Algebraic Characterizations of two Beth Definability Properties

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Abstract

In the present paper we will show that the Beth definability property corresponds to surjectiveness of epimorphisms in abstract algebraic logic. Furthermore, we will give an equally general characterization of the weak Beth definability property, hereby giving a solution to Problem 14 in [Sa 88]. Finally, we will present two counterexamples showing that two of the more striking assumptions imposed on the logics in our characterizations are *necessary*.

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Contents

1 Introduction

Abstract Algebraic Logic

The aim of this paper is to give an algebraic characterization of some Beth definability properties. In giving such a characterization we find ourselves right in the middle of a typical problem area of Abstract Algebraic Logic. Before explaining in some detail what this paper is about, let us say a few more words about the purposes and methods of Abstract Algebraic Logic.

In the middle of the nineteenth century, G. Boole initiated the investigations of a class of algebras, subsequently called *Boolean Algebras*, directly related to what is now known as *propositional logic*. Nowadays we have grown so familiar with this relation that to us, Boolean Algebras and propositional logic are, so to speak, nothing but two sides of the same coin. The use of these investigations is that it enables us to translate results about Boolean Algebras into results about propositional logic, and vice versa. At the end of the nineteenth century forms of relation algebras were developed that mirror richer system of logics, but the success of Frege's predicate logic took these out of fashion. However, the work of Boole gave an enormous impetus to the development of propositional logic. Hence, when in the middle of this century plenty of propositional logics other than classical propositional logic entered the picture, it was not surprising that investigations were started into algebraic structures corresponding to these new logics in the same way as Boolean Algebras correspond to classical propositional logic. As examples we mention Heyting Algebras corresponding in this sense to intuitionistic propositional logic, or Boolean Algebras with Operators as algebraic counterparts of modal logics. Also the original attempts to algebraize quantifier logics have been renewed. Among the algebraic structures investigated for this purpose we count Cylindric Algebras, Polyadic Algebras and Relation Algebras. For more details we refer to [Nem 91].

Eventually, these investigations lead to the development of *Abstract Algebraic Logic.*[1] In Abstract Algebraic Logic one is not so much interested in giving an algebraization for one specific logic. Rather one wants to find a *general method* for associating to *any* logic $\mathcal{L}$ a class of algebras, say $\mathsf{Alg}(\mathcal{L})$. Once a theory is developed which establishes such a general connection between Logic and Algebra, one can try to prove equivalence theorems of the following kind,

$$\text{Logic } \mathcal{L} \text{ has property } P \Longleftrightarrow \mathsf{Alg}(\mathcal{L}) \text{ has property } \mathsf{alg}(P),$$

where P is an interesting logical property, and $\mathsf{alg}(P)$ a natural algebraic property. For example, to any logic "close to" first order logic, the Craig Interpolation Property corresponds in this sense to the Amalgamation Property (D. Pigozzi, cf. [Pi 72]), and completeness theorems for logics correspond in this way to representation theorems for algebras (cf. [AKNS 94]).

The main motivation for making all these efforts, is that these equivalence theorems enable us to make use of algebraic results in solving logical problems, and vice versa. For now one can solve logical problems (like deciding whether a logic has certain properties) by translating them to Algebra, solving the problems there, and translating the solutions back to Logic. Of course, this method works both ways. So in short, it leads to an exchange of insights about Algebra and Logic, fruitful to the development of both areas.

Beth Definability Properties and Epimorphisms

In the present paper we will prove an equivalence theorem of the above kind, concerning the Beth definability property (BP for short). The latter is a distinguished property of logics and has been investigated ever since 1953 when E.W. Beth proved that ordinary first order logic has this property. The Beth definability property links two possible ways of defining relations to each other. Hence in order to get a feeling for what this property is about, let us have a quick look at how one can define a relation.

Well, for example, let us fix the language L_{PA} of Peano Arithmetic (PA for short), containing $0, +, \times$ and S, and assume that we want to define the function symbol ! (factorial) in PA. A possible definition is the following set of sentences $\Sigma(!)$ defined by,

$$\Sigma(!) \stackrel{\text{def}}{=} \{0! = S(0),\ \forall x((S(x))! = (S(x)) \times x!)\}.$$

This is a definition of ! in the sense that in any two models of $\Sigma(!)$ together with the axioms of PA, the function symbol ! has the same interpretation. Yet we say that Σ defines ! *implicitly* because we did not explicitly write down a formula without

[1] Basically, in this area one can distinguish two different approaches. First there is the work of (among others) W. Blok and D. Pigozzi, in this paper represented by [BP 89]. Second, there is the so-called Budapest School, lead by H. Andréka, I. Németi and I. Sain, cf. e.g. [NA 94], [AKNS 94], but also [HMT 85, §5.6]. In this paper we follow the second tradition, but we have tried to write this paper in such a way as to make it understandable for readers more familiar with the first approach. We will say a few more words about this in Remark A.2 in the Appendix. For a detailed comparison between the two approaches we refer to [FJ 94].

using the symbol "!", which is equivalent to the !-relation in all models of $\Sigma(!)+$ PA. If there does exist a formula for which this is the case, we say that the set of sentences $\Sigma(!)$ defines the function ! *explicitly* (cf. [ChK 85, page 90]), and we call this formula an *explicit definition* of !.

Obviously, if a set of sentences defines a relation explicitly, then it defines it implicitly. The other way around does not hold in general: We say that a logic has the *Beth definability property* when this other direction does hold. That is, when a logic $\mathcal{L}$ has the Beth definability property, then a set of sentences defines a relation explicitly in that logic iff it defines that relation implicitly. Examples of logics which have this property include ordinary first order logic, classical propositional logic, intuitionistic logic and the modal logics K, $S4$, $S5$.

Note that according to the above definition, a set of sentences $\Sigma(P)$ defines a relation P implicitly (let us say, in PA), iff each model of PA has *at most* one expansion to $L_{\mathrm{PA}} \cup P$ which is a model of $\Sigma(P)$. One could require there to be *exactly* one such expansion, which results in the notion of a *strong implicit* definition, and (in the terminology of [AKNS 94]) the *weak Beth definability property.*[2]

The algebraic property equivalent to the Beth definability property in the sense described above, is surjectiveness of epimorphisms (ES for short). Here we use the term "epimorphism" (or just "epi") in the categorical sense. That is, an algebraic homomorphism is an epimorphism iff it is right cancellative with respect to function composition. Obvious and typical examples of epis are surjections, but for certain classes of algebras there exist non-surjective epimorphisms. This is e.g. the case with the class of semigroups.

There is an extensive literature devoted to the study of the Beth definability property and surjectiveness of epimorphisms. Let us just mention some of these studies. A link between the Beth definability property and surjectiveness of epimorphisms was first laid by I. Németi (cf. [Nem 84]). Before then, some algebraic translations of the Beth definability property were known, but none of them had intrinsic algebraic interest (cf. [HMT 71, Part I, page 357]). Roughly speaking, Németi, in [Nem 84], associates with each class of algebras K a logic $\mathcal{L}(\mathsf{K})$. Next it is proved that under some weak conditions on the logic, $\mathcal{L}(\mathsf{K})$ has BP iff a certain subset of $\mathbf{SP}\mathsf{K}$ has ES. A similar result by I. Németi about surjectiveness of so-called *finite* epis can be found in [HMT 85, Theorem 5.6.10]. Since then, surjectiveness of epimorphisms has been one of the properties that has been thoroughly investigated for classes of algebras that play an important role in Algebraic Logic. As examples we mention the studies, in this connection, of classes of Polyadic, Cylindric and Relation Algebras (cf. e.g. [Sa 88] and [Sa 84]), of classes of Boolean Algebras with modal Operators (cf. e.g. [Maks 92]) or of discriminator varieties (cf. e.g. [Com 86]).

Main Results

In the present paper, we will go back to the work of I. Németi. Actually, we will

[2]At present it is not known to the author whether the weak Beth definability property is a strictly weaker property than the Beth definability property. Cf. footnote 10, page 23.

use the proof given in [Nem 84] as a starting point from which we will give a corresponding characterization *within the framework of [AKNS 94]*, the latter being the most recent and most comprehensive account of the Budapest approach. Furthermore, we will give an abstract algebraic characterization of the weak Beth definability property. This answers a question of I. Sain [Sa 88, Problem 14]. Both these characterizations were conjectured in [AKNS 94] in a slightly different form. The difference is that in our characterization certain assumptions are demanded of the logic which were not assumed in the conjectures as formulated in [AKNS 94]. More specific, our characterizations hold for *strongly nice* general logics, whereas it was conjectured that they would hold for the wider class of *structural* general logics (cf. Definition 2.4). We will prove that our stronger conditions are necessary.

Now we would like to make these results precise. At this point we have to assure the reader that all the notions introduced below, will be explained in full detail in Section 2. For the time being, we only mention the following. By a *strongly nice* logic we mean the same as in [AKNS 94]. This is basically the same as the notion of an *algebraizable* logic in the sense of Blok-Pigozzi, cf. e.g. [BP 89]. Furthermore, by $\mathsf{Alg}_{\models}(\mathbf{L})$ we mean a specific class of algebras corresponding to the logic $\mathbf{L}$ in the way described above. Intuitively, one can think of $\mathsf{Alg}_{\models}(\mathbf{L})$, as the class of Lindenbaum-Tarski algebras of $\mathbf{L}$. Finally, the *patchwork property of models* can be thought of as an amalgamation property on the meaning algebras (see Definition 2.6 for the latter, Definition 5.14 for the former), where instead of embeddings there is the real set-theoretic inclusion. Examples of strongly nice general logics for which we have checked that they have the patchwork property of models include classical propositional logic, modal logics (cf. Remark A.3 in the Appendix), difference logic, κ-times (also called stratified) logic, several versions of arrow logics and first order logic restricted to n variables.

Theorem 1 (Algebraic Characterization of the (strong) Beth Definability Property) *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$$\mathbf{L} \textit{ has the (strong) Beth definability property}$$

$$\iff$$

$$\textit{all the epimorphisms of } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ are surjective.}$$

Theorem 2 (Algebraic Characterization of the weak Beth Definability Property) *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$$\mathbf{L} \textit{ has the weak Beth definability property}$$

$$\iff$$

$$\textit{all the } \mathsf{Alg}_m(\mathbf{L})\textit{-extensible epimorphisms of } \mathsf{Alg}_{\models}(\mathbf{L}) \textit{ are surjective.}$$

Theorem 3 *There exists a non-equivalential structural general logic* $\mathbf{L}$ *which has the patchwork property of models such that* $\mathbf{L}$ *has the (strong) Beth definability property, but* not *all the epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

Theorem 4 *There exists an equivalential general logic* $\mathbf{L}$ *which has the patchwork property of models such that* $\mathbf{L}$ *does* not *have the (strong) Beth definability property, but all the epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

Section 3 is devoted to the proof of Theorem 1 and Section 4 to that of Theorem 2. In Section 5 it will be shown that the extra conditions on $\mathbf{L}$ under which Theorem 1 and Theorem 2 hold are necessary. Extra that is, as compared to those conjectured in [AKNS 94]. In order to keep the present paper self-contained, we will start in Section 2 by recalling from e.g. [AKNS 94] all the necessary definitions and preliminary results.

Prerequisites: We will assume familiarity with the basic notions of Universal Algebra (direct product, factor algebra, absolutely free algebra, lattice, etc.). These concepts are excellently explained in e.g. [Sa 94] or [BS 81]. Furthermore, some knowledge of set theory (proper class, cardinality) is presupposed. A summary of the set theory used in this paper (and much more) can e.g. be found in [ChK 85, Appendix A]. Finally, a vague familiarity with classical logic is assumed.

2 Preliminaries

2.1 Definitions

Strongly nice logic, general logic, algebraic counterpart of a logic

NOTATION 2.1 Throughout this paper we will use the notation of [AKNS 94]. The one exception that affirms this rule is that for any homomorphism f the f-image of an algebra $\mathcal{A}$ will be denoted by $f[\mathcal{A}]$ instead of by $f(\mathcal{A})$ as in [AKNS 94]. Furthermore, for clearness' sake, let us mention the following. For any two similar algebras $\mathcal{A}, \mathcal{B}$ the class of all homomorphisms from $\mathcal{A}$ into $\mathcal{B}$ will be denoted by $Hom(\mathcal{A}, \mathcal{B})$. For any algebra $\mathcal{A}$ and any congruence relation θ on $\mathcal{A}$, the factor algebra of $\mathcal{A}$ factorized by θ will be denoted by $\mathcal{A}/\theta$. An element of the universe of $\mathcal{A}/\theta$ will be denoted by a/θ. For any function f and subset X of the domain of f, $f\lceil X$ denotes the restriction of f to X, that is, $f\lceil X \stackrel{\text{def}}{=} \{\langle x, f(x)\rangle : x \in X\}$. We would also like to point out that we will apply set-theoretical notation to classes. For instance, $\bigcup C$ will denote the union of all members of C, also when C is a *proper* class. $\triangleleft$

The first step in the algebraic study of logics is to associate with each logic a class of algebras. So, the first thing to do is to define what we understand by a logic in general. Like all the other definitions given in this subsection, the following definition can also be found in any of the papers of the Budapest school, as e.g. [AKNS 94], [NA 94] or [ANS 84].

DEFINITION 2.2 (logic) By a *logic* $\mathcal{L}$ we mean an ordered quadruple[3]

$$\mathcal{L} \stackrel{\text{def}}{=} \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle,$$

such that (1-5) below hold.

1. $F_{\mathcal{L}}$ (called the set of *formulas*) is a subset of finite sequences (called *words*) over some set X (called the *alphabet* of $\mathcal{L}$).
2. $M_{\mathcal{L}}$ is a class (called the class of *models*).
3. $mng_{\mathcal{L}}$ is a function with domain $M_{\mathcal{L}} \times F_{\mathcal{L}}$ (called the *meaning function*).
4. $\models_{\mathcal{L}}$ (called the *validity relation*) is a relation between $M_{\mathcal{L}}$ and $F_{\mathcal{L}}$ that is, $\models_{\mathcal{L}} \subseteq M_{\mathcal{L}} \times F_{\mathcal{L}}$.
5. For all $\varphi, \psi \in F_{\mathcal{L}}$ and $\mathcal{M} \in M_{\mathcal{L}}$ we have

$$[mng_{\mathcal{L}}(\mathcal{M}, \varphi) = mng_{\mathcal{L}}(\mathcal{M}, \psi) \text{ and } \mathcal{M} \models_{\mathcal{L}} \varphi] \Longrightarrow \mathcal{M} \models_{\mathcal{L}} \psi. \qquad \lhd$$

In general the class $M_{\mathcal{L}}$ will be a *proper* class. At several places in this paper we will have to reckon with this fact.

DEFINITION 2.3 (class of models of Σ) Let $\mathcal{L} = \langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}} \rangle$ be a logic. For every $\Sigma \subseteq F_{\mathcal{L}}$,

$$Mod_{\mathcal{L}}(\Sigma) \stackrel{\text{def}}{=} \{\mathcal{M} \in M_{\mathcal{L}} \ : \ (\forall \sigma \in \Sigma)\ \mathcal{M} \models_{\mathcal{L}} \sigma\}.$$

$Mod_{\mathcal{L}}(\Sigma)$ is called the *class of models of* Σ. $\lhd$

Henceforth we will often omit the subscript $\mathcal{L}$ from $F_{\mathcal{L}}$, etc. when it causes no confusion.

Definition 2.2 is an attempt to capture the notion of a logic in as general a way as possible. However, for the purposes of this paper a more restricted notion of a logic is needed.

DEFINITION 2.4 (strongly nice logic, equivalential logic, structural logic) Let $\mathcal{L} = \langle F, M, mng, \models \rangle$ be a logic in the sense of Definition 2.2. We say that $\mathcal{L}$ is a *strongly nice logic* if conditions (1-5) below hold for $\mathcal{L}$.

[3] In Remark A.1 in the Appendix it will be explained why there is no mentioning of $\vdash_{\mathcal{L}}$ in this definition.

1. A set $Cn(\mathcal{L})$, called the set of *logical connectives* of $\mathcal{L}$, is fixed, together with a so-called *rank function* rank: $Cn(\mathcal{L}) \longrightarrow \omega$. For every $c \in Cn(\mathcal{L})$, rank(c) is called the *rank* of c. The set of all logical connectives of rank k is denoted by $Cn_k(\mathcal{L})$.

   Also a set P, called the set of *atomic formulas*, is fixed such that F is the smallest set satisfying (a) and (b) below,

   (a) $P \subseteq F$,

   (b) if $c \in Cn_k(\mathcal{L})$ and $\varphi_1, \ldots, \varphi_k \in F$ then $c(\varphi_1, \ldots, \varphi_k) \in F$.

   The word algebra (or absolutely free algebra) generated by P using the logical connectives from $Cn(\mathcal{L})$ as algebraic operations is denoted by $\mathcal{F}$, that is, $\mathcal{F} = \langle F, c\rangle_{c \in Cn(\mathcal{L})}$. $\mathcal{F}$ is called the *formula algebra* of $\mathcal{L}$.

2. The function $mng_{\mathcal{M}} \stackrel{\text{def}}{=} \langle mng(\varphi, \mathcal{M}) : \varphi \in F\rangle$ is a homomorphism from $\mathcal{F}$ into some algebra, for every $\mathcal{M} \in M$.

3. There are *"derived" binary connectives*[4] $\Delta_1, \ldots, \Delta_n$ of $\mathcal{L}$ (for some $n \in \omega,\ n > 0$), such that

$$(\forall \mathcal{M} \in M)(\forall \varphi, \psi \in F)[mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}}(\psi) \Leftrightarrow (\forall i \leq n)\ \mathcal{M} \models \varphi \Delta_i \psi].$$

4. There are *derived unary connectives* $\varepsilon_1, \ldots, \varepsilon_m$ and $\delta_1, \ldots, \delta_m$ of $\mathcal{L}$ (for some $m \in \omega,\ m > 0$), such that

$$(\forall \mathcal{M} \in M)(\forall \varphi \in F)[\mathcal{M} \models \varphi \Leftrightarrow (\forall j \leq m)(\forall i \leq n)\ \mathcal{M} \models \varepsilon_j(\varphi)\Delta_i \delta_j(\varphi)].$$

5. $(\forall s : P \longrightarrow F)(\forall \mathcal{M} \in M)(\exists \mathcal{N} \in M)(\forall \varphi(p_{i_0}, \ldots, p_{i_k}) \in F)$

$$mng_{\mathcal{N}}(\varphi) = mng_{\mathcal{M}}(\varphi(p_{i_0}/s(p_{i_0}), \ldots, p_{i_k}/s(p_{i_k}))),$$

   where $\varphi(p_{i_0}/s(p_{i_0}), \ldots, p_{i_k}/s(p_{i_k}))$ denotes the result of simultaneously substituting $s(p_{i_j})$ for every occurrence of p_{i_j} $(i \leq k)$ in φ.

Logics satisfying conditions 2.4.1, 2.4.2 and 2.4.5 are called *structural logics.* Structural logics satisfying condition 2.4.3 are called *equivalential logics.* $\lhd$

We note that most of the well investigated logics are strongly nice. In [AKNS 94, Exercise 3.1.4] it is shown that e.g. the following logics fall into this category: classical propositional (or sentential) logic, modal logic,[5] difference logic, κ-times (also called stratified) logic and several versions of arrow logics. The notorious absentee in this list is ordinary first order logic, for ordinary first order logic is *not* structural. This agrees with the fact that the algebraization of quantifier logics is really

[4] By "derived" we mean that Δ_i is not necessarily a member of $Cn(\mathcal{L})$. It might be "built up" from elements of $Cn(\mathcal{L})$.

[5] Cf. Remark A.3 in the Appendix.

a problem of a different character than that of algebraizing propositional logics.[6] However, the approach followed in this paper does go through for some "approximations" of ordinary first order logic: first order logic restricted to n variables *is* strongly nice,and rank free first order logic is nice (cf. [AKNS 94, Definitions 2.2.21 and 2.2.23]).

REMARK 2.5 1. An equivalent form of condition 2.4.5 is

$$(\forall \mathcal{M} \in M)(\forall h \in Hom(\mathcal{F}, mng_{\mathcal{M}}[\mathcal{F}]))(\exists \mathcal{N} \in M)\ mng_{\mathcal{N}} = h,$$

where for any homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$, $h[\mathcal{A}]$ denotes the homomorphic image of $\mathcal{A}$ along h. So, condition 2.4.5 comes down to saying that every homomorphism from a formula algebra into a meaning algebra[7] is actually the meaning function of some model.

2. As some illustrations of items 2.4.3 and 2.4.4, we mention that in case of classical propositional logic we can take $n = m = 1$, $\varepsilon_0(\varphi) = True$, $\delta_0(\varphi) = \varphi$ and $\Delta_0 = \leftrightarrow$. In $S5$ we can do the same, but we can also take $\Box(\varphi \leftrightarrow \psi)$ as $\varphi \Delta_0 \psi$. *Implicational logic* (where $\rightarrow$ is the only connective), is an example of a logic where $n > 1$ is needed. There we can take $n = 2$ and $\Delta_0 = \rightarrow$, $\Delta_1 = \leftarrow$.

3. In items 2.4.3 and 2.4.4 it is required that there exist *finitely* many derived connectives ε_j, δ_j and Δ_i with some desired properties. In [FJ 94] it is explained that one could require there to be arbitrarily (not necessarily finitely) many of those, hereby allowing for a more general theory and a wider applicability. Nevertheless, in our definition we follow [AKNS 94].

   Moreover, from now on we will assume that a strongly nice logic has *one* binary connective Δ satisfying 2.4.3, and *two* unary connectives ε, δ satisfying 2.4.4. This saves us from dragging the conditions $i \leq n$, $j \leq m$ along with us all the time. It will be easy to see that the proofs of all our results carry through to the more general case. $\lhd$

Now we are able to formulate precisely how to associate with each logic a class of algebras. Actually, we will associate with each logic *two* classes of algebras. The first class, $\mathsf{Alg}_{\models}(\mathcal{L})$, is the *abstract* class of algebras which forms the algebraic counterpart of the logic $\mathcal{L}$ as described in the Introduction. As we will see (cf. Theorem 5.20 and page 24), some semantic properties do not show up at this level of abstraction. Therefore we will also define a second, *concrete* class of algebras corresponding to the models of the logic $\mathcal{L}$. As the following definition shows, we can do all this under relatively weak conditions on the logic.

DEFINITION 2.6 (algebraic counterpart of a logic) Let $\mathcal{L} = \langle F, M, mng, \models \rangle$ be a logic satisfying conditions 2.4.1 and 2.4.2.

[6] See also the Section "Abstract Algebraic Logic" in the Introduction. For more details we refer to [Nem 91].

[7] A meaning algebra is nothing but the homomorphic image of a formula algebra under a meaning function. See also Definition 2.6.

1. Let $K \subseteq M$. We define an equivalence relation $\sim_K$ on the formula algebra $\mathcal{F}$ (cf. Definition 2.4.1) as follows,

$$(\forall \varphi, \psi \in F)\,[\varphi \sim_K \psi \overset{\text{def}}{\iff} (\forall \mathcal{M} \in K)\, mng_{\mathcal{M}}(\varphi) = mng_{\mathcal{M}}(\psi)].$$

   Since F is a set (i.e. *not* a proper class), it follows that $\{\sim_K :\ K \subseteq M\}$ is a set. By condition 2.4.2, $\sim_K$ is a congruence relation. Now,

$$\mathsf{Alg}_{\models}(\mathcal{L}) \overset{\text{def}}{=} \mathbf{I}\{\mathcal{F}/\sim_K :\ K \subseteq M\},$$

   where $\mathbf{I}$ denotes the operation of taking isomorphic copies (cf. Definition 2.16).

2. And

$$\mathsf{Alg}_m(\mathcal{L}) \overset{\text{def}}{=} \{mng_{\mathcal{M}}[\mathcal{F}] :\ \mathcal{M} \in M\}. \qquad \lhd$$

$\mathsf{Alg}_{\models}(\mathcal{L})$ is called the *algebraic counterpart* of the logic $\mathcal{L}$, and $\mathsf{Alg}_m(\mathcal{L})$ is called the class of *meaning algebras* of the logic $\mathcal{L}$. For any logic satisfying conditions 2.4.1-3 it holds that

$$(\forall K \subseteq M)(\forall \varphi, \psi \in F)[\varphi \sim_K \psi \Leftrightarrow (\forall \mathcal{M} \in Mod(Th(K)))\ \mathcal{M} \models \varphi \Delta \psi],$$

where $Th(K) \overset{\text{def}}{=} \{\varphi \in F :\ (\forall \mathcal{M} \in K)\ \mathcal{M} \models \varphi\}$. Hence for any such logic $\mathcal{L}$ the class $\mathsf{Alg}_{\models}(\mathcal{L})$ only depends on the validity relation $\models$, hereby accounting for its name.

It is important to notice here that in this definition, the class $\mathsf{Alg}_m(\mathcal{L})$ is *not* closed under taking isomorphisms. The reason for this is, as mentioned before, that $\mathsf{Alg}_m(\mathcal{L})$ is the algebraic counterpart of the *models* of the logic $\mathcal{L}$, hence we need these algebras as concrete algebras and replacing them with isomorphic copies would lead to a loss of information concerning semantic, model-theoretic matters. This fact will turn out to be crucial in the formulation of the algebraic characterization of the weak Beth definability property (see Theorem 2).

For logics satisfying condition 2.4.1 it would be more accurate to denote the set of formulas of $\mathcal{L}$ by $F^P_{\mathcal{L}}$ to express the fact that P is the set of atomic formulas used to "build up" the set of formulas of $\mathcal{L}$. This would actually yield to a family

$$\langle \langle F^P, M^P, mng^P, \models^P \rangle\ : P \text{ is a set} \rangle$$

of logics, where for each set P, $\langle F^P, M^P, mng^P, \models^P \rangle$ is a logic in the sense of Definition 2.2.

DEFINITION 2.7 (general logic) A *general logic* is a class

$$\mathbf{L} \overset{\text{def}}{=} \langle \mathcal{L}^P : P \text{ is a set} \rangle,$$

where for each set P, $\mathcal{L}^P = \langle F^P, M^P, mng^P, \models^P \rangle$ is a logic in the sense of Definition 2.2, satisfying condition 2.4.1, and P is the set of atomic formulas of logic $\mathcal{L}^P$. $\mathbf{L}$ is called a *strongly nice (resp. equivalential, structural) general logic* iff conditions (1–4) below hold for $\mathbf{L}$.

1. For each set P, $\mathcal{L}^P$ is a strongly nice (resp. equivalential, structural) logic.

2. For any sets P and Q, $Cn(\mathcal{L}^P) = Cn(\mathcal{L}^Q) \stackrel{\text{def}}{=} Cn(\mathbf{L})$. For any equivalential (or strongly nice) logic $\mathcal{L}^P$, the "special" connectives Δ_i $(i \leq n)$ (and ε_j, δ_j $(j \leq m)$) are the same (cf. Definition 2.4).

3. For any sets P and Q, if there is a bijection $f : P \longrightarrow Q$ then logic $\mathcal{L}^Q$ is an "isomorphic copy" of logic $\mathcal{L}^P$, i.e. there exists a bijection $f^M : M^P \longrightarrow M^Q$ such that for all $\varphi \in F^P$ and all $\mathcal{M} \in M^P$,
$$mng^P_{\mathcal{M}}(\varphi) = mng^Q_{f^M(\mathcal{M})}(f^F(\varphi)),$$
and
$$\mathcal{M} \models^P \varphi \;\Leftrightarrow\; f^M(\mathcal{M}) \models^Q f^F(\varphi).$$
Here f^F is the isomorphism from $\mathcal{F}^P$ onto $\mathcal{F}^Q$ extending f; this isomorphism is nothing but the simultaneous substitution of the atomic formulas $f(p) \in Q$ for each atomic formula $p \in P$.

4. For all sets $P \subseteq Q$,
$$\{mng^P_{\mathcal{M}} : \mathcal{M} \in M^P\} = \{(mng^Q_{\mathcal{M}})\lceil F^P : \mathcal{M} \in M^Q\}. \qquad \lhd$$

Item 2.7.3 can be interpreted as saying that only the cardinality of the set of atomic formulas really matters, while item 2.7.4 says that $\mathcal{L}^P$ is the natural restriction of $\mathcal{L}^Q$, if P is a subset of Q.

Extending Definition 2.6, we can now associate two classes of algebras to a *general* logic $\mathbf{L}$.

DEFINITION 2.8 (algebraic counterpart of a general logic) Let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set} \rangle$ be a structural general logic. Then
$$\mathsf{Alg}_{\models}(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup\{\mathsf{Alg}_{\models}(\mathcal{L}^P) : P \text{ is a set}\},$$
$$\mathsf{Alg}_m(\mathbf{L}) \stackrel{\text{def}}{=} \bigcup\{\mathsf{Alg}_m(\mathcal{L}^P) : P \text{ is a set}\}. \qquad \lhd$$

(Strong) Beth Definability Property, Epimorphisms

Now that we have explained in hopefully sufficient detail how to make a correspondence between Logic and Algebra (according to the Budapest approach), we can give the precise definitions, within this framework, of the notions involved in our main results. The definitions in this subsection can also be found in e.g. [HMT 85], [AKNS 94] or [Nem 84].

DEFINITION 2.9 (implicit definition, explicit definition) Let $\mathbf{L} = \langle \mathcal{L}^P : P \text{ is a set} \rangle$ be a structural general logic. Let $P \subsetneq Q$ be sets with $F^P \neq \emptyset$, and let $R \stackrel{\text{def}}{=} Q \setminus P$. We say that a set $\Sigma \subsetneq F^Q$ of formulas *defines R implicitly in Q* iff

$$(\forall \mathcal{M}, \mathcal{N} \in Mod^Q(\Sigma))\ [mng^Q_{\mathcal{M}} \lceil F^P = mng^Q_{\mathcal{N}} \lceil F^P \ \Rightarrow\ mng^Q_{\mathcal{M}} = mng^Q_{\mathcal{N}}].$$

We say that Σ *defines R explicitly in Q* iff

$$(\forall r \in R)(\exists \varphi_r \in F^P)(\forall \mathcal{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathcal{M}}(r) = mng^Q_{\mathcal{M}}(\varphi_r). \qquad \lhd$$

It is obvious that if Σ defines R explicitly in Q, then Σ defines R implicitly in Q. The converse does *not* hold in general. Logics for which it *does* hold are said to have the *Beth definability property*, named after E.W. Beth who proved that this is the case for first order logic (cf. [Beth 53]).

DEFINITION 2.10 ((strong) Beth definability property) Let $\mathbf{L}$ be a structural general logic. We say that $\mathbf{L}$ has the *(strong) Beth definability property* iff for all P, Q, R and Σ as in Definition 2.9 above

$$(\Sigma \text{ defines } R \text{ implicitly in } Q) \Longrightarrow (\Sigma \text{ defines } R \text{ explicitly in } Q). \qquad \lhd$$

The Beth definability property is a purely logical notion. As we will prove in Section 3, the algebraic equivalent of this property is surjectiveness of epimorphisms.

DEFINITION 2.11 (morphism, epimorphism) Let K be a class of similar algebras. By a *morphism of* K we understand a triple $\langle \mathcal{A}, h, \mathcal{B}, \rangle$ where $\mathcal{A}, \mathcal{B} \in \mathsf{K}$ and $h : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.
A morphism $\langle \mathcal{A}, h, \mathcal{B} \rangle$ of K is called an *epimorphism of* K iff for every $\mathcal{C} \in \mathsf{K}$ and every pair $\langle \mathcal{B}, f, \mathcal{C} \rangle$, $\langle \mathcal{B}, k, \mathcal{C} \rangle$ of morphisms we have

$$f \circ h = k \circ h \Rightarrow f = k.$$

The class of all epimorphisms of K will be denoted by Epi(K). A morphism $\langle \mathcal{A}, h, \mathcal{B} \rangle$ is called *surjective* iff h is surjective. The class of all surjections of K will be denoted by Sur(K). $\lhd$

Typical examples of epimorphisms are surjections. But for certain choices of K, there are epimorphisms of K which are not surjective.

EXAMPLE 2.12 Let K be the class of distributive lattices. Recall from e.g. [BS 81] that a *distributive lattice* is a lattice satisfying either (and hence both),

$$\text{D}_1. \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$\text{D}_2. \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let L_1 be the 3-element distributive lattice, and L_2 the 4-element distributive lattice drawn in Figure 1.

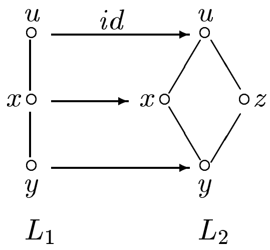

Figure 1: cf. Example 2.12.

Obviously, $\langle L_1, id, L_2\rangle$ is a non-surjective morphism of K. Now, let $L_3 \in \mathsf{K}$ be arbitrary, and let $\langle L_2, h, L_3\rangle$ and $\langle L_2, g, L_3\rangle$ be morphisms such that $h \circ id = g \circ id$. We will show that then $g(z) = h(z)$, hereby proving that $\langle L_1, id, L_2\rangle \in \mathrm{Epi}(\mathsf{K})$.

Well, $hz = hz \wedge hu = hz \wedge (gx \vee gz) = hz \wedge (hx \vee gz) \stackrel{\mathrm{D}_1}{=} (hz \wedge hx) \vee (hz \wedge gz) = hy \vee (hz \wedge gz) \stackrel{\mathrm{D}_2}{=} (hy \vee hz) \wedge (hy \vee gz) = hz \wedge gz$. Analogously, $gz = gz \wedge hz$. We can conclude that $gz = hz$. $\lhd$

The main results of this paper, stated above as Theorem 1 and Theorem 2, hold for general logics which, besides being strongly nice, have the *patchwork property of models*. This property is defined as follows.

DEFINITION 2.13 (patchwork property of models) Let $\mathbf{L}$ be a structural general logic. We say that $\mathbf{L}$ has the *patchwork property of models* iff

$(\forall \text{sets } P, Q)(\forall \mathcal{M} \in M^P)(\forall \mathcal{N} \in M^Q)$

$[F^{P\cap Q} \neq \emptyset \text{ and } mng^P_{\mathcal{M}} \lceil F^{P\cap Q} = mng^Q_{\mathcal{N}} \lceil F^{P\cap Q} \Rightarrow$

$\Rightarrow (\exists \mathcal{B} \in M^{P\cup Q})(mng^{P\cup Q}_{\mathcal{B}} \lceil F^P = mng^P_{\mathcal{M}} \text{ and } mng^{P\cup Q}_{\mathcal{B}} \lceil F^Q = mng^Q_{\mathcal{N}}].$ $\lhd$

The patchwork property is a very natural property of logics. For example all the logics mentioned before Remark 2.5 have this property. At the end of Section 5 we will come back to this. The following Theorem can be interpreted as saying that the patchwork property of models implies an amalgamation property (cf. Definition 5.14) on the meaning algebras, where instead of embeddings there is the real set-theoretic inclusion.

NOTATION 2.14 Henceforth we will use calligraphic letters like $\mathcal{A}$ and $\mathcal{B}$ to denote algebras, while their roman equivalents (i.e. A and B) will denote the universes of these algebras. $\mathcal{A}$ being a *subalgebra* of $\mathcal{B}$ will be denoted by $\mathcal{A} \subseteq \mathcal{B}$, $A \subseteq B$ indicates that A is a *subset* of B. We hope that this double use of $\subseteq$ does not cause any confusion. $\lhd$

THEOREM 2.15 *Let $\mathbf{L}$ be a structural general logic which has the patchwork property of models. Then*

$$\forall \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \in \mathsf{Alg}_m(\mathbf{L})[(\mathcal{A}_0 \subseteq \mathcal{A}_1 \textit{ and } \mathcal{A}_0 \subseteq \mathcal{A}_2) \Rightarrow$$

$$\Rightarrow \exists \mathcal{A}_3 \in \mathsf{Alg}_m(\mathbf{L})(\mathcal{A}_1 \subseteq \mathcal{A}_3 \text{ and } \mathcal{A}_2 \subseteq \mathcal{A}_3)].$$

Proof of Theorem 2.15:

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \in \mathsf{Alg}_m(\mathbf{L})$ such that $\mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A}_0 \subseteq \mathcal{A}_2$. For any set X, $|X|$ denotes the cardinality of X.

Let P and Q be sets such that $|P \cap Q| = |A_0|$, $|P| = |A_1|$ and $|Q| = |A_2|$. Now, let $f : P \longrightarrow A_1$ and $g : Q \longrightarrow A_2$ be bijections such that $f \lceil P \cap Q = g \lceil P \cap Q$ and $f[P \cap Q] = A_0$. Let $f' : \mathcal{F}^P \longrightarrow \mathcal{A}_1$ (resp. $g' : \mathcal{F}^Q \longrightarrow \mathcal{A}_2$) be the natural extension of f (resp. g) to a homomorphism from $\mathcal{F}^P$ (resp. $\mathcal{F}^Q$).

By Claim 2.19, there is an $\mathcal{M}' \in M^P$ and there is an $\mathcal{N}' \in M^Q$ such that $f' = mng^P_{\mathcal{M}'}$ and $g' = mng^Q_{\mathcal{N}'}$. Then $mng^P_{\mathcal{M}'} \lceil F^{P\cap Q} = f' \lceil F^{P\cap Q} = g' \lceil F^{P\cap Q} = mng^Q_{\mathcal{N}'} \lceil F^{P\cap Q}$. By the patchwork property of models there is a $\mathcal{B} \in M^{P\cup Q}$ such that $mng^{P\cup Q}_{\mathcal{B}} \lceil F^P = mng^P_{\mathcal{M}'}$ and $mng^{P\cup Q}_{\mathcal{B}} \lceil F^Q = mng^Q_{\mathcal{N}'}$. Hence $\mathcal{A}_1 = f'[\mathcal{F}^P] = mng^P_{\mathcal{M}'}[\mathcal{F}^P] \subseteq mng^{P\cup Q}_{\mathcal{B}}[\mathcal{F}^{P\cup Q}] \in \mathsf{Alg}_m(\mathbf{L})$. Analogously, $\mathcal{A}_2 \subseteq mng^{P\cup Q}_{\mathcal{B}}[\mathcal{F}^{P\cup Q}]$. **Q.E.D.**(Theorem 2.15)

2.2 Preliminary Results

The next two theorems can also be found in [AKNS 94, Theorem 3.1.11 and Theorem 3.2.17]. The reason for including their proofs here is the important role these theorems will play in the proofs of our main results. Let us first recall from e.g. [Sa 94] the following definitions.

DEFINITION 2.16 ($\mathbf{I}, \mathbf{S}, \mathbf{H}, \mathbf{P}, \mathbf{U}_p$) For any class of similar algebras K,

$\mathbf{I}\mathsf{K} \stackrel{\text{def}}{=} \{\mathcal{M} : \mathcal{M} \text{ is isomorphic to some } \mathcal{N} \in \mathsf{K}\}$.

$\mathbf{S}\mathsf{K} \stackrel{\text{def}}{=} \{\mathcal{M} : \mathcal{M} \text{ is a subalgebra of some } \mathcal{N} \in \mathsf{K}\}$.

$\mathbf{H}\mathsf{K} \stackrel{\text{def}}{=} \{\mathcal{M} : \mathcal{M} \text{ is a homomorphic image of some } \mathcal{N} \in \mathsf{K}\}$.

$\mathbf{P}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\{\prod_{i\in I} \mathcal{A}_i : I \text{ is a set, and } (\forall i \in I)\ \mathcal{A}_i \in \mathsf{K}\}$.

$\mathbf{U}_p\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\{\prod_{i\in I} \mathcal{A}_i/U : I \text{ is a set, } U \text{ is an ultra-filter over } I, \text{ and } (\forall i \in I)\ \mathcal{A}_i \in \mathsf{K}\}$.

$\lhd$

Note that by definition of $\mathbf{P}$ and $\mathbf{U}_p$, for any class of similar algebras K it holds that $\mathbf{PI}\,\mathsf{K} = \mathbf{P}\,\mathsf{K} = \mathbf{IP}\,\mathsf{K}$, and $\mathbf{U}_p\mathbf{I}\,\mathsf{K} = \mathbf{U}_p\,\mathsf{K} = \mathbf{IU}_p\,\mathsf{K}$. Now we can prove the following inclusions.

THEOREM 2.17 (H. Andréka, A. Kurucz, I. Németi, I. Sain) *For any logic $\mathcal{L}$ satisfying conditions 2.4.1 and 2.4.2,*

$$\mathsf{Alg}_m(\mathcal{L}) \subseteq \mathsf{Alg}_{\models}(\mathcal{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathcal{L}).$$

Proof of Theorem 2.17:

For the first inclusion it suffices to notice that for any $\mathcal{M} \in M$ the function $f : mng_{\mathcal{M}}[\mathcal{F}] \longrightarrow \mathcal{F}/{\sim_{\mathcal{M}}}$[8] defined by

$$(\forall \varphi \in F)\ f(mng_{\mathcal{M}}(\varphi)) \stackrel{\text{def}}{=} \varphi/{\sim_{\mathcal{M}}},$$

is a well-defined isomorphism by the definition of $\sim_{\mathcal{M}}$.

For the other inclusion, let $\mathcal{A} \in \mathsf{Alg}_{\models}(\mathcal{L})$. That is, $\mathcal{A} \cong \mathcal{F}/{\sim_K}$ for some class $K \subseteq M$. As pointed out in Definition 2.8, $\{\sim_K : K \subseteq M\}$ is a set. It is not difficult to see that we can find a subset $K' \subseteq K$ such that $\mathcal{F}/{\sim_K} = \mathcal{F}/{\sim_{K'}}$. Now the function $f : \mathcal{F}/{\sim_{K'}} \longrightarrow \prod_{\mathcal{M} \in K'} mng_{\mathcal{M}}[\mathcal{F}]$ defined by

$$(\forall \varphi \in F)\ f(\varphi/{\sim_{K'}}) \stackrel{\text{def}}{=} \langle mng_{\mathcal{M}}(\varphi) \ : \ \mathcal{M} \in K' \rangle$$

is a well-defined embedding. **Q.E.D.** (Theorem 2.17)

THEOREM 2.18 (H. Andréka, A. Kurucz, I. Németi, I. Sain) *For a structural general logic* $\mathbf{L}$*,*

$$\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{SP}\mathsf{Alg}_m(\mathbf{L}).$$

Proof of Theorem 2.18:
Note that $\mathsf{Alg}_{\models}(\mathbf{L}) \subseteq \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$, by Theorem 2.17. To prove the other inclusion, we need the following two claims.

Claim 2.19 *For any set P, algebra $\mathcal{A} \in \mathsf{Alg}_m(\mathbf{L})$ and homomorphism $h : \mathcal{F}^P \longrightarrow \mathcal{A}$,*

$$(\exists \mathcal{N} \in M^P)(\forall \varphi \in F^P)\ h(\varphi) = mng^P_{\mathcal{N}}(\varphi).$$

Claim 2.20 *Let $\mathcal{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ and let $h : \mathcal{F}^P \longrightarrow \mathcal{A}$ be a surjective homomorphism for some set P. Then*

$$(\exists K \subseteq M^P)\ \mathcal{A} \cong \mathcal{F}^P/{\sim_K}\ .$$

Before proving these claims, we prove Theorem 2.18 from Claim 2.20.

Let $\mathcal{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Let $h : \mathcal{F}^A \longrightarrow \mathcal{A}$ be the usual extension of the identity map of $\mathcal{A}$ to a homomorphism (cf. e.g. [BS 81, Definition 10.5 and Theorem 10.8]). By Claim 2.20, $(\exists K \subseteq M^A)\ \mathcal{A} \cong \mathcal{F}^A/{\sim_K}$, i.e. $\mathcal{A} \in \mathsf{Alg}_{\models}(\mathbf{L})$. **Q.E.D.** (Theorem 2.18)

Proof of Claim 2.19:
Let Q be a set such that $\mathcal{A} \in \mathsf{Alg}_m(\mathcal{L}^Q)$, and let $\mathcal{M} \in M^Q$ be such that $\mathcal{A} = mng_{\mathcal{M}}[\mathcal{F}]^Q$. Then

$$(\forall p \in P)(\exists \varphi_p \in F^Q)\ h(p) = mng^Q_{\mathcal{M}}(\varphi_p). \tag{1}$$

Because of condition 2.7.3 we may assume without loss of generality that either $P \subseteq Q$ or $Q \subseteq P$ holds.

[8] Strictly speaking, we should have written $\sim_{\{\mathcal{M}\}}$. Here we follow the generally accepted convention to omit these brackets around $\mathcal{M}$ when it causes no confusion.

1^{st} *case:* $Q \subseteq P$
By 2.7.4,

$$(\exists \mathcal{M}' \in M^P)(\forall p \in P)\ mng^P_{\mathcal{M}'}(\varphi_p) = mng^Q_{\mathcal{M}}(\varphi_p) = h(p). \tag{2}$$

Let $s : P \longrightarrow F^P$ be defined by $s(p) \stackrel{\text{def}}{=} \varphi_p$, for any $p \in P$. Then, by 2.4.5,

$$(\exists \mathcal{N} \in M^P)(\forall p \in P)\ mng^P_{\mathcal{N}}(p) = mng^P_{\mathcal{M}'}(\varphi_p). \tag{3}$$

Now (1),(2) and (3) together imply that $(\forall p \in P)\ h(p) = mng^P_{\mathcal{N}}(p)$.

2^{nd} *case:* $P \subseteq Q$
Let $s : Q \longrightarrow F^Q$ be defined by

$$s(p) = \begin{cases} \varphi_p, & \text{if } p \in P. \\ \text{any element of } F^Q, & \text{else.} \end{cases}$$

By 2.4.5,

$$(\exists \mathcal{M}' \in M^Q)(\forall p \in P)\ mng^Q_{\mathcal{M}'}(p) = mng^Q_{\mathcal{M}}(\varphi_p). \tag{4}$$

By 2.7.4,

$$(\exists \mathcal{N} \in M^P)(\forall p \in P)\ mng^P_{\mathcal{N}}(p) = mng^Q_{\mathcal{M}'}(p). \tag{5}$$

Then, by (1), (4) and (5), $(\forall p \in P)\ mng^P_{\mathcal{N}}(p) = h(p)$ holds again. Since h and $mng^P_{\mathcal{N}}$ are both homomorphisms, it follows that $mng^P_{\mathcal{N}} = h$. **Q.E.D.**(Claim 2.19)

Proof of Claim 2.20:
Let $\mathcal{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$. Then there exist an index set I and algebras $\mathcal{A}_i$, $i \in I$ such that $\mathcal{A} \subseteq \prod_{i\in I} \mathcal{A}_i$, where for every $i \in I$, $\mathcal{A}_i \in \mathsf{Alg}_m(\mathbf{L})$. Let π_i denote the projection function from $\mathcal{A}$ into $\mathcal{A}_i$, for each $i \in I$. By Claim 2.19, for every $i \in I$ there exists an $\mathcal{N}_i \in M^P$ such that for every $\varphi \in F^P$ $mng^P_{\mathcal{N}_i}(\varphi) = (\pi_i \circ h)(\varphi)$. Let $K \stackrel{\text{def}}{=} \{\mathcal{N}_i \ :\ i \in I\}$. It is easy to see that for any $\varphi, \psi \in F^P$,

$$h(\varphi) = h(\psi) \ \Leftrightarrow\ \varphi \sim_K \psi.$$

Hence $g : \mathcal{F}^P/{\sim_K} \longrightarrow \mathcal{A}$, defined by

$$(\forall (\varphi/{\sim_K}) \in \mathcal{F}^P/{\sim_K})\ g(\varphi/{\sim_K}) \stackrel{\text{def}}{=} h(\varphi),$$

is a well-defined isomorphism. **Q.E.D.**(Claim 2.20)

3 An Algebraic Characterization of the (strong) Beth Definability Property

Let us recall from the Introduction the first Main Theorem of this paper.

Theorem 1 (Algebraic Characterization of the (strong) Beth Definability Property) *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$\mathbf{L}$ *has the (strong) Beth definability property*

$$\Longleftrightarrow$$

all the epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

Now we are ready to turn to the proof of Theorem 1.

Proof of Theorem 1:

Let $\mathbf{L}$ be a strongly nice general logic which has the patchwork property of models. We have to show that $\mathbf{L}$ has the (strong) Beth definability property (BP for short) iff all the epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ are surjective (ES for short).

Some intuition: In reading this proof it might be helpful to keep in mind, as a guideline, that epimorphisms correspond to implicit definitions and surjections correspond to explicit definitions. The idea behind this is that when one looks at all these notions from a distance, one can observe that both epimorphisms and implicit definitions tell us that some homomorphisms, which agree on a certain subset of their domain, are equal. On the other hand, both surjections and explicit definitions tell us something about the existence of certain entities for every element of a certain set. This observation enables us to transform epimorphisms (resp. surjections) into implicit (resp. explicit) definitions, and vice versa. This will be made precise in the following proof.

ES $\Rightarrow$ BP :

Assume that all the epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ are surjective. Let $P \subsetneq Q$ be sets with $F^P \neq \emptyset$, and let $R \stackrel{\text{def}}{=} Q \setminus P$. Let $\Sigma \subseteq F^Q$ be such that

$$(\forall \mathcal{M}, \mathcal{N} \in Mod^Q(\Sigma))[mng^Q_{\mathcal{M}} \lceil F^P = mng^Q_{\mathcal{N}} \lceil F^P \ \Rightarrow \ mng^Q_{\mathcal{M}} = mng^Q_{\mathcal{N}}], i.e.$$

Σ defines R implicitly in Q. We have to show that Σ defines R explicitly in Q.

Claim 3.1($\Rightarrow$) tells us that we may assume without loss of generality that Σ looks like the set $\Sigma' \subseteq F^Q$ defined as follows:

$$\Sigma' \ \stackrel{\text{def}}{=} \ \{\varphi \Delta \psi \ : \ \varphi, \psi \in F^Q \text{ and } (\forall \mathcal{M} \in Mod^Q(\Sigma)) \ mng^Q_{\mathcal{M}}(\varphi) = mng^Q_{\mathcal{M}}(\psi)\}.$$

Claim 3.1($\Leftarrow$) will be needed later (cf. Claim 4.4).

Claim 3.1 $Mod^Q(\Sigma') = Mod^Q(\Sigma)$.

Proof of Claim 3.1:

" $\Rightarrow$ " :

Let $\mathcal{N} \in Mod^Q(\Sigma')$ and let $\varphi \in \Sigma$ be arbitrary. Then $(\forall \mathcal{M} \in Mod^Q(\Sigma))\ \mathcal{M} \models \varphi$, hence by Definition 2.4.3 and 2.4.4, $(\forall \mathcal{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathcal{M}}(\varepsilon(\varphi)) = mng^Q_{\mathcal{M}}(\delta(\varphi))$. By definition of Σ', then $\varepsilon(\varphi)\Delta\delta(\varphi) \in \Sigma'$. Since $\mathcal{N} \in Mod^Q(\Sigma')$, $\mathcal{N} \models \varepsilon(\varphi)\Delta\delta(\varphi)$. Again using Definition 2.4.4, we obtain that $\mathcal{N} \models \varphi$. Since φ was an arbitrary element of Σ, it follows that $\mathcal{N} \in Mod^Q(\Sigma)$.

" $\Leftarrow$ " :
Let $\mathcal{N} \in Mod^Q(\Sigma)$, and let $\varphi, \psi \in F^Q$ such that $(\forall \mathcal{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathcal{M}}(\varphi) = mng^Q_{\mathcal{M}}(\psi)$. Then in particular $mng^Q_{\mathcal{N}}(\varphi) = mng^Q_{\mathcal{N}}(\psi)$, hence by Definition 2.4.3, $\mathcal{N} \models \varphi\Delta\psi$. That is, $\mathcal{N} \in Mod^Q(\Sigma')$. **Q.E.D.**(Claim 3.1).

Since Σ defines R implicitly in Q, it follows from Claim 3.1($\Rightarrow$) that

$$\Sigma' \text{ defines } R \text{ implicitly in } Q. \tag{6}$$

Next let us define a homomorphism g from $\mathcal{F}^Q$ onto some algebra in $\mathsf{Alg}_{\models}(\mathbf{L})$, such that $(\forall \varphi, \psi \in F^Q)\ g(\varphi) = g(\psi)$ iff $(\forall \mathcal{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathcal{M}}(\varphi) = mng^Q_{\mathcal{M}}(\psi)$. In order to let this work we notice that although the class of models of Σ might be a *proper* class, the class $\Gamma \subseteq \wp(F^Q \times F^Q)$ defined by

$$\Gamma \stackrel{\text{def}}{=} \{\gamma \ : \ \gamma = \sim_{\mathcal{M}}, \text{ for some } \mathcal{M} \in Mod^Q(\Sigma)\},$$

is a *set* since F^Q is a set. Hence, the following is a correct definition:

$$g : \mathcal{F}^Q \longrightarrow \prod_{\gamma \in \Gamma} \mathcal{F}^Q/\gamma,$$

where

$$(\forall \varphi \in F^Q)\ g(\varphi) \stackrel{\text{def}}{=} \langle \varphi/\gamma : \ \gamma \in \Gamma \rangle.$$

The above homomorphism g is "induced" by Σ', in the sense that $(\forall \varphi, \psi \in F^Q)\ [(g(\varphi) = g(\psi)) \ \Leftrightarrow \ \varphi\Delta\psi \in \Sigma']$. That is,

$$\Sigma' = \{\varphi\Delta\psi : \ \varphi, \psi \in F^Q \text{ and } g(\varphi) = g(\psi)\}.$$

For convenience, let us fix some notation. Let $\mathcal{G} \stackrel{\text{def}}{=} g[\mathcal{F}^Q]$, $k \stackrel{\text{def}}{=} g\lceil F^P$ and $\mathcal{K} \stackrel{\text{def}}{=} k[\mathcal{F}^P]$. Let i be the identity from $\mathcal{F}^P$ into $\mathcal{F}^Q$, and let id_K be the identity from $\mathcal{K}$ into $\mathcal{G}$. Cf. Figure 2,[9] where at this point one can ignore all the functions which has not been defined yet. The function id_K will play a key role in the rest of this proof, as will be clear from the following outline.

Outline of the proof (ES $\Rightarrow$ BP): The rest of this proof is devoted to proving the following:

$$\begin{array}{ll}
\Sigma' \text{ defines } R \text{ implicitly in } Q & \stackrel{\text{Claim 3.2}}{\Longrightarrow} \\
\langle \mathcal{K}, id_K, \mathcal{G} \rangle \in \mathrm{Epi}(\mathsf{Alg}_{\models}(\mathbf{L})) & \stackrel{\text{Assumption}}{\Longrightarrow} \\
\langle \mathcal{K}, id_K, \mathcal{G} \rangle \in \mathrm{Sur}(\mathsf{Alg}_{\models}(\mathbf{L})) & \stackrel{\text{Claim 3.3}}{\Longrightarrow} \\
\Sigma \text{ defines } R \text{ explicitly in } Q. &
\end{array}$$

Now we can get to work.

[9]In the Figures, a "$*$" in the middle of a diagram indicates that this diagram *commutes*. The fact *that* these diagrams commute follows immediately from the definitions of the homomorphisms involved in the relevant diagrams.

$$\begin{tikzcd}
\mathcal{F}^P \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, phantom, "*"] & \mathcal{F}^Q \arrow[d, "g"] & \\
\mathcal{K} \arrow[r, "id_K"'] & \mathcal{G} \arrow[r, shift left, "h"] \arrow[r, shift right, "j"'] \arrow[dr, shift right, "\pi \circ j"'] \arrow[dr, shift left, "\pi \circ h"] & \mathcal{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L}) \arrow[d, "\pi"] \arrow[dl, phantom, "*"] \\
& & \mathcal{B} \in \mathsf{Alg}_m(\mathbf{L})
\end{tikzcd}$$

Figure 2: cf. Claim 3.2.

Claim 3.2 $\langle \mathcal{K}, id_K, \mathcal{G} \rangle \in Epi(\mathsf{Alg}_{\models}(\mathbf{L}))$.

Proof of Claim 3.2:

Let us say a few words about the intuition behind this proof. Roughly speaking, "Σ' defines R implicitly in Q" means that two meaning functions from $\mathcal{F}^Q$ which agree on $\mathcal{F}^P$ are equal. On the other hand, "id_K is an epimorphism" means that two homomorphisms from $g[\mathcal{F}^Q]$ which agree on $g[\mathcal{F}^P]$ are equal. What we will show is that any two homomorphisms h and j from $g[\mathcal{F}^Q]$ give rise to two meaning functions h' and j' from $\mathcal{F}^Q$ such that

$$h \lceil g[\mathcal{F}^P] = j \lceil g[\mathcal{F}^P] \Rightarrow h' \lceil \mathcal{F}^P = j' \lceil \mathcal{F}^P,$$

and

$$h' = j' \Rightarrow h = j.$$

Well, first note that $\mathcal{G} \in \mathbf{P}\mathsf{Alg}_{\models}(\mathbf{L}) = \mathsf{Alg}_{\models}(\mathbf{L})$, and $\mathcal{K} \in \mathbf{S}\mathsf{Alg}_{\models}(\mathbf{L}) = \mathsf{Alg}_{\models}(\mathbf{L})$, (cf. Theorem 2.18). Furthermore, let $\mathcal{A} \in \mathsf{Alg}_{\models}(\mathbf{L})$ and let $h : \mathcal{G} \longrightarrow \mathcal{A}$, $j : \mathcal{G} \longrightarrow \mathcal{A}$ be homomorphisms, such that $j \circ id_K = h \circ id_K$. We need to show that $h = j$.

Assume that $h \neq j$. Since $\mathcal{A} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$ there is some $\mathcal{B} \in \mathsf{Alg}_m(\mathbf{L})$ and a homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\pi \circ h \neq \pi \circ j$. This situation is sketched in Figure 2.

Since $h \circ id_K = j \circ id_K$, it holds that $\pi \circ h \circ id_K \circ k = \pi \circ j \circ id_K \circ k$. Hence $\pi \circ h \circ g \circ i = \pi \circ j \circ g \circ i$, i.e.

$$(\pi \circ h \circ g) \lceil F^P = (\pi \circ j \circ g) \lceil F^P. \tag{7}$$

On the other hand, by Claim 2.19 applied to $\pi \circ h \circ g$,

$$(\exists \mathcal{N} \in M^Q)\ \pi \circ h \circ g = mng^Q_{\mathcal{N}}.$$

Analogously,

$$(\exists \mathcal{N}' \in M^Q)\ \pi \circ j \circ g = mng^Q_{\mathcal{N}'}.$$

By (7) ,

$$mng^Q_{\mathcal{N}} \lceil F^P = mng^Q_{\mathcal{N}'} \lceil F^P.$$

We will show that $\mathcal{N}, \mathcal{N}' \in Mod^Q(\Sigma)$. Well, let $\varphi, \psi \in F^Q$ such that $g(\varphi) = g(\psi)$. Then $mng^Q_{\mathcal{N}}(\varphi) = \pi \circ h \circ g(\varphi) = \pi \circ h \circ g(\psi) = mng^Q_{\mathcal{N}}(\psi)$. By Definition 2.4.3 this means that $\mathcal{N} \models \varphi\Delta\psi$, i.e. $\mathcal{N} \in Mod^Q(\Sigma')$. In the same way it can be shown that $\mathcal{N}' \in Mod^Q(\Sigma')$. Since Σ' defines R implicitly in Q, it follows that $mng^Q_{\mathcal{N}} = mng^Q_{\mathcal{N}'}$. Hence $\pi \circ h \circ g = \pi \circ j \circ g$. Since g is surjective it follows that $\pi \circ h = \pi \circ j$. Contradiction. **Q.E.D.**(Claim 3.2)

From our assumption it now follows that $\langle \mathcal{K}, id_K, \mathcal{G}\rangle \in \mathrm{Sur}(\mathsf{Alg}_{\models}(\mathbf{L}))$.

Claim 3.3 *Σ defines R explicitly in Q.*

Proof of Claim 3.3:
Let $r \in R$. We need to find a $\varphi_r \in F^P$ such that

$$(\forall \mathcal{M} \in Mod^Q(\Sigma))(mng^Q_{\mathcal{M}}(\varphi_r) = mng^Q_{\mathcal{M}}(r)).$$

Well, let $\varphi_r \in F^P$ be such that $g(r) = id_K(k(\varphi_r))$. Such φ_r exists since id_K is surjective. Then $g(r) = id_K(k(\varphi_r)) = g(i(\varphi_r)) = g(\varphi_r)$. By definition of g, this implies that $(\forall \mathcal{M} \in Mod^Q(\Sigma))\ mng^Q_{\mathcal{M}}(r) = mng^Q_{\mathcal{M}}(\varphi_r)$. **Q.E.D.**(Claim 3.3)
Q.E.D.(ES $\Rightarrow$ BP)

BP $\Rightarrow$ ES :

Assume that $\mathbf{L}$ has the Beth definability property. Let $\bar{f} \overset{\text{def}}{=} \langle \mathcal{A}, f, \mathcal{B}\rangle \in \mathrm{Epi}(\mathsf{Alg}_{\models}(\mathbf{L}))$. We have to show that $\mathrm{Rg}(f) = B$.

First we will show that we may assume without loss of generality that there exists an algebra $\mathcal{A}' \subseteq \mathcal{B}$ such that $\bar{f}$ looks like $\langle \mathcal{A}', id_{A'}, \mathcal{B}\rangle$. Well, let $\mathcal{Q} \overset{\text{def}}{=} f[\mathcal{A}]$, and let $id_Q : \mathcal{Q} \longrightarrow \mathcal{B}$ be the identity.

Claim 3.4 $\langle \mathcal{Q}, id_Q, \mathcal{B}\rangle \in$ *Epi*$(\mathsf{Alg}_{\models}(\mathbf{L}))$.

Proof of Claim 3.4:
Note that $Q \overset{\text{def}}{=} f[\mathcal{A}] \subseteq \mathcal{B} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L}) = \mathsf{Alg}_{\models}(\mathbf{L})$. Since $\mathrm{Rg}(f) = \mathrm{Rg}(id_Q)$, and since $\bar{f}$ is an epimorphism, Claim 3.4 follows. $\square$

From now on we will use the universe Q (resp. B) of the *algebra* $\mathcal{Q}$ (resp. $\mathcal{B}$) as a set of atomic formulas, thus obtaining the *logic* $\mathcal{L}^Q$ (resp. $\mathcal{L}^B$).
Let $\widetilde{id}_B$ be the usual extension of id_B to $\mathcal{F}^B$. Define $\Sigma \subseteq F^B$ as follows:

$$\Sigma \overset{\text{def}}{=} \{\varphi\Delta\psi :\ \varphi, \psi \in F^B \text{ and } \widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)\}.$$

Let $R \overset{\text{def}}{=} B \setminus Q$, let $k \overset{\text{def}}{=} \widetilde{id}_B \lceil F^Q$, and let $i : \mathcal{F}^Q \longrightarrow \mathcal{F}^B$ be the identity. See also Figure 3.

$$
\begin{tikzcd}
 & \mathcal{F}^Q \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, phantom, "*"] & \mathcal{F}^B \arrow[d, "\widetilde{id}_B"] \\
\mathcal{A} \arrow[r, "f"] & \mathcal{Q} \arrow[r, "id_Q"'] & \mathcal{B}
\end{tikzcd}
$$

Figure 3: cf. **BP** $\Rightarrow$ **ES**.

Outline of the proof (BP $\Rightarrow$ ES):
We will show that

$$
\begin{aligned}
&\langle \mathcal{Q}, id_Q, \mathcal{B}\rangle \in \mathrm{Epi}(\mathsf{Alg}_{\models}(\mathbf{L})) && \overset{\text{Claim 3.5}}{\Longrightarrow} \\
&\Sigma \text{ defines } R \text{ implicitly in } B && \overset{\text{Assumption}}{\Longrightarrow} \\
&\Sigma \text{ defines } R \text{ explicitly in } B && \overset{\text{Claim 3.6}}{\Longrightarrow} \\
&\langle \mathcal{Q}, id_Q, \mathcal{B}\rangle \in \mathrm{Sur}(\mathsf{Alg}_{\models}(\mathbf{L})).
\end{aligned}
$$

Claim 3.5 *Σ defines R implicitly in B.*

Proof of Claim 3.5:

In this Claim we will show that the set Σ is defined in such a way that for every $\mathcal{M}, \mathcal{N} \in Mod^B(\Sigma)$ the meaning functions $mng^B_{\mathcal{M}}$ and $mng^B_{\mathcal{N}}$ give rise to homomorphisms g_m and g_n from $\mathcal{B}$ into some algebra of $\mathsf{Alg}_{\models}(\mathbf{L})$, such that

$$(mng^B_{\mathcal{M}}\lceil F^Q = mng^B_{\mathcal{N}}\lceil F^Q) \;\Rightarrow\; (g_m \circ id_Q = g_n \circ id_Q),$$

and

$$(g_m = g_n) \;\Rightarrow\; (mng^B_{\mathcal{M}} = mng^B_{\mathcal{N}}).$$

Well, let $\mathcal{M}, \mathcal{N} \in Mod^B(\Sigma)$ be such that

$$mng^B_{\mathcal{N}}\lceil F^Q = mng^B_{\mathcal{M}}\lceil F^Q. \tag{8}$$

We need to show that $mng^B_{\mathcal{N}} = mng^B_{\mathcal{M}}$.

Define $g_m \in \mathrm{Hom}(\mathcal{B}, mng^B_{\mathcal{M}}[\mathcal{F}^B])$ such that $g_m \circ \widetilde{id}_B = mng^B_{\mathcal{M}}$ as follows: $(\forall b \in B)$ $g_m(b) = mng^B_{\mathcal{M}}(\varphi_b)$, where $\varphi_b \in F^B$ is such that $\widetilde{id}_B(\varphi_b) = b$. Since $\widetilde{id}_B$ is surjective, we can always find such φ_b. Furthermore, for every $\varphi_b, \psi_b \in F^B$, if $\widetilde{id}_B(\varphi_b) = \widetilde{id}_B(\psi_b)$, then $\varphi_b \Delta \psi_b \in \Sigma$. Hence $\mathcal{M} \models \varphi_b \Delta \psi_b$, i.e. by Definition 2.4.3, $mng^B_{\mathcal{M}}(\varphi_b) = mng^B_{\mathcal{M}}(\psi_b)$. Hence g_m is well-defined. Analogously we define $g_n \in \mathrm{Hom}(\mathcal{B}, mng^B_{\mathcal{N}}[\mathcal{F}^B])$.

From Theorem 2.15 and (8) it follows that there is a $\mathcal{C} \in Alg_{\models}(\mathbf{L})$ such that $mng^B_{\mathcal{N}}[\mathcal{F}^B] \subseteq \mathcal{C}$ and $mng^B_{\mathcal{M}}[\mathcal{F}^B] \subseteq \mathcal{C}$. See also Figure 4.

$$
\begin{tikzcd}
\mathcal{F}^Q \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, phantom, "*"] & \mathcal{F}^B \arrow[d, "\widetilde{id}_B"'] \arrow[dr, shift left, "mng^B_{\mathcal{N}}"] \arrow[dr, shift right, "mng^B_{\mathcal{M}}"'] \arrow[dr, phantom, "*"] & \\
\mathcal{Q} \arrow[r, "id_Q"'] & \mathcal{B} \arrow[r, shift left, "g_m"] \arrow[r, shift right, "g_n"'] & \mathcal{C} \in \mathsf{Alg}_{\models}(\mathbf{L})
\end{tikzcd}
$$

Figure 4: cf. Claim 3.5.

$$
\begin{tikzcd}
\mathcal{F}^Q \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, phantom, "*"] & \mathcal{F}^B \arrow[d, "\widetilde{id}_B"'] \arrow[dr, "mng^B_{\mathcal{N}}"] \arrow[dr, phantom, "*"] & \\
\mathcal{Q} \arrow[r, "id_Q"'] & \mathcal{B} \arrow[r, "\pi"'] & \mathcal{C} \in \mathsf{Alg}_m(\mathbf{L})
\end{tikzcd}
$$

Figure 5: cf. Claim 3.6.

By (8), $mng^B_{\mathcal{N}} \circ i = mng^B_{\mathcal{M}} \circ i$. Hence $g_n \circ \widetilde{id}_B \circ i = g_m \circ \widetilde{id}_B \circ i$, hence $g_n \circ id_Q \circ k = g_m \circ id_Q \circ k$. Since k is surjective it follows that $g_n \circ id_Q = g_m \circ id_Q$, from which it follows that $g_n = g_m$, since $\langle \mathcal{Q}, id_Q, \mathcal{B} \rangle \in \mathrm{Epi}(\mathsf{Alg}_{\models}(\mathbf{L}))$. Hence $mng^B_{\mathcal{N}} = mng^B_{\mathcal{M}}$.
Q.E.D.(Claim 3.5)

Since $\mathbf{L}$ has the Beth definability property, we can conclude that Σ defines R explicitly in B.

Claim 3.6 $\langle \mathcal{Q}, id_Q, \mathcal{B} \rangle \in Sur(\mathsf{Alg}_{\models}(\mathbf{L}))$.

Proof of Claim 3.6 :
Let $r \in B \setminus Q \stackrel{\text{def}}{=} R$. We need to show that $r \in \mathrm{Rg}(id_Q)$. Well, since Σ defines R explicitly in B,

$$(\exists \varphi_r \in F^Q)(\forall \mathcal{M} \in Mod^B(\Sigma))(mng^B_{\mathcal{M}}(r) = mng^B_{\mathcal{M}}(\varphi_r)). \tag{9}$$

We will show that $r = id_B(\varphi_r)$. This implies that id_Q is surjective.

Assume $r \neq \widetilde{id}_B(\varphi_r)$. Just as in the proof of Claim 3.2, since $\mathcal{B} \in \mathbf{SP}\mathsf{Alg}_m(\mathbf{L})$, there is a $\mathcal{C} \in \mathsf{Alg}_m(\mathbf{L})$ and a homomorphism $\pi : \mathcal{B} \longrightarrow \mathcal{C}$ such that $\pi(\widetilde{id}_B(r)) = \pi(r) \neq \pi(\widetilde{id}_B(\varphi_r))$. By Claim 2.19, $(\exists \mathcal{N} \in M^B)(\forall \varphi \in F^B)\ mng^B_{\mathcal{N}}(\varphi) = \pi \circ \widetilde{id}_B(\varphi)$. See also Figure 5. Hence $mng^B_{\mathcal{N}}(\varphi_r) \neq mng^B_{\mathcal{N}}(r)$. By (9) it follows that $\mathcal{N} \notin Mod^B(\Sigma)$.

On the other hand, let $\varphi, \psi \in F^B$ such that $\widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)$. Then $\pi(\widetilde{id}_B(\varphi)) = \pi(\widetilde{id}_B(\psi))$, hence $mng^B_{\mathcal{N}}(\varphi) = mng^B_{\mathcal{N}}(\psi)$. That is, by Definition 2.4.3, $\mathcal{N} \models \varphi \Delta \psi$. Hence $\mathcal{N} \in Mod^B(\Sigma)$. Contradiction.

We conclude that $r = \widetilde{id}_B(\varphi_r)$, hence $r = \widetilde{id}_B(i(\varphi_r)) = id_Q(k(\varphi_r)) \in \mathrm{Rg}(id_Q)$.
Q.E.D.(Claim 3.6)

Since $\mathrm{Rg}(f) = \mathrm{Rg}(id_Q)$, we can conclude that $\langle \mathcal{A}, f, \mathcal{B} \rangle \in \mathrm{Sur}(\mathsf{Alg}_{\models}(\mathbf{L}))$.
Q.E.D.(BP $\Rightarrow$ ES)

REMARK 3.7 The property which is actually used in proving Theorem 1 (cf. Claim 3.5) is slightly weaker than the patchwork property of models (PP for short).

Let us say that a structural general logic $\mathbf{L}$ has the *weak patchwork property of models* (WPP) iff

$$(\forall \text{ set } Q)(\forall \mathcal{N}, \mathcal{M} \in M^Q)$$
$$[(\exists \text{ set } P)(\emptyset \neq P \subseteq Q \text{ and } mng^Q_{\mathcal{N}} \lceil F^P = mng^Q_{\mathcal{M}} \lceil F^P) \Rightarrow$$
$$\Rightarrow \exists \mathcal{A} \in \mathsf{Alg}_{\models}(\mathbf{L}) \text{ such that } mng^Q_{\mathcal{N}}[\mathcal{F}^Q] \subseteq \mathcal{A} \text{ and } mng^Q_{\mathcal{M}}[\mathcal{F}^Q] \subseteq \mathcal{A}].$$

From Theorem 2.15 it immediately follows that if $\mathbf{L}$ is a structural general logic which has the patchwork property of models, then $\mathbf{L}$ has the weak patchwork property of models. Since in the proof of Theorem 1 WPP is used instead of PP, this theorem already holds for strongly nice general logics satisfying this weaker condition. Nevertheless, Theorem 1 is stated in the given form since PP is a condition purely on the logic, whereas the WPP uses both logical and algebraic notions. It would be interesting to find a purely logical characterization for the WPP. This remains an open problem though. $\lhd$

4 An Algebraic Characterization of the Weak Beth Definability Property

Let us recall from Definition 2.9 that a set of formulas Σ defines a set R implicitly in a set Q iff $(\forall \mathcal{M}, \mathcal{N} \in Mod^Q(\Sigma))\ [mng^Q_{\mathcal{M}} \lceil F^P = mng^Q_{\mathcal{N}} \lceil F^P \Rightarrow mng^Q_{\mathcal{M}} = mng^Q_{\mathcal{N}}]$. Hence the fact that Σ is an implicit definition of R in Q says something about the uniqueness of certain models in $Mod^Q(\Sigma)$. However, there is nothing in the definition of implicit definition that requires the existence of such models. For some logics the absence of such models is precisely the reason for the fact that those logics do not have the Beth definability property. An example of such a logic is $\mathbf{L_2}$, i.e. first order logic restricted to 2 variables. Intuitively however, it does not seem to be a *definability* property that these logics are lacking. This is the reason why Henkin added the requirement of the existence of certain models in $Mod^Q(\Sigma)$ as an extra condition to the definition of implicit definition, thus obtaining the notion of a *strong implicit definition.*

DEFINITION 4.1 (strong implicit definition) Let $\mathbf{L}$ be a structural general logic. Let P, Q, R and Σ be as in Definition 2.9 above. We say that Σ *defines* R *implicitly in* Q *in a strong sense* iff

$$\Sigma \text{ defines } R \text{ implicitly in } Q \text{ and}$$
$$(\forall \mathcal{M} \in Mod^P(\Sigma \cap F^P))(\exists \mathcal{N} \in Mod^Q(\Sigma))\ mng^Q_{\mathcal{N}} \lceil F^P = mng^P_{\mathcal{M}}. \tag{10}$$

$\lhd$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "h"] \arrow[d, "\forall"', "f"] \arrow[dr, phantom, "*"] & \mathcal{B} \arrow[d, "\exists"', "g"] & \\ \mathcal{C} \arrow[r, phantom, "\subseteq"] & \mathcal{D} & \mathsf{K}_0 \end{tikzcd}$$

Figure 6: The definition of K_0-extensible.

This notion gives us the following definability property.

DEFINITION 4.2 (weak Beth definability property) Let $\mathbf{L}$ be a structural general logic. We say that $\mathbf{L}$ has the *weak Beth definability property* iff for all P, Q, R and Σ as in Definition 2.9 above

(Σ defines R implicitly in Q in a strong sense) $\Longrightarrow$ (Σ defines R explicitly in Q). $\lhd$

Since the weak Beth definability property is a weaker property than the Beth definability property,[10] in finding an algebraic characterization for this first notion we will somehow have to strengthen the property of being an epimorphism of $\mathsf{Alg}_m(\mathbf{L})$. A possible way to do this is the following.

DEFINITION 4.3 (K_0-extensible) Let $\mathsf{K}_0 \subseteq \mathsf{K}$ be two classes of similar algebras. Let $\langle \mathcal{A}, h, \mathcal{B} \rangle$ be a morphism of K. $\langle \mathcal{A}, h, \mathcal{B} \rangle$ is said to be K_0-*extensible* iff for every algebra $\mathcal{C} \in \mathsf{K}_0$ and every homomorphism $f : \mathcal{A} \longrightarrow \mathcal{C}$ there exists some $\mathcal{D} \in \mathsf{K}_0$ and $g : \mathcal{B} \longrightarrow \mathcal{D}$ such that $\mathcal{C} \subseteq \mathcal{D}$ and $g \circ h = f$. $\lhd$

Figure 6 illustrates this notion. It is important to emphasize that in the above definition $\mathcal{C}$ is not just required to be embeddable in $\mathcal{D}$, but $\mathcal{C}$ must be a real subalgebra of $\mathcal{D}$.

Now we are ready to prove the algebraic characterization of the weak Beth definability property as was formulated in the Introduction. Let us first bring this Theorem back to mind.

Theorem 2 (Algebraic Characterization of the weak Beth Definability Property) *Let* $\mathbf{L}$ *be a strongly nice general logic which has the patchwork property of models. Then*

$\mathbf{L}$ *has the weak Beth definability property*

[10] This naturally leads to the question whether the weak Beth definability property is a strictly weaker property than the Beth definability property. However, at present it is unknown to the author whether there exist logics which do have the weak Beth definability property, but lack the (strong) Beth definability property. A possible candidate is the above-mentioned $\mathbf{L}_2$, but whether $\mathbf{L}_2$ has the weak Beth definability property is, to my knowledge, still an open problem.

$$\Longleftrightarrow$$

all the $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ are surjective.

Note that in this characterization we need the class $\mathsf{Alg}_m(\mathbf{L})$. The reason for this is that, as was pointed out above, the definition of K_0-extensible strongly differentiates isomorphic algebras. Hence in this theorem we really need the concrete algebras.

Proof of Theorem 2:

In this proof we will closely follow the proof of the Theorem 1, making adaptions where necessary.

ES $\Rightarrow$ BP :

Assume that every $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$ is surjective. Let $P \subsetneq Q$ be sets with $F^P \neq \emptyset$ and let $R \stackrel{\text{def}}{=} Q \setminus P$. Let $\Sigma \subseteq F^Q$ be such that Σ defines R implicitly in Q in a strong sense. We have to show that Σ defines R explicitly in Q.

We will start by showing in Claim 4.4 that, just as in the proof of Theorem 1 (ES $\Rightarrow$ BP), we may assume that Σ looks like the set $\Sigma' \subseteq F^Q$ defined by,

$$\Sigma' \;\stackrel{\text{def}}{=}\; \{\varphi\Delta\psi \;:\; \varphi,\psi \in F^Q \text{ and } (\forall \mathcal{M} \in Mod^Q(\Sigma))\; mng^Q_{\mathcal{M}}(\varphi) = mng^Q_{\mathcal{M}}(\psi)\}.$$

Claim 4.4 *Σ' defines R implicitly in Q in a strong sense.*

Proof of Claim 4.4:
Above it has been shown that Σ' defines R implicitly in Q (cf. (6)). Hence it suffices to show that (10) holds.

Hereto, let $\mathcal{M} \in Mod^P(\Sigma' \cap F^P)$. We have to find an $\mathcal{N} \in Mod^Q(\Sigma')$ such that $mng^Q_{\mathcal{N}} \lceil F^P = mng^P_{\mathcal{M}}$.

Well, let $\varphi \in F^P \cap \Sigma$. Analogous to the proof of $Mod^P(\Sigma') \subseteq Mod^P(\Sigma)$ (cf. Claim 3.1($\Rightarrow$)), it can be shown that $Mod^P(\Sigma' \cap F^P) \subseteq Mod^P(\Sigma \cap F^P)$.

Since Σ defines R implicitly in Q in a strong sense, this implies that $(\exists \mathcal{N} \in Mod^Q(\Sigma))\; mng^Q_{\mathcal{N}} \lceil F^P = mng^P_{\mathcal{M}}$. By Claim 3.1($\Leftarrow$), $\mathcal{N} \in Mod^Q(\Sigma')$.
Q.E.D.(Claim 4.4)

Let $g, k, i, \mathcal{G}$ and $\mathcal{K}$ be defined as in the proof of Theorem 1 (see before Claim 3.2). Recall that

$$\Sigma' \;=\; \{\varphi\Delta\psi \;:\; \varphi,\psi \in F^Q \text{ and } g(\varphi) = g(\psi)\}.$$

Now the following extension of Claim 3.2 holds:

Claim 4.5 *$\langle \mathcal{K}, id_K, \mathcal{G}\rangle$ is an $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$.*

$$
\begin{tikzcd}
\mathcal{F}^P \arrow[r, "i"] \arrow[d, "k"] \arrow[dd, "mng^P_{\mathcal{N}}"'] \arrow[dr, phantom, "*"] & \mathcal{F}^Q \arrow[d, "g"'] \arrow[dd, "mng^Q_{\mathcal{N}'}"] \\
\mathcal{K} \arrow[r, "id_K"] \arrow[d, "f"] & \mathcal{G} \arrow[d, "h"'] \\
\mathcal{C} & mng^Q_{\mathcal{N}'}[\mathcal{F}^Q]
\end{tikzcd}
$$

Figure 7: cf. Claim 4.5.

Before proving Claim 4.5 let us notice that from this Claim it follows by our assumption that $\langle \mathcal{K}, id_K, \mathcal{G} \rangle$ is surjective. In the same way as in the proof of Theorem 1 it can be shown that this implies that Σ defines R explicitly in Q (cf. Claim 3.3), hereby proving Theorem 2 (ES $\Rightarrow$ BP).

Proof of Claim 4.5 :
By Claim 3.2 we only have to show that $\langle \mathcal{K}, id_K, \mathcal{G} \rangle$ is $\mathsf{Alg}_m(\mathbf{L})$-extensible. Well, let $\mathcal{C} \in \mathsf{Alg}_m(\mathbf{L})$, and let $f : \mathcal{K} \longrightarrow \mathcal{C}$ be a homomorphism. By Claim 2.19, $(\exists \mathcal{N} \in M^P) f \circ k = mng^P_{\mathcal{N}}$. If $\mathcal{N} \in Mod^P(\Sigma' \cap F^P)$, then Σ' being a strong implicit definition of R in Q implies that

$$\exists \mathcal{N}' \in Mod^Q(\Sigma')\ mng^Q_{\mathcal{N}'} \lceil F^P = mng^P_{\mathcal{N}}. \tag{11}$$

Well, let $\varphi, \psi \in F^P$ be such that $g(\varphi) = g(\psi)$. Since $k = g \lceil F^P$, then $k(\varphi) = k(\psi)$. Hence $mng^P_{\mathcal{N}}(\varphi) = f(k(\varphi)) = f(k(\psi)) = mng^P_{\mathcal{N}}(\psi)$. By Definition 2.4.3, this implies that $\mathcal{N} \models \varphi \Delta \psi$. Hence $\mathcal{N} \in Mod^P(\Sigma' \cap F^P)$.

Now define $h : \mathcal{G} \longrightarrow mng^Q_{\mathcal{N}'}[\mathcal{F}^Q]$ as follows:

$$(\forall x \in G)\ h(x) = mng^Q_{\mathcal{N}'}(\varphi_x),$$

where $\varphi_x \in F^Q$ is such that $g(\varphi_x) = x$. See Figure 7. Since g is surjective, there is always such φ_x. Furthermore, if $g(\varphi_x) = g(\varphi_y)$, for some $\varphi_x, \varphi_y \in F^Q$, then $\mathcal{N}' \models \varphi_x \Delta_i \varphi_y, \forall i \le n$, since $\mathcal{N}' \in Mod^Q(\Sigma')$. By Definition 2.4.3, then $mng^Q_{\mathcal{N}'}(\varphi_x) = mng^Q_{\mathcal{N}'}(\varphi_y)$. Hence h is well-defined.

Now we will prove that $h \circ id_K = f$, and that there exists a $\mathcal{D} \in \mathsf{Alg}_m(\mathbf{L})$ such that $\mathcal{C} \subseteq \mathcal{D}$ and $mng^Q_{\mathcal{N}'}[\mathcal{F}^Q] \subseteq \mathcal{D}$. This will complete the proof of Claim 4.5.

Well, let $y \in K$, i.e. $y = k(\varphi_y)$, for some $\varphi_y \in F^P$. Then $f(y) = f(k(\varphi_y)) = mng^P_{\mathcal{N}}(\varphi_y) \overset{\text{by11}}{=} mng^Q_{\mathcal{N}'}(\varphi_y) = h(g(\varphi_y)) = h(y)$. Hence $f = h \circ id_K$.

Moreover, since $mng^P_{\mathcal{N}}[\mathcal{F}^P] \subseteq \mathcal{C}$ and $mng^P_{\mathcal{N}}[\mathcal{F}^P] \subseteq mng^Q_{\mathcal{N}'}[\mathcal{F}^Q]$, it follows from Theorem 2.15 that there exists a $\mathcal{D} \in \mathsf{Alg}_m(\mathbf{L})$ such that $\mathcal{C} \subseteq \mathcal{D}$ and $mng^Q_{\mathcal{N}'}[\mathcal{F}^Q] \subseteq \mathcal{D}$.
Q.E.D.(Claim 4.5)

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "h"] \arrow[dr, "f \circ h"'] & \mathcal{Q} \arrow[r, "id_Q"] \arrow[d, "f"] & \mathcal{B} \arrow[d, "g"] \\
 & \mathcal{C} \arrow[r, phantom, "\subseteq"] & \mathcal{D}
\end{tikzcd}$$

Figure 8: cf. Claim 4.6.

Q.E.D.(ES $\Rightarrow$ BP)

BP $\Rightarrow$ ES :

Let $\bar{h} \stackrel{\text{def}}{=} \langle \mathcal{A}, h, \mathcal{B} \rangle$ be an $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$. We have to show that h is surjective.

As in the proof of Theorem 1 (BP $\Rightarrow$ ES), we will first show that we may assume $\bar{h}$ to look like $\langle \mathcal{B}', id_{B'}, \mathcal{B} \rangle$ for some algebra $\mathcal{B}' \subseteq \mathcal{B}$. Well, let $h[\mathcal{A}] \stackrel{\text{def}}{=} \mathcal{Q}$, and let $id_Q : \mathcal{Q} \longrightarrow \mathcal{B}$ be the identity.

Claim 4.6 *$\langle \mathcal{Q}, id_Q, \mathcal{B} \rangle$ is an $\mathsf{Alg}_m(\mathbf{L})$-extensible epimorphism of $\mathsf{Alg}_{\models}(\mathbf{L})$.*

Proof of Claim 4.6:
Note that by Claim 3.4 we only have to show that $\langle \mathcal{Q}, id_Q, \mathcal{B} \rangle$ is $\mathsf{Alg}_m(\mathbf{L})$-extensible. Well, let $\mathcal{C} \in \mathsf{Alg}_m(\mathbf{L})$ and let $f : \mathcal{Q} \longrightarrow \mathcal{C}$ be a homomorphism. Then $f \circ h : \mathcal{A} \longrightarrow \mathcal{C}$ is a homomorphism. See Figure 8.

Since $\langle \mathcal{A}, h, \mathcal{B} \rangle$ is $\mathsf{Alg}_m(\mathbf{L})$-extensible, there exists a $\mathcal{D} \in \mathsf{Alg}_m(\mathbf{L})$ and a $g : \mathcal{B} \longrightarrow \mathcal{D}$ such that $g \circ id_Q \circ h = g \circ h = f \circ h$ and $\mathcal{C} \subseteq \mathcal{D}$. Since $h : \mathcal{A} \longrightarrow \mathcal{Q}$ is surjective, it follows that $f = g \circ id_Q$. Hence $\langle \mathcal{Q}, id_Q, \mathcal{B} \rangle$ is $\mathsf{Alg}_m(\mathbf{L})$-extensible.
Q.E.D.(Claim 4.6)

As in the proof of Theorem 1, we will look at the universes Q and B as being sets of atomic formulas.

Let $\Sigma \subseteq F^B$ be defined as before, i.e.

$$\Sigma \stackrel{\text{def}}{=} \{\varphi \Delta \psi \ : \ \varphi, \psi \in F^B \text{ and } \widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)\}.$$

Let $B \setminus Q \stackrel{\text{def}}{=} R$.

Claim 4.7 *Σ defines R implicitly in B in a strong sense.*

Proof of Claim 4.7:
Let $\mathcal{M} \in Mod^Q(\Sigma \cap F^Q)$. By Claim 3.5 it suffices to find an $\mathcal{M}' \in Mod^B(\Sigma)$ such that $mng^B_{\mathcal{M}'} \lceil F^Q = mng^Q_{\mathcal{M}}$.

Well, by Claim 4.6 there exists some $\mathcal{N} \in \mathsf{Alg}_m(\mathbf{L})$ and there is a homomorphism $g : \mathcal{B} \longrightarrow \mathcal{N}$ such that $mng^Q_{\mathcal{M}}[\mathcal{F}^Q] \subseteq \mathcal{N}$ and $g \circ id_Q = mng^Q_{\mathcal{M}}$. By Claim 2.19, $(\exists \mathcal{M}' \in M^B)$ $g \circ \widetilde{id}_B = mng^B_{\mathcal{M}'}$. See Figure 9.

$$\begin{tikzcd}
 & \mathcal{F}^B \arrow[d, "\widetilde{id}_B"'] \arrow[dd, "mng^B_{\mathcal{M}'}"] \\
\mathcal{Q} \arrow[r, "id_Q"] \arrow[d, "mng^Q_{\mathcal{M}}"'] & \mathcal{B} \arrow[d, "g"'] \\
mng^Q_{\mathcal{M}}[\mathcal{F}^Q] \ \subseteq & \mathcal{N}
\end{tikzcd}$$

Figure 9: cf. Claim 4.7.

We will show that $\mathcal{M}'$ is the desired model. Well, let $q \in Q$. Then $mng^Q_{\mathcal{M}}(q) = g(id_Q(q)) = g(q) = g(\widetilde{id}_B(q)) = mng^B_{\mathcal{M}'}(q)$. Since $mng^B_{\mathcal{M}'}$ and $mng^Q_{\mathcal{M}}$ are homomorphisms, then $mng^Q_{\mathcal{M}} = mng^B_{\mathcal{M}'}\lceil F^Q$.

Furthermore, let $\varphi, \psi \in F^B$ such that $\widetilde{id}_B(\varphi) = \widetilde{id}_B(\psi)$. Then $mng^B_{\mathcal{M}'}(\varphi) = g(\widetilde{id}_B(\varphi)) = g(\widetilde{id}_B(\psi)) = mng^B_{\mathcal{M}'}(\psi)$. Hence $\mathcal{M}' \models \varphi\Delta\psi$. That is, $\mathcal{M}' \in Mod^B(\Sigma)$.
Q.E.D.(Claim 4.7)

By our assumption it follows that Σ defines R explicitly in B. Reasoning as in Claim 3.6, we can conclude that $\bar{h}$ is surjective.
Q.E.D.(BP $\Rightarrow$ ES)

5 Discussion

The main results of this paper are about strongly nice general logics which have the patchwork property of models. It is a natural question to ask whether these Theorems already hold for a larger class of logics. In this Section we will present two counterexamples showing that these Theorems do *not* hold when $\mathbf{L}$ is either merely a structural or an equivalential general logic which has the patchwork property of models. This justifies our use of the "special" connectives Δ, δ and ε in the proofs of Theorem 1 and Theorem 2. Another assumption that might need some justification is our assumption of the patchwork property of models. For the patchwork property of models, being some kind of amalgamation property, might at first sight seem to be a very strong assumption. In Theorem 5.20 we will show that this is not the case. Some of the results in this Section were obtained as joint results with J. Madarász.

Theorem 3 *There exists a non-equivalential structural general logic* $\mathbf{L}$ *which has the patchwork property of models such that* $\mathbf{L}$ *has the (strong) Beth definability property, but* not *all the epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

Proof of Theorem 3:

For every set P, let us define the logic $\mathcal{L}^P_{DL'}$ as follows:

1. The set of connectives of $\mathcal{L}^P_{DL'}$ is $\{\wedge, \vee, True\}$.

2. $M^P_{DL'} \stackrel{\text{def}}{=} \{\langle W, v\rangle \ : \ W$ is some fixed 2-element lattice with greatest element 1, and $v : \mathcal{F}^P \longrightarrow W$ is a homomorphism $\}$.

3. Let $\langle W, v\rangle \in M^P_{DL'}$, and $\varphi \in F^P_{DL'}$. Then $mng_{\langle W,v\rangle}(\varphi) \stackrel{\text{def}}{=} v(\varphi)$. Furthermore, $\langle W, v\rangle \models_{DL'} \varphi \stackrel{\text{def}}{\Leftrightarrow} v(\varphi) = 1$.

Note that $\mathsf{Alg}_m(\mathbf{L_{DL'}}) = \{W, \{1\}\}$. Item (2) in the above definition of $\mathcal{L}^P_{DL'}$ can now be read as saying that every homomorphism from a formula algebra to a meaning algebra of $\mathbf{L_{DL'}}$ is actually the meaning function of some model in $M^P_{DL'}$. That is, for every homomorphism $h : \mathcal{F}^P \longrightarrow W$ it holds that $h = mng_{\langle W,h\rangle}$, and by item(2) $\langle W, h\rangle \in M^P_{DL'}$. By Remark 2.5.1, this implies that $\mathbf{L_{DL'}}$ is structural. Furthermore, the following lemmas hold.

LEMMA 5.1 $\mathbf{L_{DL'}}$ *has the (strong) Beth definability property.*

LEMMA 5.2 $\mathsf{Alg}_{\models}(\mathbf{L_{DL'}})$ *is the variety of all distributive lattices with greatest element 1 (= DL^1).*

LEMMA 5.3 $\mathbf{L_{DL'}}$ *has the patchwork property of models.*

Since we know that there are non-surjective epimorphisms of DL^1 (cf. Example 2.12, take $u = 1$), Theorem 3 follows from the above lemmas. Now let us turn to the proofs of these lemmas.

Proof of Lemma 5.1:
Let I, J be some index sets, $R \stackrel{\text{def}}{=} \{r_i \ : \ i \in I\}$, $P \stackrel{\text{def}}{=} \{p_j \ : \ j \in J\}$, $R \cap P = \emptyset$ and $R \cup P \stackrel{\text{def}}{=} Q$. Let $\Sigma \subseteq F^Q_{DL'}$. We will show that either Σ defines R *ex*plicitly in Q, or Σ does *not* define R *im*plicitly in Q. In showing this, we will make use of the following claim. The proof of this claim can e.g. be found in [Mad 96, Claim 6.6.3], and will therefore be omitted.

Claim 5.4 *Every $\sigma \in \Sigma$ can be written as a finite disjunctions of conjunctions. Explicitly:*

$$\sigma = (q^1_1 \wedge \cdots \wedge q^1_{n_1}) \vee \cdots \vee (q^m_1 \wedge \cdots \wedge q^m_{n_m}),\ q^j_i \in Q.$$

By Claim 5.4 we can rewrite every $\sigma \in \Sigma$ as follows,

$$\sigma = (\underbrace{r^1_1 \wedge \cdots \wedge r^1_{n_1}}_{\in R} \wedge \underbrace{((p^1_1 \wedge \cdots \wedge p^1_{l_1}) \vee \cdots \vee (p^n_1 \wedge \cdots \wedge p^n_{l_n}))}_{\in P}) \vee \cdots$$

$$\cdots \vee (\underbrace{r^k_1 \wedge \cdots \wedge r^k_{n_k}}_{\in R} \wedge \underbrace{(\cdots)}_{\in P}),$$

such that the set of elements from R in every disjunct differs (so, in the original σ all the disjuncts with the same set of elements from R are "taken together"). Now distinguish the following two cases:

1. $(\forall r \in R)(\exists \sigma \in \Sigma)$ r occurs in every disjunct of σ.

2. $(\exists r \in R)(\forall \sigma \in \Sigma)$ there is a disjunct of σ in which r does *not* occur.

Claim 5.5 *If (1) is the case, then Σ defines R explicitly in Q.*
If (2) is the case, then Σ does not *define R implicitly in Q.*

Note that Lemma 5.1 follows from Claim 5.5.

Proof of Claim 5.5:
Case 1. Let $r \in R$. Define $\varphi_r \stackrel{\text{def}}{=} True$. Let $\mathcal{M} \in Mod^Q_{DL'}(\Sigma)$. Since we are in Case 1, $\exists \sigma \in \Sigma$ such that

$$\sigma = (r \wedge \underbrace{(\cdots)}_{\in R} \wedge \underbrace{((\cdots) \vee \cdots \vee (\cdots)))}_{\in P} \vee \cdots$$

$$\cdots \vee (r \wedge \underbrace{(\cdots)}_{\in R} \wedge \underbrace{(\cdots)}_{\in P}).$$

Since $\mathcal{M} \models_{DL'} \sigma$, it follows that $\mathcal{M} \models_{DL'} r$. Hence $mng^Q_{\mathcal{M}}(r) = 1 = mng^Q_{\mathcal{M}}(\varphi_r)$. Thus, Σ defines R explicitly in Q.
Case 2. Let $r \in R$ be such that $\forall \sigma \in \Sigma$ there is a disjunct of σ of the following kind:

$$\underbrace{(r_1 \wedge \cdots \wedge r_n}_{r \notin} \wedge \underbrace{(\cdots))}_{\in P}. \tag{12}$$

Define $\mathcal{M}, \mathcal{N} \in M^Q$ as follows: $\mathcal{M} \stackrel{\text{def}}{=} \langle W, v \rangle$, $\mathcal{N} \stackrel{\text{def}}{=} \langle W, v' \rangle$, where $v(q) = 1, \forall q \in Q$ and $v'(q) = 1$ if $q \neq r$, $v'(r) \neq 1$. Obviously, $mng^Q_{\mathcal{M}} \lceil F^P = mng^Q_{\mathcal{N}} \lceil F^P$, but $mng^Q_{\mathcal{M}}(r) \neq mng^Q_{\mathcal{N}}(r)$. Moreover, by (12) $\mathcal{M}, \mathcal{N} \in Mod^Q(\Sigma)$. Hence Σ does *not* define R implicitly in Q.
Q.E.D.(Claim 5.5)
Q.E.D.(Lemma 5.1)

Proof of Lemma 5.2:[11]

Let us recall the following well-known theorems of Birkhoff:

THEOREM 5.6 *If L is a subdirectly irreducible distributive lattice, then $|L| \leq 2$.*

THEOREM 5.7 *If K is a variety, then every member of K is isomorphic to a subdirect product of subdirectly irreducible members of* $\mathbf{K}$.

[11] In this proof we will use a little more Universal Algebra than we did before. We will assume the reader to be acquainted with notions like that of a variety or that of a subdirectly irreducible algebra. For definitions we refer to [BS 81, §8 and §9].

For the proofs of the above two theorems we refer to [BS 81, Theorem 9.6 and Exercise 8.2].

In order to prove Lemma 5.2, let us first notice that $\mathsf{Alg}_{\models}(\mathbf{L_{DL'}}) = \mathbf{SP}\,\mathsf{Alg}_m(\mathbf{L_{DL'}}) = \mathbf{SP}\,W$. Hence we have to show that $DL^1 = \mathbf{SP}\,W$. Since equations are preserved under taking subalgebras and direct products, clearly $DL^1 \supseteq \mathbf{SP}\,W$. The other inclusion is an immediate consequence of the two theorems of Birkhoff stated above.
Q.E.D.(Lemma 5.2)

Proof of Lemma 5.3:
Let P and Q be sets, and let $M = \langle W, h\rangle \in M^P$, $N = \langle W, k\rangle \in M^Q$ such that $F^{P\cap Q} \neq \emptyset$ and $mng^P_{\mathcal{M}}\lceil F^{P\cap Q} = mng^Q_{\mathcal{N}}\lceil F^{P\cap Q}$. Let $\mathcal{B} \stackrel{\text{def}}{=} \langle W, v\rangle \in M^{P\cup Q}$, where v is the unique homomorphism such that

$$v(p) = \begin{cases} h(p), & \text{if } p \in P. \\ k(p), & \text{if } p \in Q. \end{cases}$$

Note that if $r \in P \cap Q$, then $h(r) = mng^P_{\mathcal{M}}(r) = mng^Q_{\mathcal{N}}(r) = k(r)$, hence v is well-defined. Moreover, $(\forall p \in P)\ mng^{P\cup Q}_{\mathcal{B}}(p) = h(p) = mng^P_{\mathcal{M}}(p)$ and $(\forall q \in Q)\ mng^{P\cup Q}_{\mathcal{B}}(q) = k(q) = mng^Q_{\mathcal{N}}(q)$. That is, $mng^P_{\mathcal{M}}[\mathcal{F}^P] \subseteq mng^{P\cup Q}_{\mathcal{B}}[\mathcal{F}^{P\cup Q}]$ and $mng^Q_{\mathcal{N}}[\mathcal{F}^Q] \subseteq mng^{P\cup Q}_{\mathcal{B}}[\mathcal{F}^{P\cup Q}]$.
Q.E.D.(Lemma 5.3)
Q.E.D.(Theorem 3)

REMARK 5.8 It is not hard to see that the epimorphism of DL^1 defined in Example 2.12 is $\mathsf{Alg}_m(\mathbf{L_{DL'}})$-extensible. Hence the above counterexample does not only justify our use of the "special" connective Δ in the proof of Theorem 1, but also that in the proof of Theorem 2. $\lhd$

Theorem 3 shows that in proving our main results we need the logic to have a derived biconditional Δ satisfying the conditions of Definition 2.4.3. In the proofs of these Theorems (cf. Claim 3.1($\Rightarrow$) used in the proof of both Theorem 1 (ES $\Rightarrow$ BP) and Theorem 2 (BP $\Rightarrow$ ES)), we also assume the existence of connectives ε and δ as defined in Definition 2.4.4.[12] The question thus comes up whether we also need these connectives in order to prove these Theorems. The next counterexample has been developed in co-operation with J.Madarász and proves Theorem 4, hereby giving an affirmative answer to the above question.

Theorem 4 *There exists an equivalential general logic* $\mathbf{L}$ *which has the patchwork property of models such that* $\mathbf{L}$ *does* not *have the (strong) Beth definability property, but all the epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L})$ *are surjective.*

Proof of Theorem 4:

For every set P, let us define the logic $\mathcal{L}^P_{LI}$ as follows:

[12]Note that the proofs of Theorem 1 (BP $\Rightarrow$ ES) and Theorem 2 (BP $\Rightarrow$ ES) *do* go through for equivalential logics.

1. The set of connectives of $\mathcal{L}^P_{LI}$ is $\{\wedge, \vee, \leftrightarrow, True, False\}$.

2. Fix some object $\star$. Now, $M^P_{LI} \stackrel{\text{def}}{=} \{\langle M, \star, mng_{\mathcal{M}}\rangle :$ satisfying conditions (a) and (b) below$\}$.

   (a) $M = \langle uv(M), 1, 0, \wedge, \vee, \leftrightarrow\rangle$, where $\langle uv(M), 1, 0, \wedge, \vee, \rangle$ is a lattice with endpoints, $\star \in uv(M)$, $\star \neq 0$, and $\forall m, n \in uv(M)$,

   $$m \leftrightarrow n = \begin{cases} 1, & \text{if } m = n. \\ 0, & \text{otherwise.} \end{cases}$$

   (b) $mng_{\mathcal{M}} : \mathcal{F}^P \longrightarrow M$ is a homomorphism.

   Note that $\star$ is the same element for *every* model $\mathcal{M} \in M^P_{LI}$.

3. $(\forall \mathcal{M} \in M^P_{LI})(\forall \varphi \in F^P_{LI})\ \mathcal{M} \models^P \varphi \stackrel{\text{def}}{\Leftrightarrow} mng_{\mathcal{M}}(\varphi) = 1$ or $mng_{\mathcal{M}}(\varphi) = \star$.

Since for every set P, every M satisfying condition 2.a above and every homomorphism $h : \mathcal{F}^P \longrightarrow M$ it holds that $h = mng_{\mathcal{M}}$ where $\mathcal{M} \stackrel{\text{def}}{=} \langle M, \star, h\rangle \in M^P_{LI}$ (by condition 2 above), it follows that the general logic $\mathbf{L}_{LI}$ is *structural*. Moreover, the following lemmas hold.

LEMMA 5.9 $\mathbf{L}_{LI}$ *does* not *have the Beth definability property.*

LEMMA 5.10 *All the epimorphisms of* $\mathsf{Alg}_{\models}(\mathbf{L}_{LI})$ *are surjective.*

The above two lemmas together with the observation that in the same way as in Lemma 5.3 it can be shown that $\mathbf{L}_{LI}$ has patchwork property of models, constitute the proof of Theorem 4.

Proof of Lemma 5.9:
Let Q be a set, let $r \in Q$ and let $Q \setminus \{r\} \stackrel{\text{def}}{=} P \neq \emptyset$.
Let $\Sigma \stackrel{\text{def}}{=} \{r, (r \leftrightarrow True) \leftrightarrow False\}$.

Claim 5.11 Σ *defines* $\{r\}$ *implicitly in* Q.

Proof of Claim 5.11:
Let $\mathcal{M} \stackrel{\text{def}}{=} \langle M, \star, mng_{\mathcal{M}}\rangle \in Mod^Q(\Sigma)$. Then $mng_{\mathcal{M}}(r) = 1$ or $mng_{\mathcal{M}}(r) = \star$. If $mng_{\mathcal{M}}(r) = 1$, then $mng_{\mathcal{M}}((r \leftrightarrow True) \leftrightarrow False) = 0$. Since $|uv(M)| > 1$, $1 \neq 0$. Contradiction. Hence $mng_{\mathcal{M}}(r) = \star$. Since $\mathcal{M}$ was an arbitrary element of $Mod^Q(\Sigma)$, our claim follows. **Q.E.D.**(Claim 5.11)

Claim 5.12 Σ *does* not *define* $\{r\}$ *explicitly in* Q.

Proof of Claim 5.12:
Suppose there exists a $\varphi_r \in F^P$ such that $(\forall \mathcal{M} \in Mod^Q(\Sigma))\ mng_{\mathcal{M}}(r) = mng_{\mathcal{M}}(\varphi_r)$.
Define $\mathcal{M} \stackrel{\text{def}}{=} \langle M, \star, mng_{\mathcal{M}}\rangle \in M^Q$ as follows:

- $uv(M) \stackrel{\text{def}}{=} \{0, 1, \star\}$.
- $mng_{\mathcal{M}} : \mathcal{F}^Q \longrightarrow M$ is the unique homomorphism such that $mng_{\mathcal{M}}(r) = \star$ and $(\forall p \in P)\ mng_{\mathcal{M}}(p) \in \{0, 1\}$.

Then $\mathcal{M} \in Mod^Q(\Sigma)$, but $mng_{\mathcal{M}}(\varphi_r) \in \{1, 0\} \not\ni mng_{\mathcal{M}}(r)$. Contradiction.
Q.E.D.(Claim 5.12)
Q.E.D.(Lemma 5.9)

Proof of Lemma 5.10:

As in the proof of Lemma 5.2, we will suppose that the reader is familiar with some more of the basics of Universal Algebra than is needed for the rest of this paper. Moreover, we will make use of the following theorems:

THEOREM 5.13 *Let* K *be a class of similar algebras. Assume that* K *has a discriminator term. Then*

$$\mathbf{HSP}\,\mathsf{K} = \mathbf{SPU_p}\,\mathsf{K}.$$

For the definition of **H, S, P,** $\mathbf{U}_p$, see Definition 2.16. The proof of Theorem 5.13 can be found in [Sa 94, Theorem 2.4.3]. Let us recall from e.g. [Maks 92] that:

DEFINITION 5.14 A class of similar algebras K is said have the *amalgamation property* (AP for short) if for any algebras $\mathcal{A}, \mathcal{B}$ and $\mathcal{C} \in \mathsf{K}$, and for any embeddings $\alpha : \mathcal{A} \xrightarrow{1-1} \mathcal{B}$ and $\beta : \mathcal{A} \xrightarrow{1-1} \mathcal{C}$ there exists a $\mathcal{D} \in \mathsf{K}$ and embeddings $\gamma : \mathcal{B} \xrightarrow{1-1} \mathcal{D}$ and $\delta : \mathcal{C} \xrightarrow{1-1} \mathcal{D}$ such that $\gamma \circ \alpha = \delta \circ \beta$. If moreover $\gamma[\mathcal{B}] \cap \delta[\mathcal{C}] = \gamma[\alpha[\mathcal{A}]]$, then K is said to have the *strong amalgamation property* (SAP for short). $\lhd$

THEOREM 5.15 (S.D. Comer) *Let* V *be a discriminator variety. Then*

Sir(V) *has SAP* $\Rightarrow$ V *has ES.*

The proof of Theorem 5.15 can be found in [Com 86][13]. Finally, let us note the following.

LEMMA 5.16 *If* $\mathbf{L}$ *is a structural logic, then* $\mathsf{Alg}_m(\mathbf{L})$ *is closed under taking subalgebras.*

Proof of Lemma 5.16:
Let $\mathcal{A} \subseteq \mathcal{B} \in \mathsf{Alg}_m(\mathbf{L})$. We have to show that $\mathcal{A} \in \mathsf{Alg}_m(\mathbf{L})$. Well, let g be a surjection from B onto A, and let $\tilde{g}$ be the unique homomorphism from $\mathcal{F}^B$ extending g. Then $\tilde{g} \in Hom(\mathcal{F}^B, \mathcal{B})$. By Remark 2.5.1, $\tilde{g} = mng^B_{\mathcal{N}}$, for some $\mathcal{N} \in M^B$. Hence $\mathcal{A} = \tilde{g}[\mathcal{F}^B] = mng^B_{\mathcal{N}}[\mathcal{F}^B] \in \mathsf{Alg}_m(\mathbf{L})$. **Q.E.D.**(Lemma 5.16)

[13]J. Madarász proved a stronger result saying that a discriminator variety V has ES if Sir(V) has ES. This will be published in the forthcoming [Mad 9?].

Now we are ready to prove Lemma 5.10. Let us start by defining the following term $\tau(a,b,c,d)$ in the language of $\mathsf{Alg}_m(\mathbf{L}_{LI})$,

$$\tau(a,b,c,d) \stackrel{\text{def}}{=} ((a \leftrightarrow b) \wedge c) \vee ((a \odot b) \wedge d),$$

where $\forall \mathcal{A} \in \mathsf{Alg}_m(\mathbf{L}_{LI})$, $\forall a, b \in A$:

$$a \odot b \stackrel{\text{def}}{=} (a \leftrightarrow b) \leftrightarrow 0 = \begin{cases} 0, & \text{if } a = b. \\ 1, & \text{otherwise.} \end{cases}$$

It is not hard to see that τ is a *discriminator term* on $\mathsf{Alg}_m(\mathbf{L}_{LI})$. That is,

$$\mathbf{HSP}\,\mathsf{Alg}_m(\mathbf{L}_{LI}) \text{ is a discriminator variety.} \tag{13}$$

Moreover, by Theorem 5.13,

$$\mathbf{HSP}\,\mathsf{Alg}_m(\mathbf{L}_{LI}) = \mathbf{SPU}_p\,\mathsf{Alg}_m(\mathbf{L}_{LI}). \tag{14}$$

Note that $\mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$ is axiomatizable by first order formulas, and hence is closed under taking ultra products. Since by definition of $\mathbf{U}_p$, $\mathbf{U}_p\,\mathsf{Alg}_m(\mathbf{L}_{LI}) = \mathbf{U}_p\,\mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$, it follows from (14) that

$$\mathbf{H}\,\mathsf{Alg}_{\models}(\mathbf{L}_{LI}) = \mathsf{Alg}_{\models}(\mathbf{L}_{LI}). \tag{15}$$

Theorem 5.15 together with (13), (15) and Claim 5.17 and Claim 5.18 below implies that $\mathsf{Alg}_{\models}(\mathbf{L}_{LI})$ has ES.

Claim 5.17 *$Sir(\mathsf{Alg}_{\models}(\mathbf{L}_{LI}))$* $= \mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$.

Proof of Claim 5.17:
Since $\mathsf{Alg}_m(\mathbf{L}_{LI})$ has a discriminator term, every $\mathcal{A} \in \mathsf{Alg}_m(\mathbf{L}_{LI})$ is simple. Hence every $\mathcal{A} \in \mathsf{Alg}_m(\mathbf{L}_{LI})$ is subdirectly irreducible. Hence $\mathsf{Alg}_m(\mathbf{L}_{LI}) \subseteq \mathrm{Sir}(\mathsf{Alg}_{\models}(\mathbf{L}_{LI}))$. It is not hard to see that this implies that $\mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI}) \subseteq \mathrm{Sir}(\mathsf{Alg}_{\models}(\mathbf{L}_{LI}))$.

To prove the other inclusion, let $\mathcal{A} \in \mathrm{Sir}(\mathsf{Alg}_{\models}(\mathbf{L}_{LI}))$. Since $\mathcal{A} \in \mathbf{SP}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$, there is an index set I and algebras $\mathcal{A}_i \in \mathsf{Alg}_m(\mathbf{L}_{LI})$, $i \in I$, such that $\mathcal{A} \subseteq \prod_{i\in I} \mathcal{A}_i$. Let $\pi_i : \prod_{i\in I} \mathcal{A}_i \longrightarrow \mathcal{A}_i$ denote the projection on the i^{th} coordinate. Then $\langle \mathcal{A} \xrightarrow{\pi_i\lceil A} \pi_i[\mathcal{A}]\rangle_{i\in I}$ is a subdirect decomposition of $\mathcal{A}$. Since $\mathcal{A}$ is subdirectly irreducible, there is an $i \in I$ such that $\pi_i\lceil A$ is injective. That is, there is an $i \in I$ such that $\mathcal{A} \cong \pi_i[\mathcal{A}] \subseteq \mathcal{A}_i \in \mathbf{S}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$. that is, $\mathcal{A} \in \mathbf{IS}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$. Since $\mathbf{L}_{LI}$ is structural, $\mathsf{Alg}_m(\mathbf{L}_{LI})$ is closed under taking subalgebras (cf. Lemma 5.16). Hence $\mathcal{A} \in \mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$.
Q.E.D.(Claim 5.17)

Claim 5.18 $\mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$ *has SAP.*

$$\begin{tikzcd}
& \mathcal{M}_1 \arrow[dr, "\gamma"] & \\
\mathcal{M}_0 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & * & \mathcal{M}_3 \in \mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI}) \\
& \mathcal{M}_2 \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Figure 10: In the proof of Claim 5.18 it will be shown that such $\mathcal{M}_3$, γ and δ exist.

In the following proof, L_{01} denotes the variety of lattices with endpoints. It is well-known (cf. e.g. [KMPT 83]) that L_{01} has SAP.

Proof of Claim 5.18:
Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \in \mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$, and let $\alpha : \mathcal{M}_0 \xrightarrow{1-1} \mathcal{M}_1$ and $\beta : \mathcal{M}_0 \xrightarrow{1-1} \mathcal{M}_2$ be embeddings. See also Figure 10.

For every $\mathcal{M} \in \mathsf{Alg}_m(\mathbf{L}_{LI})$, let $\mathcal{M}\lceil\mathbf{L}_{01}$ denote the reduct $\langle uv(M), 1, 0, \wedge, \vee\rangle \in \mathbf{L}_{01}$ of $\mathcal{M}$. Since $\mathbf{L}_{01}$ has SAP, there exists an $\mathcal{M}_3' \in \mathbf{L}_{01}$ and embeddings $\gamma' : \mathcal{M}_1\lceil\mathbf{L}_{01} \xrightarrow{1-1} \mathcal{M}_3'$, $\delta' : \mathcal{M}_2\lceil\mathbf{L}_{01} \xrightarrow{1-1} \mathcal{M}_3'$ such that $\gamma' \circ (\alpha\lceil(\mathcal{M}_0\lceil\mathbf{L}_{01})) = \delta' \circ (\beta\lceil(\mathcal{M}_0\lceil\mathbf{L}_{01}))$, and $\gamma'[\mathcal{M}_1\lceil\mathbf{L}_{01}] \cap \delta'[\mathcal{M}_2\lceil\mathbf{L}_{01}] = \gamma'[\alpha[\mathcal{M}_0\lceil\mathbf{L}_{01}]]$. Define " $\leftrightarrow$ " on $\mathcal{M}_3'$ as follows:

$$\forall m, n \in M_3', \qquad m \leftrightarrow n = \begin{cases} 1, & \text{if } m = n. \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}_3 \stackrel{\text{def}}{=} \langle \mathcal{M}_3', \leftrightarrow\rangle \in \mathbf{I}\,\mathsf{Alg}_m(\mathbf{L}_{LI})$. Moreover, $\gamma : \mathcal{M}_1 \xrightarrow{1-1} \mathcal{M}_3$ and $\delta : \mathcal{M}_2 \xrightarrow{1-1} \mathcal{M}_3$ defined by $(\forall x \in M_1)\ \gamma(x) = \gamma'(x)$ and $(\forall x \in M_2)\ \delta(x) = \delta'(x)$ are the desired embeddings (cf. Definition 5.14).
Q.E.D.(Claim 5.18)
Q.E.D.(Theorem 4)

REMARK 5.19 From the above counterexample it does not only follow that in the proof of Theorem 1 the connectives ε and δ are needed, but also that this is the case for the proof of Theorem 2. For it can easily be proved that Σ defines $\{r\}$ implicitly in Q in a strong sense (cf. Claim 5.11), where Σ, r and Q are defined as in Lemma 5.9. Just define for every $\mathcal{M} \in Mod^Q(\Sigma \cap F^P)$, the model $\mathcal{N} \in M_{LI}^Q$ as follows: $\mathcal{N} \stackrel{\text{def}}{=} \langle \mathcal{M}, \star, mng_{\mathcal{N}}\rangle$, where $mng_{\mathcal{N}}^Q\lceil P = mng_{\mathcal{M}}^P$ and $mng_{\mathcal{N}}^Q(r) = \star$. Then $\mathcal{N} \in Mod^Q(\Sigma)$ and $mng_{\mathcal{N}}^Q\lceil F^P = mng_{\mathcal{M}}^P$. $\lhd$

In the remaining part of this Section we would like to discuss the patchwork property of models. As pointed out before (cf. Theorem 2.15), this property can be thought of as an amalgamation property on the meaning algebras. In view of the (traditional) connections between amalgamation properties and surjectiveness of epimorphisms, this might lead the reader to believe that the assumption that the logic $\mathbf{L}$ has the PP is a very strong one when proving that $\mathsf{Alg}_{\models}(\mathbf{L})$ has ES.[14] The next Theorem asserts that this is not the case.

THEOREM 5.20 *For every strongly nice general logic* $\mathbf{L}$, *there exists a strongly nice general logic* $\mathbf{L}'$, *such that*

$$\mathsf{Alg}_{\models}(\mathbf{L}) = \mathsf{Alg}_{\models}(\mathbf{L}'), \textit{ and } \mathbf{L}' \textit{ has the patchwork property of models.}$$

Proof of Theorem 5.20:
Let $\mathbf{L}$ be a strongly nice general logic. Let $X \subseteq \mathsf{Alg}_{\models}(\mathbf{L})$ be a class such that

$$(\forall \mathcal{A} \in \mathsf{Alg}_{\models}(\mathbf{L}))(\exists! \mathcal{B} \in X)\ \mathcal{A} \cong \mathcal{B},$$

and

$$(\forall \mathcal{A}, \mathcal{B} \in X)[\mathcal{A} \neq \mathcal{B} \Rightarrow A \cap B = \emptyset].$$

Note that $\mathsf{Alg}_{\models}(\mathbf{L}) = \mathbf{I}\, X$.

Now, let for every set P the logic $\mathcal{L}'^P$ be defined as $\mathcal{L}^P$ with the exception that

$$M^P_{\mathcal{L}'} \stackrel{\text{def}}{=} \{\langle \mathcal{A}, h\rangle \ :\ \mathcal{A} \in X,\ h : \mathcal{F}^P \longrightarrow \mathcal{A} \text{ is a homomorphism } \}.$$

Obviously, $\mathbf{L}'$ is a strongly nice general logic. As the proof of Lemma 5.16 shows, this implies that $\mathsf{Alg}_m(\mathbf{L}') = \mathbf{S}\, X$. Hence $\mathsf{Alg}_{\models}(\mathbf{L}') = \mathbf{SPS}\, X = \mathbf{SSP}\, X = \mathbf{SP}\, \mathsf{Alg}_{\models}(\mathbf{L}) = \mathsf{Alg}_{\models}(\mathbf{L})$.

Finally we will show that $\mathbf{L}'$ has the patchwork property of models. Let $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{A}, h\rangle \in M^P_{\mathcal{L}'}$, $\mathcal{N} \stackrel{\text{def}}{=} \langle \mathcal{B}, g\rangle \in M^Q_{\mathcal{L}'}$ such that $F^{P\cap Q} \neq \emptyset$ and

$$h \lceil F^{P\cap Q} = g \lceil F^{P\cap Q}. \tag{16}$$

By (16), $A \cap B \neq \emptyset$, hence by the construction of $\mathbf{L}'$, $\mathcal{A} = \mathcal{B}$. Now $\langle \mathcal{A}, f\rangle \in M^{P\cup Q}_{\mathcal{L}'}$, where $f : \mathcal{F}^{P\cup Q} \longrightarrow \mathcal{A}$ is the unique homomorphism such that

$$f(p) = \begin{cases} h(p), & \text{if } p \in P. \\ g(p), & \text{if } p \in Q. \end{cases}$$

is the desired amalgamant.
Q.E.D.(Claim 5.20)

The importance of Theorem 5.20 is twofold. First this theorem assures us that the patchwork property of models does not affect the purport of our main results. That

[14] Note that in the proof of Theorem 1 we only need the patchwork property of models for proving that all the epimorphisms of $\mathsf{Alg}_{\models}(\mathbf{L})$ are surjective, on the assumption that $\mathbf{L}$ has the Beth definability property (cf. Claim 3.5); the other direction goes through for strongly nice general logics *without* the PP. In the proof of Theorem 2 we use the PP in *both* directions.

is, this theorem shows that it is not because of some strong assumption on the logic that the Beth definability property corresponds to surjectiveness of epimorphisms. Rather it is because of some intrinsic resemblance between these two properties, as was our intention to show.

Second, Theorem 5.20 proves that semantic properties like the patchwork property of models do *not* show up on the level of abstraction of the abstract class $\mathsf{Alg}_{\models}(\mathbf{L})$. This shows us once again[15] that for algebraizing semantics we really need the concrete class $\mathsf{Alg}_m(\mathbf{L})$.

The above Theorem, together with the fact that all the well-known logics we came a'cross (e.g. all the logics mentioned in [AKNS 94, Definitions 2.2.21-31]) have the patchwork property of models, indicates that this property is very weak. This leads to the question whether the assumption that the logic $\mathbf{L}$ has the patchwork property of models, is *necessary* in order to prove Theorem 1 and Theorem 2. However, at present this is unknown to the author.

A Appendix

REMARK A.1 In Definition 2.2 we define a logic as some special quadruple. Another option frequently encountered in the literature, is to define a logic as a *five*-tuple $\langle F_{\mathcal{L}}, M_{\mathcal{L}}, mng_{\mathcal{L}}, \models_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$, where $\vdash_{\mathcal{L}} \subseteq \wp(F_{\mathcal{L}}) \times F_{\mathcal{L}}$ is called the *provability relation* of $\mathcal{L}$. Intuitively, for $\Gamma \subseteq F_{\mathcal{L}}$ and $\varphi \in F_{\mathcal{L}}$, $\Gamma \vdash_{\mathcal{L}} \varphi$ means here that φ is provable from Γ by means of the syntactic inference system of $\mathcal{L}$.

However, as outlined in [AKNS 94, page 8] this would, for the type of result discussed here, *not* be any real generalization. The reason for this is that our validity relation $\models_{\mathcal{L}} \subseteq M_{\mathcal{L}} \times F_{\mathcal{L}}$ induces a relation $\models_{\mathcal{L}} \subseteq \wp(F_{\mathcal{L}}) \times F_{\mathcal{L}}$, called the *semantical consequence relation*, as follows: $(\forall \varphi \in F_{\mathcal{L}})(\forall \Gamma \subseteq F_{\mathcal{L}})\,[\Gamma \models_{\mathcal{L}} \varphi \overset{\text{def}}{\Leftrightarrow} \forall M \in M_{\mathcal{L}}[(\forall \gamma \in \Gamma)\ M \models_{\mathcal{L}} \gamma \Rightarrow M \models_{\mathcal{L}} \varphi]]$. Although replacing $\vdash_{\mathcal{L}}$ with $\models_{\mathcal{L}}$ would be at many places (e.g. completeness theorems) a grave oversimplification, all the theorems in [AKNS 94] proved for $\models_{\mathcal{L}}$ *do* carry over to $\vdash_{\mathcal{L}}$, whenever these theorems are not about connections between $\vdash_{\mathcal{L}}$ and $\models_{\mathcal{L}}$. This will be discussed in more detail in the forthcoming [ABNPS 9?]. For this reason, it was decided in [AKNS 94] to restrict the attention to the four-tuple as stated in Definition 2.2. Here we follow this decision. In doing so, we find ourselves in the so-called *semantical tradition*. Another possibility would be to call the syntactical part $\mathcal{L} = \langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$ a logic. For this *syntactical approach*, cf. e.g. [BP 89]. $\triangleleft$

REMARK A.2 Readers more familiar with the Blok-Pigozzi approach can read this paper by simply replacing every occurrence of the notion of a strongly nice logic by that of an *algebraizable* logic (see e.g. [BP 89, Definition 2.10]). The reason for this is that the $\langle F_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$-part of a strongly nice logic $\mathcal{L}$ corresponds closely to an algebraizable logic $\langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$ in the sense of [BP 89], where $\models_{\mathcal{L}}$ denotes the semantical consequence relation induced by the validity relation of $\mathcal{L}$ (cf. Remark A.1).

[15] See also Section 4, page 24.

The main difference is that the consequence relation $\vdash_{\mathcal{L}}$ is finitary in the sense that $\forall\Gamma \subseteq F_{\mathcal{L}}$, $\forall\varphi \in F_{\mathcal{L}}$, if $\Gamma \vdash_{\mathcal{L}} \varphi$, then $(\exists\Gamma' \subseteq_{\omega} \Gamma)\ \Gamma' \vdash_{\mathcal{L}} \varphi$, whereas $\models_{\mathcal{L}}$ need not be finitary. Logics which *do* have this property are called *consequence compact* in [AKNS 94] (see [AKNS 94, Definition 3.2.10]). It turns out that the $\langle F_{\mathcal{L}}, \models_{\mathcal{L}}\rangle$-part of a consequence compact, strongly nice logic is always an algebraizable logic and, the other way round; every algebraizable logic $\langle F_{\mathcal{L}}, \vdash_{\mathcal{L}}\rangle$ can always naturally be expanded to a consequence compact, strongly nice logic. This is explained in [AKNS 94, page 36]. Since the present paper does not speak about compactness, one can safely read "algebraizable" where "strongly nice" is stated. For a detailed comparison between the two approaches, we refer to [FJ 94]. $\lhd$

REMARK A.3 As far as modal logic is concerned, is it necessary to distinguish between a *global* and a *local* interpretation of the validity relation. In the global way of thinking, models of a modal logic $\mathcal{L}$ are structures $\langle\langle W, R\rangle, V\rangle$, where W is a set of states, R an accessibility relation, V a valuation, and $(\forall\varphi \in F_{\mathcal{L}})\langle\langle W, R\rangle, V\rangle \models \varphi \Leftrightarrow (\forall w \in W)\ w \models \varphi$. Under this interpretation *all* normal modal logics are strongly nice; one can simply take $n = m = 1$, $\varepsilon_0(\varphi) = True$, $\delta_0(\varphi) = \varphi$ and $\Delta_0 = \leftrightarrow$. The Budapest school adheres to this way of thinking.

It is an entirely different question whether in the local way of thinking modal logics are strongly nice. In this style, models of a modal logic $\mathcal{L}$ also contain a distinguished state s, and $(\forall\varphi \in F_{\mathcal{L}})\ \langle\langle W, R, s\rangle, V\rangle \models \varphi \Leftrightarrow s \models \varphi$. Actually, under this interpretation modal logics are not equivalential, unless they have transitive frames. This can be solved by allowing the set of derived connectives Δ to be *infinite* (cf. Remark 2.5.3). Then the most obvious candidate for $\varphi\Delta\psi$ becomes $\{\Box^n(\varphi \leftrightarrow \psi) : n \in \omega\}$. Yet in this case the most obvious candidates for ε_0 and δ_0 (i.e. $\varepsilon_0(\varphi) = True$, $\delta_0(\varphi) = \varphi$) fail. Interesting as it is, the question of strong niceness of modal logics under this interpretation falls outside the scope of the present paper, and we can make no claim about the validity of our results concerning Beth's Theorem for modal logics in this local set-up. $\lhd$

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