Algebraic logic

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Algebraic logic can be divided into two major parts: (i) abstract (or universal) algebraic logic and (ii) "concrete" algebraic logic (or algebras of relations of various ranks).

(1) Abstract algebraic logic is built around a duality theory which associates, roughly speaking, quasi-varieties of algebras to logical systems (logics for short) and vice versa. After the duality theory is elaborated, *characterization theorems* follow which characterize distinguished logical properties of a logic L in terms of natural algebraic properties of the algebraic counterpart $Alg(L)$ of L.

A *logic* is, usually, a tuple

$$\mathcal{L} = \langle Fm_{\mathcal{L}}, Mod_{\mathcal{L}}, \models_{\mathcal{L}}, mng_{\mathcal{L}}, \vdash_{\mathcal{L}} \rangle$$

where Fm is the set of formulas of $\mathcal{L}$, Mod is the class of models of $\mathcal{L}$, $\models_{\mathcal{L}} \subseteq Mod \times Fm$ is the validity relation, $mng : Mod \times Fm \to Sets$ is the semantical meaning (or denotation) function of $\mathcal{L}$ and $\vdash$ is the syntactical provability relation of $\mathcal{L}$.

More generally, a *general logic* consists of a class $Voc_{\mathcal{L}}$ of vocabularies and then to each vocabulary $\tau \in Voc_{\mathcal{L}}$, $\mathcal{L}$ associates a logic, i.e. a 5-tuple $\mathcal{L}(\tau) = \langle Fm_\tau, Mod_\tau, \models_\tau, mng_\tau, \vdash_\tau \rangle$ as indicated above. As an example, first-order logic is a general logic in the sense that to any collection of predicate symbols it associates a concrete first-order language built up from those predicate symbols (i.e. vocabulary).

Of course, there are some conditions which logics and general logics have to satisfy, otherwise any "crazy" odd 5-tuple would count as a logic, which one wants to avoid. We do not recall the conditions on logics and general logics, instead we refer to [2], or [16], or [11], or for the case of logics without semantics (i.e. without $Mod_{\mathcal{L}}$) to [4]. These conditions go back to pioneering papers of Tarski.

To each logic and general logic there is a set $Cnn_{\mathcal{L}}$ of *logical connectives* specified in such a way that $Fm_{\mathcal{L}}$ or Fm_{τ} becomes an absolutely free algebra generated by the atomic formulas of τ and $\mathcal{L}$ and using $Cnn_{\mathcal{L}}$ as algebraic operations. Hence we can view $Cnn_{\mathcal{L}}$ as the similarity type of the algebras Fm_{τ}. Using the algebras Fm_{τ} and the provability relation $\vdash_{\tau}$ one can associate a class $\mathsf{Alg}_{\vdash}(\mathcal{L})$ of algebras to $\mathcal{L}$. Each of these algebras corresponds to a syntactical theory of $\mathcal{L}$. Using Fm_{τ} together with mng_{τ}, $\models_{\tau}$ one can associate a second class $\mathsf{Alg}_{\models}(\mathcal{L})$ of algebras to $\mathcal{L}$. $\mathsf{Alg}_{\models}(\mathcal{L})$ represents semantical aspects of $\mathcal{L}$, e.g. each model $\mathfrak{M} \in Mod_{\tau}$ corresponds to an algebra in $\mathsf{Alg}_{\models}(\mathcal{L})$. Often, the members of $\mathsf{Alg}_{\models}(\mathcal{L})$ are called representable algebras or meaning algebras of $\mathcal{L}$. Under mild conditions on $\mathcal{L}$, one can prove that $\mathsf{Alg}_{\vdash}(\mathcal{L})$ is a quasi-variety and that $\mathsf{Alg}_{\models}(\mathcal{L}) \subseteq \mathsf{Alg}_{\vdash}(\mathcal{L})$, cf. e.g. [2].

Examples: If $\mathcal{L}$ is propositional logic, then $\mathsf{Alg}_{\vdash}(\mathcal{L}) = \mathsf{Alg}_{\models}(\mathcal{L})$ is the class BA of Boolean algebras. Let $n \in \omega$. For the n-variable fragment L_n of first order logic, $\mathsf{Alg}_{\vdash}(L_n)$ is the class CA_n of cylindric algebras of dimension n, while $\mathsf{Alg}_{\models}(L_n)$ is the class RCA_n of representable cylindric algebras. For a certain variant L_ω of first-order logic, $\mathsf{Alg}_{\vdash}(L_\omega)$ is the class RCA_ω of representable CA_ω's. L_ω is called full restricted first order language in [6], cf. also [2, §6] and Appendix C in [4]. For the algebraic counterparts of other logics (as well as other versions of first order logic) we refer to [2].

Let us take the logic L_n as an example. The algebraic counterparts of theories of L_n are exactly the algebras in CA_n and the interpretations between theories correspond exactly to the homomorphisms between CA_n's. Further, axiomatizable classes of models of L_n correspond to RCA_n's and (semantic) interpretations between such classes of models correspond to special homomorphisms called base-homomorphisms between RCA_n's, cf. [6]p.170. Individual models of L_n correspond to simple RCA_n's and elementary equivalence of models corresponds to isomorphism of RCA_n's. The elements of an RCA_n corresponding to a model $\mathfrak{M}$ are best thought of as the relations definable in $\mathfrak{M}$.

Of the duality theory between logics and their algebraic counterparts we discussed only the translation

$$\mathsf{Alg} : \text{“logics”} \longrightarrow \text{“pairs of classes of algebras”}.$$

The other direction can also be elaborated (and then we can have a two-sided duality like Stone duality between BA's and certain topological spaces); we refer to [4, p.21] for more on such a two-sided duality between logics and quasi-varieties of algebras.

Some of the equivalence theorems. We use the above outlined duality theory for characterizing logical properties of $\mathcal{L}$ with algebraic properties of $\mathsf{Alg}_{\models}(\mathcal{L})$, $\mathsf{Alg}_{\vdash}(\mathcal{L})$ (under some mild assumptions on $\mathcal{L}$). E.g. the deduction property of $\mathcal{L}$ is equivalent with $\mathsf{Alg}_{\vdash}(\mathcal{L})$ having equationally definable principal congruences. The Beth definability property for $\mathcal{L}$ is equivalent with surjectiveness of all epimorphisms in $\mathsf{Alg}_{\models}(\mathcal{L})$. The various definability properties (weak Beth, local Beth etc.) and interpolation properties are equivalent with distinguished versions of the amalgamation property and surjectiveness of epimorphisms, respectively, in $\mathsf{Alg}_{\vdash}(\mathcal{L})$ or $\mathsf{Alg}_{\models}(\mathcal{L})$. A kind of completeness theorem for $\mathcal{L}$ is equivalent with finite axiomatizability of $\mathsf{Alg}_{\models}(\mathcal{L})$. Compactness of $\mathcal{L}$ is equivalent with $\mathsf{Alg}_{\models}(\mathcal{L})$ being closed under ultraproducts. The above (and further) equivalence theorems are elaborated e.g. in [2]. Further such results are in e.g. [5], the works of Blok and Pigozzi, e.g. in [4], [10], [18], [8], works of Czelakowski, Maksimova, and the references in Studia Logica Vol. 65 No 1.

(2) Concrete algebraic logic investigates those classes of algebras which arise in the algebraization of the most frequently used logics. Below we restrict attention to algebras of classical quantifier logics, of the finite variable fragments L_n of these logics, relativized versions of these logics, e.g. the guarded fragment, and logics of the dynamic trend whose algebras are relation algebras or relativized relation algebras. Here we want to "algebraize" logics which extend classical propositional logic. The algebras of this propositional logic are Boolean algebras (BA's for short). BA's are natural algebras of unary relations. We expect the algebras of the extended logics to be extensions of BA's to algebras of relations of higher ranks. The elements of a BA are sets of points; we expect the elements of the new algebras to be sets of sequences (since relations are sets of sequences).

n-ary representable *cylindric algebras* (RCA_n's) are algebras of n-ary relations. They correspond to the n-variable fragment L_n of first-order logic. The new operations are cylindrifications c_i $(i < n)$. If $R \subseteq {}^nU$ is a relation defined by a formula $\varphi(v_0, \ldots, v_{n-1})$, then $c_i(R) \subseteq {}^nU$ is the relation defined by the formula $\exists v_i \varphi(v_0, \ldots, v_{n-1})$. (To be precise, we should write c_i^U for c_i.) Assume $n = 2$, $R \subseteq U \times U$. Then $c_0(R) = U \times \mathrm{Rng}(R)$ and $c_1(R) = \mathrm{Dom}(R) \times U$. This shows that c_i is a natural and simple operation on n-ary relations: it simply abstracts from the i-th argument of the relation. Let $i < n$, $R \subseteq {}^nU$. Then

$$c_i(R) = \{\langle b_0, \ldots, b_{i-1}, a, b_{i+1}, \ldots \rangle : b \in R \text{ and } a \in U\}.$$

$\mathfrak{P}(U) = \langle \mathcal{P}(U), \cap, \cup, - \rangle$ denotes the Boolean algebra of all subsets of U. The algebra of n-ary relations over U is

$$\mathfrak{Rel}_n(U) = \langle \mathfrak{P}({}^nU), c_0, \ldots, c_{n-1}, Id \rangle$$

where the constant operation Id is the n-ary identity relation $Id = \{\langle a, \ldots, a \rangle : a \in U\}$ over U. E.g. the smallest subalgebra of $\mathfrak{Rel}_2(U)$ has ≤ 2 atoms while that of $\mathfrak{Rel}_n(U)$ has $\leq 2^{(n^2)}$ atoms. The class RCA_n of n-ary *representable cylindric algebras* is defined as

$$\mathsf{RCA}_n = \mathbf{SP}\{\mathfrak{Rel}_n(U) : U \text{ is a set}\}$$

where $\mathbf{S}$ and $\mathbf{P}$ are the operators on classes of algebras corresponding to taking isomorphs of subalgebras and direct products, respectively.

Let $n > 2$. Then RCA_n is a discriminator variety, with an undecidable but recursively enumerable equational theory. RCA_n is not finitely axiomatizable and fails to have almost any form of the amalgamation property and has non-surjective epimorphisms. Almost all of these theorems remain true if we throw away the constant Id (from RCA_n) and close up under $\mathbf{S}$ to make it a universally axiomatizable class. These properties imply theorems about L_n via the duality theory between logics and classes of algebras elaborated in abstract algebraic logic (cf. way above). Further, usual set theory can be built up in L_3 (and even in the equational theory of CA_3). Hence L_3 (and CA_3) have the "Gödel incompleteness property", cf. [13].

For first-order logic L_ω with infinitely many variables (cf. e.g. Appendix C of [4]) the algebraic counterpart is RCA_ω (algebras of ω-ary relations). To generalize RCA_n to RCA_ω we need only one nontrivial step; we have to brake up our single constant Id to a set of constants $Id_{ij} = \{q \in {}^\omega U : q_i = q_j\}$, with $i, j \in \omega$. Now

$$\mathsf{RCA}_\omega = \mathbf{SP}\{\langle \mathfrak{P}({}^\omega U), c_i, Id_{ij} \rangle_{i,j \in \omega} : U \text{ is a set }\}.$$

The definition of RCA_α with α an arbitrary ordinal is practically the same. RCA_α is an arithmetical variety, not axiomatizable by any set Σ of formulas involving only finitely many individual variables. Most of the theorems we mentioned about RCA_n carry over to RCA_α.

The greatest element of a "generic" RCA_α was required to be a Cartesian space ${}^\alpha U$. If we remove this condition and replace ${}^\alpha U$ with an arbitrary α-ary relation $V \subseteq {}^\alpha U$ in the definition, we obtain the important generalization Crs_α of RCA_α. Many of the negative properties of RCA_α disappear in Crs_α. E.g. the equational theory is decidable, is a variety generated by its finite members, enjoys the super-amalgamation property (hence strong amalgamation property (SAP), too), etc. Recent logic applications of Crs_α abound, cf. e.g. [3], [20], [9], [15].

Since RCA_α is not finite schema axiomatizable, a finitely schematizable approximation $\mathsf{CA}_\alpha \supsetneq \mathsf{RCA}_\alpha$ was introduced by Tarski. There are theorems to the effect that CA's approximate RCA's well, cf. [6, §3.2].

The above illustrates the flavor of the theory of algebras of relations; important kinds of algebras which we did not mention include relation algebras and quasi-polyadic algebras, cf. e.g. [6], [19], [2], [14], [17], [12]. The theory of the latter two is analogous with that of RCA_α's. We did not have space to mention the category theoretic approaches, but cf. [2] and the references therein.

There are many open problems in this area (cf. e.g. [14], [7], [1, pp. 727-745]). To mention one, is there a variety $V \subseteq CA_\alpha$ having SAP but not super-amalgamation property?

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