ON THE SEARCH FOR A FINITIZABLE ALGEBRAIZATION OF FIRST ORDER LOGIC

ILDIKÓ SAIN[1]
Mathematical Institute of the Hungarian Academy of Sciences
Budapest, PF.127, H-1364, Hungary
e-mail: sain@math-inst.hu

May 20, 1997 (first version: 1987)

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Abstract

We give an algebraic version of first order logic *without equality* in which the class of representable algebras forms a finitely based equational class. Further, the representables are defined in terms of set algebras, and all operations of the latter are permutation invariant. The algebraic form of this result is Theorem 1 (a concrete version of which is given by Theorems 1.8 and 3.2), while its logical form is Corollary 4.2.

For first order logic *with equality* we give a result weaker than the one for first order logic without equality. Namely, in this case — instead of finitely axiomatizing the corresponding class of all representable algebras — we finitely axiomatize only the equational theory of that class. See subsection 5.1, especially Remark 5.5 there.

The proof of Theorem 1 is elaborated in sections 2 and 3. These sections contain theorems which are interesting of their own rights, too, e.g. Theorem 3.2 is a purely semigroup theoretic result. Cf. also "Further main results" in the Introduction.

Keywords: algebraic logic, finitization problem, representation problem, finite axiomatization, completeness problem, ultraproducts, formula schemes, cylindric algebras, polyadic algebras, algebras of relations, finite presentation of semigroups of relations.

AMS subject classification:

03G15	(CA, PA, RA)
03G25	(other algebras to logic)
03G99	(none of the above)
03B10	(classical first order logic)
03C20	(ultraproducts and related constructions)
20M05	(free semigroups, generators and relations etc.)
20M20	(semigroups of transformations)

[1]Research supported by Hungarian National Foundation for Scientific Research grant No's T16448, T7567, F17452.

Contents

INTRODUCTION

The problem

In the present paper we address a problem of logic the algebraic form of which is formulated in Monk [Mo70, p.20^{4-17}] as follows.

"Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based equational class."

(An equational class [i.e., variety] is called finitely based iff it is axiomatizable by finitely many equations.)

We refer to the problem quoted above as finding a *"finitizable" algebraization of first order logic*, or briefly, the *finitization problem*. A more concrete version of the finitization problem is the following (cf. e.g. [Né91]). To any logic (or logical system) $\mathcal{L}$, a class $\mathsf{Alg}(\mathcal{L})$ of algebras is associated in [AKNS] as well as in [NA94], cf. also [HMT §5.6 pp.255–259]. Essentially, $\mathsf{Alg}(\mathcal{L})$ is the class of all (semantical) Lindenbaum–Tarski algebras of all theories of $\mathcal{L}$.[2] Now, the task is to find such a complete logic $\mathcal{L}$ whose expressive power is greater than or equal to that of first order logic such that $\mathsf{Alg}(\mathcal{L})$ is a finitely axiomatizable variety or quasi–variety (quasi–equational class). Here $\mathcal{L}$ has to satisfy the axioms of abstract model theory, in particular, the semantics of $\mathcal{L}$ should be invariant under taking isomorphisms of models.[3] We will present a possible solution for this form of the problem in section 4.

The finitization problem has a long history and many people worked on it, e.g., A. Tarski worked on it already in the 1940's.[4] For a further formulation of the problem see e.g. Henkin–Monk [HM74]. More recent detailed expositions of the finitization problem can be found in [Si91] and in [SGy96] — the latter deals with a schema variant of the problem quoted above. The most exhaustive account (known for the author) of this and related problems can be found in the blocks of text marked "Finitization" in Németi [Né91].

In [HM74] and [Mo70] (as well as in the standard literature of algebraic logic, e.g. in [HMT], [HMTAN], [TG]) an algebra is called *representable* iff it is isomorphic to a subdirect product of *set algebras*. (Examples for set algebras are: Boolean set algebras, i.e. Boolean fields of sets; cylindric set algebras — which are Boolean algebras of relations, augmented with some extra–Boolean set–operations.) Therefore the key step in solving the finitization problem is finding an appropriate notion of set algebra. In devising such definitions, usually, an ordinal α is fixed in advance, and then the elements of a set algebra are (certain) α–ary relations over some base set U. So, an element X of a set algebra $\mathfrak{A}$ is usually of the form $X \subseteq {}^{\alpha}U$. One piece of the connection with logic is that U corresponds to the universe of that model of first order logic from which $\mathfrak{A}$ is obtained (see [HMTII, §§4.3, 5.6] for details). Tarski called an operation $f^{\mathfrak{A}} : {}^{n}A \longrightarrow A$ of such a set algebra $\mathfrak{A}$ *logical* iff it is invariant under permutations of U. That is, using the notation of [HMTII], $f^{\mathfrak{A}}$ is logical iff for every permutation $p : U \rightarrowtail\!\!\!\!\rightarrow U$ and $X_1, \ldots, X_n \in A$, we have $\tilde{p}\left(f^{\mathfrak{A}}(X_1, \ldots, X_n)\right) = f^{\mathfrak{A}}(\tilde{p}X_1, \ldots, \tilde{p}X_n)$, where $\tilde{p}$ is the permutation of $\mathcal{P}\left({}^{\alpha}U\right)$ induced by p the natural way[5]. The same property is called *"(permutation) invariant"* in Jónsson [Jó84]. In this paper we study that version of the above quoted problem of Henkin and Monk in which all operations of the set algebras are required to be logical. This requirement was explicitly indicated among the conditions in the book Tarski–Givant [TG86] when formulating related problems (e.g. on p.60 there). When formulating the relation algebraic counterpart of this problem, Bjarni Jónsson

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[2] In this version of a Lindenbaum–Tarski algebra, every connective of the logic $\mathcal{L}$ appears as an operation.

[3] Familiarity with $\mathsf{Alg}(\mathcal{L})$ is not needed for the rest of this paper.

[4] The problem can be traced back even to E. Schröder (in implicit form) of the last century.

[5] Recall from [HMTII] that for any $p : U \rightarrowtail\!\!\!\!\rightarrow U$ and $X \subseteq {}^{\alpha}U$, $\tilde{p}(X) = \{p \circ q : q \in X\}$.

also emphasized the requirement that the operations should be logical.[6] The logical counterpart of this requirement is that isomorphic models satisfy the same formulas — this is one of the basic axioms of abstract model theory.

In Corollary 3.3 in section 3 of the present paper, we present an algebraic version of first order logic without equality in which the class of representable algebras forms a finitely based equational class. Further, the representable algebras are defined in terms of set algebras, and all operations of the latter are logical in Tarski's sense (recalled above). For first order logic without equality, therefore, Corollary 3.3 provides a possible solution to the Henkin–Monk problem quoted above.

The results

One of our main theorems is Theorem 1 below. Recall from, e.g., [HMT] that a *subreduct* is a subalgebra of a reduct. ω denotes the first infinite cardinal (i.e., the set of all natural numbers).

THEOREM 1. *There is a finitely based variety* V *satisfying conditions (i)–(iii) below.*

(i) V *consists of all homomorphic images of subreducts of (representable) polyadic algebras of dimension* ω (PA_ω*'s); of course,* V *has only finitely many operation symbols. This statement in symbols reads as:*
$$\mathsf{V} = \mathbf{HSRd}\,\mathsf{PA}_\omega\,.$$

(ii) *Every element of* V *is representable as an algebra whose elements are* ω*–ary relations, and whose operations are among the natural ones on these relations (analogously to the representability of* PA_ω*'s).*

(iii) *The class of the* (ω*–dimensional) quasi–polyadic subreducts of members of* V *coincides with the class of the representable quasi–polyadic algebras. (Here by "reduct" we mean generalized reducts where term functions of* V *become operations of the reduct.) In symbols:*
$$\mathsf{RQPA}_\omega = \mathbf{SRd}_{\mathsf{QPA}_\omega}\mathsf{V}\,.$$

On the **proof**: In the above mentioned Corollary 3.3 we concretely give a finitely based variety V satisfying conditions (i)–(iii) in Theorem 1. See the last part of subsection 3.1. ∎

We note that an *equivalent* form of (iii) would say:

(iii$^+$) The class of ω–dimensional representable quasi–polyadic algebras coincides with the class of all subreducts of V^0 for some variety V^0 term definitionally equivalent[7] with V. In symbols:
$$\mathsf{RQPA}_\omega = \mathbf{SRd}\,\mathsf{V}^0 \quad \text{for some } \mathsf{V}^0 \equiv_{\text{term}} \mathsf{V}\,.$$

(Here subreducts are understood in the strict sense.)

From Theorem 1 immediately follows that there is an algebraic version V of first order logic without equality such that the representable members of V form a finitely based equational class, namely V itself. Further, all the operations of V are logical (cf. item (i)). The rôle of item (i) in Theorem 1 is to provide us with some kind of "control" over what kind of operations may show up as operations of V (e.g., they all remain logical). (Adequateness of V for first order logic without equality is ensured by (iii) of Theorem 1, since quasi–polyadic algebras are well known to be

[6] In personal communication, the originators of the problem expressed their opinion that this condition is necessary for a satisfactory solution. The condition itself goes back to Lindenbaum–Tarski [LT36], and was extensively discussed in Tarski [Ta66] (where it was suggested that the condition is essential for investigations of a truly logical nature).

[7] Cf. e.g. McKenzie–McNulty-Taylor [MMT87, Def.4.1.38, p.246] (where term–definitional equivalence is called simply "equivalence").

adequate for that logic. Actually, that is what they were designed for.) Therefore, by Theorem 1, the above quoted Henkin–Monk problem admits a positive solution for logic without equality.[8] The purely logical consequences will be elaborated in section 4. Namely, in section 4 we will show that the version of Theorem 1 we prove here has the following equivalent form. There is an extension (with new logical connectives) $\mathcal{L}$ of first order logic without equality such that the usual algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ of $\mathcal{L}$ coincides with V in Theorem 1. Further, $\mathcal{L}$ is complete, and the formula schemas valid in $\mathcal{L}$ admit a strictly finite axiomatization. (The latter is not true for ordinary first order logic in place of $\mathcal{L}$.)

First order logic *with equality* seems to behave in a radically different way. Namely, as a contrast with Theorem 1 above, in section 5 we will see that Theorem 1 cannot be extended to polyadic and quasi–polyadic algebras *with equality* (PEA's and QPEA's, respectively). More concretely, if in Theorem 1 we replace PA with PEA and quasi–polyadic algebra with QPEA everywhere, then the theorem becomes false (cf. Theorem 5.1, Remark 5.4). It remains false even if we replace the condition that V is finitely axiomatizable with the weaker one that V is term definitionally equivalent with a finitely axiomatizable class. For more cf. subsection 5.1, where we also state open problems in this direction. Other forms of this kind of "negative" behavior of logic with equality were also investigated in Craig [Cr74], Johnson [Jo69], Németi–Sági [NS96, NS96a], see also [HMTII, item 5.4.41]. Further negative results are in section 5.2 which[9] is, basically, a more detailed and technical version of Henkin [He87]. Actually, we conjecture that under the restriction that the operations of the set algebras have to be logical, the answer to the finitization problem will be negative for logic *with* equality.[10] If this would turn out to be the case then algebraic logic could provide "pure" logic with a significant property distinguishing these two versions of first order logic. We should note that in the above conjecture the condition requiring that the operations are logical is of crucial importance. We think that the second part of [Bi87] as well as [Md89a, §3, Def.7 and Thm.9] indicate the possibility of easy positive solutions for the problem without this condition.

Although we said that Theorem 1 does not extend to the case with equality, a certain weaker form does: In subsection 5.1 we also look at the weaker form of Theorem 1 where we finitely axiomatize only the equational theory $\mathbf{Eq}(\mathsf{V})$ of V instead of axiomatizing V itself. We will see that in this weaker form Theorem 1 *does* extend to (algebras of) logic with equality. The price we have to pay for having equality is that we can axiomatize only the theory of V but not V itself. For cylindric algebras this result means that the class RCA_ω of representable cylindric algebras (of dimension ω) can be expanded with new logical operations such that RCA_ω becomes finitely axiomatizable w.r.t. using these new operations (but the new operations themselves are not axiomatized). In the logic applications of this we obtain a completeness theorem (for a certain logic with equality). The price we have to pay for having equality is that this completeness theorem is "completeness without compactness". That is, $(\Sigma \models \varphi \not\Rightarrow \Sigma \vdash \varphi)$ for some Σ.

We note that the finitization problem exists in a finite variable (or finite dimensional) form too, cf. e.g. Németi [Né91], [TG87]. The finite variable version seems to be dominated by negative results so far. A strong result in this direction is in Biró [Bi87]. Biró's negative result was considerably

[8] Richard J. Thompson informed us that, independently of the work reported here, he had found a finitely based variety V_T which, at least in a sense, could be considered as an algebraic device for studying first order logic without equality. However, the adequateness of V_T for first order logic was somewhat problematic. Namely, the algebraic counterpart c_i of the quantifier $\exists v_i$ was not definable in V_T. The precise formulation of that result is not yet published, and is not known for the present author, but hopefully it will be published sometime.

[9] For stronger results in the line of section 5.2 see [MSS90], [SS90].

[10] This conjecture concerns only the *strong* version of the finitization problem when we try to axiomatize V itself. Cf. Open Problem 5.7 in subsection 5.1. The weak (or "Tarski–Givant style") version asks for a finite axiomatization of the equational theory $\mathbf{Eq}(\mathsf{V})$ of V only. This weaker problem can be solved for the case with equality. We will discuss such a solution in section 5 herein. For more datail see [SGy96] and [Sa92].

strengthened by H. Andréka and I. Németi, cf. [Né91, Remark 2, footnote 65 (in section 6)] and [An91, Thm.4 and Remark 6].

Let us turn to (parts of) the subject matter of this paper on a somewhat more technical level.

Further main results

Besides Theorem 1, the following somewhat more concrete results can also be considered as main results of the present paper.

Daigneault and Monk [DM63] proved that the class RPA_ω of representable polyadic algebras is axiomatizable by a kind of finite schemas (introduced by Paul Halmos). The extra Boolean operations of RPA_ω are $\{\mathrm{C}_{(\Gamma)}, \mathrm{S}_\tau : \Gamma \subseteq \omega \text{ and } \tau \in {}^\omega\omega\}$. We extend this result in directions (i)–(iv) below.

(i) If we throw away the $\mathrm{C}_{(\Gamma)}$'s (keeping only C_0), the class of representables remains *finite schema axiomatizable*, cf. Theorem 2.15.
(ii) This remains true if in addition we keep only some of the S_τ's as far as the semigroup they generate remains "rich" (in the sense of our definition at the beginning of subsection 1.3). Cf. Theorem 2.15.
(iii) The class of representable algebras remains a quasi–variety even if we keep only an arbitrary subset of $\mathrm{C}_{(\Gamma)}$'s and S_τ's as extra Boolean operators (see Lemma 2.5 in subsection 2.1).
(iv) The Daigneault–Monk schematization of RPA_ω was shown "non-algorithmic" (or "non-effective") in Sági [Sá96], cf. also Németi–Sági [NS96, NS96a][11]. We provide here "algorithmic" schematizations for certain variants of RPA_ω (cf. Theorems 1.8, 2.17).

Finally, assume we keep only the C_i's and some S_τ's. Let Σ_G be the set of true equations involving the S_τ's only. (By "true" we mean valid in the representable algebras.) Then the class of representable algebras is finite schema axiomatizable over Σ_G (for a precise formulation see Theorem 2.17).

In many respects, the present work is a direct continuation of a research direction started in the 60's by W. Craig, J. D. Monk, and their followers. In this direction, we will try to make explicit at least some of the connections with Craig [Cr74], Demaree [De69], McKenzie–Thompson [MT73], and Monk [Mo70]. (An old version [Sa87a] of this paper contains more material on these connections. In particular, some of its parts form a direct continuation of research started in Craig [Cr74], while others of that in Demaree [De69].)

On the structure of the paper

Sections 1, 2 of this paper contain general "meta–theorems" as tools for finding and/or simplifying solutions for the above quoted Henkin–Monk problem. These theorems reduce the problem to a purely semigroup theoretic one. (This kind of approach to algebraic logic originates with W. Craig, see e.g. [Cr74a].) It is our hope that these meta–theorems will be used in the future to obtain simpler and nicer algebraizations of logic (by finding "nice" semigroups satisfying the conditions of the theorem).

Section 3 presents a proof for a semigroup theoretic theorem (Theorem 3.2) needed for applying the above "meta–theorems" to obtain a finitizable algebraization of predicate logic (Corollary 3.3).

[11] Here an axiom schema Σ is called non–algorithmic iff the consequences (or equivalently, instances) of Σ are not Turing enumerable, no matter how we generalize Turing enumerability to uncountable sets (assuming the generalization satisfies some "reasonable" postulates).

This semigroup theoretic theorem is based on works and ideas of R. J. Thompson (cf. [MT73], [Th80][12]), I. Németi and J. X. Madarász.

The fact that the finitely based variety presented in section 3 is indeed adequate for the algebraization of first order logic (without equality) will be shown in section 4. Some applications to "pure" logic are in that section, too.

Considerations on the case of logic with equality are in section 5. (Many of these are based on ideas originating with J. D. Monk from the 1960's.)

[12]In section 3 full references are given to Thompson's works on this subject.

1. REDUCTION TO SEMIGROUP PROBLEMS

1.1. Notation

1.1.1. Throughout this paper, we use the (rather standard set theoretical and algebraic) notation of [HMT], except for minor deviations. Such deviations are the following.

- We often write $A \setminus B$ for the *set theoretic difference* of A and B instead of $A \sim B$ as in [HMT]. That is,
$$A \setminus B = A \sim B = \{x \in A : x \notin B\}.$$
- The *powerset* of a set H will often be denoted as $\mathcal{P}(H)$ (instead of $Sb(H)$ as in [HMT]).
- We denote the *restriction* of a function f to a set H by $f \restriction H$ (instead of $H \upharpoonleft f$ as in [HMT]).
- We denote the f–image of a set H by $f''(H)$ (instead of $f^*(H)$ as in [HMT]). That is, for any function f and set H,
$$f''(H) = \{f(a) : a \in H\}.$$
- For a class K of similar structures (i.e., models of first order logic), $Th(\mathsf{K})$ denotes the set of all first order formulas valid in K.
- The basic operations of a(n abstract) Boolean algebra will often be denoted by $+, \cdot, -, 0, 1$ (standing for union, intersection, complementation, empty set, largest element respectively, in Boolean fields of sets).

1.1.2. Let I be an arbitrary set. Recall that ${}^II$ is the set of all transformations (i.e. functions) from I into I. We introduce some notation for a few such transformations. The *identity transformation* $\{\langle i, i\rangle : i \in I\}$ is denoted by Id_I or simply Id when no confusion will arise. Let $i, j \in I$. The permutation of I that simply transposes i and j is called a *transposition*, and is denoted by $[i, j]$. The transformation that maps i to j and is the identity on $I \setminus \{i\}$ is called a (simple) *replacement* and is denoted by $[i/j]$. That is:
$$[i/j](x) \stackrel{\text{def}}{=} \begin{cases} j & \text{if } x = i \\ x & \text{otherwise}. \end{cases}$$
A transformation $\tau \in {}^II$ is called a *finite transformation* if $\tau(i) \neq i$ for at most finitely many $i \in I$. As it is well known, the semigroup of all finite transformations is generated (using composition $\circ$) by the set of simple replacements and transpositions.

If $G \subseteq {}^II$ then $[G]$ denotes the subsemigroup generated by G in $\langle {}^II, \circ\rangle$.

Abusing terminology a bit, we shall refer to a subset $T \subseteq {}^II$ closed under functional composition $\circ$ (i.e., $\tau \circ \nu \in T$ whenever $\tau, \nu \in T$) as a *semigroup of transformations* or simply a *semigroup*.

Let $f \in {}^AB$, that is f is a function from A into B, further let $a \in A$ and b any set. Then
$$f^a_b(x) = f[a/b](x) \stackrel{\text{def}}{=} \begin{cases} b & \text{if } x = a \\ f(x) & \text{otherwise}. \end{cases}$$
Thus, e.g., $[i/j] = \mathrm{Id}^i_j$ (for every $i, j \in I$). Let further $H \subseteq A$ and g be a function such that $Dom(g) \supseteq H$. Then
$$f^H_g(x) = f[H/g](x) \stackrel{\text{def}}{=} \begin{cases} g(x) & \text{if } x \in H \\ f(x) & \text{otherwise}. \end{cases}$$
That is, $f^H_g = \big(f \restriction (A \setminus H)\big) \cup g \restriction H$. We note that f^H_g is a generalization of f^a_b.[13]

[13] The alternative notation $f[H/g]$ is taken from [HMTAN, p.132].

1.1.3. Next we recall some concepts from [HMT]. Most of these will concern natural operations on α–ary relations for an arbitrary ordinal α. (α may be, e.g., 2, 3, or ω.) To be able to have operations like Boolean complementation, we look at subrelations of a fixed biggest relation V.

For any set V, *complementation* relative to V is denoted by ${}_V\!\sim$, i.e., ${}_V\!\sim X \stackrel{\text{def}}{=} V \setminus X (= V \sim X)$.

Let U be a set and α an arbitrary ordinal. Let $p, q \in {}^{\alpha}U$ and $\Gamma \subseteq \alpha$. We say that *p and q are Γ–neighbors* (or that *p is a Γ–neighbor of q*) if p and q differ at most on Γ, that is, if $\{i \in \alpha : p_i \neq q_i\} \subseteq \Gamma$. If $\Gamma = \{i\}$ for some $i \in \alpha$, then we will simply say "i–neighbor" instead of "$\{i\}$–neighbor".

Let $V \subseteq {}^{\alpha}U$, $X \subseteq V$, and $i \in \alpha$. Then the i–th *cylinder* $\mathrm{C}_i^{[V]}(X)$ or briefly $\mathrm{C}_i(X)$ over X relative to V is defined as follows:

$$\begin{aligned}\mathrm{C}_i(X) = \mathrm{C}_i^{[V]}(X) &\stackrel{\text{def}}{=} \{q \in V : q \text{ is an } i\text{–neighbor of some element of } X\}\\ &= \{q \in V : (\exists u \in U) q^i_u \in X\}\,.\end{aligned}$$

C_i is called the i-th *cylindrification* (relative to V). Let $\Gamma \subseteq \alpha$. Then

$$\begin{aligned}\mathrm{C}_{(\Gamma)}(X) = \mathrm{C}_{(\Gamma)}^{[V]}(X) &\stackrel{\text{def}}{=} \{q \in V : q \text{ is a } \Gamma\text{–neighbor of some element of } X\}\\ &= \{q \in V : (\exists u \in {}^{\Gamma}U) q^{\Gamma}_u \in X\}\,.\end{aligned}$$

$\mathrm{C}_{(\Gamma)}$ is called a *generalized cylindrification*. Now let $\tau \in {}^{\alpha}\alpha$. Then

$$\begin{aligned}\mathrm{S}_\tau(X) = \mathrm{S}_\tau^{[V]}(X) &\stackrel{\text{def}}{=} \{q \in V : q \circ \tau \in X\}\\ &= \{\langle q_0, q_1, \ldots, q_n, \ldots\rangle \in V : \langle q_{\tau 0}, q_{\tau 1}, \ldots, q_{\tau n}, \ldots\rangle \in X\}\,.\end{aligned}$$

S_τ is called the *substitution* operation belonging to the transformation τ relative to V. Distinguished (often used) kinds of substitutions are $\mathrm{S}_{[i/j]}$ and $\mathrm{S}_{[i,j]}$ ($i, j \in \alpha$). We recall from the literature the special notation for them:

$$\mathrm{S}^i_j \stackrel{\text{def}}{=} \mathrm{S}_{[i/j]}, \qquad \mathrm{P}_{ij} \stackrel{\text{def}}{=} \mathrm{S}_{[i,j]}, \qquad \text{for any } i, j \in \alpha\,.$$

1.1.4. Let $G \subseteq {}^{\omega}\omega$ and let T be the subsemigroup generated by G in $\langle {}^{\omega}\omega, \circ\rangle$. We use the convention that $\tau \circ \nu(x) = \tau(\nu(x))$.

G^* denotes the set of words (i.e., finite sequences) over G. Then G^* is also the free semigroup generated by G, and we let $hat : G^* \longrightarrow T$ be the natural anti–homomorphism extending the identical embedding $\mathrm{Id} : G \rightarrowtail T$, and we let $\hat{w} \stackrel{\text{def}}{=} hat(w)$ for any $w \in G^*$. That is, $\hat{\gamma} = \gamma$ and $(\gamma v)\hat{} = \hat{v} \circ \gamma$ for any $\gamma \in G$ and $v \in G^*$. See Figure 1.1.

G^* $\qquad$ hat $\qquad$ $T = [G]$

$\langle \tau, \nu \rangle$ $\qquad$ $\nu \circ \tau$

$\boxed{G}$ —— Id_G ——→ $\boxed{G}$

Figure 1.1

If $w \in G^*$ then $w = \langle w_1, \ldots, w_n \rangle$ for some $w_1, \ldots, w_n \in G$. Instead of $\langle w_1, \ldots, w_n \rangle$, we sometimes write $w_1 w_2 \ldots w_n$. Further, if $u = \langle w_1, \ldots, w_i \rangle$ and $v = \langle w_{i+1}, \ldots, w_n \rangle$ then we sometimes write $w = uv$ (pretending that $\langle \langle w_1, \ldots, w_i \rangle, \langle w_{i+1}, \ldots, w_n \rangle \rangle = \langle w_1, \ldots, w_n \rangle$, as it is often done).

1.1.5. Throughout this paper, BA denotes the set of all **B**oolean **a**lgebras, and also abbreviates the expression "Boolean algebra". Similarly, CA, PA, and QPA abbreviate "**c**ylindric **a**lgebra", "**p**olyadic **a**lgebra", and "**q**uasi–**p**olyadic **a**lgebra", respectively. For any ordinal α, CA_α, PA_α, and QPA_α denote, respectively, the class of all CA's, PA's, and QPA's of dimension α. (For a definition of QPA's, see e.g. [HMTII, p.266].) Representable PA's and representable QPA's are understood in the sense of [HMTII, p.235] (i.e., the standard "subdirect product of set algebras" sense). However, we think of QPA_ω's as algebras of the form

$$\langle \mathfrak{B}, \mathrm{C}_i, \mathrm{S}^i_j, \mathrm{P}_{ij} \rangle_{i,j \in \omega}$$

where $\mathfrak{B} \in \mathsf{BA}$. Obviously, our version is term-definitionally (i.e., polynomially) equivalent with

$$\langle \mathfrak{B}, \mathrm{C}_{(\Gamma)}, \mathrm{S}_\tau : \omega \supset \Gamma, \Gamma \text{ is finite, and } \tau \in {}^\omega\omega \text{ is a finite transformation} \rangle$$

where $\mathfrak{B} \in \mathsf{BA}$. The latter version was used in [HMTII], in Halmos [Ha62, p.120], [Ha54], Daigneault–Monk [DM63]. For development of the theory of the former version of QPA (with C_i, $\mathrm{S}_{[i/j]}$ etc.) and for connections between the two versions see Sain–Thompson [ST91], Németi [Né91].

1.2. Generalized weak set G–algebras (definition)

We turn to defining the kind of algebras that will play the most important rôle in this paper. The elements of these algebras are ω–ary relations (that is, sets of ω–sequences). The operations of these algebras are the Boolean ones together with C_0 and *some* of the S_τ's. Therefore, the choice of those τ's (from ${}^\omega\omega$) for which S_τ is included among the operations, determines the kind of algebra we work with. Let $G \subseteq {}^\omega\omega$ denote the set of those τ's for which we want to include S_τ. Then we call our algebras *set G–algebras*, to distinguish them from other possible choices of the parameter G.

Our algebras will be variants of cylindric algebras (their clones share both some of the C_i's and some of the S^i_j's with them[14]). In cylindric algebra theory, an algebra whose elements are ω–ary relations is called a set algebra only if its greatest element is a Cartesian space, i.e., is of the form ${}^\omega U$ for some set U. However, this restriction is too strong (e.g., these algebras are not closed under direct products), and therefore both in cylindric algebras and in related branches of algebras of relations, this restriction on the greatest relation is often replaced with some weaker ones. We will adopt the one which is, perhaps, most widely used in [HMTII]. The algebras we are referring to are called generalized weak cylindric set algebras (Gws's) in [HMTII]. The adjective "*generalized weak*" refers to the permission that the greatest relation need not be of the form ${}^\omega U$.

In [HMTII], if α is an ordinal (like our ω is) then Gws_α abbreviates Gws consisting of α–ary relations. An α–ary relation V is called a Gws_α–unit if it is suitable for being the greatest element of a Gws_α. We will build up our definition in a manner strongly parallel with the one just quoted from [HMTII]. Namely, as we said, our algebras will also be special algebras of α–ary (in fact, ω–ary) relations, and we also will make the same kind of restrictions on the greatest element as [HMTII] does in Gws_α's. Since our operations are slightly different, our conditions will be slightly different, too. More concretely, the crucial operations in the cylindric algebraic case were the C_i's. If the transformation τ of ω moves infinitely many points, then S_τ is an essentially new operation (even when added to the cylindric algebraic ones). Therefore we will have new conditions associated with these S_τ's. Because of the similarities with Gws_α's in the sense of [HMTII], and for staying consistent with the literature, we will call our algebras *generalized weak set G–algebras*.

Following [HMT], we call the greatest element of an algebra its *unit*. Now let us turn to selecting the kind of unit we are going to allow in our algebras. First come some preparations.

Let us recall from [HMTII, p.6] that a set W of ω–sequences is called a *compressed space* iff for every $p, q \in W$ and $i, j \in \omega$, $p[i/q_j] \in W$.

FACT 1.1.

(i) *W is a compressed space iff*

$$W \subseteq {}^\omega U \text{ for some set } U, \text{ and } \mathrm{C}_i^{[{}^\omega U]}(W) = W \text{ for all } i \in \omega\,.$$

(ii) *${}^\omega U$ is a compressed space for any set U.*

(iii) *Let $p \in {}^\omega U$ for some set U. We put*

$${}^\omega U^{(p)} \stackrel{\text{def}}{=} \{q \in {}^\omega U : \{i \in \omega : q(i) \neq p(i)\} \text{ is finite}\}\,.$$

${}^\omega U^{(p)}$ is a compressed space for any set U and $p \in {}^\omega U$.

The verification of Fact 1.1 is left to the reader. ∎

[14] In the important cases we have all of both as term functions.

We call an ω–ary relation V a *normal* Gws_ω*–unit* iff it is a union $\bigcup_{i\in I} W_i$ of compressed spaces W_i such that $\mathrm{base}(W_i)\cap\mathrm{base}(W_j)=\emptyset$ for all $i,j\in I$, $i\neq j$, where $\mathrm{base}(W)\stackrel{\mathrm{def}}{=}\bigcup\{Rng(q): q\in W\}$.

Following [HMTII, pp.5-6], we call the set ${}^\omega U^{(p)}$ of ω–sequences defined in (iii) of Fact 1.1 a *weak (Cartesian) space* (of dimension ω, determined by p, with base U).

FACT 1.2. *V is a normal Gws_ω–unit iff $V=\bigcup_{i\in I}{}^\omega U_i^{(p_i)}$ for some sets I, U_i and functions $p_i\in{}^\omega U_i$ $(i\in\omega)$ such that $U_i\cap U_j=\emptyset$ or $U_i=U_j$ for all $i,j\in I$, $i\neq j$.*

The proof of Fact 1.2 is left to the reader. ∎

The *base* of a Gws_ω–unit V is defined to be $\bigcup\{Rng(q): q\in V\}$. By [HMTII, L.1.3.1], there is only one way of decomposing V to $\bigcup_{i\in I}{}^\omega U_i^{(p_i)}$ as in Fact 1.2 above. The U_i's of this decomposition are called the *subbases* of V.

For later use we state and prove the following:

LEMMA 1.3. *Let V be a set of ω–sequences. Then V is a normal Gws_ω–unit iff V satisfies condition (1.1) below.*

$$(\forall p,q\in V)\big(Rng(p)\cap Rng(q)\neq\emptyset\Longrightarrow(\forall i,j\in\omega)p[i/q(j)]\in V\big)\,. \tag{1.1}$$

Proof: First assume that V is a normal Gws_ω–unit. Let $p,q\in V$ be such that $Rng(p)\cap Rng(q)\neq\emptyset$. Let $i,j\in\omega$ be arbitrary. We want to prove that $p[i/q(j)]\in V$.

Since V is a normal Gws_ω–unit, V is the disjoint union of some compressed spaces $\{W_i: i\in I\}$. Since $p,q\in V$, $p\in W_k$ and $q\in W_\ell$ for some $k,\ell\in I$. Since $Rng(p)\cap Rng(q)\neq\emptyset$, $k=\ell$. Thus $p,q\in W_k$, where W_k is a compressed space. Then, by the definition of a compressed space, $p[i/q(j)]\in W_k\subseteq V$. Thus V satisfies (1.1).

Next assume that V satisfies (1.1). For every $q\in V$, we let

$$W_q\stackrel{\mathrm{def}}{=}\{p\in V: Rng(p)\cap Rng(q)\neq\emptyset\}\,.$$

By (1.1), W_q is a compressed space. Putting $I\stackrel{\mathrm{def}}{=}\{W_q: q\in V\}$, we have that $V=\bigcup_{i\in I}W_i$ and $(\forall i,j\in I)(i\neq j\Rightarrow W_i\cap W_j=\emptyset)$. ∎

DEFINITION 1.4: Let $G\subseteq{}^\omega\omega$ be fixed.

By a Gws_G*–unit* we understand a normal Gws_ω–unit V such that $\mathsf{S}_\tau^{[V]}(V)=V$ for every $\tau\in G$.

By a <u>g</u>*eneralized* <u>w</u>*eak* <u>s</u>*et G–algebra*, Gws_G for short, we understand a subalgebra

$$\mathfrak{A}\subseteq\langle\mathcal{P}(V),\cup,{}_V\!\sim,\mathsf{C}_0^{[V]},\mathsf{S}_\tau^{[V]}:\tau\in G\rangle$$

where V is a Gws_G–unit.

V is called the *unit* (or greatest element) *of* $\mathfrak{A}$, and is often denoted by $1^{\mathfrak{A}}$. To the base and subbases of V we will also refer to as those of $\mathfrak{A}$. ◀

We note that the condition $\mathsf{S}_\tau^{[V]}(V)=V$ above is equivalent with the following one:

$$(\forall q\in V)(\forall\tau\in G)q\circ\tau\in V\,.$$

$\mathfrak{A}\in\mathsf{Gws}_G$ is called a *full* Gws_G iff $A=\mathcal{P}\left(1^{\mathfrak{A}}\right)$.

We have chosen the strongly parallel abbreviation Gws_G with the one Gws_α in [HMTII] because of the strong similarity of the two concepts (Gws_G is a variant of Gws_α). For unique readability we point out that when the index of Gws is an ordinal (denoted by a Greek letter) then it denotes cylindric algebras, while when it is a set G of transformations, then it denotes generalized set G–algebras.

By a *set G–algebra* we understand a Gws_G whose unit (greatest element) is ${}^\omega U$ for some set U. We denote the class of all set G–algebras by Cs_G. Notice that $\mathbf{S}\mathsf{Cs}_G = \mathsf{Cs}_G$ but $\mathbf{P}\mathsf{Cs}_G \supsetneqq \mathsf{Cs}_G$.

Remark 1.5 (discussion of the definition of Gws_G–units)**:**
The reasons why we defined Gws_G–units as we did are among the following.

(1) We want to choose our Gws_G–units V such that

(1.2) equations valid for C_i and S_τ in standard situations would remain valid in all algebras whose greatest element is V.

Can then V be a completely arbitrary set of sequences? If, e.g., we would choose

$$V = \{\langle 0,0,\ldots,0,\ldots\rangle, \langle 1,0,\ldots,0,\ldots\rangle, \langle 1,1,\ldots,0,\ldots\rangle\},$$

then in the powerset algebra $\langle \mathcal{P}(V), \cup, {}_V\!\sim, \mathrm{C}_0^{[V]}, \mathrm{S}_\tau^{[V]} : \tau \in G\rangle$ of V, cylindrifications would behave strangely, e.g. $\mathrm{C}_0\mathrm{C}_1(X) = \mathrm{C}_1\mathrm{C}_0(X)$ would not be valid. So, a completely arbitrary choice of V could distort the behavior of our natural operations (on relations) C_i and S_τ.

For satisfying (1.2) above, it is sufficient to ensure that a "relativization" of a standard situation with V remains a homomorphism (i.e., we would have $V \subseteq {}^\omega U$ for some U such that $\mathrm{C}_i(X) = V \cap \mathrm{C}_i^{[{}^\omega U]}(X)$ and $\mathrm{S}_\tau(X) = V \cap \mathrm{S}_\tau^{[{}^\omega U]}(X)$ would hold for every $X \subseteq V$, $i \in \omega$, $\tau \in G$). The Gws_G–algebras are normal Boolean Algebras with Operators (BAO's) which is the same as normal Bo_α's in the sense of [HMT]. By Németi [Né80 p.46, Thm.1(i)] or [Né81], in any normal BAO $\mathfrak{A}$ and for any operator f of $\mathfrak{A}$ and $V \in A$, it is sufficient and necessary for $X \mapsto X \cap V$ to preserve f if $f(-V) \subseteq -V$. In the light of this, we can "derive" the definition of a Gws_G–unit from our choice of operations as follows.

(i) Contribution of C_i to our definition:
It is easy to see that $\mathrm{C}_i(-X) \subseteq -X$ is equivalent with $\mathrm{C}_i(X) = X$, hence we want the following condition

(1.3) $\qquad V$ is a fixed point of $\mathrm{C}_i^{[{}^\omega U]}$ for some U with $V \subseteq {}^\omega U$, for all $i \in \omega$.

As this was already observed above Definition 1.4, (1.3) holds iff V is a *compressed space*. So the "compressed space condition" in the definition comes from the fact that, in all the applications we are interested in, the C_i's will be term–definable operations in our algebras.

(ii) Contribution of the S_τ's to our definition:
The above considerations yield the condition that $\mathrm{S}_\tau^{[{}^\omega U]}(-V) \subseteq -V$ for some U and for all $\tau \in G$. But this condition is easily seen to be equivalent with $\mathrm{S}_\tau^{[V]}(V) = V$, since S_τ is always a Boolean homomorphism. Thus the condition $\mathrm{S}_\tau^{[V]}(V) = V$ originates from the presence of S_τ as an operation of our algebras.

(2) At this point the question arises "are we done?" That is, would the class K of algebras with compressed spaces satisfying $\mathrm{S}_\tau^{[V]}(V) = V$ as units be a nice class of algebras of relations? The answer is "not yet". Let us recall from the Introduction that we would like to obtain a class of set algebras forming a *variety*. For our present K, $\mathbf{I}\mathsf{K}$ may not be closed under direct products (and therefore it would not have a chance for being even a quasi–variety). To see that this is so, let $\tau = \omega \times \{0\}$, i.e., $(\forall i \in \omega)\tau(i) = 0$. Then $(\mathrm{C}_0\mathrm{S}_\tau(X) = 0 \vee \mathrm{C}_0\mathrm{S}_\tau(X) = 1)$ is a formula valid in K but obviously not valid in $\mathbf{P}\mathsf{K}$, where $\mathbf{P}\mathsf{K}$ stands for the class of isomorphic copies of direct products of members of K. In Algebraic Logic, there is a standard way of alleviating this problem as follows (cf. [HMTAN], [HMTI, II]). Besides units of members of K, we also permit *disjoint unions* of units of members of K as our Gws_G's. By Stone duality theory (for BAO's, cf. e.g. [HMT]), taking disjoint unions of units corresponds to taking direct products of algebras, e.g., if $\mathfrak{A}, \mathfrak{B}$ are BAO's, then $\mathfrak{A} \times \mathfrak{B}$ is isomorphic to a BAO with unit $1^{\mathfrak{A}} \dot\cup 1^{\mathfrak{B}}$. In our special case, for $\mathfrak{A} \times \mathfrak{B}$ to be isomorphic to an algebra with unit $1^{\mathfrak{A}} \dot\cup 1^{\mathfrak{B}}$, we have to make this union "as disjoint as possible", that is, we have to ensure that already the bases of $1^{\mathfrak{A}}$ and $1^{\mathfrak{B}}$ are disjoint. Therefore, ensuring that the units of Gws_G's are closed under special disjoint unions will in turn ensure that $\mathbf{I}\mathsf{Gws}_G$ is closed under direct products. Certainly this step (of allowing "very disjoint" unions of old units) does not destroy the results of step (1) above in that it will not distort equational or even quasi–equational properties of our operations (since products preserve quasi–equations).

Steps (1)–(2) above give us back our definition of a Gws_G–unit as it is in Definition 1.4, saying that a Gws_G–unit is a union of compressed spaces with disjoint bases satisfying $\mathrm{S}_\tau^{[V]}(V) = V$. ◀

DEFINITION 1.6: Let $G \subseteq {}^\omega\omega$ and $T = [G] \subseteq \langle {}^\omega\omega, \circ\rangle$ (subsemigroup generated by G).

(i) In harmony with the notation of [HMT], we use $+,-,\mathsf{c}_0,\mathsf{s}_\tau$ ($\tau \in G$) to denote the function symbols of the similarity type (or language) of Gws_G's. Of course, $+,-,\mathsf{c}_0,\mathsf{s}_\tau$ correspond, respectively, to $\cup, {}_V\!\sim, \mathrm{C}_0^{[V]}, \mathrm{S}_\tau^{[V]}$ in any Gws_G with unit V.

Let $V \subseteq {}^\omega U$ for some set U. For every $\tau \in G^*$ we define $\mathrm{S}_\tau^{[V]} : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$ by recursion, as follows:

- If $\tau \in G$ then $\mathrm{S}_\tau^{[V]}$ was defined at the end of subsection 1.1.3.
- If $w = uv$ for some $u, v \in G^*$ then $\mathrm{S}_w^{[V]} \stackrel{\text{def}}{=} \mathrm{S}_v^{[V]} \circ \mathrm{S}_u^{[V]}$.

(This is a sound definition because function composition is associative.)

If V is the unit of some Gws_G $\mathfrak{A}$ and $w \in G^*$ then we denote the Gws_G–term naming $\mathrm{S}_w^{[V]}$ by s_w, thus $\mathsf{s}_w^{\mathfrak{A}} = \mathrm{S}_w^{[V]}$.

For example, if $w = \sigma\tau\pi$ with $\sigma, \tau, \pi \in G$ then $\mathrm{S}_w = \mathrm{S}_\pi \circ \mathrm{S}_\tau \circ \mathrm{S}_\sigma$, and $\mathsf{s}_w^{\mathfrak{A}}(x) = \mathsf{s}_\pi^{\mathfrak{A}}\mathsf{s}_\tau^{\mathfrak{A}}\mathsf{s}_\sigma^{\mathfrak{A}}(x) = \mathrm{S}_\pi^{[V]}\mathrm{S}_\tau^{[V]}\mathrm{S}_\sigma^{[V]}(x) = \mathrm{S}_w^{[V]}(x)$ for every $x \in A$.

Let

$$\Sigma_G \stackrel{\text{def}}{=} \{\mathsf{s}_w(x) = \mathsf{s}_u(x) : u, w \in G^* \text{ and } \hat{u} = \hat{w}\}$$

(for $\hat{}$ see subsection 1.1.4). In passing we note that, though Σ_G denotes a set of equations, it is not a schema in the sense of [Mo69].

(ii) Whenever Σ_G holds in a subclass K of the similarity class of Gws_G, the following notation is legitimate in K:

$$\mathrm{S}_{\hat{w}}(x) \stackrel{\text{def}}{=} \mathrm{S}_w(x) \quad \text{for any } w \in G^*\,.$$

Notice that by this, in K, S_τ has been defined as a *term function* in the language of Gws_G, for *every* $\tau \in T$. Namely, if $\tau \in T$ then $\tau = \tau_1 \circ \cdots \circ \tau_n$ for some $\tau_1, \ldots, \tau_n \in G$, and for any $\mathfrak{A} \in$ K with unit V, $\mathrm{S}_\tau^{[V]}(x) = (\mathsf{s}_{\tau_1}\mathsf{s}_{\tau_2}\ldots\mathsf{s}_{\tau_n})^{\mathfrak{A}}(x)$ for any $x \in A$.

For any $\tau \in T$, s_τ is the Gws_G–term defining S_τ.

Instead of $\mathsf{s}_{[i/j]}$ and $\mathsf{s}_{[i,j]}$ "naming", respectively, S_j^i and P_{ij} in the language of Gws_G, we will often write s_j^i and p_{ij} respectively ($i, j \in \omega$). That is,

$$\mathsf{s}_{[i/j]} = \mathsf{s}_j^i \quad \text{and} \quad \mathsf{s}_{[i,j]} = \mathsf{p}_{ij}\,.$$

(iii) In a similar spirit, for all $i \in \omega$,

$$\mathrm{C}_i(x) \stackrel{\text{def}}{=} \mathrm{S}_{[0,i]}\mathrm{C}_0\mathrm{S}_{[0,i]}(x) = \mathrm{P}_{0i}\mathrm{C}_0\mathrm{P}_{0i}(x)\,.$$

Thus C_i ($i \in \omega$) is also term definable in the language of Gws_G whenever P_{0i} is term definable. Namely, for any Gws_G $\mathfrak{A}$ with unit V, $\mathrm{C}_i^{[V]}(x) = (\mathsf{s}_{[0,i]}\mathsf{c}_0\mathsf{s}_{[0,i]})^{\mathfrak{A}}(x) = (\mathsf{p}_{0i}\mathsf{c}_0\mathsf{p}_{0i})^{\mathfrak{A}}(x)$ for any $x \in A$.

From now on, c_i ($i \in \omega$) denotes the term function $\mathsf{p}_{0i}\mathsf{c}_0\mathsf{p}_{0i}$. ◀

1.3. Main theorem

Let $T \subseteq \langle {}^\omega\omega, \circ\rangle$ be a semigroup. We say that T is *rich* if T satisfies conditions (1.4), (1.5) below. Recall from subsection 1.1 that if $H \subseteq \omega$ and $\tau, \nu \in {}^\omega\omega$, then $\tau[H/\nu] \stackrel{\text{def}}{=} (\tau \restriction (\omega \sim H)) \cup (\nu \restriction H)$.

(1.4) $(\forall i, j \in \omega)(\forall \tau \in T)\tau[i/j] \in T$.

(1.5) $(\exists \sigma, \pi \in T)\big(\pi \circ \sigma = \mathrm{Id}, Rng(\sigma) \neq \omega,$ and $(\forall \tau \in T)(\sigma \circ \tau \circ \pi)\left[(\omega \setminus Rng(\sigma))/\mathrm{Id}\right] \in T\big)$.

Clearly, $\langle {}^\omega\omega, \circ\rangle$ itself is rich, but, for example, if we take successor $\{\langle n, n+1\rangle : n \in \omega\}$ for σ and predecessor $\{\langle 0, 0\rangle\} \cup \{\langle n+1, n\rangle : n \in \omega\}$ for π then the subsemigroup of $\langle {}^\omega\omega, \circ\rangle$ generated by $\{\sigma, \pi\}$ satisfies $\pi \circ \sigma = \mathrm{Id}$ and $Rng(\sigma) \neq \omega$ but not the rest of (1.4) and (1.5), therefore it is not rich.

LEMMA 1.7. *If $T \subseteq \langle {}^\omega\omega\rangle$ is a rich semigroup then (i)–(v) below hold.*

(i) $\mathrm{Id} \in T$*;*
(ii) $[i/j], [i,j] \in T$ *for every* $i, j \in \omega$*;*
(iii) *every finite transformation belongs to* T*;*
(iv) σ *(as in (1.5)) is one–one;*
(v) *if T is generated by some G then C_i, S^i_j and P_{ij} are term–definable in Gws_G, for every $i, j \in \omega$.*

The **proof** of this lemma is straightforward from the definition of a rich semigroup, and is left to the reader. ∎

One of the main results of the present paper, Theorem 1 formulated in the Introduction, will be proved via Theorem 1.8 below (see the last part of subsection 3.1).

THEOREM 1.8. *Assume that $T \subseteq \langle {}^\omega\omega, \circ\rangle$ is a finitely presented rich semigroup. Let $G \subseteq T$ denote a finite generator set of T. Then (i) and (ii) below hold.*

(i) $\mathbf{I}\mathsf{Gws}_G$ *is a* finitely based *variety.*
(ii) *The subalgebras of the quasi–polyadic reducts of $\mathbf{I}\mathsf{Gws}_G$'s coincide with the representable quasi–polyadic algebras. Taking quasi–polyadic reducts makes sense because the functions C_i, S^i_j, P_{ij} $(i, j \in \omega)$ are term–definable in $\mathbf{I}\mathsf{Gws}_G$.*

Proof of Theorem 1.8 (ii): Let G be as in the formulation of the theorem. We need to prove that $\mathbf{SRd}_{\mathsf{QPA}_\omega}\mathbf{I}\mathsf{Gws}_G = \mathsf{RQPA}_\omega$.

Let $(\mathsf{Gws}_\omega + \mathsf{p}_{ij})$ denote the expansion of Gws_ω with the set $\{\mathsf{p}_{ij} : i, j \in \omega\}$ of permutations. The definitions of the classes $(\mathsf{Gs}_\omega + \mathsf{p}_{ij})$ and $(\mathsf{Cs}_\omega + \mathsf{p}_{ij})$ are analogous. It is easy to see that [HMTII, Cor.3.1.107, p.45] generalizes to $(\mathsf{Gws}_\omega + \mathsf{p}_{ij})$ (from Gws_ω), thus

$$\mathbf{I}(\mathsf{Gws}_\omega + \mathsf{p}_{ij}) = \mathbf{I}(\mathsf{Gs}_\omega + \mathsf{p}_{ij}) = \mathbf{ISP}(\mathsf{Cs}_\omega + \mathsf{p}_{ij}) = \mathsf{RQPA}_\omega\,. \tag{1.6}$$

To prove $\mathbf{SRd}_{\mathsf{QPA}_\omega}\mathbf{I}\mathsf{Gws}_G \subseteq \mathsf{RQPA}_\omega$, let $\mathfrak{A} \in \mathbf{SRd}_{\mathsf{QPA}_\omega}\mathbf{I}\mathsf{Gws}_G$ be arbitrary but fixed. Then $\mathfrak{A} \in \mathbf{SI}(\mathsf{Gws}_\omega + \mathsf{p}_{ij}) = \mathbf{I}(\mathsf{Gws}_\omega + \mathsf{p}_{ij}) = \mathsf{RQPA}_\omega$ by (1.6).

Next assume $\mathfrak{A} \in \mathsf{RQPA}_\omega$. Then $\mathfrak{A}$ can be embedded into some full $\mathfrak{B} \in \mathsf{RQPA}_\omega$. Since $\mathfrak{B}$ is a full $(\mathsf{Gs}_\omega + \mathsf{p}_{ij})$, $\mathfrak{B} \in \mathbf{Rd}_{\mathsf{QPA}_\omega}\mathbf{I}\mathsf{Gws}_G$. Thus $\mathfrak{A} \in \mathbf{SRd}_{\mathsf{QPA}_\omega}\mathbf{I}\mathsf{Gws}_G$. ∎

Idea of the proof of Theorem 1.8 (i) (the full proof will be elaborated in section 2): The reader not familiar with concepts of cylindric algebra theory mentioned but not recalled in the following “idea of proof” may safely skip it, since we will not rely on these concepts in the detailed proof (in section 2). However, even if not familiar with cylindric algebras, the reader may find it useful to “skim through” the idea outlined below because some of the general intuition might be understandable even without knowing all the definitions.

One of the main representation methods in the theories of algebras of not necessarily binary relations (e.g. cylindric algebras, Craig's algebras, Pinter's algebras, polyadic algebras, see e.g. in Németi [Né91]) is based on the so called *Neat Embedding Theorem* in [HMT] (to be recalled below). Recall that CA_α denotes the class of α–ary (or α–dimensional) cylindric algebras, i.e., algebras satisfying a prespecified finite schema of equations trying to force CA's to consist of algebras of true α–ary relations only. Let $\mathfrak{B} \in \mathsf{CA}_\beta$ for some ordinal β and let $\alpha \leqslant \beta$. Recall from [HMTI, Def.2.6.28, p.401] that a CA_α $\mathfrak{C}$ is called the *neat α–reduct* of $\mathfrak{B}$ iff $C = \{b \in B : (\forall i \in \beta \setminus \alpha) \mathsf{c}_i^{\mathfrak{B}}(b) = b\}$, and the operations of $\mathfrak{C}$ are those of $\mathfrak{B}$ restricted to C. Let $\mathfrak{A} \in \mathsf{CA}_\alpha$. We say that $\mathfrak{A}$ is *neatly embeddable* in $\mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{C} \in \mathsf{CA}_\alpha$ for some neat α–reduct $\mathfrak{C}$ of $\mathfrak{B}$. If this is the case then $e : \mathfrak{A} \rightarrowtail \mathfrak{C}$ embedding $\mathfrak{A}$ into $\mathfrak{C}$ is called a *neat embedding*. The Neat Embedding Theorem (cf. [HMTII, Thm.3.2.10, p.63]) says that

whenever $\mathfrak{A} \in \mathsf{CA}_\alpha$ is neatly embeddable into some $\mathsf{CA}_{\alpha+n}$ for every $n \in \omega$,
then $\mathfrak{A}$ is representable

(is isomorphic to an algebra of true α–ary relations). The proof of the Neat Embedding Theorem (originally given by L. Henkin in 1954, cf. [HMTI, p.401]) can be pushed through for many variants of cylindric algebras, e.g. to quasi-polyadic algebras, quasi–polyadic equality algebras, polyadic algebras, Pinter's substitution–cylindric algebras.[15] (For the most well known variants of CA's see [HMTII] and Németi [Né91].)

We are working with algebras that are variants of CA's. Actually, we are *choosing* the variant we want to work with, keeping in mind the restriction that this variant should be adequate for algebraizing first order logic as mentioned in the Introduction.[16] We want to use the Neat Embedding Theorem. Therefore, while choosing or modifying our variant of CA's, we always make sure that the Neat Embedding Theorem remains true for our variant. (Keeping in mind the above list of positive examples can give us some guiding light in this "navigation". The Neat Embedding Theorem is not true for e.g. diagonal–free cylindric algebras or polyadic equality algebras, so there are certain combinations to avoid, cf. [HMTII, 5.4.41 together with 5.6.15].)

The *outline* of the present proof is the following.

(1) We collect enough equations true in Gws_G into Σ such that the proof of Neat Embedding Theorem goes through for $Mod(\Sigma)$. Then we try to find a transformation $\sigma \in G$ such that for any $\mathfrak{A} \in Mod(\Sigma)$, S_σ neatly embeds $\mathfrak{A}$ into *itself* (as opposed to embedding $\mathfrak{A}$ into some other algebra). If S_σ neatly embeds $\mathfrak{A}$ into itself with $n > 0$ extra dimensions, e.g. by $\sigma : \omega \rightarrowtail (\omega \setminus n)$, then we can repeat application of S_σ (because it embeds $\mathfrak{A}$ into itself and not somewhere else), and this way we obtain embeddability into arbitrary large (finite) extra dimensions. (This is a typical "Hilbert hotel" type argument.) But then we can apply the Neat Embedding Theorem to conclude that $\mathfrak{A}$ is representable.

(2) To implement (1), a large amount of fine–tuning is required, to such an extent that the original idea is no longer apparent in the final proof (as can be seen by reading section 2). The easy part of the fine tuning is that at each step we have to add equations to Σ to make the tools we want to use available, e.g. we add equations to ensure that S_σ is a neat embedding. Another kind of fine tuning is needed in making our intuitive ideas, like neatly embedding $\mathfrak{A}$ into itself, precise. E.g. we cannot write down $\mathsf{S}_\sigma : \mathfrak{A} \longrightarrow \mathfrak{A}$, because it is not literally true. (If $\mathfrak{B}\mathfrak{l}\mathfrak{A}$ denotes the Boolean

[15] [HMTII, Thm.5.6.17 (p.263)] states this, roughly speaking, for those "algebraic logics", where the locally finite dimensional algebras are representable (a condition which is usually easy to meet), and the representables form a universally axiomatizable class. (The question of how much of these conditions is needed, is investigated in [ANS84] and in the references there.) Cf. also [AGN77, Coroll.'s 3.17, 3.18, 3.22].

[16] The reason why we say that we are "choosing" our variant of CA's, though it is given in the statement of our theorem, is the following. We would like to do more here than just prove Theorem 1.3. We would also like to explain how we arrived at the conditions of that theorem. So let us imagine for a while that we do not have the theorem yet, and we want to *modify* CA's such that its generalized weak set algebras would become a finitely axiomatizable variety.

reduct of $\mathfrak{A}$, then $\mathsf{S}_\sigma : \mathfrak{B}\restriction\mathfrak{A} \longrightarrow \mathfrak{B}\restriction\mathfrak{A}$ is true, but $\mathsf{S}_\sigma : \langle \mathfrak{B}\restriction\mathfrak{A}, \mathsf{C}_0 \rangle \longrightarrow \langle \mathfrak{B}\restriction\mathfrak{A}, \mathsf{C}_0 \rangle$ is not. However, the following variant of the latter is true: $\mathsf{S}_\sigma : \langle \mathfrak{B}\restriction\mathfrak{A}, \mathsf{C}_0 \rangle \longrightarrow \langle \mathfrak{B}\restriction\mathfrak{A}, \mathsf{C}_{\sigma(0)} \rangle$. This points in the direction of renaming the operations on the range side of S_σ.) We have to generalize the operator $\mathfrak{Rd}$ of taking reducts of algebras such that, besides forgetting operations, it also renames some of them. After that $\mathsf{S}_\sigma : \mathfrak{A} \rightarrowtail \mathfrak{Rd}\mathfrak{A}$ becomes at least meaningful. So the task is to choose $\mathfrak{Rd}$ such that $\mathsf{S}_\sigma : \mathfrak{A} \rightarrowtail \mathfrak{Rd}\mathfrak{A}$ becomes a (non–trivial) neat embedding. This task involves a piece of self–reference too, since S_σ plays a double rôle here: it is both a homomorphism on $\mathfrak{A}$ and an operation of $\mathfrak{A}$. Therefore we have to ensure that S_σ will be a neat embedding w.r.t. *itself* as an operation, too. With some more work we will be able to ensure this. (This is where the second part of condition (1.5) in the definition of a rich semigroup comes from.) When all these subtle adjustments are taken care of, we transformed the above idea (1) to the actual proof found in section 2.

(3) The proof outline given so far yields axiomatizability of $\mathbf{I}\mathsf{Gws}_G$ by some set Σ of equations. In the final step of the proof (beginning with Theorem 2.17 in subsection 2.3) we will cut down the size of Σ to finite. For this we will need the condition that G is finitely presented. ∎

As mentioned before formulating Theorem 1.8, (ii) of it implies that $\mathbf{I}\mathsf{Gws}_G$ is adequate for the algebraization of first order logic (in the same sense as quasi–polyadic algebras are adequate for this). This adequateness is discussed and its implications are elaborated in section 4 of this paper.

For a discussion of the conditions of Theorem 1.8 see subsection 2.4.

For completeness, we note the following. As we mentioned in the "Idea of the proof of Theorem 1.8 (i)" above, the Neat embedding Theorem fails for both diagonal–free cylindric algebras (DFCA's) and polyadic equality algebras (PEA's).

2. PROOF OF THEOREM 1.8 (i)

Besides giving a complete proof of Theorem 1.8 in this section, in each of subsections 2.1–2.3 we prove results which are of interest in themselves (independently of their rôles in proving Theorem 1.8). Usually these results are more general than what is needed for Theorem 1.8. (Such are e.g. Theorems 2.1, 2.15.)

2.1. $\mathbf{I}\mathsf{Gws}_G$ is a variety if the C_i's are term–definable in it

In this subsection we will prove that $\mathbf{I}\mathsf{Gws}_G$ is a variety under a very mild condition on G. This is stated precisely in Theorem 2.1 below. In the proof of Theorem 2.1 we will use three lemmas: Lemmas 2.5, 2.9 and 2.10 to be formulated and proved below. The main tool used in the proof is a representation of ultraproducts of Gws_G's. The kind of representing function defined in Definition 2.3 below was introduced first in [AGN77, p.38] (for a class of algebras slightly different from Gws_G). A similar representing function is defined in [HMTAN], [HMTII, 3.1.89] for cylindric algebras. The concept in [HMTAN], [HMTII], however, is different from ours in that it does not work for cylindric–like algebras with infinitary substitutions (like our Gws_G); (on the other hand, our concept cannot handle diagonal elements).

Our Lemma 2.5 below establishes the fact that rep_F as defined in our Definition 2.3 really gives a representation of Gws_G–ultraproducts as Gws_G's, thus $\mathbf{I}\mathsf{Gws}_G$ is closed under taking ultraproducts. Lemma 2.9 states that in a Gws_G $\mathfrak{A}$, the relativization function rl_Z with $Z \overset{\mathrm{def}}{=} \bigcap L$ where L is a filter of $\mathfrak{A}$, is a homomorphism into some Gws_G. This is used in the proof of Lemma 2.10, proving that Gws_G is closed under taking homomorphic images. The proof of Lemma 2.10 is the hardest part of this section. After having Lemma 2.10, the proof of Theorem 2.1 is relatively easy.

At the end of section 2.1, we will claim that if we replace ω with an arbitrary infinite ordinal α throughout section 2.1, then we will get true statements.

THEOREM 2.1. *Let $G \subseteq {}^{\omega}\omega$ be such that C_i is term–definable in Gws_G, for every $i \in \omega$. Then $\mathbf{I}\mathsf{Gws}_G$ is a variety.*

In section 2.4 we will show that the assumption of Theorem 2.1 cannot be omitted.

COROLLARY 2.2. *Let $G \subseteq {}^{\omega}\omega$ be arbitrary. Then any of the assumptions (1)–(2) below is enough for $\mathbf{I}\mathsf{Gws}_G$ to be a variety.*

(1) *For each $n \in \omega$ there is a $\Gamma \subseteq \omega$ with $n \in \Gamma$ such that $\mathrm{C}_{(\Gamma)}$ is term–definable in Gws_G. (E.g. the condition $(\forall i \in \omega)[i, 0] \in [G]$ is sufficient for this.)*

(2) *We change the definition of Gws_G such that for each $i \in \omega$, we include C_i as a basic operation in the definition of Gws_G.*

We will return to proving Corollary 2.2 after the proof of Theorem 2.1.

Before proving Theorem 2.1, we formulate and prove Lemmas 2.5, 2.9 and 2.10. Before doing this, we define the representation function mentioned above.

DEFINITION 2.3: Let I be an arbitrary set and let $\mathfrak{A}_i \in \mathsf{Gws}_G$ with base U_i for every $i \in I$. Let F be an ultrafilter over I. We define a *representation function*

$$rep_F : \Pi_{i\in I}\mathfrak{A}_i/F \longrightarrow \mathcal{P}\left({}^{\omega}\left(\Pi_{i\in I}U_i\right)\right)$$

belonging to F as follows. Let $a \in \Pi_{i\in I}A_i$ be arbitrary. Then

$$rep_F(a/F) \stackrel{\text{def}}{=} \{q \in {}^{\omega}\left(\Pi_{i\in I}U_i\right) : \{i \in I : \langle q_n(i)\rangle_{n\in\omega} \in a_i\} \in F\} . \quad \blacktriangleleft$$

We will use the following notation[17] . If $q \in {}^{\omega}\left(\Pi_{i\in I}U_i\right)$ then $q^+ \stackrel{\text{def}}{=} \langle\langle q_n(i)\rangle_{n\in\omega} : i \in I\rangle$. (Here, of course, $q^+(i) = \langle q_n(i)\rangle_{n\in\omega} \in {}^{\omega}U_i$ for each $i \in I$. Then $q^+(i)(n) = q_n(i)$ for each $n \in \omega$ and $i \in I$.) Using this notation,

$$rep_F(a/F) \stackrel{\text{def}}{=} \left\{q \in {}^{\omega}\left(\Pi_{i\in I}U_i\right) : \left\{i \in I : q^+(i) \in a_i\right\} \in F\right\} .$$

Remark 2.4: For the ultrapower case $rep_F : {}^{I}\mathfrak{A}/F \longrightarrow \mathcal{P}\left({}^{\omega}\left({}^{I}U\right)\right)$, rep_F is illustrated on Figure 2.4 way below. Here is some explanation to Figure 2.4.

The sets U, I and ω are illustrated by the three line segments as indicated on Figure 2.1. We imagine that U, I and ω are coordinate axes, pairwise perpendicular to each other. Thus we have a 3–dimensional coordinate system on Figure 2.1, the coordinates being U, I and ω:

U

I

ω

Figure 2.1

Then we can draw curves to represent functions in this coordinate system, and "tube–like" shapes to represent relations. For example, a curve in the $U \times I$ plane represents a function from I to U (the third coordinate of each point of which being 0). E.g., the elements of the direct power ${}^{I}U$ are such curves:

U

I

ω

Figure 2.2

[17] This notation is taken from [Mo91].

An element of A, which is an ω–ary relation over U by definition, is illustrated by a tube–like shape "parallel to" the $U \times \omega$ plane. An element of such a relation (which is an ω–ary relation over U) is illustrated as a curve parallel to the $U \times \omega$ plane and staying inside the tube (illustrating the relation):

U

I

ω

Figure 2.3

For example, $\langle q_n(i)\rangle_{n\in\omega}$ on Figure 2.4 is an element of the relation $a_i \in A$ (drawn as a tube). The sequence of tubes on Figure 2.4 illustrates the element $a = \langle a_i\rangle_{i\in I}$ of ${}^IA$; a is a representative of the equivalence class a/F. $rep(a/F)$ is illustrated on Figure 2.4 as an ω–sequence of curves a member of which is the curve q_n.

U_j U_i

$\langle q_n(i)\rangle_{n\in\omega} = q^+(i)$

a_i

j i I

q_n

n

ω

Figure 2.4

The function rep_F gives a representation (as a set algebra in the usual sense) of the ultraproduct $\Pi_{i\in I}\mathfrak{A}_i/F$. In the present paper we will need the ultrapower case only. Notice that, when representing an ultrapower ${}^I\mathfrak{A}/F$ by rep_F, the target algebra $rep_F''\left({}^I\mathfrak{A}/F\right)$ is a set algebra of base ${}^IU$, where U is the base of $\mathfrak{A}$. For any $a \in {}^IA$, the elements of $rep_F\,(a\,/F)$ can be visualized as surfaces of three dimensions, cf. Figure 2.4 above. Such a surface can be determined by its cut–curves of direction I, as well as by those of direction ω.

Let δ denote the diagonal embedding of $\mathfrak{A}$ into ${}^I\mathfrak{A}/F$. Then $rp_F \stackrel{\text{def}}{=} rep_F \circ \delta$ is a new representation of $\mathfrak{A}$. See Figure 2.5.

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\delta"] \arrow[rr, bend right, "rp_F"'] & {}^I\mathfrak{A}/F \arrow[r, "rep_F"] & rep''_F\left({}^I\mathfrak{A}/F\right) \end{tikzcd}$$

Figure 2.5

For any singleton $x \in A$, the cut–curve of the surface $rp_F(x)$ with the plane $y = n$ (parallel to the $U \times I$ plane in Figure 2.4) is a constant function, for every $n \in \omega$.

In Lemma 2.5 we will see that $rep''_F\left({}^I\mathfrak{A}/F\right)$ is a Gws_G. ◀

Recall that Cs_G was defined in section 1.2 (below Definition 1.4).

LEMMA 2.5. *Let $G \subseteq {}^\omega\omega$, $\mathfrak{A}_i \in \mathsf{Gws}_G$ for every $i \in I$ for some set I, and let F be an ultrafilter over I. Then (i)—(iii) below hold.*

(i) $rep_F : \Pi_{i\in I}\mathfrak{A}_i/F \rightarrowtail \mathfrak{B} \in \mathsf{Gws}_G$ *is an isomorphic embedding, for some* $\mathfrak{B}$.

(ii) *If* $1^{\mathfrak{A}_i} = {}^\omega U_i$ *for every* $i \in I$, *then* $1^{\mathfrak{B}} = {}^\omega\left(\Pi_{i\in I}U_i\right)$ *in (i) above. That is, denoting* $V_i = {}^\omega U_i$ *and* $V = {}^\omega\left(\Pi_{i\in I}U_i\right)$,

$$rep_F : \Pi_{i\in I}\langle \mathcal{P}(V_i), \cup, {}_{V_i}\!\sim, \mathsf{C}_0^{[V_i]}, \mathsf{S}_\tau^{[V_i]} : \tau \in G\rangle/F \rightarrowtail \langle \mathcal{P}(V), \cup, {}_V\!\sim, \mathsf{C}_0^{[V]}, \mathsf{S}_\tau^{[V]} : \tau \in G\rangle$$

is an isomorphism taking any ultraproduct of Cs_G*'s into a* Cs_G.

(iii) *Both* $\mathbf{I}\mathsf{Gws}_G$ *and* $\mathbf{I}\mathsf{Cs}_G$ *are closed under taking ultraproducts.*

Proof of Lemma 2.5: Let U_i denote the base of $\mathfrak{A}_i$ for every $i \in I$. For brevity, we write *rep* for rep_F in this proof.

(i) From the definition of *rep* it follows that $rep : \Pi_{i\in I}\mathfrak{A}_i/F \rightarrowtail \mathcal{P}\left({}^\omega\left(\Pi_{i\in I}U_i\right)\right)$ is one–one. Let $V \stackrel{\text{def}}{=} rep\left(1^{(\Pi_{i\in I}\mathfrak{A}_i/F)}\right)$. Then, by the definition of *rep*, $V \subseteq {}^\omega(\Pi_{i\in I}U_i)$. Let

$$\mathfrak{B} \stackrel{\text{def}}{=} \langle \mathcal{P}(V), \cup, {}_V\!\sim, \mathsf{C}_0^{[V]}, \mathsf{S}_\tau^{[V]} : \tau \in G\rangle .$$

We will prove that $rep : \Pi_{i\in I}\mathfrak{A}_i/F \longrightarrow \mathfrak{B}$ is a Gws_G–homomorphism.

Checking that *rep* is a Boolean homomorphism is left to the reader.

The following derivation proves that *rep* is a C_0–homomorphism. (This proof is illustrated on the top diagram of Figure 2.6.) For any $q \in {}^\omega(\Pi_{i\in I}U_i)$ and $a \in \Pi_{i\in I}A_i$,

$$\begin{aligned} q \in rep\left(\langle \mathsf{C}_0^{\mathfrak{A}_i}a_i\rangle_{i\in I}/F\right) &\iff \left\{i \in I : q^+(i) \in \mathsf{C}_0^{\mathfrak{A}_i}a_i\right\} \in F \\ &\iff \left\{i \in I : (\exists u_i \in U_i)\left(q^+(i)\right)^0_{u_i} \in a_i\right\} \in F \\ &\iff \{i \in I : (\exists u_i \in U_i)\langle u_i, q_1(i), \dots, q_n(i), \dots\rangle \in a_i\} \in F \\ &\iff (\exists u \in \Pi_{i\in I}U_i)\,\{i \in I : \langle u_i, q_1(i), \dots, q_n(i), \dots\rangle \in a_i\} \in F \\ &\iff (\exists u \in \Pi_{i\in I}U_i)\langle u, q_1, \dots, q_n, \dots\rangle \in rep\left(\langle a_i\rangle_{i\in I}/F\right) \\ &\iff (\exists u \in \Pi_{i\in I}U_i)q^0_u \in rep\left(\langle a_i\rangle_{i\in I}/F\right) \\ &\iff q \in \mathsf{C}_0^{\mathfrak{B}}\left(rep\left(\langle a_i\rangle_{i\in I}/F\right)\right) . \end{aligned}$$

$$
\begin{tikzcd}[column sep=huge]
\langle a_i \rangle_{i\in I}/F \arrow[r, "rep"] \arrow[d, "\mathrm{C}_0^{\Pi_{i\in I}\mathfrak{A}_i/F}"'] & rep\,(\langle a_i \rangle_{i\in I}/F\,) \arrow[d, "\mathrm{C}_0^{\mathfrak{B}}"] \\
\mathrm{C}_0^{\Pi_{i\in I}\mathfrak{A}_i/F}\,(\langle a_i \rangle_{i\in I}/F\,) & \mathrm{C}_0^{\mathfrak{B}} rep\,(\langle a_i \rangle_{i\in I}/F\,) \arrow[d, phantom, "?"] \\
\langle \mathrm{C}_0^{\mathfrak{A}_i} a_i \rangle_{i\in I}/F \arrow[r, "rep"] & rep\left(\langle \mathrm{C}_0^{\mathfrak{A}_i} a_i \rangle_{i\in I}/F\right)
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\langle a_i \rangle/F \arrow[r, "rep"] \arrow[d, "\mathrm{S}_\tau^{\Pi\mathfrak{A}_i/F}"'] & rep\,(\langle a_i \rangle/F\,) \arrow[d, "\mathrm{S}_\tau^{\mathfrak{B}}"] \\
 & \mathrm{S}_\tau^{\mathfrak{B}} rep\,(\langle a_i \rangle/F\,) \arrow[d, phantom, "?"] \\
\bullet \arrow[r, "rep"] & rep\,(\langle \mathrm{S}_\tau^{\mathfrak{A}_i} a_i \rangle/F\,)
\end{tikzcd}
$$

Figure 2.6

Next we prove that *rep* is an S_τ–homomorphism, for any $\tau \in {}^\omega\omega$. (This is illustrated on the bottom part of Figure 2.6.) Let $\tau \in {}^\omega\omega$, $q \in {}^\omega(\Pi_{i\in I}U_i)$, $a \in \Pi_{i\in I}A_i$. Then

$$
\begin{aligned}
q \in rep\,(\langle \mathrm{S}_\tau^{\mathfrak{A}_i} a_i\rangle_{i\in I}/F\,) &\iff \{i \in I : q^+(i) \in \mathrm{S}_\tau^{\mathfrak{A}_i} a_i\} \in F\\
&\iff \{i \in I : q^+(i)\circ\tau \in a_i\} \in F\\
&\iff \{i \in I : \langle q_{\tau 0}(i), q_{\tau 1}(i), q_{\tau 2}(i), \ldots\rangle \in a_i\} \in F\\
&\iff \langle q_{\tau 0}, q_{\tau 1}, q_{\tau 2}, \ldots\rangle \in rep\,(\langle a_i\rangle_{i\in I}/F\,)\\
&\iff q\circ\tau \in rep\,(\langle a_i\rangle_{i\in I}/F\,)\\
&\iff q \in \mathrm{S}_\tau^{\mathfrak{B}} rep\,(\langle a_i\rangle_{i\in I}/F\,)\ .
\end{aligned}
$$

So far we have proved that $rep : \Pi_{i\in I}\mathfrak{A}_i/F \longrightarrow \mathfrak{B}$ is a one–one homomorphism. We turn to checking that $\mathfrak{B} \in \mathsf{Gws}_G$.

To see that $\mathfrak{B} \in \mathsf{Gws}_G$, it is enough to prove that $1^{\mathfrak{B}}(= V = rep\,(1^{(\Pi_{i\in I}\mathfrak{A}_i/F\,)}))$ is a Gws_G–unit. To see that $1^{\mathfrak{B}}$ is a normal Gws_ω–unit, by Lemma 1.3, it is enough to prove that

$$(\forall p, q \in 1^{\mathfrak{B}})\big(Rng(p)\cap Rng(q) \neq \emptyset \Longrightarrow (\forall i, j \in \omega)p[i/q(j)] \in 1^{\mathfrak{B}}\big)\ . \tag{2.0}$$

To prove (2.0), let $p, q \in 1^{\mathfrak{B}}$ be arbitrary but fixed, and assume $Rng(p)\cap Rng(q) \neq \emptyset$. Then $p_\ell = q_\ell$ for some $\ell \in \omega$. Let $M = \{k \in I : p^+(k) \in 1^{\mathfrak{A}_k}$ and $q^+(k) \in 1^{\mathfrak{A}_k}\}$. So $M \in F$. Suppose that

$k \in M$. Then $p_\ell(k) = q_\ell(k)$, hence $p^+(k)_\ell = q^+(k)_\ell$. So $Rng\,(p^+(k)) \cap Rng\,(q^+(k)) \neq \emptyset$. Since $1^{\mathfrak{A}_k}$ is a normal Gws_ω–unit, $p^+(k)[i/q^+(k)_j] \in 1^{\mathfrak{A}_k}$ for each $i, j \in \omega$, by Lemma 1.3. Now

$$(p[i/q(j)])^+(k) = p^+(k)[i/q^+(k)_j]\,. \tag{2.1}$$

For, if $n \in \omega$ then $(p[i/q(j)])^+(k)(n) = (p[i/q(j)])_n(k)$. If $n \neq i$, then this is $p_n(k) = p^+_k(n) = p^+(k)[i/q^+(k)_j](n)$. If $n = i$, it is $q(j)(k) = q^+(k)_j = p^+(k)[i/q^+(k)_j](n)$. So (2.1) holds. Thus $(p[i/q(j)])^+(k) \in 1^{\mathfrak{A}_k}$. Since $k \in M$, $\{k \in I : (p[i/q(j)])^+(k) \in 1^{\mathfrak{A}_k}\} \in F$. So $p[i/q(j)] \in 1^{\mathfrak{B}}$. This completes the proof of (2.0).

Since $\mathrm{S}_\tau^{[1^{\mathfrak{A}_k}]}\left(1^{\mathfrak{A}_k}\right) = 1^{\mathfrak{A}_k}$ for every $\tau \in G$ and $k \in I$, and since *rep* is a homomorphism, $\mathrm{S}_\tau^{[1^{\mathfrak{B}}]}\left(1^{\mathfrak{B}}\right) = 1^{\mathfrak{B}}$, completing the proof of the fact that $1^{\mathfrak{B}}$ is a Gws_G–unit. We have proved (i).

(ii) To prove (ii), it is enough to check that if $V_i = {}^\omega U_i$ then $rep\,(\langle V_i\rangle_{i\in I}/F) = {}^\omega\,(\Pi_{i\in I}U_i)$. Inclusion "$\subseteq$" is immediate from the definition of *rep*. For proving the other direction, let $q \in {}^\omega\,(\Pi_{i\in I}U_i)$ be arbitrary. Now for any $i \in I$, $\langle q_n(i) : n \in \omega\rangle \in V_i = {}^\omega U_i$ obviously holds, proving "$\supseteq$".

(iii) is a straightforward consequence of (i) and (ii). ∎

Corollary 2.6 (i) below says that Lemma 2.5 (i) remains true for that *variant* of Gws_G where we drop the condition $\mathrm{S}_\tau^{[V]}(V) = V$ from the definition of V's being a Gws_G–unit. (ii) of Corollary 2.6 is a variant of (i).

COROLLARY 2.6.

(i) *Let $G \subseteq {}^\omega\omega$, let I be a set, and for each $i \in I$, let W_i be a Gws_ω–unit. Let $\mathfrak{A}_i = \langle \mathcal{P}\,(W_i)\,, \cup, {}_{W_i}\!\sim, \mathrm{C}_0^{[W_i]}, \mathrm{S}_\tau^{[W_i]} : \tau \in G\rangle$. Let F be an ultrafilter over I. Then rep_F : $\Pi_{i\in I}\mathfrak{A}_i\,/F \rightarrowtail \mathfrak{B}$ is an embedding for some $\mathfrak{B} = \langle \mathcal{P}\,(W)\,, \cup, {}_W\!\sim, \mathrm{C}_0^{[W]}, \mathrm{S}_\tau^{[W]} : \tau \in G\rangle$ with W a Gws_ω–unit.*

(ii) *Let everything be as in (i) above. Assume in addition that the Gws_ω–units W_i are normal. Then W is a normal Gws_ω–unit.*

Proof: This is a corollary of the proof (and not of the statement) of Lemma 2.5. Namely, the fact that rep_F is an injective homomorphism was proven in the proof of Lemma 2.5 without referring to the units of the $\mathfrak{A}_i$'s, therefore that proof works under the present conditions as well.

W's being a Gws_ω–unit can be proved, e.g., by using [HMTII, Thm.3.1.31(iv) (p.21)], as follows. By the quoted theorem, for any $\mathfrak{A} \in \mathsf{Cs}_\omega$,

$$X \in A \text{ is a } \mathsf{Gws}_\omega\text{–unit} \;\text{ iff }\; X = \mathsf{s}^i_j X \cap \mathsf{s}^j_i X \text{ for every } i, j \in \omega. \tag{2.2}$$

To use (2.2), using the notation introduced in the proof of Lemma 2.5, for every $k \in I$ we let $\mathfrak{C}_k \stackrel{\text{def}}{=} \langle \mathcal{P}\,({}^\omega U_k)\,, \cap, \mathsf{s}^i_j\rangle_{i,j\in\omega}$, and $W \stackrel{\text{def}}{=} \Pi_{k\in I}U_k$. Let $\mathfrak{D}$ denote the full powerset algebra with unit element ${}^\omega W$ and similar to the $\mathfrak{C}_k$'s. Then $V_k \stackrel{\text{def}}{=} 1^{\mathfrak{A}_k} \in C_k$ and $1^{\mathfrak{B}} = rep\,(\langle V_k\rangle_{k\in I}/F) \in D$, see Figure 2.7 below. The proof of (2.2) works for these reducts, too.

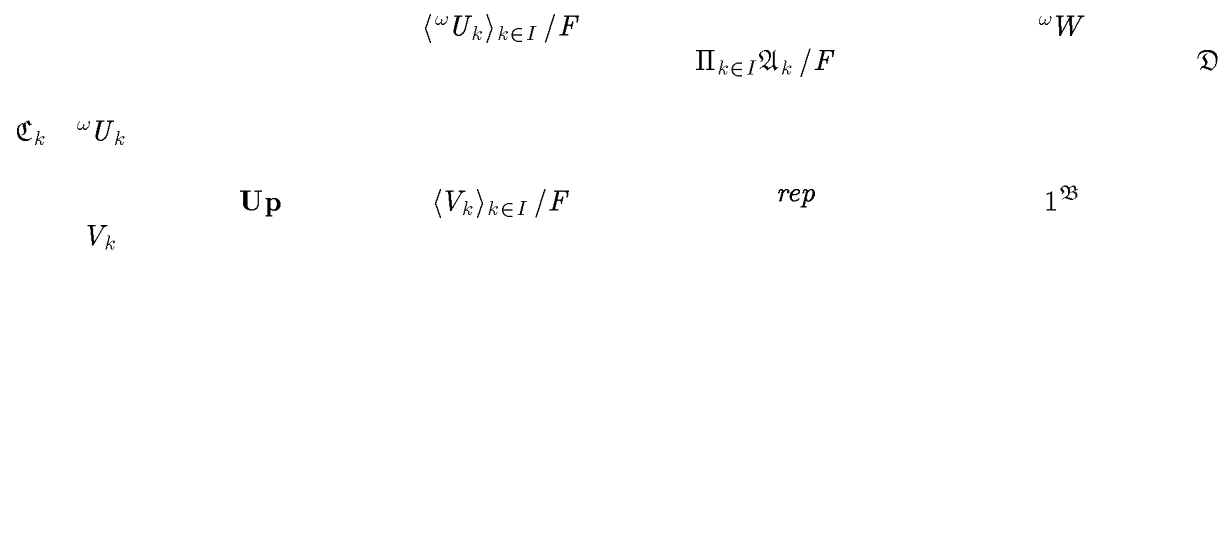

Figure 2.7

By (2.2), $\mathfrak{C}_k \models V_k = \mathsf{s}^i_j V_k \cap \mathsf{s}^j_i V_k$ for every $i, j \in \omega$ and $k \in I$ (since V_k is a Gws_ω–unit). Since ultraproducts preserve first order formulas, and *rep* is a homomorphism preserving $\cap$ and the s^i_j's by the first part of the proof of Lemma 2.5, $\mathfrak{D} \models 1^{\mathfrak{B}} = \mathsf{s}^i_j 1^{\mathfrak{B}} \cap \mathsf{s}^j_i 1^{\mathfrak{B}}$ for every $i, j \in \omega$. Applying (2.2) again, but now in the other direction, we get that $1^{\mathfrak{B}}$ is a Gws_ω–unit. We have proved (i).

The proof of Lemma 2.5 contains a part proving explicitly that the (*rep*–image of the) unit of the ultraproduct algebra is a normal Gws_ω–unit. This part does not refer to any operation of the algebras involved, therefore it proves that our present W is a normal Gws_ω–unit. This completes the proof of Corollary 2.6. **QED (Corollary 2.6)** ∎

COROLLARY 2.7. *Corollary 2.6 remains true even if we replace* C_0 *with* $\mathrm{C}_{(\Gamma)}$ *in it, for any* $\Gamma \subseteq \omega$.

Proof: The first computation in the proof of Lemma 2.5 easily generalizes to $\mathrm{C}_{(\Gamma)}$ from C_0, as follows.

$$
\begin{aligned}
q \in rep\left(\langle \mathrm{C}^{\mathfrak{A}_i}_{(\Gamma)} a_i\rangle_{i\in I}/F\right) &\iff \left\{i \in I : q^+(i) \in \mathrm{C}^{\mathfrak{A}_i}_{(\Gamma)} a_i\right\} \in F \\
&\iff \left\{i \in I : (\exists u_i \in {}^{\Gamma}U_i)\left(q^+(i)\right)^{\Gamma}_{u_i} \in a_i\right\} \in F \\
&\iff \left(\exists u \in {}^{\Gamma}(\Pi_{i\in I}U_i)\right)\left\{i \in I : \left(q^+(i)\right)^{\Gamma}_{u_i} \in a_i\right\} \in F \\
&\iff \left(\exists u \in {}^{\Gamma}(\Pi_{i\in I}U_i)\right) q^{\Gamma}_u \in rep\left(\langle a_i\rangle_{i\in I}/F\right) \\
&\iff q \in \mathrm{C}^{\mathfrak{B}}_{(\Gamma)}\left(rep\left(\langle a_i\rangle_{i\in I}/F\right)\right). \qquad \blacksquare
\end{aligned}
$$

Remark 2.8: We will see in subsection 5.1 (see Lemma 5.2 there) that no similar proof can be given for a version of Lemma 2.5 in which Gws_G is replaced with the modification Gws^+_G of Gws_G such that Gws^+_G includes *diagonal elements* (i.e. the algebraic counterparts of equality as logical symbol) as well. (Diagonal elements will be defined in subsection 5.1.) Namely, we will see that if we want *rep* to "preserve diagonals" as well then we cannot allow infinitary substitutions like S_{suc} among our operations.

For cylindric algebras (and similar kinds of algebras containing diagonal elements but not containing infinitary substitutions) the *rep* function can be modified such that its new version rep^+ preserves all the operations of the algebras in question. For this definition see e.g. [HMTAN, sections I.7 and II.7], [HMTII] or [Mo91] or [Né79]. ◄

If $\mathfrak{A} \in \mathsf{Gws}_G$ for some G then by a Gws_G*–filter* we mean a congruence class of $1^{\mathfrak{A}}$. The rôle of Gws_G–filters in Gws_G–theory is similar to the rôle of ideals in ring theory or that of normal subgroups in group theory. Namely, the Gws_G–filters and the Gws_G–congruences determine each other. More precisely, if $\mathfrak{A} \in \mathbf{I}\mathsf{Gws}_G$, then there is a one–one correspondence between the two sets $Fl(\mathfrak{A}) \stackrel{\text{def}}{=} \{L : L \text{ is a } \mathsf{Gws}_G\text{–filter of } \mathfrak{A}\}$ and $Con(\mathfrak{A}) \stackrel{\text{def}}{=} \{\theta : \theta \text{ is a } \mathsf{Gws}_G\text{–congruence of } \mathfrak{A}\}$. As a consequence of [Sa82a p.9 Prop.7.4], we have that $L \in Fl(\mathfrak{A})$ iff

L is a BA–filter ($x, y \in L \Rightarrow x \cdot y \in L$ and $x \in L, y \geqslant x \Rightarrow y \in L$), and

$x \in L \Rightarrow \left(\mathrm{C}_0^{\partial} x \stackrel{\text{def}}{=} -\mathrm{C}_0(-x) \in L \text{ and } \mathrm{S}_\tau(x) \in L\right)$.

LEMMA 2.9. *Let $G \subseteq {}^{\omega}\omega$ be such that C_i is term–definable in Gws_G, for every $i \in \omega$. Let $\mathfrak{A} \in \mathsf{Gws}_G$, L a Gws_G–filter on $\mathfrak{A}$, and let $Z \stackrel{\text{def}}{=} \bigcap L$. Then the relativization function $rl_Z \stackrel{\text{def}}{=} \{\langle a, a \cap Z\rangle : a \in A\}$ is a homomorphism $rl_Z : \mathfrak{A} \longrightarrow \mathfrak{B} \in \mathsf{Gws}_G$ for some $\mathfrak{B}$.*

Proof of Lemma 2.9: Let $G, \mathfrak{A}, L, Z, rl_Z$ be as in the formulation of the lemma. Let V denote the unit of $\mathfrak{A}$ (i.e., $V = 1^{\mathfrak{A}}$). By hypothesis, C_i is term–definable in $\mathfrak{A}$, for every $i \in \omega$. First we prove that

$$(\forall i \in \omega)\mathrm{C}_i^{\mathfrak{A}} Z \big(= \mathrm{C}_i^{[V]} Z\big) = Z\,. \tag{2.3}$$

Let $i \in \omega$ be arbitrary but fixed. Let us recall that the set of all C_i–closed elements is coinitial in L, that is,

$$(\forall x \in L)(\exists y \in L)(y \leqslant x \text{ and } \mathrm{C}_i y = y) \tag{2.4}$$

(choose $y = \mathrm{C}_i^{\partial} x = -\mathrm{C}_i(-x)$; this y works, because $y \in L$ since L is a filter, and by $(\forall z \in A)\mathrm{C}_i^{\partial} z \leqslant z$ we have $y \leqslant x$; further $\mathrm{C}_i y = \mathrm{C}_i \mathrm{C}_i^{\partial} x = \mathrm{C}_i(-\mathrm{C}_i(-x)) = -\mathrm{C}_i(-x) = \mathrm{C}_i^{\partial} x = y$). Therefore

$$Z = \bigcap L = \bigcap \{x \in L : \mathrm{C}_i x = x\} \tag{2.5}$$

since $\bigcap\{x \in L : \mathrm{C}_i x = x\} \supseteq \bigcap L$ holds by definition, and the other inclusion follows from (2.4). Therefore $\mathrm{C}_i Z = Z$, since an arbitrary intersection of C_i–closed elements is C_i–closed.

Now we let $\mathfrak{B} \stackrel{\text{def}}{=} \langle \mathcal{P}(Z), \cup, {}_Z\!\sim, \mathrm{C}_0^{[Z]}, \mathrm{S}_\tau^{[Z]}\rangle_{\tau \in G}$. We claim that $\mathfrak{B} \in \mathsf{Gws}_G$. To see this, it is enough to show that Z is a Gws_G–unit.

The fact that Z is a Gws_ω–unit follows from our assumption that all the C_i's are term–definable in Gws_G, as follows. By Lemma 1.3, it is enough to prove that

$$(\forall p, q \in Z)(\forall i, j \in \omega)\,(Rng(p) \cap Rng(q) \neq \emptyset \Longrightarrow q[i/p(j)] \in Z)\,.$$

To see this, let $p, q \in Z$, $i, j \in \omega$. Assume $Rng(p) \cap Rng(q) \neq \emptyset$. Since $p, q \in V$ and V is a normal Gws_ω–unit, $q[i/p(j)] \in V$ by Lemma 1.3. Then $q[i/p(j)] \in \mathrm{C}_i^{[V]}(Z)$. By (2.3), $\mathrm{C}_i^{[V]}(Z) = Z$, and thus $q[i/p(j)] \in Z$.

Now let $\tau \in G$. To see that $\mathrm{S}_\tau^{[Z]}(Z) = Z$, it is enough to prove $\mathrm{S}_\tau^{[Z]}(Z) \supseteq Z$ (because the other direction is obvious).

$$\begin{aligned}
\mathrm{S}_\tau^{[Z]}(Z) = \mathrm{S}_\tau^{[Z]}\left(\bigcap L\right) &= \left\{q \in Z : q \circ \tau \in \bigcap L\right\} \\
&= \{q \in Z : (\forall x \in L) q \circ \tau \in x\} \\
&= \left\{q \in Z : (\forall x \in L) q \in \mathrm{S}_\tau^{[Z]} x\right\} \\
&= \bigcap \left\{\mathrm{S}_\tau^{[Z]} x : x \in L\right\} \\
&= \bigcap \big(\mathrm{S}_\tau^{[Z]}\big)''(L) \supseteq \bigcap L = Z\,.
\end{aligned}$$

By this we have completed the proof of $\mathfrak{B} \in \mathsf{Gws}_G$.

Finally we prove that $rl_Z : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a homomorphism.[18] Let $a \in A$ and $\tau \in {}^{\omega}\omega$. Then

$$\begin{aligned}
rl_Z\left(\mathrm{C}_0^{[V]}(a)\right) &= \mathrm{C}_0^{[V]}(a) \cap Z && \text{by the definition of } rl_Z \\
&= \mathrm{C}_0^{[V]}(a) \cap \mathrm{C}_0^{[V]}(Z) && \text{by (2.3)} \\
&= \mathrm{C}_0^{[V]}\left(a \cap \mathrm{C}_0^{[V]}(Z)\right) \cap Z && \text{by } \mathsf{Gws}_G \models \mathsf{c}_0 x \cdot \mathsf{c}_0 y = \mathsf{c}_0(x \cdot \mathsf{c}_0 y) \text{ (this is [HMT, } C_3\text{])} \\
&= \mathrm{C}_0^{[V]}(a \cap Z) \cap Z && \text{by (2.3)} \\
&= \mathrm{C}_0^{[Z]}\left(rl_Z(a)\right) && \text{by the definition of } rl_Z.
\end{aligned}$$

$$rl_Z\left(\mathrm{S}_\tau^{[V]}(a)\right) = \mathrm{S}_\tau^{[V]}(a) \cap Z = \mathrm{S}_\tau^{[V]}(a) \cap \mathrm{S}_\tau^{[Z]}(Z) \cap Z = \mathrm{S}_\tau^{[V]}(a \cap Z) \cap Z = \mathrm{S}_\tau^{[Z]}\left(rl_Z(a)\right).$$

Checking that rl_Z is a BA–homomorphism is left to the reader. QED (Lemma 2.9) ∎

We call $\mathfrak{B}$ in the formulation of Lemma 2.9 the *relativization* of $\mathfrak{A}$ to Z.

LEMMA 2.10. *Assume that* C_i *is term–definable in* Gws_G*, for every* $i \in \omega$*. Then* $\mathbf{H}\mathsf{Gws}_G = \mathsf{Gws}_G$.

Proof of Lemma 2.10: Let $\mathfrak{A} \in \mathsf{Gws}_G$ and let L be a Gws_G–filter on $\mathfrak{A}$. For proving Lemma 2.10, it is enough to show that $\mathfrak{A}/L \in \mathbf{I}\mathsf{Gws}_G$.

The idea of the proof is the following. Let $Z \stackrel{\text{def}}{=} \bigcap L$. ($Z \notin A$ in general.) Let $\mathfrak{Rl}_Z\mathfrak{A}$ denote the relativization of $\mathfrak{A}$ to Z. By Lemma 2.9, $\mathfrak{Rl}_Z\mathfrak{A}$ is a homomorphic image of $\mathfrak{A}$, and is a Gws_G. It is easy to see that $\mathfrak{Rl}_Z\mathfrak{A}$ is a homomorphic image of $\mathfrak{A}/L$, e.g. as follows. The Gws_G–filter corresponding to the relativization function rl_Z is $H \stackrel{\text{def}}{=} \{y \in A : Z \subseteq y\}$. Therefore $\mathfrak{A}/H \cong \mathfrak{Rl}_Z\mathfrak{A}$, further $\mathfrak{A}/H$ is a homomorphic image of $\mathfrak{A}/L$ by $L \subseteq H$. See Figure 2.8.

$$\begin{array}{ccccc}
 & \mathfrak{A} & & & \\
\mathfrak{A}/L & & rl_Z & & \\
\mathfrak{A}/H & \cong & \mathfrak{Rl}_Z\mathfrak{A} & \in & \mathsf{Gws}_G
\end{array}$$

Figure 2.8

Thus it would be enough to prove that $H = L$. But this is not the case in general, since it may be the case that $Z \subseteq y \notin L$ for some $y \in A$. To fix this, we will switch for such a new representation[19] $rp''(\mathfrak{A})$ of $\mathfrak{A}$ in which $(\forall x \in A \smallsetminus L)(\exists q \in \bigcap rp''(L))q \notin rp(x)$. Then, using the shorthands $\mathfrak{A}^+ = rp''(\mathfrak{A})$, $L^+ = rp''(L)$ and $Z^+ = \bigcap L^+$, we will have that

$$\mathfrak{A}/L \cong \mathfrak{A}^+/L^+ \cong \mathfrak{Rl}_{Z^+}\mathfrak{A}^+ \in \mathbf{I}\mathsf{Gws}_G\,,$$

completing the proof.

Thus, for proving Lemma 2.10, it is enough to give a representation of $\mathfrak{A}$ such that

$$(\forall x \in A \smallsetminus L)\exists q(\forall z \in L)\big(q \in rp(z) \& q \notin rp(x)\big).$$

See Figure 2.9.

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[18] Another proof of this goes by proving $\mathrm{C}_0^{\mathfrak{A}}(Z) = Z$ and $\mathrm{S}_\tau^{\mathfrak{A}}(Z) \supseteq Z$, and using [Né80, Thm.1(i)].

[19] This method is called the "*change of base technique*" in the literature of algebraic logic, cf. e.g. [HMTAN], [HMTII], [Mo91].

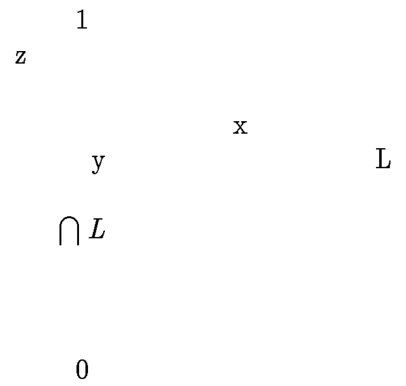

Figure 2.9

Let F be an ultrafilter over (the index set) L containing the set $\{\{x \in L : x \leqslant z\} : z \in L\}$ (which clearly has the finite intersection property). Let $rp \stackrel{\text{def}}{=} rep_F \circ \delta$, where rep_F is as in Definition 2.3, and $\delta : \mathfrak{A} \rightarrowtail {}^L\mathfrak{A}/F$ is the diagonal embedding. Notice that, by the definition of rp, the base of $rp''(\mathfrak{A})$ is ${}^LU$, where U is the base of $\mathfrak{A}$; further, for any $x \in A$, the n–th "coordinate" or "cut–curve" of $rp(x)$ is a constant function, for any $n \in \omega$.

Now, let $x \in A \smallsetminus L$. To any $z \in L$, let $\langle q_n(z) : n \in \omega\rangle \in z \smallsetminus x$. Such sequences exist, since $z \smallsetminus x \neq \emptyset$ for any $z \in L$ (since $x \notin L$). Let $q \stackrel{\text{def}}{=} \langle\langle q_n(z) : z \in L\rangle : n \in \omega\rangle \in {}^{\omega}\left({}^LU\right)$. Then $q \notin rp(x)$ by the definition of rp. Let $z \in L$ be arbitrary but fixed. We will prove that $q \in rp(z)$. Let $Y \stackrel{\text{def}}{=} \{y \in L : y \subseteq z\}$. Then $Y \in F$. Further, $(\forall y \in Y)\,(\langle q_n(y) : n \in \omega\rangle \in y \subseteq z)$. Thus $(\forall y \in Y)\,(\langle q_n(y) : n \in \omega\rangle \in z)$ where $Y \in F$, proving that $q \in rp(z)$.

We have seen that, for any $x \in A \smallsetminus L$, there is a q such that $q \notin rp(x)$ and $(\forall z \in L) q \in rp(z)$, which completes the proof. QED (Lemma 2.10) ▮

Proof of Theorem 2.1: Let $G \subseteq {}^{\omega}\omega$ be arbitrary. Assume that for every $i \in \omega$, C_i is term–definable in Gws_G. To see that $\mathbf{I}\mathsf{Gws}_G$ is a variety, it is enough to prove that it is closed under the operations $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ (by a famous theorem of G. Birkhoff). By Lemma 2.10, $\mathbf{H}\mathsf{Gws}_G = \mathsf{Gws}_G$ (because the C_i's are term–definable in Gws_G), and $\mathbf{S}\mathsf{Gws}_G = \mathsf{Gws}_G$ follows from the definition of Gws_G. $\mathbf{P}\mathsf{Gws}_G = \mathsf{Gws}_G$ remains to prove.

This can be done e.g. by using a statement analogous to [HMTII, 3.1.76]. Another way of proving that $\mathbf{I}\mathsf{Gws}_G$ is closed under $\mathbf{P}$ is a relatively easy generalization of the proof for Boolean algebras (BA's).

Indeed, by Stone Duality theory, *direct product* of Boolean algebras corresponds to *disjoint union* of their unit elements (because the Stone functor is contravariant). Hence, if $\mathfrak{A}, \mathfrak{B} \in \mathsf{BA}$ with unit elements $1^{\mathfrak{A}}$ and $1^{\mathfrak{B}}$, then the unit element of $\mathfrak{A} \times \mathfrak{B}$ is $1^{\mathfrak{A}} \mathbin{\dot\cup} 1^{\mathfrak{B}}$. Actually, if $\mathfrak{A} = \langle \mathcal{P}(V), \cup, \sim\rangle$ and $\mathfrak{B} = \langle \mathcal{P}(W), \cup, \sim\rangle$ then $\mathfrak{A} \times \mathfrak{B} \cong \langle \mathcal{P}\left(V \mathbin{\dot\cup} W\right), \cup, \sim\rangle$.

To generalize this to Gws_G, let $\mathfrak{A}_0, \mathfrak{A}_1 \in \mathsf{Gws}_G$ with units V_0, V_1, respectively. We may assume that each subbase of V_0 is disjoint from each subbase of V_1. Then $V_0 \cup V_1$ is a Gws_G–unit, because it is clearly a normal Gws_ω–unit, further, for any $\tau \in G$,

$$\begin{aligned}
\mathrm{S}_\tau^{[V_0\cup V_1]}(V_0 \cup V_1) &= \mathrm{S}_\tau^{[V_0\cup V_1]}(V_0) \cup \mathrm{S}_\tau^{[V_0\cup V_1]}(V_1) && \text{by } \mathrm{S}_\tau \text{ being a BA–homomorphism}\\
&= \mathrm{S}_\tau^{[V_0]}(V_0) \cup \mathrm{S}_\tau^{[V_1]}(V_1) && \text{by disjointness of subbases}\\
&= V_0 \cup V_1 && \text{because } V_0, V_1 \text{ are } \mathsf{Gws}_G\text{–units.}
\end{aligned}$$

Now we represent $\mathfrak{A}_0 \times \mathfrak{A}_1$ as a subalgebra of the Gws_G

$$\mathfrak{C} \stackrel{\text{def}}{=} \langle \mathcal{P}(V_0 \cup V_1), \cup, {}_{V_0\cup V_1}\!\sim, \mathrm{C}_0^{[V_0\cup V_1]}, \mathrm{S}_\tau^{[V_0\cup V_1]}\rangle_{\tau\in G}.$$

It is enough to check that the standard BA–embedding $\langle a, b\rangle \mapsto (a \dot{\cup} b)$ works here. To this, it is enough to check that, for $i < 2$,

$$rl_{V_i} : \mathfrak{C} \longrightarrow \mathfrak{B}_i \stackrel{\text{def}}{=} \langle \mathcal{P}(V_i), \cup, {}_{V_i}\!\sim, \mathrm{C}_0^{[V_i]}, \mathrm{S}_\tau^{[V_i]}\rangle_{\tau \in G}$$

is a Gws_G–homomorphism, where $rl_{V_i}(x) = x \cap V_i$ for all $x \in C$. Indeed, if rl_{V_i} $(i < 2)$ is a homomorphism, then $\langle rl_{V_0}, rl_{V_1}\rangle$ provide a direct decomposition of $\mathfrak{C}$ into the direct factors $\langle \mathfrak{B}_0, \mathfrak{B}_1\rangle$. Since $\mathfrak{A}_i$ $(i < 2)$ is a subalgebra of $\mathfrak{B}_i$, both having the same unit V_i, we are done.

By [Né80, Thm.1(i)] and by the fact that Gws_G is a subclass of Boolean algebras with operators (BAO's), for rl_{V_i} $(i < 2)$ to be a homomorphism, it is enough to prove that

$$\mathrm{S}_\tau^{[V_0 \cup V_1]}({}_{V_0 \cup V_1}\!\sim V_i) \subseteq {}_{V_0 \cup V_1}\!\sim V_i \quad \text{and} \quad \mathrm{C}_0^{[V_0 \cup V_1]}(V_i) = V_i \quad (i < 2)\,. \tag{2.6}$$

By our definition of a Gws_G–unit, $\mathrm{S}_\tau^{[V_i]}(V_i) = V_i$, hence $\mathrm{S}_\tau^{[V_0 \cup V_1]}(V_i) \supseteq V_i$ which, by S_τ being a Boolean homomorphism, yields

$$\mathrm{S}_\tau^{[V_0 \cup V_1]}({}_{V_0 \cup V_1}\!\sim V_i) \subseteq {}_{V_0 \cup V_1}\!\sim V_i\,.$$

Showing $\mathrm{C}_0^{[V_0 \cup V_1]}(V_i) = V_i$ is easy and is left to the reader (can also be found in [HMTII]).[20] By these we have proved (2.6), completing the proof of the fact that $\mathfrak{A}_0 \times \mathfrak{A}_1 \in \mathsf{Gws}_G$.

The generalization of the above proof from two components to an arbitrary system $\langle \mathfrak{A}_i : i \in I\rangle$ is straightforward and is left to the reader. Hence $\mathbf{P}\mathsf{Gws}_G \subseteq \mathbf{I}\mathsf{Gws}_G$ as was desired. QED (Theorem 2.1) ∎

Proof of Corollary 2.2: The implication '(2) $\Rightarrow$ $\mathbf{I}\mathsf{Gws}_G$ is a variety' follows from Theorem 2.1; while a proof for '(1) $\Rightarrow$ $\mathbf{I}\mathsf{Gws}_G$ is a variety' can easily be obtained from the proof of Theorem 2.1. Namely, the condition of Theorem 2.1 that all the C_i's are term–definable in Gws_G was needed only in proving that $\mathbf{H}\mathsf{Gws}_G = \mathsf{Gws}_G$ (see Lemma 2.10). This condition, in fact, was made use of in proving Lemma 2.9 (which was then used in Lemma 2.10). Now, replacing term–definability of the C_i's with (1) of Corollary 2.2 gives us a true lemma again, the proof of which is very similar to the proof of Lemma 2.9 (and which is left to the reader). Using this new lemma in place of Lemma 2.9, proving $\mathbf{H}\mathsf{Gws}_G = \mathbf{I}\mathsf{Gws}_G$ from this new lemma and leaving all the rest of the proof of Theorem 2.1 unchanged gives us thus a proof for Corollary 2.2. ∎

Our Theorem 2.1, as well as the lemmas we used in its proof, have the following straightforward generalization: throughout, instead of ω, we can use an arbitrary infinite ordinal α. Let thus α be an arbitrary infinite ordinal. ${}_\alpha\mathsf{Gws}_G$ denotes the class of *α–dimensional generalized weak set G–algebras*, the definition of which we can get from Definition 1.4 by replacing ω with α (thus $G \subseteq {}^\alpha\alpha$ etc.). Then the following theorem holds:

COROLLARY 2.11. *Let α be an arbitrary infinite ordinal. Let $G \subseteq {}^\alpha\alpha$ be such that C_i is term–definable in ${}_\alpha\mathsf{Gws}_G$ for every $i \in \alpha$. Then $\mathbf{I}_\alpha\mathsf{Gws}_G$ is a variety.* ∎

[20] This is a special case of a more general result about direct decompositions also given in [Né80]. The reader not familiar with this result should be able to prove it without looking up items [HMT, 2.7.31 (p.452), 2.3.26(i) (p.291)], from which the proof can be recovered especially in view of the last line of p.452 (finding a direct proof seems to be easier than pasting these pieces together).

2.2. Finite schema axiomatizing $\mathbf{I}\mathsf{Gws}_G$

In subsection 1.3 we described the intuitive idea of the proof of Theorem 1.8 (i). Now we would like to tie up that "idea of proof" with the actual content of the present subsection. The main bulk of what we described in the "idea of proof" is carried through in the proof of Theorem 2.15 below. At the end of the "idea of proof" we wrote about reducing the size of the axiomatization Σ of $\mathbf{I}\mathsf{Gws}_G$ to finite. This will be done in subsection 2.3.

Theorem 2.15 below states that if some of the conditions of Theorem 1.8 are satisfied then $\mathbf{I}\mathsf{Gws}_G$ can be axiomatized by 7 axiom schemata. Before formulating the theorem, we give explicitly these as schemata $(*1)-(*7)$ in Definition 2.13 below.

Convention 2.12. *Throughout, $T \subseteq {}^\omega\omega$ denotes a semigroup under $\circ$, unless stated otherwise. Further, T is generated by G (so $G \subseteq T$ etc.), that is, $T = [G]$ (using the notation introduced in subsection 1.1.2).* ◀

Let us recall from subsection 1.1.4 that for any $G \subseteq {}^\omega\omega$, G^* denotes the set of words (i.e., finite sequences) over G, and for any $w \in G^*$, $\hat{w}$ is the image of w at the natural anti–homomorphism *hat* extending $\mathrm{Id} : G \rightarrowtail [G]$; further

$$\Sigma_G = \{\mathsf{s}_w(x) = \mathsf{s}_u(x) : u, w \in G^* \text{ and } \hat{u} = \hat{w}\} ,$$

where s_w, for $w = \langle\tau_1, \ldots, \tau_n\rangle \in G^*$, was defined to be $\mathsf{s}_{\tau_n} \ldots \mathsf{s}_{\tau_1}(x)$ (cf. Definition 1.6). In this section we will work with algebras in the similarity class of Gws_G for some fixed G, and we will assume or prove that Σ_G holds for all our algebras (not only for the ones in $\mathbf{I}\mathsf{Gws}_G$). Therefore, in all our algebras with unit V, $\mathrm{S}_\tau^{[V]}$ will be definable by the Gws_G term s_τ for every $\tau \in [G]$, cf. Definition 1.6 in subsection 1.2. Recall from Definition 1.6 that s_j^i is an alternative notation for $\mathsf{s}_{[i/j]}$, and $\mathsf{c}_i = \mathsf{s}_{[0,i]}\mathsf{c}_0\mathsf{s}_{[0,i]}$.

DEFINITION 2.13: Let $T \subseteq {}^\omega\omega$ be a subsemigroup generated by some $G \subseteq {}^\omega\omega$. Below we define postulates $(*1)$–$(*7)$. They are formulated in the equational language of Gws_G. The postulates are stated for all $\tau, \nu \in T$ and for those $i, j, k, \ell \in \omega$ for which $\mathsf{c}_i, \mathsf{c}_j, \mathsf{c}_k, \mathsf{s}_\ell^k$ are term definable in Gws_G.

$(*1)$ A finite set of equations axiomatizing Boolean algebras, further
$x \leqslant \mathsf{c}_i x = \mathsf{c}_i\mathsf{c}_i x,\ \ \mathsf{c}_i(x+y) = \mathsf{c}_i x + \mathsf{c}_i y,\ \ \mathsf{c}_i(-\mathsf{c}_i x) = -\mathsf{c}_i x,\ \ \mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$
(that is, c_i is an additive complemented closure operator, and $\mathsf{c}_i, \mathsf{c}_j$ commute).[21]

$(*2)$ $\mathsf{s}_\tau(x+y) = \mathsf{s}_\tau x + \mathsf{s}_\tau y$, $\mathsf{s}_\tau(-x) = -\mathsf{s}_\tau x$ (that is, s_τ is a Boolean homomorphism).

$(*3)$ Σ_G. Hence, in particular,	$\mathsf{s}_\tau\mathsf{s}_\nu x = \mathsf{s}_{\tau\circ\nu}x$.
$(*4)$ $\mathsf{s}_{\mathrm{Id}}x = x$	if $\mathrm{Id} \in T$.
$(*5)$ $\mathsf{s}_\tau\mathsf{c}_i x = \mathsf{s}_{\tau[i/j]}\mathsf{c}_i x$	if $\tau[i/j] \in T$.
$(*6)$ $\mathsf{s}_\tau\mathsf{c}_i x = \mathsf{c}_j\mathsf{s}_\tau x$	if $(\forall k \in \omega)(\tau(k) = j \leftrightarrow k = i)$.
$(*7)$ $\mathsf{c}_k\mathsf{s}_\ell^k x = \mathsf{s}_\ell^k x$	if $k \neq \ell$ and $[k/\ell] \in T$. ◀

Remark 2.14: In Theorem 2.17 below, conditions $(*1)$–$(*7)$ can be replaced with their simpler instances, where c_i and c_j are both c_0. (So for example we may use (2.20) way below instead of $(*6)$.) This follows from the proof of Lemma 2.18. ◀

The proof of Theorem 2.15 below is based partly on generalizations of ideas in Daigneault–Monk [DM63] or in [AGN77]. If we would have not decided to allow at most the C_i's ($i \in \omega$) instead of allowing at least some of the $\mathrm{C}_{(\Gamma)}$'s ($\Gamma \subseteq \omega$) in our algebras, then Theorem 2.15 could be called a generalization of a result in [DM63]. (However, as a consequence of our decision to stick with the C_i's even if $G = {}^\omega\omega$, our algebras remain different from polyadic algebras treated in [DM63].)

[21] These postulates are the same as $(C_0 - C_4)$ of [HMTI, p.162].

Recall the definition of a rich semigroup from the beginning of subsection 1.3.

THEOREM 2.15. *Let $G \subseteq {}^{\omega}\omega$ and $T = [G]$ (the semigroup generated by G in $\langle {}^{\omega}\omega, \circ\rangle$). Assume that T is rich. Then*

$$Mod\,((*1) - (*7)) = \mathbf{I}\mathsf{Gws}_G\,.$$

That is, (∗1)-(∗7) is a complete axiomatization of $\mathbf{I}\mathsf{Gws}_G$.

On the idea of the proof of Theorem 2.15:

Here we would like to refer back to the "idea of proof" outlined below the formulation of Theorem 1.8 (in subsection 1.3). There we wrote that we would construct a neat embedding $\mathrm{S}_\sigma : \mathfrak{A} \rightarrowtail \mathfrak{N}\mathfrak{d}\mathfrak{A}$ for some choice of $\mathfrak{N}\mathfrak{d}$. This construction is in Step 2 below. (Actually, $\mathfrak{N}\mathfrak{d}\mathfrak{A}$ is denoted as $\mathfrak{N}_1$ in Step 2.) We also wrote that, for arbitrarily large $k \in \omega$, by iterating the above step, we will neatly embed $\mathfrak{A}$ into some algebra $\mathfrak{N}_k$ with k extra dimensions. Indeed, this "embedding algebra" is denoted as $\mathfrak{N}_k$ in Step 2 below. (Cf. Claim 2.15.2 for stating that $(\mathrm{S}_\sigma)^k$ is a neat embedding of $\mathfrak{A}$ into $\mathfrak{N}_k$.)

We will have to do some extra work not mentioned in the "idea of proof" in subsection 1.3. Namely, the standard proof of the Neat Embedding Theorem (quoted there) goes by first proving that if $\mathfrak{A}$ is neatly embeddable "into k extra dimensions" for each $k \in \omega$, then $\mathfrak{A}$ is neatly embeddable into ω extra dimensions, too. The proof goes by ultraproducts.

We will have to invent a fine–tuned version of this proof because the ω extra dimensions should behave in such a way that we can finish the representability proof by using them. This problem was not there in the proof of the original Neat Embedding Theorem, because in CA's the operations are simpler than here. This step (neatly embedding into ω extra dimensions which behave nicely) is also in Step 2. (The algebra "ω–neat embedding" $\mathfrak{A}$ is denoted by $\mathfrak{N}$ there.)

As we wrote in subsection 1.3, the next step is reproving that version of the Neat Embedding Theorem which says that neat embeddability into ω extra dimensions implies representability. The earliest proof of the original Neat Embedding Theorem goes by using locally finite dimensional CA's and their representability. This route does not seem to work for our kind of algebras. However, a different method does work, as we will see in Step 3 way below.

Proof of Theorem 2.15:

Assume that G and T are as in the formulation of Theorem 2.15. Then it is straightforward to check that $\mathbf{I}\mathsf{Gws}_G \models (*1) - (*7)$. Therefore it is enough to prove that $\mathfrak{A} \models (*1) - (*7)$ implies $\mathfrak{A} \in \mathbf{I}\mathsf{Gws}_G$.

So we assume $\mathfrak{A} \models (*1) - (*7)$. By $(*3)$, $\mathfrak{A}$ is polynomially equivalent with an algebra similar to Gws_T's. (Cf. Definition 1.6 in subsection 1.2.) Further, clearly, Gws_G and Gws_T are polynomially equivalent. Therefore in proving this theorem, we may pretend that $G = T$. This is what we will do throughout this proof. Assume that the semigroup T is rich, and that the algebra $\mathfrak{A} = \langle A, +, -, \mathsf{c}_0, \mathsf{s}_\tau\rangle_{\tau\in T}$ satisfies $(*1) - (*7)$. We want to show that $\mathfrak{A}$ is representable, that is that $\mathfrak{A} \in \mathbf{I}\mathsf{Gws}_T$.

Step 1. We may assume $|A| \leqslant \omega$ and $|T| \leqslant \omega$ because of the following standard universal algebraic argument. Let

$$\mathcal{T} \overset{\text{def}}{=} \{S \subseteq T : S \text{ is a countable rich subsemigroup of } T\}\,, \text{ and let}$$
$$I \overset{\text{def}}{=} \{\langle \mathfrak{B}, S\rangle : S \in \mathcal{T}, \mathfrak{B} \subseteq \langle A, +, -, \mathsf{c}_0, \mathsf{s}_\tau\rangle_{\tau\in S}, |B| \leqslant \omega\}\,.$$

We will prove that

$$(\forall \langle \mathfrak{B}, S\rangle \in I)\, \mathfrak{B} \in \mathbf{I}\mathsf{Gws}_S \quad \text{implies} \quad \mathfrak{A} \in \mathbf{I}\mathsf{Gws}_T\,.$$

So assume that $\mathfrak{B} \in \mathbf{I}\mathsf{Gws}_S$ for each $\langle \mathfrak{B}, S \rangle \in I$. Let $\langle \mathfrak{B}, S \rangle \in I$ be arbitrary but fixed. Let W denote the unit of $\mathfrak{B}$. Let $\mathfrak{B}^+ = \langle \mathcal{P}(W), \cup, {}_W\!\sim, \mathrm{C}_0^{[W]}, \mathrm{S}_\tau^{[W]} : \tau \in T \rangle$. Then $\mathfrak{B}^+$ has the similarity type of Gws_T's, and $\mathfrak{B}$ is isomorphically embeddable into a reduct of $\mathfrak{B}^+$. Let F be any ultrafilter on I such that

$$\text{for each } \langle \mathfrak{B}, S \rangle \in I \text{ we have } \{\langle \mathfrak{C}, Q \rangle \in I : \mathfrak{B} \subseteq \mathfrak{C}, S \subseteq Q\} \in F\,.$$

For any $i = \langle \mathfrak{B}, S \rangle \in I$, $\mathfrak{B}_i \stackrel{\text{def}}{=} \mathfrak{B}^+$. Let $\mathfrak{P} \stackrel{\text{def}}{=} \mathbf{P}_{i \in I} \mathfrak{B}_i / F$. Then $\mathfrak{A} \rightarrowtail \mathfrak{P}$, e.g. the embedding $a \mapsto \bar{a}/F$ ($a \in A$, and $\bar{a}_{\langle \mathfrak{B}, S \rangle} = \begin{cases} a \text{ if } a \in B \\ 0 \text{ otherwise} \end{cases}$) is an isomorphic embedding, because for any finite subsets $A_0 \subseteq A$ and $T_0 \subseteq T$ there exists $\langle \mathfrak{B}, S \rangle \in I$ such that $A_0 \subseteq B$ and $T_0 \subseteq S$.

Corollary 2.6 (in subsection 2.1) yields that $\mathfrak{P}$ is embeddable into some $\mathfrak{P}^+ = \langle \mathcal{P}(V), \cup, {}_V\!\sim, \mathrm{C}_0^{[V]}, \mathrm{S}_\tau^{[V]} : \tau \in T \rangle$, where V is a Gws_ω–unit. Next we prove that V is a Gws_T–unit. For this it is sufficient to check that $\mathrm{S}_\tau^{[V]}(V) = V$ for each $\tau \in T$. This holds, because for any $\tau \in T$, $\{i \in I : \mathfrak{B}_i \models \mathsf{s}_\tau 1 = 1\} \in F$, therefore, by Łos' lemma, we have $\mathfrak{P} \models \mathsf{s}_\tau 1 = 1$, and thus $\mathrm{S}_\tau^{[V]}(V) = V$.

Thus $\mathfrak{A} \rightarrowtail \mathfrak{P} \in \mathbf{I}\mathsf{Gws}_T$, and it is closed under taking isomorphic copies of subalgebras by the definition of Gws_G, thus $\mathfrak{A} \in \mathbf{I}\mathsf{Gws}_T$.

Step 2. We construct an algebra $\mathfrak{R}$ and a neat embedding $e : \mathfrak{A} \longrightarrow \mathfrak{R}$; $\mathfrak{R}$ and e will be used for representing $\mathfrak{A}$ in Step 3.

For an arbitrary set X disjoint from ω and for any $\tau \in {}^\omega\omega$, we define $bar(\tau, X) \stackrel{\text{def}}{=} \tau \cup \mathrm{Id}_X$. When X is known from context, we simply write $\bar{\tau}$ instead of $bar(\tau, X)$ for brevity.

Let $\alpha \stackrel{\text{def}}{=} \omega + \omega$, $M \stackrel{\text{def}}{=} \alpha \smallsetminus \omega$, $\bar{\tau} = bar(\tau, M) = \tau \cup \mathrm{Id}_M$ for every $\tau \in T$. Thus $\bar{\tau} \in {}^\alpha\alpha$. We drew such a $\bar{\tau}$ in two different ways on Figure 2.10.

$$\bar{\tau} = \tau \cup \mathrm{Id}_M \in {}^\alpha\alpha, \qquad \tau \in T$$

a)

α

M ω

Id_M τ

M ω

---

b)

Id_M τ

M ω

α

Figure 2.10

Let Q be the subsemigroup generated by $\left(\{\bar\tau : \tau \in T\} \cup \bigcup_{i,j\in\alpha}\{[i/j],[i,j]\}\right)$ in $\langle {}^{\alpha}\alpha, \circ\rangle$. It is easy to see that for any $\gamma \in Q$, $\gamma \restriction M$ is "almost everywhere" identity, moreover,

$$(\forall\gamma \in Q)\,|\,\{m \in M : \gamma(m) \neq m \text{ or } (\exists k \neq m)\gamma(k) = m\}\,| < \omega\,. \tag{2.7}$$

We start constructing an algebra $\mathfrak{R} = \langle R, +, -, \mathsf{c}_0^{\mathfrak{R}}, \mathsf{s}_\gamma^{\mathfrak{R}}\rangle_{\gamma\in Q}$ with the following properties:

(1) $\mathfrak{A}$ is a subalgebra of $\langle R, +, -, \mathsf{c}_0^{\mathfrak{R}}, \mathsf{s}_{\bar\tau}^{\mathfrak{R}}\rangle_{\tau\in T}$,
(2) $(\forall x \in A)(\forall m \in M)\mathsf{c}_m^{\mathfrak{R}}x = x$ whenever c_m is definable from the basic operations of $\mathfrak{R}$,
(3) $\mathfrak{R}$ satisfies $(*1) - (*7)$ formulated for α instead of ω, and A generates $\mathfrak{R}$.

First, to arbitrary $n \in \omega$, we construct an $\mathfrak{R}_n$ with properties (1)–(3), but "using $\omega + n$ instead of $\omega + \omega$". Then we will construct $\mathfrak{R}$ from these $\mathfrak{R}_n$'s, using ultraproduct.

Remember that, since we assumed that T is rich, we have special elements $\sigma, \pi \in T$ with property (1.5) (see the beginning of subsection 1.3). Let $R_n \stackrel{\text{def}}{=} \omega \smallsetminus Rng(\sigma^n)$, for any $n \in \omega$. Then $0 \subset R_1 \subset R_2 \subset \ldots$ by our hypothesis $Rng(\sigma) \neq \omega$ inside of (1.5), hence $|R_n| \geqslant n$.

Let $n \in \omega$ be arbitrary but fixed. Let $\alpha_n \stackrel{\text{def}}{=} \omega + |R_n|$, $M_n \stackrel{\text{def}}{=} \alpha_n \smallsetminus \omega$, $\bar\tau = bar(\tau, M_n) = \tau \cup \mathrm{Id}_{M_n}$ for every $\tau \in T$. Let $k_n : \alpha_n \rightarrowtail\!\!\!\!\rightarrow \omega$ be such that $k_n \restriction \omega = \sigma^n$, see Figure 2.11. (Such a k_n exists because σ^n is one–one by Lemma 1.7 (iv).)

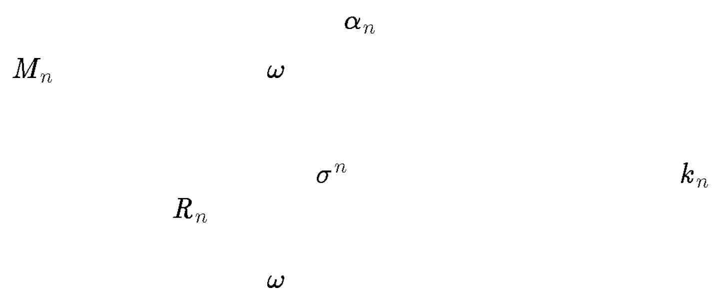

Figure 2.11

Let Q_n be the subsemigroup generated by $\left(\{\bar\tau : \tau \in T\} \cup \bigcup_{i,j\in\alpha_n}\{[i/j],[i,j]\}\right)$ in $\langle {}^{\alpha_n}\alpha_n, \circ\rangle$. For any $\nu \in Q_n$ we define

$$\nu' \stackrel{\text{def}}{=} k_n \circ \nu \circ k_n^{-1}\,.$$

See Figure 2.12.

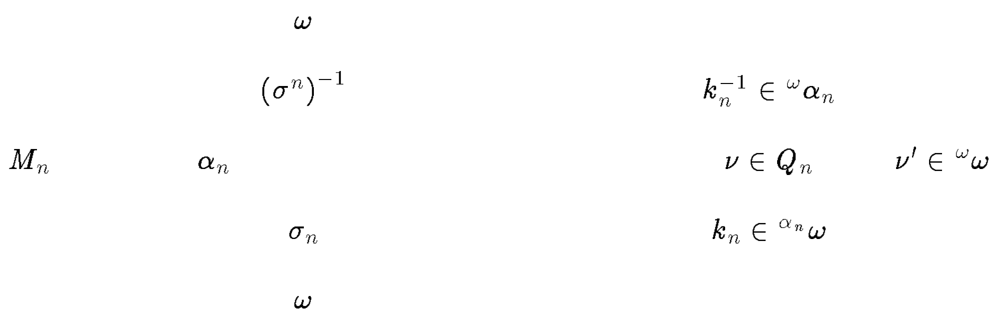

Figure 2.12

Claim 2.15.1. *The function $'$ is a homomorphism between $\langle Q_n, \circ\rangle$ and $\langle T, \circ\rangle$; k_n is an isomorphic embedding of $\langle \alpha_n, \nu\rangle_{\nu\in Q_n}$ into $\langle \omega, \nu'\rangle_{\nu\in Q_n}$ (ν corresponds to ν'). In particular, (i), (ii), moreover, (iii), (iv) below hold for every $\nu, \mu \in Q_n$ and $i, j \in \alpha_n$.*

(i) $\nu' \in T$.

(ii) $(\nu \circ \mu)' = \nu' \circ \mu'$.

(iii) *Recall from subsection 1.1.2 that*

$$\nu[i/j](x) = \begin{cases} j & \text{if } x = i \\ \nu(x) & \text{otherwise}. \end{cases}$$

Now $(\nu[i/j])' = \nu'[k_n(i)/k_n(j)]$.

(iv) $\sigma^n \circ \tau = \bar{\tau}' \circ \sigma^n$ *if* $\tau \in T$.

Proof: (ii)–(iv) are easy to check, using the definition of $'$. To show (i), it is enough to show that $\nu' \in T$ for every generator ν of Q_n, by (ii).

Let $\tau \in T$. $\bar{\tau}' = (\tau \cup \mathrm{Id}_{M_n})'$ is illustrated on Figure 2.13 (for $n = 2$). It is easy to check that $\bar{\tau}' = (\sigma^n \circ \tau \circ \pi^n)[R_n/\mathrm{Id}_{M_n}]$, and the latter belongs to T by our hypothesis (1.5) on T ((1.5) is in the definition of a rich semigroup).

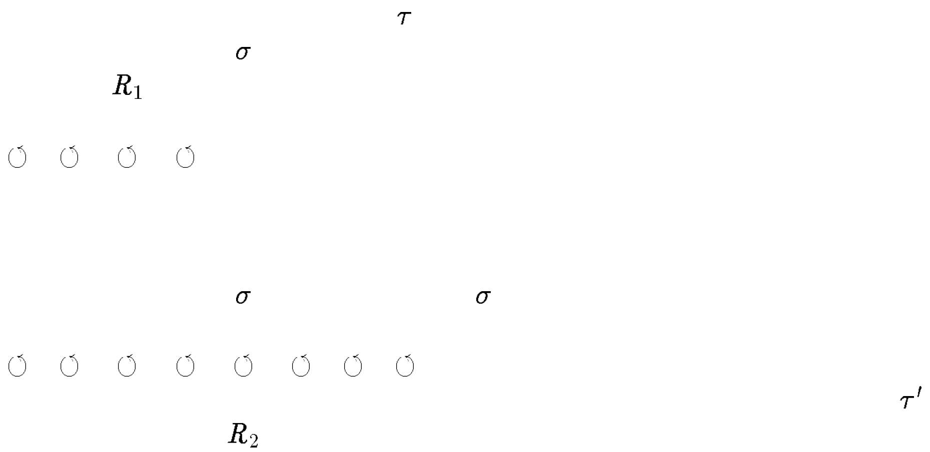

Figure 2.13

If $i, j \in \alpha_n$ then $[i/j]' = k_n \circ [i/j] \circ k_n^{-1} = [k_n(i)/k_n(j)] \in T$ by (1.4) (in the definition of a rich semigroup). We have proved (i).

Now, $'$: $\langle Q_n, \circ\rangle \longrightarrow \langle T, \circ\rangle$ is a homomorphism by (i) and (ii). k_n is one–one by definition, further $\nu'(k_n(i)) = k_n \circ \nu \circ k_n^{-1}(k_n(i)) = k_n\left(\nu\left(k_n^{-1}(k_n(i))\right)\right) = k_n(\nu(i))$, proving that k_n is a homomorphism. QED (Claim 2.15.1) ∎

We define $\mathfrak{R}_n \stackrel{\text{def}}{=} \langle A, +, -, \mathsf{c}_0^{\mathfrak{R}_n}, \mathsf{s}_\tau^{\mathfrak{R}_n}\rangle_{\tau\in Q_n}$ as follows:

$$\mathsf{c}_0^{\mathfrak{R}_n} \stackrel{\text{def}}{=} \mathsf{c}_{k_n 0}^{\mathfrak{A}} \quad \text{and}$$
$$\mathsf{s}_\tau^{\mathfrak{R}_n} \stackrel{\text{def}}{=} \mathsf{s}_{\tau'}^{\mathfrak{A}} \quad \text{for any } \tau \in Q_n\,.$$

Claim 2.15.2.

(i) $\mathsf{c}_i^{\mathfrak{R}_n} = \mathsf{c}_{k_n i}^{\mathfrak{A}}$ *for all* $i \in \alpha_n$.

(ii) *Let* $e_n \stackrel{\text{def}}{=} \mathsf{s}_{\sigma^n}^{\mathfrak{A}}$. *Then* $e_n : \mathfrak{A} \rightarrowtail \langle A, +, -, \mathsf{c}_0^{\mathfrak{R}_n}, \mathsf{s}_{\bar{\tau}}^{\mathfrak{R}_n} \rangle_{\tau \in T}$ *is such that*

$$(\forall m \in M_n)(\forall x \in A)\mathsf{c}_m^{\mathfrak{R}_n} e_n x = e_n x \, .$$

(iii) $\mathfrak{R}_n \models (*1) - (*7)$.

Proof of (i): Recall from Definition 1.6 (ii) (see the end of subsection 1.2) that $\mathsf{p}_{ij} \stackrel{\text{def}}{=} \mathsf{s}_{[i,j]}$ $(i, j \in \omega)$. Instead of p_{ij}, we sometimes write $\mathsf{p}_{i,j}$ for better readability.

$$\mathsf{c}_i^{\mathfrak{R}_n} x \underset{\substack{\uparrow \\ \text{def. of } \mathsf{C}_i}}{=} \mathsf{p}_{0i}^{\mathfrak{R}_n} \mathsf{c}_0^{\mathfrak{R}_n} \mathsf{p}_{0i}^{\mathfrak{R}_n} x \underset{\substack{\uparrow \\ \text{def. of } \mathfrak{R}_n}}{=} \mathsf{p}_{k_n 0, k_n i}^{\mathfrak{A}} \mathsf{c}_{k_n 0}^{\mathfrak{A}} \mathsf{p}_{k_n 0, k_n i}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ (*6)}}{=} \mathsf{c}_{k_n i}^{\mathfrak{A}} \mathsf{p}_{k_n 0, k_n i}^{\mathfrak{A}} \mathsf{p}_{k_n 0, k_n i}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ (*3),(*4)}}{=} \mathsf{c}_{k_n i}^{\mathfrak{A}} x.$$

(ii): e_n is a Boolean homomorphism by $(*2)$.

$$e_n \left(\mathsf{c}_0^{\mathfrak{A}} x \right) = \mathsf{s}_{\sigma^n}^{\mathfrak{A}} \mathsf{c}_0^{\mathfrak{A}} x \underset{\substack{\uparrow \\ (*6)}}{=} \mathsf{c}_{k_n 0}^{\mathfrak{A}} \mathsf{s}_{\sigma^n}^{\mathfrak{A}} x = \mathsf{c}_0^{\mathfrak{R}_n} e_n x \, .$$

We could use $(*6)$ because σ^n is one–one by Lemma 1.7 (ii).

$$e_n \left(\mathsf{s}_\tau^{\mathfrak{A}} x \right) \underset{\substack{\uparrow \\ \text{def.}}}{=} \mathsf{s}_{\sigma^n}^{\mathfrak{A}} \mathsf{s}_\tau^{\mathfrak{A}} x \underset{\substack{\uparrow \\ (*3)}}{=} \mathsf{s}_{\sigma^n \circ \tau}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \text{Cl.2.15.1(iv) and } (*3)}}{=} \mathsf{s}_{\bar{\tau}'}^{\mathfrak{A}} \mathsf{s}_{\sigma^n}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \text{def.}}}{=} \mathsf{s}_{\bar{\tau}'}^{\mathfrak{A}} e_n x = \mathsf{s}_{\bar{\tau}}^{\mathfrak{R}_n} e_n x \, .$$

We have seen that e_n is a homomorphism. By (1.5) we have $\pi^n \circ \sigma^n = \mathrm{Id}$, hence

$$\mathsf{s}_{\pi^n} e_n x = \mathsf{s}_{\pi^n} \mathsf{s}_{\sigma^n} x \underset{\substack{\uparrow \\ (*3),(*4)}}{=} x \, ,$$

thus by $(*2)$ we have $e_n x \neq 0$ if $x \neq 0$, i.e. e_n is one–one.

Let $m \in M_n$. Then

$$\begin{aligned} \mathsf{c}_m^{\mathfrak{R}_n} e_n x &\underset{\substack{\uparrow \\ \text{Cl.2.15.2(i)}}}{=} \mathsf{c}_{k_n m}^{\mathfrak{A}} \mathsf{s}_{\sigma^n}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ k_n m \notin Rng(\sigma^n) \Rightarrow \\ [k_n m / j] \circ \sigma^n = \sigma^n, \\ (*3)}}{=} \mathsf{c}_{k_n m}^{\mathfrak{A}} \left(\mathsf{s}_{\sigma^n 0}^{k_n m} \right)^{\mathfrak{A}} (\mathsf{s}_{\sigma^n})^{\mathfrak{A}} x \underset{\substack{\uparrow \\ (*7), k_n m \neq \sigma^n 0}}{=} \\ &= \left(\mathsf{s}_{\sigma^n 0}^{k_n m} \right)^{\mathfrak{A}} (\mathsf{s}_{\sigma^n})^{\mathfrak{A}} x \underset{\substack{\uparrow \\ k_n m \notin Rng(\sigma^n) \Rightarrow \\ [k_n m / j] \circ \sigma^n = \sigma^n, \\ (*3)}}{=} e_n x \, . \end{aligned}$$

(iii): $(*1), (*2)$ hold for $\mathfrak{R}_n$, since they hold for $\mathfrak{A}$, by Claim 2.15.2 (i) and by using the definition of $\mathfrak{R}_n$. E.g.,

$$\mathsf{c}_i^{\mathfrak{R}_n} \mathsf{c}_j^{\mathfrak{R}_n} x \underset{\substack{\uparrow \\ \text{Cl.2.15.2(i)}}}{=} \mathsf{c}_{k_n i}^{\mathfrak{A}} \mathsf{c}_{k_n j}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \mathfrak{A} \models (*1)}}{=} \mathsf{c}_{k_n j}^{\mathfrak{A}} \mathsf{c}_{k_n i}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \text{Cl.2.15.2(i)}}}{=} \mathsf{c}_j^{\mathfrak{R}_n} \mathsf{c}_i^{\mathfrak{R}_n} x \, .$$

$(*3)$:
$$\mathsf{s}_\tau^{\mathfrak{R}_n} \mathsf{s}_\nu^{\mathfrak{R}_n} x \underset{\substack{\uparrow \\ \text{def. of } \mathfrak{R}_n}}{=} \mathsf{s}_{\tau'}^{\mathfrak{A}} \mathsf{s}_{\nu'}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \mathfrak{A} \models (*3)}}{=} \mathsf{s}_{\tau' \circ \nu'}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \text{Cl.2.15.1(ii)}}}{=} \mathsf{s}_{(\tau \circ \nu)'}^{\mathfrak{A}} x \underset{\substack{\uparrow \\ \text{def. of } \mathfrak{R}_n}}{=} \mathsf{s}_{\tau \circ \nu}^{\mathfrak{R}_n} x \, .$$

$(*4)$ holds since $(\mathrm{Id}_{\alpha_n})' = \mathrm{Id}_\omega$.
$(*5)$:

$$\mathsf{s}^{\mathfrak{N}_n}_\tau \mathsf{c}^{\mathfrak{N}_n}_i x \underset{\text{def. and Cl.2.15.2(i)}}{\underset{\uparrow}{=}} \mathsf{s}^{\mathfrak{A}}_{\tau'} \mathsf{c}^{\mathfrak{A}}_{k_n i} x \underset{\mathfrak{A}\models(*5)}{\underset{\uparrow}{=}} \mathsf{s}^{\mathfrak{A}}_{\tau'[k_n i/k_n j]} \mathsf{c}^{\mathfrak{A}}_{k_n i} x \underset{\text{Cl.2.15.1(iii)}}{\underset{\uparrow}{=}} \mathsf{s}^{\mathfrak{A}}_{(\tau[i/j])'} \mathsf{c}^{\mathfrak{A}}_{k_n i} x \underset{\text{def.}}{\underset{\uparrow}{=}} \mathsf{s}^{\mathfrak{N}_n}_{\tau[i/j]} \mathsf{c}^{\mathfrak{N}_n}_i x\,.$$

$(*6)$: Assume τ, i and j are such that $(\forall \ell \in \alpha_n)\big(\tau(\ell) = j \leftrightarrow \ell = i\big)$.
Then $\tau', k_n i$ and $k_n j$ also satisfy this. Now

$$\mathsf{s}^{\mathfrak{N}_n}_\tau \mathsf{c}^{\mathfrak{N}_n}_i x \underset{\text{def. and Cl.2.15.2(i)}}{\underset{\uparrow}{=}} \mathsf{s}^{\mathfrak{A}}_{\tau'} \mathsf{c}^{\mathfrak{A}}_{k_n i} x \underset{\mathfrak{A}\models(*6)}{\underset{\uparrow}{=}} \mathsf{c}^{\mathfrak{A}}_{k_n j} \mathsf{s}^{\mathfrak{A}}_{\tau'} x \underset{\text{def. and Cl.2.15.2(i)}}{\underset{\uparrow}{=}} \mathsf{c}^{\mathfrak{N}_n}_j \mathsf{s}^{\mathfrak{N}_n}_\tau x\,.$$

$(*7)$: $$\mathsf{c}^{\mathfrak{N}_n}_i \left(\mathsf{s}^i_j\right)^{\mathfrak{N}_n} x \underset{\text{def. and Cl.2.15.2(i)}}{\underset{\uparrow}{=}} \mathsf{c}^{\mathfrak{A}}_{k_n i} \left(\mathsf{s}^{k_n i}_{k_n j}\right)^{\mathfrak{A}} x \underset{\mathfrak{A}\models(*7)}{\underset{\uparrow}{=}} \left(\mathsf{s}^{k_n i}_{k_n j}\right)^{\mathfrak{A}} x \underset{\text{def. and Cl.2.15.1(iii)}}{\underset{\uparrow}{=}} \left(\mathsf{s}^i_j\right)^{\mathfrak{N}_n} x\,.$$

QED (Claim 2.15.2) ∎

Now we start constructing $\mathfrak{N}$ from the $\mathfrak{N}_n$'s. Let $n \in \omega$. Recall that we defined Q at the beginning of Step 2. For any $\tau \in Q$ we define

$$[\tau]_n \stackrel{\text{def}}{=} \tau \restriction (M_n \cup \omega)\,.$$

Now, by using property (2.7) of Q, one can easily check the following:

(2.8) $(\forall \tau \in Q)(\exists n \in \omega)(\forall m \geqslant n)[\tau]_m \in Q_m$ and
$$(\forall \tau, \nu \in Q)(\exists n \in \omega)(\forall m \geqslant n)\big([\tau]_m \circ [\nu]_m = [\tau \circ \nu]_m\big)\,.$$

For every $n \in \omega$ let $\mathfrak{N}^+_n = \langle A, +, -, \mathsf{c}^{\mathfrak{N}_n}_0, \mathsf{s}^n_\tau\rangle_{\tau\in Q}$ be an algebra such that $\mathsf{s}^n_\tau = \mathsf{s}^{\mathfrak{N}_n}_{[\tau]_n}$ if $[\tau]_n \in Q_n$. (Note that this means that if $[\tau]_n \notin Q_n$ then s^n_τ is arbitrary but fixed.) Let F be any nonprincipal ultrafilter on ω, and let

$$\mathfrak{N}^+ \stackrel{\text{def}}{=} \mathbf{P}_{n\in\omega}\mathfrak{N}^+_n/F\,.$$

Now $\mathfrak{N}^+$ satisfies $(*1) - (*7)$, and there is an

$$e : \mathfrak{A} \rightarrowtail \langle R^+, +, -, \mathsf{c}^{\mathfrak{N}^+}_0, \mathsf{s}^{\mathfrak{N}^+}_{\bar\tau}\rangle_{\tau\in T}$$

such that $\mathsf{c}^{\mathfrak{N}^+}_m ex = ex$ for all $x \in A$ and $m \in M$, by Claim 2.15.2 (ii), (2.8), and the definition of $\mathfrak{N}^+_n$. Let $\mathfrak{N}$ be the subalgebra of $\mathfrak{N}^+$ generated by the e–image of A. Clearly, we may assume that $\mathfrak{A}$ is a subalgebra of the appropriate reduct $\mathfrak{R}\mathfrak{d}\mathfrak{N}$ of $\mathfrak{N}$ (instead of $\mathfrak{A} \rightarrowtail \mathfrak{R}\mathfrak{d}\mathfrak{N}$), see Figure 2.14. Further $\mathfrak{N}$ satisfies the other prescribed properties too, that is $\mathfrak{N} \models (*1) - (*7)$, $(\forall x \in A)(\forall m \in M)\mathsf{c}^{\mathfrak{N}}_m x = x$, and $|R| \leqslant \omega$. We have completed Step 2.

1

$\mathfrak{R}^+ = \mathbf{P}\mathfrak{R}_n^+/F$

$\mathfrak{R}$

$\mathfrak{A}$

0

Figure 2.14

For any $x \in R$, let Δx denote the *dimension set* of x (cf. [HMT, Def.1.6.1, p.199]), that is

$$\Delta x \stackrel{\text{def}}{=} \{i \in \alpha : \mathsf{c}_i x \neq x \text{ and } \mathsf{c}_i \text{ is definable in } \mathfrak{R}\}\,.$$

In Step 3 we will use the following consequence of properties (1)–(3) of $\mathfrak{R}$:

COROLLARY 2.15.3. *Let $\mathfrak{R}$ be as in Step 2. Then $(\forall x \in R)|M \cap \Delta x| < \omega$.*

Proof: Since, by (2), we have $(\forall x \in A)|M \cap \Delta x| < \omega$, it is enough to prove that for any operation f of $\mathfrak{R}$,

$$|M \cap \Delta x| < \omega \Longrightarrow |M \cap \Delta(fx)| < \omega \tag{2.9}$$

by (3).

So let $x \in R$ be arbitrary but fixed. Assume $|M \cap \Delta x| < \omega$.

- Let $\tau \in Q$. Let $D_\tau \stackrel{\text{def}}{=} \{m \in M : \tau(m) \neq m \text{ or } (\exists k \neq m)\tau(k) = m\}$. Then $|D_\tau| < \omega$ by (2.7). By $|M \cap \Delta x| < \omega$, $|(M \cap \Delta x) \cup D_\tau| < \omega$ as well. Thus, to see $|M \cap \Delta(\mathsf{s}_\tau x)| < \omega$, it is enough to show that $M \cap \Delta(\mathsf{s}_\tau x) \subseteq (M \cap \Delta x) \cup D_\tau$.

  Assume $i \in M$ is such that $i \notin (M \cap \Delta x) \cup D_\tau$. Then $i \in (M \smallsetminus D_\tau) \smallsetminus \Delta x$. Then $\tau(i) = \tau^{-1}(i) = i$, and thus, applying $(*6)$, we get

$$\mathsf{c}_i \mathsf{s}_\tau x \underset{\substack{\uparrow \\ (*6)}}{=} \mathsf{s}_\tau \mathsf{c}_i x \underset{\substack{\uparrow \\ i \notin \Delta x}}{=} \mathsf{s}_\tau x\,,$$

  proving that $i \notin M \cap \Delta(\mathsf{s}_\tau x)$. This proves (2.9) for the case $f = \mathsf{s}_\tau$.
- Let $j \in \alpha$. If $i \in M$ is such that $i \notin \Delta x$ then

$$\mathsf{c}_i \mathsf{c}_j x \underset{\substack{\uparrow \\ (*1)}}{=} \mathsf{c}_j \mathsf{c}_i x \underset{\substack{\uparrow \\ i \notin \Delta x}}{=} \mathsf{c}_j x\,,$$

  proving $i \notin \Delta(\mathsf{c}_j x)$, and thus (2.9) for $f = \mathsf{c}_j$.
- Using $(*2)$, (2.9) is easy to prove for the Boolean operations as well.

QED (Corollary 2.15.3) ∎

Step 3. We will now represent $\mathfrak{A}$ by using the operations of $\mathfrak{R}$. Recall that $\alpha = \omega + \omega$. To any $\tau \in T$, ${}^{\omega}(\alpha)^{(\tau)}$ denotes the weak Cartesian space $\left\{\nu \in {}^{\omega}\alpha : \{i \in \omega : \nu(i) \neq \bar{\tau}(i)\} \text{ is finite}\right\}$. Let

$$V \stackrel{\text{def}}{=} \bigcup \left\{ {}^{\omega}(\alpha)^{(\tau)} : \tau \in T \right\} .$$

Claim 2.15.4. $(\forall \tau \in V)\bar{\tau} \in Q$.

Proof: Let $\nu \in V$, say $\nu \in {}^{\omega}(\alpha)^{(\tau)}$ for some $\tau \in T$. Let

$$J \stackrel{\text{def}}{=} \{j \in \omega : \nu_j \neq \tau_j\}, \qquad J_1 \stackrel{\text{def}}{=} \{j \in J : \nu_j \in \omega\}, \qquad J_2 \stackrel{\text{def}}{=} \{j \in J : \nu_j \notin \omega\} .$$

Let $\mu \stackrel{\text{def}}{=} \tau[j/\nu_j : j \in J_1]$. Then $\mu \in T$ by (1.4), and $\nu = \mu[j/\nu_j : j \in J_2]$. Now $\bar{\nu} = \bar{\mu} \circ [j_1/\nu_{j_1}] \circ \cdots \circ [j_k/\nu_{j_k}]$, where $J_2 = \{j_1, \ldots, j_k\}$. QED (Claim 2.15.4) ∎

Claim 2.15.5. *Let $a \in A$ be arbitrary but fixed. Then there exists an ultrafilter $H_a \subseteq R$ such that $a \in H_a$ and*

$$(\forall i \in \alpha)(\forall x \in R)\left(\mathsf{c}_i x \in H_a \iff (\exists j \in \alpha)\left(\mathsf{s}_j^i\right)^{\mathfrak{R}} x \in H_a\right) . \tag{2.10}$$

See Figure 2.15.

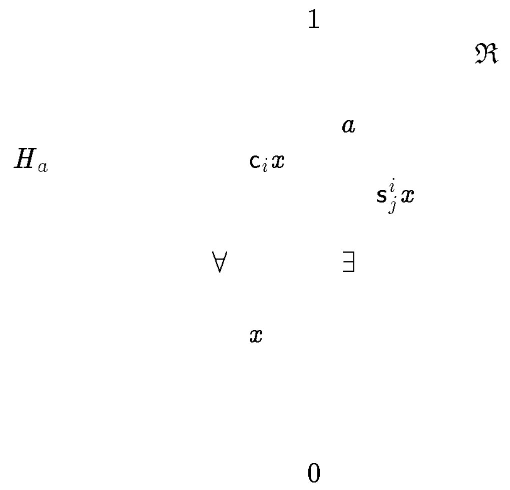

Figure 2.15

The proof below is analogous to the corresponding part of the proof of Thm.A.3 in [AGN77, pp.41–43]. Instead of [AGN77], we could also start out from the ultrafilter P constructed on p.173 of Daigneault–Monk [DM63], see conditions (5), (7) there, which are strongly related to our (2.10) above.

Proof of Claim 2.15.5: Let $a \in A$, $a \neq 0$. Let $x \in R, i \in \alpha$ be arbitrary but fixed.

Claim 2.15.5.1. $(\exists j \in \alpha)(\mathsf{c}_i x \geqslant a \Longrightarrow \mathsf{s}^i_j x \cap a \neq 0)$.

Proof of Claim 2.15.5.1: Let $j \notin \Delta a \cup \Delta x$. Such a $j \in \alpha$ exists by Corollary 2.15.3.
First we prove

$$\mathsf{c}_j \mathsf{s}^i_j \mathsf{c}_j x = \mathsf{c}_i \mathsf{c}_j x \tag{2.11}$$

as follows.

$$\mathsf{s}^i_j \mathsf{c}_j x \underset{\substack{\uparrow \\ (*5),\, \mathsf{p}_{ij} = \mathsf{s}^i_j[j/i], \\ T \text{ is rich}}}{=} \mathsf{p}_{ij} \mathsf{c}_j x \underset{\substack{\uparrow \\ (*6)}}{=} \mathsf{c}_i \mathsf{p}_{ij} x\,,$$

thus

$$\mathsf{c}_j \mathsf{s}^i_j \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i \mathsf{p}_{ij} x \underset{\substack{\uparrow \\ (*6)}}{=} \mathsf{p}_{ij} \mathsf{c}_i \mathsf{c}_j x \underset{\substack{\uparrow \\ (*5)}}{=} \mathsf{c}_i \mathsf{c}_j x\,.$$

Now assume that $\mathsf{c}_i x \geqslant a$ and $\mathsf{s}^i_j x \cap a = 0$. Then

$$\begin{array}{rcl}
\mathsf{s}^i_j x \cap a = 0 & & \\
\Downarrow & \longleftarrow & \mathsf{c}_j 0 = 0 \text{ by } (*2) \\
\mathsf{c}_j(\mathsf{s}^i_j x \cap a) = 0 & & \\
\Downarrow & \longleftarrow & \mathsf{c}_j a = a \text{ and } \mathsf{c}_j(y \cap \mathsf{c}_j z) = \mathsf{c}_j y \cap \mathsf{c}_j z \text{ by } (*1) \\
\mathsf{c}_j \mathsf{s}^i_j x \cap a = 0 & & \\
\Downarrow & \longleftarrow & x = \mathsf{c}_j x \\
\mathsf{c}_j \mathsf{s}^i_j \mathsf{c}_j x \cap a = 0 & & \\
\Downarrow & \longleftarrow & (2.11) \\
\mathsf{c}_i \mathsf{c}_j x \cap a = 0\,. & &
\end{array}$$

Now $0 = \mathsf{c}_i \mathsf{c}_j x \cap a \underset{\substack{\uparrow \\ (*1)}}{\geqslant} \mathsf{c}_i x \cap a = 0$, contradicting $0 < a \leqslant \mathsf{c}_i x$. QED (Claim 2.15.5.1) ∎

We continue proving Claim 2.15.5:
Let $h : \omega \rightarrowtail\!\!\!\!\rightarrow A \times \alpha$ be an arbitrary but fixed bijection (exists since $|A| = \omega$, $|\alpha| = \omega$, and thus $|A \times \alpha| = \omega$). For any $n \in \omega$, we denote $h(n)$ by $\langle x_n, i_n \rangle$ ($x_n \in A, i_n \in \alpha$).

Let $a_0 \overset{\text{def}}{=} a$. To the triple $\langle a_0, x_0, i_0 \rangle$, we associate $a_1 \in A$ as follows. If $\mathsf{c}_{i_0} x_0 \geqslant a_0$ then $(\exists j \in \alpha) \mathsf{s}^{i_0}_j x_0 \cap a_0 \neq 0$ by Claim 2.15.5.1. Let j_0 denote the smallest such j, and let

$$a_1 \overset{\text{def}}{=} \begin{cases} a_0 \cap \mathsf{s}^{i_0}_{j_0} x_0 & \text{if } \mathsf{c}_0 x_0 \geqslant a_0 \\ a_0 \cap -\mathsf{c}_{i_0} x_0 & \text{otherwise.} \end{cases}$$

We repeat this step for $\langle a_1, x_1, i_1 \rangle$ instead of $\langle a_0, x_0, i_0 \rangle$, obtaining an a_2. Then we repeat it again for $\langle a_2, x_2, i_2 \rangle$, and so on. This way we obtain a decreasing ω–chain $a_0 \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant \ldots$ of elements of A. Let F_a be the filter generated by $\{a_i : i \in \omega\}$ (exists since $\{a_i : i \in \omega\}$ has the finite intersection property by Claim 2.15.5.1). Then we have that $a \in F_a$ and $(\forall \langle x, i \rangle \in A \times \alpha)(-\mathsf{c}_i x \in F_a$ or $\mathsf{s}^i_j x \in F_a$ for some $j \in \alpha)$. Therefore any ultrafilter H_a containing F_a satisfies (2.10).
QED (Claim 2.15.5) ∎

We are ready to define our representation. Let $a \in A$ and let $H_a \subseteq R$ be an ultrafilter as in Claim 2.15.5 above. We define, for every $x \in A$,

$$rep_a(x) \stackrel{\text{def}}{=} \left\{\tau \in V : \mathsf{s}^{\mathfrak{R}}_\tau x \in H_a\right\} .$$

Claim 2.15.6. *$rep_a : \mathfrak{A} \longrightarrow \mathfrak{P}V \stackrel{\text{def}}{=} \langle \mathcal{P}(V), \cup, {}_V\!\sim, \mathrm{C}_0^{[V]}, \mathrm{S}_\tau^{[V]}\rangle_{\tau\in T}$ is a homomorphism.*

Proof: rep_a is a Boolean homomorphism (that is $rep_a(x+y) = rep_a(x) \cup rep_a(y)$, $rep_a(-x) = V \smallsetminus rep_a(x)$), since $\mathfrak{R} \models (*2)$ and H_a is an ultrafilter.

$rep_a\left(\mathsf{s}^{\mathfrak{A}}_\tau x\right) = \mathrm{S}_\tau rep_a(x)$ for $\tau \in T$ follows from $\mathfrak{R} \models (*3)$:

$$\nu \in rep_a\left(\mathsf{s}^{\mathfrak{A}}_\tau x\right) \underset{\substack{\uparrow \\ \text{def.}}}{\iff} \mathsf{s}^{\mathfrak{R}}_\nu \mathsf{s}^{\mathfrak{R}}_\tau x \in H_a \underset{\substack{\uparrow \\ (*3) \\ \text{and def. of } ^-}}{\iff} \mathsf{s}^{\mathfrak{R}}_{\overline{\nu\circ\tau}} x \in H_a \underset{\substack{\uparrow \\ \text{def. of} \\ rep_a}}{\iff}$$

$$\nu\circ\tau \in rep_a(x) \underset{\substack{\uparrow \\ \text{def. of} \\ \mathrm{S}_\tau}}{\iff} \nu \in \mathrm{S}_\tau rep_a(x) .$$

It remains to show that $rep_a\left(\mathsf{c}^{\mathfrak{A}}_0 x\right) = \mathrm{C}_0 rep_a(x)$. Now $rep_a(x) \subseteq rep_a\left(\mathsf{c}^{\mathfrak{A}}_0 x\right) = \mathrm{C}_0 rep_a\left(\mathsf{c}^{\mathfrak{A}}_0 x\right)$ by $(*1)$, $(*3)$, $(*5)$, hence $\mathrm{C}_0 rep_a(x) \subseteq rep_a\left(\mathsf{c}_0 x\right)$. Next we will show $rep_a\left(\mathsf{c}_0 x\right) \subseteq \mathrm{C}_0 rep_a(x)$.

Assume $\nu \in rep_a\left(\mathsf{c}_0 x\right)$. Then $\mathsf{s}^{\mathfrak{R}}_\tau \mathsf{c}^{\mathfrak{R}}_0 x \in H_a$, where $\tau \stackrel{\text{def}}{=} \bar{\nu}$. We have to show $(\exists j \in \alpha)\mathsf{s}_{\tau[0/j]} x \in H_a$, since then $\nu[0/j] \in \mathrm{C}_0 rep_a(x)$. Let $m \in \alpha$ be such that $\mathsf{c}_m x = x$ and $(\forall \ell \in \alpha)\big(\tau(\ell) = m \leftrightarrow \ell = m\big)$. Such an m exists, because $x \in A$ and A is "neatly embedded" into a wider algebra of dimension $\alpha = \omega + \omega$.

There is $k \neq m$ such that $\tau[0/m, m/k] \in Q$ can be seen as follows. Let $\ell \in \alpha$, $\ell \neq 0$, $\ell \neq m$, and let $k \stackrel{\text{def}}{=} \tau(\ell)$. Then $k \neq m$ by $\ell \neq m$ and $\tau[0/m, m/k] = \tau \circ [0/m] \circ [m/\ell] \in Q$ by $\tau \in Q$. Also, $\tau[0/m] \in Q$ by $\tau[0/m] = \tau \circ [0/m]$. Now

$$\mathsf{s}_\tau \mathsf{c}_0 x \underset{\substack{\uparrow \\ (*5)}}{=} \mathsf{s}_{\tau[0/m]} \mathsf{c}_0 x \underset{\substack{\uparrow \\ x = \mathsf{c}_m x, \\ (*1),(*5)}}{=} \mathsf{s}_{\tau[0/m,m/k]} \mathsf{c}_0 x \underset{\substack{\uparrow \\ (*6)}}{=} \mathsf{c}_m \mathsf{s}_{\tau[0/m,m/k]} x \in H_a .$$

Then by (2.10), there is a $j \in \alpha$ such that $\mathsf{s}^m_j \mathsf{s}_{\tau[0/m,m/k]} x \in H_a$, and then

$$\mathsf{s}^m_j \mathsf{s}_{\tau[0/m,m/k]} x \underset{\substack{\uparrow \\ (*3)}}{=} \mathsf{s}_{\tau[0/j,m/k]} x \underset{\substack{\uparrow \\ x = \mathsf{c}_m x, (*5) \\ \tau[0/j] \in Q}}{=} \mathsf{s}_{\tau[0/j]} x .$$

In the last step we used $\tau[0/j] \in Q$. This latter can be seen as follows. $\tau = \bar{\nu}$, where $\nu \in V$. Then $\nu[0/j] \in V$ by the definition of V, and then $\overline{\nu[0/j]} \in Q$ by Claim 2.15.4. But $\tau[0/j] = \overline{\nu[0/j]}$ by $\tau = \bar{\nu}$. QED (Claim 2.15.6) ∎

So far we have seen that $rep_a : \mathfrak{A} \longrightarrow \mathfrak{P}V$. Also, $rep_a(a) \neq 0$ by $(*4)$ and $a \in H_a$. Since $a \neq 0$ was chosen arbitrarily, and since V is a Gws_G–unit, this shows that $\mathfrak{A}$ is representable.

QED Theorem 2.15 ∎

Remark 2.16: Theorem 2.15 above implies representability of infinite dimensional polyadic algebras with only finite quantifiers. This is so because in that case $G = {}^I I$ obviously satisfies our conditions, and we did not use the fact that $I = \omega$ in our proof.

Let $|\alpha| \geqslant \kappa > \omega$, κ a cardinal. Consider κ–quasi polyadic algebras introduced in Daigneault–Monk [DM63, p.169], but again restricted to finite quantifiers. Then, by Theorem 2.15 above, these algebras are representable, too. Note that for $\kappa = \omega$ this result becomes false, since not all quasi–polyadic algebras are representable. (Actually, the representable quasi–polyadic algebras are not even finite schema axiomatizable. This was proved in Sain–Thompson [ST91].)

The proof of Theorem 2.15 can be "pushed through" for the cases when infinitary quantifiers are added, but discussing that here would be too much digression (since infinitary quantifiers seem to be quite irrelevant to the subject matter of this paper). ◀

2.3. Finite set of equations axiomatizing $\mathbf{I}\mathsf{Gws}_G$

Next we prove the derivability of conditions $(*1)$–$(*7)$ used in Theorem 2.15 from a finite set Σ of equations. This will be done in Lemma 2.18 below. We highlight the first step of the proof of Lemma 2.18 as Theorem 2.17 since it is interesting in itself as well.

Investigations and results strongly related to Theorem 2.17 below can be found in Craig [Cr74a]; there, more operations are associated to G than here. Therefore the results sometimes have a different character. (Roughly speaking, the extra operations considered in [Cr74a] but not here are the so called conjugates of our s_τ's with $\tau \in G$.)

Let us recall from subsection 1.1.4 that, for $G \subseteq {}^\omega\omega$, Σ_G denotes the set of equations

$$\{\mathsf{s}_w(x) = \mathsf{s}_u(x) : u, w \in G^* \text{ and } \hat{u} = \hat{w}\} .$$

THEOREM 2.17. *Let $G \subseteq {}^\omega\omega$ be finite, and let T be generated by G. Assume that T is rich. Assume further that*

$$\textit{the subsemigroup } T^0 = \{\tau \in T : (\forall i \in \omega)\,(\tau(i) = 0 \leftrightarrow i = 0)\} \textit{ is finitely generated.}$$

Then $\mathbf{I}\mathsf{Gws}_G$ is finitely axiomatizable over Σ_G. That is, there is a finite set Σ_0 of equations such that

$$\mathbf{I}\mathsf{Gws}_G = Mod\,(\Sigma_G \cup \Sigma_0)\ .$$

Proof: Recall that, by Theorem 2.15, $(*1) - (*7)$ is a complete axiomatization of $\mathbf{I}\mathsf{Gws}_G$. The only problem is that $(*1) - (*7)$ is an infinite set. Therefore it is enough to prove that $(*1) - (*7)$ is finitely axiomatizable over Σ_G. Since $\Sigma_G \vdash (*3)$ is obvious, it is enough to prove that $\{(*1), (*2), (*4) - (*7)\}$ is finitely axiomatizable over Σ_G. This will be proved in the proof of Lemma 2.18 below, the following way.

The proof of Lemma 2.18 consists of two parts. The first part is proving item (2.15) there, which amounts to proving that Σ_G is finitely axiomatizable. Let us call this finite axiomatization Σ_1 (a notation *not* introduced in the proof of Lemma 2.18). The rest of the proof of Lemma 2.18 consists of collecting another finite set, call it Σ_0, of axioms such that $\Sigma_1 \cup \Sigma_0 \vdash (*1) - (*7)$. The only assumption of Lemma 2.18 not assumed in Theorem 2.17 (this is (2.12)) is used only in establishing that Σ_1 is finite. That is, the proof of $\Sigma_1 \cup \Sigma_0 \vdash \{(*1), (*2), (*4) - (*7)\}$ in Lemma 2.18 does not use more than the assumptions of Theorem 2.17, which in turn implies that, writing Σ_G in place of Σ_1, that proof yields $\Sigma_G \cup \Sigma_0 \vdash (*1) - (*7)$ under the present assumtions. Since Σ_0 is finite, we are done. ∎

LEMMA 2.18. *Let $T \subseteq {}^\omega\omega$ be a semigroup containing all finite transformations of ω. Assume*

(2.12) *T is finitely presented via some generator set G, and*

(2.13) *the subsemigroup $T^0 = \big\{\tau \in T : (\forall i \in \omega)\big(\tau(i) = 0 \leftrightarrow i = 0\big)\big\}$ is finitely generated.*

Then there is a finite set Σ of equations valid in Gws_G such that

$$\Sigma \models (*1) - (*7)\,.$$

Proof: Assume $T \subseteq {}^\omega\omega$ satisfies all the conditions of Lemma 2.18. Let G be the finite generator set in (2.12).

By our assumption, $[i/j], [i,j] \in T$, $\mathsf{s}^i_j = \mathsf{S}_{[i/j]}$, and $\mathsf{p}_{ij} = \mathsf{S}_{[i,j]}$ are term definable in Gws_G for $i, j \in \omega$. We will use these two kinds of terms. Let us recall that $\mathsf{c}_i x = \mathsf{p}_{0i}\mathsf{c}_0\mathsf{p}_{0i}x$. Thus c_i is a term function, too. Let us postulate

$$\mathsf{c}_0 x = \mathsf{s}^0_k \mathsf{c}_0 x \qquad \text{and} \qquad \mathsf{c}_0 \mathsf{s}^0_k x = \mathsf{s}^0_k x, \qquad \text{for } 0 < k \leqslant 5. \tag{2.14}$$

This means finitely many axioms only, so we are allowed to do this. (During the proof, this "trick" of postulating finitely many equations true in Gws_G is applied several times.) For every defining relation $w = u$ in the finite presentation of T, which exists by (2.12), we postulate $\mathsf{s}_w(x) = \mathsf{s}_u(x)$. Of course, then

(2.15) for any equality $\hat{w} = \hat{u}$ over G true in $T, \mathsf{s}_w(x) = \mathsf{s}_u(x)$ will be provable from our axioms.

Let us check derivability of the most basic cylindric or quasi–polyadic axioms. Let i, j be arbitrary. Then there is $k \notin \{i, j, 0\}$ with $k \leqslant 5$.

$$\mathsf{s}^i_j \mathsf{c}_i x = \mathsf{s}^i_j \mathsf{p}_{i0}\mathsf{c}_0\mathsf{p}_{i0}x \underset{\substack{\uparrow \\ (2.14)}}{=} \mathsf{s}^i_j \mathsf{p}_{i0}\mathsf{s}^0_k\mathsf{c}_0\mathsf{p}_{i0}x \underset{\substack{\uparrow \\ \text{semigroup} \\ \text{equations} \\ (2.15)}}{=} \mathsf{p}_{i0}\mathsf{s}^0_k\mathsf{c}_0\mathsf{p}_{i0}x \underset{\substack{\uparrow \\ (2.14)}}{=} \mathsf{p}_{i0}\mathsf{c}_0\mathsf{p}_{i0}x = \mathsf{c}_i x. \tag{2.16}$$

$$\mathsf{c}_i \mathsf{s}^i_j x = \mathsf{p}_{0i}\mathsf{c}_0\mathsf{p}_{0i}\mathsf{s}^i_j x \underset{\substack{\uparrow \\ \text{semigroup} \\ \text{equations} \\ (2.15)}}{=} \mathsf{p}_{0i}\mathsf{c}_0\mathsf{s}^0_k\mathsf{p}_{0i}\mathsf{s}^i_j x \underset{\substack{\uparrow \\ (2.14)}}{=} \mathsf{p}_{0i}\mathsf{s}^0_k\mathsf{p}_{0i}\mathsf{s}^i_j x \underset{\substack{\uparrow \\ \text{semigroup} \\ \text{equations} \\ (2.15)}}{=} \mathsf{s}^i_j x. \tag{2.17}$$

We postulate the Boolean axioms, and for each $\tau \in G$, that s_τ is a Boolean endomorphism. Further, we postulate that c_0 is an additive and complemented (i.e., $\mathsf{c}_0 - \mathsf{c}_0 x = -\mathsf{c}_0 x$) closure operator. Hence

(2.18) $\quad \mathsf{c}_i$ is an additive and complemented closure operator, for every $i \in \omega$

(that is, $(C_0) - (C_3)$ of [HMTI] holds). This is easy to prove. Also

(2.19) $\quad \mathsf{s}_w$ is a Boolean endomorphism for each $w \in G^*$.

To each word $w \in G^*$, let $\hat{w} \in T$ be the element of $T \subseteq {}^\omega\omega$ associated naturally to w. Let G^0 be the finite generator set of T^0 (cf. condition (2.13) of the theorem). To each $\tau \in G^0$ we postulate $\mathsf{c}_0\mathsf{s}_\tau x = \mathsf{s}_\tau\mathsf{c}_0 x$. Then we prove that

(2.20) for each word $w \in G^*$ with $(\forall i \in \omega)\big(\hat{w}(i) = 0 \leftrightarrow i = 0\big)$
$\mathsf{c}_0\mathsf{s}_w x = \mathsf{s}_w\mathsf{c}_0 x$ follows from our finite set of postulates,

as follows. Let $w \in G^*$ with $(\forall i \in \omega)(\hat{w}(i) = 0 \leftrightarrow i = 0)$. Then $\hat{w} \in T^0$, and thus $\hat{w} = g_1 \circ \cdots \circ g_n$ for some $g_1, \ldots, g_n \in G^0$ (by $T^0 = [G^0]$). For every $i \in \{1, \ldots, n\}$, $g_i = \hat{w}_i$ for some $w_i \in G^*$. Thus

$$\hat{w} = \hat{w}_1 \circ \cdots \circ \hat{w}_n. \tag{2.21}$$

Now

$$\mathsf{s}_w x \underset{\substack{\uparrow \\ (2.15)}}{=} \mathsf{s}_{\hat{w}} x \underset{\substack{\uparrow \\ \hat{} \text{ is a} \\ \text{homom.}}}{=} \mathsf{s}_{hat(w_1 \ldots w_n)} x \underset{\substack{\uparrow \\ (2.21)}}{=} \mathsf{s}_{\hat{w}_1 \circ \cdots \circ \hat{w}_n} x \underset{\substack{\uparrow \\ \text{def. of} \\ \mathsf{s}_{\hat{w}_i}}}{=} \mathsf{s}_{\hat{w}_n} \ldots \mathsf{s}_{\hat{w}_1} x \underset{\substack{\uparrow \\ g_i = \hat{w}_i}}{=} \mathsf{s}_{g_n} \ldots \mathsf{s}_{g_1}\,. \tag{2.22}$$

Then

$$\mathsf{c}_0\mathsf{s}_w x \underset{(2.22)}{\underset{\uparrow}{=}} \mathsf{c}_0\mathsf{s}_{g_n}\dots\mathsf{s}_{g_1}x \underset{\substack{\mathsf{c}_0\mathsf{s}_{g_i}=\mathsf{s}_{g_i}\mathsf{c}_0\\ \text{by } g_i\in G^0}}{\underset{\uparrow}{=}} \mathsf{s}_{g_n}\dots\mathsf{s}_{g_1}\mathsf{c}_0 x \underset{(2.22)}{\underset{\uparrow}{=}} \mathsf{s}_w\mathsf{c}_0 x\,.$$

Finally, we add to our finite set of axioms

$$\mathsf{s}_{\mathrm{Id}}x = x. \tag{2.23}$$

So far we have constructed a finite set Σ of equations such that $\mathsf{Gws}_G \models \Sigma$ and $\Sigma \models$ (2.15)–(2.23), where statements (2.15)–(2.23) are formulated above. Therefore it is enough to show that (2.15)–(2.23) $\models (*1) - (*7)$.

Proof of (2.15)–(2.23) $\models (*6)$**:** Let $\tau \in T$, $i,j\in\omega$ with $(\forall k)\big(\tau(k) = j \leftrightarrow k = i\big)$. Let $\nu \stackrel{\text{def}}{=} [j,0]\circ\tau\circ[i,0]$. Clearly, ν satisfies the condition of (2.20), hence (2.20) $\models (\mathsf{c}_0\mathsf{s}_\nu x = \mathsf{s}_\nu\mathsf{c}_0 x)$. Then

$$\mathsf{s}_\tau\mathsf{c}_i x \underset{\text{def. of } \mathsf{c}_i}{\underset{\uparrow}{=}} \mathsf{s}_\tau\mathsf{p}_{i0}\mathsf{c}_0\mathsf{p}_{i0}x \underset{(2.15)}{\underset{\uparrow}{=}} \mathsf{p}_{j0}\underbrace{\mathsf{p}_{j0}\mathsf{s}_\tau\mathsf{p}_{i0}}_{\mathsf{s}_\nu}\mathsf{c}_0\mathsf{p}_{i0}x = \mathsf{p}_{j0}\mathsf{c}_0\mathsf{p}_{j0}\mathsf{s}_\tau\mathsf{p}_{i0}\mathsf{p}_{i0}x = \mathsf{c}_j\mathsf{s}_\tau x$$

is a derivation of $(*6)$ from (2.15)–(2.20). **End of proving** $(*6)$**.**

Proof of (2.15)–(2.23) $\models (*5)$**:** Assume $\tau, \tau[i/j]\in T$ (for some fixed $i,j\in\omega$).

Case 1. $j \in Rng(\tau)$. Then $j = \tau(k)$ for some k. Now

$$\mathsf{s}_\tau\mathsf{c}_i x \underset{(2.16)}{\underset{\uparrow}{=}} \mathsf{s}_\tau\mathsf{s}^i_k\mathsf{c}_i x \underset{(2.15)}{\underset{\uparrow}{=}} \mathsf{s}_{\tau[i/\tau k]}\mathsf{c}_i x \underset{j=\tau(k)}{\underset{\uparrow}{=}} \mathsf{s}_{\tau[i/j]}\mathsf{c}_i x\,.$$

Case 2. $j\notin Rng(\tau)$.

Case 2.1. $(\exists k\neq i)\tau(i) = \tau(k)$. Let $\tau' \stackrel{\text{def}}{=} \tau[i/j]$ and $i^+ \stackrel{\text{def}}{=} \tau(i)$. Then $i^+\in Rng(\tau')$ and $\tau = \tau'[i/i^+]$, hence Case 1 is applicable (since we did assume $\tau[i/j]\in T$). $\mathsf{s}_{\tau'}\mathsf{c}_i x = \mathsf{s}_{\tau'[i/i^+]}\mathsf{c}_i x = \mathsf{s}_\tau\mathsf{c}_i x$.

Case 2.2. $(\nexists k\neq i)\tau(i) = \tau(k)$. Then letting $i^+ = \tau(i)$, the condition of $(*6)$ holds for τ, i, and i^+. Thus $\mathsf{s}_\tau\mathsf{c}_i x = \mathsf{c}_{i^+}\mathsf{s}_\tau x$ by $(*6)$. Also $\tau[i/j] = [i^+/j]\circ\tau$. Now

$$\mathsf{s}_\tau\mathsf{c}_i x = \mathsf{c}_{i^+}\mathsf{s}_\tau\mathsf{c}_i x \underset{(2.16)}{\underset{\uparrow}{=}} \mathsf{s}^{i^+}_j\mathsf{c}_{i^+}\mathsf{s}_\tau\mathsf{c}_i x = \mathsf{s}^{i^+}_j\mathsf{s}_\tau\mathsf{c}_i x = \mathsf{s}_{\tau[i/j]}\mathsf{c}_i x\,.$$

End of proving $(*5)$**.**

Now, $(*1), (*3), (*4), (*7)$ are immediate by (2.18)+(2.19), (2.15), (2.23), (2.17) respectively (in this order, e.g. $(*3)$ follows from (2.15)), *except that* part $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$ of $(*1)$ (i.e. (C_4) of [HMTI]) does not follow from (2.18).

To prove $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$ from (2.15)–(2.23), we will use the results from Pinter [Pn73]. Namely, $(q_1) - (q_9)$ of Def.2.1 in [Pn73] follow from (2.15)–(2.23), since (q_9) is $(*7)$ which we already proved, and the others are parts of statements in (2.15)–(2.23), hence (2.15)–(2.23) $\models \mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$ by [Pn73, Lemma 2.2 (iii)]. For completeness, we give the details of this proof, as follows.

Let $i,j\in\omega$, $i\neq j$. First we prove

$$\mathsf{c}_i\mathsf{c}_j\mathsf{c}_i x = \mathsf{c}_j\mathsf{c}_i x \tag{2.24}$$

as follows. Let $k\notin\{i,j\}$.

$$\mathsf{c}_i\mathsf{c}_j\mathsf{c}_i x \underset{(2.16)}{\underset{\uparrow}{=}} \mathsf{c}_i\mathsf{c}_j\mathsf{s}^i_k\mathsf{c}_i x \underset{\substack{j\notin\{i,k\}\\(*6)}}{\underset{\uparrow}{=}} \mathsf{c}_i\mathsf{s}^i_k\mathsf{c}_j\mathsf{c}_i x \underset{(2.17)}{\underset{\uparrow}{=}} \mathsf{s}^i_k\mathsf{c}_j\mathsf{c}_i x \underset{(*6)}{\underset{\uparrow}{=}} \mathsf{c}_j\mathsf{s}^i_k\mathsf{c}_i x \underset{(2.16)}{\underset{\uparrow}{=}} \mathsf{c}_j\mathsf{c}_i x\,.$$

Now

$$\mathsf{c}_i\mathsf{c}_j x \underset{\substack{\uparrow \\ \boxed{\substack{x\leqslant \mathsf{c}_i x \text{ and} \\ z\leqslant y \to \mathsf{c}_i z \leqslant \mathsf{c}_i y \\ \text{by (2.15)}}}}}{\leqslant} \mathsf{c}_i\mathsf{c}_j\mathsf{c}_i x \underset{\substack{\uparrow \\ (2.24)}}{=} \mathsf{c}_j\mathsf{c}_i x\,.$$

QED (Lemma 2.18) ∎

We are ready for turning to the proof of Theorem 1.8. Below we reformulate briefly Theorem 1.8, and then prove it.

Reformulation of Theorem 1.8. *Let $T \subseteq \langle {}^\omega\omega, \circ\rangle$ be a rich semigroup, and assume that T is finitely presented via some generator set G. Then conclusions (i) and (ii) of the first formulation of Theorem 1.8 hold. (Briefly, $\mathbf{I}\mathsf{Gws}_G$ is a finitely based variety the appropriate reduct of which coincides with "Representable QPA".)*

Proof: Assume the conditions of the "Reformulation ... ", that is assume (1.4), (1.5) and (2.12). We will prove (2.13) (formulated in Lemma 2.18 above).

By (2.12), there is a finite generator set G for T. We will not use the rest of (2.12). Let $\pi, \sigma \in T$ as in (1.5). Let T^0 be as in (2.13). We may assume $0 \notin Rng(\sigma)$. If it were otherwise, then we would just rename the elements of ω. Let $\tau' \stackrel{\text{def}}{=} (\sigma \circ \tau \circ \pi)[(\omega \smallsetminus Rng(\sigma))/\mathrm{Id}]$ for any $\tau \in T$. By (1.5), $\tau' \in T$. Let $\sigma^0 \stackrel{\text{def}}{=} \sigma[0/0]$ and $\pi^0 \stackrel{\text{def}}{=} ([0/\sigma(0)] \circ \pi)[0/0]$. Then $\sigma^0, \pi^0 \in T^0$; moreover, for any $\tau \in T^0$,

$$\pi^0 \circ \tau' \circ \sigma^0 = \tau. \tag{2.25}$$

(The latter holds because $\pi^0\tau\sigma^0(0) = \pi^0\tau'(0) = \pi^0(0) = 0$ by $0 \notin Rng(\sigma)$, and if $i > 0$ then $\tau \in T^0 \Rightarrow \tau(i) \neq 0 \Rightarrow \sigma\tau(i) \neq \sigma(0) \Rightarrow 0 \neq \pi\sigma\tau(i)$. Then $\pi^0\tau'\sigma^0(i) = \pi^0\tau'\sigma(i) = \pi^0\sigma\tau\pi\,(\sigma(i)) = \pi^0\sigma\tau(i) = \pi\sigma\tau(i) = \tau(i)$, by $\sigma\tau(i) \neq 0$.)

Let $T' \stackrel{\text{def}}{=} \{\tau' : \tau \in T\}$, and $G' \stackrel{\text{def}}{=} \{\tau' : \tau \in G\}$. Then $T' \subseteq T^0$ and G' generates T'. The latter follows from the fact that $\tau \mapsto \tau'$ is an endomorphism of T (i.e., $(\tau \circ \nu)' = \tau' \circ \nu'$; to see the latter, note that $\left(R \cap Rng(g) = 0 \Rightarrow (f \circ g)^R_{\mathrm{Id}} = f^R_{\mathrm{Id}} \circ g^R_{\mathrm{Id}}\right)$ for $f, g \in {}^\omega\omega$, $R \subseteq \omega$). Assume $\tau \in T^0$. Then G' generates $\tau' \in T'$. By (2.25), $\tau = \pi^0 \circ \tau' \circ \sigma^0$ proving that $G^0 \stackrel{\text{def}}{=} G' \cup \{\pi^0, \sigma^0\}$ generates T^0. By this, $\left((1.4), (1.5), (2.12)\right) \Rightarrow (2.13)$ has been proved.

For possible later applications, we formulate the following:

COROLLARY 2.19. *Assume (1.5). Assume $\sigma^0, \pi^0 \in T$, where σ^0, π^0 are defined above. Assume that T is finitely generated. Then (2.13) holds.*

Proof of Corollary 2.19: The above proof proves just this. ∎

We continue proving Theorem 1.8. We have proved (2.13). By (1.4) and (2.12), Lemma 2.18 is applicable. Hence there is a finite Σ with $\mathsf{Gws}_G \models \Sigma$ and $\Sigma \models (*1) - (*7)$. By Theorem 2.17 then $\mathbf{I}\mathsf{Gws}_G = Mod(\Sigma)$. This proves (i) of Theorem 1.8. (ii) of Theorem 1.8 was proved in subsection 1.3. **QED (Theorem 1.8)** ∎

2.4. Discussion

In section 3 we will prove that there is a transformation semigroup T satisfying all the conditions of Theorem 1.8, that is, there exists a finitely presented and rich semigroup (in the sense of subsection 1.3). We will prove this by giving a concrete example for such a semigroup. It would be interesting to find *some more* finitely presented and rich semigroups. The following Conjecture 2.20 is concerned with this issue.

Conjecture 2.20:

(i) One possible candidate, suggested in an early version of [Sa87a], for a semigroup T satisfying all the conditions of Theorem 1.8 is the subsemigroup of ${}^{\mathbb{Z}}\mathbb{Z}$ ($\mathbb{Z}$ denotes the set of all integers) generated by Gsc to be defined below. Consider the functions:

$$\begin{aligned} &\mathit{shift} && \stackrel{\text{def}}{=} \langle z+1 : z \in \mathbb{Z}\rangle, \\ &\mathit{shift}^{-1}, && \\ &\mathit{suc} && \stackrel{\text{def}}{=} \mathrm{Id} \restriction (\mathbb{Z} \setminus \omega) \cup (\mathit{shift} \restriction \omega), \text{ and} \\ &\mathit{pred} && \stackrel{\text{def}}{=} \mathit{suc}^{-1} \cup \{\langle 0,0\rangle\}. \end{aligned}$$

Note that $\mathit{pred}(1) = 0 = \mathit{pred}(0)$. Let $\mathsf{Gsc} \stackrel{\text{def}}{=} \{\mathit{shift}, \mathit{shift}^{-1}, \mathit{suc}, \mathit{pred}, [0,1]\}$ and $T = [\mathsf{Gsc}]$ (the semigroup generated by Gsc in $\langle {}^{\mathbb{Z}}\mathbb{Z}, \circ\rangle$).

We *conjecture* that either T or a relatively natural modification of T is finitely presentable, and satisfies the conditions of Theorem 1.8. If T would turn out to be finitely presentable, then, together with Theorem 1.8, it would yield an alternative solution to the problem of Henkin–Monk [HM74], for first order logic without equality. This is so because in an early version of [Sa87a] it was proved that T satisfies all the conditions of Theorem 1.8 except perhaps finite presentability.

Scrap denotes the class $\mathsf{Gws}_{\mathsf{Gsc}}$.[22]

Let T^- be the semigroup generated by $\{\mathit{pred}, \mathit{suc}, [i/j] : i,j \in \omega\}$. Then $T^- \subset T$. We did not consider T^- as a candidate for satisfying all the conditions of Theorem 1.8, because T^- is not finitely generated. (We added *shift* and its inverse to T^- to make it finitely generated. On the other hand, the Copeland–Demaree semigroup generated by $\{\mathit{shift}, \mathit{shift}^{-1}, [i/j], [i,j] : i,j \in \omega\}$ is finitely generated, but, by Demaree's construction [$\mathfrak{B}_n$ on p.652 of Demaree [De69]], is not finitely presented. We should note that adding *suc* to the latter semigroup rules out using Demaree's construction for a negative proof, moreover, it seems to do so essentially.)

(ii) Let I consist of three disjoint copies of ω, say N_0, N_1, N_2. Let $\mathit{suc}_i, \mathit{pred}_i$ be the usual *suc* and *pred* on N_i and the identity on the rest of I. Let $0_i \in N_i$ be the zero of N_i. Let

$$H \stackrel{\text{def}}{=} \{\mathit{suc}_i, \mathit{pred}_i, [0_0, 0_1], [0_1, 0_2] : i < 3\}.$$

It is not hard to see that $[H]$ is rich. We conjecture that [H] is finitely presentable. ◀

Remark 2.21: The condition $(\exists\sigma,\pi \in T)(\pi \circ \sigma = \mathrm{Id}$ and σ is not onto $\omega)$ (this is part of (1.5)) seems to be needed in Theorem 1.8 because of the following. Let $G_0 \stackrel{\text{def}}{=} \{\mathit{shift}, \mathit{shift}^{-1}, [0,1], [0/1]\}$ $(\subseteq {}^{\mathbb{Z}}\mathbb{Z})$. Let T be generated by G_0 in $\langle {}^{\mathbb{Z}}\mathbb{Z}, \circ\rangle$. Then T satisfies all the conditions in Theorem 1.8 except for $(\exists\sigma,\pi \in T)(\pi \circ \sigma = \mathrm{Id}$ and σ is not onto $\omega)$, and for being finitely presented. We conjecture that there is an example which is even finitely presented, i.e., it satisfies all the conditions except the existence of σ and π, and for which the conclusion of Theorem 1.8 fails. Concerning G_0, we have the following. (Let us recall the definition of Σ_G from subsection 1.1.4.)

[22] The name "Scrap" was suggested by R. J. Thompson referring to the names of Craig and Copeland.

THEOREM 2.21.1 (Andréka–Németi–Sain). *Neither* $\mathbf{I}\mathsf{Gws}_{G_0}$ *nor* $\mathbf{Eq}(\mathsf{Gws}_{G_0})$ *are finitely axiomatizable over* Σ_{G_0}.

In order to keep the present paper streamlined, we omit the **proof** which in turn is made available from I. Németi. The idea of this proof is similar to the one given in [ST91] for the choice $G' \stackrel{\mathrm{def}}{=} \{[i/j], [i,j] : i,j \in \omega\}$. ▮

We note that Demaree [De69] proved that this Gws_{G_0} is not finitely axiomatizable. What more Theorem 2.21.1 above says is that Gws_{G_0} is not finitely axiomatizable even over Σ_{G_0}. Intuitively, Theorem 1.8 says that to axiomatize Gws_G, it is enough to axiomatize its substitutions (i.e., Σ_G), if G satisfies certain conditions. In contrast, Theorem 2.21.1 says that this becomes false if we drop the condition that G has a "successor–like" element.

The reason why we feel that Theorem 2.21.1 implies that the conditions $Rng(\sigma) \neq \omega$ and $\pi \circ \sigma = \mathrm{Id}$ are needed in Theorem 1.8 is the following. G_0 (in Theorem 2.21.1) satisfies all the other conditions of Theorem 1.8, except that we do not know whether T is finitely presented. This latter property can be eliminated, however, from Theorem 1.8, as follows. The proof of Theorem 1.8 proceeds by first proving (Theorem 2.17 in subsection 2.3, saying) that $\mathbf{I}\mathsf{Gws}_G$ is finitely axiomatizable over Σ_G *even if* T is *not finitely* presented. Then from finite presentability of T we derive finite axiomatizability of Σ_{G_0}. This together with the first result yields finite axiomatizability of $\mathbf{I}\mathsf{Gws}_G$. Therefore Theorem 2.21.1 above shows that, without the condition on σ and π, Theorem 2.17 (i.e. finite axiomatizability over Σ_G, which might be of interest in itself), fails.

The algebras discussed above constitute the substitution–cylindric version (in Pinter's sense or in the sense of [AGN77]) of Copeland set algebras, see [HMTII, p.264]. So the above theorem says that the substitution–cylindric version of Copeland set algebras is not finitely axiomatizable either, extending the results of Demaree quoted in [HMTII, p.264–265].[23]

End of Remark 2.21. ◀

Following the terminology of [HMTII], by a Gs–*unit* V we understand a disjoint union $V = {}^{\omega}U_1 \dot\cup {}^{\omega}U_2 \dot\cup \ldots$ of Cartesian spaces ${}^{\omega}U_i$, $i \in I$. Here, of course, $I = 1$ is permitted. Recall that by a Cs_G we understand a Gws_G with unit ${}^{\omega}U$ for some set U.

THEOREM 2.22. *Let* $G \subseteq {}^{\omega}\omega$ *be* arbitrary. *Then*

(i) Gws_G*'s with* Gs*–units form a quasi–variety (up to isomorphism of course).*
(i) $\mathbf{SP}\mathsf{Cs}_G$ *is a quasi–variety (i.e., is closed under ultraproducts, or equivalently, is axiomatizable by quasi–equations).*

The **proof** is based on Lemma 2.5 and on the "**SP**–part" of the proof of Theorem 2.1 in subsection 2.1. ▮

Remark 2.23: Below Theorem 1 (in the Introduction) we explained how unlikely it is to obtain a positive result for quasi–polyadic equality algebras (QPEA_ω's). However, we would like to point out that using the proof methods for Theorem 1, one can prove the following variant (2.26) of Theorem 1 for QPEA_ω's.

(2.26) There is a finitely axiomatizable variety V satisfying conditions $(i)^* - (iii)^*$ below.

$(i)^*$ V consists of homomorphic images of subreducts of representable polyadic equality algebras. That is,

$$\mathsf{V} = \mathbf{HSRd}\,\mathsf{RPEA}_\omega\,.$$

[23] The negative results in Demaree [De69] are in interesting contrast with the "algebras with equality" section of [SGy96], where finite schema axiomatization is given for the equational theory of a "Coopeland–style" class of algebras with equality. (Demaree proved the opposite of this for the original class.)

This implies that

$(ii)^*$ V is generated as a variety by its representable members (where representable was explained in Theorem 1 (ii)).

$(iii)^*$ The class of quasi–polyadic subreducts of members of V coincides with the class of representable quasi–polyadic equality algebras. That is,

$$\mathsf{RQPEA}_\omega = \mathbf{SRd}\mathsf{V}\,.$$

In connection with the proof of (2.26) see Remark 5.5 in subsection 5.1, and [Sa87].

We will briefly return to this subject in subsection 5.1. The problem solved by (2.26) above can be regarded as the infinite–dimensional analog of the (3–dimensional) problem formulated in [TG87, p.62, lines 9–10].

Comparing statement (2.26) with Theorem 1, we should point out the following three things.

(1) Statement (2.26) is weaker than Theorem 1 in that $(ii)^*$ does not claim that every element of V would be representable (though the "important operations" can be represented in every element of V, by $(iii)^*$). Indeed, strengthening $(ii)^*$ in this direction is not possible, because the representable part RV of V is not closed under taking ultraproducts.

(2) Statement (2.26) implies that RQPEA_ω can be expanded to a representable class RV such that the equational theory of RV is finitely axiomatizable. (To be precise, we are talking about a term–definitionally equivalent version of RV.)

(3) Theorem 1.8 admits a reformulation (with adjoining the diagonal constant d_{01} to Gws_G) such that this reformulation implies statement (2.26) above. ◀

DISCUSSION 2.24 (On the conditions of Theorems 1.8, 2.15, 2.17, 2.1):

(i) *Condition (1.5) (formulated at the beginning of section 1.3) is needed in Theorem 2.15.*

Indeed, let $G = \{[i/j], [i,j] : i, j \in \omega\}$. Then $\mathbf{I}\mathsf{Gws}_G$ coincides with the class RQPA_ω of representable quasi–polyadic algebras. It was proved in Sain–Thompson [ST91] that RQPA_ω is not axiomatizable by a finite schema of equations. The proof there easily yields the (weaker) fact that $Mod\big((*1)-(*7)\big) \supsetneqq \mathsf{RQPA}_\omega = \mathbf{I}\mathsf{Gws}_G$ or in other words, that there are non–representable quasi–polyadic algebras (QPA_ω's). (The fact that $\mathsf{QPA}_\omega \neq \mathsf{RQPA}_\omega$ must have been known well before [ST91] was written.) This shows that Theorem 2.15 becomes false if we drop (1.5).

(ii) To see that *condition (1.5) is needed in Theorem 2.17*, we need a new semigroup, because that of QPA's is not finitely generated.

As in the case of our first candidate **Scrap** for a simple semigroup satisfying the conditions of Theorems 1.8, 2.17, we will use $\mathbb{Z}$ as an index set instead of ω. Using the same notation as in the definition of **Scrap** (see Conjecture 2.20), let $G = \{\mathit{shift}, \mathit{shift}^{-1}, [0,1], [0/1]\}$. Then G satisfies all the conditions of Theorem 2.17 except (1.5). At the same time, the conclusion of Theorem 2.17 fails for Gws_G. That is, there is no finite set Σ_0 of equations such that $\mathbf{I}\mathsf{Gws}_G = Mod\{\Sigma_0 + \Sigma_G\}$ would hold, where Σ_G is as in Definition 1.6 (ii). Our proof of this result is too complicated for publishing. It is an open problem to find a publishable proof of this result. The following observation might lead to such a proof. Let $\Sigma_\infty \stackrel{\text{def}}{=} \{\mathrm{C}_0\mathrm{S}_{[i,i+1]} = \mathrm{S}_{[i,i+1]}\mathrm{C}_0 : 0 < i \in \omega\}$. Then it seems likely that, by choosing the n–th non–representable algebra $\mathfrak{A}_n$ to deviate from Gws_G's only in $\mathsf{c}_0^{\mathfrak{A}_n} \stackrel{\text{def}}{=} \mathrm{C}_{(\{n\cdot k : k\in\omega\})}$, we obtain a sequence of algebras such that their ultraproduct is in $\mathbf{I}\mathsf{Gws}_G$.

(iii) *We do not know whether the part* $(\sigma \circ \tau \circ \pi)[(\omega \smallsetminus Rng(\sigma))/\mathrm{Id}] \in T$ *of condition (1.5) is needed.* (If $\omega \smallsetminus Rng(\sigma)$ is finite, then it is not needed.)

(iv) *If we drop both (1.4) and (1.5) then Theorems 1.8, 2.15 and 2.17 become false.*

To see this, let $G \stackrel{\text{def}}{=} \{[0/1]\} \cup \{[i,j] : i, j < 4\}$. Then the subsemigroup $[G]$ of $\langle {}^\omega\omega, \circ\rangle$ is finite, hence is finitely presented, too. By Johnson [Jo69], the class RPA_4 of representable polyadic algebras of dimension 4 is not finitely axiomatizable. From this it is fairly easy to prove that $\mathbf{I}\mathsf{Gws}_G$ is not finitely axiomatizable either, showing that the conclusion of Theorem 1.8 fails. Extending this argument to Theorems 2.15 and 2.17 is left to the reader. (Since in these theorems we have C_i for all $i \in \omega$, it is easier to use Johnson's proof instead of his theorem.)

There is another counter–example for proving the necessity of some conditions, besides $[G]$ being finitely presentable, even if $[G]$ is infinite. Namely, let $G \stackrel{\text{def}}{=} \{\mathit{shift}, \mathit{shift}^{-1}\}$ on the index set $\mathbb{Z}$ of **Scrap**. Then $[G]$ is well known to be finitely presentable. However, it is again an *open problem* to find a relatively simple proof for the fact that Gws_G is not finitely axiomatizable. (A set of equations which is not the consequence of any finite set of true equations if $\{\mathsf{c}_0\mathsf{c}_i x = \mathsf{c}_i\mathsf{c}_0 x : i \in \omega\}$, where $\mathsf{c}_n x = (\mathit{shift})^n\, \mathsf{c}_0\, (\mathit{shift}^{-1})^n$ for each $n \in \omega$. This example illustrates the price of coding infinitely many operations c_i $(i \in \omega)$ with finitely many ones $(\mathsf{c}_0, \mathit{shift}, \mathit{shift}^{-1})$; namely, what used to be a simple finite schema $\{\mathsf{c}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{c}_i : i, j \in \omega\}$, became a set of longer and longer equations. So, in a sense, after our first step (coding of operations) in finitization, we are farther removed from having a finitary axiom system than we were to begin with.)

(v) *The condition of Theorem 2.1 cannot be omitted. Namely, the condition that* C_i *is term definable in* Gws_G *is needed in the proof of Lemma 2.10 (*$\mathbf{H}\mathsf{Gws}_G = \mathbf{I}\mathsf{Gws}_G$*).* (In the proof of Theorem 2.1, this is the only point where it is needed. It is not needed in the **SP** part.)

Indeed, let $U = \omega$ and $V = \mathrm{C}_0\{\langle 0, 0, \ldots, 0, \ldots\rangle\}$. Let $G = \{[i/j] : i, j \in \omega \text{ and } j \neq 0\}$. Let $\mathfrak{A}$ be the full Gws_G with unit ${}^{\omega}U$ (i.e., $A = \mathcal{P}({}^{\omega}U)$). Then $rl_V : A \longrightarrow \mathcal{P}(V)$ is a surjective homomorphism (since $\mathrm{C}_0 V = V$ and $\mathrm{S}_\tau V \supseteq V$ for any $\tau \in G$). Say, $rl_V : \mathfrak{A} \longrightarrow \mathfrak{B} = \langle \mathcal{P}(V), \ldots\rangle$. Clearly $\mathfrak{B} \notin \mathsf{Gws}_G$ since V is not a Gws_G unit. What is less obvious, but true, is that $\mathfrak{B}$ is not even isomorphic to a Gws_G. To see this, let $b = \{\langle 0, 0, \ldots, 0, \ldots\rangle\}$. Now, in $\mathfrak{B}$, we have: $\mathrm{C}_0 b = 1$, $b = \mathsf{s}^i_j b$, $1 = \mathsf{s}^0_2\mathsf{s}^2_3 b$, $\mathsf{s}^0_i b = 1$, $\mathsf{s}^0_i(-b) = 0$. Assume that $\mathfrak{B}$ is isomorphic to a Gws_G with unit W. By the last two equations, $b = \{q \in W : q_0 = q_1\}$. Therefore $\mathsf{s}^0_2\mathsf{s}^2_3 b \subseteq \{q \in W : q_1 = q_2\} \neq W$, a contradiction, since $b \neq 1$.

It might be instructive to observe that rl_V would not be a homomorphism if $\mathrm{S}_{[1/0]}$ would be an operation. That is, $\mathrm{S}^1_0(V \cap (-V)) \neq V \cap \mathrm{S}^1_0(-V)$, or equivalently, $\mathrm{S}^1_0 V \not\supseteq V$.

We note that Lemma 2.5 is true for arbitrary G. (We did not use expressibility of C_i there.)

(vi) *In Theorem 1.8 and Lemma 2.18, the assumption that* $[G]$ *is finitely presented, cannot be replaced by the weaker one* $|G| < \omega$. This is well known, so we do not discuss it.

(vii) *The possibility of replacing condition (2.13) in Lemma 2.18 with natural conditions* which are required in other parts anyway is discussed in Corollary 2.19. It was exactly because of this that we did not check whether (2.13) is needed in Lemma 2.18.

(viii) Let $G \subseteq {}^{\omega}\omega$ be arbitrary. Let Gws^*_G denote the class of algebras which can be defined by the following modification of Definition 1.4: Put "Gws_ω–unit" in place of "normal Gws_ω–unit" (in the first line). *While Lemmas 2.5, 2.9, 2.10 remain true for* Gws^*_G *in place of* Gws_G*, Theorems 2.1, 2.15 become false.* However, if C_i is term–definable in Gws_G (for every $i \in \omega$) then $\mathbf{I}\mathsf{Gws}^*_G$ remains a variety (by the proof of Theorem 2.1).

Proof: It is not hard to modify the proofs of Lemmas 2.5, 2.9, 2.10 such that the modified versions work for the Gws^*_G case. A counterexample, due to Viktor Gyuris, showing that the "Gws^*_G–version" of Theorem 2.15 fails is the following.

Let $V \stackrel{\text{def}}{=} {}^{\omega}2^{\langle 0,1,0,1,\ldots\rangle} \cup {}^{\omega}3^{\langle 0,0,0,0,\ldots\rangle}$, $\tau(i) \stackrel{\text{def}}{=} 2i$ for every $i \in \omega$, and assume that $\tau \in G \subseteq {}^{\omega}\omega$. Then $\mathfrak{A} = \langle \mathcal{P}(V), \cup, {}_V\!\sim, \mathrm{C}_0, \mathrm{S}_\tau \ldots\rangle \in \mathsf{Gws}^*_G$, but $\mathfrak{A} \not\models (*6)$. Namely, for $x \stackrel{\text{def}}{=} \{\langle 2, 0, 0, 0, \ldots\rangle\}$, $\mathfrak{A} \not\models \mathrm{C}_0\mathrm{S}_\tau(x) = \mathrm{S}_\tau\mathrm{C}_0(x)$.

Next we prove that term–definability of the C_i's is necessary for varietyhood of Gws_G and/or Gws^*_G. This can be proved e.g. by using the result saying that

$$\mathbf{SP}\{\langle \mathfrak{P}({}^{\alpha}U), \cup, -, \mathsf{s}^i_j\rangle_{i,j\in\alpha} : U \text{ a set}\}$$

is only a quasi–variety but not a variety. A proof of the latter can be found in Andréka [An91]. ∎

By Lemma 2.5 (ii), $\mathbf{I}\mathsf{Cs}_G = \mathbf{SUp}\mathsf{Cs}_G$, that is, Cs_G is a class axiomatizable by universally quantified formulas.

OPEN PROBLEM 2.25. *Let G be as in Theorem 1.8. Is then $\mathbf{I}\mathsf{Cs}_G$ finitely axiomatizable?*[24] Here cf. items 2.16, 2.18, 2.19 in [SGy96]. ∎

Remark 2.26: If $(\forall\tau \in G)\ \omega \smallsetminus \mathit{Rng}(\tau)$ is finite, then $\mathbf{I}\mathsf{Cs}_G$ is a quasivariety. In particular, recall the generators Gsc of the semigroup "**Scrap**" from Conjecture 2.20. Now, $\mathbf{I}\mathsf{Cs}_{\mathsf{Gsc}}$ is a quasi–variety. On the other hand, if $\omega \times \{0\} \in G$ then $\mathbf{I}\mathsf{Cs}_G$ is not closed under $\mathbf{P}$. ∎

OPEN PROBLEM 2.27. *Assume the subsemigroup $[G]$ of $\langle {}^{\omega}\omega, \circ\rangle$ generated by G is finitely presented. Assume that G satisfies the condition of the "meta–theorem", in particular, that G has a "suc"–like element and a "pred"–like one. (Presence of all finite transformations in $[G]$ seems to be unnecessary for the "meta–theorem", it would be good to check this.) Is then $\mathbf{SP}\mathsf{Cs}_G$ finitely axiomatizable?*[25] *(We know that $\mathbf{SP}\mathsf{Cs}_G = \mathbf{I}\mathsf{Gs}_G$ is a quasi–variety.)*[26] ∎

[24] The proof of $\mathbf{SP}\mathsf{Cs}_G = \mathbf{I}\mathsf{Gs}_G$ goes the same way as the ones given for analogous results in [HMT] and [Né91]. Hence, to prove that $\mathbf{I}\mathsf{Gs}_G$ is a quasi–variety, it is enough to prove that $\mathbf{Up}\mathsf{Gs}_G = \mathbf{I}\mathsf{Gs}_G$. This can be proved by using the proof of Lemma 2.5.

[25] Compare with [SGy96, Thm.'s 2.16, 2.17 (the joint result with Németi)].

[26] Compare with [SGy96, Thm.'s 2.16, 2.17 (the joint result with Németi)].

3. ALGEBRAIC LOGIC WITH A FINITELY BASED VARIETY AS ITS REPRESENTABLES

3.1. Formulating the result

Let us return to the problem in Monk [Mo70] and Henkin–Monk [HM74] quoted in the Introduction. In this section we propose a solution for the case of first order logic without equality.[27] Algebraization of first order logic without equality has been extensively investigated, e.g., polyadic algebras were introduced particularly for this purpose (see e.g. Halmos [Ha54] or [HMTII, Chap.5.4]). Thus the content of the present section 3 might be relevant to polyadic algebras,[28] while that of section 5, treating first order logic with equality, might be more relevant to cylindric and relation algebras.

The solution proposed here is stated as Corollary 3.3 at the end of this subsection. Corollary 3.3 is based on Theorem 1.8 (stated in subsection 1.3 and proved in section 2) and on Theorem 3.2 below. The latter is an appropriately modified version of semigroup–theoretic results of Richard J. Thompson, cf. [Th80], adapted to the needs of the present paper. We include here a self–contained proof because Thompson's publications do not provide a sufficient number of "Lemmas" which would support a proof significantly shorter than the one here.[29]

DEFINITION 3.1 ($\mathbb{G}$ and $\mathbb{T}$):

(i) We define the functions $\bar{\sigma}, \pi, \mu, \eta \in {}^{\omega}\omega$ as follows. For all $n \in \omega$, we let

$$\bar{\sigma}(2n) \stackrel{\text{def}}{=} 4n, \qquad \bar{\sigma}(2n+1) \stackrel{\text{def}}{=} 2n+1,$$
$$\pi(2n) \stackrel{\text{def}}{=} \pi(2n+1) \stackrel{\text{def}}{=} n,$$
$$\mu(2n) \stackrel{\text{def}}{=} 2n+1, \qquad \mu(2n+1) \stackrel{\text{def}}{=} 2n,$$
$$\eta(4n+1) \stackrel{\text{def}}{=} 4n+2, \quad \eta(4n+2) \stackrel{\text{def}}{=} 4n+1, \quad \eta(4n+k) \stackrel{\text{def}}{=} 4n+k \quad \text{for } k \in \{0,3\}.$$

[27] Our solution does not apply to the important case of that special version L_n of first order logic in which only the first n variables are allowed with $n \in \omega$ fixed in advance. In cylindric algebraic terms this means that our solution does not apply to the finite dimensional case. A different kind of solution, related to Scrap (cf. subsection 3.7 later), is suggested in [SGy96], cf. e.g. Thm's 2.5, 2.7 (4), 2.15 there.

[28] However, we should note that Charles Pinter's substitution–cylindric algebras (SC_α's for short) were also introduced to handle first order logic without equality. SC_α's form probably the most economical (and streamlined or elegant) algebraization of logic without equality. They enjoy the further advantage of being a rather natural subreduct class of cylindric algebras and therefore most of the material in [HMT] applies to them. The results on Gws_G seem to be as closely related to substitution–cylindric algebras as to polyadic ones.

[29] Historical note: In 1987, István Németi and Richard J. Thompson started to work together on solving a conjecture in an earlier version of [Sa87a] (containing the mathematical content of our present sections 1 and 2, among others). During their cooperation, Thompson suggested adding [0/1] to the generators of his semigroup $Ft(K)$, thoroughly studied in [Th80, MT73]. Finite presentability of $Ft(K)$ was already proved in [Th80]. The idea here is to rely on the good properties (cf. [Th80, p.415, below Prop.2.7]) of $Ft(K)$ established in [Th80] in generalizing finite presentability from $Ft(K)$ to the extended (with [0/1]) semigroup. Thompson proved so many positive properties of $Ft(K)$ in [Th80] (e.g. $Ft(K)$ is a group) that they support workability of this approach. Later, Thompson provided extensive help in generalizing his result from $Ft(K)$ to what we could denote as $(Ft(K) + [0/1])$. Indeed, roughly speaking, the semigroup $\mathbb{T}$ in Theorem 3.2 here coincides with $(Ft(K) + [0/1])$, provided that we choose K to be the set of all ω–sequences of 0,1 containing only finitely many 1's. Let $\mathbb{T}^-$ be the subsemigroup of $\mathbb{T}$ generated by $\{\bar{\sigma}, \pi, \mu, \eta\}$ (cf. Definition 3.1). Then $\mathbb{T}^-$ is almost the same as Thompson's $Ft(K)$, cf. [Th80, p.405, below Prop.0.8 and Thm.1.4]. (However, Thompson's generators for $Ft(K)$ are different from our ones.) Actually, $Ft(K)$ is the permutational (or group) part of $\mathbb{T}^-$. The proof obtained along these lines was further improved by Judit X. Madarász, I. Németi and the present author.
The present proof of Theorem 3.2 together with the material in [Th80] and Sain–Thompson [ST91] represent the starting point of a long term joint research project of Thompson and the present author. The purpose of this project is to find a simple semigroup with a simple set of axioms satisfying the conditions of Theorem 1.8 here.

(ii) We let

$$\mathbb{G} \stackrel{\text{def}}{=} \{\bar{\sigma}, \pi, \mu, \eta, [0/1]\}.$$

(iii) $\mathbb{T}$ denotes the subsemigroup of $\langle {}^{\omega}\omega, \circ \rangle$ generated by $\mathbb{G}$. ◀

THEOREM 3.2. *The semigroup $\mathbb{T}$ defined above satisfies all the conditions of Theorem 1.8 (formulated in subsection 1.3). That is, $\mathbb{T}$ is finitely presented and rich (in the sense described at the beginning of subsection 1.3).*

More specifically, $\mathbb{T}$ satisfies conditions (i)–(iii) below.

(i) *$\mathbb{T}$ is finitely presented, and the generator set for a presentation can be chosen to be $\mathbb{G}$.*

(ii) *There exists a $\sigma \in \mathbb{T}$ for which*
$R \stackrel{\text{def}}{=} \omega \setminus Rng(\sigma)$ *is nonempty,*
$\pi \circ \sigma = \mathrm{Id}$, *and*
$(\sigma \circ \tau \circ \pi)^{R}_{\mathrm{Id}} \in \mathbb{T}$ *for all* $\tau \in \mathbb{T}$.

(iii) $\tau^{i}_{j} \in \mathbb{T}$ *for all* $\tau \in \mathbb{T}$ *and* $i, j \in \omega$.

The **proof** of Theorem 3.2 will be given in later subsections (3.3–3.7). Subsection 3.2 contains preparations for the proof of Theorem 3.2.

COROLLARY 3.3.

(i) $\mathbf{I}\mathsf{Gws}_{\mathbb{G}}$ *is a finitely based variety.*

(ii) *The quasi–polyadic subreducts of* $\mathbf{I}\mathsf{Gws}_{\mathbb{G}}$ *coincide with the representable* QPA*'s.*

Proof: Immediate by Theorems 1.8 and 3.2. ∎

Now we have all the ingredients for proving Theorem 1:

Proof of Theorem 1: We need to verify the existence of a finitely based variety V satisfying (i)–(iii) below.

(i) $\mathsf{V} = \mathbf{HSRd}\,\mathsf{PA}_{\omega}$;

(ii) every member of V is representable as a set algebra of ω–ary relations with "natural" operations;

(iii) $\mathbf{SRd}_{\mathsf{QPA}_{\omega}}\mathsf{V} = \mathsf{RQPA}_{\omega}$.

Let us choose $\mathsf{V} = \mathbf{I}\mathsf{Gws}_{G}$ with $[G]$ finitely presentable and rich (such a G exists by Theorem 3.2). Then V is a finitely based variety by Theorem 1.8 (i).

(ii) above is satisfied by the definition of Gws_{G}, and (iii) is the same as Theorem 1.8 (ii) (which is the same as Corollary 3.3 (ii)). Therefore it remains to prove (i).

$\mathbf{HSRd}\,\mathsf{PA}_{\omega} \subseteq \mathsf{V}$ holds by $\mathbf{Rd}\,\mathsf{PA}_{\omega} \subseteq \mathbf{I}\mathsf{Gws}_{G} = \mathsf{V}$.

For proving the other direction $\mathsf{V} \subseteq \mathbf{HSRd}\,\mathsf{PA}_{\omega}$, let $\mathfrak{A} \in \mathbf{I}\mathsf{Gws}_{G}$ be arbitrary but fixed. Then $\mathfrak{A} \cong \mathfrak{A}' \in \mathsf{Gws}_{G}$ for some $\mathfrak{A}'$. Let V denote the unit of $\mathfrak{A}'$. Then $V = \bigcup_{i\in I} {}^{\omega}U_i^{p_i}$ for some I, U_i's and p_i's, here the U_i's are mutually disjoint. Let $W \stackrel{\text{def}}{=} \bigcup_{i\in I} {}^{\omega}U_i$. Let $\mathfrak{B}$ denote the full Gws_{G} with unit W. Then $V \in B$, and $rl_V : \mathfrak{B} \longrightarrow \mathfrak{P}(V)$ (where $\mathfrak{P}(V)$ denotes the full Gws_{G} with unit V) is a homomorphism (the proof of this is similar to that of Lemma 2.9, namely, for any Gws_{G}–unit V, $\mathsf{S}^{[V]}_{\tau}(V) = V$ and $\mathsf{C}^{[V]}_{0}(V) = V$ by definition). Finally, $\mathfrak{A}'$ is clearly a subalgebra of $\mathfrak{P}(V)$. Thus $\mathfrak{A}' \in \mathbf{SHRd}\,\mathsf{PA}_{\omega} \subseteq \mathbf{HSRd}\,\mathsf{PA}_{\omega}$ by $\mathbf{SH} \leqslant \mathbf{HS}$, see any textbook on universal algebra, e.g. [HMTI, Chap.0]. This proves $\mathsf{V} \subseteq \mathbf{HSRd}\,\mathsf{PA}_{\omega}$ because $\mathfrak{A} \cong \mathfrak{A}'$ was chosen arbitrarily. ∎

3.2. Representing (and drawing) elements of $\mathbb{T}$ and some notation

There is a natural bijection between the set ω of all natural numbers and the set of all ω–sequences of 0,1 containing only finitely many 1's. Since we shall use this correspondence extensively, we fix now some notation concerning sequences.

If α is an ordinal and H is a set then an α–sequence of elements of H (or a sequence of length α consisting of elements of H) is a function from α into H, and also, α is the set of all ordinals smaller than α. Thus, for example, $2 = \{0,1\}$ and ${}^{\omega}2$ is the set of all ω–sequences of 0,1. Recall from subsection 1.1.4 that H^* denotes the set of all finite sequences of elements of H, that is,

$$H^* = \bigcup\{{}^{n}H : n \in \omega\}.$$

$\bar{0}$ denotes the ω–sequence consisting of 0's only, that is, $\bar{0}$ is the only element of ${}^{\omega}\{0\}$, thus $\bar{0} = \langle 0, 0, \ldots\rangle$. Let $n \in \omega$, $s \in {}^{n}2$ and $q \in {}^{\omega}2$. Then sq denotes the ω–sequence obtained from s and q by writing "q after s". (This is often denoted by $s^{\frown}q$ in the literature, and $^{\frown}$ is called concatenation.) That is,

$$(sq)(k) = s(k) \text{ if } k < n \quad \text{and} \quad (sq)(n+k) = q(k) \text{ for all } k \in \omega.$$

If, e.g., $s = \langle 0,0,1,0\rangle$ then we just write 0010 instead of s (that is, we omit the angle brackets and the commas). Thus, e.g., $011\bar{0}$ denotes the ω–sequence beginning with 011 and ending with 0's. $K \stackrel{\text{def}}{=} {}^{\omega}2^{(\bar{0})}$ denotes the set of all ω–sequences of 0,1 containing only finitely many 1's.

Now we can return to our one–one correspondence between ω and $K = {}^{\omega}2^{(\bar{0})}$.[30] Let $n \in \omega$. We define its corresponding ω–sequence, $tr(n) \in K$, as follows. Write n in diadic form (that is, write it in number system of base 2). Then we obtain a finite sequence of 0's and 1's beginning with 1 if $n \neq 0$. Now reverse the order of this finite sequence and extend it to an ω–sequence by writing 0's to its end. For example,

$$\begin{aligned} tr(0) &= \bar{0}, \\ tr(1) &= 1\bar{0}, \\ tr(2) &= 01\bar{0}, \\ tr(3) &= 11\bar{0}, \\ tr(4) &= 001\bar{0}, \\ tr(5) &= 101\bar{0}, \\ tr(6) &= 011\bar{0}. \end{aligned}$$

Then *n is even iff $tr(n)$ begins with 0*, and *n is of form $4k+1$ iff $tr(n)$ begins with 10.* Let us recall from number theory that congruence classes like $(2/mod\,4)$ of ω are defined as $(2/mod\,4) = \{4k+2 : k \in \omega\}$, more generally, $(r/mod\,m) = \{m \cdot k + r : k \in \omega\}$ for $0 \leqslant r < m$. Then the "tr–translations" of the congruence classes $(r/mod\,2^{\ell})$ ($\ell \in \omega$, $0 \leqslant r < 2^{\ell}$) will be sets of sequences beginning with a fixed finite sequence. For example, using the notation $tr"S \stackrel{\text{def}}{=} \{tr(s) : s \in S\}$ for any $S \subseteq \omega$,
$tr"(0/mod\,2) = \{0q : q \in K\}$, $\quad tr"(1/mod\,2) = \{1q : q \in K\}$,
$tr"(0/mod\,4) = \{00q : q \in K\}$, $\quad tr"(1/mod\,4) = \{10q : q \in K\}, \ldots, tr"(3/mod\,4) = \{11q : q \in K\}$,
$tr"(0/mod\,8) = \{000q : q \in K\}$, $\ldots$
etc.

We can now go even further. We can consider an ω–sequence $q \in K$ as a real number $im(q)$ between 0 and 1 by taking $im(q) \stackrel{\text{def}}{=} 0.q$, where the latter number is considered to be in number system of base 2. This is convenient, because then we can represent ω–sequences in K as points on the interval [0,1), see Figure 3.1 below.

[30] It is this choice of the parameter K which we need to substitute into $Ft(K)$ as defined in Thompson [Th80] to ensure applicability of [Th80] to the present concepts.

$\bar{0}$	$01\bar{0}$	$1\bar{0}$	$11\bar{0}$	
|	|	|	|	|
0				1

Figure 3.1

This representation of elements of K as points on the interval [0,1) preserves the "structure" that is interesting for us, namely, the sequences beginning with 0 (i.e., translations of the even numbers) lie on the first half of the interval, sequences beginning with 1 (translations of the odd numbers) lie on the second half of the interval, and similarly, sequences beginning with 00 (translations of numbers from $(0/mod\,4)$) lie on the first fourth of the interval, and so on, see Figure 3.2 below.

$\bar{0}$ $1\bar{0}$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{sequences beginning with 0}} \quad \underbrace{\qquad\qquad\qquad\qquad}_{\text{sequences beginning with 1}}$$

00 01 10 11

$$\underbrace{\qquad\qquad}_{\text{sequences beginning with 00}} \quad \underbrace{\qquad\qquad}_{\text{sequences beginning with 01}} \quad \underbrace{\qquad\qquad}_{\text{sequences beginning with 10}} \quad \underbrace{\qquad\qquad}_{\text{sequences beginning with 11}}$$

$tr"(0/mod\,2)$ $tr"(1/mod\,2)$

$tr"(0/mod\,4)$ $tr"(2/mod\,4)$ $tr"(1/mod\,4)$ $tr"(3/mod\,4)$

Figure 3.2

Thus we have a representation of ω–sequences by functions from the set [0,1) of real numbers between 0 and 1 into itself. We can use this representation to "draw" the elements of $\mathbb{G}$ (and of $\mathbb{T}$). Figure 3.3 below consists of this kind of illustrations of $\bar{\sigma}$, π, μ, η and $[0/1]$. We will then use such pictures as illustrations throughout the proof of Theorem 3.2 in later subsections.

Notice that from the elements of $\mathbb{G}$, μ and η are permutations of ω of order 2 (that is, $\mu \circ \mu = \eta \circ \eta = \mathrm{Id}$), such that μ interchanges the classes $(0/mod\,2)$ and $(1/mod\,2)$, and η interchanges the classes $(1/mod\,4)$ and $(2/mod\,4)$ and leaves $(0/mod\,4)$ and $(3/mod\,4)$ unchanged. Also, $\bar{\sigma}$ is deviding by 2 on $(0/mod\,2)$ and identity on $(1/mod\,2)$. See Figure 3.3 below.

$\bar{\sigma}$

$\bar{\sigma}$ "compresses" the interval $[\bar{0}, 1\bar{0})$ into the interval $[\bar{0}, 01\bar{0})$ and is identity on $[1\bar{0}, 1)$.

π

π "expands" $[\bar{0}, 1\bar{0})$ to $[0, 1)$ and $[1\bar{0}, 1)$ to $[0, 1)$.

μ

μ "interchanges" the intervals $[\bar{0}, 1\bar{0})$ and $[1\bar{0}, 1)$.

η

η "interchanges" the intervals $[01\bar{0}, 1\bar{0})$ and $[1\bar{0}, 11\bar{0})$, and leaves the rest fixed.

$[0/1]$

$[0/1]$ takes $\bar{0}$ to $1\bar{0}$ and leaves the rest fixed

Figure 3.3

3.3. "Table characterization" of $\mathbb{T}$ and $\mathbb{T}^-$; Proof of (ii) and (iii) of Theorem 3.2[31]

Convention 3.4: Let us recall from subsection 3.2 that $K = {}^{\omega}2^{(\bar{0})}$. From now on we will work "in K" instead of "in ω". By this we mean that the bijection $tr : \omega \rightarrowtail\!\!\!\!\twoheadrightarrow K$ introduced in subsection 3.2 induces an isomorphism between the semigroups $\langle {}^{\omega}\omega, \circ\rangle$ and $\langle {}^{K}K, \circ\rangle$, namely, the image of a function $\tau \in {}^{\omega}\omega$ is $(tr \circ \tau \circ tr^{-1})$ according to this isomorphism. In the rest of section 3, we will not distinguish between τ and $(tr \circ \tau \circ tr^{-1})$. Specifically, by $\bar{\sigma}, \pi, \mu, \eta, [0/1]$ we will always mean their isomorphic images $(tr \circ \bar{\sigma} \circ tr^{-1}), (tr \circ \pi \circ tr^{-1})$ etc. ◀

Let $\mathbb{G}^- \stackrel{\text{def}}{=} \{\bar{\sigma}, \pi, \mu, \eta\}$ and let $\mathbb{T}^-$ be the subsemigroup of $\langle {}^{\omega}\omega, \circ\rangle$ generated by $\mathbb{G}^-$. Along with $\mathbb{T}$, we will investigate $\mathbb{T}^-$ as well.

In the present subsection we give syntactic characterizations for the elements of $\mathbb{T}$ and $\mathbb{T}^-$, this will be Lemma 3.9 way below. Then we will use Lemma 3.9 for proving (ii) and (iii) of Theorem 3.2. We need to start with some definitions.

Notice that any $\tau \in \mathbb{G}$ is such that for any $s \in K$, $\tau(s)$ can be calculated by looking at a finite intial segment of s. For example, if $s = s_0 s_1 s_2 \ldots$ then
$\pi(s) = s_1 s_2 s_3 \ldots$
(we get $\pi(s)$ by omitting the first "letter" s_0 of s), and
$$\mu(s) = \begin{cases} 1 s_1 s_2 \ldots & \text{if } s_0 = 0 \\ 0 s_1 s_2 \ldots & \text{if } s_0 = 1 \end{cases}$$
(we replaced the first letter of s by its complement). This property of transformations in ${}^{K}K$ is formulated in the following definition.

DEFINITION 3.5: Let $\tau \in {}^{K}K$ be an arbitrary function.

(i) Let $m \in \omega$. We say that "*τ is defined by a table of length m*" iff
$$(\forall w \in {}^{m}\{0,1\})(\exists u \in \{0,1\}^*)(\forall q \in K)\, \tau(wq) = uq\,.$$
(Here u depends on w; thus a more precise notation would be u_w for u.)[32] We say that "*τ is defined by a forking table of length m*" iff
$$(\forall w \in {}^{m}\{0,1\})(\exists u_1, u_2 \in \{0,1\}^*)(\forall q \in K)\, \tau(wq) = \begin{cases} u_1 q & \text{if } q = \bar{0} \\ u_2 q & \text{if } q \neq \bar{0}\,. \end{cases}$$

(ii) We say that "*τ is defined by a (forking) table*" iff, for some $m \in \omega$, τ is defined by a (forking) table of length m.

(iii) Assume that τ is defined by a (forking) table. Then $m(\tau)$ denotes the smallest $m \in \omega$ such that τ is defined by a (forking) table of length m. ◀

If $\tau \in {}^{K}K$ is defined by a table of length m in the sense above and we are given the function ${}^{m}\{0,1\} \ni w \mapsto u_w \in \{0,1\}^*$ concretely then we can prepare a real "table" defining τ, as follows. Let the table consist of (finitely many) lines; let each line have two parts separated by the symbol $\longmapsto$; on the left hand sides of the lines let us list all the elements of ${}^{m}\{0,1\}$, and to any $w \in {}^{m}\{0,1\}$, put u_w to the right hand side. Forking tables defining elements of ${}^{K}K$ can be prepared similarly, see the examples below.

[31] The table notation presented in this subsection is taken from [MT73, pp.460–461] and [Th80, items 1.1–1.4].

[32] E.g. μ is defined by a table of length 1. In particular, $u_0 = 1$ and $u_1 = 0$.

EXAMPLES, NOTATION, CONVENTIONS 3.6:

If, e.g., $\tau(00q) = 1q$, $\tau(01q) = q$, $\tau(10q) = 000q$ and $\tau(11q) = 11q$ for all $q \in K$ then we say that *τ is defined by the table*

$$\begin{array}{rl} 00 \longmapsto & 1 \\ 01 \longmapsto & \\ 10 \longmapsto & 000 \\ 11 \longmapsto & 11\,. \end{array}$$

(This is a table of length 2.)

By

$$\text{"}\gamma \textit{ is defined by the forking table} \quad \begin{array}{rl} 0 \longmapsto & 0 \\ 1 \longmapsto & \left\{ \begin{array}{l} 110 \\ 1 \end{array} \right. \end{array} \text{"}$$

we mean that $\gamma \in {}^K K$ is such that

$$\begin{aligned} \gamma(0q) &= 0q \qquad \text{and} \\ \gamma(1q) &= \begin{cases} 110q & \text{if } q = \bar{0} \\ 1q & \text{if } q \neq \bar{0} \end{cases} \end{aligned}$$

for all $q \in K$.

We can abbreviate the table $\begin{array}{rl} 00 \longmapsto & w0 \\ 01 \longmapsto & w1 \\ 10 \longmapsto & v \\ 11 \longmapsto & u \end{array}$ by the shorter one $\begin{array}{rl} 0 \longmapsto & w \\ 10 \longmapsto & v \\ 11 \longmapsto & u \end{array}$. Thus, in the shorter table, words on the left hand side of $\longmapsto$ do not have the same length m ($m = 2$ in the present example), but, still, all the possible permutations with (possible) repetitions of $\{0, 1\}$ of length m are represented. Further,

$$\begin{array}{rl} 00 \longmapsto & 01 \\ 01 \longmapsto & 00 \end{array} \quad \text{stands for the table} \quad \begin{array}{rl} 00 \longmapsto & 01 \\ 01 \longmapsto & 00 \\ 1 \longmapsto & 1 \end{array} \quad \text{as a shorthand,}$$

that is, we may drop the lines of the form $w \longmapsto w$. ◀

Claim 3.7. *$\bar{\sigma}, \pi, \mu, \eta, [0/1]$ are defined by the following forking tables:*

$$\bar{\sigma} : 0 \longmapsto 00\,, \quad \pi : \begin{array}{rl} 0 \longmapsto & \\ 1 \longmapsto & \end{array}, \quad \mu : \begin{array}{rl} 0 \longmapsto & 1 \\ 1 \longmapsto & 0 \end{array}, \quad \eta : \begin{array}{rl} 01 \longmapsto & 10 \\ 10 \longmapsto & 01 \end{array}, \quad [0/1] : \longmapsto \left\{ \begin{array}{l} 1 \\ \ \end{array} \right. .$$

Proof: This follows directly from the definitions of $\bar{\sigma}, \pi, \mu, \eta, [0/1]$. ∎

Notation 3.8: If $g, h \in {}^K K$ then gh denotes their "relation composition" $g|h$ in the sense of, e.g., [HMT], that is,

$$gh \stackrel{\text{def}}{=} g|h \stackrel{\text{def}}{=} h \circ g\,. \quad ◀$$

In what follows, we will use the transformations to be defined in items (3.1), (3.3), (3.5), (3.7) below. First, we let

$$\sigma \stackrel{\text{def}}{=} \bar{\sigma}\mu\bar{\sigma}\mu\eta \ . \tag{3.1}$$

It is not difficult to check that

$$\sigma(q) = 0q \quad \text{for all } q \in K \ ,$$

that is, the table of σ is: $\longmapsto$ 0. See Figure 3.4 below.

$$\begin{array}{lcc} & 0q & 1q \\ \bar{\sigma}: & \downarrow & \downarrow \\ & 00q & 1q \\ \mu: & \downarrow & \downarrow \\ & 10q & 0q \\ \bar{\sigma}: & \downarrow & \downarrow \\ & 10q & 00q \\ \mu: & \downarrow & \downarrow \\ & 00q & 10q \\ \eta: & \downarrow & \downarrow \\ & 00q & 01q \end{array}$$

Figure 3.4

We note that, in the proof of Theorem 3.2, σ is going to play the rôle of the distinguished element σ according to the definition of a rich semigroup (the definition was given at the beginning of subsection 1.3), for $\mathbb{T}$, and its half–inverse will turn out to be π. Clearly,

$$\pi \circ \sigma = \text{Id} \quad \text{and} \quad \sigma \text{ is not onto.} \tag{3.2}$$

Next we define

$$\xi \stackrel{\text{def}}{=} \sigma\mu \, . \tag{3.3}$$

Clearly,

$$\xi(q) = \sigma\mu(q) = 1q \quad \text{for all } q \in K \, , \tag{3.4}$$

thus ξ is defined by the table $\longmapsto$ 1.

Let $R \stackrel{\text{def}}{=} K \smallsetminus Rng(\sigma)$. Then $R = \{1q : q \in K\}$. For any $\tau \in \mathbb{T}$, we define

$$\overset{\vee}{\tau} \stackrel{\text{def}}{=} (\tau)^{\vee} \stackrel{\text{def}}{=} (\sigma \circ \tau \circ \pi)^{R}_{\text{Id}} \, . \tag{3.5}$$

Then

$$\overset{\vee}{\tau}(1q) = 1q \ \text{ and } \ \overset{\vee}{\tau}(0q) = 0\tau(q) \quad \text{ for all } q \in K \, . \tag{3.6}$$

That is, the affect of $^{\vee}$ to τ is that it compresses τ down to the first half of the interval [0,1) and puts identity to the second half. Figure 3.5 shows an example for τ and $\overset{\vee}{\tau}$.

Figure 3.5

While $^\vee$ compresses τ to the first half, the next new function we introduce, $\tilde{}$, compresses τ to the second half of [0,1):

$$\tilde{\tau} \stackrel{\text{def}}{=} \mu\check{\tau}\mu \quad \text{for all } \tau \in \mathbb{T}\,. \tag{3.7}$$

Clearly,

$$\tilde{\tau}(0q) = 0q \quad \text{and} \quad \tilde{\tau}(1q) = 1\tau(q) \quad \text{for all } q \in K\,. \tag{3.8}$$

Let us recall the definitions of $\mathbb{T}$ and $\mathbb{T}^-$ from subsection 3.1 and from the beginning of the present subsection, respectively.

LEMMA 3.9. $\mathbb{T}^- = \{\tau \in {}^K K : \tau$ *is defined by a table*$\}$ *and*
$\mathbb{T} \;= \{\tau \in {}^K K : \tau$ *is defined by a forking table*$\}$.

In the proof[33] of Lemma 3.9 we will use Lemma 3.10. Therefore we formulate and prove Lemma 3.10 before proving Lemma 3.9.

For every $n \in \omega$ we let

$$\begin{aligned}
Tb_n &\stackrel{\text{def}}{=} \{\tau \in {}^K K : \tau \text{ is defined by a table of length } n\}\,,\\
Ftb_n &\stackrel{\text{def}}{=} \{\tau \in {}^K K : \tau \text{ is defined by a forking table of length } n\}\,,\\
Tb &\stackrel{\text{def}}{=} \bigcup\{Tb_n : n \in \omega\}\,,\\
Ftb &\stackrel{\text{def}}{=} \bigcup\{Ftb_n : n \in \omega\}\,.
\end{aligned}$$

For any $S \subseteq {}^K K$ we let

$$\check{S} \stackrel{\text{def}}{=} \{\check{\tau} : \tau \in S\}\,.$$

LEMMA 3.10. *For every $n \in \omega$, statements (P_n) and (F_n) below hold.*

$$Tb_n \cup \check{Tb}_n \subseteq \mathbb{T}^-\,, \tag{P_n}$$

$$Ftb_n \cup \check{Ftb}_n \subseteq \mathbb{T}\,. \tag{F_n}$$

We need some preparations for proving Lemma 3.10. We let

$$\rho \stackrel{\text{def}}{=} \bar{\sigma}\eta\bar{\sigma}\pi \qquad \text{and} \qquad \rho^- \stackrel{\text{def}}{=} \check{\mu}\mu\rho\check{\mu}\mu\,.$$

---

[33] A large part of the proof of Lemma 3.9 is already in Thompson [Th80]. For completeness, we include a full proof here (instead of using [Th80]).

Figure 3.6

Then it is easy to check that ρ is defined by the table

$$\begin{aligned} 0 &\longmapsto 00 \\ 10 &\longmapsto 01 \\ 11 &\longmapsto 1 \end{aligned}$$

and that ρ^{-} is the inverse of ρ. The following claim is not difficult to check either (e.g., by using tables).

Claim 3.11. *Statements (3.9–3.11) below hold. Further, (3.12) is a corollary of (3.9–3.11).*

$$\begin{aligned}
&(3.9) && \overset{\vee}{\mu} = \nu\bar{\sigma}\nu\tilde{\sigma}\pi\mu \quad \text{where } \nu = \eta\mu\eta \,. \\
& && \overset{\vee}{\pi} = \rho^{-}\bar{\sigma}\pi \,, \\
& && [0/1]^{\vee} = \eta[0/1]\eta \,, \\
& && \overset{\vee}{\sigma} = \bar{\sigma} \,. \\
&(3.10) && \overset{\vee}{\overset{\vee}{\tau}} = \rho^{-}\overset{\vee}{\tau}\rho \quad \textit{for all } \tau \in {}^{K}K \,. \\
&(3.11) && (\tau_1 \ldots \tau_n)^{\vee} = \overset{\vee}{\tau}_1 \ldots \overset{\vee}{\tau}_n \quad \textit{for all } \tau_1, \ldots, \tau_n \in {}^{K}K \,. \\
&(3.12) && \{\xi, \overset{\vee}{\xi}, \sigma, \overset{\vee}{\sigma}, \mu, \overset{\vee}{\mu}, \pi, \overset{\vee}{\pi}, \rho, \rho^{-}\} \subseteq \mathbb{T}^{-} \,. \quad \blacksquare
\end{aligned}$$

Claim 3.12. *If $\tau \in Tb_0$ then $\tau \in \{\sigma, \xi, \pi\}^*$. In particular,*

$$\begin{aligned} \tau = \mathrm{Id} &\Rightarrow \tau = \xi\pi \\ \tau \neq \mathrm{Id} &\Rightarrow \tau \in \{\sigma, \xi\}^* \,. \end{aligned}$$

Proof of Claim 3.12: Let us recall that ξ is defined by the table $\longmapsto 1$ and σ is defined by the table $\longmapsto 0$. Then clearly, $\xi\pi = \mathrm{Id}$.

If $\tau \in Tb_0$ then τ is defined by some table $\longmapsto u$ (where $u \in \{0,1\}^*$). The proof goes by induction on the length of u.

- If $\mathrm{length}(u) = 0$ then $\tau = \mathrm{Id}$ and thus $\tau = \xi\pi$ as we have seen above.
- If $\mathrm{length}(u) = 1$ then

$$\tau = \begin{cases} \xi & \text{if } u = 1 \\ \sigma & \text{if } u = 0 \,. \end{cases}$$

- Assume that for any $\tau' \in Tb_0$, if τ' is defined by the table $\longmapsto u'$ with $\mathrm{length}(u') = n$, $0 < n \in \omega$, then $\tau' \in \{\sigma, \xi\}^*$. If τ is defined by $\longmapsto u$ where $\mathrm{length}(u) = n + 1$ then $u = xu'$ with

$x \in \{0,1\}$ and length$(u') = n$. $\longmapsto$ u' defines some τ'. By the induction hypothesis, $\tau' \in \{\sigma, \xi\}^*$. Further, $\tau = \xi\tau'$ or $\tau = \sigma\tau'$, completing the proof of Claim 3.12. ∎

Proof of Lemma 3.10: We will proceed by induction on the length of the defining table.

Proof of (P_0): To prove (P_0), let $\tau \in Tb_0$. Then

$$\tau \in \{\sigma, \xi, \pi\}^* \subseteq \mathbb{T}^- \tag{3.13}$$

by Claim 3.12 and item (3.12) (in Claim 3.11).

By Claim 3.12, $\tau = \tau_0 \ldots \tau_n$ for some $n \in \omega$ and $\tau_0, \ldots, \tau_n \in \{\sigma, \xi, \pi\}$. $\check{\tau} = \check{\tau}_0 \ldots \check{\tau}_n$ by (3.11). $\{\check{\sigma}, \check{\xi}, \check{\pi}\} \subseteq \mathbb{T}^-$ by (3.12). Thus

$$\check{\tau} \in \{\check{\sigma}, \check{\xi}, \check{\pi}\}^* \subseteq \mathbb{T}^- . \tag{3.14}$$

Now (3.13) and (3.14) complete the proof of (P_0).

Proof of $(P_n) \Rightarrow (P_{n+1})$ and $(F_n) \Rightarrow (F_{n+1})$: Let τ be any function defined by a forking table of length $n+1$, say by

$$\begin{array}{l}
0w_1 \longmapsto \begin{cases} v_1^1 \\ v_1^2 \end{cases} \\
\ldots \\
0w_k \longmapsto \begin{cases} v_k^1 \\ v_k^2 \end{cases} \\
1u_1 \longmapsto \begin{cases} z_1^1 \\ z_1^2 \end{cases} \\
\ldots \\
1u_k \longmapsto \begin{cases} z_k^1 \\ z_k^2 \end{cases} \quad .
\end{array}$$

Let κ and χ be the functions defined, respectively, by the forking tables

$$\begin{array}{lcl}
w_1 \longmapsto \begin{cases} v_1^1 \\ v_1^2 \end{cases} & & u_1 \longmapsto \begin{cases} z_1^1 \\ z_1^2 \end{cases} \\
\ldots & \text{and} & \ldots \\
w_k \longmapsto \begin{cases} v_k^1 \\ v_k^2 \end{cases} & & u_k \longmapsto \begin{cases} z_k^1 \\ z_k^2 \end{cases}
\end{array}$$

of length n. (Thus $\kappa = \sigma\tau$ and $\chi = \xi\tau$.) Then it is not difficult to check that

$$\tau = \check{\kappa}\tilde{\chi}\pi = \check{\kappa}\mu\check{\chi}\mu\pi . \tag{3.15}$$

The second equality is clear by the definition of $\tilde{\chi}$. To see $\tau = \check{\kappa}\tilde{\chi}\pi$, notice that $Rng(\tau) \restriction [\bar{0}, 1\bar{0}) = Rng(\kappa)$, $Rng(\tau) \restriction [1\bar{0}, \bar{1}) = Rng(\chi)$; remember that $\check{\kappa}$ is the "compressed version" of κ to $[\bar{0}, 1\bar{0})$, and $\tilde{\chi}$ is the "compressed version" of χ to $[1\bar{0}, \bar{1})$; also notice that applying π after $\check{\kappa}$ and $\tilde{\chi}$ "decompresses" these compressed versions.

By our hypothesis (P_n) (or (F_n)), we have that $\check\kappa, \check\chi \in \mathbb{T}^-$ (or, respectively, $\check\kappa, \check\chi \in \mathbb{T}$). Then $\tau, \check\tau \in \mathbb{T}^-$ (or, resp., $\tau, \check\tau \in \mathbb{T}$) by (3.10–3.12), since

$$\check\tau = \check{\check\kappa}\check\mu\check{\check\chi}\check\mu\check\pi = \rho^-\check\kappa\rho\check\mu\rho^-\check\chi\rho\check\mu\check\pi\,.$$

Thus $(P_n) \Rightarrow (P_{n+1})$ and $(F_n) \Rightarrow (F_{n+1})$ have been proved.

By (P_0) then we have

$$(\forall n \in \omega)(P_n)\,.$$

Proof of (F_0): Let τ be a function defined by a forking table of length 0, say let τ be defined by

$$\longmapsto \begin{cases} w \\ v \end{cases}.$$

Let κ be defined by

$$\begin{aligned} 0 &\longmapsto v \\ 1 &\longmapsto w\,. \end{aligned}$$

Then it is easy to check that

$$\tau = \sigma[0/1]\kappa\,,$$

see Figure 3.7.

$$\begin{array}{cccc}
 & & q & \\
\sigma : & & \downarrow & \\
 & & 0q & \\
[0/1] : & q = \bar{0} \swarrow & & \searrow q \neq \bar{0} \\
 & 1q & & 0q \\
\kappa : & \downarrow & & \downarrow \\
 & vq & & wq
\end{array}$$

Figure 3.7

Then $\tau \in \mathbb{T}$ by $\kappa \in Tb \subseteq \mathbb{T}^-$, $\sigma \in \mathbb{T}^-$ and $[0/1] \in \mathbb{G}$; and $\check\tau \in \mathbb{T}$ by (3.9–3.12) and (P_1). This proves (F_0). Finally, $(\forall n \in \omega)(F_n)$ by induction, (F_0) and $\forall n\,((F_n) \Rightarrow (F_{n+1}))$. ∎

Proof of Lemma 3.9: We want to show that

$$\mathbb{T}^- = Tb \qquad \text{and} \qquad \mathbb{T} = Ftb\,.$$

By Lemma 3.10, $\mathbb{T}^- \supseteq Tb$ and $\mathbb{T} \supseteq Ftb$. It remains to show the inclusions in the other direction. By Claim 3.7, $\mathbb{G}^- \subseteq Tb$ and $\mathbb{G} \subseteq Ftb$. Further, shortly we will prove that Tb and Ftb are closed under composition of functions. This together with $\mathbb{G}^- \subseteq Tb$ and $\mathbb{G} \subseteq Ftb$ imply $\mathbb{T}^- \subseteq Tb$ and $\mathbb{T} \subseteq Ftb$, completing the proof of Lemma 3.9.

It remains to prove that Tb and Ftb are closed under composition. It is enough to prove that Ftb is closed (closedness of Tb is a special case of this). To this end, let $n, k \in \omega$, $\tau \in Ftb_n$ and $\nu \in Tb_k$ be arbitrary but fixed. We will prove that $\tau\nu \in Ftb_{n+k}$. To see the latter, let $w \in {}^{n+k}\{0,1\}$ be arbitrary but fixed. Then $w = w_1 w_2$ for some $w_1 \in {}^n\{0,1\}$ and $w_2 \in {}^k\{0,1\}$. We may assume that

$$\text{(3.16)}\qquad \text{the forking table of } \tau \text{ contains:}\quad w_1 \longmapsto \begin{cases} r_0 \\ r \end{cases} ; \qquad \text{and}$$

$$\text{(3.17)}\qquad \text{the forking table of } \nu \text{ contains:}\quad rw_2 \longmapsto \begin{cases} s_0 \\ s \end{cases} \qquad \text{and} \qquad r_0 w_2 \longmapsto \begin{cases} t_0 \\ t \end{cases}$$

because $|w_1| = n$, $|rw_2| \geqslant |w_2| = k$, $|r_0 w_2| \geqslant |w_2| = k$. We claim that

$$\text{(3.18)}\qquad \text{the forking table of } \tau\nu \text{ at } w \text{ is} \quad \begin{cases} w \longmapsto \begin{cases} s_0 \\ s \end{cases} & \text{if 1 occurs in } w_2 \\ w \longmapsto \begin{cases} t_0 \\ s \end{cases} & \text{otherwise.} \end{cases}$$

To prove (3.18), let first $\bar{0} \neq q \in K$. Then

$$\begin{aligned} \tau\nu(wq) &= \nu(\tau(w_1 w_2 q)) && \text{by } w = w_1 w_2 \\ &= \nu(r w_2 q) && \text{by } w_2 q \neq \bar{0} \text{ and (3.16)} \\ &= sq && \text{by } q \neq \bar{0} \text{ and (3.17).} \end{aligned}$$

In checking the case when $q = \bar{0}$, we have the following two cases.

Case 1: 1 occurs in w_2. Then

$$\begin{aligned} \tau\nu(w\bar{0}) &= \nu(\tau(w_1 w_2 \bar{0})) && \text{by } w = w_1 w_2 \\ &= \nu(r w_2 \bar{0}) && \text{by } w_2 \bar{0} \neq \bar{0} \text{ and (3.16)} \\ &= s_0 \bar{0} && \text{by (3.17).} \end{aligned}$$

Case 2: 1 does not occur in w_2. Then

$$\begin{aligned} \tau\nu(w\bar{0}) &= \nu(\tau(w_1 w_2 \bar{0})) && \text{by } w = w_1 w_2 \\ &= \nu(r_0 w_2 \bar{0}) && \text{by } w_2 \bar{0} = \bar{0} \text{ and (3.16)} \\ &= t_0 \bar{0} && \text{by (3.17).} \end{aligned}$$

We have proved (3.18), and thus completed the proof of Lemma 3.9. ∎

COROLLARY 3.13. $(\forall \tau \in \mathbb{T}^-)\, \overset{\vee}{\tau}, \tilde{\tau} \in \mathbb{T}^-$ *and* $(\forall \tau \in \mathbb{T})\, \overset{\vee}{\tau}, \tilde{\tau} \in \mathbb{T}$.

Proof: Assume $\tau \in \mathbb{T}^-$. Then τ is defined by a table by Lemma 3.9, and $\overset{\vee}{\tau} \in \mathbb{T}^-$ by $(\forall n \in \omega)(P_n)$. Thus, clearly, $\tilde{\tau} = \mu \overset{\vee}{\tau} \mu \in \mathbb{T}^-$, too. The proof of the statement for $\mathbb{T}$ is analogous, and is left to the reader. ∎

Proof of (ii) and (iii) of Theorem 3.2: Clearly, the first two conditions listed in (ii) of Theorem 3.2 are satisfied by our concrete σ and π (defined in (3.1) above and in Definition 3.1 in subsection 3.1, respectively), see (3.2) above. To complete the proof of (ii), it is enough to show that $\check{\mathbb{T}} \subseteq \mathbb{T}$. By Lemma 3.9, $\mathbb{T} = Ftb$; by Lemma 3.10, $\check{Ftb} \subseteq \mathbb{T}$. Thus we have $\check{\mathbb{T}} = \check{Ftb} \subseteq \mathbb{T}$, proving (ii).

For proving (iii), let $\tau \in \mathbb{T}$ and $p, q \in K$ be arbitrary. We want to show that $\tau^p_q \in \mathbb{T}$. By Lemma 3.9, there is an $m \in \omega$ such that τ is defined by a forking table of length m and $p = w\bar{0}$, $q = v\bar{0}$ for some $w, v \in {}^m\{0,1\}$. Then there is a line

$$w \longmapsto \begin{cases} u_1 \\ u_2 \end{cases}$$

in the forking table defining τ. It is not difficult to check that if we replace this line in the table with the line

$$w \longmapsto \begin{cases} v \\ u_2 \end{cases}$$

then the resulting new forking table will define τ^p_q. Thus $\tau^p_q \in \mathbb{T}$ by Lemma 3.10. We have proved (ii) and (iii) of Theorem 3.2. ∎

Remark 3.14: It would be interesting to find other, simple generator sets, too. In connection with this we note the following (i) and (ii).

(i) It would be nice to have σ in place of $\bar{\sigma}$ among the generator elements of $\mathbb{T}^-$. However, J. X. Madarász proved that the set $\{\sigma, \pi, \mu, \eta\}$ does not generate $\bar{\sigma}$.

(ii) We let $\mathbb{G}_1 \stackrel{\text{def}}{=} \{\sigma, \pi, \rho, \check{\sigma}, \check{\mu}\}$. Then $\mathbb{G}_1$ generates $\mathbb{T}^-$.

Proof (J. X. Madarász): By Lemma 3.9, $Tb = \mathbb{T}^-$. Therefore, to see $\mathbb{T}^- \subseteq [\mathbb{G}_1]$, it is enough to prove that $Tb \subseteq [\mathbb{G}_1]$. To see this, first we show

$$\{\xi, \check{\xi}, \sigma, \check{\sigma}, \mu, \check{\mu}, \pi, \check{\pi}, \rho, \rho^-\} \subseteq [\mathbb{G}_1] \tag{3.19}$$

as follows. It is easy to see that

$$\begin{aligned} \xi &= \sigma\sigma\check{\mu}\pi \\ \check{\xi} &= \check{\sigma}\check{\mu} \\ \mu &= \sigma\check{\mu}\pi\,. \end{aligned}$$

Further,

$$\begin{aligned} \rho^- &= \check{\mu}\mu\rho\mu\check{\mu} && \text{by definition, and} \\ \check{\pi} &= \rho^-\check{\sigma}\pi && \text{by (3.9) in Claim 3.11.} \end{aligned}$$

Now we can proceed along the lines of the proof of Lemma 3.9, as follows. $Tb_0 \subseteq [\mathbb{G}_1]$ by (3.19) and Claim 3.12.

Proof of $\left(Tb_n \cup \check{Tb}_n \subseteq [\mathbb{G}_1] \Longrightarrow Tb_{n+1} \cup \check{Tb}_{n+1} \subseteq [\mathbb{G}_1]\right)$:

Assume $Tb_n \cup \check{Tb}_n \subseteq [\mathbb{G}_1]$. Let $\tau \in Tb_{n+1}$, $\kappa, \chi \in Tb_n$ be as in the "Proof of $(P_n) \Rightarrow (P_{n+1})$ and $(F_n) \Rightarrow (F_{n+1})$" (inside of the proof of Lemma 3.10). So $\tau = \check{\kappa}\mu\check{\chi}\mu\pi$ (this is (3.15)). Here $\check{\kappa}, \check{\chi} \in \check{Tb}_n \subseteq [\mathbb{G}_1]$ and $\mu, \pi \in [\mathbb{G}_1]$ (by (3.19)). Thus $\tau \in [\mathbb{G}_1]$. $\check{\tau} = \check{\check{\kappa}}\check{\mu}\check{\check{\chi}}\check{\mu}\check{\pi}$. $\check{\check{\kappa}} = \rho^-\check{\kappa}\rho$ and $\check{\check{\chi}} = \rho^-\check{\chi}\rho$ by (3.10) in Claim 3.11. Thus $\check{\tau} = \rho^-\check{\kappa}\rho\check{\mu}\rho^-\check{\chi}\rho\check{\mu}\check{\pi}$, so $\check{\tau} \in [\mathbb{G}_1]$ by (3.19) and by $\check{\kappa}, \check{\chi} \in \check{Tb}_n \subseteq [\mathbb{G}_1]$. ∎

Hence any $\mathbb{G}_2 \subseteq \mathbb{T}^-$ generating $\mathbb{G}_1$ is such that $\mathbb{G}_2 \cup \{[0/1]\}$ generates $\mathbb{T}$. E.g., $\{\sigma, \pi, \check{\mu}, \theta\}$ is a generator set for $\mathbb{T}^-$, too, where θ is defined by

$$\begin{aligned} 001 &\longmapsto 01 \\ 01 &\longmapsto 001 \end{aligned}$$

(since $\bar{\sigma} = \sigma\sigma\theta\pi$, $\rho = \sigma\check{\mu}\theta\pi\check{\mu}$). Thompson's original generator set is given in [Th80, Thm.1.4][34] .
◀

[34] It was this generator set which he recommended us for being extended with [0/1] (so that his original proof methods could be generalized to the extended generator set).

3.4. Defining relations for the monoids $\mathbb{T}$ and $\mathbb{T}^-$

Recall that $\mathbb{T}$ is not only a semigroup but it is a *monoid* as well (with distinguished identity element Id). We will use the same letter $\mathbb{T}$ to refer to the monoid, too. In particular, the defining relations for $\mathbb{T}$ and $\mathbb{T}^-$ collected below will be in the *monoid language* (we will use Id as the distinguished constant symbol of monoids).

In subsections 3.4–3.6 we will prove finite presentability of the monoid $\mathbb{T}$ (and in the meantime, we will prove finite presentability of $\mathbb{T}^-$, too). Our plan is the following.

Let us recall from Convention 3.4 that we identify the isomorphic semigroups $\langle {}^{\omega}\omega, \circ\rangle$ and $\langle {}^{K}K, \circ\rangle$. If $w \in \mathbb{G}^*$ then we define $hat(w) = \hat{w} \in {}^{K}K$ as follows:

If w is the empty word λ then $\hat{w} \stackrel{\text{def}}{=} \mathrm{Id}_K \in {}^{K}K$.
If $w \in \mathbb{G}$ then $\hat{w} \stackrel{\text{def}}{=} w \in {}^{K}K$.
If $w = \gamma u$ for some $\gamma \in \mathbb{G}$ then $hat(\gamma u) \stackrel{\text{def}}{=} \hat{\gamma}\hat{u} (= \gamma\hat{u}) \in {}^{K}K$.

Equivalently, $hat : \mathbb{G}^* \longrightarrow \mathbb{T}$ is the natural anti–homomorphism extending the identical embedding $\mathrm{Id}_{\mathbb{G}} : \mathbb{G} \rightarrowtail \mathbb{T}$ to the free *monoid* $\mathbb{G}^*$ (cf. subsection 1.4.4). Then, clearly, $\hat{\lambda} = \mathrm{Id}_K$ (λ and Id_K are the unit elements of the two monoids).

Thus $\hat{w}$ is the function naturally denoted (or named) by the word w. We will often write w instead of $\hat{w}$ when no confusion is likely to arise.

Next we recall the definition of *finite presentability of a monoid*. Let M denote the variety of all monoids and let $\mathfrak{M} \in \mathsf{M}$ be such that $\mathfrak{M}$ can be generated by some set G. Let

$$\mathrm{Term}^G \stackrel{\text{def}}{=} \{t(g_1, \ldots, g_n) : g_1, \ldots, g_n \in G,\ t \text{ an } n\text{–ary term function in the language of monoids},\ n \in \omega\}.$$

Let $E \subseteq \mathrm{Term}^G \times \mathrm{Term}^G$. Let V denote the following variety of similarity type $G \cup$ "monoid symbols": $V \stackrel{\text{def}}{=} Mod(E) \cap \mathsf{M}$. (Thus the elements of G are constant symbols of V, and $\mathsf{V} \models E$.) We say that $\langle G, E\rangle$ is a *presentation of* $\mathfrak{M} \in \mathsf{M}$ iff $\mathfrak{M}$ is the monoid reduct of the $\emptyset$–generated free algebra of V. In this case E is called the set of *defining relations* with parameters from G. We say that $\mathfrak{M}$ has a *finite presentation* iff there exists a presentation $\langle G, E\rangle$ of $\mathfrak{M}$ with both G and E finite.

Now let us return to our concrete monoid $\mathbb{T}$. Let E be a set of defining relations in the monoid language with parameters from $\mathbb{G}$. If $u, w \in \mathbb{G}^*$ then we define

$$\models u = w \stackrel{\text{def}}{\Longleftrightarrow} \hat{u} = \hat{w} \qquad \text{and} \qquad \vdash u = w \stackrel{\text{def}}{\Longleftrightarrow} E \vdash u = w\,.$$

Here "$E \vdash u = w$" means derivability of the equation $u = w$ from E, in the equational logic of semigroups.

We will give a finite set E of defining relations and prove that

$$(\forall u, w \in \mathbb{G}^*)\big(\models u = w \Longleftrightarrow \vdash u = w\ \big)\,,$$

for this particular set E. This implies finite presentability of $\mathbb{T}$. Next we start implementing this plan.

DEFINITION 3.15: We say that the function $wd : \mathbb{T} \longrightarrow \mathbb{G}^*$ is a *naming function* for $\mathbb{G}^*$ iff wd satisfies conditions (i)–(iii) below.

(i) For every $\tau \in \mathbb{T}$, $wd(\tau) \in hat^{-1}(\tau)$;
(ii) $wd(\mathrm{Id}_K) = \lambda$;
(iii) for every $\tau \in \mathbb{T}^-$, $wd(\tau) \in \mathbb{G}^{-*}$. ◄

Conditions (i) and (ii) in Definition 3.15 make sense because for every $\tau \in \mathbb{T}$, $hat^{-1}(\tau) = \{w \in \mathbb{G}^* : \hat{w} = \tau\} \neq \emptyset$ (because $hat : \mathbb{G}^* \longrightarrow \mathbb{T}$ was defined to be a *surjective* function); and for every $\tau \in \mathbb{T}^-$, $hat^{-1}(\tau) \cap \mathbb{G}^{-*} \neq \emptyset$.

Convention 3.16:

(i) From now on, let wd be a fixed naming function for $\mathbb{G}$ such that $wd(\gamma) = \gamma$ for every $\gamma \in \mathbb{G}$.[35] (Otherwise wd is arbitrary.)

(ii) From now on, instead of $wd(\nu)$, we will often write ν only, in particular we will consider σ, ξ to be words over $\mathbb{G}^-$. As we already did in subsection 1.1.4, again, we will use the usual identification of $(\mathbb{G}^{-*})^*$ with $\mathbb{G}^{-*}$, that is, if $w_1, \ldots, w_n \in \mathbb{G}^{-*}$ then we consider the word $w_1 \ldots w_n$ $(= \langle w_1, \ldots, w_n \rangle)$ to be an element of $\mathbb{G}^{-*}$ by identifying $w_1 \ldots w_n$ with $w_1{}^\frown \ldots {}^\frown w_n$. Then $\{\sigma, \xi\}^* \subseteq \mathbb{G}^{-*}$. For example, by $\sigma\xi$ we mean the word $wd(\sigma)^\frown wd(\xi)$ (e.g., $\bar{\sigma}\mu\bar{\sigma}\mu\eta\bar{\sigma}\mu\bar{\sigma}\mu\eta\mu$, if we choose $wd(\sigma) = \bar{\sigma}\mu\bar{\sigma}\mu\eta$, $wd(\xi) = wd(\sigma)\mu$). Similarly, for all $m \in \omega$, ${}^m\{\sigma, \xi\} \subseteq \mathbb{G}^{-*}$. If $v \in {}^m\{\sigma, \xi\}$ then we will write $|v| = m$. ◀

We have already defined $\overset{\vee}{\gamma}$ and $\tilde{\gamma}$ for functions $\gamma \in {}^K K$. In the next definition we expand $^\vee$ and $\tilde{}$ to words $w \in \mathbb{G}^*$, too (the natural way). Recall from Corollary 3.13 (from the previous subsection) that $(\forall \tau \in \mathbb{T})\overset{\vee}{\tau} = (\sigma \circ \tau \circ \pi)^R_{\mathrm{Id}} \in \mathbb{T}$.

DEFINITION 3.17: We define $^\vee : \mathbb{G}^* \longrightarrow \mathbb{G}^*$ and $\tilde{} : \mathbb{G}^* \longrightarrow \mathbb{G}^*$ as follows. For the empty word λ,

$$\overset{\vee}{\lambda} \overset{\text{def}}{=} \lambda = wd(\mathrm{Id}_K)\,.$$

For all $\gamma \in \mathbb{G}$, we let

$$\overset{\vee}{\gamma} \overset{\text{def}}{=} wd(\overset{\vee}{\gamma}) = wd\big((\sigma \circ \gamma \circ \pi)^R_{\mathrm{Id}}\big) = (\sigma \circ \gamma \circ \pi)^R_{\mathrm{Id}}$$

(see Convention 3.16 for wd and recall that $R = K \setminus Rng(\sigma) = \{1q : q \in K\}$),
and for all $n \in \omega$ and $\gamma_1, \ldots, \gamma_n \in \mathbb{G}$ we define

$$(\gamma_1 \ldots \gamma_n)^\vee \overset{\text{def}}{=} \overset{\vee}{\gamma}_1 \ldots \overset{\vee}{\gamma}_n\,.$$

For all $w \in \mathbb{G}^*$ we define

$$\tilde{w} \overset{\text{def}}{=} \mu\overset{\vee}{w}\mu\,. \quad ◀$$

Claim 3.18.

(i) $(\forall w \in \mathbb{G}^{-*})\hat{w} \in \mathbb{T}^-$.

(ii) $(\forall w \in \mathbb{G}^*)\overset{\vee}{\hat{w}} = \overset{\hat{\vee}}{w}$.

(iii) $(\forall u, w \in \mathbb{G}^*) \models (uw)^\vee = \overset{\vee}{u}\overset{\vee}{w}$.

Proof: These statements are straightforward consequences of the definitions involved, therefore checking them is left to the reader. ▮

Next we are giving a set E of defining relations for $\mathbb{T}$. A subset of E will serve to be a set of defining relations for $\mathbb{T}^-$.

[35]This requirement is not necessary for our purposes, but makes some calculations simpler.

DEFINITION 3.19: Let $^\vee : \mathbb{G}^* \longrightarrow \mathbb{G}^*$, $\tilde{} : \mathbb{G}^* \longrightarrow \mathbb{G}^*$ be as in Definition 3.17. Recall that $\sigma = \bar{\sigma}\mu\bar{\sigma}\mu\eta$; $\xi = \sigma\mu$; $\rho = \bar{\sigma}\eta\bar{\sigma}\pi$ and $\rho^- = \check{\mu}\mu\rho\check{\mu}\mu$.

We let (F1–F7) be the following groups of defining relations. For all $\gamma, \chi \in \mathbb{G}$:

(F1) $\mu\mu = \eta\eta = \rho\rho^- = \rho^-\rho = \sigma\pi = \xi\pi = \check{\sigma}\tilde{\xi}\pi = \mathrm{Id}\,,$ $\bar{\sigma} = \check{\sigma}\,.$

(F2) $\xi\bar{\sigma} = \xi\,,$ $\xi\xi\eta = \xi\xi\,,$ $\sigma\sigma\eta = \sigma\sigma = \sigma\check{\sigma}\,,$
$\xi\mu = \sigma\,,$ $\xi\sigma\eta = \xi\xi\,,$ $\sigma\xi\eta = \xi\eta\,.$

(F3) For all $\gamma, \chi \in \mathbb{G}$:

$$\check{\check{\gamma}} = \rho^-\check{\gamma}\rho\,, \qquad \check{\gamma}\tilde{\chi} = \tilde{\chi}\check{\gamma}\,, \qquad \check{\gamma}\tilde{\gamma}\pi = \pi\gamma\,.$$

(F4) $\sigma[0/1]\pi = \mathrm{Id}\,.$

(F5) $\xi[0/1] = \xi\,,$ $\xi\sigma[0/1] = \xi\sigma\,.$

(F6) $[0/1]\check{[0/1]} = [0/1]\,,$ $\sigma[0/1][0/1]\tilde{} = \sigma[0/1]\tilde{\xi}\,.$

(F7) $\check{\sigma}[0/1] = [0/1]\check{\sigma}\,,$ $\sigma[0/1]\tilde{\sigma} = \sigma[0/1]\,.$

We let $E_1 \stackrel{\text{def}}{=} (\text{F1–F7})$ and $E \stackrel{\text{def}}{=} E_1 \cup \{\check{u} = \check{w} : (u = w) \in E_1\}$. ◀

In E we use only the elements of $\mathbb{G}$ as parameters, because $(\forall\gamma \in \mathbb{G})\check{\gamma} \in \mathbb{G}^*$ by Corollary 3.13.

PROPOSITION 3.20 (Soundness of defining relations). *For every $e \in E$, $\mathbb{T} \models e$.*

Proof: The proof is simple semantical checking and is left to the reader. ∎

3.5. Finite presentation of $\mathbb{T}^-$

Recall that in Definition 3.19 we gave the finite set E of defining relations in terms of the elements of $\mathbb{G}$. The following theorem uses a (proper) subset of E using only elements of $\mathbb{T}^-$ as its parameters.

THEOREM 3.21 (Finite presentation of $\mathbb{T}^-$).
(F1–F3) *in Definition 3.19 together with* $\{\overset{\vee}{u} = \overset{\vee}{w} : (u = w) \in \text{(F1–F3)}\}$ *constitute a finite presentation for* $\mathbb{T}^-$.

The rest of the present subsection is devoted to the proof of Theorem 3.21.

We define

$$Norm^- \overset{\text{def}}{=} \{\sigma, \xi\}^* .$$

We would like to state that every element w of ${\mathbb{G}^-}^*$ "*has a normal form*", that is, that every $w \in {\mathbb{G}^-}^*$ is provably equal to some $v' \in Norm^-$. Instead of this, in Lemma 3.22 below, we can only state the following weaker statement. For every $w \in {\mathbb{G}^-}^*$ there is a number $m \in \omega$ such that whenever v is a $\{\sigma, \xi\}$–sequence of length m then vw is provably equal to some $v' \in Norm^-$.

LEMMA 3.22 (Normal form lemma for $\mathbb{T}^-$).
For every $w \in {\mathbb{G}^-}^*$ there exists an $m \in \omega$ satisfying conditions (i) and (ii) below.

(i) $(\forall v \in {}^m\{\sigma, \xi\})(\exists v' \in Norm^-) \vdash vw = v'$.
(ii) $(\forall v \in {}^m\{\sigma, \xi\})(\exists v' \in Norm^-) \models vw = v'$.

Proof: By induction on the length of w:

- If $w = \lambda$ then $m = 0$ works ($\models \lambda = \lambda$ and $\vdash \lambda = \lambda$ by equational logic).
- If $w \in \mathbb{G}^- = \{\bar{\sigma}, \pi, \mu, \eta\}$ then let

$$\begin{aligned} H = \{ \quad & \sigma\bar{\sigma} = \sigma\sigma, \quad \xi\bar{\sigma} = \xi, \\ & \sigma\pi = \lambda, \quad \xi\pi = \lambda, \\ & \sigma\mu = \xi, \quad \xi\mu = \sigma, \\ & \sigma\sigma\eta = \sigma\sigma, \quad \sigma\xi\eta = \xi\sigma, \quad \xi\sigma\eta = \sigma\xi, \quad \xi\xi\eta = \xi\xi \quad \} . \end{aligned}$$

  Then for every $e \in H$, $\vdash e$, since e occurs among the defining relations listed in Definition 3.19; and $\models e$ by Proposition 3.20.
- Assume (i) and (ii) hold for w, that is, assume

$$\text{(IH)} \qquad (\exists m \in \omega)\,(\forall v \in {}^m\{\sigma, \xi\})\,(\exists v' \in Norm^-)\,(\vdash vw = v' \text{ and } \models vw = v') .$$

  Let $m \in \omega$ be such. Let $\gamma \in \mathbb{G}^-$. We are proving (i) and (ii) for γw.

  Notice that if (i) and (ii) hold for some number $k \in \omega$ then they hold for every $k' > k$ as well. Therefore, and by the previous step of this proof, we have

$$(3.20) \qquad (\forall v \in {}^2\{\sigma, \xi\})\,(\exists v' \in Norm^-)\,(\vdash v\gamma = v' \text{ and } \models v\gamma = v') .$$

  Let $u \in {}^{m+2}\{\sigma, \xi\}$ be arbitrary but fixed. Then $u = u'ab$ for some $a, b \in \{\sigma, \xi\}$. Thus

$$\begin{aligned} u\gamma w &= u'ab\gamma w \\ &= u'u"w && \text{for some } u" \in Norm^-, \text{ by (3.20)} \\ &= v' && \text{for some } v' \in Norm^- \text{ by (IH) and by } |u'u"| \geqslant m . \end{aligned}$$

  The steps made in the above calculation are correct both in the semantical ($\models$) and syntactical ($\vdash$) sense. Therefore

$$\vdash u\gamma w = v' \qquad \text{and } \models u\gamma w = v' .$$

  Since $u \in {}^{m+2}\{\sigma, \xi\}$ was arbitrary, the above line proves (i) and (ii) for γw. This completes our induction. ∎

COROLLARY 3.23. *Let $u, w \in \mathbb{G}^{-*}$. Then*

$$\models u = w \ \textit{ implies } \ (\exists m \in \omega)(\forall v \in {}^m\{\sigma, \xi\}) \vdash vu = vw\,.$$

Proof: Let $u, w \in \mathbb{G}^{-*}$ and assume $\models u = w$. Let $\Phi(m, w)$ denote the statement in Lemma 3.22 (i). Then, by Lemma 3.22 (i), there exist $m_1, m_2 \in \omega$ such that $\Phi(m_1, u)$ and $\Phi(m_2, w)$ hold for our u and w. As we already observed in the proof of Lemma 3.22, $\Phi(k, v)$ implies $\Phi(k', v)$ for all $k' \geqslant k$ and for any v. Therefore, for $m \stackrel{\text{def}}{=} m_1 + m_2$ and for any $v \in {}^m\{\sigma, \xi\}$ we have

$$\vdash vu = t_1 \quad \text{and} \quad \vdash vw = t_2 \qquad \text{for some } t_1, t_2 \in \{\sigma, \xi\}^* \tag{3.21}$$

by Lemma 3.22 (i). Since $\models u = w$ by hypothesis, we have $\models vu = vw$. Therefore $\models t_1 = t_2$. But then t_1 and t_2 are the same word, by the definitions of σ and ξ. Thus $\vdash t_1 = t_2$. This implies $\vdash vu = vw$ by (3.21). ∎

LEMMA 3.24. $(\forall u, w \in \mathbb{G}^*)\big((\vdash \sigma u = \sigma w \ \textit{ and } \ \vdash \xi u = \xi w) \Longrightarrow\ \vdash u = w \ \big)$.

The idea of proving Lemma 3.24 is to write up w as a term of σw and ξw, see Claim 3.25 below. Therefore, to prove Lemma 3.24, first we prove the technical Claim 3.25.

Claim 3.25. *Let $u, w \in \mathbb{G}^*$ be arbitrary. Then (i) and (ii) below hold.*
 (i) $\vdash w = (\sigma w)^\vee (\xi w)^\sim \pi$
 (ii) $\vdash u = w \Longrightarrow (\vdash \overset{\vee}{u} = \overset{\vee}{w}$ *and* $\vdash \tilde{u} = \tilde{w})$.

Proof: *Proof of (i):* Let $w = \gamma_1 \ldots \gamma_n$ for $n \in \omega$ and $\gamma_1, \ldots, \gamma_n \in \mathbb{G}$. Then the following derivation is a proof for $(\sigma w)^\vee (\xi w)^\sim \pi$.

$$\begin{aligned}
(\sigma w)^\vee (\xi w)^\sim \pi &= (\overset{\vee}{\sigma}\overset{\vee}{\gamma}_1 \ldots \overset{\vee}{\gamma}_n)(\mu\overset{\vee}{\xi}\overset{\vee}{\gamma}_1 \ldots \overset{\vee}{\gamma}_n\mu)\pi && \text{by the def's of } {}^\vee \text{ and } {}^\sim \\
&= (\overset{\vee}{\sigma}\overset{\vee}{\gamma}_1 \ldots \overset{\vee}{\gamma}_n)(\mu\overset{\vee}{\xi}\mu\mu\overset{\vee}{\gamma}_1\mu \ldots \mu\overset{\vee}{\gamma}_n\mu)\pi && \text{by } \ (\mu\mu = \mathrm{Id}) \in (\mathrm{F1}) \\
&= (\overset{\vee}{\sigma}\overset{\vee}{\gamma}_1 \ldots \overset{\vee}{\gamma}_n)(\tilde{\xi}\tilde{\gamma}_1 \ldots \tilde{\gamma}_n)\pi && \text{by the definition of } {}^\sim \\
&= \overset{\vee}{\sigma}\tilde{\xi}\overset{\vee}{\gamma}_1 \ldots \overset{\vee}{\gamma}_n\tilde{\gamma}_1 \ldots \tilde{\gamma}_n\pi && \text{by } \ (\overset{\vee}{\gamma}\tilde{\chi} = \tilde{\chi}\overset{\vee}{\gamma}) \in (\mathrm{F3}) \\
&= \overset{\vee}{\sigma}\tilde{\xi}\overset{\vee}{\gamma}_1\tilde{\gamma}_1 \ldots \overset{\vee}{\gamma}_n\tilde{\gamma}_n\pi && \text{by } \ (\overset{\vee}{\gamma}\tilde{\chi} = \tilde{\chi}\overset{\vee}{\gamma}) \in (\mathrm{F3}) \\
&= \overset{\vee}{\sigma}\tilde{\xi}\pi\gamma_1 \ldots \gamma_n && \text{by } \ (\overset{\vee}{\gamma}\tilde{\gamma}\pi = \pi\gamma) \in (\mathrm{F3}) \\
&= \gamma_1 \ldots \gamma_n && \text{by } \ (\overset{\vee}{\sigma}\tilde{\xi}\pi = \mathrm{Id}) \in (\mathrm{F1}) \\
&= w && \text{by the definition of } w.
\end{aligned}$$

Proof of (ii): By definition, clearly, if $\vdash \overset{\vee}{u} = \overset{\vee}{w}$ then $\vdash \tilde{u} = \tilde{w}$. Thus it is enough to prove that $\vdash \overset{\vee}{u} = \overset{\vee}{w}$ for any $u, w \in \mathbb{G}^*$. First we show that

$$(\forall (u = w) \in E) \vdash \overset{\vee}{u} = \overset{\vee}{w} \quad \Longrightarrow \quad (\forall u, w \in \mathbb{G}^*)(\vdash u = w \Rightarrow \vdash \overset{\vee}{u} = \overset{\vee}{w})\,. \tag{3.22}$$

Namely, let $u, w \in \mathbb{G}^*$ be arbitrary but fixed. Assume $\vdash u = w$. Then $(u = w)$ has a derivation in the equational logic of monoids, from E. Let $pr = \langle s_0 = t_0, \ldots, s_n = t_n \rangle$ be such a derivation. We are building a derivation of $(\overset{\vee}{u} = \overset{\vee}{w})$ from pr as follows. First we expand the function ${}^\vee : \mathbb{G}^* \longrightarrow \mathbb{G}^*$ to all terms of monoids by postulating $\overset{\vee}{x} \stackrel{\text{def}}{=} x$. Now consider $\overset{\vee}{pr} \stackrel{\text{def}}{=} \langle \overset{\vee}{s}_0 = \overset{\vee}{t}_0, \ldots, \overset{\vee}{s}_n = \overset{\vee}{t}_n \rangle$. We will prove that $\overset{\vee}{pr}$ is a derivation of $(\overset{\vee}{u} = \overset{\vee}{w})$ from E in the equational logic of monoids, by checking that all the rules of equational logic "respect $^\vee$".

Since pr is a derivation from E in the equational logic of monoids,

(i) $\langle s_0, t_0\rangle$ is

- either a pair of identical terms; then $\langle \overset{\vee}{s_0}, \overset{\vee}{t_0}\rangle$ is a pair of identical terms again,
- or an element of E or a monoid axiom. Then $(\overset{\vee}{s_0} = \overset{\vee}{t_0}) \in E$ in the first case, and $(\overset{\vee}{s_0} = \overset{\vee}{t_0})$ is a monoid theorem in the second case, by Claim 3.18 (iii).

(ii) For $i \in \{1, \dots, n\}$, $(s_i = t_i)$ can be derived from the previous members of pr using the rules of equational logic. Now there are 4 cases:

- $(t_i = s_i)$ occurs among the members of pr preceding $(s_i = t_i)$. Then we can use the derivation $\dfrac{\overset{\vee}{t_i} = \overset{\vee}{s_i}}{\overset{\vee}{s_i} = \overset{\vee}{t_i}}$ in $\overset{\vee}{pr}$.
- $(s_i = t_i)$ was included in pr by deriving $\dfrac{s_i = q,\ q = t_i}{s_i = t_i}$ for some term q with $\langle s_i, q\rangle = \langle s_k, t_k\rangle$ and $\langle q, t_i\rangle = \langle s_\ell, t_\ell\rangle$ for some $k < i$, $\ell < i$. Then $\dfrac{\overset{\vee}{s_i} = \overset{\vee}{q},\ \overset{\vee}{q} = \overset{\vee}{t_i}}{\overset{\vee}{s_i} = \overset{\vee}{t_i}}$ derives $(\overset{\vee}{s_i} = \overset{\vee}{t_i})$.
- Leibniz rule: $t_i = (s_i)^p_q$ for some terms p, q such that p is a subterm of s_i and $\langle p, q\rangle = \langle s_k, t_k\rangle$ for some $k < i$. Then $\dfrac{\overset{\vee}{p} = \overset{\vee}{q}}{\overset{\vee}{s_i} = \overset{\vee}{t_i}}$ because $\big((s_i)^p_q\big)^\vee = (\overset{\vee}{s_i})^{\overset{\vee}{p}}_{\overset{\vee}{q}}$.
- Rule of substitution: $s_i = p^{\bar{x}}_{\bar{r}}$ and $t_i = q^{\bar{x}}_{\bar{r}}$ for some $\langle p, q\rangle = \langle s_k, t_k\rangle$ with $k < i$, for variables $\bar{x} = \langle x_1, \dots, x_m\rangle$ occurring in p and terms $\bar{r} = \langle r_1, \dots, r_m\rangle$. Then $\dfrac{\overset{\vee}{p} = \overset{\vee}{q}}{\overset{\vee}{s_i} = \overset{\vee}{t_i}}$ because $\overset{\vee}{s_i} = (\overset{\vee}{p})^{\overset{\vee}{\bar{x}}}_{\overset{\vee}{\bar{r}}}$ and $\overset{\vee}{t_i} = (\overset{\vee}{q})^{\overset{\vee}{\bar{x}}}_{\overset{\vee}{\bar{r}}}$ with $\overset{\vee}{\bar{x}} = \langle \overset{\vee}{x_1}, \dots, \overset{\vee}{x_m}\rangle = \langle x_1, \dots, x_m\rangle$ and $\overset{\vee}{\bar{r}} = \langle \overset{\vee}{r_1}, \dots, \overset{\vee}{r_m}\rangle$.

By (i) and (ii) above, $\overset{\vee}{pr}$ is a derivation of $(\overset{\vee}{u} = \overset{\vee}{w})$. This proves (3.22).

So next we are aiming at proving $\big(\forall(u = w) \in E\big) \vdash \overset{\vee}{u} = \overset{\vee}{w}$. First we show that

$$\vdash \overset{\vee\vee}{w} = \rho^- \overset{\vee}{w} \rho \quad \text{for all } w \in \mathbb{G}^*. \tag{3.23}$$

Cf. (3.10) in Claim 3.11 above. A derivation for $\overset{\vee\vee}{w} = \rho^- \overset{\vee}{w} \rho$ is the following:

$$\begin{aligned}
\overset{\vee\vee}{w} &= \overset{\vee\vee}{\gamma_1} \dots \overset{\vee\vee}{\gamma_n} && \text{by the definition of } {}^\vee \\
&= (\rho^- \overset{\vee}{\gamma_1}\rho)(\rho^- \overset{\vee}{\gamma_2}\rho)\dots(\rho^- \overset{\vee}{\gamma_n}\rho) && \text{by } (\overset{\vee\vee}{\gamma} = \rho^- \overset{\vee}{\gamma}\rho) \in (\mathrm{F3}) \\
&= \rho^- \overset{\vee}{\gamma_1}\dots\overset{\vee}{\gamma_n}\rho && \text{by } (\rho\rho^- = \mathrm{Id}) \in (\mathrm{F1}) \\
&= \rho^- \overset{\vee}{w}\rho && \text{by the definition of } {}^\vee.
\end{aligned}$$

We have proved (3.23).

Let now $(u = w) \in E$. We want to show $\vdash \overset{\vee}{u} = \overset{\vee}{w}$. If $(u = w) \in E_1$ then we are done by the definition of E. Assume that $(\overset{\vee}{u} = \overset{\vee}{w})$ is an instance of $E(\smallsetminus E_1)$. Then

$$\begin{aligned} \overset{\vee\vee}{u} &= \rho^- \overset{\vee}{u} \rho && \text{by (3.23)} \\ &= \rho^- \overset{\vee}{w} \rho && \text{by } (\overset{\vee}{u} = \overset{\vee}{w}) \in E \\ &= \overset{\vee\vee}{w} && \text{by (3.23).} \end{aligned}$$

Thus $\vdash \overset{\vee\vee}{u} = \overset{\vee\vee}{w}$. ∎

Proof of Lemma 3.24: Assume $\vdash \sigma v = \sigma w$ and $\vdash \xi v = \xi w$. Then $\vdash (\sigma v)^{\vee} = (\sigma w)^{\vee}$ and $\vdash (\xi v)^{\smile} = (\xi w)^{\smile}$ by Claim 3.25 (ii), hence $\vdash w = (\sigma w)^{\vee}(\xi w)^{\smile}\pi = (\sigma v)^{\vee}(\xi v)^{\smile}\pi = v$ by Claim 3.25 (i). ∎

Proof of Theorem 3.21: It is enough to prove that

$$(\forall u, w \in \mathbb{G}^{-*})(\models u = w \iff \vdash u = w)\,.$$

Direction "$\Longleftarrow$" follows from Proposition 3.20 and the soundness of equational logic. To prove

$$(\forall u, w \in \mathbb{G}^{-*})(\models u = w \Longrightarrow \vdash u = w)\,,$$

let $u, w \in \mathbb{G}^{-*}$ be arbitrary but fixed, and assume

$$\models u = w\,.$$

Then, by Corollary 3.23, there exists an $m \in \omega$ such that

$$(\forall v \in {}^{m}\{\sigma, \xi\}) \vdash vu = vw\,. \tag{3.24}$$

Next we prove, by induction on m, that for *any* $m \in \omega$,

$$(3.24) \Longrightarrow \vdash u = w\,. \tag{3.25}$$

- (3.25) trivially holds for $m = 0$.
- Assume (3.25) holds for m and assume $(\forall v \in {}^{m+1}\{\sigma, \xi\}) \vdash vu = vw$. By the latter, for any $t \in {}^{m}\{\sigma, \xi\}$ we have

  $$\vdash \sigma t u = \sigma t w \qquad \text{and} \qquad \vdash \xi t u = \xi t w\,.$$

  By Lemma 3.24 then

  $$\vdash tu = tw\,.$$

  Since t was arbitrary, we have

  $$(\forall t \in {}^{m}\{\sigma, \xi\}) \vdash tu = tw\,.$$

  Then $\vdash u = w$ by the induction hypothesis. This completes the proof of $(\forall m \in \omega)(3.25)$.

$(\forall m \in \omega)(3.25)$ completes the proof of Theorem 3.21, for let m be a fixed number satisfying (3.24). Then by (3.25) we can conclude $\vdash u = w$.

Notice that in the proofs of Corollary 3.23 and Lemma 3.24 we did not use more than the assumption of Theorem 3.21. This completes the proof of Theorem 3.21. ∎

We will often refer to the property

$$(\forall u, w \in \mathbb{G}^{-*})(\models u = w \Longrightarrow \vdash u = w)\,.$$

of $\mathbb{T}^-$ proved by Theorem 3.21 as "*completeness of* $\mathbb{T}^-$".

COROLLARY 3.26. *$\mathbb{T}^-$ is complete in the above sense.*

Proof: This was proved in the proof of Theorem 3.21. ∎

3.6. Finite presentation of $\mathbb{T}$

Recall that in Definition 3.19 (in subsection 3.4) we gave the finite set E of defining relations in terms of the elements of $\mathbb{G}$. Also recall that $\mathbb{T}$ is the subsemigroup of transformations generated by $\mathbb{G}$ in $\langle {}^{\omega}\omega, \circ\rangle$ (see Definition 3.1 in subsection 3.1).

THEOREM 3.27 (Finite presentation of $\mathbb{T}$).
The set E of defining relations given in Definition 3.19 constitute a finite presentation for $\mathbb{T}$.

The rest of the present subsection is devoted to the proof of Theorem 3.27.

First we turn to proving Lemma 3.28 below, which is analogous to Lemma 3.22, but it refers to *all* elements of $\mathbb{G}^*$ (instead of those of ${\mathbb{G}^-}^*$ only). First we define the set *Norm* of "normal forms" that will play that rŏle in Lemma 3.28 which was played by $Norm^-(=\{\sigma,\xi\}^*)$ in Lemma 3.22.

Recall from Definition 3.5 (iii) that if $\gamma\in\mathbb{T}^-$ then $m(\gamma)$ denotes the smallest number $m\in\omega$ such that γ is defined by a table of length m. Thus if $m(\gamma)=1$ then γ is defined by a table of form $\begin{matrix} 0\longmapsto u \\ 1\longmapsto v\end{matrix}$. We define

$$Norm \stackrel{\text{def}}{=} \{\sigma[0/1]\gamma : \gamma\in{\mathbb{G}^-}^* \text{ and } m(\hat{\gamma})\leqslant 1\}\,.$$

We would like to state that every element w of $\mathbb{G}^*$ has a "*normal form*", that is, that every $w\in\mathbb{G}^*$ is provably equal to some $v'\in Norm$. Instead of this, in Lemma 3.28 below, we can only state the following weaker statement. For every $w\in\mathbb{G}^*$ there is a number $m\in\omega$ such that whenever v is a $\{\sigma,\xi\}$–sequence of length m then vw is provably equal to some $v'\in Norm$.

LEMMA 3.28 (Normal form lemma for $\mathbb{T}$). $(\forall w\in\mathbb{G}^*)(\exists m\in\omega)$

$$(\forall v\in {}^{m}\{\sigma,\xi\})(\exists v'\in Norm)\vdash vw=v'\,.$$

Before proving Lemma 3.28, first we show that it is a generalization of Lemma 3.22 (i), namely:

Claim 3.29. $(\forall v\in Norm^-)(\exists v'\in Norm)\vdash v=v'\,.$

Proof: $\vdash v=\sigma[0/1]\pi v$ with $m(\pi v)\leqslant 1$ by $(\sigma[0/1]\pi=\mathrm{Id})\in(\mathrm{F4})$. ∎

In the proof of Lemma 3.28 we will use Claims 3.31–3.33 below. All of them state Lemma 3.28 for special elements $w\in\mathbb{G}^*$.

Convention 3.30. $\sigma^0=\mathrm{Id}$, $\tilde{\sigma}^0=\mathrm{Id}$, $\overset{\vee}{\sigma}{}^0=\mathrm{Id}$. ◀

Claim 3.31. *If* $v\in\{\sigma,\xi\}^*\smallsetminus\{\sigma\}^*$ *then* $\vdash v[0/1]=v\,.$

Proof: Let $v\in\{\sigma,\xi\}^*\smallsetminus\{\sigma\}^*$. Then $v=v'\xi\sigma^k$ for some $v'\in\{\sigma,\xi\}^*$ and $k\in\omega$.
<u>*Case 1:*</u> $k=0$. Then $v=v'\xi$, and we derive:

$$\begin{aligned}\vdash v[0/1] &= v'\xi[0/1] \\ &= v'\xi && \text{by } (\xi[0/1]=\xi)\in(\mathrm{F5}) \\ &= v\,.\end{aligned}$$

<u>*Case 2:*</u> $k = 1$. Then $v = v'\xi\sigma$, and we derive:

$$\begin{aligned} \vdash v[0/1] &= v'\xi\sigma[0/1] \\ &= v'\xi\sigma && \text{by } (\xi\sigma[0/1] = \xi\sigma) \in \text{(F5)} \\ &= v\,. \end{aligned}$$

<u>*Case 3:*</u> $k > 1$. Then

$$\begin{aligned} \vdash v[0/1] &= v'\xi\sigma^k[0/1] \\ &= v'\xi\sigma\check{\sigma}^{k-1}[0/1] && \text{by } (\sigma\sigma = \sigma\check{\sigma}) \in \text{(F2) (and induction)} \\ &= v'\xi\sigma[0/1]\check{\sigma}^{k-1} && \text{by } (\check{\sigma}[0/1] = [0/1]\check{\sigma}) \in \text{(F7)} \\ &= v'\xi\sigma\check{\sigma}^{k-1} && \text{by } (\xi\sigma[0/1] = \xi\sigma) \in \text{(F5)} \\ &= v'\xi\sigma^k && \text{by } (\sigma\sigma = \sigma\check{\sigma}) \in \text{(F2)} \\ &= v\,. \end{aligned}$$

Cases 1–3 prove Claim 3.31. ∎

DEFINITION 3.32: Let $w \in \mathbb{G}^*$ and $m \in \omega$. We say that "*m is good for w*" iff

$$(\forall v \in {}^m\{\sigma,\xi\})(\exists v' \in \mathit{Norm}) \vdash vw = v'\,. \quad \blacktriangleleft$$

Using the terminology just introduced, Lemma 3.28 above states that

$$(\forall w \in \mathbb{G}^*)(\exists m \in \omega)(m \text{ is good for } w)\,.$$

Claim 3.33. *Let $w \in \mathbb{G}^*$ and $m, m' \in \omega$. Then*

$$(m \textit{ is good for } w \textit{ and } m' \geqslant m) \Longrightarrow m' \textit{ is good for } w\,.$$

Proof: Assume m is good for $w \in \mathbb{G}^*$ and $m' \geqslant m$. We want to prove that

$$(\forall v \in {}^{m'}\{\sigma,\xi\})(\exists v' \in \mathit{Norm}) \vdash vw = v'\,. \tag{3.26}$$

So let $v \in {}^{m'}\{\sigma,\xi\}$ be arbitrary. Then $v = v_2v_1$ with $|v_1| = m$ by $m' \geqslant m$. Then $\vdash vw = v_2v_1w = v_2w'$ for some $w' \in \mathit{Norm}$ because m is good for w by hypothesis. Therefore, to prove (3.26), it is enough to prove that

$$(\forall v \in \{\sigma,\xi\}^*)(\forall w \in \mathit{Norm})(\exists v' \in \mathit{Norm}) \vdash vw = v'\,. \tag{3.27}$$

To prove (3.27), let $v \in \{\sigma,\xi\}^*$ and $w \in \mathit{Norm}$ be arbitrary. Thus $\vdash w = \sigma[0/1]\gamma$ for some $\gamma \in \mathbb{G}^-$ with $m(\gamma) \leqslant 1$.

<u>*Case 1:*</u> $v \in \{\sigma,\xi\}^* \smallsetminus \{\sigma\}^*$. Then, by Claim 3.31,

$$\vdash vw = v\sigma[0/1]\gamma = v\sigma\gamma\,. \tag{3.28}$$

By $m(\gamma) \leqslant 1$, it is easy to see that $m(v\sigma\gamma) = 0$. Thus

$$\vdash v\sigma\gamma = t \tag{3.29}$$

for some $t \in \{\sigma, \xi\}^*$. By (3.28) and (3.29) then $\vdash vw = t$. By Claim 3.29, $\vdash t = v'$ for some $v' \in \mathit{Norm}$. Thus $\vdash vw = v'$, completing the proof of Claim 3.33 in Case 1.

<u>*Case 2:*</u> $v \in \sigma^n$ for some $n \in \omega$. Then

$$\begin{aligned} \vdash vw &= \sigma^n\sigma[0/1]\gamma \\ &= \sigma\check{\sigma}^n[0/1]\gamma && \text{by } (\sigma\sigma = \sigma\check{\sigma}) \in \text{(F2)} \\ &= \sigma[0/1]\check{\sigma}^n\gamma && \text{by } (\check{\sigma}[0/1] = [0/1]\check{\sigma}) \in \text{(F7)}. \end{aligned}$$

By $m(\gamma) \leqslant 1$, it is easy to see that $m(\check{\sigma}^n\gamma) \leqslant 1$, completing the proof of Claim 3.33. ∎

Proof of Lemma 3.28: Let $w \in \mathbb{G}^*$ be arbitrary but fixed. We will prove that

$$(\exists m \in \omega)(m \text{ is good for } w)\,,$$

by induction of the length of w.

Base step: If $w = \mathrm{Id}$ then $\vdash w = \sigma[0/1]\pi$ by $(\mathrm{Id} = \sigma[0/1]\pi) \in$ (F4).

Induction step: Assume that $u \in \mathbb{G}^*$ is such that

$$\text{(IH)} \qquad (\exists n \in \omega)(n \text{ is good for } u)\,.$$

Let $n \in \omega$ be such. Let $\delta \in \mathbb{G}$ be arbitrary but fixed. We will prove that $n+2$ is good for δu.

2 is good for δ: by Lemma 3.22 if $\delta \in \mathbb{G}^{-*}$; and by Claim 3.33 plus the fact that 1 is good for $[0/1]$ if $\delta = [0/1]$. (1 is good for $[0/1]$ by $\vdash \sigma[0/1] = \sigma[0/1]\mathrm{Id}$, $\vdash \xi[0/1] = \xi$ $((\xi[0/1] = \xi) \in \text{(F5)})$, and Claim 3.29).

Let $v \in {}^{n+2}\{\sigma, \xi\}$ be arbitrary but fixed. Then $v = v_1 v_0$ for some v_1, v_0 with $|v_1| = n$, $|v_0| = 2$. Now

$$\text{(3.30)} \qquad \begin{aligned} \vdash v\delta u &= v_1 v_0 \delta u \\ &= v_1\sigma[0/1]\gamma u \end{aligned}$$

for some $\gamma \in \mathbb{G}^{-*}$ with $m(\hat{\gamma}) \leqslant 1$, because $|v_0| = 2$ and because 2 is good for δ.

<u>*Case 1:*</u> $v_1 \in {}^n\{\sigma, \xi\} \setminus \{\sigma\}^*$. Then

$$\begin{aligned} \vdash v\delta u &= v_1\sigma[0/1]\gamma u && \text{by (3.30)} \\ &= v_1\sigma\gamma u && \text{because } \vdash \sigma[0/1] = \sigma \text{ by Claim 3.31} \\ &= v_1 t u && \text{for some } t \in \{\sigma, \xi\}^* \text{ by } m(\sigma\gamma) = 0 \text{ and Lemma 3.22 (i)} \\ &= \sigma[0/1]\gamma_1 && \text{for some } \gamma_1 \in \mathbb{G}^{-*} \text{ with } m(\hat{\gamma}_1) \leqslant 1 \text{ by (IH), } |v_1 t| \geqslant n \text{ and Claim 3.33.} \end{aligned}$$

We have seen, in this case, that $n+2$ is good for δu.

<u>*Case 2:*</u> $v_1 = \sigma^n$. Then first we prove

$$\text{(3.31)} \qquad \vdash v\delta u = \sigma[0/1](\sigma^n\sigma\gamma u)^{\vee}(\sigma^n\xi\gamma u)^{\smile}\pi$$

by the following computation.

$$
\begin{aligned}
\vdash v\delta u &= v_1\sigma[0/1]\gamma u && \text{by (3.30)}\\
&= \sigma^{n+1}[0/1]\gamma u && \text{by } v_1 = \sigma^n\\
&= \sigma^{n+1}[0/1](\sigma\gamma u)^{\vee}(\xi\gamma u)^{\sim}\pi && \text{by Claim 3.25 (i)}\\
&= \sigma\overset{\vee}{\sigma}{}^{n}[0/1](\sigma\gamma u)^{\vee}(\xi\gamma u)^{\sim}\pi && \text{by } (\sigma\sigma = \sigma\overset{\vee}{\sigma}) \in \text{(F2) (and induction)}\\
&= \sigma[0/1](\sigma^n\sigma\gamma u)^{\vee}(\xi\gamma u)^{\sim}\pi && \text{by } (\overset{\vee}{\sigma}[0/1] = [0/1]\overset{\vee}{\sigma}) \in \text{(F7) (and induction)}\\
&= \sigma[0/1]\tilde{\sigma}^n(\sigma^n\sigma\gamma u)^{\vee}(\xi\gamma u)^{\sim}\pi && \text{by } (\sigma[0/1]\tilde{\sigma} = \sigma[0/1]) \in \text{(F7) (and induction)}\\
&= \sigma[0/1](\sigma^n\sigma\gamma u)^{\vee}(\sigma^n\xi\gamma u)^{\sim}\pi && \text{by (F3) (and induction).}
\end{aligned}
$$

We have proved (3.31). Now

$$
\vdash \sigma^n\sigma\gamma u = \sigma[0/1]\gamma_1 \quad \text{and} \quad \vdash \sigma^n\xi\gamma u = \sigma[0/1]\gamma_2 \tag{3.32}
$$

for some $\gamma_1, \gamma_2 \in \mathbb{G}^{-*}$ with $m(\hat{\gamma_1}) \leqslant 1$ and $m(\hat{\gamma_2}) \leqslant 1$, because of the following. $m(\sigma\gamma) = m(\xi\gamma) = 0$ by $m(\gamma) \leqslant 1$. Thus, by Lemma 3.22 (i),

$$
\vdash \sigma\gamma = t_1 \quad \text{and} \quad \vdash \xi\gamma = t_2
$$

for some $t_1, t_2 \in \{\sigma, \xi\}^*$. Now

$$
\vdash \sigma^n\sigma\gamma u = \sigma^n t_1 u = \sigma[0/1]\gamma_1 \quad \text{and} \quad \vdash \sigma^n\xi\gamma u = \sigma^n t_2 u = \sigma[0/1]\gamma_2
$$

for some $\gamma_1, \gamma_2 \in \mathbb{G}^{-*}$ with $m(\gamma_1) \leqslant 1$ and $m(\gamma_2) \leqslant 1$ by (IH), $|\sigma^n t_1| \geqslant n$, $|\sigma^n t_2| \geqslant n$ and Claim 3.33. This completes the proof of (3.32). Now

$$
\begin{aligned}
\vdash v\delta u &= \sigma[0/1](\sigma^n\sigma\gamma u)^{\vee}(\sigma^n\xi\gamma u)^{\sim}\pi && \text{this is (3.31)}\\
&= \sigma[0/1](\sigma[0/1]\gamma_1)^{\vee}(\sigma[0/1]\gamma_2)^{\sim}\pi && \text{by (3.32) and by Claim 3.25 (ii)}\\
&= \sigma[0/1]\overset{\vee}{\sigma}\overset{\vee}{[0/1]}\overset{\vee}{\gamma_1}\tilde{\sigma}\widetilde{[0/1]}\tilde{\gamma}_2\pi && \text{by Claim 3.18 (iii)}\\
&= \sigma\overset{\vee}{\sigma}[0/1]\overset{\vee}{[0/1]}\overset{\vee}{\gamma_1}\tilde{\sigma}\widetilde{[0/1]}\tilde{\gamma}_2\pi && \text{by } (\overset{\vee}{\sigma}[0/1] = [0/1]\overset{\vee}{\sigma}) \in \text{(F7)}\\
&= \sigma\overset{\vee}{\sigma}[0/1]\overset{\vee}{\gamma_1}\tilde{\sigma}\widetilde{[0/1]}\tilde{\gamma}_2\pi && \text{by } ([0/1]\overset{\vee}{[0/1]} = [0/1]) \in \text{(F6)}\\
&= \sigma[0/1](\sigma\gamma_1)^{\vee}\tilde{\sigma}\widetilde{[0/1]}\tilde{\gamma}_2\pi && \text{by } (\overset{\vee}{\sigma}[0/1] = [0/1]\overset{\vee}{\sigma}) \in \text{(F7)}\\
&= \sigma[0/1]\tilde{\sigma}\widetilde{[0/1]}(\sigma\gamma_1)^{\vee}\tilde{\gamma}_2\pi && \text{by (F3) and induction}\\
&= \sigma[0/1]\widetilde{[0/1]}(\sigma\gamma_1)^{\vee}\tilde{\gamma}_2\pi && \text{by } (\sigma[0/1]\tilde{\sigma} = \sigma[0/1]) \in \text{(F7)}\\
&= \sigma[0/1]\tilde{\xi}(\sigma\gamma_1)^{\vee}\tilde{\gamma}_2\pi && \text{by } \sigma[0/1]\widetilde{[0/1]} = \sigma[0/1]\tilde{\xi}\\
&= \sigma[0/1](\sigma\gamma_1)^{\vee}(\xi\gamma_2)^{\sim}\pi && \text{by (F3).}
\end{aligned}
$$

Here $m(\sigma\gamma_1) = m(\sigma\gamma_2) = 0$ by $m(\gamma_1) \leqslant 1$ and $m(\gamma_2) \leqslant 1$. Thus $m((\sigma\gamma_1)^{\vee}(\xi\gamma_2)^{\sim}\pi) \leqslant 1$, completing the proof of the fact that $n+2$ is good for δu in case of $v_1 = \sigma^n$ as well.

Cases 1 and 2 together complete the induction step. ∎

COROLLARY 3.34.

$$(\forall u, w \in \mathit{Norm})(\models u = w \Longrightarrow \vdash u = w)\,.$$

Proof: Let $u, w \in \mathit{Norm}$, say $u = \sigma[0/1]\gamma_1$ with $m(\hat{\gamma_1}) \leqslant 1$ and $w = \sigma[0/1]\gamma_2$ with $m(\hat{\gamma_2}) \leqslant 1$. Assume $\models \sigma[0/1]\gamma_1 = \sigma[0/1]\gamma_2$. Then $\gamma_1 \restriction [\bar{0}, 1\bar{0}] = \gamma_2 \restriction [\bar{0}, 1\bar{0}]$. It is easy to see that from the latter and from $m(\hat{\gamma_1}) \leqslant 1$ and $m(\hat{\gamma_2}) \leqslant 1$ it follows that, for some $k \in \omega$,

$$\hat{\gamma_1} = \hat{\tilde{\sigma}^k}\hat{\gamma_2} \quad \text{or} \quad \hat{\gamma_2} = \hat{\tilde{\sigma}^k}\hat{\gamma_1}\,.$$

We may assume $\hat{\gamma_1} = \hat{\tilde{\sigma}^k}\hat{\gamma_2}$. By the completeness of $\mathbb{T}^-$ then

$$\vdash \gamma_1 = \tilde{\sigma}^k\gamma_2\,.$$

Thus

$$\begin{aligned} \vdash \sigma[0/1]\gamma_1 &= \sigma[0/1]\tilde{\sigma}^k\gamma_2 \\ &= \sigma[0/1]\gamma_2 \qquad\qquad \text{by (F7).} \quad \blacksquare \end{aligned}$$

Proof of Theorem 3.27: We need to prove that

$$(\forall u, w \in \mathbb{G}^*)(\models u = w \Longrightarrow \vdash u = w)\,.$$

Let $u, w \in \mathbb{G}^*$. By Lemma 3.28 and Claim 3.33, there is an $m \in \omega$ which is good for both u and w. Then for all $v \in {}^m\{\sigma, \xi\}$ there are $v', v" \in \mathit{Norm}$ such that $\vdash vw = v'$ and $\vdash vu = v"$. By $\models u = w$ we have then $\models v' = v"$. By Corolary 3.34 then $\vdash v' = v"$. Thus $\vdash vu = vw$ (by equational logic). From this, by Lemma 3.24, we obtain $\vdash u = w$. $\blacksquare$

Analogously to the terminology introduced at the end of subsection 3.5, we refer to the property

$$(\forall u, w \in \mathbb{G}^*)(\models u = w \Longrightarrow \vdash u = w)\,.$$

as "*completeness of* $\mathbb{T}$".

COROLLARY 3.35. *$\mathbb{T}$ is complete in the sense above.*

Proof: This was proved in the proof of Theorem 3.27. $\blacksquare$

OPEN PROBLEM 3.36: Replace the set E of defining relations of $\mathbb{T}$ by a simple and natural set of defining relations (over $\mathbb{G}$ or perhaps over an even simpler generator set). This would simplify the (finite) axiomatization of $\mathbf{I}\mathsf{Gws}_{\mathbb{G}}$. ◀

OPEN PROBLEM 3.37 (joint with R. J. Thompson): Find a semigroup "more natural" than $\mathbb{T}$ satisfying the conditions of Theorem 1.8. Cf. Conjecture 2.20 in subsection 2.4. ◀

4. APPLICATION TO LOGIC

Let $\langle \mathbf{R}, r\rangle$ be a the vocabulary (or similarity type) that is let $\mathbf{R}$ be a set (the elements of which are called relation symbols); and let $r : \mathbf{R} \longrightarrow [(\omega+1)\smallsetminus 1]$ that is for every $R \in \mathbf{R}$ let $0 < r(R) \leqslant \omega$. $r(R)$ is called the **r***ank* or *arity* of the relation symbol R.

Next we define our set Fm of formulas: Fm is defined to be the smallest set satisfying conditions (i)–(ii) below.

(i) $R(v_i)_{i<r(R)} \in Fm$ for all $R \in \mathbf{R}$.

(ii) $\{\neg\varphi, \exists v_0\varphi, \mathrm{S}_\tau\varphi, (\varphi\wedge\psi)\} \subseteq Fm$ if $\varphi, \psi \in Fm$ and $\tau \in \mathbb{G}$.

(Recall from section 3 that $\mathbb{G} = \{\overline{\sigma}, \pi, \mu, \eta, [0/1]\}$ and $\mathbb{T}$ is the subsemigroup of $\langle {}^\omega\omega, \circ\rangle$ generated by $\mathbb{G}$.)

Let *Mod* denote the class of our models, that is

$$Mod \overset{\text{def}}{=} \left\{\mathfrak{M} : \mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle_{R\in\mathbf{R}} \text{ such that } M \text{ is any nonempty set and } R^{\mathfrak{M}} \subseteq {}^{r(R)}M\right\} .$$

Now we define validity. Let $\mathfrak{M} \in Mod$, $\varphi, \psi \in Fm$, $\tau \in \mathbb{G}$, $R \in \mathbf{R}$ and $k \in {}^\omega M$ be arbitrary. Then

$$\begin{aligned}
\mathfrak{M} \models R(v_i)_{i<r(R)}[k] &\quad\overset{\text{def}}{\Longleftrightarrow}\quad k \restriction r(R) \in R^{\mathfrak{M}}\\
\mathfrak{M} \models \mathrm{S}_\tau\varphi[k] &\quad\overset{\text{def}}{\Longleftrightarrow}\quad \mathfrak{M} \models \varphi[k\circ\tau]\\
\mathfrak{M} \models \exists v_0\varphi[k] &\quad\overset{\text{def}}{\Longleftrightarrow}\quad (\exists a \in M)\mathfrak{M} \models \varphi[k[0/a]]\\
\mathfrak{M} \models \neg\varphi[k] &\quad\overset{\text{def}}{\Longleftrightarrow}\quad \mathfrak{M} \not\models \varphi[k]\\
\mathfrak{M} \models (\varphi\wedge\psi)[k] &\quad\overset{\text{def}}{\Longleftrightarrow}\quad (\mathfrak{M} \models \varphi[k] \text{ and } \mathfrak{M} \models \psi[k]) \ .\\
\mathfrak{M} \models \varphi &\quad\overset{\text{def}}{\Longleftrightarrow}\quad (\forall k \in {}^\omega M)\mathfrak{M} \models \varphi[k]\,,\\
\models \varphi &\quad\overset{\text{def}}{\Longleftrightarrow}\quad (\forall\mathfrak{M} \in Mod)\mathfrak{M} \models \varphi\,.
\end{aligned}$$

If $\models \varphi$ then we say that φ is *valid*.

Next we define a proof system for this logic (we shall see in Corollary 4.2 below that this system is complete). We construct our axioms for this proof system from the finite set Σ of equations in Lemma 2.13 (in section 2), where the finite presentation E of $\mathbb{T}$ (see section 3) is used, too. We do not polish our axioms here, our only goal is to show that from the finite base Σ of $\mathbf{I}\mathsf{Gws}_{\mathbb{G}}$ we can obtain a complete proof system for our logic.

Next we recall that Σ was the following set (recall that the finite G^0 was defined in section 3).

(1) Boolean equations

(2) C_0 is an additive, complemented closure operator

(3) S_τ is a Boolean endomorphism (for $\tau \in \mathbb{G}$)
S_{Id} is the identity
$\mathrm{S}_u = \mathrm{S}_v$ if $\langle u, v\rangle \in E$

(4) $\mathrm{C}_0 = \mathrm{S}_{[0/k]}\mathrm{C}_0$, $\mathrm{S}_{[0/k]} = \mathrm{C}_0\mathrm{S}_{[0/k]}$ for $k = 1, 2, \ldots, 5$
$\mathrm{C}_0\mathrm{S}_\tau = \mathrm{S}_\tau\mathrm{C}_0$ for $\tau \in G^0$.

Convention 4.1: For any $\tau \in \mathbb{T}$, let us choose $w \in (\mathbb{G})^*$ such that $\hat{w} = \tau$. Then, if $w = g_1\cdots g_n$ for $g_1, \ldots, g_n \in (\mathbb{G})^*$, then for any formula $\varphi \in Fm$,

$$\mathrm{S}_\tau\varphi \text{ denotes } \mathrm{S}_{g_1}\cdots\mathrm{S}_{g_n}\varphi\,. \quad ◀$$

Let $\sigma : \omega \longrightarrow \omega$ and $s : \omega \longrightarrow \omega$ be defined by

$$\sigma(k) \overset{\text{def}}{=} 2k \quad \text{and} \quad s(k) \overset{\text{def}}{=} 2k+1$$

for all $k \in \omega$. Then $\sigma, s \in \mathbb{T}$.

Our logical axioms are the following:

(1) propositional tautologies
(2) $\exists v_0(\varphi \vee \psi) \leftrightarrow (\exists v_0 \varphi \vee \exists v_0 \psi)$, $\exists v_0 \neg \exists v_0 \varphi \leftrightarrow \neg \exists v_0 \varphi$, $\varphi \to \exists v_0 \varphi$, $\exists v_0 \exists v_0 \varphi \leftrightarrow \exists v_0 \varphi$
(3) $\mathrm{S}_\tau(\varphi \vee \psi) \leftrightarrow (\mathrm{S}_\tau \varphi \vee \mathrm{S}_\tau \psi)$, $\mathrm{S}_\tau(\neg\varphi) \leftrightarrow \neg \mathrm{S}_\tau \varphi$ for all $\tau \in \mathbb{G}$,
$\mathrm{S}_e \mathrm{S}_e \varphi \leftrightarrow \varphi$,
$\mathrm{S}_{g_1} \cdots \mathrm{S}_{g_n} \varphi \leftrightarrow \mathrm{S}_{h_1} \cdots \mathrm{S}_{h_n} \varphi$ if $\langle g_1, \ldots, g_n, h_1, \ldots, h_n \rangle \in E$, $g_1, \ldots, g_n, h_1, \ldots, h_n \in \mathbb{G}$, for example
$\mathrm{S}_\pi \mathrm{S}_{[0/1]} \varphi \leftrightarrow \mathrm{S}_\pi \varphi$
(4) $\exists v_0 \varphi \leftrightarrow \mathrm{S}_{[0/k]} \exists v_0 \varphi$,
$\mathrm{S}_{[0/k]} \varphi \leftrightarrow \exists v_0 \mathrm{S}_{[0/k]} \varphi$ for $k = 1, 2, \ldots, 5$
$\exists v_0 \mathrm{S}_\tau \varphi \leftrightarrow \mathrm{S}_\tau \exists v_0 \varphi$ for $\tau \in G^0$
(5) $\mathrm{S}_{\sigma[n/\mathrm{Id}]} R(v_i)_{i<r(R)} \leftrightarrow R(v_i)_{i<r(R)}$, $\mathrm{S}_{s[n/\mathrm{Id}]} R(v_i)_{i<r(R)} \leftrightarrow R(v_i)_{i<r(R)}$ if $R \in \mathbf{R}$ and $n = r(R)$.

Our inference rules are
Modus Ponens,
$(\varphi \to \psi) \vdash (\exists v_0 \varphi \to \exists v_0 \psi)$, and
$(\varphi \to \psi) \vdash (\mathrm{S}_\tau \varphi \to \mathrm{S}_\tau \psi)$ (for $\tau \in \mathbb{G}$).

The above axiom system is finite if $\mathbf{R}$ is finite. (In "finiteness" here we disregard the number of substitution instances of the formula variables φ and ψ. Without this, even propositional logic is not finitizable. The point is that we got rid of metavariables ranging over logical connectives or over individual variables.) We note that in the above system, (1)–(4) correspond to the equational base Σ of $\mathbf{I}\mathsf{Gws}_{\mathbb{G}}$, and (5) is included to take care of the similarity type (rank function). Equational derivations can be simulated by using our rules of inference.

$\vdash \varphi$ denotes that φ is derivable by the above proof system. The next corollary states that the valid and the derivable formulas coincide.

COROLLARY 4.2 (Completeness). *For every $\varphi \in Fm$ we have*

$$\models \varphi \qquad \textit{iff} \qquad \vdash \varphi\,.$$

Proof: It is enough to show $\nvdash \varphi \Rightarrow \not\models \varphi$ (since the other direction is easy to show). To every $\psi \in Fm$ we associate a term $\overline{\psi}$ as follows (we will consider the elements of $\mathbf{R}$ as variables in these terms):

$$\overline{R(v_i)_{i<r(R)}} \stackrel{\mathrm{def}}{=} R \text{ if } R \in \mathbf{R}, \quad \overline{\neg\psi} \stackrel{\mathrm{def}}{=} -\overline{\psi}, \quad \overline{\psi \wedge \delta} \stackrel{\mathrm{def}}{=} \overline{\psi} \cdot \overline{\delta}, \quad \overline{\exists v_0 \psi} = \mathsf{c}_0 \overline{\psi}, \quad \overline{\mathrm{S}_\tau \psi} = \mathrm{S}_\tau \overline{\psi}\,.$$

Then it is not difficult to see (our original axioms were designed so) that (i) and (ii) below are equivalent.

(i) $\nvdash \varphi$
(ii) $(\exists \mathfrak{A} \in Mod(\Sigma))(\exists h : \mathbf{R} \longrightarrow A)$
$(\mathfrak{A} \not\models \overline{\psi} = 1[h]$ and $(\forall R \in \mathbf{R}) \mathrm{S}_{\sigma[r(R)/\mathrm{Id}]} h(R) = \mathrm{S}_{s[r(R)/\mathrm{Id}]} h(R) = h(R)$ in $\mathfrak{A})$.

Assume $\nvdash \varphi$. Let $\mathfrak{A} \in Mod(\Sigma)$ and $h : \mathbf{R} \longrightarrow A$ be as in (ii). We may assume $|A| \leqslant \omega$ (since in φ only finitely many relation symbols occur). By $\mathfrak{A} \models \Sigma$ and the proofs of Theorem 2.12, Lemma 2.13 in section 2, we may also assume that

$$\mathfrak{A} \subseteq \mathfrak{N} \stackrel{\mathrm{def}}{=} \langle \mathcal{P}(()V), \cup, {}_V\!\sim, \mathrm{C}_0^{[V]}, \mathrm{S}_\tau^{[V]} : \tau \in \mathbb{G} \rangle$$

where $V = \bigcup \{ {}^{\omega}U^{(\tau)} : \tau \in \mathbb{T} \}$ and $U = \omega + \omega$. (Namely, by the proofs of Theorem 2.12 and Lemma 2.13, there is a homomorphism $rep : \mathfrak{A} \longrightarrow \mathfrak{N}$ such that $\mathfrak{A}' \stackrel{\mathrm{def}}{=} rep"\mathfrak{A}$ and $h' \stackrel{\mathrm{def}}{=} rep \circ h$ still satisfy the conditions in (ii) herein, that is $\mathfrak{A}' \not\models \overline{\psi} = 1[h']$ etc.)

Let $\mathfrak{M} \stackrel{\text{def}}{=} \langle U, R^{\mathfrak{M}}\rangle_{R\in\mathbf{R}}$ where $R^{\mathfrak{M}} = \{s \restriction r(R) : s \in h(R)\}$ for all $R \in \mathbf{R}$. Then $\mathfrak{M} \in Mod$. We will show that $\mathfrak{M} \not\models \varphi$. For every $\psi \in Fm$ let $\psi^{\mathfrak{M}} \stackrel{\text{def}}{=} \{k \in {}^{\omega}U : \mathfrak{M} \models \psi[k]\}$, and $\overline{\psi}^{\mathfrak{N},h}$ denotes the term $\overline{\psi}$ computed in $\mathfrak{N}$ under the evaluation h of the variables.

We will show that

$$V \cap \psi^{\mathfrak{M}} = \overline{\psi}^{\mathfrak{N},h} \tag{4.1}$$

for every $\psi \in Fm$. Since by our assumption $\mathfrak{A} \not\models \overline{\psi} = 1[h]$ and by $\mathfrak{A} \subseteq \mathfrak{N}$ we have $\overline{\psi}^{\mathfrak{N},h} \subsetneqq V$, (4.1) will imply $\mathfrak{M} \not\models \varphi$.

We will prove here (4.1) by induction on ψ, but the main idea is that if $W \stackrel{\text{def}}{=} {}^{\omega}U$ and $\mathfrak{W} \stackrel{\text{def}}{=} \langle \mathcal{P}\left(()\, W\right), \cup, \ldots\rangle$ then $\psi^{\mathfrak{M}} = \overline{\psi}^{\mathfrak{N},h}$ (cf. [HMTII, section 4.3]), and by $\mathrm{C}_0^{[W]}V = V$, $\mathrm{S}_{\tau}^{[W]}V = V$ we have that relativizing $\mathfrak{W}$ with V is a homomorphism, that is if $rl_V(x) \stackrel{\text{def}}{=} V \cap x$ then $rl_V : \mathfrak{W} \longrightarrow \mathfrak{N}$ is a homomorphism.

We start proving (4.1) by induction on ψ. First we will show that

$$V \cap \left(R(v_i)_{i<r(R)}\right)^{\mathfrak{M}} = h(R), \text{ for any } R \in \mathbf{R}\,. \tag{4.2}$$

Let $n = r(R)$ and $\delta \stackrel{\text{def}}{=} R(v_i)_{i<r(R)}$. If $n = \omega$ then (4.2) clearly holds. Assume $n \in \omega$. Clearly, $h(R) \subseteq V \cap \delta^{\mathfrak{M}}$. Let $z \in V \cap \delta^{\mathfrak{M}}$. Then $z \in V$, and by $z \in \delta^{\mathfrak{M}}$ there is $s \in h(R)$ such that $s \restriction r(R) = z \restriction r(R)$. We define $z' \in {}^{\omega}U$ as follows. For any $k \in \omega$ let

$$z'(k) \stackrel{\text{def}}{=} \begin{cases} z(k) \text{ if } k < n \text{ or } k \text{ is odd,} \\ s(k) \text{ otherwise.} \end{cases}$$

Let $\sigma' \stackrel{\text{def}}{=} \sigma[n/\mathrm{Id}]$, $s' \stackrel{\text{def}}{=} s[n/\mathrm{Id}]$. Then

$$s \circ \sigma' = z' \circ \sigma' \text{ and } z' \circ s' = z \circ s'\,. \tag{4.3}$$

By $\mathrm{S}_{\sigma'}h(R) = \mathrm{S}_{s'}h(R) = h(R)$ we have that for any $q \in V$,

$$q \in h(R) \qquad \text{iff} \qquad q \circ \sigma' \in h(R) \qquad \text{iff} \qquad q \circ s' \in h(R)\,,$$

thus by $s \in h(R)$, $z \in V$ and by (4.3) we obtain $z \in h(R)$.

By (4.2) then (4.1) is true for the formulas $R(v_i)_{i<r(R)}$. Clearly if (4.1) is true for ψ, η then (4.1) if true for $\neg\psi$ and $(\psi \wedge \eta)$ as well. Assume that (4.1) is true for ψ, that is $V \cap \psi^{\mathfrak{M}} = \overline{\psi}^{\mathfrak{N},h}$.

$$\begin{aligned}
V \cap (\exists v_0\psi)^{\mathfrak{M}} &= \{k \in V : (\exists a \in U)k[0/a] \in \psi^{\mathfrak{M}}\} \\
&= \mathrm{C}_0^{\mathfrak{N}}(V \cap \psi^{\mathfrak{M}}) \qquad (\text{by } k \in V \Rightarrow k[0/a] \in V) \\
&= \mathrm{C}_0^{\mathfrak{N}}\overline{\psi}^{\mathfrak{N},h} \qquad (\text{by inductive hypothesis}) \\
&= \overline{\exists v_0\psi}^{\mathfrak{N},h} \qquad (\text{by } \overline{\exists v_0\psi} = \mathrm{C}_0\overline{\psi})\,. \\
V \cap (\mathrm{S}_{\tau}\psi)^{\mathfrak{M}} &= \{k \in V : k \circ \tau \in \psi^{\mathfrak{M}}\} \\
&= \mathrm{S}_{\tau}^{\mathfrak{N}}(V \cap \psi^{\mathfrak{M}}) \qquad \text{by } k \in V \Rightarrow k \circ \tau \in V \\
&= \mathrm{S}_{\tau}^{\mathfrak{N}}\overline{\psi}^{\mathfrak{N},h} \qquad (\text{induction hypothesis}) \\
&= \overline{\mathrm{S}_{\tau}\psi}^{\mathfrak{N},h} \qquad (\text{by } \overline{\mathrm{S}_{\tau}\psi} = \mathrm{S}_{\tau}\overline{\psi})\,.
\end{aligned}$$

By the above we proved (4.1) for every $\psi \in Fm$. Thus we proved $\mathfrak{M} \not\models \varphi$, that is $\not\vdash \varphi \Longrightarrow \not\models \varphi$.

This completes the proof of Corollary 4.2. ∎

Next we turn to showing further positive properties of our new logic. We will show that ultraproducts can be extended to this logic such that Łos lemma holds for our logic. This leads to compactness which in turn yields strong completeness (from Corollary 4.2 [Completeness Theorem] above). The method we use is the same as the one in Sain [Sa82], where it was used for establishing similar positive results for topics very closely related to the present one.

Below we translate our definitions and results concerning ultraproducts (Def. 2.2, Lemma 2.4) from algebra (that is from §2) to logic along the lines of the standard connection between algebraic logic and logic described in [HMTII, §4.3 and in the first half (pp. 255–263) of §5.6]. However, familiarity with [HMTII] is not required from the reader.

Let $\mathfrak{M}_i \in Mod$ for each $i \in I$ where the vocabulary of Mod is $\langle \mathbf{R}, r\rangle$. That is, $\mathfrak{M}_i = \langle M_i, R^{\mathfrak{M}_i}\rangle_{R\in\mathbf{R}}$ (as at the beginning of this section). Let F be a filter over I. Then the reduced product $\mathfrak{M} \stackrel{\text{def}}{=} \mathbf{P}_{i\in I}\mathfrak{M}_i/F$ is defined the following way.
$\mathfrak{M} = \langle M, R^{\mathfrak{M}}\rangle_{R\in\mathbf{R}}$ where $M = \mathbf{P}_{i\in I}M_i$ and
$R^{\mathfrak{M}} \stackrel{\text{def}}{=} \left\{q \in {}^{r(R)}M : \{i \in I : \langle q_j(i) : j < r(R)\rangle \in R^{\mathfrak{M}_i}\} \in F\right\}$. Note that our definition differs only at two points from the standard one, namely (i) M is a direct product and *not* an F–quotient of the direct product (of the M_i's), and (ii) the rank $r(R)$ of the relation $R^{\mathfrak{M}}$ might be infinite, which forced us to spell out some details more carefully than in the ordinary case.

One can check that our definition of $\mathbf{P}_{i\in I}\mathfrak{M}_i/F$ is the translation of Definition 2.2 to logic (from algebraic logic), as we promised before stating the definition. A similar translation of Lemma 2.4 to logic will yield that Łos lemma holds for our logic, implying e.g. that, for any $\varphi \in Fm$ we have

$$\mathbf{P}_{i\in I}\mathfrak{M}_i/F \models \varphi \qquad \text{iff} \qquad \{i \in I : \mathfrak{M}_i \models \varphi\} \in F\,,$$

if F is an ultrafilter.

PROPOSITION 4.3 (Generalized Łos lemma). *Let F be an ultrafilter on I and $\mathfrak{M} = \mathbf{P}_{i\in I}\mathfrak{M}_i/F$ with $\mathfrak{M}_i \in Mod$. Let $k \in {}^{\omega}M$ be an evaluation of variables. Let $pj_i : \mathbf{P}_{i\in I}M_i \longrightarrow M_i$ be the usual projection function, thus $(pj_i \circ k) \in {}^{\omega}M_i$. Let $\varphi \in Fm$. Then*

$$\mathfrak{M} \models \varphi[k] \qquad \textit{iff} \qquad \{i \in I : \mathfrak{M}_i \models \varphi[pj_i \circ k]\} \in F\,.$$

Proof: The proof is immediate by repeating the proof of Lemma 2.4 in the present context. Indeed, this is only a slight extension of the standard proof of the usual version of Łos lemma, and the points where we need to do some extra checking are all identified (and the extra checking is elaborated) in the proof of Lemma 2.4. ∎

We are ready for concluding compactness.

Let $\mathfrak{M} \in Mod$, $k \in {}^{\omega}M$ and $\Gamma \subseteq Fm$. We say that $\langle \mathfrak{M}, k\rangle$ *satisfies* Γ iff $\mathfrak{M} \models \varphi[k]$ for every $\varphi \in \Gamma$. Further, Γ is *satisfiable* iff some choice of $\langle \mathfrak{M}, k\rangle$ satisfies Γ.

COROLLARY 4.4 (Compactness). *Let $\Gamma \subseteq Fm$. Then Γ is satisfiable iff every finite subset of Γ is satisfiable.*

Proof: From the above Generalized Łos Lemma the proof is exactly the same as the standard proof of compactness of usual first order logic from usual Łos lemma. ∎

In passing we note that one can obtain different compactness properties from our Generalized Łos Lemma; for example, Γ is valid in some model iff every finite subset of Γ is valid in some model. We will not need these here, therefore we do not go into further detail (but these can be useful to be aware of e.g. when applying or studying further our logic).

In the remark below we combine Compactness (proved above) with our Completeness Theorem (Corollary 4.2 at the beginning of this section) to obtain a stronger completeness result.

Remark 4.5 (STRONG COMPLETENESS): Here we show that the usual argument works for our present logic too, for establishing strong completeness.

Let $\Gamma \subseteq Fm$ and $\varphi \in Fm$. We will define the binary semantical consequence relation "$\models$" imitating the style of Ebbinghaus et al. [EFF, Definitions 1.3, 3.1, 4.1 (pp. 27–32)] or equivalently of Bell–Machover [BM, Def.15 (p.53)].

DEFINITION 4.5.1: Let $\Gamma \subseteq Fm$ and $\varphi \in Fm$. We define $\Gamma \models \varphi$ to hold iff for every $\mathfrak{M} \in Mod$ and every $k \in {}^{\omega}M$, whenever $\langle \mathfrak{M}, k\rangle$ satisfies Γ then also $\mathfrak{M} \models \varphi[k]$.

DEFINITION 4.5.2: Let $\Gamma \subseteq Fm$ and $\varphi \in Fm$. Then $\Gamma \vdash_0 \varphi$ means that, using our axioms and rules, there is such a derivation of φ from Γ in which the inference rules other than Modus Ponens are used only for deriving validities. In other words, a $\vdash_0$ derivation consists of a first part and a second part. In the first part we do not use non–logical axioms and we are allowed to use the full power of $\vdash$. In the second part we use the formulas derived in the first part, we also use the members of Γ as non–logical axioms but the only rule of inference rule we use is Modus Ponens. (We note that the style of our definition of $\vdash_0$ is quite standard in the logic literature; and that as usual, it is possible to eliminate the "two parts" character, but then the list of axioms gets longer. Another equivalent formulation says that a $\vdash_0$ derivation of φ from Γ is a $\vdash$ derivation of $(\wedge\Gamma_0) \to \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.)

COROLLARY 4.5.3. *$\vdash_0$ is (sound) and strongly complete for our logic. That is,*

$$\Gamma \models \varphi \qquad \textit{iff} \qquad \Gamma \vdash_0 \varphi$$

for any $\Gamma \cup \{\varphi\} \subseteq Fm$.

Proof: We will follow the standard textbook pattern. Soundness is easy, therefore we omit it. To see strong completeness, assume $\Gamma \models \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is not satisfiable. Hence by our Compactness Corollary, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \cup \{\neg\varphi\}$ is not satisfiable. This means $\models (\wedge\Gamma_0) \to \varphi$. Then by our Corollary 4.2, $\vdash (\wedge\Gamma_0) \to \varphi$. Since we did not use non–logical axioms, this implies $\vdash_0 (\wedge\Gamma_0) \to \varphi$. By Modus Ponens we obtain $(\wedge\Gamma_0) \vdash_0 \varphi$. By propositional logic then $\Gamma_0 \vdash_0 \varphi$. Therefore $\Gamma \vdash_0 \varphi$, since $\Gamma_0 \subseteq \Gamma$. This completes the proof of strong completeness.
End of Remark 4.5 (Strong Completeness). ∎

We note that while Corollary 4.2 can be extended to an expansion of our logic *with equality*, as a contrast, compactness and strong completeness (above) cannot.

Finally we show that our logic here is at least as expressive as "normal" first order logic without equality. Assume that the vocabulary $\langle \mathbf{R}, r\rangle$ is such that $r(R) \in \omega$ for all $R \in \mathbf{R}$. Let $R \in \mathbf{R}$, $n \overset{\text{def}}{=} r(R)$, $j_1, \ldots, j_n \in \omega$, $i \in \omega$ and $\varphi \in Fm$ be arbitrary. Let $\tau \in {}^{\omega}\omega$ be such that

$$\tau(k) \overset{\text{def}}{=} \begin{cases} j_{k+1} & \text{if } k < n, \\ k & \text{if } k \geqslant n. \end{cases}$$

Then $\tau \in \mathbb{T}$. Let

$$tr\,(R(v_{j_i} \cdots v_{j_n})) \overset{\text{def}}{=} \mathrm{S}_{\tau} R(v_i)_{i<r(R)}, \quad \text{and}$$

$$tr(\exists v_i \varphi) \overset{\text{def}}{=} \mathrm{S}_{[0,i]} \exists v_0 \mathrm{S}_{[0,i]} \varphi\,.$$

Along these lines we can define a "translation" function tr mapping the "normal" formulas without equality into Fm. Then it is not difficult to check that, for χ in the domain of tr, $tr(\chi) \in Fm$ and "χ and $tr(\chi)$ means the same", that is for any model $\mathfrak{M} \in Mod$ and evaluation $k \in {}^{\omega}M$,

$$\mathfrak{M} \models \chi[k] \text{ (in the "normal" sense) iff}$$
$$\mathfrak{M} \models tr(\chi)[k] \text{ (in the sense defined here in section 4).}$$

The above ideas induce a brief return to algebras. Recall from subsection 1.2 that

$$\mathsf{Cs}_G \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{Gws}_G : 1^{\mathfrak{A}} \text{ is a Cartesian space, that is } 1^{\mathfrak{A}} = {}^{\omega}U \text{ for some set } U\}\,.$$

The following corollary is related to the corresponding theorems in [HMTII, section 3.1] and in [HMTAN], see e.g. Thm.3.1.103 in [HMTII].

COROLLARY 4.6. *Assume that $G \subseteq {}^{\omega}\omega$ satisfies the conditions of Lemma 2.10 (in section 2). Then*

$$\mathbf{I}\mathsf{Gws}_G = \mathbf{HSP}\mathsf{Cs}_G\,.$$

Proof: By $\mathsf{Cs}_G \subseteq \mathsf{Gws}_G$ and Theorem 2.1 in section 2, it is enough to prove that $\mathsf{Gws}_G \subseteq \mathbf{HSP}\mathsf{Cs}_G$. To this end, it is enough to show that if some equation fails in a Gws_G then it also fails in a Cs_G. Let $\mathfrak{A} \in \mathsf{Gws}_G$, let e be an equation in the language of Gws_G, and assume that $\mathfrak{A} \not\models e$. We may assume that $|A| \leqslant \omega$, $|G| \leqslant \omega$. Then by the proof of Lemma 2.10, $\mathfrak{A}$ has a homomorphic image $\mathfrak{A}'$ with unit $W = \bigcup\left\{{}^{\omega}(\omega+\omega)^{(\tau)} : \tau \in [G]\right\}$, such that $\mathfrak{A}' \not\models e$ still holds. Then, by the proof of Corollary 4.2, $\mathfrak{A}'$ is a homomorphic image of some $\mathfrak{A}" \in \mathsf{Gws}_G$ with unit ${}^{\omega}(\omega+\omega)$. Then $\mathfrak{A}" \in \mathsf{Cs}_G$ and $\mathfrak{A}" \not\models e$ by $\mathfrak{A}' \not\models e$, that is $\mathsf{Cs}_G \not\models e$. ∎

5. EQUALITY

In sections 5.1 and 5.2 we are looking at two existing approaches to "finitizing" logic *with* equality.

We will outline facts supporting the conjecture that the Henkin–Monk problem (quoted in the Introduction) might *not* admit a positive solution for logic with equality. More precisely, for logic with equality, only the weaker, "Tarski–Givant style" version[36] of the problem admits a positive solution, which positive solution in turn can be found in [Sa92, section 6] and in [SGy96, section 2.3] (cf. Remark 5.5 here and the paragraph following it). In this connection we feel that algebraic logic might give some useful information back to logic. Namely, an essential difference between the behaviors of these two versions (with and without equality) of first order logic may have been found now (by means of algebraic logic).

We will also look briefly at related positive results, in the direction of finitizing logic with equality, and will give references to literature dealing with them. The price of obtaining positive results using the approach outlined in subsection 5.1 is weakening the original problem, while using the approach sketched in subsection 5.2, for obtaining positive results, the underlying set theory had to be modified.

5.1. Adding diagonal elements

A very natural approach to extending the present work to logic with equality consists of adding the diagonal element $\mathsf{d}_{01} = \mathrm{D}_{01} = \mathrm{D}_{01}^{[V]} \stackrel{\text{def}}{=} \{q \in V : q_0 = q_1\}$ as a distinguished constant to Gws_G (for Gws_G see Definition 1.4 in section 1.2). Such an approach was elaborated in a great extent in, e.g., Craig [Cr74] (for different purposes). However, as we will see below Lemma 5.2, the set algebras in this case do not form a variety or even an axiomatizable (elementary) class.

The above fact, namely that adding d_{01} to Gws_G "does not work", does not prove that some other approach will not work either for finitizing (algebraic) logic with equality. So it would be instructive to see some more general negative result. Our Theorem 1 in the Introduction was an existence statement only, therefore its negation would provide us with a more general non–existence statement (than the negation of Corollary 3.3).[37]

Recall from subsection 1.1.5 that, for an ordinal α, PA_α and QPA_α denote the varieties of polyadic algebras and quasi–polyadic algebras of dimension α, respectively. PEA_α and QPEA_α denote, respectively, the classes of PA_α's and QPA_α's *with equality*; cf. e.g. [HMTII] or [Né91] for detailed definitions. For a similarity type t, $\mathbf{Rd}_t$ denotes the operator (on classes of algebras) of taking *reducts* to the language t. For a class V of algebras, $\mathbf{Rd}_{\mathsf{V}}$ is defined to be $\mathbf{Rd}_t$, where t is the similarity type of V. For example, $\mathbf{Rd}_{\mathsf{QPA}_\omega}(\mathsf{PA}_\omega) \subsetneqq \mathsf{QPA}_\omega$.[38] Let t_0 denote the similarity type of QPA_ω. Let $\mathsf{V}_1 \equiv_{\text{term}} \mathsf{V}_2$ denote *term definitional equivalence* of the varieties V_1 and V_2.

Recall that Theorem 1 stated the existence of a *finitely axiomatizable* variety V and a variety V_1 such that (5.1) (i–iv) below hold.[39]

(5.1) (i) $\text{Representable}\mathsf{QPA}_\omega = \mathbf{SRd}_{t_0}\mathsf{V}_1$,
(ii) $\mathsf{V}_1 \equiv_{\text{term}} \mathsf{V}$,
(iii) $\mathsf{V} = \mathbf{HSRd}_{\mathsf{V}}(\text{Representable}\mathsf{PA}_\omega)$[40] , and
(iv) all elements of V are *representable*

[36] Cf. [TG87 paperback edition, p.62, lines 9-10].
[37] Of course, in both cases "negation" is understood as a consequence of adding a new operation like d_{01} to our algebras.
[38] This follows from the result of [DM63] quoted in the previous footnote and from [ST91] where we proved that there exist non–representable QPA's.
[39] As a curiosity, we note that for some choices of t, $\mathbf{SRd}_t\mathsf{PA}_\omega$ is not a variety, but, e.g., $\mathbf{SRd}_{t_0}\mathsf{PA}_\omega$ is a variety.
[40] $\text{Representable}\mathsf{PA}_\omega = \mathsf{PA}_\omega$, but we do not want to burden the reader with using this result (of A. Daigneault and J. D. Monk).

(with the natural logical, that is permutation invariant operations). In (5.1) above, condition (iv) is essential in the sense that without (iv), (5.1) becomes true even for QPEA_ω, PEA_ω and for some finitely axiomatizable V, as we will see in Remark 5.5 below.

The following stronger form (5.1^+) of (5.1) also follows from the results of this paper. We let **rl** denote the operator (on classes of algebras) of *relativizing* with elements such that the relativization is a homomorphism. So $\mathfrak{Rl}_b\mathfrak{A} \in \mathbf{rl}\{\mathfrak{A}\} = \mathbf{rl}\mathfrak{A}$ if $\langle a \cap b : a \in A\rangle$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{Rl}_b\mathfrak{A}$. Then

(5.1^+)
(i) $\mathrm{Representable}\mathsf{QPA}_\omega = \mathbf{SRd}_{t_0}\mathsf{V}_1$,
(ii) $\mathsf{V}_1 \equiv_{\mathrm{term}} \mathsf{V}$,
(iii) $\mathsf{V} = \mathbf{rlSRd}_{\mathsf{V}}(\mathrm{Representable}\mathsf{PA}_\omega)$ and
(iv) all elements of V are *representable*.

Replacing QPA_ω and PA_ω in (5.1^+) (or even in the weaker (5.1)) with QPEA_ω and PEA_ω, respectively, gives us a false statement, for any variety V *term definitionally equivalent* with a *finitely axiomatizable* one (by Theorem 5.1 below). Note that the latter statement is stronger than the negation of Theorem 1 because there V itself was required to be finitely axiomatizable. Since $\mathbf{Rd}_{\mathsf{CA}_\omega}\mathsf{QPEA}_\omega \subsetneq \mathsf{CA}_\omega$, we obtain an even stronger negative result if we replace QPEA_ω with CA_ω in the just mentioned strong negation of Theorem 1. This is Theorem 5.1 below.

Let t be a similarity type consisting of a subset of the operations $\{+,-\} \cup \{\mathsf{c}_i : i \in \omega\} \cup \{\mathsf{d}_{ij} : i,j \in \omega\} \cup \{\mathsf{s}_\tau : \tau \in {}^\omega\omega\}$ with the usual arities, and let $\mathfrak{A}$ be of type t. We say that $\mathfrak{A}$ is *representable as a relativized set algebra* if there is some U and $V \subseteq {}^\omega U$ such that $\mathfrak{A} \subseteq \langle \mathcal{P}(V), f^{[V]}\rangle_{f\in t}$, where $f^{[V]}$ denotes the natural interpretation of the operation f in $\mathcal{P}(V)$.

THEOREM 5.1. *Let $\alpha > 2$ be an ordinal. Then for any three classes V, V_1, V_2 satisfying* (5.2) *below, V is not axiomatizable by finitely many first order formulas. The same applies to* (5.2^+) *in place of* (5.2).

(5.2)
(i) $\mathrm{Representable}\mathsf{CA}_\alpha = \mathbf{HSPRd}_{\mathsf{CA}_\alpha}\mathsf{V}_1$,
(ii) $\mathsf{V}_1 \equiv_{\mathrm{term}} \mathsf{V} \equiv_{\mathrm{term}} \mathsf{V}_2$,
(iii) $\mathsf{V}_2 \subseteq \mathbf{HSPRd}_{\mathsf{V}_2}(\mathsf{PEA}_\alpha)$, and
(iv) every element of V_2 is representable as a *relativized* set algebra.

(5.2^+)
(i) $\mathrm{Representable}\mathsf{CA}_\alpha = \mathbf{HSPRd}_{\mathsf{CA}_\alpha}\mathsf{V}_1$,
(ii) $\mathsf{V}_1 \equiv_{\mathrm{term}} \mathsf{V} \equiv_{\mathrm{term}} \mathsf{V}_2$,
(iii) $\mathsf{V}_2 \subseteq \mathbf{SPrlRd}_{\mathsf{V}_2}(\mathsf{PEA}_\alpha)$.

We note that the form of item (5.2) was made more permissive than a strict transcription of (5.1), in order to make Theorem 5.1 stronger than a simple negation of Theorem 1 for algebras with equality. (In principle, it could be that Theorem 1 does not extend to algebras with equality only because the formulation of (5.1) is stronger than needed for solving the finitization problem. Theorem 5.1 shows that this is not the case.)

For proving Theorem 5.1, we will need Lemma 5.2 below, which might be of interest in itself (cf. the last sentence of the first paragraph of the present subsection). Concrete instances of Lemma 5.2 appeared, e.g., in Craig [Cr74] and Johnson [Jo69]. In its present form, the lemma was brought to our attention by R. J. Thompson (we include it here with his permission).

LEMMA 5.2. *Let $\alpha \geqslant \omega$. Let $\tau \in {}^\alpha\alpha$ move infinitely many points. (That is, assume $\{i \in \alpha : \tau(i) \neq i\}$ is infinite.) Let $|U| > 1$. Let $\mathfrak{A} = \langle \mathcal{P}({}^\alpha U), \cap, \mathrm{S}_\tau, \mathrm{D}_{01}^{[U]}, \mathsf{s}_j^i\rangle_{i,j\in\alpha}$. Then $\mathfrak{A}$ has an ultrapower $\mathfrak{B}$ which has the following strong non–representability property:*

$$\text{For any } W \text{ and } V \subseteq {}^\alpha W, \quad \mathfrak{B} \notin \mathbf{SP}\left\{\langle \mathcal{P}(V), \cap, \mathrm{S}_\tau^{[V]}, \mathrm{D}_{01}^{[V]}, \mathsf{s}_j^i\rangle_{i,j\in\alpha}\right\}.$$

Proof: This is an adaptation of a proof of J. D. Monk and S. J. Johnson, cf. [Jo69].

Let $\mathfrak{A}$ be as in the formulation of the lemma. Let $H \stackrel{\mathrm{def}}{=} \{i \in \alpha : \tau(i) \neq i\}$. By hypothesis, $|H| \geqslant \omega$. Let $W \subseteq H$ be such that $W \cap (\tau'' W) = 0$ and $|W| = \omega$. (Exists.) Let $h = \{\langle i, m\rangle \in \alpha \times 2 : m = 1 \text{ if } i \in W \text{ and } m = 0 \text{ if } i \notin W\}$. Then $h \in {}^{\alpha}2$. We may assume $2 \subseteq U$. For all $j \in \alpha$, let $k_j \stackrel{\mathrm{def}}{=} h \restriction (\alpha \setminus j) \cup \{\langle i, 0\rangle : i < j\} \in {}^{\alpha}2$, and $X_j \stackrel{\mathrm{def}}{=} \{k_j\}$.

Claim 5.2.1. $\mathrm{S}_\tau\{X_j\} \cap X_j = 0$ *for any fixed* $j \in \alpha$.

Proof: Let $j < i \in W$. Then $q \in \mathrm{S}_\tau(X_j)$ implies $q \circ \tau = k_j$, hence $q_{\tau i} = (q \circ \tau)_i = k_j(i)$. But then, by $j < i \in W$, we have $q_{\tau i} = k_j(i) = h(i) = 1$. However, if $g \in X_j$ then $g_{\tau i} = 0$ because by $\tau(i) \notin W$ we have $h_{\tau i} = 0$. Thus $g \in X_j \Rightarrow g_{\tau i} = 0 \neq 1 = q_{\tau i}$, proving $q \notin X_j$.
QED (Claim 5.2.1) ∎

Let F be a nonprincipal ultrafilter on α. Let $X = \langle X_j : j \in \alpha\rangle / F$. Then $\mathrm{S}_\tau(X) \cap X = 0$ by Claim 5.2.1, and $(\forall i \in \alpha) X \leqslant \mathrm{D}_{0i}$ by Łos lemma, where $\mathrm{D}_{0i} = \mathrm{s}^1_i \mathrm{D}_{01}$. Further $X > 0$, hence $X \nleqslant \mathrm{S}_\tau(X)$. These together imply nonrepresentability, since in set algebras $0 < X \Rightarrow \exists q \forall i\, (q \in X < \mathrm{D}_{0i})$, thus $\forall i\, (q_0 = q_i)$, which implies $q \circ \tau = q \in X$, hence $X \leqslant \mathrm{S}_\tau(X)$ should be the case, but it is not.
QED (Lemma 5.2) ∎

We reached the point where the claim at the beginning of the present subsection saying that "$\mathsf{Gws}_G + \mathsf{d}_{01}$ does not work", can be justified. In this case, for a finitization of (the similarity type of) our algebras, we need at least one $\tau \in {}^{\omega}\omega$ which is not a finite transformation. But then, Lemma 5.2 above says that if we include one of the D_{ij}'s (to represent equality), we end up with a strongly non–axiomatizable situation. Namely, then there is a "Cs–type" (cf. the definition of Cs_G after Definition 1.4 in subsection 1.2) algebra an ultrapower of which is not representable even in a "relativized Cs–type" ("Crs–type") way. (To capture this phenomenon, we do not need the full Boolean algebraic power of Gws_G's, it is enough to have the ordering, which can be captured by "$\cap$". We do not need C_i's either.)

Next we turn to outlining the proof of Theorem 5.1.

Outline of proof of Theorem 5.1: The $\alpha < \omega$ case is straightforward by [An91, Thm.5], cf. also [HMTII, 5.4.40].

Therefore we may assume $\alpha \geqslant \omega$.

As in [HMT], D_{ij} denotes the algebraic counterpart of (logical) equality $v_i = v_j$. In [An91] there is a relatively simple proof for the version of Theorem 5.1 with CA in place of PEA. The present proof goes by generalizing Andréka's just quoted proof from CA to PEA. We will outline here the generalization only for the QPEA $\mapsto$ PEA change, since the CA $\mapsto$ QPEA change is straightforward.[41]

The proof goes by first observing that for any $H \subseteq \mathcal{P}(\alpha)$, in [[An91, Thm.1] or in [Mo91], or in updated version of [Né91, Thm.5.1]], the "simple proof for Monk's non–finite schematizability of RCA_ω" goes through for the expansion $\mathsf{QPEA}_{\alpha,H}$ of QPEA_α's with the operations $\{\mathrm{C}_{(\Gamma)} : \Gamma \in H\}$ (in place of CA_ω). This proof shows that the complement of $\mathsf{RQPEA}_{\alpha,H}$ (representable $\mathsf{QPEA}_{\alpha,H}$'s) is not closed under ultraproducts. Let $\mathsf{V}, \mathsf{V}_1, \mathsf{V}_2$ be as in Theorem 5.1. If $\mathsf{V}_1 = \mathsf{RQPEA}_{\alpha,H}$ for some H, then it is very easy to prove that the complement of V is not ultraproduct-closed either, since $\mathsf{V}_1 \equiv_{\mathrm{term}} \mathsf{V}$. Therefore we may assume $\mathsf{V}_1 \neq \mathsf{RQPEA}_{\alpha,H}$ for any $H \subseteq \mathcal{P}(\alpha)$. Then $\mathsf{V}_1 \supsetneq \mathsf{RQPEA}_{\alpha,H}$, hence, by $\mathsf{V}_2 \subseteq \mathbf{HSPRd}_{\mathsf{V}_2}\mathsf{PEA}_\alpha$ there must be an S_τ in the language of V_2 but not in the language of QPEA_α. Then τ must move infinitely many points in α. But then having the $\mathrm{D}_{i,j}$'s and S_τ together implies that V_2 is not closed under ultraproducts, by Lemma 5.2. By $\mathsf{V} \equiv_{\mathrm{term}} \mathsf{V}_2$ then V cannot be closed under ultraproducts either. ∎

[41] Actually, by analyzing Andréka's proof further, the result can be generalized to RCA_3 in place of RCA_α, that is, sandwiching (via subreducts) V in between RCA_3 and PA_α, where RCA means representable CA. We will not go into this here.

Conjecture 5.3. *We conjecture that Theorem 5.1 above remains true if, in* (5.2), *instead of* $\mathsf{V}_2 \subseteq \mathbf{HSPRd}_{\mathsf{V}_2}\mathsf{PEA}_\alpha$ *we require only the existence of a class* K *with* $\mathsf{V}_2 \subseteq \mathbf{HSPRd}_{\mathsf{V}_2}\mathsf{K}$ *and* $\mathsf{K} \equiv_{\mathrm{term}} \mathsf{PEA}_\alpha$. ◀

A proof of the above conjecture would be a step, more convincing than the results obtained so far, in the direction of justifying the general conjecture that logic with equality does not admit a finitizable algebraization.

Remark 5.4: Theorem 5.1 can be strengthened the following way. Let RPEA^+_α denote the class obtained from RepresentablePEA_α by expanding it with the generalized (infinitary) diagonal constants D_E for each equivalence relation E on α, where $\mathrm{D}_E \stackrel{\mathrm{def}}{=} \bigcap\{\mathrm{D}_{ij} : iEj \text{ and } i,j \in \alpha\}$. RPEA^+ is known as Lucas' expanded representable PEA's, see e.g. [HMTII, p.267 (item 12)].

(5.3) Theorem 5.1 remains true if we replace PEA_α in it with RPEA^+_α.

The idea of proof given for Theorem 5.1 works for proving (5.3), too. Actually, we can add all infinitary Boolean combinations of the D_{ij}'s to RPEA^+_α as constants, and the theorem remains true for the so obtained even stronger class $\mathsf{RPEA}^{++}_\alpha$. This class is basically the same as what we get by adding the constants $\bar{d}R$ for all $R \subseteq \alpha \times \alpha$ as defined in [HMTI] to RPEA^+_α. In $\mathsf{RPEA}^{++}_\alpha$, almost *all* of the *generalized quantifiers* are expressible, so adding them to PEA_α does not help in trying to avoid the negative result Theorem 5.1. Probably Andréka's new methods for proving non–finitizability reported in [Né91] and partially in [An91] could extend our negative theorems to all generalized quantifiers and perhaps to modalities (unary operations distributing over "$\vee$") too; but we did not check this. ◀

Let us turn to seeing what kind of *positive* solution is available for logic *with equality* (for the finitization problem).

Item (5.4) below shows that the condition in (5.2), saying that the elements of V_1 are representable, is necessary for Theorem 5.1 to be true.

Remark 5.5: There is a finitely axiomatizable variety V and a V_1 such that

(5.4) (i) Representable$\mathsf{CA}_\omega = \mathbf{SRd}_{\mathsf{CA}_\omega}\mathsf{V}_1$,
(ii) $\mathsf{V}_1 \equiv_{\mathrm{term}} \mathsf{V}$,
(iii) $\mathsf{V} = \mathbf{HSRd}_{\mathsf{V}}(\text{Representable}\mathsf{PEA}_\omega)$.

The same applies to RepresentableQPEA_ω in place of RepresentableCA_ω, cf. item (2.26) in subsection 2.4.

Let $(\mathsf{Gws}_G + \mathsf{d}_{01})$ denote the class Gws_G (as introduced in Definition 1.4 in subsection 1.2) expanded with the extra constant $\mathsf{d}_{01} = \mathrm{D}_{01}$. (The members of Gws_G not containing d_{01} are thrown away.) Then

(5.5) The equational theory $\mathbf{Eq}(\mathsf{Gws}_G + \mathsf{d}_{01})$ is axiomatizable by a finite set of equations.

(5.6) Statement (5.5) remains true even if we restrict ourselves to those members of Gws_G whose unit elements are Cartesian spaces.

On the proofs of items (5.4–5.6): First we note that (5.4) follows from (5.5). The proof of (5.5) is parallel with that of [SGy96, Thm.2.24], in the following way. [SGy96] first proves a positive theorem for a class (called Gwdf_G) analogous to our Gws_G. Then d_{01} is added to this class and the theorem is pushed through for the variety generated by the so obtained $(\mathsf{Gwdf}_G + \mathsf{d}_{01})$. Following the same pattern, in our case, our present positive result for Gws_G can be pushed through for obtaining a finite axiomatization of the equational theory of $(\mathsf{Gws}_G + \mathsf{d}_{01})$.

(5.6) follows from the following. $\mathsf{Gws}_G \subseteq \mathbf{HSP}\mathsf{Cs}_G$, in fact, $\mathsf{Gws}_G \subseteq \mathbf{H}\mathsf{Cs}_G$ because the definition of a Gws_G–unit was chosen such that relativization (of a Cs_G) with such a unit is always a homomorphism. ◀

Consider the problem in [TG87 paperback edition, p.62, lines 9-10]. The algebraic counterpart of this problem, in terms of relation algebras, can be formulated as follows. *Is it possible to expand representable relation algebras (*RRA*) with finitely many permutation invariant operations obtaining the class* RRA^+ *such that the equational theory* $\mathbf{Eq}(\mathsf{RRA}^+)$ *of* RRA^+ *becomes finitely axiomatizable?*[42] The important difference with what we call here "finitization problem" is that here we require finite axiomatization of $\mathbf{Eq}(\mathsf{RRA}^+)$ only, instead of axiomatizing RRA^+ itself. The CA-theoretic counterpart of this problem goes as follows.

(5.7) *Is it possible to expand* RCA_ω *with finitely many permutation invariant operations*[43] *obtaining* RCA_ω^+ *such that* $Mod\mathbf{Eq}(\mathsf{RCA}_\omega^+) = \mathbf{Var}\mathsf{RCA}_\omega^+(= \mathbf{HSP}(\mathsf{RCA}_\omega^+))$ *is term definitionally equivalent with a finitely axiomatizable variety* V*? Here* RCA_ω^+ *should be such that* $\mathsf{RCA}_\omega = \mathbf{SRd}(\mathsf{RCA}_\omega^+)$.

To see the point here, by a *reduct-finite* axiomatization of a variety V we understand a finite axiomatization of a reduct of V term definitionally equivalent with V. The point in (5.7) is that we add new operations to RCA_ω such that we can use the new operations to provide a reduct-finite axiomatization of the *old operations only*, in the following sense. We are asking for a reduct-finite axiomatization of $\mathbf{HSP}(\mathsf{RCA}_\omega^+)$, for some RCA_ω^+, while $\mathsf{RCA}_\omega = \mathbf{SRd}(\mathbf{HSP}\mathsf{RCA}_\omega^+)$. The logical counterpart of solving this weaker version of the finitization problem amounts to providing a completeness theorem without compactness (for the logic in question) as was proved in [ANSK, **Thm.3.2.4**].

Our (5.7) is exactly the problem which was discussed in Remark 5.5. So this problem admits a positive solution.

The logical meaning of items (5.1), (5.4) is the following.

$L_{\omega\omega}$ denotes first order logic with equality.

There is an expansion L_1 of $L_{\omega\omega}$ (that is, L_1 is obtained from $L_{\omega\omega}$ by adding new logical connectives) such that L_1 (or an equivalent formulation of L_1) admits a truly finite *weakly* complete[44] inference system, say $\vdash_1$. This $\vdash_1$ is completely similar to what we did for $L_{\omega\omega}$ *without* equality in section 4.

At the same time, if the extra connectives of L_1 correspond to operations of RPEA_α^+ then the inference system $\vdash_1$ of L_1 cannot be strongly complete for L_1. This negative statement follows from items (5.1), (5.3) the following way. (We give only an intuitive outline since for more we would need Universal Algebraic Logic, cf. [ANS84] or [AKNS].) The (Hilbert style) finite axiomatization of $\vdash_1$ comes from the finite equational (or quasi–equational if any) axiomatization of V or of $\mathbf{Eq}(\mathsf{Gws}_G + \mathsf{d}_{01})$ in Remark 5.5, exactly as we obtained our finite axiomatization of logic in section 4 from the finite axiomatization of the variety $\mathbf{I}\mathsf{Gws}_G$. By Theorem 5.1, if V is finitely axiomatizable then some elements of V are not representable. But "Representable V" is closed under $\mathbf{SP}$, hence it is an infinitary quasi–variety. Therefore there exists an infinitary quasi–equation $\bigwedge_{i\in I} e_i \to e$ valid in Representable V but not in V. Let φ_i be the standard logical counterpart of e_i and φ be that of e. Then by Universal Algebraic Logic, "Representable V" $\models \bigwedge_{i\in I} e_i \to e$ is equivalent with $\{\varphi_i : i \in I\} \models \varphi$ holding for L_1. But since our inference system $\vdash_1$ is based on (the axiomatization

[42] Note that in this case, $\mathsf{RRA} = \mathbf{SRd}\,Mod(Ax)$ for the finite set Ax of equations axiomatizing $\mathbf{Eq}(\mathsf{RRA}^+)$. Further we note that, strictly speaking, [TG] formulates the problem for the one-variable equations axiomatizing $\mathbf{Eq}(\mathsf{RRA}^+)$ in place of $\mathbf{Eq}(\mathsf{RRA}^+)$. Actually, the problem asks for an expansion of the "concrete class" of relation set algebras (or proper relation algebras) in place of the abstract class RRA.

[43] More precisely, we are asking for an expansion of the class Cs_ω of cylindric set algebras (in place of RCA_ω).

[44] Weakly complete means (roughly) "complete but not necessarily compact". More precisely, $\models \varphi \Leftrightarrow \vdash \varphi$, but not necessarily $\Sigma \models \varphi \Leftrightarrow \Sigma \vdash \varphi$ (for an arbitrary set Σ of formulas).

of) V, and $\mathsf{V} \not\models \bigwedge_{i\in I} e_i \to e$, we cannot have $\{\varphi_i : i \in I\} \vdash_1 \varphi$. This means that $\vdash_1$ is not strongly complete for $\models_1$.

For more about the connections between the existence or nonexistence of a finitely axiomatizable V satisfying (5.2) and purely logical completeness issues (that is existence and properties of L_1 and $\vdash_1$ as discussed above) see Simon [Si92, Appendix A].

Open Problem 5.6: Is the quasi–equational theory of $\mathsf{Cs}_{\mathbb{G}}$ *with equality* finitely axiomatizable? If not then is the quasi–equational theory of Cs_G *with equality* finitely axiomatizable for some choice of G which makes all the c_i's and d_{ij}'s term definable? Cf. Németi–Sági [NS96] for results that may be relevant. ◀

Open Problem 5.7: Is there an expansion RCA^+_ω of RCA_ω satisfying all the conditions in item (5.7) such that RCA^+_ω itself would be a finitely axiomatizable variety (or quasi–variety)?

A negative answer to this problem would prove (a tangible version of) the conjecture (cf. Monk [Mo70]) that first order logic with equality is not finitizable in the strong sense. We note that Theorem 5.1 already excludes a large number of possible choices of RCA^+_ω. ◀

5.2. True pairing algebras and related attempts to finitize first order logic with equality

This approach seems to be similar to that of Craig and Vaught [CV58], and was suggested in the lecture [Ma87] and independently (in different form) in Veloso–Haeberer [VH91], [VH91a]. It is based on Tarski's Q–relation algebras (QRA's for short), more precisely, only on the "standard" QRA's enriched with extra constant symbols. (For QRA see [TG87, p.242], and for the ones we call "standard", [TG87, item 4.1 (iv) on p.96, pp. 251–254], Németi [Né86, Né97, Né97a], Andréka–Jónsson–Németi [AJN88].) QRA's were used by Tarski in 1953 to prove that set theory was finitizable, see [TG87]. The presently discussed approach would try to finitize not only set theory but also logic itself. The above mentioned standard QRA's enriched with extra constant symbols are called *true pairing algebras* (TPA's) in the literature ([Md89a etc.]). We do not recall the definition of QRA's, but that of TPA's will be recalled in Definition 5.6 below.

There are various difficulties with this approach, e.g. the basic operations of TPA's are not permutation invariant. We do not go into discussing whether or not this difficulty can be overcome because of Theorem 5.7 below, saying that the equational theory **Eq**(TPA) of TPA is not only not recursively enumerable but its degree of unsolvability is exactly Π^1_1. The finitization problem was asking for a finitely based variety. Π^1_1 is so far from finitely axiomatizable that, to our minds, Theorem 5.7 in itself seems to exclude applicability of this approach to the finitization problem.

Let us recall from, e.g. [HMTII, Def.5.3.1 p. 211] that a *relation algebra* (RA) is a system $\mathfrak{A} = \langle A, +, \dots, -, 0, 1, \circ, ^{-1}, 1'\rangle$ such that $\langle A, +, \cdot, -, 0, 1\rangle$ is a BA, $\circ$ is a binary operation on A (called composition), $^{-1}$ is a unary operation on A (called converse), $1' \in A$ (called identity), and $\mathfrak{A}$ satisfies certain conditions.

$\mathfrak{Re}(U)$ denotes the full relation set algebra with base U in the sense of [HMTII, Def. 5.3.2]. Thus the greatest element of $\mathfrak{Re}(U)$ is $U \times U$, and all its elements are binary relations on U.

DEFINITION 5.8 ([Ma81], [Ma87], [Ma89, §4 Def. 20]):

(1) $\mathfrak{A}$ is a *full true pairing algebra* (full TPA) iff $\mathfrak{A} = \langle \mathfrak{R}, p, q\rangle$, where $\mathfrak{R} = \mathfrak{Re}(U)$ for some set U; p, q are distinguished constants of $\mathfrak{A}$;

$$p = \{\langle\langle x, y\rangle, x\rangle : x, y \in U\}, \quad q = \{\langle\langle x, y\rangle, y\rangle : x, y \in U\},$$

further U satisfies condition (5.8) below.

(5.8) There is a sequence $\langle W_n : n \in \omega\rangle$ such that $W_0 \subseteq U$, no element of W_0 is an ordered pair, $W_{n+1} = W_n \cup (W_n \times W_n)$ for all $n \in \omega$, and $U = \bigcup\{W_n : n \in \omega\}$.

p and q are called *projection elements* of $\mathfrak{A}$.

(2) TPA = **SP**full TPA, that is, $\mathfrak{A} \in$ TPA iff $\mathfrak{A}$ is isomorphic to a subalgebra of a direct product of full TPA's. ◀

In formulating Theorem 5.7 below, we use the terminology of recursion theory for clarifying the "uncomputability" of non-computable functions. The complexity classes are denoted by Π^k_n and Σ^k_n $(n, k \in \omega)$, where if $k > 0$, then the set in question is so far from being computable that it is called non–arithmetical. (This implies that such sets are not definable in set theory without the Axiom of Infinity.)

THEOREM 5.9 (joint with A. Simon). *The set* **Eq**(TPA) *of all equations valid in* TPA *is* Π^1_1 *complete (in the analitical hierarchy). That is, the "degree of unsolvability" or measure of undefinability of* **Eq**(TPA) *is exactly* Π^1_1.

Proof: This theorem is fully proved in [SS92]; somewhat less detailed proofs are given in [SN96] and [Né97a]. ∎

For completeness we note the following. A class PRA of algebras, which can be viewed as a variant of TPA, was introduced in [Né86], [AJN88]. The definition of PRA is obtained from Definition 5.8 above by deleting condition (5.8). It was proved in [Né97, Né97a] that in a non–well–founded set theory, PRA becomes a finitely axiomatizable variety, while in usual set theory, Theorem 5.9 applies to PRA, too. Therefore, if one is willing to work in a non–well–founded set theory, then Theorem 5.9 can be evaded, and PRA becomes a possible vehicle for attacking the finitization problem. Indeed, this kind of possibility was recently studied in Németi [Né96], Simon [Si96], Salibra–Goldblatt [SG95].

ACKNOWLEDGEMENT:

The author is grateful to Richard J. Thompson, whose kind interest inspired her to dust off her half forgotten notes containing the mathematical content of the present sections 1 and 2, in 1986–87, during his scientific visit in Budapest (Hungary). He should also be credited for ideas simplifying some of the proofs in section 2 (which were much clumsier in the old version). Further, we would like to thank him for providing extensive help, advice and ideas for modifying his semigroup theoretic result in [MT73], [Th80] to satisfy the needs of the present paper.

The author would like to express her deep gratitude to William Craig, Leon Henkin, J. Donald Monk, Hajnal Andréka and István Németi for encouragement, suggestions, and extensive help in various form (e.g. reading previous versions, pointing out what is new and what is "not quite so", indicating which parts needed improvement).

Some of the applications to logic were obtained in the Alfred Tarski Room at University of California Berkeley (summer 1987) with helpful guidance from W. Craig and L. Henkin.

Special thanks are due to Steven Givant for reading a previous version of the paper, and giving very useful advice.

The author feels indebted to the anonymous referee for his/her very useful suggestions.

Many of the revisions based on the 1988 version of this paper were done on the basis of W. Craig's suggestions during the Banach Semester "Applications of universal algebra and algebraic logic in computer science" (Warsaw, Poland, fall 1991) and on the basis of feedbacks from members (mainly from J. D. Monk) of the Logic Seminar of University of Colorado Dept. of Mathematics (Boulder, Colorado, October–December 1994) where the author got the opportunity for reporting the results of this paper in a series of talks.

Thanks are due to Don Pigozzi for extensive help in obtaining the result reported in section 5.2. Thanks are due to Judit Madarász for substantially simplifying the 1996 version of the proof of Theorem 3.2. Thanks are due to Yde Venema for his kind interest and for suggestions concerning presentation. Finally, thanks are due to the following phd students of "Logic Graduate School Budapest": Viktor Gyuris, Eva Hoogland, Ágnes Kurucz, Judit Madarász, Szabolcs Mikulás and András Simon for reading and checking parts of the paper.

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