-1}(0) = \{0\}$ so $f' \in T_0$ and if $g \in T_0$ then $(\underline{\mathrm{pred}} \circ g \circ \underline{\mathrm{suc}})' = g$). Finally, we note that the set of transformations named by the elements of $G_0$ coincide with the $'$ image of the set of those transformations that are named by the elements of $G$.

We note that for any $\tau \in G_0$, $\mathbf{s}_\tau\mathbf{c}_0 x = \mathbf{c}_0\mathbf{s}_\tau x$. This follows from Lemma 3.14 since all the extra equations listed in the formulation of the lemma are consequences of $\mathrm{Ax}_G$, as it was proved up till this point. In passing, we mention that the same reasoning shows that Claim 3.16 proves that the definition of $\mathbf{c}_i$ $(i > 0)$ is implied by $\mathrm{Ax}_G$. Turning our attention again to (*6), suppose $\tau \in G^*$, $\underline{\tau}^{-1}(j) = \{i\}$. Then $f = \underline{[j,0]} \circ \underline{\tau} \circ \underline{[i,0]} \in T_0$. Using the previous observation, $f$ is a composition of functions named by elements of $G_0$ say, $f = \underline{\nu_1} \circ \ldots \circ \underline{\nu_n}$. (3), (4) and (6) imply that

$$\mathbf{s}_{[j,0]\tau[i,0]}\mathbf{c}_0 x = \mathbf{s}_{\nu_1\ldots\nu_n}\mathbf{c}_0 x = \mathbf{s}_{\nu_1}\ldots\mathbf{s}_{\nu_n}\mathbf{c}_0 x = \mathbf{c}_0\mathbf{s}_{\nu_1}\ldots\mathbf{s}_{\nu_n} x = \mathbf{c}_0\mathbf{s}_{[j,0]\tau[i,0]} x.$$

Finally we derive

$$\begin{aligned} \mathbf{s}_\tau \mathbf{c}_i x &\overset{\dagger}{=} \mathbf{s}_\tau \mathbf{s}_{[i,0]}\mathbf{c}_0\mathbf{s}_{[i,0]} x = \mathbf{s}_{[j,0]}\mathbf{s}_{[j,0]\tau[i,0]}\mathbf{c}_0\mathbf{s}_{[i,0]} x = \\ &= \mathbf{s}_{[j,0]}\mathbf{c}_0\mathbf{s}_{[j,0]\tau[i,0]}\mathbf{s}_{[i,0]} x \overset{\dagger}{=} \mathbf{c}_j\mathbf{s}_\tau x. \end{aligned}$$

(In † we used that the definition of $\mathbf{c}_i$ and $\mathbf{c}_j$ is already proven from $\mathrm{Ax}_G$.)

It is left to prove $\mathbf{c}_i\mathbf{c}_j x = \mathbf{c}_j\mathbf{c}_i x$ in (*1). For that we need $x \le y \Rightarrow \mathbf{c}_i x \le \mathbf{c}_i y$ and $\mathbf{c}_i\mathbf{c}_j\mathbf{c}_i x = \mathbf{c}_j\mathbf{c}_i x$. $\mathbf{c}_i\mathbf{c}_j\mathbf{c}_i x \overset{6}{=} \mathbf{c}_i\mathbf{c}_j\mathbf{s}_{[i/k]}\mathbf{c}_i x \overset{*6}{=} \mathbf{c}_i\mathbf{s}_{[i/k]}\mathbf{c}_j\mathbf{c}_i x \overset{6}{=} \mathbf{s}_{[i/k]}\mathbf{c}_j\mathbf{c}_i x \overset{*6}{=} \mathbf{c}_j\mathbf{s}_{[i/k]}\mathbf{c}_i x \overset{6}{=} \mathbf{c}_j\mathbf{c}_i x$.
Now $\mathbf{c}_i\mathbf{c}_j x \le \mathbf{c}_i\mathbf{c}_j\mathbf{c}_i x = \mathbf{c}_j\mathbf{c}_i x$ and symmetrically $\mathbf{c}_j\mathbf{c}_i x \le \mathbf{c}_i\mathbf{c}_j x$ giving $\mathbf{c}_j\mathbf{c}_i x = \mathbf{c}_i\mathbf{c}_j x$. ∎

In the previous two subsections we distinguished the symbol $\tau$ from the transformation $\underline{\tau}$ denoted by it. This distinction is not necessary anymore since axioms (*3), (*4) of Definition 3.3 hold in all algebras studied hereafter. So we return to the original notation where $\tau$ denotes *both* the symbol and the corresponding transformation hoping that context will help.

### 3.3 Theorem 2.11: compressing the unit element

**Proof of Theorem 2.11 (ii)** $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}} = \mathbf{H}\mathrm{Cdf}_H$ can be verified as follows. Suppose that $\mathcal{A} \in \mathrm{Gwdf}_H^{\mathrm{cm}}$. Put $V$ to be the unit element, $A$ the universe of $\mathcal{A}$. Since $V$ is a compressed $\mathrm{Gws}_\omega$–unit, every weak space in $V$ has the same base, say $U$. Put $B = \{x \subseteq {}^\omega U : x \cap V \in A\}$. Clearly, $\mathcal{B} \overset{\mathrm{def}}{=} \langle B, \cup, -, \mathbf{s}_\tau, \mathbf{c}_k\rangle_{\tau\in H, k\in\omega}$ is in $\mathrm{Cdf}_H$ and the relativization of $\mathcal{B}$ with the set $V$ is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$. This gives that $\mathcal{A} \in \mathbf{H}\mathrm{Cdf}_H$.

We need to show that $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}} = \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}}$. The $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}} \supseteq \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}}$ part is obvious so it is left to check that $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}} \subseteq \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}}$.

Let $\mathcal{A}$ be an algebra from $\mathrm{Gwdf}_H^{\mathrm{nm}}$. The unit element $V$ of $\mathcal{A}$ is a normal $\mathrm{Gws}_\omega$–unit therefore it is a disjoint union of weak spaces. Denote the set of weak spaces building $V$ by $\Lambda$. We put $U$ to be the union of the bases of elements of $\Lambda$. We define an accessibility relation $R$ on $\Lambda$ as: for any $\lambda_1, \lambda_2 \in \Lambda$

$$\lambda_1 R \lambda_2 \quad \text{iff} \quad (\exists q \in \lambda_1)(\exists \sigma \in H)\; q \circ \sigma \in \lambda_2.$$

It follows that $\lambda_1 R \lambda_2 \Rightarrow base(\lambda_1) = base(\lambda_2)$. (Here we used that $V$ is a normal $\mathrm{Gws}_\omega$–unit.) Let $R^*$ be the transitive, symmetric, reflexive closure of $R$. Clearly, $R^*$ is an equivalence relation. We have that $\lambda_1 R^* \lambda_2 \Rightarrow base(\lambda_1) = base(\lambda_2)$.

For each $\chi \in \Lambda/R^*$ we put $U_\chi$ to be $base(\lambda)$ for some $\lambda \in \chi$, and we fix a function $f_\chi : U \to U_\chi$ that is the identity on $U_\chi$. Now we can map the sequences of $V$ into subsets of ${}^\omega U$ in a following way. For any $q \in \lambda \in \chi \in \Lambda/R^*$

$$m(q) \stackrel{\text{def}}{=} \{s \in {}^\omega U \mid f_\chi \circ s = q,\ s \approx q\}.$$

cll1

**Claim 3.19** *The statements below hold. For all $q, q', s \in V$*

*(i). $m(q)$ and $m(q')$ are disjoint if $q$ and $q'$ are distinct.*

*(ii). $s \in m(q) \Rightarrow s[i/u] \in m(q[i/f_\chi(u)])$ if $u \in U$.*

*(iii). $s \in m(q) \Rightarrow s \circ \sigma \in m(q \circ \sigma)$ if $\sigma \in H$.*

**Proof** (i): Suppose that $s \in m(q) \cap m(q')$. Then $q \approx s \approx q'$ so $q$ and $q'$ are in the same weak space of $V$, say in $\lambda$. Since $\lambda \in \chi$ for some $\chi \in \Lambda/R^*$ we infer that $q = f_\chi \circ s = q'$.

(ii): Suppose that $s \in m(q)$, $q \in \lambda \in \chi \in \Lambda/R^*$. Since $q \approx q[i/f_\chi(u)]$ we have $q[i/f_\chi(u)] \in \lambda \in \chi$. Now $f_\chi \circ s[i/u] = q[i/f_\chi(u)]$, $s[i/u] \approx s \approx q \approx q[i/f_\chi(u)]$ shows that (ii) holds.

(iii): Suppose that $s \in m(q)$, $q \in \lambda \in \chi \in \Lambda/R^*$. If $\lambda'$ is the weak space of $q \circ \sigma$ then $\lambda R \lambda'$ and so $\lambda R^* \lambda'$, $\lambda' \in \chi$. We have $s \approx q$, $f_\chi \circ s = q$ and therefore $f_\chi \circ s \circ \sigma = q \circ \sigma$ and $s \circ \sigma \approx q \circ \sigma$ (we used that $\sigma$ is bounded). ∎

We put $V^* = \bigcup_{q \in V} m(q)$. (ii) of the previous claim shows that $V^*$ is a compressed Cartesian space, (iii) gives that $\mathbf{s}_\sigma(V^*) = V^*$ whenever $\sigma \in H$.

cll2

**Claim 3.20** *The statements below hold.*

*(i).* $\{s \in V^* \mid \exists u \in U\ s[i/u] \in m(q)\} = \bigcup_{q' \in \mathbf{c}_i\{q\}} m(q')$.

*(ii).* $\{s \in V^* \mid s \circ \sigma \in m(q)\} = \bigcup_{q' \in \mathbf{s}_\sigma\{q\}} m(q')$.

**Proof** (i): If $s[i/u] \in m(q)$ then $s = s[i/u][i/s(i)] \stackrel{3.19(ii)}{\in} m(q[i/f_\chi(s(i))]) \subseteq \bigcup_{q' \in \mathbf{c}_i\{q\}} m(q')$. On the other hand, if $s \in m(q')$ for some $q' \in \mathbf{c}_i\{q\}$ then $s[i/q(i)] \stackrel{3.19(ii)}{\in} m(q'[i/f_\chi(q(i))]) = m(q)$ since $f_\chi(q(i)) = q(i), q'[i/q(i)] = q$.

(ii): If $s \in V^*, s \circ \sigma \in m(q)$ then $s \in m(q')$ for some $q' \in V$. By Claim 3.19 (iii), $s \circ \sigma \in m(q \circ \sigma)$. Claim 3.19 (i) implies $m(q) \cap m(q' \circ \sigma) \neq \emptyset \Rightarrow q' \circ \sigma = q$.

So $q' \in \mathbf{s}_\sigma\{q\}$. On the other hand, if $s \in m(q')$ for some $q' \in \mathbf{s}_\sigma\{q\}$ then $s \circ \sigma \in m(q' \circ \sigma) = m(q)$. ∎

We arrived to the point when the algebra in which $\mathcal{A}$ is embeddable and the representation function *rep* can be defined:

$$\mathcal{A}' = \underline{\mathcal{P}}(V^*) \in \mathrm{Gwdf}_H^{\mathrm{cm}},$$

$$\text{for any } a \in A, \quad rep(a) = \bigcup_{q \in a} m(q).$$

Clearly, *rep* is a Boolean embedding of $\mathcal{A}$ into $\mathcal{A}'$ since $m(q)$ is not empty ($q \in m(q)$), and $m(q)$ and $m(q')$ are disjoint when $q$ and $q'$ are distinct (see Claim 3.19(i)).

To see that *rep* preserves the extra Boolean operations we compute:

$\mathbf{c}_i rep(a) = \mathbf{c}_i \bigcup_{q \in a} m(q) = \bigcup_{q \in a} \mathbf{c}_i m(q) \overset{3.20(i)}{=} \bigcup_{q \in a} \bigcup_{q' \in \mathbf{c}_i\{q\}} m(q') = \bigcup_{q' \in \mathbf{c}_i a} m(q') = rep(\mathbf{c}_i a)$.

$\mathbf{s}_\sigma rep(a) = \mathbf{s}_\sigma \bigcup_{q \in a} m(q) = \bigcup_{q \in a} \mathbf{s}_\sigma m(q) \overset{3.20(ii)}{=} \bigcup_{q \in a} \bigcup_{q' \in \mathbf{s}_\sigma\{q\}} m(q') = \bigcup_{q' \in \mathbf{s}_\sigma a} m(q') = rep(\mathbf{s}_\sigma a)$. ∎

Before proving Theorem 2.11 (iii) we state an important result.

l-qb

**Lemma 3.21** *Let $H$ be a set of quasi-bounded transformations, $\mathcal{A}$ be a* $\mathrm{Gwdf}_H$*, $\lambda_1$ and $\lambda_2$ be weak spaces of $\mathcal{A}$. If $(\exists q \in \lambda_1)(\exists \sigma \in H)\ q \circ \sigma \in \lambda_2$, then $base(\lambda_1) \subseteq base(\lambda_2)$.*

**Proof** Suppose that $u \in base(\lambda_1)$, $q \in \lambda_1$, $\sigma \in H$, $|\sigma^{-1}(\sigma(k))| < \omega$ (there is a proper $k \in \omega$ while $\sigma$ is quasi-bounded), $q \circ \sigma \in \lambda_2$. Since $\lambda_1$ is a weak space, $q[\sigma(k)/u] \in \lambda_1$. But then $\{i \in \omega \mid q[\sigma(k)/u] \circ \sigma(i) \neq q \circ \sigma(i)\} \subseteq \sigma^{-1}(\sigma(k))$ is finite so $q[\sigma(k)/u] \circ \sigma \approx q \circ \sigma \in \lambda_2$. The sequence $q[\sigma(k)/u] \circ \sigma$ of $\lambda_2$ takes the value $u$ at $k$ so $u \in base(\lambda_2)$. ∎

**Proof of Theorem 2.11 (iii)** We need to show that under the conditions formulated in the theorem $\mathrm{Gwdf}_H \subseteq \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$.

Let $\mathcal{A}$ be an algebra from $\mathrm{Gwdf}_H$. As we did in the proof of (ii) we put $V$ to be the unit element of $\mathcal{A}$, $\Lambda$ to be the set of weak spaces of $\mathcal{A}$, $R$ to be the accessibility relation on $\Lambda$, $R^*$ to be the equivalence relation generated by $R$.

We show that $\lambda R \lambda_0 \Rightarrow base(\lambda) = base(\lambda_0)$ (and so $\lambda R^* \lambda_0 \Rightarrow base(\lambda) = base(\lambda_0)$). Suppose that $\lambda R \lambda_0$. Then $(\exists q \in \lambda)(\exists \sigma_0 \in H)\ q \circ \sigma_0 \in \lambda_0$. We know that $\sigma_0$ has a quasi-inverse in $[H]$, say $\sigma_1 \circ \ldots \circ \sigma_n$. For any $i \in n+1$ we put $\lambda_i$ to be the weak space of $q \circ \sigma_0 \circ \ldots \circ \sigma_i$. Lemma 3.21 implies that $base(\lambda_i) \subseteq base(\lambda_{i+1})$ $(i \in n)$. If we note that $\sigma_0 \circ \sigma_1 \circ \ldots \circ \sigma_n \approx Id$, and so $\lambda = \lambda_n$ then we have $base(\lambda) \subseteq base(\lambda_0) \subseteq base(\lambda_1) \subseteq \ldots \subseteq base(\lambda_n) = base(\lambda)$.

Now we can define the set $U_\chi$ to be $base(\lambda)$ for some $\lambda \in \chi \in \Lambda/R^*$. Surely, the definition is independent from the choice of $\lambda$. We put $f_\chi : U_\chi \to U_\chi \times \{\chi\}$ to be the natural embedding ($f_\chi(u) = \langle u, \chi\rangle$), and $\chi : V \to \Lambda/R^*$ to be the function that associates the $R^*$ class of the weak space of $q$ to a $q \in V$. We define

$$\text{for any } a \in A, \quad rep(a) = \{f_{\chi(q)} \circ q \mid q \in a\}$$
$$A' = \{rep(a) \mid a \in A\}$$
$$\mathcal{A}' = \langle A', \cup, -, \mathbf{s}_\tau, \mathbf{c}_k\rangle_{\tau \in H, k \in \omega}$$

It is straightforward to verify that $\mathcal{A}' \in \mathrm{Gwdf}_H^{\mathrm{nm}}$ and *rep* is an isomorphism between $\mathcal{A}$ and $\mathcal{A}'$. ∎

## 3.4 Quasi-axiomatizing $\mathbf{I}\mathrm{Cdf}_H$

To find a proof to Theorem 2.15, we examine the situation described in Example 2.13. The key idea is that the presence of a constant sequence can be captured by a quasi-equation. Among those $\mathrm{Gwdf}_G$'s that have constant sequences in their universes, the simplest in structure are the ones with unit elements of the form ${}^\omega U^{(q)}$ for some set $U$ and constant sequence $q$. A unit element of this form will be called *horizontal*. (It is easy to see that $\mathbf{s}_\tau({}^\omega U^{(q)}) = {}^\omega U^{(q)}$.)

Any algebra from $\mathrm{Gwdf}_H$ that has a constant sequence $q$ in it's universe can be homomorphically mapped into a $\mathrm{Gwdf}_H$ with horizontal unit element. (Indeed, the relativization with ${}^\omega U^{(q)}$ — the function that maps the element $x$ of the algebra into $x \cap {}^\omega U^{(q)}$ — will be a satisfactory homomorphism.) So any $\mathrm{Gwdf}_H$ that is isomorphic to a $\mathrm{Cdf}_H$ has a homomorphism into some $\mathrm{Gwdf}_H$ with horizontal unit element. The next theorem states that this requirement is not only necessary but also sufficient to force a $\mathrm{Gwdf}_H$ to be isomorphic to a $H$–cylindric set algebra.

th-csg

**Theorem 3.22** *Let $H$ be an arbitrary subset of ${}^\omega\omega$ with the property that for any $\sigma \in H$, $\sigma$ is bounded and has a right quasi-inverse in ${}^\omega\omega$. Then any algebra $\mathcal{A} \in \mathrm{Gwdf}_H$, that has a homomorphism into some $\mathrm{Gwdf}_H$ with horizontal unit element, is isomorphic to a $H$–cylindric set algebra. That is,*

$$(\exists \textit{ set } U)(\exists u_0 \in U)(\exists h : \mathcal{A} \to \underline{\mathcal{P}}({}^\omega U^{(\langle u_0, u_0, \ldots\rangle)})) \Rightarrow \mathcal{A} \in \mathbf{I}\mathrm{Cdf}_H.$$

Before the proof of the Theorem above we present an important lemma.

lm-d->s

**Lemma 3.23** *Suppose $U$ is a set, $u_0 \in U$. There is a cardinal $\kappa$ for that if $\underline{\mathcal{P}}(W')$ is a full compressed $H$–cylindric set algebra with base $U'$ of cardinality bigger than $\kappa$ then there is a homomorphism $h' : \underline{\mathcal{P}}({}^\omega U^{(\langle u_0, u_0, \ldots\rangle)}) \to \underline{\mathcal{P}}(W')$.*

**Proof** Let $\kappa$ be a cardinal bigger than $\omega$, the cardinality of $U$ and that of $H$.

For the algebra $\underline{\mathcal{P}}(W')$, we define the set $\Lambda$ of weak spaces, the accessibility relations $R$ and $R^*$ on $\Lambda$ as we defined in the proof of Theorem 2.11 (ii). For each weak space $\lambda \in \Lambda$ we fix a sequence $q_\lambda$ in $\lambda$. We put $U'_\chi$ $(\chi \in \Lambda/R^*)$ to be the set $\{q_\lambda(i) \mid \lambda \in \chi, i \in \omega\}$.

**Claim 3.24** *The cardinality of* $|U'_\chi| + |U|$ *is less than* $\kappa$. cl-card

**Proof** For a given $\lambda_1 \in \Lambda$ and $\sigma \in H$ there is exactly one $\lambda_2 \in \Lambda$ with $\lambda_1 R \lambda_2$. This can verified by noting that from $q \circ \sigma \in \lambda_2$, $q \in \lambda_1$ we can infer $q' \circ \sigma \in \lambda_2$ when $q' \in \lambda_1$ ($\sigma$ is bounded so $q \approx q' \Rightarrow q \circ \sigma \approx q' \circ \sigma$ and $q \circ \sigma \in \lambda_2 \cap \lambda'_2 \Rightarrow \lambda_2 = \lambda'_2$).

For any $\sigma \in H$, $\sigma$ is right quasi-invertible in ${}^\omega\omega$ so there is a transformation $f_\sigma \in {}^\omega\omega$ with $\sigma \circ f_\sigma \approx Id$. Suppose for contradiction that there is a $k \in \omega$ for that $|f_\sigma^{-1}(k)| = \omega$. Then the transformation $\sigma \circ f_\sigma$ takes the value $\sigma(k)$ on any coordinate of the infinite set $f_\sigma^{-1}(k)$. But this is impossible since $\sigma \circ f_\sigma$ takes the value $i$ on coordinate $i$ for all but finitely many $i \in \omega$. This reasoning shows that $f_\sigma$ is bounded.

For a given $\lambda_2 \in \Lambda$ and $\sigma \in H$ there is exactly one $\lambda_1 \in \Lambda$ with $\lambda_1 R \lambda_2$. Suppose that $\lambda_1, \lambda'_1 \in \Lambda$, $q \in \lambda_1$, $q' \in \lambda'_1$ and $q \circ \sigma \in \lambda_2 \ni q' \circ \sigma$. Then $q \circ \sigma \approx q' \circ \sigma$ and $q \circ \sigma \circ f_\sigma \approx q' \circ \sigma \circ f_\sigma$ since $f_\sigma$ is bounded. But this implies that $q \approx q \circ \sigma \circ f_\sigma \approx q' \circ \sigma \circ f_\sigma \approx q'$, $q \in \lambda_1 \cap \lambda'_1$ so $\lambda_1 = \lambda'_1$.

For a given $\lambda \in \Lambda$, $|\{\lambda' \in \Lambda \mid \lambda R \lambda' \text{ or } \lambda' R \lambda\}| \leq 2 \cdot |H|$. For a given $\lambda \in \Lambda$, we have that $\lambda' \in \Lambda$ and $\lambda$ sit in the same $R^*$ class if and only if there is a finite series $\lambda_0, \dots \lambda_n$ of weak spaces with $\lambda_0 = \lambda$, $\lambda_n = \lambda'$ and $i \in n \Rightarrow (\lambda_i R \lambda_{i+1} \text{ or } \lambda_{i+1} R \lambda_i)$. Therefore, the size of a $R^*$ class is less than $1 + 2 \cdot |H| + (2 \cdot |H|)^2 + \ldots \leq \omega \cdot |H|$. Finally, $|U'_\chi| \leq \omega \cdot |\chi| \leq \omega \cdot \omega \cdot |H| = \omega \cdot |H|$, $|U'_\chi| + |U| < \kappa$. ∎

Let $f_\chi$ $(\chi \in \Lambda/R^*)$ be any function mapping $U'$ onto $U$ with the property

$$(\forall i \in \omega)(\forall \lambda \in \chi) \;\; f_\chi(q_\lambda(i)) = u_0.$$

Such an $f_\chi$ exists since not more than $|U'_\chi|$-many elements of $U'$ are to be mapped on $u_0$ so there are enough elements left to cover the set $U - \{u_0\}$.

With the help of $f_\chi$, for each sequence $q$ of ${}^\omega U^{(\langle u_0, u_0, \dots \rangle)}$ a subset of $W'$ can be associated.

$$m(q) = \{s \in W' \mid f_\chi \circ s = q, s \in \lambda \in \chi \in \Lambda/R^*\}.$$

As it was presented in the more complicated situation in the proof of Theorem 2.11 (ii) the function $h'(a) = \bigcup_{q \in a} m(q)$ is a homomorphism. (Claims 3.19 and 3.20 hold here as well since the original proofs can be adapted to the present situation. ) ∎

**Proof of Theorem 3.22** In Theorem 2.11 (ii) and (iii) $\mathbf{I}\mathrm{Gwdf}_H = \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{cm}}$ was proved under the conditions that for any $\sigma \in H$, $\sigma$ is quasi-bounded and it has a quasi-inverse in $[H]$. This holds in our situation so we may assume that $\mathcal{A} \in \mathrm{Gwdf}_H^{\mathrm{cm}}$. The unit element of $\mathcal{A}$ is $V \subseteq {}^{\omega}U^*$ for some set $U^*$. Let $U'$ be a set with cardinality bigger than the cardinality of $U^*$ and the cardinal that corresponds to the set $U$ according to Lemma 3.23.

Let $f$ be a function mapping $U'$ onto $U^*$. With the help of $f$, for each sequence $q$ of ${}^{\omega}U^*$ a subset of ${}^{\omega}U'$ can be associated.

$$m(q) = \{s \in {}^{\omega}U' \mid f \circ s = q\}.$$

Let $V' = \bigcup\limits_{q \in V} m(q)$. As it was presented in the proof of Theorem 2.11 (ii) the function $r_1 : \mathcal{A} \to \underline{\mathcal{P}}(V')$, $\quad r_1(a) = \bigcup\limits_{q \in a} m(q)$ is an embedding.

Let $W' = {}^{\omega}U' \setminus V'$. Just like $V'$, $W'$ is a $\mathrm{Gws}_{\omega}$–unit and $\mathbf{s}_{\sigma}(W') = W'$ for any $\sigma \in H$. So $W'$ can be a unit element of a $\mathrm{Gwdf}_H^{\mathrm{cm}}$. According to Lemma 3.23 there is a homomorphism $h'$ mapping $\underline{\mathcal{P}}({}^{\omega}U'^{(\langle u_0, u_0, \ldots\rangle)})$ into $\underline{\mathcal{P}}(W')$. Let $r_2$ be the composition of the homomorphisms $h'$ and $h$ ($r_2 = h' \circ h$) mapping $\mathcal{A}$ into $\underline{\mathcal{P}}(W')$. Since $\underline{\mathcal{P}}({}^{\omega}U') = \underline{\mathcal{P}}(V') \times \underline{\mathcal{P}}(W')$ the function *rep* defined as *rep*$(a) = r_1(a) \cup r_2(a)$ is an embedding and it demonstrates that $\mathcal{A} \in \mathbf{I}\mathrm{Cdf}_H$. ∎

In the second part of this subsection we show that if an algebra in $\mathrm{Gwdf}_H$ satisfies Qx then it has a homomorphism to $\mathrm{Gwdf}_H$ with horizontal unit element. We assume that $H$ satisfies the assumption stated in Theorem 2.19.

ll1

**Lemma 3.25** *If $\mathcal{A} \in \mathrm{Gwdf}_H$ and $\mathcal{A} \models \mathrm{Qx}$ then there is an ultrafilter $U$ of $\mathcal{A}$ with the property that $-\mathbf{c}_{(\Gamma)}x \in U$ holds whenever $x \cap \mathbf{s}_{\tau}x = 0$ for some $\Gamma \subseteq_{\omega} \omega$ $\tau \in [H]$ with $\tau\lceil_{\Gamma} = Id\lceil_{\Gamma}$.*

cl1l1

**Claim 3.26** $\forall \sigma, \tau_1, \ldots, \tau_n \in [H]\ \exists \sigma', \tau_1', \ldots, \tau_n' \in [H] : \bigwedge\limits_{i \leq n} \sigma' \circ \tau_i = \tau_i' \circ \sigma.$ *Furthermore, if $\sigma\lceil_{\Gamma} = \tau_1\lceil_{\Gamma} = \cdots = \tau_n\lceil_{\Gamma} = Id\lceil_{\Gamma}$ then the same holds to $\sigma', \tau_1', \ldots, \tau_n'$ ($\Gamma \subseteq_{\omega} \omega$).*

**Proof** We prove by induction. If $n = 0$ then $\sigma' = \sigma$. Suppose that the statement holds for $n-1$. Let $\sigma'', \tau_1'', \ldots, \tau_{n-1}''$ be associated to $\sigma, \tau_1, \ldots, \tau_{n-1}$. Let $\sigma^*$ and $\tau_n^*$ be the left quasi-inverse of $\sigma'' \circ \tau_n$ and $\sigma$ respectively. Since $[H]$ contains all finite transformations we can assume that $\sigma^* \circ \sigma'' \circ \tau_n(k) = \left\{ \begin{array}{ll} k & \in \Gamma \\ 0 & k \in m_0 - \Gamma \\ k & m_0 \leq k \end{array} \right\} = \tau_n' \circ \sigma(k)$ for some $m_0 \in \omega$. Setting $\sigma' = \sigma^* \circ \sigma''$, $\tau_i' = \sigma^* \circ \tau_i''$ ($i = 1, \ldots, n$) gives the required result. ∎

**Proof of Lemma 3.25** Let $B^* = \{-\mathbf{c}_{(\Gamma)}x : \Gamma \subseteq_\omega \omega, \exists K \subseteq_\omega [H]\ (\tau \in K \to \tau\lceil_\Gamma = Id\lceil_\Gamma), \bigcap_{\tau\in K} \mathbf{s}_\tau x = 0\}$. Qx states that $B^*$ has the finite intersection property. We put $B$ to be $B^*$'s filter closure, that is, $B = \{y \in \mathcal{A} : \exists X \subseteq_\omega B^*\ y \geq \bigcap X\}$.

cl112

**Claim 3.27** *$B$ is closed under $\mathbf{s}_\nu$ for all $\nu \in [H]$.*

**Proof** Suppose $-\mathbf{c}_{(\Gamma)}x \in B^*$, $\bigcap_{\tau\in K} \mathbf{s}_\tau x = 0$ for some $\Gamma \subseteq_\omega \omega, K \subseteq_\omega [H]$, $\tau \in K \to \tau\lceil_\Gamma = Id\lceil_\Gamma$. Let $\Gamma' = \{i \in \omega - \Gamma : \nu(i) \in \Gamma, \forall j \in i - \Gamma\ (\nu(i) \neq \nu(j))\}$. $\nu$ is bounded so $\Gamma'$ is finite. If $\Gamma' = \{\gamma_1, \ldots, \gamma_n\}$ then let $\nu_0 = [\gamma_1, \nu(\gamma_1)] \circ \ldots \circ [\gamma_n, \nu(\gamma_n)]$. Let $\Gamma'' = \Gamma' \cup (\Gamma - \{\nu(\gamma) : \gamma \in \Gamma'\})$, $\nu_1 = (\nu_0 \circ \nu)[\Gamma/Id]$. Now we have that $\nu_0 \circ \nu_1\lceil_{\omega - \Gamma} = \nu\lceil_{\omega - \Gamma}$ and for all $i \in \Gamma$ $\nu_1^{-1}(i) = \{i\}$. Using Claim 3.26, we pick corresponding transformations $\nu_1'$ and $\tau'$ to $\nu$ and $\tau$ $(\tau \in K)$. Then $0 = \mathbf{s}_{\nu_0}\mathbf{s}_{\nu_1'} \bigcap_{\tau\in K} \mathbf{s}_\tau x = \bigcap_{\tau\in K} \mathbf{s}_{\nu_0}\mathbf{s}_{\nu_1'}\mathbf{s}_\tau x = \bigcap_{\tau\in K} \mathbf{s}_{\nu_0}\mathbf{s}_{\tau'}\mathbf{s}_{\nu_1} x = \bigcap_{\tau\in K} \mathbf{s}_{\nu_0\circ\tau'\circ\nu_0}\mathbf{s}_{\nu_0}\mathbf{s}_{\nu_1} x$. Noting that $\nu_0\circ\tau'\circ\nu_0\lceil_{\Gamma''} = Id\lceil_{\Gamma''}$ we have $-\mathbf{c}_{(\Gamma'')}\mathbf{s}_{\nu_0}\mathbf{s}_{\nu_1}x \in B^*$. So $-\mathbf{c}_{(\Gamma'')}\mathbf{s}_{\nu_0}\mathbf{s}_{\nu_1}x \overset{*6}{=} -\mathbf{s}_{\nu_0}\mathbf{c}_{(\Gamma)}\mathbf{s}_{\nu_1}x \overset{*6}{=} -\mathbf{s}_{\nu_0}\mathbf{s}_{\nu_1}\mathbf{c}_{(\Gamma)}x \overset{*5}{=} -\mathbf{s}_\nu\mathbf{c}_{(\Gamma)}x = \mathbf{s}_\nu(-\mathbf{c}_{(\Gamma)}x)$ gives $\mathbf{s}_\nu(-\mathbf{c}_{(\Gamma)}x) \in B^*$.

Finally, suppose that $x \in B$. Then there are $x_1, \ldots, x_n \in B^*$ with $x \geq \bigcap_1^n x_i$. So $\mathbf{s}_\nu x \geq \mathbf{s}_\nu \bigcap_1^n x_i = \bigcap_1^n \mathbf{s}_\nu x_i \in B$ giving $\mathbf{s}_\nu x \in B$. ∎

Let $U$ be a maximal filter extending $B$ with the property that it is closed under $\mathbf{s}_\nu$ for all $\nu \in [H]$. We can use Zorn's lemma to construct such a filter. We show that $U$ is indeed an ultrafilter. Suppose not. Then there is a $y$ such that neither $y$ nor $-y$ is in $U$. $y$ cannot be added to $U$ so there are $z \in U$, $K \subseteq_\omega [H]$ such that $z \cap \bigcap_{\tau\in K} \mathbf{s}_\tau y = 0$. Then $0 = z \cap \bigcap_{\tau\in K} \mathbf{s}_\tau y \supseteq \bigcap_{\tau\in K} \mathbf{s}_\tau(z \cap y)$ giving $-(y \cap z) \in B$. Similarly, $-(-y \cap z') \in B$. Therefore $U \ni -(-y \cap z') \cap -(y \cap z) = -((y \cup z') \cap (z \cup z') \cap (z \cup -y))$. But all the three terms in the last expression are in $U$ so their intersection is in $U$ giving that $U$ contains an element and its complement. ∎

Our algebra $\mathcal{A} \in \mathrm{Gwdf}_H$ is a *normal Boolean algebra with operators.* (Cf. e.g. [HMTI] Definition 2.7.1.) We can define its *canonical embedding algebra* or *ultrafilter extension* $\mathcal{A}^*$ as follows (see Definition 2.7.4. in [HMTI]). Let $V$ be the set of ultrafilters of $\mathcal{A}$. If $O$ is any of the operators $\{\mathbf{s}_\tau : \tau \in H\} \cup \{\mathbf{c}_i : i \in \omega\}$ and $X \subseteq V$ then we define

$$OX = \{U \in V : (\exists S \in X)(\forall x \in U)\ Ox \in S\}.$$

Then $\mathcal{A}^*$ is $\langle \mathcal{P}(V), \cup, -, \mathbf{s}_\tau, \mathbf{c}_i \rangle_{\tau\in H, i\in\omega}$.

As stated in Corollary 3.5, $\mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$ is axiomatized by (*1)–(*7) (defined in Definition 3.3) together with the definition of $\mathbf{c}_i$ ($i \in \omega$). All that axioms but $\mathbf{c}_i(-\mathbf{c}_i x) = -\mathbf{c}_i x$ and $\mathbf{s}_\sigma(-x) = -\mathbf{s}_\sigma x$ are positive, that is, they do not contain any occurance of the symbol $-$ for complementation. But the two exceptions are equivalent (in the presence of the others) with some positive formulas:

$$\begin{aligned} \mathbf{c}_i(-\mathbf{c}_i x) = -\mathbf{c}_i x \quad &\text{iff} \quad \mathbf{c}_i 0 = 0, \mathbf{c}_i(x \wedge \mathbf{c}_i y) = \mathbf{c}_i x \wedge \mathbf{c}_i y \\ \mathbf{s}_\sigma(-x) = -\mathbf{s}_\sigma x \quad &\text{iff} \quad \mathbf{s}_\sigma(x \wedge y) = \mathbf{s}_\sigma x \wedge \mathbf{s}_\sigma y. \end{aligned}$$

Since positive equations valid in a Boolean algebra with operators still hold in the ultrafilter extension of the algebra, (see e.g. [HMTI] Remark 2.7.15) we can conclude that the ultrafilter extension $\mathcal{A}^*$ of $\mathcal{A} \in \mathrm{Gwdf}_H^{\mathrm{nm}}$ is in $\mathrm{Gwdf}_H^{\mathrm{nm}}$. Furthermore, under the hypothesis of Theorem 2.19, $\mathbf{I}\mathrm{Gwdf}_H = \mathbf{I}\mathrm{Gwdf}_H^{\mathrm{nm}}$ holds, so the conclusion stated above holds to the class $\mathrm{Gwdf}_H$ as well.

Finally, we note that every Boolean algebra with operators is embeddable into its ultrafilter extension. (The map that associates $\{U \in V : x \in U\}$ to an element $x$ in $\mathcal{A}$ will suffice.)

ll2

**Lemma 3.28** *If $\mathcal{A} \in \mathrm{Gwdf}_H$ and $\mathcal{A}^*$ is its ultrafilter extension then the atom $a = \{U\}$ has the property $x \cap \mathbf{c}_{(\Gamma)} a \subseteq \mathbf{s}_\tau x$ ($\Gamma \subseteq_\omega \omega, \tau \in [H], \tau\lceil_\Gamma = Id\lceil_\Gamma, x \in \mathcal{A}^*$) provided that $U$ is an ultrafilter of $\mathcal{A}$ that satisfies the property stated in Lemma 3.25.*

**Proof** It is enough to show that the condition $x \cap \mathbf{c}_{(\Gamma)} a \subseteq \mathbf{s}_\tau x$ holds for all atoms of $\mathcal{A}^*$. Suppose $x = \{F\}$. If $F \notin \mathbf{c}_{(\Gamma)} a$ then there is nothing to show. If $F \in \mathbf{c}_{(\Gamma)} a$ then $\forall y \in F\ \mathbf{c}_{(\Gamma)} y \in U$. Suppose for contradiction that $x \not\subseteq \mathbf{s}_\tau x$, that is, $\exists y \in F\ \mathbf{s}_\tau y \notin F$. Then $-\mathbf{s}_\tau y \in F$, $w = y \cap -\mathbf{s}_\tau y \in F$. We can conclude that $\mathbf{c}_{(\Gamma)} w \in U$. On the other hand, $w \cap \mathbf{s}_\tau w = 0$ gives $-\mathbf{c}_{(\Gamma)} w \in U$ and that is a contradiction. ∎

ll3

**Lemma 3.29** *If $\mathcal{A}^* \in \mathrm{Gwdf}_H^{\mathrm{cm}}$ and $\mathcal{A}^*$ has an atom $a$ with $x \cap \mathbf{c}_{(\Gamma)} a \subseteq \mathbf{s}_\tau x$ whenever $\Gamma \subseteq_\omega \omega, \tau \in [H], \tau\lceil_\Gamma = Id\lceil_\Gamma, x \in \mathcal{A}^*$ then $\mathcal{A}^*$ has a homomorphism to a $\mathrm{Gwdf}_H$ with horizontal unit element.*

**Proof** We denote the base of $\mathcal{A}^*$ by $U'$ and the universe of $\mathcal{A}^*$ by $A^*$. Let $q$ be a sequence in $a$, $V = \{q \circ \tau : \tau \in [H]\}$, $V^* = \bigcup_{m \in \omega} \mathbf{c}_{(m)} V$. Put $U_0$ to be the range of $q$. Fix an arbitrary element $u \in U_0$. Denote $U' - U_0 \cup \{u\}$ by $U$. Let $f$ be the function that is the identity on $U' - U_0$ and the constant $u$ on $U_0$.

clll3

**Claim 3.30** $x \in A^*, s, s' \in V^*, s \in x, f \circ s = f \circ s' \to s' \in x.$

**Proof** Let $\Gamma = \{i \in \omega : s(i) \notin U_0\}$. Since $s \approx q \circ \tau$ for some $\tau \in [H]$, $\Gamma$ is finite. Clearly, $\Gamma = \{i \in \omega : s'(i) \notin U_0\}$ so $s\lceil_\Gamma = s'\lceil_\Gamma$. We have that $s' \approx q \circ \tau'$ for some

$\tau' \in [H]$. $s = q \circ \tau[\Gamma/s]$, $s' = q \circ \tau'[\Gamma/s]$. Let $\nu$ and $\nu'$ be the right quasi-inverses of $\tau$ and $\tau'$. Since all the finite transformations are in $[H]$, we can choose $\nu$ and $\nu'$ that they satisfy the following additional properties: $\nu\lceil_{\Gamma} = \nu'\lceil_{\Gamma} = Id\lceil_{\Gamma}$, $\tau \circ \nu\lceil_{\omega - \Gamma} = \tau' \circ \nu'\lceil_{\omega - \Gamma}$. Then $s \circ \nu = (q \circ \tau[\Gamma/s]) \circ \nu = q \circ \tau \circ \nu[\Gamma/s] = q \circ \tau' \circ \nu'[\Gamma/s] = (q \circ \tau'[\Gamma/s]) \circ \nu' = s' \circ \nu'$. Since $s \in x \cap \mathbf{c}_{(\Gamma)} a \subseteq \mathbf{s}_{\nu} x$, we get that $s' \circ \nu' = s \circ \nu \in x$. (We used that $a \subseteq \mathbf{s}_{\tau} a$ and so $q \circ \tau \in a$.) If we suppose for contradiction that $s' \notin x$ then $s' \in -x \cap \mathbf{c}_{(\Gamma)} a \subseteq \mathbf{s}_{\nu'} - x$ and so $s' \circ \nu' \in -x$ but that cannot be the case. ∎

Now we can define a homomorphism that maps $\mathcal{A}$ into $\underline{\mathcal{P}}({}^{\omega}U^{(\langle u,u,\ldots\rangle)})$:

$$h(x) = \{f \circ s : s \in x \cap V^*\}.$$

It is straightforward to verify that $h$ preserves $\cap$, while Claim 3.30 proves that it preserves complementation. The following two claims prove that $h$ preserves the extra Boolean operations.

cll14

**Claim 3.31** $\mathbf{c}_i h(x) = h(\mathbf{c}_i x)$.

**Proof** $r \in \mathbf{c}_i h(x) \Rightarrow r = (f \circ s)[i/r(i)]$ for some $s \in x$. But $r = (f \circ s)[i/r(i)] = f \circ (s[i/r(i)]) \in h(\mathbf{c}_i x)$ since $s[i/r(i)] \in \mathbf{c}_i x$.

On the other hand, $r \in h(\mathbf{c}_i x) \Rightarrow r = f \circ s$ for some $s \in \mathbf{c}_i x$. So there is $s' \in x$ with $s'[i/s(i)] = s$. But $r = f \circ s = f \circ (s'[i/s(i)]) = (f \circ s')[i/f(s(i))]) \in \mathbf{c}_i h(x)$ ∎

cll15

**Claim 3.32** $\mathbf{s}_{\tau} h(x) = h(\mathbf{s}_{\tau} x)$.

**Proof** $r \in \mathbf{s}_{\tau} h(x) \Leftrightarrow r \circ \tau \in h(x) \Leftrightarrow \exists s' \in x\ f \circ s' = r \circ \tau$. $r = f \circ s$ for some $s \in V^*$ so $f \circ s' = r \circ \tau = f \circ s \circ \tau$. Using Claim 3.30 we have that $s' \in x \Leftrightarrow s \circ \tau \in x$. But $s \circ \tau \in x \Leftrightarrow s \in \mathbf{s}_{\tau} x \Leftrightarrow r = f \circ s \in h(\mathbf{s}_{\tau} x)$ completes the proof. ∎

∎

Summarizing the results of this subsection, we prove Theorem 2.19.

**Proof of Theorem 2.19** Suppose that the algebra $\mathcal{A}$ is a model of Qx $\cup$ $Eq(\mathbf{I}\mathrm{Gwdf}_H)$. Lemmas 3.25 and 3.28 state that the ultrafilter extension $\mathcal{A}^*$ of $\mathcal{A}$ has an atom with certain properties. Since the ultrafilter extension satisfies all the positive equations valid in the algebra, we have that $\mathcal{A}^* \models Eq(\mathbf{I}\mathrm{Gwdf}_H)$. Using Theorem 2.11 we conclude that $\mathcal{A}^*$ is isomorphic to an algebra $\mathcal{B} \in \mathrm{Gwdf}_H^{\mathrm{cm}}$. Lemma 3.29 implies that $\mathcal{B}$ has a homomorphism to a $\mathrm{Gwdf}_H$ with horizontal unit element. Since $\mathcal{A}$ is embeddable into its ultrafilter extension and, in that way, into $\mathcal{B}$, $\mathcal{A}$ also has a homomorphism to a $\mathrm{Gwdf}_H$ with horizontal unit element.

Finally, Theorem 3.22 states that the above mentioned result is sufficient to show that $\mathcal{A} \in \mathbf{I}\mathrm{Cdf}_H$. ∎

### 3.5 Theorem 2.30: the cylindric reduct of $Mod(H1, H2)$

The proof will have the following structure. Any model of H1 is actually an extension of a model of $Eq\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$. But Theorem 2.11 (i) imply that a model of $Eq\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$ is isomorphic to a $\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$ We will examine the extra abstract constant $\mathbf{d}_{01}$. (In the presence of the substitutions the other $\mathbf{d}_{ij}$'s are term definable so it is enough to concentrate on $\mathbf{d}_{01}$.) Finally, the proper behaviour of $\mathbf{d}_{01}$ will lead us to a representation function that proves our theorem.

Recall from Notation 2.8 that if $q_1$ are $q_2$ sequences then $q_1 \approx q_2$ indicates that they agree on all but finitely many coordinates.

nt-stau

**Notation 3.33** Let $H$ be a set of transformations of $\omega$. Suppose that $\tau = \langle \sigma_1 \dots \sigma_n \rangle \in H^*$. We introduce the notation $\mathbf{s}_\tau x$ for the term $\mathbf{s}_{\sigma_1} \dots \mathbf{s}_{\sigma_n} x$. □

rm-H4

**Remark 3.34** Let $H$ be a set of transformations of $\omega$ with the property that $[\bar{H}]$ is rich. (Recall that $[\bar{H}]$ is the subsemigroup of $\langle {}^\omega\omega, \circ \rangle$ generated by $H \cup \{[i/j] \mid i, j \in \omega\}$.)

In the definition of a rich semigroup the existence of two special elements $\hat{s}$ and $\hat{p}$ were stated. We fix two elements of $[\bar{H}]$ with the required property and denote them by $\hat{s}$ and $\hat{p}$. We also fix $\mathbf{s}_{\hat{s}}$ and $\mathbf{s}_{\hat{p}}$ to stand for the terms $\mathbf{s}_{\tau_1}$ and $\mathbf{s}_{\tau_2}$ respectively where $\tau_1 = \hat{s}$, $\tau_2 = \hat{p}$, and $\tau_1, \tau_2 \in \bar{H}^*$. It is provable from H1 that for any choice of $\tau_1 \in \bar{H}^*$ with $\tau_1 = \hat{s}$, $\mathbf{s}_{\tau_1}$ is the same operation of the algebra so $\mathbf{s}_{\hat{s}}$ (and similarly $\mathbf{s}_{\hat{p}}$) is well defined.

$\hat{s}(i) \neq \hat{s}(j)$ if $i \neq j$ since $\hat{p} \circ \hat{s} = Id$. There is a permutation $f$ of $\omega$ that takes $\hat{s}(0)$ to 0 and $\hat{s}(1)$ to 1 and fixes all the elements of $\omega - \{0, 1, \hat{s}(0), \hat{s}(1)\}$. ($f$ is one of $[0, \hat{s}(0)] \circ [1, \hat{s}(1)]$, $[1, \hat{s}(1)] \circ [0, \hat{s}(0)]$ or $[0, 1]$ depending on whether any of $\hat{s}(0) = 0$, $\hat{s}(0) = 1$, $\hat{s}(1) = 0$ or $\hat{s}(1) = 1$ hold.) Since $f$ is finite $f \in [\bar{H}]$. So there is a $\tau_H \in \bar{H}^*$ for that $\tau_H = f$.

We formalize a set of equations that are all consequences of H1,H2.

$$
\begin{array}{rcl}
\mathbf{s}_{[0/1]} x & = & \mathbf{c}_0(\mathbf{d}_{01} \wedge x), \\
\mathbf{s}_{[1/0]} x & = & \mathbf{c}_1(\mathbf{d}_{01} \wedge x), \\
\mathbf{c}_i \mathbf{d}_{01} & = & \mathbf{d}_{01} \quad \text{whenever } i \geq 2, \\
\mathbf{s}_{[0,1]} \mathbf{d}_{01} & = & \mathbf{d}_{01}, \\
\mathbf{s}_{\tau_H} \mathbf{s}_{\hat{s}} \mathbf{d}_{01} & = & \mathbf{d}_{01}.
\end{array}
\tag{H4}
$$

Here $\mathbf{s}_{[i,j]} x$ stands for the term $\mathbf{s}_\tau x$ where $\tau \in \bar{H}^*$ and $\tau = [i, j]$. Since $[\bar{H}]$ is rich, all the finite transformations (including $[i, j]$) belong to $[\bar{H}]$.

lm-2

**Lemma 3.35** *Let $H$ be a set of transformations of $\omega$ with the property that $[\bar{H}]$ is rich. Suppose that $\mathcal{A}$ is a $\mathrm{Gwdf}_{\bar{H}}$ extended with the abstract*[18] *constant* $\mathbf{d}_{01}$

[18] Here the adjective "abstract" means that this constant is NOT necessarily the set theoretically definable diagonal element, but any constant that satisfies the required equations. It will be shown that this "abstract" constant need to have a nice structure similar to that of the set theoretic diagonal element.

*that satisfies* H4. *If $x$ is an element of $\mathcal{A}$ and $\langle u_0, u_1, u_2, \ldots\rangle$ is a sequence from the universe of $\mathcal{A}$ then $\langle u_0, u_1, u_2, \ldots\rangle \in \mathbf{d}_{01} \wedge x$ implies*

$$\begin{aligned} \langle u_0, u_0, u_2, \ldots\rangle &\in x, \\ \langle u_1, u_0, u_2, \ldots\rangle &\in x, \\ \langle u_1, u_1, u_2, \ldots\rangle &\in x. \end{aligned}$$

**Proof** Suppose that $\langle u_0, u_1, u_2, \ldots\rangle$ satisfies the premise of the lemma. Then $\langle u_0, u_1, u_2, \ldots\rangle \in \mathbf{c}_0(\mathbf{d}_{01} \wedge x) \overset{H4}{=} \mathbf{s}_{[0/1]}x$. This implies $\langle u_0, u_1, u_2, \ldots\rangle \circ [0/1] = \langle u_1, u_1, u_2, \ldots\rangle \in x$.

Using $\mathbf{s}_{[1/0]}x \overset{H4}{=} \mathbf{c}_1(\mathbf{d}_{01} \wedge x)$ we can get $\langle u_0, u_0, u_2, \ldots\rangle \in x$.

$\mathbf{s}_{[0,1]}\mathbf{d}_{01} \overset{H4}{=} \mathbf{d}_{01}$ implies $\langle u_1, u_0, u_2, \ldots\rangle \in \mathbf{d}_{01}$. If we suppose for contradiction that $\langle u_1, u_0, u_2, \ldots\rangle \notin x$ then $\langle u_1, u_0, u_2, \ldots\rangle \in -x$ and according to the previous observation we can conclude that $\langle u_1, u_1, u_2, \ldots\rangle \in -x$ and that is impossible. ∎

lm-3

**Lemma 3.36** *Let $H, \mathcal{A}$ and $\mathbf{d}_{01}$ be as in Lemma 3.35. If $u$ and $v$ are the first two elements of a sequence $q$ of $\mathbf{d}_{01}$ then any sequence $r$ of an arbitrary $x \in A$ can be modified in any of its positions from $u$ to $v$ provided that $q$ and $r$ are in the same weak space. In other words, if $q \in \mathbf{d}_{01}$, $r \in x$ for some $x \in A$, $q \approx r$, $r(k) = q(0) = u$ for some $k \in \omega$ and $q(1) = v$ then $r[k/v] \in x$.*

**Proof** First suppose that $k \neq 1$. Let $q^* = (r \circ [0,k])[1/v]$. Clearly, $q^*$ is in the weak space of $q$ (and of $r$). Therefore, $q^* \in \mathbf{c}_{i_1} \ldots \mathbf{c}_{i_n}\mathbf{d}_{01}$ where $\Gamma = \{i_1, \ldots i_n\}$ is the set where $q$ and $q^*$ do not agree (it is finite). Now $q^*(0) = (r \circ [0,k])(0) = r(k) = u$ and $q^*(1) = v$ so $0, 1 \notin \Gamma$ and $\mathbf{c}_i\mathbf{d}_{01} \overset{H4}{=} \mathbf{d}_{01}$ whenever $i \geq 2$. We can conclude that $q^* \in \mathbf{d}_{01}$.

We put $z$ to be an element of $\omega - Rng(\hat{s})$. $[\bar{H}]$ is rich so $\omega - Rng(\hat{s})$ is not empty. $(q^* \circ \hat{p})[z/r(1)] \circ \hat{s} = q^* \in \mathbf{d}_{01}$ implies that $(q^* \circ \hat{p})[z/r(1)] \in \mathbf{s}_{\hat{s}}\mathbf{d}_{01}$. $(q^* \circ \hat{p})[z/v][\hat{s}(1)/r(1)] \circ \hat{s} = r \circ [0,k] \in \mathbf{s}_{[0,k]}x$, $(q^* \circ \hat{p})[z/v][\hat{s}(1)/r(1)] \in \mathbf{s}_{\hat{s}}\mathbf{s}_{[0,k]}x$. Furthermore, $(q^* \circ \hat{p})[z/r(1)] = (q^* \circ \hat{p})[z/v][\hat{s}(1)/r(1)] \circ [z, \hat{s}(1)]$. We put $r^* \overset{\text{def}}{=} (q^* \circ \hat{p})[z/r(1)] \circ \tau_H^{-1}$.

Summarizing all the results we get that $r^* \in \mathbf{s}_{\tau_H}\mathbf{s}_{\hat{s}}\mathbf{d}_{01} \overset{H4}{=} \mathbf{d}_{01}$ and $r^* \in \mathbf{s}_{\tau_H}\mathbf{s}_{[z,\hat{s}(1)]}\mathbf{s}_{\hat{s}}\mathbf{s}_{[0,k]}x$, $r^*(0) = q^*(0) = u$, $r^*(1) = q^*(1) = v$. Using Lemma 3.35 we conclude that $r^*[0/v] \in \mathbf{s}_{\tau_H}\mathbf{s}_{[z,\hat{s}(1)]}\mathbf{s}_{\hat{s}}\mathbf{s}_{[0,k]}x$. This means that $r^*[0/v] \circ \tau_H \circ [z, \hat{s}(1)] \circ \hat{s} \circ [0,k] \in x$. Then $r^*[0/v] \circ \tau_H \circ [z, \hat{s}(1)] \circ \hat{s} \circ [0,k] = ((q^* \circ \hat{p})[z/r(1)] \circ \tau_H^{-1} \circ \tau_H)[\hat{s}(0)/v] \circ [z, \hat{s}(1)] \circ \hat{s} \circ [0,k] = (q^* \circ \hat{p} \circ [z, \hat{s}(1)])[\hat{s}(1)/r(1)][\hat{s}(0)/v] \circ \hat{s} \circ [0,k] = (q^* \circ \hat{p} \circ \hat{s})[1/r(1)][0/v] \circ [0,k] = ((r \circ [0,k])[1/v][1/r(1)] \circ [0,k])[k/v] = (r \circ [0,k] \circ [0,k])[k/v] = r[k/v] \in x$.

If $k = 1$ then $r \in x \;\Rightarrow\; r \circ [1,2] \in \mathbf{s}_{[1,2]}x \;\Rightarrow\; (r \circ [1,2])[2/v] \in \mathbf{s}_{[1,2]}x \;\Rightarrow$ $(r \circ [1,2])[2/v] \circ [1,2] = r[1/v] \in x$. ∎

**Proof of Theorem 2.27 (1)** Surely, any representable cylindric set algebra is a subalgebra of a reduct of a model of H1,H2.

To see the other direction, suppose that $\mathcal{A}'$ is a model of H1,H2. $\mathcal{A}' \models H1$ means that all the valid equations of $\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$ hold in $\mathcal{A}'$. Naturally, some operations of $t_{\bar{H}}$ are term functions in $\mathrm{Gws}_H$. Since $\mathbf{I}\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$ is a variety, the $t_{\bar{H}}$–reduct of $\mathcal{A}'$ is isomorphic to a $\mathrm{Gwdf}_{\bar{H}}^{\mathrm{nm}}$, say to $\mathcal{A}$. We will use only that $\mathcal{A} \in \mathbf{I}\mathrm{Gwdf}_{\bar{H}}$. The image of $\mathbf{d}_{01}$ under the isomorphism is an element of $A$ not necessary the set theoretic (0,1)–diagonal. The image of any other diagonal constant can be computed from the image of $\mathbf{d}_{01}$ since $\mathbf{d}_{ij}$ is term definable from $\mathbf{d}_{01}$ in $\mathrm{Gwdf}_{\bar{H}}$. This shows that to represent $\mathcal{A}'$, it is enough to represent $\mathcal{A}$ that is to find a isomorphism *rep* mapping the cylindric reduct of $\mathcal{A}$ into a $\mathrm{RCA}_\omega$.

The universe $V$ of $\mathcal{A}$ is a $\mathrm{Gws}_\omega$–unit therefore it is a (disjoint) union of weak spaces. Denote the set of weak spaces building $\mathcal{A}$ by $\Lambda$. For any weak space $\lambda$ from $\Lambda$, $base(\lambda)$ is the base set of $\lambda$. $\mathbf{d}_{01}$ defines an equivalence relation on $base(\lambda)$ in a following way: for any $u, u' \in base(\lambda)$

$$u \sim_\lambda u' \ \text{ iff } \ (\exists q \in \mathbf{d}_{01} \cap \lambda) \ \ q(0) = u, \ q(1) = u'$$

cl-1

**Claim 3.37** $\sim_\lambda$ *is reflexive.*

**Proof** Suppose $u \in base(\lambda)$. Let $\langle u_0, u_1, u_2, \ldots \rangle \in \lambda$ arbitrary. It is obvious that $\langle u, u, u_2, \ldots \rangle \in \lambda$. Now $\langle u, u, u_2, \ldots \rangle \in V = \mathbf{c}_1\mathbf{d}_{01}$. So $(\exists v \in base(\lambda))$ $\langle u, v, u_2, \ldots \rangle \in \mathbf{d}_{01}$. Applying Lemma 3.35 to the case when $x = \mathbf{d}_{01}$ it gives that $\langle u, u, u_2, \ldots \rangle \in \mathbf{d}_{01}$ ∎

cl-2

**Claim 3.38** $\sim_\lambda$ *is symmetric.*

**Proof** Follows from Lemma 3.35 when applied to the case $x = \mathbf{d}_{01}$. ∎

cl-3

**Claim 3.39** $\sim_\lambda$ *is transitive.*

**Proof** Follows from Lemma 3.36 when applied to the case $x = \mathbf{d}_{01}$, $k = 0$. ∎

Our aim is to embed the cylindric reduct of the not fully represented algebra $\mathcal{A}$ into a $\mathrm{Gws}_\omega$. For saving energy and not losing what we have already achieved, we are looking for a representation function *rep* that acts not only on the algebra level but deeper, on the sequence level hiding in the elements of $\mathcal{A}$. In other words,

$$rep(x) = \{f(q) \mid q \in x\}$$

where $f$ if the sequence level function mapping $V$ into the universe $V^*$ of some algebra $\mathcal{A}^*$. To ensure that $f(\mathbf{d}_{01})$ is the set theoretic constant we want for any $q, q' \in \lambda \in \Lambda$ that $(\forall i \in \omega)$ $q(i) \sim_\lambda q'(i)$ should imply $f(q) = f(q')$. This observation gives us a hint for an adequate definition of $f$.

For every $\lambda \in \Lambda$, we define $U_\lambda$ to be the set $(base(\lambda)/\!\sim_\lambda) \times \{\lambda\}$. $U_\lambda$ corresponds to the $\sim_\lambda$ classes of $base(\lambda)$. Clearly, if $\lambda$ and $\lambda'$ are distinct then $U_\lambda$ and $U_{\lambda'}$ are disjoint. For every $q \in \lambda$, we define

$$f(q) = \langle \langle q(0)/\!\sim_\lambda, \lambda\rangle, \langle q(1)/\!\sim_\lambda, \lambda\rangle, \langle q(2)/\!\sim_\lambda, \lambda\rangle, \ldots \rangle$$

The $f$ image of a $\lambda \in \Lambda$ is also a weak space in ${}^\omega U_\lambda$. Let it be denoted by $\lambda^*$. Now we can define the image algebra $\mathcal{A}^* \in \mathrm{Gws}_\omega$:

$$\begin{aligned} V^* &= \bigcup_{\lambda \in \Lambda} \lambda^*, \\ A^* &= \{rep(x) \mid x \in A\}, \\ \mathcal{A}^* &= \langle A^*, \cap, -, \mathbf{c}_i, \mathbf{d}_{ij}\rangle. \end{aligned}$$

One can easily verify that the above defined *rep* is an isomorphism between the cylindric reduct of $\mathcal{A}$ and a generalised weak cylindric set algebra. Finally, the result $\mathbf{I}\mathrm{Gws}_\omega \subseteq \mathbf{I}\mathrm{Gs}_\omega = \mathrm{RCA}_\omega$ published in [HMTII] Theorem 3.1.103. completes the proof. ∎

## 3.6 Further observations on $Mod(H1, H2)$

We have seen that a model $\mathcal{A}'$ of H1,H2 is isomorphic to an algebra $\mathcal{A}$ whose $t_{\bar{H}}$ reduct is a $\mathrm{Gwdf}_{\bar{H}}$. In the previous proof another isomorphism *rep* was defined that mapped the cylindric reduct of $\mathcal{A}$ into $\mathcal{A}^* \in \mathrm{Gws}_\omega$. $\mathcal{A}^*$ can be expanded with the abstract operations $\mathbf{s}_\sigma$ $(\sigma \in H)$ as follows

$$\mathbf{s}_\sigma y \stackrel{\text{def}}{=} rep\, \mathbf{s}_\sigma rep^{-1} y.$$

The expanded algebra is denoted by $\mathcal{A}^*$ as well.

We examine the behaviour of the new, abstract operations. In this subsection we collect plenty of the characteristic features that will be useful in the proof of Theorem 2.30 (3). All the notations $(\mathcal{A}, \mathcal{A}^*, \Lambda, \sim_\lambda, f, rep)$ that were introduced in the previous subsection are kept. The set of weak spaces of $\mathcal{A}^*$ is denoted by $\Lambda^*$. $\Lambda$ and $\Lambda^*$ is in a one-to-one correspondence in a natural way. We have that $base(\lambda^*) = (base(\lambda)/\!\sim_\lambda) \times \{\lambda\}$.

¿From now we suppose that all the elements of $H$ are bounded transformations of $\omega$.

cl-bound

**Claim 3.40** *If $\sigma \in H$ (and so $\sigma$ is bounded) and $\lambda_1, \lambda_2 \in \Lambda$ then the statements below hold.*

- $\mathbf{s}_\sigma \lambda_1 \supseteq \lambda_2$ *or* $\mathbf{s}_\sigma \lambda_1 \cap \lambda_2 = \emptyset$.
- $\mathbf{s}_\sigma \lambda_1 \supseteq \lambda_2$ *iff* $(\exists q \in \lambda_2)(q \circ \sigma \in \lambda_1)$.

- $(\forall\lambda_2 \in \Lambda)(\exists!\lambda_1 \in \Lambda)(\mathbf{s}_\sigma\lambda_1 \supseteq \lambda_2)$. *(Here $\exists!$ stands for 'there exists exactly one'.)*

**Proof** Suppose that $q \in \mathbf{s}_\sigma\lambda_1 \cap \lambda_2$. For all $q' \in \lambda_2$, $q' \circ \sigma \in \lambda_1$ since $q \circ \sigma \in \lambda_1$ and $\{i \in \omega \mid q \circ \sigma(i) \neq q' \circ \sigma(i)\} \subseteq \sigma^{-1}\{i \in \omega \mid q(i) \neq q'(i)\}$ is finite. So $\lambda_2 \subseteq \mathbf{s}_\sigma\lambda_1$.

To prove the last statement suppose that $q \in \lambda_2 \in \Lambda$. Put $\lambda_1$ to be the weak space of $q \circ \sigma$. Clearly, $q \circ \sigma \in \lambda_1$ and so $\lambda_2 \subseteq \mathbf{s}_\sigma\lambda_1$. If $\lambda' \in \Lambda$ and $\lambda_2 \subseteq \mathbf{s}_\sigma\lambda'$ then $\mathbf{s}_\sigma\lambda_1 \cap \mathbf{s}_\sigma\lambda' = \mathbf{s}_\sigma(\lambda_1 \cap \lambda') \supseteq \lambda_2$ shows that $\lambda_1 \cap \lambda' \neq 0$ and so $\lambda_1 = \lambda'$. ∎

That claim shows that the following definition is explicit.

d-sonL

**Definition 3.41** For each $\sigma \in H$, we define a transformation $\hat\sigma$ of $\Lambda$:

$$\hat\sigma(\lambda_2) = \lambda_1 \text{ if } \lambda_2 \subseteq \mathbf{s}_\sigma\lambda_1.$$

Since $\Lambda$ and $\Lambda^*$ are in one-to-one correspondence, $\hat\sigma$ can be viewed as a transformation of $\Lambda^*$: $\lambda_1, \lambda_2 \in \Lambda$, $\hat\sigma(\lambda_2^*) = \lambda_1^*$ if $\hat\sigma(\lambda_2) = \lambda_1$. □

We present an important property of these transformations.

cl-hatjo

**Claim 3.42** *Suppose that $\sigma_1, \ldots, \sigma_n$ and $\sigma'_1, \ldots, \sigma'_m$ are elements of $H$. If $\sigma_n \circ \ldots \circ \sigma_1 \approx \sigma'_m \circ \ldots \circ \sigma'_1$ then $\hat\sigma_1 \circ \ldots \circ \hat\sigma_n$ and $\hat\sigma'_1 \circ \ldots \circ \hat\sigma'_m$ are the same transformations of $\Lambda$ (and of $\Lambda^*$ as well).*

**Proof** Suppose that $\lambda \in \Lambda$. $\hat\sigma_1 \circ \ldots \circ \hat\sigma_n(\lambda) = \lambda'$ if $\lambda \subseteq \mathbf{s}_{\sigma_n} \ldots \mathbf{s}_{\sigma_1}\lambda'$. Then for any $q \in \lambda$, $q \circ \sigma_n \circ \ldots \circ \sigma_1 \in \lambda'$. Since $\sigma_n \circ \ldots \circ \sigma_1 \approx \sigma'_m \circ \ldots \circ \sigma'_1$ and they are bounded, $q \circ \sigma_n \circ \ldots \circ \sigma_1 \approx q \circ \sigma'_m \circ \ldots \circ \sigma'_1$. Then $q \circ \sigma'_m \circ \ldots \circ \sigma'_1 \in \lambda'$, $\lambda \subseteq \mathbf{s}_{\sigma'_m} \ldots \mathbf{s}_{\sigma'_1}\lambda'$, $\hat\sigma'_1 \circ \ldots \circ \hat\sigma'_m(\lambda) = \lambda'$ and this was to prove. ∎

r-ph

**Remark 3.43** The domain of the function $\hat{}$ defined in Definition 3.41 is $H$. The previous claim implies that $\hat{}$ can be defined on $[\bar H]$ as well. If $\nu \in [\bar H] \subseteq {}^\omega\omega$ then we set

$$\hat\nu \stackrel{\text{def}}{=} \hat\sigma_1 \circ \ldots \circ \hat\sigma_n$$

where $\nu \approx \sigma_n \circ \ldots \circ \sigma_1$. Clearly, $\hat\nu$ does not depend on the choice of $\sigma_1, \ldots, \sigma_n$. The new $\hat{}: [\bar H] \to {}^{\Lambda^*}\Lambda^*$ has the following properties:

- If $\nu, \mu \in [\bar H], \nu \approx \mu$ then $\hat\nu = \hat\mu$.
- If $\nu, \mu \in [\bar H]$ then $\widehat{\nu \circ \mu} = \hat\mu \circ \hat\nu$.

cl-base

**Claim 3.44** *When $\hat\sigma(\lambda_2) = \lambda_1$ for some $\sigma \in H, \lambda_1, \lambda_2 \in \Lambda$ then*

- *$base(\lambda_2) \subseteq base(\lambda_1)$,*
- *$\sim_{\lambda_2} = \sim_{\lambda_1} \cap\ {}^2 base(\lambda_2)$.*

**Proof** Suppose that $\lambda_2 \subseteq \mathbf{s}_\sigma \lambda_1$, $q \in \lambda_2$. For all $u \in \mathit{base}(\lambda_2)$, $q[\sigma(0)/u] \in \lambda_2$ and so $q[\sigma(0)/u] \circ \sigma \in \lambda_1$. But $(q[\sigma(0)/u] \circ \sigma)(0) = u \in \mathit{base}(\lambda_1)$. It shows that $\mathit{base}(\lambda_2) \subseteq \mathit{base}(\lambda_1)$.

To prove the second statement suppose that $u, u' \in \mathit{base}(\lambda_2)$, $u \sim_{\lambda_2} u'$. There is a $q_2 \in \mathbf{d}_{01} \cap \lambda_2$ for that $q_2(0) = u$, $q_2(1) = u'$. We fix a $k \in \omega$ for that $\sigma(0) \neq \sigma(k)$ and a finite transformation $h$ of $\omega$ that maps $\sigma(0)$ to 0 and $\sigma(k)$ to 1. Since $[\bar{H}]$ is rich $h \in [\bar{H}]$ and so there is $\tau \in \bar{H}^*$ for that $\tau = h$. We put $q_1 \stackrel{\text{def}}{=} q_2 \circ h \circ \sigma \circ [1,k]$. $q_2 \in \mathbf{d}_{01}$ and $\mathbf{s}_\tau \mathbf{s}_\sigma \mathbf{s}_{[1,k]} \mathbf{d}_{01} = \mathbf{d}_{h(\sigma([1,k](0))),h(\sigma([1,k](1)))} = \mathbf{d}_{01}$ implies that $q_1 \in \mathbf{d}_{01}$. $q_2 \in \lambda_2$ and $\lambda_2 \subseteq \mathbf{s}_\sigma \lambda_1$ gives that $q_1 \in \lambda_1$. Finally, $q_1(0) = u$, $q_1(1) = u'$ shows that $u \sim_{\lambda_1} u'$.

For the other direction suppose that $u, u' \in \mathit{base}(\lambda_2)$, $u \sim_{\lambda_1} u'$. There is a $q \in \lambda_1 \cap \mathbf{d}_{01}$ for that $q(0) = u$, $q(1) = u'$. Pick any $q_2 \in \lambda_2$ with $q_2(0) = u$, $q_2(1) = u'$. We put $q_1 \stackrel{\text{def}}{=} q_2 \circ h \circ \sigma \circ [1,k]$. As in the previous case $q_1 \in \lambda_1$. $q \approx q_1$ so they disagree on a finite set $\Gamma = \{i_1, \ldots, i_n\}$. $0, 1 \notin \Gamma$. We conclude that $q_1 \in \mathbf{c}_{i_1} \ldots \mathbf{c}_{i_n} \mathbf{d}_{01} = \mathbf{d}_{01}$. Using again $\mathbf{s}_\tau \mathbf{s}_\sigma \mathbf{s}_{[1,k]} \mathbf{d}_{01} = \mathbf{d}_{01}$ and $q_1 \in \mathbf{d}_{01}$ we get that $q_2 \in \mathbf{d}_{01}$. This proves that $u \sim_{\lambda_2} u'$. ∎

d-em

**Definition 3.45** We define the function

$$b : \bigcup_{\sigma \in H, \lambda^* \in \Lambda^*} \{\langle \sigma, \lambda^* \rangle\} \times \mathit{base}(\lambda^*) \to \bigcup_{\lambda^* \in \Lambda^*} \mathit{base}(\lambda^*)$$

as follows. $b_{\sigma,\lambda^*}(\langle \alpha, \lambda \rangle) = \langle \alpha', \hat{\sigma}(\lambda) \rangle$ if $\alpha' \in \mathit{base}(\hat{\sigma}(\lambda))/\!\sim_{\hat{\sigma}(\lambda)}$ and $\alpha \subseteq \alpha'$ holds. The previous claim proved that $b$ is well defined. □

We collect some properties of the function $b$ defined above.

cl-propb

**Claim 3.46** *(1).* $q \in \lambda^* \in \Lambda^* \;\Rightarrow\; b_{\sigma,\lambda^*} \circ q \circ \sigma \in V^*$.

*(2).* $q, q' \in \lambda^* \in \Lambda^* \;\Rightarrow\; b_{\sigma,\lambda^*} \circ q \circ \sigma \approx b_{\sigma,\lambda^*} \circ q' \circ \sigma$.

*(3).* $b_{\sigma,\lambda^*}$ *is injective.*

*(4).* $(\forall \lambda \in \Lambda^*)(\exists! \lambda' \in \Lambda^*)(\forall q \in \lambda)\; b_{\sigma,\lambda} \circ q \circ \sigma \in \lambda'$.

**Proof** (1): Since $q \in V^*$, there is a $s \in V$ for that $f(s) = q$. The weak space of $s$ in $\mathcal{A}$ is $\lambda$, and that of $q$ in $\mathcal{A}^*$ is $\lambda^*$. We show that $b_{\sigma,\lambda^*} \circ q \circ \sigma = f(s \circ \sigma) \in V^*$ by checking that they take the same value on an arbitrary $i \in \omega$. $(b_{\sigma,\lambda^*} \circ q \circ \sigma)(i) = b_{\sigma,\lambda^*}(q(\sigma(i))) = b_{\sigma,\lambda^*}(f(s)(\sigma(i))) = b_{\sigma,\lambda^*}(\langle s(\sigma(i))/\!\sim_\lambda, \lambda \rangle) = \langle s(\sigma(i))/\!\sim_{\hat{\sigma}(\lambda)}, \hat{\sigma}(\lambda) \rangle = f(s \circ \sigma)(i)$

To prove (2) we need only that $\sigma$ is bounded. (3) follows from the definition, and (4) from (2). ∎

rm-sigmab

**Remark 3.47** If a function $g$ satisfies condition (4) of Claim 3.46 then for each $\sigma \in H$, a transformation $\hat{\sigma}_g$ of $\Lambda^*$ can be defined:

$$\hat{\sigma}_g(\lambda) = \lambda' \text{ iff } (\forall q \in \lambda)\ g_{\sigma,\lambda} \circ q \circ \sigma \in \lambda'.$$

When $g$ coincides with $b$ then we get back $\hat{\sigma}$, that is $\hat{\sigma}_b = \hat{\sigma}$.

cl-prop5

**Claim 3.48** *Let $\lambda^* \in \Lambda^*$, $\sigma_1, \ldots, \sigma_n$ and $\sigma'_1, \ldots, \sigma'_m$ be elements of $H$.*

*(5). If $\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n(\lambda^*) = \lambda^*$ then $b_{\sigma_1,\hat{\sigma}_2(\ldots\hat{\sigma}_n(\lambda^*)\ldots)} \circ b_{\sigma_2,\hat{\sigma}_3(\ldots\hat{\sigma}_n(\lambda^*)\ldots)} \circ \ldots \circ b_{\sigma_n,\lambda^*}$ is the identity mapping of base$(\lambda^*)$.*

*(6). If $\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n(\lambda^*) = \hat{\sigma}'_1 \circ \ldots \circ \hat{\sigma}'_m(\lambda^*)$ then*

$$b_{\sigma_1,\hat{\sigma}_2(\ldots\hat{\sigma}_n(\lambda^*)\ldots)} \circ b_{\sigma_2,\hat{\sigma}_3(\ldots\hat{\sigma}_n(\lambda^*)\ldots)} \circ \ldots \circ b_{\sigma_n,\lambda^*} = \\ = b_{\sigma'_1,\hat{\sigma}'_2(\ldots\hat{\sigma}'_m(\lambda^*)\ldots)} \circ b_{\sigma'_2,\hat{\sigma}'_3(\ldots\hat{\sigma}_m(\lambda^*)\ldots)} \circ \ldots \circ b_{\sigma'_m,\lambda^*}$$

**Proof** First we note that the expression formulated above is correct. The range of $b_{\sigma_i,\hat{\sigma}_{i+1}(\ldots\hat{\sigma}_n(\lambda)\ldots)}$ is contained in $base(\hat{\sigma}_i(\ldots\hat{\sigma}_n(\lambda)\ldots))$ and that is the domain of $b_{\sigma_{i+1},\hat{\sigma}_i(\ldots\hat{\sigma}_n(\lambda)\ldots)}$.

To check the validity of the expression in (6) we evaluate the two sides of the equation on an arbitrary element of $base(\lambda^*)$, say an $\langle u/\sim_\lambda, \lambda\rangle$. When the left side of the equation is applied to that element we get $\langle u/\sim_{(\hat{\sigma}_1\circ\ldots\circ\hat{\sigma}_n)(\lambda)}, (\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n)(\lambda)\rangle$ and that is the same point as the image of $\langle u/\sim_\lambda, \lambda\rangle$ under the function that appear on the right side of the equation.

(5) follows from the definition. ∎

nt-circ

**Notation 3.49** We define a new composition operation $\bullet$ mapping $V^* \times H$ into $V^*$:

$$q \bullet \sigma \stackrel{\text{def}}{=} b_{\sigma,[q]} \circ q \circ \sigma$$

where $[q]$ is the weak space of $q$ in $\mathcal{A}^*$. □

Now a direct definition can be given to $\mathbf{s}_\sigma$ ($\sigma \in H$).

cl-newd2

**Claim 3.50** *For all $\sigma \in H, y \in A^*$*

$$\mathbf{s}_\sigma y = \{q \in V^* \mid q \bullet \sigma \in y\}$$

**Proof** As a side result, in the proof of Claim 3.46 (1) we showed that if $s \in V, q \in V^*, f(s) = q$ then $f(s \circ \sigma) = q \bullet \sigma$.

Now we can compute that $q \in \mathbf{s}_\sigma y \Leftrightarrow s \in \mathbf{s}_\sigma rep^{-1} y \Leftrightarrow s \circ \sigma \in rep^{-1} y \Leftrightarrow f(s \circ \sigma) \in y \Leftrightarrow q \bullet \sigma \in y$. ∎

We arrived to the point when a new class of set algebras can be introduced. A new algebra has a deeper structure than a $\mathrm{Gws}_H$ does and its operations are computable from the structure of the algebra.

df-nonst

**Definition 3.51** An algebra $\mathcal{A} = \langle A, \cup, -, \mathbf{c}_k, \mathbf{d}_{kl}, \mathbf{s}_\sigma \rangle_{k,l \in \omega, \sigma \in H}$ is called a *non-standard generalised weak $H$–cylindric set algebra with equality* (or briefly a $\mathrm{nstGws}_H$) if $\langle A, \cup, -, \mathbf{c}_k, \mathbf{d}_{kl} \rangle_{k,l\in\omega}$ is a generalised weak cylindric set algebra and there is a function $b$ with the following properties:

- Let $V$ be the unit element of $\mathcal{A}$. We put $\Lambda$ to be the set of weak spaces building $V$. The domain of $b$ is $\bigcup\limits_{\sigma \in H, \lambda \in \Lambda} \{\sigma, \lambda\} \times \mathit{base}(\lambda)$, the range is a subset of $\bigcup\limits_{\lambda\in\Lambda} \mathit{base}(\lambda)$.
- $b$ satisfies the properties (1)-(6) listed in Claim 3.46 and Claim 3.48 where $\hat{\sigma} = \hat{\sigma}_b$ is defined as in Remark 3.47.
- We put $\bullet : V \times H \to V$ to be the function $q \bullet \sigma \stackrel{\text{def}}{=} b_{\sigma,[q]} \circ q \circ \sigma$. Clearly, the properties of $b$ gives that $b_{\sigma,[q]} \circ q \circ \sigma$ is an element of $V$. Then $\mathbf{s}_\sigma y = \{q \in V \mid q \bullet \sigma \in y\}$.

If $\mathcal{A}$ is a $\mathrm{nstGws}_H$ and $b$ is a function with the required properties then $b$ is called a *base transformation function of $\mathcal{A}$*. Note that the base transformation function is not uniquely determined. $\square$

The theorem below is a summary of the results that have achieved so far in this subsection.

th-nonst

**Theorem 3.52** *Any model of H1,H2 is isomorphic to a* $\mathrm{nstGws}_H$.

th-st-nst

**Theorem 3.53** $\mathrm{Gws}_H \subseteq \mathrm{nstGws}_H$.

**Proof** Any $\mathrm{Gws}_H$ is a $\mathrm{nstGws}_H$ with the identity as base transformation function. $(\forall \sigma \in H)(\lambda \in \Lambda)(\forall u \in \mathit{base}(\lambda))\ \ b_{\sigma,\lambda}(u) \stackrel{\text{def}}{=} u$. $\blacksquare$

cl-hatjo2

**Claim 3.54** *Suppose that $\mathcal{A} \in \mathrm{nstGws}_H$. Then Claim 3.42 is true in $\mathcal{A}$.*

**Proof** As usual, $\Lambda$ is the set of weak spaces of $\mathcal{A}$. Pick an arbitrary element of $\Lambda$, say $\lambda$, and an arbitrary sequence $q$ in $\lambda$. Using the definition of $\hat{\sigma}$ in Remark 3.47 we have that $\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n(\lambda)$ and $\hat{\sigma}'_1 \circ \ldots \circ \hat{\sigma}'_m(\lambda)$ are the weak spaces of $b_{\sigma_1, \hat{\sigma}_2(\ldots\hat{\sigma}_n(\lambda)\ldots)} \circ b_{\sigma_2,\hat{\sigma}_3(\ldots\hat{\sigma}_n(\lambda)\ldots)} \circ \ldots \circ b_{\sigma_n,\lambda} \circ q \circ \sigma_n \circ \ldots \circ \sigma_1 = q \bullet \sigma_n \bullet \ldots \bullet \sigma_1$ and $b_{\sigma'_1,\hat{\sigma}'_2(\ldots\hat{\sigma}'_m(\lambda)\ldots)} \circ b_{\sigma'_2,\hat{\sigma}'_3(\ldots\hat{\sigma}_m(\lambda)\ldots)} \circ \ldots \circ b_{\sigma'_m,\lambda} \circ q \circ \sigma'_m \circ \ldots \circ \sigma'_1 = q \bullet \sigma'_m \bullet \ldots \bullet \sigma'_1$ each respectively. Since $\sigma_n \circ \ldots \circ \sigma_1 \approx \sigma'_m \circ \ldots \circ \sigma'_1$, we have that $q \circ \sigma_n \circ \ldots \circ \sigma_1 \approx q \circ \sigma'_m \circ \ldots \circ \sigma'_1$. This fact and Claim 3.48 implies that $q \bullet \sigma_n \bullet \ldots \bullet \sigma_1 \approx q \bullet \sigma'_m \bullet \ldots \bullet \sigma'_1$ and so they are in the same weak space, that is $\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n(\lambda) = \hat{\sigma}'_1 \circ \ldots \circ \hat{\sigma}'_m(\lambda)$. $\blacksquare$

r-pb

**Remark 3.55** As we did in Remark 3.43, we can introduce the function $\hat{}$ : $[\bar{H}] \to {}^{\Lambda}\Lambda$ as $\hat{\nu} \stackrel{\text{def}}{=} \hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n$ when $\nu \in [\bar{H}], \sigma_1, \ldots, \sigma_n \in H$ and $\nu \approx \sigma_n \circ \ldots \circ \sigma_1$. In the same sense, the domain of $b$ can be expanded

$$b : \bigcup_{\nu \in [\bar{H}], \lambda \in \Lambda} \{\nu, \lambda\} \times \mathit{base}(\lambda) \to \bigcup_{\lambda \in \Lambda} \mathit{base}(\lambda)$$

$$b_{\nu,\lambda} \stackrel{\text{def}}{=} b_{\sigma_1, \hat{\sigma}_2(\ldots \hat{\sigma}_n(\lambda)\ldots)} \circ b_{\sigma_2, \hat{\sigma}_3(\ldots \hat{\sigma}_n(\lambda)\ldots)} \circ \ldots \circ b_{\sigma_n, \lambda}$$

where $\nu \in [\bar{H}], \sigma_1, \ldots, \sigma_n \in H, \lambda \in \Lambda$ and $\nu \approx \sigma_n \circ \ldots \circ \sigma_1$. If $\sigma'_1, \ldots, \sigma'_m \in H$ and $\nu \approx \sigma'_m \circ \ldots \circ \sigma'_1$ then $\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n = \hat{\nu} = \hat{\sigma}'_1 \circ \ldots \circ \hat{\sigma}'_m$ so $\hat{\sigma}_1 \circ \ldots \circ \hat{\sigma}_n(\lambda) = \hat{\sigma}'_1 \circ \ldots \circ \hat{\sigma}'_m(\lambda)$. This gives that (6) of Claim 3.48 is applicable showing that the above definition of $b$ does not depend on the choice of $\sigma_1, \ldots, \sigma_n$.

We collect some properties of our new $\hat{}$ functions. $\nu, \mu \in [\bar{H}], \lambda \in \Lambda$.

(1). $\nu \approx \mu$ implies $\hat{\nu} = \hat{\mu}$.

(2). $\widehat{\nu \circ \mu} = \hat{\mu} \circ \hat{\nu}$.

(3). $\nu \approx Id$ implies that $b_{\nu,\lambda}$ is the identity mapping of $\mathit{base}(\lambda)$.

(4). $b_{\nu \circ \mu, \lambda} = b_{\mu, \hat{\nu}(\lambda)} \circ b_{\nu, \lambda}$.

(5). $\hat{\nu}(\lambda) = \hat{\mu}(\lambda)$ implies $b_{\nu,\lambda} = b_{\mu,\lambda}$.

(1) and (2) can be found in Remark 3.43, (3) and (5) are only reformulations of (5) and (6) of Claim 3.48, and (4) is a consequence of (6) of Claim 3.48.

th-subd

**Lemma 3.56** *Let $\mathcal{A}$ be a subdirect irreducible model of H1,H2,H3. Then either* $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ *or* $-\mathbf{c}_0(-\mathbf{d}_{01}) = 0$.

**Proof** Suppose that in a model $\mathcal{A}$ of H1,H2,H3 neither $-\mathbf{c}_0(-\mathbf{d}_{01}) = 0$ nor $\mathbf{c}_0(-\mathbf{d}_{01}) = 0$ holds. We show that the relativizations with $-\mathbf{c}_0(-\mathbf{d}_{01})$ and $\mathbf{c}_0(-\mathbf{d}_{01})$ are homomorphisms, but not isomorphisms, and $\mathcal{A}$ is a subdirect product of the two image algebras. This proves that $\mathcal{A}$ cannot be subdirectly irreducible.

A relativization with some element $R$ is a homomorphism if and only if $R$ is closed under the operations of the algebra. H3 implies that $-\mathbf{c}_0(-\mathbf{d}_{01})$ and $\mathbf{c}_0(-\mathbf{d}_{01})$ are closed. In the image algebras $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ or $\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ will hold showing that they cannot be isomorphic to $\mathcal{A}$. Finally, an element of $\mathcal{A}$ is solely determined by its two image under the two relativizations. ∎

l-extr

**Lemma 3.57** *If* $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ *holds in* $\mathcal{A} \in \mathrm{nstGws}_H$ *then* $\mathcal{A}$ *is a* $\mathrm{Gs}_H$.

**Proof** Let $A$ be the universe, $V$ the unit element, $\Lambda$ the set of weak spaces of $\mathcal{A}$. $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ shows that for all $\lambda \in \Lambda$, $|base(\lambda)| = 1$.

Introduce the set theoretical operations on $A$:

$$\tilde{\mathbf{s}}_\sigma x \stackrel{\text{def}}{=} \{q \in V \mid q \circ \sigma \in x\}.$$

Then $\mathcal{A}' \stackrel{\text{def}}{=} \langle A, \cup -, \mathbf{c}_k, \mathbf{d}_{kl}, \tilde{\mathbf{s}}_\sigma \rangle_{k,l \in \omega, \sigma \in H} \in \mathrm{Gs}_H$. We show that $\tilde{\mathbf{s}}_\sigma = \mathbf{s}_\sigma$ and so $\mathcal{A} = \mathcal{A}' \in \mathrm{Gs}_H$.

$\mathbf{s}_\sigma x = x$ since the axiom $\mathbf{s}_\sigma(x - \mathbf{c}_0(-\mathbf{d}_{01})) = x - \mathbf{c}_0(-\mathbf{d}_{01})$ in H3 can be reduced to that form using $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$. On the other hand, $\tilde{\mathbf{s}}_\sigma x = x$ is a result of that fact that in $V$ all the sequences are of the form $\langle u, u, u, \ldots \rangle$ for some $u$. ∎

## 3.7 The witness tree method

In this subsection we build a machinery called the witness tree method to prove Theorem 2.30. The essence of the method is the following. The failure of an equation in a given algebra can be captured by a tree structure that reflects the local nature of the algebra. This tree can be cut out from the algebra and moved to another one with better properties to testify the failure of the same equation therein. In our case we will show that an equation that is not satisfied in a $\mathrm{nstGws}_H$ cannot hold in all the $\mathrm{Gws}_H$s.

d-wt

**Definition 3.58** Let $H$ be a set of transformations of $\omega$. We say that $\Omega = \langle V, E, r, U, \tau, \varepsilon, l \rangle$ is a *witness tree of type $H$* if the following requirements are fulfilled.

(i). $\langle V, E \rangle$ is a directed tree. $r \in V$. The edges are directed from the root $r$ towards the leaves.

(ii). $U, \tau$ and $\varepsilon$ are functions with domain $V$. If $v \in V$ then $U_v$ is a set, $\tau_v$ is a term of type $t_H^{=}$, and $\varepsilon_v$ is either $\ni$ or $\not\ni$. (Note that we used the less conventional notation $U_v$ instead of $U(v)$.)

(iii). $l$ is a function with domain $E$. If $e = \vec{vv'} \in E$ is an edge then $l_e$ is one of '=', '$[k/u]$' or '$\sigma$' where $k \in \omega$, $u \in U_v$, $\sigma \in H$. We will call $l_e$ the *label* of the edge $e$.

(iv). If $e = \vec{vv'} \in E$ is an edge then $U_v \subseteq U_{v'}$. Furthermore, if $l_e$ is not '$\sigma$' for some $\sigma \in H$ then $U_v = U_{v'}$.

(v). For every vertex $v$, $\tau_v$, $\varepsilon_v$, the number of children of $v$, the labels of the edges pointing to the children of $v$, and the value of $\tau$ and $\varepsilon$ on of the children vertexes of $v$ are in a connection indicated in the following table.

| $\tau_v$, $\varepsilon_v$ | The number of children | The label of an edge pointing to a child | The value of $\tau$, $\varepsilon$ on a child vertex | |
|---|---|---|---|---|
| $(-\varrho), \ni$ | 1 | $=$ | $\varrho, \not\ni$ | |
| $(-\varrho), \not\ni$ | 1 | $=$ | $\varrho, \ni$ | |
| $(\varrho_1 \cup \varrho_2), \ni$ | 1 | $=$ | $\varrho_i, \ni$ | where $i$ is 1 or 2 |
| $(\varrho_1 \cup \varrho_2), \not\ni$ | 2 | $=$ | $\varrho_i, \not\ni$ | for $i \in \{1,2\}$ |
| $\mathbf{c}_k\varrho, \ni$ | 1 | $[k/u]$ | $\varrho, \ni$ | where $u \in U_v$ |
| $\mathbf{c}_k\varrho, \not\ni$ | $\lvert U_v\rvert$ | $[k/u]$ | $\varrho, \not\ni$ | for each $u \in U_v$ |
| $\mathbf{s}_\sigma\varrho, \ni$ | 1 | $\sigma$ | $\varrho, \ni$ | |
| $\mathbf{s}_\sigma\varrho, \not\ni$ | 1 | $\sigma$ | $\varrho, \not\ni$ | |
| $X, \ni$ | 0 | $-$ | $-$ | |
| $X, \not\ni$ | 0 | $-$ | $-$ | |
| $\mathbf{d}_{kl}, \ni$ | 0 | $-$ | $-$ | |
| $\mathbf{d}_{kl}, \not\ni$ | 0 | $-$ | $-$ | |

$\square$

d-nuv

**Definition 3.59** Let $\Omega$ be a witness tree. We define a function $\nu : V \to {}^\omega\omega$ by recursion. For the root $r$, $\nu_r = Id$. If $e = v_1\vec{v}_2 \in E$ and $\nu_{v_1}$ is already defined then $\nu_{v_2}$ is set to $\nu_{v_1}$ when $l_e$ is '$=$' or '$[k/u]$' for some $k \in \omega, u \in U_{v_1}$ and to $\nu_{v_1} \circ \sigma$ when $l_e$ is '$\sigma$'. $\square$

d-Mr

**Definition 3.60** Let $\varrho$ be a term. Then $M_\varrho$ denotes the set of those transformations $f$ of $\omega$ for that exist $\sigma_1, \ldots, \sigma_n \in H$ with the property that $f = \sigma_1 \circ \ldots \sigma_n$ and $\mathbf{s}_{\sigma_1}, \ldots, \mathbf{s}_{\sigma_n}$ are letters of $\varrho$. We allow $\sigma_i$ to be equal with $\sigma_j$ $(i \neq j)$ but in that case $\mathbf{s}_{\sigma_i}$ and $\mathbf{s}_{\sigma_j}$ refer to the multiple occurrence of the same letter in $\varrho$. Clearly, $n$ is smaller than or equal to the length of term $\varrho$. $\square$

cl-Mfin

**Claim 3.61** *The set $M_\varrho$ defined in the previous definition is finite.*

**Proof** The term $\varrho$ has only finite number of letters so there is only finitely many possible choices of $\sigma_1, \ldots, \sigma_n$. $\blacksquare$

d-M

**Definition 3.62** Let $\Omega$ be a witness tree. Then $M_\Omega$ denotes $M_{\tau_r}$. $\square$

cl-nuM

**Claim 3.63** *For each vertex $v$ of $\Omega$, $\nu_v \in M_\Omega$.*

d-M=

**Definition 3.64** $M_\Omega^=, M_\Omega^{\neq}$ denotes the following subsets of $M_\Omega \times M_\Omega$.

$$M_\Omega^= \stackrel{\text{def}}{=} \{\langle \nu, \nu' \rangle \mid \nu \approx \nu \wedge (\exists \text{ vertex } v, v')(\nu = \nu_v \wedge \nu' = \nu_{v'})\}$$
$$M_\Omega^{\neq} \stackrel{\text{def}}{=} \{\langle \nu, \nu' \rangle \mid \nu \not\approx \nu' \wedge (\exists \text{ vertex } v, v')(\nu = \nu_v \wedge \nu' = \nu_{v'})\}$$

Both $M^{=}_{\Omega}$ and $M^{\neq}_{\Omega}$ are finite. □

d-degen

**Definition 3.65** Let $\Omega$ be a witness tree. We say the $\Omega$ is *not degenerated* if

$$(\forall \text{ vertex } v_1, v_2)(\forall \mu \in [\bar{H}])\ \ \nu_{v_1} \circ \mu \approx \nu_{v_2} \Rightarrow U_{v_1} \subseteq U_{v_2} \qquad (\ddagger)$$

holds. □

d-validq

**Definition 3.66** Let $\Omega$ be a witness tree. We say that a function $Q$ is a *valuation of the vertex set of* $\Omega$ if the conditions below hold.

(1). The domain of $Q$ is the vertex set of $\Omega$.

(2). For all vertex $v$, $Q_v$ is an $\omega$-sequence over $U_v$.

(3). If $v_1$ and $v_2$ are two adjacent vertexes of $\Omega$ then the connection between $Q_{v_1}$ and $Q_{v_2}$ is determined by the label of the edge that joins $v_1$ and $v_2$.

- If the label is '=' then $Q_{v_2} = Q_{v_1}$.
- If the label is '$[k/u]$' then $Q_{v_2} = Q_{v_1}[k/u]$.
- If the label is '$\sigma$' then $Q_{v_2} = Q_{v_1} \circ \sigma$.

□

cl-vq

**Claim 3.67** *A valuation $Q$ of the vertex set of a witness tree $\Omega$ and the value of $Q$ on the root of $\Omega$ uniquely determine each other.*

**Proof** If we know the value of $Q$ on the root then we can compute the values of $Q$ on other vertexes going step-by-step towards the leaves using (3) of Definition 3.66. To verify (2), we need only that $U_{v_1} \subseteq U_{v_2}$ whenever $v_1\vec{v}_2$ is an edge. (See the table in Definition 3.58.) ■

cl-c1

**Claim 3.68** *Let $\Omega$ be a witness tree with root $r$, $Q$ be a valuation of the vertex set. Then*

$$(\exists c_1 \in \omega)(\forall \textit{ vertex } v)(\forall i \in \omega)\ \ Q_v(i) \neq (Q_r \circ \nu_v)(i) \Rightarrow i \leq c_1.$$

**Proof** To get an insight into the relation of $Q_v$ and $Q_r \circ \nu_v$, we take an example first. Suppose that we find the sequence $\sigma_1$, $[k_1/u_1]$, $\sigma_2$, $[k_2/u_2]$, $\sigma_3$, $\sigma_4$, $[k_3/u_3]$, $\sigma_5$ of labels on the $r \to v$ path. Then

$$Q_v(i) = \begin{cases} u_3 & \text{if } i \in \sigma_5^{-1}(k_3), \\ u_2 & \text{if } i \in (\sigma_3 \circ \sigma_4 \circ \sigma_5)^{-1}(k_2) - \sigma_5^{-1}(k_3), \\ u_1 & \text{if } i \in (\sigma_2 \circ \sigma_3 \circ \sigma_4 \circ \sigma_5)^{-1}(k_1) - B, \\ Q_r \circ \nu_v(i) & \text{otherwise} \end{cases}$$

where $B = (\sigma_3 \circ \sigma_4 \circ \sigma_5)^{-1}(k_2) \cup \sigma_5^{-1}(k_3)$.

This example shows that for any $v$, $Q_v$ and $Q_r \circ \nu_v$ are equal everywhere but on the union of finitely many sets of the form $(\sigma_1 \circ \ldots \circ \sigma_n)^{-1}(k)$ where $\mathbf{s}_{\sigma_1}, \ldots, \mathbf{s}_{\sigma_n}, \mathbf{c}_k$ are letters of the root term. In that case $(\sigma_1 \circ \ldots \circ \sigma_n)^{-1}(k)$ is finite since $\sigma_i$ is bounded $(i \leq n)$ and $\sigma_1 \circ \ldots \circ \sigma_n \in M_\Omega$. We can conclude that the set $\bigcup_{\substack{\nu \in M_\Omega, \\ \mathbf{c}_k \text{ is a letter of the root term}}} \nu^{-1}(k)$ is finite and $c_1$ can be set to be its maximum. ∎

cl-c2

**Claim 3.69** *Let $\Omega$ be a witness tree with root $r$, $Q$ be a valuation of the vertex set. Then*

$$(\exists c_2 \in \omega)(\forall \text{ vertex } v, v')(\forall i \in \omega)\ \nu_v \approx \nu_{v'} \wedge (Q_r \circ \nu_v)(i) \neq (Q_r \circ \nu_{v'})(i) \Rightarrow i \leq c_2.$$

**Proof** Put $c_2 \stackrel{\text{def}}{=} \max \bigcup_{\langle \nu, \nu' \rangle \in M_\Omega^{=}} \{i \in \omega \mid \nu(i) \neq \nu'(i)\}$. This definition is transparent since the set, we have to take the maximum of is finite. ($M_\Omega^{=}$ is finite, $\{i \in \omega \mid \nu(i) \neq \nu'(i)\}$ is finite if $\nu \approx \nu'$.) ∎

l-c

**Lemma 3.70** *Let $\Omega$ be a witness tree with root $r$, $Q$ be a valuation of the vertex set. Then*

$$(\exists c \in \omega)(\forall \text{ vertex } v, v')(\forall i \in \omega)\ \ \nu_v \approx \nu_{v'} \wedge Q_v(i) \neq Q_{v'}(i) \Rightarrow i < c.$$

**Proof** Put $c = \max\{c_1, c_2\} + 1$. Claims 3.68 and 3.69 will prove the statement. ∎

l-S

**Lemma 3.71** *Suppose that $U_r$ is a set with cardinality bigger than 2. Then there is a sequence $S \in {}^\omega U_r$ with the property that for all $\mu, \nu \in [\bar{H}]$*

$$\mu \not\approx \nu \ \Rightarrow\ S_r \circ \mu \not\approx S_r \circ \nu$$

To prove this lemma we will need the following claim.

cl-h1

**Claim 3.72** *There is a set $D \subseteq \mathcal{P}(\omega)$ for that*

- $\alpha \in D \ \Rightarrow\ |\alpha| = 2$,
- $\alpha, \beta \in D \wedge \alpha \neq \beta \ \Rightarrow\ \alpha \cap \beta = \emptyset$,
- $\mu, \nu \in [\bar{H}], \mu \not\approx \nu \ \Rightarrow\ |D \cap \{\{\mu(i), \nu(i)\} \mid i \in \omega\}| = \omega$.

**Proof** Let $\langle \mu_i, \nu_i \rangle$ $(i \in \omega)$ be a series of pairs of transformations of $\omega$ with the property that for any $\mu, \nu \in [H]$ with $\mu \not\approx \nu$ we have that $\langle \mu_i, \nu_i \rangle = \langle \mu, \nu \rangle$ for infinitely many $i \in \omega$. A series of that kind exists since $|H| \leq \omega, |\bar{H}| \leq \omega, |[\bar{H}]| \leq |\bar{H}^*| \leq \omega, |[\bar{H}] \times [\bar{H}]| \leq \omega$.

For all $k \in \omega$ we define the set $\alpha_k$ recursively:

- $\alpha_0 = \{\mu_0(i), \nu_0(i)\}$ for an $i \in \omega$ with $\mu_0(i) \neq \nu_0(i)$.
- $\alpha_k = \{\mu_k(i), \nu_k(i)\}$ for an $i \in \omega$ with $\mu_k(i) \neq \nu_k(i)$ and $i \notin \mu_k^{-1}(\bigcup_{j<k} \alpha_j) \cup \nu_k^{-1}(\bigcup_{j<k} \alpha_j)$.

$\alpha_0$ exists since $\mu_0(i) \neq \nu_0(i)$ for all but finitely many $i \in \omega$. $\alpha_k$ can be chosen since $|\{i \in \omega \mid \mu_k(i) \neq \nu_k(i)\}| = \omega$ and $|\mu_k^{-1}(\bigcup_{j<k} \alpha_j) \cup \nu_k^{-1}(\bigcup_{j<k} \alpha_j)| < \omega$ (both $\mu_k$ and $\nu_k$ are bounded). Finally, we put $D$ to be $\{\alpha_k \mid k \in \omega\}$. ∎

**Proof of Lemma 3.71** Suppose that $D$ is a subset of $\mathcal{P}(\omega)$ with the properties listed in Claim 3.72, $u_1$ and $u_2$ are distinct elements of $U_r$. We put $S_r \in {}^{\omega}\{u_1, u_2\}$ to be

$$S_r(i) = \begin{cases} u_1 & (\exists j \in \omega)\ \{i,j\} \in D \wedge i < j, \\ u_2 & \text{otherwise.} \end{cases}$$

Clearly, for all $\{i,j\} \in D$, $S_r(i) \neq S_r(j)$. If $\mu, \nu \in [\bar{H}]$, $\mu \not\approx \nu$ then there is infinitely many $i \in \omega$ for that $\{\mu(i), \nu(i)\} \in D$ and so $S_r(\mu(i)) \neq S_r(\nu(i))$ and this is what we stated in the formulation of the lemma. ∎

d-cons

**Definition 3.73** Let $\Omega$ be a witness tree of type $H$ with root $r$, $Q$ be a valuation of the vertex set of $\Omega$. We say that $Q$ is *consistent* if (1), (2), (3) below hold.

(1). If $v$ is a leaf, $\tau_v = \mathbf{d}_{ij}$ and $\varepsilon_v = \ni$ (or $\varepsilon_v = \not\ni$) then $Q_v(i) = Q_v(j)$ (or $Q_v(i) \neq Q_v(j)$ respectively).

(2). If $v_1$ and $v_2$ is a pair of leaves, $\tau_{v_1} = X = \tau_{v_2}$ and $\varepsilon_{v_1} \neq \varepsilon_{v_2}$ then $Q_{v_1} \neq Q_{v_2}$.

(3). If $v_1, v_2$ are two vertexes of $\Omega$, $\mu \in [\bar{H}]$ and $Q_r \circ \nu_{v_1} \circ \mu \approx Q_r \circ \nu_{v_2}$ then $U_{v_1} \subseteq U_{v_2}$.

We say that $Q$ is *semi–consistent* if (1) above and (2'), (3') below hold.

(2'). If for a pair of leaves $v_1$ and $v_2$, we have that $\tau_{v_1} = X = \tau_{v_2}$, $\varepsilon_{v_1} \neq \varepsilon_{v_2}$ and $\nu_{v_1} \approx \nu_{v_2}$ then $Q_{v_1} \neq Q_{v_2}$.

(3'). $\Omega$ is not degenerated.

Note that if $Q$ is consistent then it is semi–consistent but it does not hold the other way round. □

1-P

**Lemma 3.74** *Let $\Omega$ be a witness tree with root $r$, $Q$ be a semi–consistent valuation of the vertex set of $\Omega$. Suppose that $|U_r| \geq 2$. Then there is a consistent valuation $P$ of the vertex set.*

**Proof** Since $Q$ is semi–consistent, we have that $\Omega$ is not degenerated. Lemma 3.71 is applicable, therefore there is a sequence $S_r$ with the property that $(\forall \mu, \nu \in [\bar{H}])\ \mu \not\approx \nu \Rightarrow S_r \circ \mu \not\approx S_r \circ \nu$. $S_r$ uniquely determines a valuation $S$ of the vertex set of $\Omega$ (Claim 3.67).

With the help of a merge function $m \in {}^{\omega}\{0,1\}$ the merge $P$ of $Q$ and $S$ can be defined: for all $i \in \omega$, vertex $v$

$$P_v(i) \stackrel{\text{def}}{=} \begin{cases} S_v(i) & \text{if} \quad m \circ \nu_v(i) = 0 \\ Q_v(i) & \text{if} \quad m \circ \nu_v(i) = 1. \end{cases}$$

We show that there is a merge function, such that $P$ is a consistent valuation of the vertex set.

For any merge function $m$, $P$ is a valuation of the vertex set. We check the conditions formulated in Definition 3.66. The domain of $P$ is the vertex set of $\Omega$, $P_v$ is an $\omega$-sequence over $U_v$.

If vertexes $v_1$ and $v_2$ are connected with an edge labelled by '$=$' (or '$[k/u]$') then $m \circ \nu_{v_1} = m \circ \nu_{v_2}$ so $P_{v_2} = P_{v_1}$ (or $P_{v_2} = P_{v_1}[k/u]$). If the edge is labelled by '$\sigma$' then $P_{v_2}(i) = \begin{cases} S_{v_2}(i) & \text{if} \quad m \circ \nu_{v_2}(i) = 0, \\ Q_{v_2}(i) & \text{if} \quad m \circ \nu_{v_2}(i) = 1. \end{cases}$ Since $S_{v_2} = S_{v_1} \circ \sigma$, $Q_{v_2} = Q_{v_1} \circ \sigma$ and $\nu_{v_2} = \nu_{v_1} \circ \sigma$ we get $P_{v_2}(i) = \begin{cases} S_{v_1}(\sigma(i)) & \text{if} \quad m \circ \nu_{v_1}(\sigma(i)) = 0, \\ Q_{v_1}(\sigma(i)) & \text{if} \quad m \circ \nu_{v_1}(\sigma(i)) = 1 \end{cases} = P_{v_1}(\sigma(i)) = P_{v_1} \circ \sigma(i)$.

The following claims give sufficient conditions on $m$ to ensure that $P$ is consistent.

cl-s1

**Claim 3.75** *There is a finite set $D_1 \subseteq \omega$ with the property that if $m(i) = 1$ for all $i \in D_1$ then $P$ satisfies (1) of the definition of consistency.*

**Proof** Put $D_1 \stackrel{\text{def}}{=} \{\nu(i), \nu(j) \mid \nu \in M_\Omega, \mathbf{d}_{ij}$ is in one of the leaves$\}$. Since there are only finitely many $\mathbf{d}_{ij}$'s that appear as leaf label term, $D_1$ is finite.

Suppose that a for a leaf $v$ $\tau_v = \mathbf{d}_{ij}$. Since $\nu_v(i), \nu_v(j) \in D_1$, $m \circ \nu_v(i) = m \circ \nu_v(j) = 1$, $P_v(i) = Q_v(i)$, $P_v(j) = Q_v(j)$. Now $P_v(i) = P_v(j)$ if and only if $Q_v(i) = Q_v(j)$, but this holds exactly when $\varepsilon_v$ is '$\ni$'. ∎

cl-s2

**Claim 3.76** *There is a finite set $D_2 \subseteq \omega$ with the property that if $m(i) = 1$ for all $i \in D_1$ and $m \approx \langle 0, 0, 0, \ldots \rangle$ then $P$ satisfies (2) of the definition of consistency.*

**Proof** Put $D_2 \stackrel{\text{def}}{=} \{\nu(i) \mid \nu \in M_\Omega,\ i < c\}$ where $c$ is the constant defined in Lemma 3.70. Suppose that for a pair of leaves $v_1$ and $v_2$ we have that $\tau_{v_1} = X = \tau_{v_2}$ and $\varepsilon_{v_1} \neq \varepsilon_{v_2}$.

If $\nu_{v_1} \not\approx \nu_{v_2}$ then $S_{v_1} \not\approx S_{v_2}$ since $S_{v_1} \approx S_r \circ \nu_{v_1} \not\approx S_r \circ \nu_{v_2} \approx S_{v_2}$. Furthermore, from the facts that $m \approx \langle 0,0,0,\ldots\rangle$ and $\nu_{v_1}$ and $\nu_{v_2}$ are bounded we can infer that $m \circ \nu_{v_1} \approx \langle 0,0,0,\ldots\rangle \approx m \circ \nu_{v_2}$. It gives that $P_{v_1} \approx S_{v_1}$ and $P_{v_2} \approx S_{v_2}$ and so $P_{v_1} \not\approx P_{v_2}$. But then $P_{v_1} \neq P_{v_2}$.

Now suppose that $\nu_{v_1} \approx \nu_{v_2}$. In that case there is an $i \in \omega$ for that $Q_{v_1}(i) \neq Q_{v_2}(i)$ since $Q$ was semi–consistent and (2') was applicable. Lemma 3.70 implies that $i < c$. Since $\nu_{v_1}, \nu_{v_2} \in M_\Omega$ we have that $\nu_{v_1}(i), \nu_{v_2}(i) \in D_2$ and so $m \circ \nu_{v_1}(i) = m \circ \nu_{v_2}(i) = 1$. This gives $P_{v_1}(i) = Q_{v_1}(i) \neq Q_{v_2}(i) = P_{v_2}(i)$. But then $P_{v_1} \neq P_{v_2}$. ∎

**Claim 3.77** *If $m \approx \langle 0,0,0,\ldots\rangle$ then $P$ satisfies (3) of the definition of consistency.* cl-s3

**Proof** Suppose that $v_1, v_2$ are two vertexes of $\Omega$, $\mu \in [\bar{H}]$ and $P_r \circ \nu_{v_1} \circ \mu \approx P_r \circ \nu_{v_2}$. $m \approx \langle 0,0,0,\ldots\rangle$ implies that $P_r \approx S_r$. Using the fact that $\nu_{v_1} \circ \mu$ and $\nu_{v_2}$ are bounded, we conclude that $S_r \circ \nu_{v_1} \circ \mu \approx S_r \circ \nu_{v_2}$. Referring to the property of $S$, we infer that $\nu_{v_1} \circ \mu \approx \nu_{v_2}$. $\Omega$ is not degenerated so we derive that $U_{v_1} \subseteq U_{v_2}$. ∎

If we put $m(i) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if} \quad i \in D_1 \cup D_2 \\ 0 & \text{if} \quad i \notin D_1 \cup D_2 \end{cases}$ then $m$ satisfies the conditions described in the previous claims so $P$ is consistent. ∎

**Definition 3.78** Let $\Omega$ be a witness tree of type $H$, $\mathcal{A}$ be a nstGws$_H$ with base transformation function $b$. Let $\Lambda$ be the set of weak spaces building $\mathcal{A}$. We say that a function $a : \bigcup_{\text{vertex } v} \{v\} \times U_v \to \bigcup_{\lambda \in \Lambda} base(\lambda)$ is an *adaptation* of $\Omega$ in $\langle \mathcal{A}, b\rangle$ if d-adapt

- $(\exists \lambda \in \Lambda)(\forall \text{ vertex } v)$ $a_v$ is a bijection between $U_v$ and $base(\hat{\nu}_v(\lambda))$,
- for any pair of vertexes $v_1, v_2$, and $\mu \in [\bar{H}]$ we have that $\nu_{v_1} \circ \mu \approx \nu_{v_2} \Rightarrow a_{v_2}\lceil_{U_{v_1}} = b_{\mu, \hat{\nu}_{v_1}(\lambda)} \circ a_{v_1}$.

□

**Definition 3.79** Let $\Omega$ be a witness tree of type $H$, $\mathcal{A}$ be a nstGws$_H$, $b$ be a base transformation function of $\mathcal{A}$, $Q$ be a valuation of the vertex set of $\Omega$, and $a$ be an adaptation of $\Omega$ in $\langle \mathcal{A}, b\rangle$. Suppose that $w$ is a function mapping the set of those variables that occur in a term of a vertex label into the universe of $\mathcal{A}$. d-validw

- $\langle Q, w, a, b\rangle$ is said to be a *valuation of* $\Omega$ *into* $\mathcal{A}$ if for all vertex $v$, $a_v \circ Q_v$ is a sequence from the unit element of $\mathcal{A}$, $a_v$ is a bijection between $U_v$ and the base of that weak space of $\mathcal{A}$ that contains $a_v \circ Q_v$.

- We say that a valuation $\langle Q, w, a, b\rangle$ of $\Omega$ into $\mathcal{A}$ *makes the label of a vertex* $v$ *true (or satisfies* $v$*)* if $\tau_v[w]\ \varepsilon_v\ a_v \circ Q_v$ holds in $\mathcal{A}$. (Depending on $\varepsilon_v$, it is $\tau_v[w] \ni a_v \circ Q_v$ or $\tau_v[w] \not\ni a_v \circ Q_v$.)

- And finally, a valuation $\langle Q, w, a, b\rangle$ of $\Omega$ into $\mathcal{A}$ is *valid* if it makes all the vertex labels of $\Omega$ true.

In the special situation, when $\mathcal{A} \in \mathrm{Gws}_H$ we say that $\langle Q, w\rangle$ is a (valid) valuation of $\Omega$ into $\mathcal{A}$ if $\langle Q, w, a, b\rangle$ is a (valid) valuation, where $a$ and $b$ are the identity mappings. □

`cl-leaves`

**Claim 3.80** *A valuation $\langle Q, w, a, b\rangle$ of $\Omega$ into $\mathcal{A}$ is valid if and only if it makes all the leaf labels of $\Omega$ true.*

**Proof** The requirements formulated in Definition 3.58 and 3.66 were designed to ensure that the validity of a vertex label could be proved from the validity of the labels of the children of that vertex. ■

`cl-sound`

**Claim 3.81** *If $\langle Q, w, a, b\rangle$ is a valid valuation of the witness tree $\Omega$ into some algebra $\mathcal{A} \in \mathrm{nstGws}_H$ then $Q$ is semi–consistent.*

**Proof** To prove (1) we pick an arbitrary leaf $v$ with $\tau_v = \mathbf{d}_{ij}, \varepsilon_v = \ni$. $\langle Q, w, a, b\rangle$ satisfies this vertex so $\mathbf{d}_{ij} \ni a_v \circ Q_v$ and that is $a_v \circ Q_v(i) = a_v \circ Q_v(j)$. Since $a_v$ is a bijection, it is implied that $Q_v(i) = Q_v(j)$. In a similar way, we can infer that $Q_v(i) \neq Q_v(j)$ whenever $\tau_v = \mathbf{d}_{ij}, \varepsilon_v = \not\ni$.

To verify (2') we suppose that $v_1$ and $v_2$ are vertexes with $\tau_{v_1} = X = \tau_{v_2}$, $\varepsilon_{v_1} \neq \varepsilon_{v_2}$ and $\nu_{v_1} \approx \nu_{v_2}$. Using the property (3) of the base transformation (Remark 3.55) and that of the adaptation we get that $a_{v_1} = a_{v_2}$. Since $\langle Q, w, a, b\rangle$ satisfies $v_1$ and $v_2$ we have that $w(X) \ni a_{v_1} \circ Q_{v_1}$, $w(X) \not\ni a_{v_2} \circ Q_{v_2}$ and so $a_{v_1} \circ Q_{v_1} \neq a_{v_2} \circ Q_{v_2}$. Now using that $a_{v_1} (= a_{v_2})$ is a bijection, we get $Q_{v_1} \neq Q_{v_2}$ as desired.

Finally, (3') is an immediate consequence of the last property of the adaptation. Suppose that for a pair of vertexes $v_1, v_2$, and $\mu \in [\bar{H}]$ we have that $\nu_{v_1} \circ \mu \approx \nu_{v_2}$. Then $a_{v_2} \lceil_{U_{v_1}} = b_{\mu, \hat{\nu}_{v_1}(\lambda)} \circ a_{v_1}$. This shows that the domain of $a_{v_1}$, that is $U_{v_1}$, is contained in that of $a_{v_2}$ so $U_{v_1} \subseteq U_{v_2}$. ■

`cl-compl`

**Claim 3.82** *Let $\Omega$ be witness tree with root $r$, $P$ be a consistent valuation of the vertex set of $\Omega$. Then there is an $\mathcal{A} \in \mathrm{Gws}_H$ and a function $w$ for that $\langle P, w\rangle$ is a valid valuation of $\Omega$ into $\mathcal{A}$.*

**Proof** For all vertex $v$, $U_r \subseteq U_v$ and so $s \in {}^\omega U_r \Rightarrow s \in {}^\omega U_v$. Furthermore, $s \in {}^\omega U_r \Rightarrow s \circ f \in {}^\omega U_v$, where $f$ is any transformation of $\omega$.

We define the set $U_s$ for all $s \in {}^\omega U_r$ as follows.

$$U_s \stackrel{\text{def}}{=} \bigcup \{U_v \mid \ v \text{ is vertex of } \Omega, \, (\exists \mu \in [\bar{H}]) \ P_v \circ \mu \approx s\}$$

If no vertex satisfies the condition, then $U_s$ is set to the empty set. We put $V$ to be $\bigcup\{{}^\omega U_s^{(s)} \mid s \in {}^\omega U_r\}$, $\mathcal{A}$ to be the full generalised weak $H$–cylindric set algebra with equality over $V$.

To see that $V$ is a $\mathrm{Gws}_\omega$–unit we need to show that $(\forall s_1, s_2 \in {}^\omega U_r)({}^\omega U_{s_1}^{(s_1)} \cap {}^\omega U_{s_2}^{(s_2)} \neq \emptyset \Rightarrow {}^\omega U_{s_1}^{(s_1)} = {}^\omega U_{s_2}^{(s_2)})$. Suppose that $q \in {}^\omega U_{s_1}^{(s_1)} \cap {}^\omega U_{s_2}^{(s_2)}$. Then $s_1 \approx q \approx s_2$. This gives that $(\forall \text{ vertex } v)(\forall \mu \in [\bar{H}]) P_v \circ \mu \approx s_1 \Leftrightarrow P_v \circ \mu \approx s_2$ and so $U_{s_1} = U_{s_2}$, ${}^\omega U_{s_1}^{(s_1)} = {}^\omega U_{s_2}^{(s_2)}$.

To ensure that $V$ can be a unit element of a $\mathrm{Gws}_H$ we need that $(\forall \sigma \in H)\ \mathbf{s}_\sigma V = V$, that is $(\forall \sigma \in H)(\forall q \in V)\ q \circ \sigma \in V$. Suppose that $q \in {}^\omega U_s^{(s)} \subseteq V, \sigma \in H$. $U_{s\circ\sigma} \supseteq U_s$ since for any vertex $v$, $\mu \in [\bar{H}]$, $P_v \circ \mu \approx s \Rightarrow P_v \circ \mu \circ \sigma \approx s \circ \sigma$. (We used that $\sigma$ is bounded.) Now $q \circ \sigma \in {}^\omega U_s^{(s\circ\sigma)} \subseteq {}^\omega U_{s\circ\sigma}^{(s\circ\sigma)} \subseteq V$.

An important result is that for any vertex $v$, $U_v = U_{P_r \circ \nu_v}$. Clearly, $P_v \circ Id = P_v \approx P_r \circ \nu_v$ so $U_v \subseteq U_{P_r \circ \nu_v}$. On the other hand, if for some vertex $v'$, $\mu \in [\bar{H}]$, $P_{v'} \circ \mu \approx P_r \circ \nu_v$ holds then $P_r \circ \nu_{v'} \circ \mu \approx P_r \circ \nu_v$ and so $U_{v'} \subseteq U_v$. (Property (3) of consistency.) This gives that $U_v \supseteq U_{P_r \circ \nu_v}$.

We have that for every vertex $v$, $P_v \in V$ since $P_v \in {}^\omega U_{P_r \circ \nu_v}^{(P_r \circ \nu_v)} \subseteq V$.

Finally we can define function $w$. For any variable $X$ that appear in $\tau_v$ for some vertex $v$, we set

$$w(X) \stackrel{\text{def}}{=} \{P_v \mid v \text{ is a vertex}, \tau_v = X, \varepsilon_v = \ni\}.$$

The previous observations shows that $\langle P, w\rangle$ is a valuation of $\Omega$ into $\mathcal{A}$. To see that $\langle P, w\rangle$ is valid we need to check that it satisfies the leaves. Property (1) of consistency can be used to verify that a leaf $v$ with $\tau_v = \mathbf{d}_{ij}$ is satisfied. If $\tau_v = X, \varepsilon_v = \ni$ then $P_v \in X[w] = w(X)$. Finally, if $\tau_v = X, \varepsilon_v = \not\ni$ then there is no other vertex $v'$ with $\tau_{v'} = X, \varepsilon_{v'} = \ni$ and have $P_v = P_{v'}$ in the same time. (Property (2) of consistency.) Therefore we have that $P_v \notin X[w] = w(X)$. ∎

In the following corollary we collect all the results we have achieved so far.

cr-help

**Corollary 3.83** *Let $\Omega$ be a witness tree with root $r$. Suppose that the cardinality of $U_r$ is bigger than 1 and $\Omega$ has a valid valuation into a* $\mathrm{nstGws}_H$. *Then $\Omega$ has a valid valuation into a* $\mathrm{Gws}_H$.

**Proof** If $\Omega$ has a valid valuation, say $\langle Q, w, a, b\rangle$ to a $\mathrm{nstGws}_H$ then $Q$ is semi–consistent (Claim 3.81). Lemma 3.74 proves that in that situation, there is a consistent valuation of the vertex set. Finally, using Claim 3.82 we conclude that $\Omega$ has a valid valuation into a $\mathrm{Gws}_H$. ∎

**Definition 3.84** We say that a witness tree $\Omega$ with root $r$ witnesses the fact that the equation $\varrho = 0$ does not hold in an algebra $\mathcal{A} \in \mathrm{nstGws}_H$ if $\tau_r = \varrho, \varepsilon_r = \ni$ and $\Omega$ has a valid valuation into $\mathcal{A}$. $\square$ d-wing

**Lemma 3.85** *Let $\mathcal{A}$ be a* $\mathrm{nstGws}_H$*, $\varrho$ be a term. Then $\mathcal{A} \not\models \varrho = 0$ if and only if there is a witness tree $\Omega$ that witnesses that fact.* l-wing

**Proof** If $\Omega$ witnesses that $\varrho = 0$ does not hold in $\mathcal{A}$, then there is a valid valuation $\langle Q, w, a, b\rangle$ that makes all the labels true. Concentrating on the root label we have that $\varrho[w] \ni a_r \circ Q_r$, and so $\mathcal{A} \not\models \varrho = 0[w]$, $\mathcal{A} \not\models \varrho = 0$.

To prove the other direction suppose that $\mathcal{A} \in \mathrm{nstGws}_H$, $\mathcal{A} \not\models \varrho = 0$. Let $b$ be a base transformation function of $\mathcal{A}$. There is a valuation $w$ of the variables of $\varrho$ for that $\varrho[w]$ is not empty. We fix $w$ and a sequence $q$ in $\varrho[w]$. We denote the weak space of $q$ by $\lambda$.

**Claim 3.86** $(\exists \xi \in [\bar{H}])(\forall \nu \in M_\varrho)(\exists \nu^* \in [\bar{H}])\ \ \nu \circ \nu^* \approx \xi$*, that is, $M_\varrho$ has an upper bound. ($M_\varrho$ is defined in Definition 3.60).* c-w1

**Proof** We prove that the statement holds for an arbitrary, but finite $M \subseteq [\bar{H}]$. We prove it by induction on the size of $M$. If $|M| = 1$ then there is nothing to verify.

Suppose that $M \subseteq [\bar{H}]$ is finite, $\nu_0, \xi \in [\bar{H}]$, $\nu_0 \notin M$ and $(\forall \nu \in M)(\exists \nu^* \in [\bar{H}])\ \ \nu \circ \nu^* \approx \xi$. Using the property of $[\bar{H}]$ formulated in Theorem 2.30 (3), we get that there are transformations $\nu_0^*, \xi^*$ with $\nu_0 \circ \nu_0^* \approx \xi \circ \xi^*$. Now for the set $H' = H \cup \{\nu_0\}$, $\xi \circ \xi^*$ is an appropriate choice to be its upper bound. Indeed, $(\forall \nu \in M)\ \ \nu \circ \nu^* \circ \xi^* \approx \xi \circ \xi^*$. (We used that $\xi^*$ is bounded.) $\blacksquare$

For every $\nu \in M_\varrho$ we set

$$\begin{aligned} U_\nu &= \{b_{\nu^*, \hat{\nu}(\lambda)}(u) \mid u \in \mathit{base}(\hat{\nu}(\lambda))\}, \\ a_\nu &= b^{-1}_{\nu^*, \hat{\nu}(\lambda)}. \end{aligned}$$

Clearly, $a_\nu$ is a bijection between $U_\nu$ and $\mathit{base}(\hat{\nu}(\lambda))$.

**Claim 3.87** *If $\nu_1 \approx \nu_2 \circ \mu$ then $a_{\nu_1}\lceil_{U_{\nu_2}} = b_{\mu, \hat{\nu}_2(\lambda)} \circ a_{\nu_2}$ ($\nu_1, \nu_2 \in M_\varrho, \mu \in [\bar{H}]$).* c-w2

**Proof** $\nu_1 \approx \nu_2 \circ \mu$ and $\nu_2 \circ \nu_2^* \approx \xi \approx \nu_1 \circ \nu_1^*$ gives that $\nu_2 \circ \nu_2^* \approx \nu_2 \circ \mu \circ \nu_1^*$ and so $\hat{\nu}_2^* \circ \hat{\nu}_2 = \widehat{\mu \circ \nu_1^*} \circ \hat{\nu}_2$, $\hat{\nu}_2^*(\hat{\nu}_2(\lambda)) = \widehat{\mu \circ \nu_1^*}(\hat{\nu}_2(\lambda))$. Property (5) of Remark 3.55 and then (4) gives that $b_{\nu_2^*, \hat{\nu}_2(\lambda)} = b_{\mu \circ \nu_1^*, \hat{\nu}_2(\lambda)} = b_{\nu_1^*, \hat{\mu} \circ \hat{\nu}_2(\lambda)} \circ b_{\mu, \hat{\nu}_2(\lambda)}$. Using property (2), from $\nu_1 \approx \nu_2 \circ \mu$ we can infer that $\hat{\mu} \circ \hat{\nu}_2 = \hat{\nu}_1$. This gives $b_{\nu_2^*, \hat{\nu}_2(\lambda)} = b_{\nu_1^*, \hat{\nu}_1(\lambda)} \circ b_{\mu, \hat{\nu}_2(\lambda)}$ so $a_{\nu_2}^{-1} = a_{\nu_1}^{-1} \circ b_{\mu, \hat{\nu}_2(\lambda)}$. But then $a_{\nu_1}\lceil_{U_{\nu_2}} = b_{\mu, \hat{\nu}_2(\lambda)} \circ a_{\nu_2}$. $\blacksquare$

We construct a witness tree $\Omega$ that witnesses $\mathcal{A} \models \varrho \neq 0$ together with an adaptation $a$ and a valid valuation $Q$ of the vertex set by induction. To give name to the interior stages of the construction we introduce the notion of a unsaturated witness tree and that of an unsaturated system.

$\Omega$ is an *unsaturated witness tree* if it satisfies all but the last requirement listed in Definition 3.58 and it satisfies the following one.

(v'). For every vertex $v$, that is not a leaf, $\tau_v$, $\varepsilon_v$, the number of children of $v$, the labels of the edges pointing to the children of $v$, and the value of $\tau$ and $\varepsilon$ on of the children vertexes of $v$ are in a connection indicated in the table given in Definition 3.58.

The level (or depth) of saturation is the minimal length of a path in the tree that joins the root with a vertex that does not satisfy (v) of Definition 3.58. The level of saturation is set to $\omega$ if every vertex satisfies (v), that is, the tree is a witness tree. We also call a tree with that property saturated. If the saturation level is bigger than the length of the root term then the tree is saturated i.e. it is a witness tree. Clearly, on a path starting form the root every term written on a vertex is a subterm of the previous one and so the length of a path cannot be longer than the length of the root term.

An *unsaturated system* is a triple $\langle \Omega, Q, a\rangle$ where $\Omega$ is an unsaturated witness tree, $Q$ is a valuation of the vertex set of $\Omega$, $a$ is an adaptation of $\Omega$ in $\langle \mathcal{A}, b\rangle$ and $\langle Q, w, a, b\rangle$ is a valid valuation.

We construct a series of unsaturated systems. Every system is an extension of the previous one, that is, $V^{n+1} \supseteq V^n, E^{n+1} \supseteq E^n, r^{n+1} = r^n$, the functions $U^{n+1}, \tau^{n+1}, \varepsilon^{n+1}, l^{n+1}, Q^{n+1}, a^{n+1}$ are extensions of the corresponding functions of the $n$-th system. The root term of the first system, and so that of all systems, is $\varrho$. The saturation level of the witness trees in the series is strictly increasing. This proves that after a certain index, say $n_0$ all the systems are saturated in the series. So $\Omega^{n_0}$ is a witness tree and $\langle Q^{n_0}, w, a^{n_0}, b\rangle$ proves that $\Omega^{n_0}$ witnesses the fact that $\mathcal{A} \not\models \varrho = 0$ as desired.

**Construction of the first system:** Let r be any point. Then we set $V^1 \stackrel{\text{def}}{=} \{r\}, E^1 \stackrel{\text{def}}{=} \emptyset, U^1_r \stackrel{\text{def}}{=} U_{\text{Id}}, \tau^1_r \stackrel{\text{def}}{=} \varrho, \varepsilon^1_r = \ni, a^1_r \stackrel{\text{def}}{=} a_{\text{Id}}, Q^1_r \stackrel{\text{def}}{=} a^{-1}_{\text{Id}} \circ q$.

**Construction of the $n$-th system from the $(n-1)$-th:** We add children vertexes to those leaves of $\Omega^{n-1}$ that do not satisfy (v). If leaf $v \in V^{n-1}$ does not satisfy (v) and $\tau^{n-1}_v, \varepsilon^{n-1}_v$ has the value indicated in column C1 then we add as many children vertexes as given in column C2, we label the edges pointing to the new vertexes according to column C3, and we set the values of $\tau^n, \varepsilon^n, Q^n, a^n$ on the new vertexes as indicated in column C4.

| $C1$ | $C2$ | $C3$ | $C4$ | |
|---|---|---|---|---|
| $\tau_v^{n-1}, \varepsilon_v^{n-1}$ | | $l^n$ | $U^n, \tau^n, \varepsilon^n, Q^n, a^n$ | |
| $(-\varrho), \ni$ | 1 | $=$ | $U_v^{n-1}, \varrho, \not\ni, Q_v^{n-1}, a_v^{n-1}$ | |
| $(-\varrho), \not\ni$ | 1 | $=$ | $U_v^{n-1}, \varrho, \ni, Q_v^{n-1}, a_v^{n-1}$ | |
| $(\varrho_1 \cup \varrho_2), \ni$ | 1 | $=$ | $U_v^{n-1}, \varrho_i, \ni, Q_v^{n-1}, a_v^{n-1}$ | (*) where $i \in \{1,2\}$ and $\varrho_i[w] \not\ni a_v \circ Q_v$ |
| $(\varrho_1 \cup \varrho_2), \not\ni$ | 2 | $=$ | $U_v^{n-1}, \varrho_i, \not\ni, Q_v^{n-1}, a_v^{n-1}$ | for $i \in \{1,2\}$ |
| $\mathbf{c}_k\varrho, \ni$ | 1 | $[k/u]$ | $U_v^{n-1}, \varrho, \ni, Q_v^{n-1}[k/u], a_v^{n-1}$ | (*) where $u \in U_v$ and $\varrho \ni a_v \circ (Q_v[k/u])$ |
| $\mathbf{c}_k\varrho, \not\ni$ | $\lvert U_v \rvert$ | $[k/u]$ | $U_v^{n-1}, \varrho, \not\ni, Q_v^{n-1}[k/u]_v, a_v^{n-1}$ | for each $u \in U_v$ |
| $\mathbf{s}_\sigma\varrho, \ni$ | 1 | $\sigma$ | $U_{\nu_v \circ \sigma}, \varrho, \ni, Q_v^{n-1} \circ \sigma, a_{\nu_v \circ \sigma}$ | (**) |
| $\mathbf{s}_\sigma\varrho, \not\ni$ | 1 | $\sigma$ | $U_{\nu_v \circ \sigma}, \varrho, \not\ni, Q_v^{n-1} \circ \sigma, a_{\nu_v \circ \sigma}$ | (**) |

In (*) we used that $\varrho_1 \cup \varrho_2[w] \not\ni a_v \circ Q_v$ implies that there is an $i \in \{1,2\}$ with that property. Similarly, if $\mathbf{c}_k\varrho[w] \ni a_v \circ Q_v$ then there is an $u \in U_v$ with $\varrho \ni a_v \circ (Q_v[k/u])$. To see (**) we note that $\nu_v \circ \sigma \in [\bar{H}]$ so $a_{\nu_v \circ \sigma}$ and $U_{\nu_v \circ \sigma}$ is defined some pages before.

The verification of the facts that the newly defined $Q^n$ is indeed a valuation of the vertex set of the expanded tree, $a^n$ is an adaptation and $\langle Q^n, w, a^n, b\rangle$ is a valid valuation is straightforward. We need to examine the table above, the definitions of the notions in question and to note that for every vertex $v$ $a_v^n = a_{\nu_v}, U_v^n = U_{\nu_v}$. ∎

Before turning to prove Theorem 2.30 (3) we state a final corollary of the previous lemma.

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**Corollary 3.88** *Suppose that the equation $\varrho = 0$ fails in an algebra $\mathcal{A} \in$ $\mathrm{nstGws}_H$, and $\mathcal{A} \models -\mathbf{c}_0(-\mathbf{d}_{01}) = 0$. Then* $\mathrm{Gws}_H \not\models \varrho = 0$.

**Proof** If $\mathcal{A} \models -\mathbf{c}_0(-\mathbf{d}_{01}) = 0$ then none of the weak spaces of $\mathcal{A}$ can have a base of cardinality 1.

If $\varrho = 0$ fails in $\mathcal{A}$, then there is a witness tree $\Omega$ with root $r$ that witnesses that fact. $\Omega$ has a valid valuation into $\mathcal{A}$. $|U_r| > 1$. Corollary 3.83 is applicable, therefore $\Omega$ has a valid valuation into a $\mathrm{Gws}_H$. Using again Lemma 3.85 we get that $\varrho = 0$ fails in that algebra of $\mathrm{Gws}_H$. ∎

**Proof of Theorem 2.30 (3)** Since both $\mathbf{H}\mathrm{Gws}_H$ and $Mod(H1, H2, H3)$ are varieties, we need to show that they satisfy the same set of equations.

First we show that $\mathrm{Gws}_H \models H1, H2, H3$. The only equation that needs explanation is the first one in H3. First suppose that $q \in x - \mathbf{c}_0(-\mathbf{d}_{01})$. Then the weak space of $q$ has a base of cardinality 1, so $q = \langle u, u, u, \ldots\rangle$. But in that case $q = q \circ \sigma$, $q \in \mathbf{s}_\sigma(x - \mathbf{c}_0(-\mathbf{d}_{01}))$. Now suppose that $q \in \mathbf{s}_\sigma(x - \mathbf{c}_0(-\mathbf{d}_{01}))$. Then $q \circ \sigma \in x - \mathbf{c}_0(-\mathbf{d}_{01})$, so the weak space of $q \circ \sigma$ has a base of cardinality 1. Referring to Claim 3.44 we have that the base of the weak space of $q$ is contained in that of $q \circ \sigma$ so it is of cardinality 1 as well. Hence, $q = q \circ \sigma$, $q \in x - \mathbf{c}_0(-\mathbf{d}_{01})$.

For the other direction we show that any equation $e$ that fails in an $\mathcal{A} \in Mod(H1, H2, H3)$ also fails in some $\mathcal{B} \in \mathrm{Gws}_H$. First we simplify the situation by putting assumptions on $\mathcal{A}$ and $e$ and then we will apply Corollary 3.88 or Lemma 3.57 to prove the theorem.

If $e$ is of the form $\tau = \tau'$ then the equation $\tau \oplus \tau' = 0$ is satisfied in exactly those algebras where $e$ ($\oplus$ is the symmetric difference operator). For that reason, without loss of generality we may assume that $e$ of the form $\varrho = 0$.

Every algebra is a subalgebra of a product of subdirect irreducible algebras. If an equation fails in a subdirect product, then it fails in some of the factors. So we may assume that $\mathcal{A}$ is a subdirectly irreducible algebra.

Furthermore, referring to Theorem 3.52 we may put the assumption that $\mathcal{A}$ is a $\mathrm{nstGws}_H$. According to Theorem 3.56 either $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ or $-\mathbf{c}_0(-\mathbf{d}_{01}) = 0$ holds in $\mathcal{A}$. In the first case Lemma 3.57, in the second Corollary 3.88 provides the required $\mathcal{B}$ in which $\varrho = 0$ fails. ∎

**Proof of Theorem 2.30 (4)** The proof of Theorem 2.30 (3) is applicable with some adjustments. We present the required modifications here.

Theorem 3.56 remains valid.

In the proof of Lemma 3.57 one can note that the algebra $\mathcal{A}'$ defined therein is a direct product of some two element algebras. (Clearly, any weak space with a base of cardinality 1 is a $H$-cylindric set algebra with equality in itself.) Hence, this lemma can be reformalized as: *If* $-\mathbf{c}_0(-\mathbf{d}_{01}) = 1$ *holds in* $\mathcal{A} \in \mathrm{nstGws}_H$ *then* $\mathcal{A}$ *is a* $\mathbf{P}\mathrm{Cs}_H$.

Under the conditions listed in Theorem 2.30 (4), we have that: *If* $\Omega$ *is a not degenerated witness tree then for every vertex* $v$ *we have that* $U_v = U_r$. Suppose that $v$ is a vertex. Put $\nu_v^*$ and $\nu_r^*$ to be the right quasi-inverse of $\nu_v$ and $\nu_r$ respectively. Then $\nu_v \circ \nu_v^* \circ \nu_r \approx \nu_r$ and $\nu_r \circ \nu_r^* \circ \nu_v \approx \nu_v$ shows that $U_v \supseteq U_r$ and $U_r \supseteq U_v$ giving $U_v = U_r$. (We used that $\nu_r$ and $\nu_v$ are bounded transformations.)

Claim 3.82 can be modified to: *Let* $\Omega$ *be witness tree with root* $r$, $P$ *be a consistent valuation of the vertex set of* $\Omega$. *Then there is a function* $w$ *for that* $\langle P, w\rangle$ *is a valid valuation of* $\Omega$ *into the full* $H$*–cylindric set algebra with equality over the base set* $U_r$. To prove that statement we need to skip the argumentation about the algebra itself in the original proof and start reading from the point where $w$ is defined.

Using that modified Claim we get: *Suppose that the equation* $\varrho = 0$ *fails in an algebra* $\mathcal{A} \in \mathrm{nstGws}_H$, *and* $\mathcal{A} \models -\mathbf{c}_0(-\mathbf{d}_{01}) = 0$. *Then* $\mathrm{Cs}_H \not\models \varrho = 0$.

Following the argument described in the proof of Theorem 2.30 (3) and inserting the modifications listed above we get that the statement in Theorem 2.30 (4) holds. ∎

## Historical remark

The early versions of the ideas presented in this work can be found in [Sa87] but the investigations here go much further than the ones therein. Many of our results *for the special choice* $H = G$ are already stated in [Sa87] in less developed form. The case without equality is fully proved there, but our axiom system Ax is considerably simpler than the one in [Sa87]. (Cf. around the end of Part I and beginning of Part II of [Sa87].) The *Tarski–Givant style* part of the case with equality is implicit in [Sa87a]. These research directions are not investigated in [Sa93].

The works of Sándor Csizmazia (e.g. [Cs93], [Cs95]) were based on the early version [Sa87] of the present ideas.


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