Logic and Relativity

(in the light of definability theory)

Matematika Doktori Iskola, Elméleti Matematika Program

Iskolavezető: Laczkovich Miklós

Programvezető: Szenthe János

Témavezető: Németi István

Kutatóhely: MTA Rényi Alfréd Matematikai Kutatóintézet

Logic and Relativity
(in the light of definability theory)

Judit X. Madarász

PhD-dissertation

February 26, 2002

Advisers: István Németi,
Hajnal Andréka

Contents

Gödel and Tarski

Gödel and Einstein

Figure 1: An "amalgamation diagram" in the sense of e.g. Madarász [164], [165] (illustration for the first sentences of §1.)

On the length of the dissertation: It was our ambition to make this work pleasant reading for readers from different research fields, hence having perhaps strongly different backgrounds. For this reason, we included a lot of pictures, intuitive explanation and historical background. Further, to reduce the need for consulting the literature, we included parts which do not count as main results of the dissertation but which help understanding the dissertation and putting it into perspective. (Much of these parts also originate with the author and were originally made available e.g. in the book [18] on the Internet.) Let us call these parts "parts of an appendix character". Chapter 2 is *only* of an *appendix* character (pp.1–105). Because of this appendix character, most of the proofs are omitted from Chapter 2. (Chapter 2 recalls e.g. the logical machinery which we use here from e.g. [18].) Exceptions in Chapter 2 are §§ 2.6, 2.7, 2.8.3; these are not of appendix character. §§ 4.2.3–4.2.5 (pp.182-215) are also of appendix character. The same applies to the various appendices and lists making up the last 59 pages. So, *the main text of the dissertation is about 250 pages long*. Much of this space is taken up by pictures, intuitive motivation, historical background material designed to improve readability and provide perspective. Summing it up, the lengthiness of this work is a consequence of our ambition to assist the reader.

Convention: We will often refer to the background material Andréka-Madarász-Németi [18] of the present work. Therefore, we will abbreviate it as AMN [18]. The other background materials Andréka-Madarász-Németi [16] and [21] are also important and they will be abbreviated as AMN [16], AMN [21]. The main results of AMN [18], [21] are intended to constitute part of this dissertation. However, for lack of space, they will be recalled only very briefly and partially cf. e.g. §4.6 herein. The results quoted in this work from the above mentioned background materials [18], [16], [21] were obtained by the present author.

On footnotes: Unlike in other works, here the *footnotes* often contain *more important* information than the main text. Certain texts are put into footnotes not for reasons of importance but for reasons of composition. We suggest *first* reading each section completely *ignoring* the footnotes. Then we suggest a second reading with paying attention to the footnotes. The footnotes will become particularly important from the introduction §4.1 of Chapter 4 (Geometry) on. Cf. the note about this on p.128.

1 Introduction

Many people think that Gödel and Tarski were the greatest logicians who ever lived. Gödel in turn was a close friend and collaborator of Einstein and in connection with their discussions of relativity theory, Gödel made discoveries about relativity and cosmology whose impact is being more and more appreciated nowadays, cf. e.g. Yourgrau [270], Friedman [91], or (the "accompanying" papers by present day leading physicists in) Gödel [99, 100]. Tarski, too, wanted to use logic as a foundation for science in general, and for relativity theory (and related areas) in particular.[1]

The present work intends to apply mathematical logic to relativity in the spirit of the just quoted tradition.[2] Besides using mathematical logic as a foundation for special relativity, we

[1] This is how the series "Studies in Logic and the Foundations of Mathematics" came into being.

[2] Connections with Tamás Matolcsi's school are summarized in Andai-Andréka-Madarász-Németi [7].

have more ambitious goals in mind.[3] Such are e.g. using logic for a deeper understanding of relativity, using logic for bringing "modularity" or a so-called "lego-toy-world" character into relativity by making it decomposable into a hierarchy of weaker, simpler theories (with clear intuitive meanings) which in turn can be re-assembled in many different ways. This way, among other things, we would like to make relativity more flexible and more easily combinable with other theories. The main goals of this work are summarized in more detail in AMN [18, §1.1], [7]. We do not recall them, but very roughly they are: deeper (and clearer) understanding, insights, logical analysis, addressing the so-called "why-type" questions[4], and eliminating the so-called tacit assumptions.

Besides the famous predecessors Gödel, Einstein and Tarski we have to mention Reichenbach who as early as 1924 began his relativity book by emphasizing the importance of the branch of logic called definability theory for relativity theory, cf. Reichenbach [218, pp.3-11]. (Reichenbach wrote important monographs on relativity as well as on logic.)

1.1 Historical perspective and some of our goals

More motivation, and a careful, more clearly elaborated formulation of our goals (in the present work) is available in AMN [18, Chap. 1] (the introduction of [18]), in the introduction to Chapter 4 (i.e. §4.1) herein, and in the introduction of Tőke [262]. We think that Andai-Andréka-Madarász-Németi [7] may be particularly helpful in this connection. Below we recall only a subset of the motivating ideas/goals in the above quoted works and only briefly.

Tarski formalized geometry as a theory of first-order logic. The point here is to use *only* first-order logic; no external "devices" or tacit assumptions are allowed to enter the picture. Motivated by Tarski, P. Suppes [242] raised the problem of formalizing the theory of special relativity as a theory purely in first-order logic.[5] This problem was studied by Ax, Goldblatt and others. In the present work we want to work on a "programme" which is related to the just quoted one (e.g. insist on using only first-order logic) but is slightly more general (than the quoted one) in various respects, e.g. in the following one. A possible approach to axiomatizing special relativity in first-order logic (FOL) would be the following: Axiomatize, first, Minkowskian geometry in FOL and then try to build a relativity theory on top of that. Here, we want to develop a different, in some sense a more ambitious, approach. Namely,

[3] If we wanted only to build up special relativity in first-order logic, that could be done in less than 150 pages. The reader can convince himself of this by leafing through Chapter 2 of the present work together with AMN [16]. (In an appendix of [102], Goldblatt already formalized in pure first-order logic an "only-the-heart" fragment of special relativity, where see p.vii for "only-the-heart" approaches.) We are mentioning this to illustrate that besides formalizing special relativity and some of its generalizations in first-order logic, we have more ambitious goals in mind here.

[4] For a mathematical logical exposition/investigation of why-type questions with applications to foundation of science cf. e.g. Hintikka-Halonen [128]. We note that the volume [76] by distinguished physicist Earman also discusses why-questions in connection with foundation of physics, and so does David Deutsch [71] (in a slightly different sense).

[5] There are certain methodological reasons why we want to stick with (possibly many-sorted and perhaps modal) first-order logic (FOL) as opposed to using higher-order logic with its standard semantics. These reasons are connected with the fact that higher-order logic is not absolute in the set theoretic sense cf. e.g. Barwise-Feferman [45], e.g. p.33, below item 2.1.1, and §XVII.2.1., and therefore no effective complete proof system can exist for higher-order logic. Putting it more bluntly: There is *no completeness* theorem for higher-order logic, moreover it is *impossible* to obtain a completeness theorem for higher-order logic (this follows e.g. from Gödel's incompleteness theorem). The above mentioned reasons for sticking with FOL were presented at various logic conferences in Amsterdam (during the period 1994-1998) and can be (partially) recovered from Sain [228], cf. also Johan van Benthem [266]. We collected, explained and elaborated several of these reasons in the Appendix entitled "Why first-order logic?" of AMN [18, pp.1245–1252].

primarily (or firstly) we want to write up a natural and convincing axiomatization, call it **Specrel**, of special relativity in FOL, and then we want to study and develop this first-order theory **Specrel**, so that studying **Specrel** would lead us to "deriving" something like Minkowskian geometry as a "theoretical construct" (i.e. Minkowskian geometry will show up as a "theoretical consequence" of our "primary" theory **Specrel**). One of our reasons for this preference is that we want to start out with axioms about the subject matter of special relativity (i.e. motion etc.) which are *self-evident* (in some sense). In other words, we would like to *derive* (in some sense) relativity theory from easily comprehensible, natural axioms which are convincing (and acceptable) even for the outsider (who does not know anything about relativity). All discussions will be in terms of simple concepts. When formalizing our (language and) axioms we will confine ourselves to a very plain language,[6] using such easily comprehensible concepts as "bodies" or "observers". Whenever we need more complex concepts like "energy", "entropy" or "curvature of space-time", we will first define these, as a logician would do, in terms of our plain language. This allows us to make the *axioms* with which we started *subject to debate*: both because of the plain language in which they are expressed and because of the purely logical nature of our reasoning.

Sometimes physical theories are formalized in the following style: Only the "heart" (in some sense) of the theory is formalized; and then the so obtained formal theory comes together with a *non-formalized*, natural language explanation of how to use the formal theory. This natural language text is often called the "*interpretation*" of the formal theory. An example of such an *"only-the-heart" approach* would be formalizing e.g. Minkowskian geometry in first-order logic and then writing an explanation in natural language on how to use Minkowskian geometry for solving problems in special relativity. In the present work we intend to formalize the whole theory and *not* only the heart. In particular, we want to obtain a formalized theory which contains its own "interpretation" (where the word "interpretation" is used in the above sense). Efforts will be made to keep the axioms both "observational" and simple; and to maintain a standard of discussing, analyzing and refining their intuitive meanings. At the same time, theoretical concepts etc. will also be studied, but they will be left to be discussed when we feel that we can tell *why* we introduce them.[7]

After formalizing the theory we also develop it to some extent and then use the formalized version to analyze the logical structure of the theory. If someone wants to build up special relativity in first-order logic[8], that can be done in a little fraction of the size of the present work (cf. footnote 3 on p.vi). The purposes of the present work are much more ambitious than just formalizing special relativity theory. Here we want to do things with relativity theory, using the tools of modern mathematical logic, that could not be done without mathematical logic. E.g. we want to analyze the logical structure of relativity theory, to see why relativity theory is built up the way it is and what would happen if we built it up differently. One of our further aims has been to build up a version of relativity which has a *modular* and "*lego-toy*" character. For an explanation of this goal we refer to §3.1 (p.105).

Another motivation for building up relativity theory in logic, which we quote from Chap. 1 of AMN [18], is the following. First we quote from the book Matolcsi [187, p.11].

[6] Just as the language of set theory is very simple: it contains only one binary relation symbol for the set theoretic membership relation.

[7] It is the above described only-the-heart approach which has lead to the so-called interpretational problems in some parts of physics, cf. e.g. Tőke [262].

[8] analogously to as e.g. set theory is built up in first-order logic

> "Mathematics reached a crisis at the end of the last century when a number of paradoxes came to light. Mathematicians surmounted the difficulties by revealing the origin of the troubles: the obscure notations, the inexact definitions; then the modern mathematical exactness was created and all the earlier notions and results were reappraised. After this great work nowadays mathematics is firmly based upon its exactness.
>
> Theoretical physics — in quantum field theory — reached its own crisis in the last decades. The reason of the troubles is the same. Earlier physics has treated common, visible and palpable phenomena, everything has been obvious."
>
> ...
>
> "It is quite evident, that we have to follow a way similar to that followed by mathematicians to create a firm theory based on mathematical exactness; having mathematical exactness as a guiding principle, we must reappraise physics, its most common, most visible and most palpable notions as well. Doing so we can hope we shall be able to overcome the difficulties."

Mathematics solved the above problem by using *logic*. Here we will experiment with doing the same in relativity theory, that is, build up (at least parts of) relativity theory in first-order logic.[9]

We also note that several physicists and cosmologists[10] suggested using a branch of *algebraic logic* called topos theory as a better framework for a clearer understanding of certain issues in relativity theory, cosmology and related areas. Cf. e.g. Isham-Butterfield [137], Markopoulou [182], Crane [63], Smolin [236], Smolin [237, pp.27-31, 46-47, 219] where e.g. the expression "cosmological logic" is extensively used (cf. e.g. p.30, line 5 bottom up).

1.2 Some connections with the literature, related work and predecessors

To our knowledge, the first attempt at a deductive treatment of relativity is due to Reichenbach [218], but we mention also Robb [223] which is earlier but which seems to be an "only-the-heart" approach. Although no explicit logical framework is present in [218], [218] can be considered a second-order logic approach analogous with Hilbert's second-order logic discussion of Euclidean geometry in [125].[11] The requirement of using basic, observation-oriented terms as primitives is made explicit by Reichenbach in his general philosophy of natural sciences.

The first logic-oriented results related to relativity are due to Robb [223], who aimed at deriving the geometrical structure induced (in some sense) by the binary relation *'being after'* over events (in the sense indicated above). Despite the apparent similarity of Minkowskian geometry to Euclidean geometry, the absence of a comprehensive axiomatization allowing foundational and metamathematical discussions of the former is pointed out by Suppes [242], who proposes the idea of a first-order formalization of Minkowskian geometry. (He might also be interpreted as proposing a broader project of a first-order axiomatization of special relativity. The identification of special relativity with its "heart" [or theoretical core], i.e. Minkowskian geometry, is not rare in the literature. As we have already mentioned, we consider this identification as unfortunate.[12]) Such a treatment of Minkowskian geometry was provided in turn by Goldblatt

[9] The foundation of mathematics (i.e. axiomatic set theory) is also formalized in first-order logic.

[10] e.g. L. Smolin, F. Markopoulou, C. J. Isham, J. Butterfield, L. Crane and others

[11] Some definitions and axioms in Reichenbach's work suggest the impossibility of a first-order translation. Reichenbach did not aim at a first-order logic formalization of relativity.

[12] Such identifications are typical examples of what we call "only-the-heart" approach.

[102]. From the point of view of special relativity as a comprehensive physical theory, Goldblatt's study can be regarded as an "only-the-heart" approach. We mention also Schutz [231], whose axiomatization is in second-order language, but is distinguished by the discussion of the independence of its axioms; Ax [36], which aims at deriving Minkowskian geometry from observational primitives similar to those in Reichenbach's approach; and Mundy [196], who presents a systematically simplified second-order axiomatization related to Robb's treatment. We should also mention Montague [195, §11] which represents a *model theoretical* (hence also logical) approach to physical theories of motion. (Montague was a student of Tarski and became famous for successfully applying the methodology of model theory [and logic] outside of pure mathematics.) Friedman [91] is strongly related to the present work in several ways. E.g. it uses model theory (of logic) just as we do. Also, it puts much effort into logical analysis of relativity theories like we do. The present list of references to related work is far from being complete. Further references can be found in the bibliographies of the works we quoted. For further motivation and related work we refer to AMN [18], Ax [36], van Benthem [264], Busemann [55], Friedman [91], Goldblatt [102], Matolcsi [187], Mundy [196], Rakić [215], Reichenbach [218], Schutz [231] and Suppes [242]. For further related logic-oriented approaches to axiomatizing relativity we refer to the references in the introduction of Schutz [231]. Reichenbach [218] is a rather important reference in this direction. In our list of references we include further related work.

The question naturally arises: What is new in the present work (relative to the above references)? A short answer is that we continue where our predecessors stopped. The idea of starting theory building from the observational side (of the observational/theoretical distinction), mentioned above briefly but outlined in more detail in AMN [§1.1][18], already appears in Reichenbach's work but is not implemented there in first-order logic. The idea of restricting our tools strictly and consistently to (many-sorted) first-order logic is carried through in Goldblatt [102], but he does not go beyond the "only-the-heart" approach.

There seems to be a point where most of the above quoted authors seem to stop. This is, more or less, the following.[13] Roughly speaking, they write up axiom systems, then prove that the axiom systems have certain desirable properties.[14] But sooner or later they seem to stop. With some exaggeration one might say that in the present work the real fun begins after we have written up some suitable axiom systems and after we have proved that these have the desirable properties.

In connection with the above we would like to point out the following. If we want to do the logical analysis of a theory (which is not yet in logical form), say of special relativity, the first step is to build an axiom system in the language of the logic we have chosen, which will be our "logicized" version of the theory in question. Then we prove that this "logicized" theory is indeed about the subject matter we wanted to analyze (and not about something else). Let us call this Step 2. However, it is only *after* Step 2 that we can really start applying the methods of mathematical logic to analyze the so obtained logic-based theory of whatever we wanted to study, e.g. that of special relativity. In passing we note that during this analysis, among other things, we will probably experiment with changing the axioms, so e.g. we end up with having

[13] We quoted so many works that it is hard to make categorical statements about them. Therefore what we write here is intended to be a "general impression" only, allowing exceptions etc. and not a careful critical study of the literature.

[14] E.g. if the author's aim was to axiomatize Minkowskian geometry, he proves, say, that every model of the axiom system is representable by a Minkowskian geometry over some real closed field.

several concurrent logic-based versions of special relativity.[15] [16]

1.3 On the results of this dissertation

Without aiming for completeness, below we list a few of the results of the dissertation (and related works by the author) to which we would like to draw the reader's attention.

(i) In section §4.3 ("definability") we further develop a well established branch of mathematical logic called *definability theory*. The application of definability theory in relativity was already emphasized in the classic relativity book Reichenbach [218], this emphasis on definability has been growing ever since as illustrated e.g. by Friedman [91] or Rakić [215, §2.5, p.39].[17] The classical theory of definability is restricted (mainly) to one-sorted logic and the new things that can be defined are new relations (or functions) between old individuals. In §4.3 of the present work we extend the theory to many-sorted logic, and besides new relations (between "old" individuals) we also allow defining new *universes of individuals*, i.e. new sorts. We extend various ones of the central theorems of definability to the new situation, e.g. we have a new, extended version of Beth's theorem on the eliminability of implicit definitions (cf. Corollary 4.3.49, p.268). This part of the dissertation is strongly related to works of Shelah, Pillay, Hodges and others, cf. e.g. pp.245, 268 herein, § 12.5 in Hodges [130] and items (1)-(3) on p.169 in Hodges-Hodkinson-Macpherson [131].[18]

Definability theory usually involves the so-called Beth properties together with the Craig (interpolation) properties, the two of which are strongly "interwound". As usual, we refer to these two properties and their variants as "the definability properties" of logics. We indicate at the end of §4.3 and in §A.3[19] that the definability results reported in this work form only

[15] An example of what we are saying is Tarski's and his followers' (Tarski et al.'s for short) first-order-logic-based approach to geometry. They, too, begin with writing up axiom systems for geometry and proving so-called representation theorems (which prove that the axiom systems describe the mathematical structures that the authors wanted to study). This is what we called Step 2 above. Indeed, it is only *after* this Step 2 (and on the basis of Step 2) that the main bulk (the main results etc.) of the theory developed by Tarski et al. unfolds (or in other words, is developed). Further, this mathematical logic-based theory of geometry (initiated by Tarski et al.) is not finished or "closed down" even today; it is still under development; it continues to provide new insights into the original subject matter (and into related subjects).

Another example is provided by Tarski's theory of cylindric algebras. Tarski wrote up the axioms of this theory a long time ago, and then he proved a representation theorem, saying that locally finite cylindric algebras are exactly those structures which he originally wanted to axiomatize, cf. [120, Part I]. This part of the theory could be written up and fully proved in not more than 50 pages. However, the main bulk of the theory of cylindric algebras came into existence *after* these Step 2-type results were obtained, and in 1985 they already filled two volumes, which together make up almost 1000 pages (cf. [120, Parts I, II], [121]). Ever since then new results have been added to the theory of cylindric algebras leading to deeper and deeper understanding of the subject matter for which Tarski initiated the study. (Very roughly, this subject matter can be summarized as the development and understanding of the theories of quantifier logics in a structuralist (or algebraic) perspective.)

[16] For completeness, we note that the following works are also connected to the present one: [171], [15], [57], [66], [77], [92] [133], [270].

[17] The logical theory of definability (e.g. the notions of implicit and explicit definitions) can be traced back to Hilbert [126] (Hilbert calls implicit definitions "axioms", and explicit definitions "explanations"), while it was Tarski's 1936 paper which gave the field its first big impetus. (Indeed, definability remained one of Tarski's main interests all his life.)

[18] The fact that we allow definitions of new *sets of entities* (i.e. new universes or sorts) besides new relations between old entities renders generalization of definability theory to the new situation not easy, e.g. we need to use recent results of Shelah, Pillay, Hodges and others which are based more of less on a definability result of Michael Makkai and C. C. Chang, cf. [60, Thm.5.3.6], or Hodges [130, Thm.12.4.1].

[19] and at other parts of the "duality theory" section §4.5

a part of a broader perspective of definability investigations conducted by the present author. E.g. Tarski and his co-workers Henkin and Pigozzi introduced and started to study the schema version of first-order logic in the late 1950's (cf. e.g. Henkin-Tarski [123]) the definability properties of which remained open problems summarized in algebraic form in Pigozzi [212]. In papers related to the present work the present author answers all of these problems,[20] cf. e.g. Madarász [167], [163], [178]. For proving these results she elaborated a duality theory of the kind presented (for relativistic purposes) in §4.5 herein, cf. Madarász [164], [166], [170], [163].[21]

(ii) Using Tarski's *elimination of quantifiers* for real-closed fields, in Chapter 4 we prove certain properties of the relations that are definable in our relativistic models and in relativistic geometries (cf. Chapter 4, proof of Thm.4.2.23, pp.168-174).[22]

(iii) Roughly, there are two main approaches to relativity. The first one uses observation-oriented models based on observers, clocks, coordinate frames, photons etc. This approach is preferred e.g. by the logical positivists, cf. e.g. Reichenbach [218]. For brevity, we will call this approach "observational", and its models $\mathfrak{M}$ observational models. The other approach starts out at the other side of the observational/theoretical duality:[23] it defines abstract, streamlined mathematical objects called geometries, and interprets the ideas of relativity in terms of these geometries. We use the German letter $\mathfrak{G}$ to denote such a geometry. We use our new definability theory mentioned in item (i) above to prove that these two worlds of relativity theory are *definitionally equivalent*. I.e. we prove that the world of observational models, the $\mathfrak{M}$'s, and the world of geometries, the $\mathfrak{G}$'s, are definitionally equivalent, cf. Thm.4.3.38, p.261. Actually, we do much more than this: Instead of a single relativity theory, in this work we investigate a hierarchy of progressively weaker relativity theories. The word "weaker" can be interpreted here as "more general" or "more flexible", too. In Thm.4.3.38 (p.261) we prove that under some mild assumptions on our relativity theory Th, the class $\mathsf{Mod}(Th)$ of observational models associated to Th is definitionally equivalent with the class $\mathsf{Ge}(Th)$ of geometries associated to Th.[24] In our definability section §4.3 we show that definitional equivalence is an extremely strong kind of equivalence which means that under the conditions of the above theorem, $\mathsf{Mod}(Th)$ is the same thing as $\mathsf{Ge}(Th)$, the only difference between the two being merely notational. This also implies that the two approaches to relativity quoted from the literature are not so incompatible as one might think, one can switch from one to the other and back without any danger of loss of mathematical precision.

We also established a strong connection between the two worlds $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ of relativity, which does not require the conditions of Thm.4.3.38. This is done in the duality theory section (§4.5) where we elaborate *strong duality theories* between the observational world of relativity and the theoretical world of relativity. This is done e.g. in Thm.4.5.57 (p.328), items A.2.9-A.2.12 (p.A-13), Thm.4.5.43, p.315. One way of summarizing (some of) our duality results is the following: They establish the existence of strong category theoretic adjoint functor pairs acting between the observational world $\mathsf{Mod}(Th)$ and the geometrical (or theoretical) world $\mathsf{Ge}(Th)$, under practically no condition on Th. Here the worlds $\mathsf{Mod}(Th)$

[20] we mean the ones which were not solved earlier

[21] Some of the problems we have in mind were formulated by Tarski (and his co-authors) in the 1950's, were studied in the meantime by several authors, e.g. by Comer, Pigozzi, Andréka, Sain, Németi, and were finally solved in Madarász [167].

[22] Or equivalently, we prove that certain relations cannot be defined.

[23] For the observational/theoretical distinction cf. the introduction of Chapter 4 here, and §1.1 in AMN [18].

[24] For motivation to studying $\mathsf{Mod}(Th), \mathsf{Ge}(Th)$ and their interconnections we refer the reader to the introduction of Chapter 4.

and $\mathsf{Ge}(Th)$ are regarded as categories. If we add conditions on Th, the adjoint situation becomes nicer and nicer until it becomes an equivalence of these categories. Actually, we present two different kinds of dualities (between "observational" and "theoretical"), these are the $(\mathcal{G}, \mathcal{M})$ duality of Thm.4.5.11, while the other one is the $(\mathcal{G}o, \mathcal{M}o)$ duality of items A.1.10-A.1.11, A.2.9(ii). Each of these has its advantages. E.g. $(\mathcal{G}o, \mathcal{M}o)$ abstracts from the units of measurement and in "return" has a nicer mathematical theory. Cf. also Appendix A on farther reaching implications (and connections) of this "duality"-approach.

(iv) Usually, relativity theory is derived from various "postulates" formulated in natural language and from tacit assumptions which are not stated but only used. This gives the theory a kind of pseudo-axiomatic flavour. Among other things, we replace these postulates and tacit assumptions by precisely formulated axioms in the language of first-order logic.[25] One of the usual postulates is that the world-view transformations between (world-views of) observers are collineations.[26] In Thm.3.2.6 we prove that this postulate (or axiom) is superfluous because it is provable from the rest of the usual assumptions (which are of a much more basic and natural character). Moreover, we prove a statement stronger than this, namely, we prove that this *collineation property follows* from a very little fraction **Pax** of the tacit assumptions which are taken for granted in every presentation of every version of special relativity. This very weak system of axioms **Pax** is formalized on p.109,[27] and Thm.3.2.6 says **Pax** $\vdash$ *"the world-view transformations are collineations"*. The point in this theorem is not the strength of its conclusion, but the weakness of its assumption, **Pax**.

A further axiom of relativity, more often debated and criticized, is the *no FTL axiom* which says that observers cannot move faster than light (relative to each other). The usual justification of this axiom is highly speculative and has been criticized by leading scientists, like Kurt Gödel or David Lewis [157]. Cf. also Gödel [99],[100]. The axiom itself has been much debated e.g. by the so-called tachyon theorists (cf. e.g. Davies [69], Feinberg [85], Mocanu [193], [220], [219], Gott [106] or Andai [5]), and also by recent papers based on new solutions of Maxwell's equations, cf. e.g. Matolcsi-Rodrigues [188], Donelly-Ziolkowski [74], [43], [224].[28] At the same time, such extremely careful works as Reichenbach's book [218] on axiomatization of relativity assume this statement as an axiom. In Chapter 3, we *prove* that the *no FTL* observer axiom is superfluous in the sense that it is provable from a very small fragment $\mathbf{Bax}^{-\oplus}$ of the remaining axioms which in turn are always assumed in the literature. $\mathbf{Bax}^{-\oplus}$ consists of **Pax** together with some extremely mild assumptions on photons. In Thm.3.2.13 (p.118) we prove

$\mathbf{Bax}^{-\oplus} \vdash$ *"no FTL observers exist"*.

Again, the point is that our assumption part $\mathbf{Bax}^{-\oplus}$ is very weak and intuitively convincing. Moving in the direction of general relativity, we also introduce the "local" version $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$ of $(\mathbf{Bax}^{-\oplus})$. This local version is even more general, i.e. weaker than $\mathbf{Bax}^{-\oplus}$ in that, as in general relativity, we make assumptions about observers etc. only "locally" (in some sense). In Thm.3.2.15 we push through our no FTL theorem for this local theory $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$, too.[29]

[25] Then we analyze the so obtained formal theory of first-order logic in various directions, in various ways, asking various kinds of questions etc, cf. the introductions of Chapter 3 and Chapter 4 for some of these research directions.

[26] Cf. e.g. Einstein [80, Appendix for §1.1 (p.125 in the Hungarian translation)], Nagy [200, p.233, lines 28-30], or Friedman [91, p.139 lines 1-3]. Usually, the tacit justification for this postulate is that it is easy to work with linear transformations and that this postulate does not lead to contradictions.

[27] **Pax** is already assumed in the Newtonian and Galilean theories of motion (besides the modern ones).

[28] Cf. also the notes and footnotes in §2.7 ("FTL in two dimensions"), pp.70-73.

[29] For this theorem we had to localize the conclusion (no FTL), too, since the existence of certain cosmological models shows that in general relativity the global form of no FTL fails.

In Thm.3.2.15 we establish, under some mild assumptions, that[30]

$\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) \vdash$ *"no FTL observers exist"*.

We also prove that within the non-local paradigm, Thm.3.2.13 is "best possible" in the sense that if we weaken $\mathbf{Bax}^{-\oplus}$, but stay in our hierarchy of non-local theories, the conclusion does not follow, cf. Thm.3.2.14; this negative result is improved in AMN [18, §4.8], e.g. in Thm.4.8.12 therein (cf. (E6) on p.641 there).

(v) As mentioned earlier, several of our studies are motivated by preparing the road for generalizing towards accelerated observers[31] and eventually general relativity. Two examples are the operator **Loc**(−) in Chapter 3 (fully reported on in §4.9 of AMN [18]); and §4.7 herein devoted to geodesics. The results in §4.7 show that our first-order logic based approach is suitable for studying *geodesics*. Among other things, we exhibit a strange property of geodesics which shows up even in the most classical (relativistic) situations. Namely, if we use (one of) the usual (mathematically precise) definition(s) of geodesics (cf. e.g. Busemann [55] or the book "Geometry of geodesics" [54]), in the Robb planes all sorts of strange curves will count as geodesics. Actually, all curves in any Robb plane count as geodesics. Cf. Thm.4.7.12, Corollary 4.7.13 (pp.361-363). Then we suggest an extra condition (Condition ($\ast\ast\ast$) on p.352) which, when added to the definition of geodesics, removes this anomaly (but does not remove the curves that are desirable as geodesics).

(vi) The approach elaborated in the present dissertation makes it possible to have a precise mathematical comparison of the three competing relativity theories, the Einsteinean one, the Reichenbachian one, and the Lorentzian one, cf. Szabó [243] for these and the debates they have provoked. The idea is outlined in Chapter 3 and is fully presented in AMN [18, §4.5]. There a completely transparent mathematical connection is elaborated between the first two of these theories, so that they need not compete: they can cooperate. E.g. a so-called transfer principle[32] is elaborated there by which each model of the Reichenbachian theory can be obtained from an Einsteinean model by an algebraic operation called relativization. In the other direction, all Einsteinean models are also Reichenbachian. The results in AMN [18, Chapter 4] give us practically full mathematical control over the connections between the first two of these three distinguished approaches to relativity.[33] We plan to extend this investigation to the Lorentzian version of relativity, too.

(vii) Throughout the dissertation we tried to keep our assumptions (explicit and tacit alike) at the barest minimum for proving (hopefully) strong and interesting results. We feel that at most places we managed to prove from a small number of transparent and clear assumptions interesting and exotic predictions of relativity.[34] This effort is also relevant to the goal of

[30] An interesting feature of Thm.3.2.15 is that (roughly, it is even stronger than Thm.3.2.13 and) because of the local nature of our assumptions, there seems to be no chance [not even a remote one] of trying to obtain an Alexandrov-Zeeman type of alternative proof for it. (Cf. AMN [18, Remark 3.1.14] for Alexandrov-Zeeman type proofs in this context; cf. also AMN [18, §6.7.2(I) on p.1156].

[31] Cf. AMN et al. [25], [26].

[32] transfer principles are special tools of mathematical logic

[33] In algebraic logic, there is an especially successful method called relativization which goes back to early work of Henkin and Tarski. We found two natural counterparts of this method for studying relativity. These counterparts are connected to the operators **Reich**(−) and **Loc**(−) introduced in Chapter 3 and further elaborated in AMN [18, §§4.5, 4.9]. Cf. Andréka-van Benthem-Németi [32], Andréka-Goldblatt-Németi [11], Németi [203], Madarász [169]. This fact illustrates a further connection between the present work and Madarász [160], [161], [164], [166], [176], [23], [20].

[34] We did not even use the assumption that the field of quantities we use is Archimedean or even Euclidean.

answering the "why"-type questions summarized in AMN [18, §1.1 items (I), (III), (V), (VI), (IX) therein].

(viii) One can test the power of the methodology for logical analysis of relativity by trying to derive the main predictions of relativity from as little as possible. Some motivation for doing this was already mentioned above. Below we summarize a similar experiment in a different direction. Following suggestions from Gyula Dávid[35], the present author used the methods elaborated here to *derive by purely logical methods* the predictions of *special relativity from* fairly natural *assumptions not involving photons* or anything remotely related to electrodynamics. It is important here to emphasize that the logical language we used did not allow mentioning even implicitly anything related to electrodynamics (such as e.g. photons). Another thing to emphasize is that the number of axioms was small and they were natural, transparent and convincing axioms. The set of axioms used can be found herein in the List of Axioms under the name **Relnoph** (relativity with no photons) on p.A-30. Further, the derivation used only pure logic and nothing like, say, "physical intuition" or common sense. The final result of the derivation said, roughly, that the world is either Newtonian, or it satisfies special relativity. The details are in AMN [18, Chapter 5] and special thanks are due to Gyula Dávid for ideas, suggestions and encouragement.[36] His version of a similar paper is in preparation, cf. Dávid [68]. A related idea is in the paper Gnädig et al. [49] which avoids photons just as Dávid and we did but does not avoid electrons or the fact that two wires carrying electric currents attract each other (if the currents are parallel).

(ix) Recently there has been an extensive debate, in the literature of relativity theory and related areas, concerning the connections between relativity (and its possible variants) and *Gödel's incompleteness* theorems.[37] These debates were triggered by the programme of searching for a "final theory" (or sometimes T.O.E.) proposed by Hawking, Weinberg and others. Cf. e.g. Hájek [112, p.291], Stöltzner [240], Dyson [75, p.53], Regge [216, p.296] for criticism using Gödel's incompleteness theorem as a "weapon" against the "final theory". We investigate the issue and answer some questions in AMN [18, §3.8 (pp.294-346)], §4.5.5 herein, in [16], in [17], in Chapter 7 of [19]. Cf. also the "laws of nature" part of Chapter 6 in AMN [18].

On the methods of the dissertation. The theory of definability is one of the unifying themes of the publications quoted herein by the author, including her first refereed publication [161] and the present work (It is only fitting to note here that the theory of definability was initiated by Hans Reichenbach in 1920-21 when writing one of the first books on relativity [218].) For more on the methods of the dissertation (and related results of the author in refereed journals) we refer to e.g. Appendix A.

1.4 Outline of the dissertation and on some of our aims

As already indicated, all our axioms will be formulas of first-order logic. We do not want to make our axioms generate a complete theory.[38] Our purpose is the opposite: we want to make

[35] Department of Physics, ELTE University, Budapest

[36] At the end of AMN [18, §5.2 (p.751)] we trace the history of the basic idea back for a considerable time, but none of these historical papers were free of tacit assumptions and they did not attempt a purely logical derivation.

[37] There are several of these varying in strength. Therefore there are several Gödel incompleteness properties of theories.

[38] A theory T is called *complete* iff for every sentence φ in the language of T, exactly one of φ and its negation $(\neg\varphi)$ follows from T.

our axioms as weak (and intuitively acceptable and convincing) as possible while still strong enough to prove interesting theorems of relativity theory.[39]

When introducing a new axiom, say **Ax**, we will investigate why **Ax** is plausible, why we (or a student) should believe in **Ax**, why we need it, and what would happen if we omitted it. This way we will obtain a relatively small set, called **Basax** (for basic axioms), of convincing (almost self-evident) axioms. **Basax** will be our *first* "possible" axiom system. Later, as a result of studying **Basax**, we will introduce and study a *hierarchy* of axiom systems (or of possible special relativity theories) in which hierarchy **Basax** will be neither the strongest nor the weakest theory. In AMN [18], the present author investigates how many different complete theories $Th \supseteq$ **Basax** exist, which are possible consistent extensions of **Basax**.[40]

Chapter 2 begins with introducing the logical framework (e.g. langauge) for the whole of this work. Different logical "vocabularies" will appear only in Chapter 4 (where refinements of the idea of a logical framework for relativity will also be discussed, cf. e.g. Remark 4.7.3). Then we introduce and discuss **Basax**, our basic axiom system for special relativity theory. We also study it there to some extent, e.g. we prove that **Basax** is consistent, that in dimension two it permits faster than light (FTL) observers, which in turn leads to time-travel-like phenomena, and that the latter does *not* lead to logical paradoxes i.e. "**Basax** + there are FTL observers" is consistent (in dimension 2). In this chapter we also prove from **Basax** what we call the "paradigmatic effects" of relativity: moving clocks slow down, moving clocks get out of synchronism, and moving meter-rods shrink. In section 2.8 we experiment with adding a strong symmetry axiom, **Ax(symm)**, to **Basax**. Roughly, this symmetry axiom can be considered as an "instance" of Einstein's (special) principle of relativity, cf. §2.8.3 (p.84). We will find that adding a very natural and transparent axiom to **Basax** yields a theory which completely reproduces the usual, standard version of special relativity (i.e. the one based on Minkowskian geometry).[41] Actually, this extra axiom will be a version of **Ax(symm)**. We call this extended theory **Specrel**.

In **Chapter 3**, we elaborate the "lego-toy-world" character (or modular character) of our logic based approach to relativity. We start out from the theory **Specrel**, and decompose it into a hierarchy of more refined, flexible and general theories. Since **Specrel** has only a small number of axioms, we can get a really big and really fine-tunable hierarchy of theories only if we break up the axioms of **Specrel** to a great number of weaker, more refined new axioms. Some of the so obtained theories are so general that they are "local" in the sense that observers use only a subset of our usual Cartesian coordinate system $\mathfrak{R}^4$ instead of the whole of $\mathfrak{R}^4$. This is a generalization which will be useful when moving towards theories of accelerated observers or eventually towards general relativity. Then we use this hierarchy to get a handle on the why-type questions, namely, we take interesting predictions of relativity and try to

[39] The situation is somewhat analogous with the difference between classical number theory studying the standard model $\underline{Z} = \langle Z, +, \cdot, -, 0, 1\rangle$ consisting of the set Z of integers, and theory of numbers as a part of abstract algebra, e.g. ring theory (or the theory of fields) where we study a broad class K of all rings of which $\underline{Z}$ is only a very special element. Sometimes when we prove theorems about K, we say (or feel) that we understand more (or better) why that theorem is true for $\underline{Z}$. In this analogy, classical, standard special relativity is analogous with the complete theory of $\underline{Z}$ while the version we are describing here is analogous with the algebraic theory of K. (We note, however, that this analogy is imperfect, as if often happens with analogies.)

[40] We also give a structural description of the essentially different kinds of models of **Basax**, in AMN [18, Chap.3].

[41] The connections between standard Minkowskian geometry and our more flexible (or more general) versions of relativity are discussed in greater detail in Chapter 4, but cf. also AMN [18, §3.9 ("Symmetry axioms")].

isolate the weakest theory in the lattice from which the prediction in question is provable. An example is the "no faster than light observers" theorem (no FTL observers for short) for which the weakest theory is identified in Theorems 3.2.13-3.2.15. This kind of investigation, including a pursuit of why-type questions, is pushed much further in AMN [18, Chap.s 4,5], Madarász [168], [172], [173], [174]. More generally, the subject matter of Chapter 3 is pursued in greater detail (with more results, more historical motivation etc.) in AMN [18, Chap's 4,5]. At the end of Chapter 3 we arrange the theories in our above mentioned hierarchy into a fairly natural lattice. We refer the reader to the introduction of Chapter 3 for more intuitive motivation.

In **Chapter 4**, we "discover" that there is an "observer independent" geometry sitting inside each model $\mathfrak{M}$ of, say, **Bax**; where **Bax** is one of the weak theories introduced in Chapter 3. (In particular, **Basax** $\models$ **Bax**.) If $\mathfrak{M}$ is a model of **Specrel** mentioned above, this geometry agrees with the standard Minkowskian geometry. Further, we elaborate a so-called duality theory acting between the "world of certain kinds of geometries" on the one part, and the world of our observation-oriented models $\mathfrak{M}$ on the other. We could call this duality a duality between the so-called "observational worlds" (the $\mathfrak{M}$'s) and the "theoretical worlds" (the geometries). This duality theory works not only for the "simplest" theory **Basax** we started with, but also for practically all the (more general and more refined) theories introduced in Chapters 3,4. Extensive intuitive motivation for Chapter 4 is presented in its introduction §4.1. This motivation explains and uses ideas of Einstein, Gödel, Mach, Reichenbach and sketches a historical background ranging from William Occam (14th century) through Leibniz, Kant and the logical positivists to followers of Gödel and others. These ideas together with the above mentioned duality theory lead to a series of questions in definability theory (a branch of mathematical logic). Therefore a major section, §4.3, of Chapter 4 is devoted to new results in definability theory, making the advancement of this branch of logic to be one of the major themes of Chapter 4. The results are then applied to relativity. E.g. we prove that all our "theoretical" concepts are definable from "observational" ones. Further, we prove that the above mentioned observational world (the $\mathfrak{M}$'s) and theoretical world (the $\mathfrak{G}$'s) are definitionally equivalent, under some assumptions. We close Chapter 4 with defining and investigating geodesics in our new, logic based framework. (Geodesics play a role in further, more general theories.) For the outline of Chapter 4 we refer the reader to §4.1 (p.129).

At the end of this work there are various appendices, lists (of e.g. axioms, definitions) and similar items designed to assist the reader in various ways.

The introduction §4.1 of Chapter 4 contains extensive material[42] which is useful not only as an introduction to Chapter 4 but also as an introduction to the whole dissertation. Therefore we think it would be useful for the reader to skim through §4.1 before starting to read Chapter 2.

Map of this work to save time

The directed graph in Figure 2 intends to assist the reader in several ways:

(i) It helps to save time in two ways, cf. the "Explanation for Figure 2".

(ii) It provides a "weighted map"[43] of this work helping the reader to decide which parts he wants to read first.

[42] ranging from intuitive motivation through historical background to related work

[43] by "weighted" we mean the information on importance given by thickness of lines

(iii) It represents the structure of this work. This might be useful e.g. in forming a "general perspective" after a first reading.

Explanation for Figure 2, interconnections of sections. The thick arrows on the figure represent a minimalist first reading of the "backbone" of this work. We prepared this for the case the reader would prefer first to leaf through the "core" part of this work in order to form a general impression, before reading the whole work. An even more minimalist first reading would involve §§2.1–2.4, 3, 4.1, 4.2.1, first 5 pages of §4.2.3, §4.3, the introduction to §4.5, §§4.5.1–4.5.4, the Gödel incompleteness part of §4.5.5, §4.7. Taking into account that §§2.1–2.9 are regarded as an appendix (cf. p.v), the remaining part of this "more minimalist reading" involves approx. 225 pages.

Assume "a", "b" are (sub-)section names. Then an arrow $a \to b$ means that reading "a" is a prerequisite for "b". A broken arrow between "a" and "b" means that leafing through the main definitions and main ideas in "a" is desirable before reading "b". Further, sub-sections connected by thick arrows are usually important; if "a" is boxed, then "a" is important, if "a" is in a broken box, then "a" is fairly important. If "a" is in a thick box, then "a" is very important.

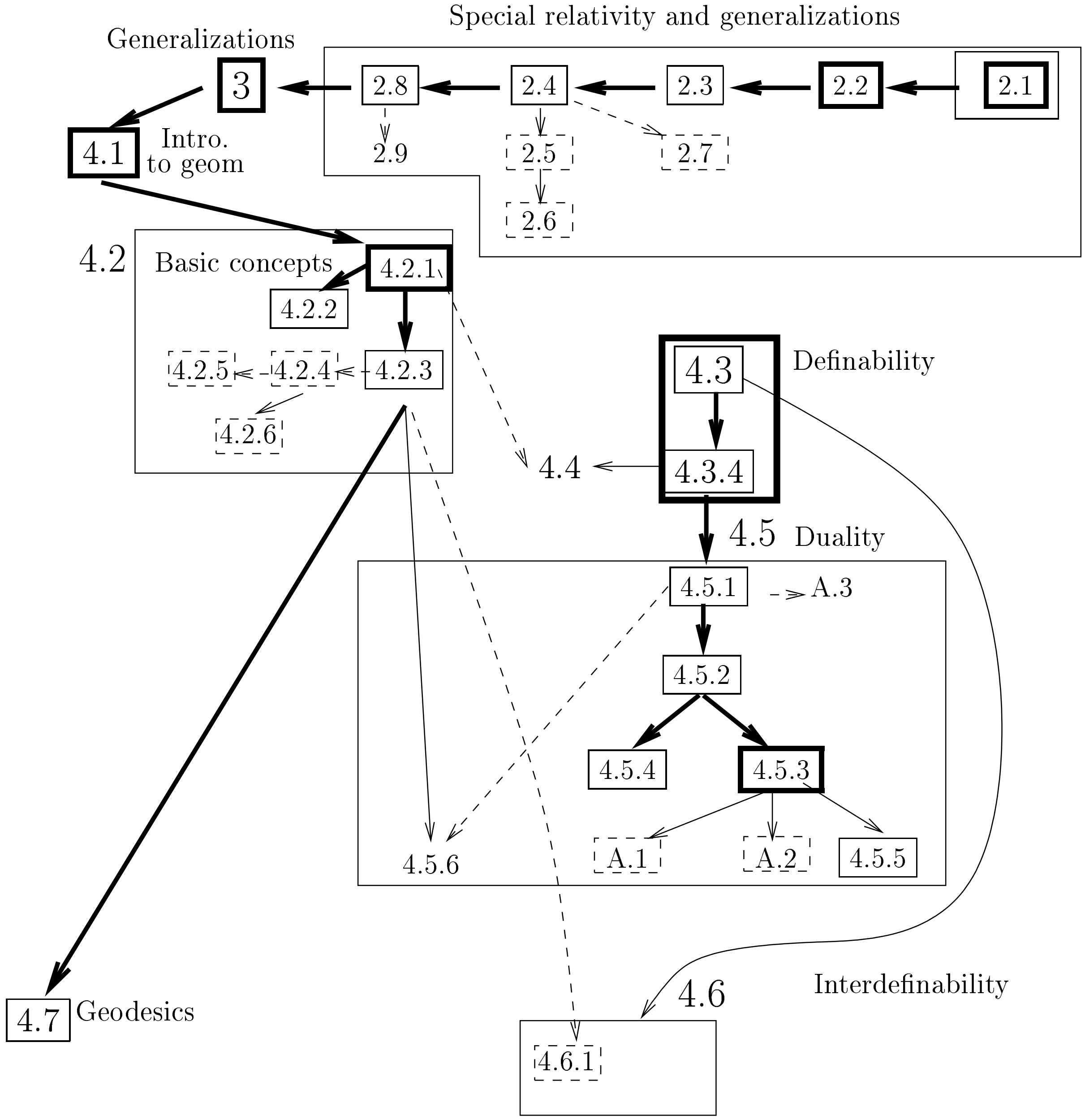


Figure 2: a→b means that reading (sub-)section "a" is a prerequisite for "b". Further, the dashed (broken) arrows a→b mean that leafing through the definitions and main ideas in (sub-)section "a" is desirable before reading "b".

2 Special Relativity (background)

Since the present chapter (Chapter 2) is of an appendix character, in this chapter we omit almost all the proofs. These can be found in our background book AMN [18]. Some proofs are omitted from AMN [18]; these are available from Judit Madarász.

2.1 Frame language of relativity theory; world-view function

Some set theoretical notation and convention:

ω denotes the set of all *natural numbers* $\{0, 1, \dots, n, \dots\}$. We use von Neumann's concept of natural numbers, that is,

$0 \stackrel{\text{def}}{=} \emptyset$ ($\emptyset$ denotes the empty set) and

$n+1 \stackrel{\text{def}}{=} n \cup \{n\} = \{0, \dots, n\}$ for every $n \in \omega$. Therefore, in this spirit we will often write $i \in n$ for $i < n$, where $i, n \in \omega$.

R denotes the set of all *real numbers*, and

$\mathfrak{R} = \langle \mathrm{R}, +, \cdot, \leq \rangle$ denotes the *ordered field of real numbers* (where $+, \cdot, \leq$ are the usual ones).

Z denotes the set of all *integers*.

For any set H, $\mathcal{P}(H)$ denotes the *powerset* of H, that is,

$\mathcal{P}(H) = \{ X : X \subseteq H \}$.

If R is a binary relation, i.e. set of (ordered) pairs, $Dom(R)$ and $Rng(R)$ denote its *domain* and *range*, respectively. That is:

$Dom(R) \stackrel{\text{def}}{=} \{ a : \exists b \, \langle a, b \rangle \in R \}$ and

$Rng(R) \stackrel{\text{def}}{=} \{ b : \exists a \, \langle a, b \rangle \in R \}$.

A function is a binary relation f with the property that for each $x \in Dom(f)$ there is only one y such that $\langle x, y \rangle \in f$. As usual, $f(x)$ denotes this unique y.

$f : A \longrightarrow B$ or $A \xrightarrow{f} B$ denote that f is a function, $Dom(f) = A$ and $Rng(f) \subseteq B$.

For an arbitrary set H and $n \in \omega$, we often identify the set

${}^nH \stackrel{\text{def}}{=} \{ f : (f : n \longrightarrow H) \}$ with the Cartesian power

$\underbrace{H \times \ldots \times H}_{n\text{-times}} \stackrel{\text{def}}{=} \{ \langle h_0, \dots, h_{n-1} \rangle : (\forall i < n) h_i \in H \}$. Thus, in particular,

${}^2H = H \times H$.

If R and S are binary relations, their composition $R \circ S$ is defined as

$R \circ S \stackrel{\text{def}}{=} \{ \langle a, b \rangle : (\exists c)[\langle a, c \rangle \in R \ \wedge \ \langle c, b \rangle \in S] \}$.

Therefore, in particular if f and g are functions with $Rng(f) \subseteq Dom(g)$, we write their *composition* the following way[44]:

$(f \circ g)(x) \stackrel{\text{def}}{=} g\Big(f(x)\Big)$ for every $x \in Dom(f)$.

For a binary relation R and a set X, the R-image $R[X]$ of X is defined as

$R[X] \stackrel{\text{def}}{=} \{ b : (\exists a \in X) \, \langle a, b \rangle \in R \}$.

Therefore in particular for a function f,

$f[X] = \{ f(x) : x \in Dom(f) \cap X \}$.

For a binary relation R, its inverse is

---

[44] This is usually used in the reverse order in the literature.

$R^{-1} \stackrel{\text{def}}{=} \{\, \langle b, a\rangle \ : \ \langle a, b\rangle \in R \,\}$.

$\mathrm{Id}_A \stackrel{\text{def}}{=} \{\, \langle x, x\rangle \ : \ x \in A \,\}$ is the *identity function* on A, for any set A. When A is understood from the context we will write Id in place of Id_A.
The following is a notation for defining functions. Let $expr(x)$ be an expression involving x, and let D be a set. Then

$\langle\, expr(x) \ : \ x \in D \,\rangle \stackrel{\text{def}}{=} \{\, \langle x, expr(x)\rangle \ : \ x \in D \,\}$.

$f \restriction C \stackrel{\text{def}}{=} \{\, \langle x, y\rangle \in f \ : \ x \in C \,\}$ is the restriction of the function f to the set C, for any function f and set C.

$A \setminus B \stackrel{\text{def}}{=} \{a \in A : a \notin B\}$.

$\lhd$

CONVENTION 2.1.1 $f : A \twoheadrightarrow B$ denotes that f is a surjective function from A onto B. Further $f : A \rightarrowtail B$ denotes that f is an injective function from A into B. I.e. $\twoheadrightarrow$ denotes surjectiveness, while $\rightarrowtail$ denotes injectiveness. (If we combine the two, we obtain $\rightarrowtail\!\!\!\twoheadrightarrow$ denoting bijectiveness.) When used between german letters, i.e. structures, they denote injectiveness or surjectiveness of homomorphisms the natural way. Cf. Def.4.5.3(i) on p.284. $\lhd$

Before giving the definition of our frame-language, we recall from [60] some of the standard notation and terminology used in (many-sorted) first-order logic.

By a *first-order language* we understand a language[45] of first-order logic. Similarly for *3-sorted first-order language* or *many-sorted first-order language*. We will often use the word *vocabulary* instead of a first-order language (to avoid ambiguity arising from the fact that "language" could also refer to the set of all formulas of some theory). A vocabulary is a collection of sort-symbols, relation-symbols and function-symbols.

In the present work we will use *many-sorted first-order logic*. We hope that the reader having some familiarity with one-sorted first-order logic will find the transition from one-sorted to many-sorted easy to make. Indeed, throughout the literature it is emphasized that many-sorted (first-order) logic is only a convenient "notational dialect" of one-sorted first-order logic and that anyone familiar with the one-sorted version will easily understand the many-sorted version without studying it separately.

By *many-sorted logic* we understand the many-sorted version of first-order logic. I.e. for brevity, we will omit the adjective "first-order" (so in this work many-sorted automatically implies first-order). Many-sorted logic is so close to one-sorted first-order logic, that most logic books study and discuss the one-sorted case first and then they formulate the generalization to the many-sorted case as an *exercise* left to the reader. Of course, for this exercise they explain how many-sorted logic can be reduced to the one-sorted case. (The fact that here we allow finitely many sorts only makes this reduction easier and "stronger", cf. footnote 55 on p.6.)

For an introduction to many-sorted logic and for its reduction to one-sorted first-order logic we refer to almost any logic book, e.g. to Enderton [82, §4.3, pp.277-281 (but the whole of §4 will be useful later)] or Manzano [181] or Monk [194]. We note that the whole book Meinke-Tucker [190] is devoted to many-sorted logic and its connections with higher-order logic. For completeness, we note that further useful information on this subject is available in the book

[45] Let us recall from the literature of logic that a *language of (many-sorted) first-order logic* or a "*vocabulary*" or a "*similarity type*" are different names for the same thing. The details can be found in any logic book e.g. in Monk [194, p.14] or Enderton [82]. Cf. §4.3 for more information on this.

Barwise & Feferman [45] on pp.25-27, pp.33-34, and item 7.1.2 (p.68).[46] We would like to reassure the reader that for understanding the present work, looking into [45] is not a prerequisite. (At a second reading of the present work, the just quoted parts of [45] might improve appreciation of certain "fine details".) Looking into [45] might also help seeing the connections of our approach with second-order logic. At this point we would like to emphasize that throughout the present work we are staying *strictly within first-order* logic. More precisely, all our relativity theories, e.g. **Basax**, **Newbasax**, ..., **Loc**(**Flxbasax**), **Reich**(**Bax**) will be strictly first-order ones. Further, the logical, model theoretic machinery, like the semantic consequence relation "$\models$" etc, applied to them will also be that of first-order logic.[47] At the same time, when we formulate a theorem (about these axiom systems or their models etc), then the statement of the theorem need not be translatable to a formula of FOL. (E.g. the well known theorem about FOL stating that elementarily equivalent models have isomorphic ultrapowers is not a first-order formula. Similar examples are theorems involving the cardinalities of models of first-order theories. With these examples we wanted to illustrate the fact that when studying FOL, we need not confine ourselves to expressing our ideas in the language of FOL.)

Let Fm and M denote, respectively, the set of all formulas and the class of all models of an arbitrary first-order language.

Then $\models$ $(\subseteq \mathsf{M} \times Fm)$ denotes the *validity relation* of this language. We extend $\models$ to $\mathcal{P}(\mathsf{M}) \times \mathcal{P}(Fm)$ the usual way: Let $\mathsf{K} \subseteq \mathsf{M}$ and $\Sigma \subseteq Fm$. Then

$$\mathsf{K} \models \Sigma \quad \text{iff} \quad (\forall \mathfrak{M} \in \mathsf{K})(\forall \varphi \in \Sigma)\mathfrak{M} \models \varphi\,.$$

We will write $\mathsf{K} \models \varphi$ in place of $\mathsf{K} \models \{\varphi\}$ and $\mathfrak{M} \models \Sigma$ when $\mathsf{K} = \{\mathfrak{M}\}$.

$$\mathsf{Th}(\mathsf{K}) \stackrel{\text{def}}{=} \{\, \varphi \in Fm \;:\; \mathsf{K} \models \varphi \,\}$$

is the *theory* of K, and

$$\mathsf{Mod}(\Sigma) \stackrel{\text{def}}{=} \{\, \mathfrak{M} \in \mathsf{M} \;:\; \mathfrak{M} \models \Sigma \,\}$$

is the *class of all models* of Σ. Let $\varphi \in Fm$. Then we say that φ is a *semantical consequence* of Σ, in symbols $\Sigma \models \varphi$, iff $\mathsf{Mod}(\Sigma) \models \varphi$.

$$\mathsf{Th}(\mathfrak{M}) \stackrel{\text{def}}{=} \mathsf{Th}(\{\mathfrak{M}\})$$

is the (first-order) theory of the model $\mathfrak{M}$.

We will start our formal exposition of relativity theory with fixing a 3-sorted first-order language. We will call this language the *frame-language of relativity theory*.[48] We will use this language for formulating our first axiom systems for special relativity (this way producing our first formalized versions of the theory).[49]

[46] For that book page numbers are important because it has no index and is 893 pages long.

[47] This is important because the issue of our staying within first-order logic [or in one of its equivalent forms] is an important one from our methodological point of view. For reasons see the Appendix of AMN [18] entitled "Why FOL?". Cf. Enderton [82, §4] or Manzano [181] for reducing higher-order logic to *first-order* many-sorted one.

[48] Later we will expand our frame-language with e.g. a kind of *pseudo-metric* $d : {}^nF \times {}^nF \longrightarrow F$, also called *distance*, see §8.1 of a future edition [19] of AMN [18]. Our choice of language is re-considered in AMN [18, §6.9 ("On what we learned (so far) about choosing our first-order language for relativity")], but cf. also Remark 4.7.3 on p.354.

[49] Because of the purposes explained in the Introductions §1.4 (p.xiv) and §3.1 (p.105), in later chapters we develop several axiom systems.

Before introducing the formal language, we explain our intuition behind the symbols of our frame-language.

* * *

So let's get started. We want to develop a kinematics.[50]

– What is kinematics?
– A theory of *motion*.
– What moves?
– Idealization: We assume that there are things called *bodies* (like "heavenly bodies") and they move.
– How do bodies move?
– Idealization: They change their (spatial) locations.
– What does change of location mean?
– At different *time instances* the same body has different *locations*.

OK, then there are time instances and locations involved (whatever they are). Let us fix that. Our paradigm says that time instances and locations are only relative to something which we will call *observers*.[51] So we assume that there are observers (special bodies). Given an observer m, time instance t and location s, observer m may "observe" a certain body b as being present at $\langle t, s\rangle$ while m may observe other bodies b_1 as not being present at $\langle t, s\rangle$. This simply means that from the point of view of m, b is present at location s at time t. We treat this concept of observing as a primitive and denote it as $b \in w_m(t, s)$. That is, $w_m(t, s)$ is defined to be the set of bodies present at $\langle t, s\rangle$ from the point of view of m. We should emphasize that this kind of observing has (almost) nothing to do with the intuitive notion of observing in the form of, say, seeing optically.

– What are time instances t and locations s?
– Our first answer is that they are "labels" used by observers. But sooner or later we will have to be more specific. So let us see what t is.

We agree that, for an observer m, time instances are "quantities", like 100, 500 or 1/2. To be faithful to the spirit of the axiomatic method, we *do not decide* what quantities are, we only postulate that they satisfy some simple axioms which in themselves are intuitively convincing. Namely, we assume that quantities form an ordered field $\mathfrak{F} = \langle F, +, \cdot, \leq\rangle$, that is, $\mathfrak{F}$ satisfies the usual axioms of ordered fields. The time scale of observer m is simply $\mathfrak{F}$ itself, the neutral element 0 of $\mathfrak{F}$ means "*now*", $t > 0$ represents "*future*" and $t < 0$ represents "*past*". For simplicity, we agree that locations s are represented by triplets of quantities $s = \langle s_1, s_2, s_3\rangle \in {}^3F$.

So far, we agreed on representing locations by triplets of quantities, or by triplets of "coordinates" from the field $\mathfrak{F}$. It is pairs $p = \langle t, s\rangle$ of time instances and locations for which

[50] For simplicity, we concentrate on kinematics of relativity, but by the same methods one can extend the investigations to, say, mechanics. A motivation for sticking with kinematics is that by using *only* kinematics we can prove things which are usually proved by using notions, e.g. mass.

[51] We use the expression "*observer*" in the sense of the physics book d'Inverno [73, pp.17,21]. So, in our sense, an observer "*coordinatizes*" the set of events and as we will later (in Chapter 4) call it, an observer coordinatizes what will be called there "space-time". Other books (e.g. Hraskó [135, p.32], Landau-Lifsic [152], Misner-Thorne-Wheeler [192, p.327]) use the expression "*reference frame*" for what we call observer. Still other books use a more abstract notion of observer such that for them "reference frame" = "observer + coordinatization" becomes the case. For us, this is *only* a matter of choosing words, no issue of ideology is involved; and since we had to make a choice, we decided to follow d'Inverno's terminology where "observer" is basically the same as "reference frame". In passing we note that it is our impression that Einstein used the word "observer" in the same sense as d'Inverno does and we do, cf. [80, §9]. Cf. also Taylor-Wheeler [257, §I.4 (the definition of observers)]. Cf. also Remark 2.2.5 on p.25.

we say that a body b occurs there (at $\langle t, s\rangle$) for observer m. We call such pairs points of our *coordinate-system* ${}^4\mathfrak{F}$, which we also denote by ${}^4F$. Therefore points of our coordinate-system are of the form $p = \langle p_0, p_1, p_2, p_3\rangle \in {}^4F$. We call p_0 the *time coordinate* and $\langle p_1, p_2, p_3\rangle$ the *space coordinates* of p.

Although our coordinate-system is four-dimensional, many of the ideas (and proofs) can be illustrated already in two or three dimensions. We will try to keep our presentation as simple as possible. Therefore we will sometimes *pretend* that our coordinate-system is 2-dimensional *but* we will go up to 3 or 4 dimensions as soon as the higher dimensional case would behave differently.

As we said, to each point $p \in {}^4F$ of our coordinate-system, an observer m associates a set $w_m(p)$ of bodies which, for m, are present at point p. Therefore, to each observer m, we associate a so-called world-view function $w_m : {}^4F \longrightarrow \mathcal{P}(B)$ mapping our coordinate-system ${}^4F$ into the powerset $\mathcal{P}(B)$ of the set B of bodies. We call the elements of $\mathcal{P}(B)$ "*events*". Matolcsi [187] calls them occurrences. For us an event is nothing but information telling us which bodies are present and which are absent.[52] (This is why [187] calls them *occurrences*.) Therefore we can identify an event by a subset of B.

On terminology: Sometimes we might write sloppily *space-time* for our coordinate-system ${}^4\mathfrak{F}$. However we need to reserve the expression "space-time" for a similar but slightly different structure. Namely, in Chapter 4, we will use the word space-time for a structure whose *elements* are the *events* (roughly, the universe of this structure is $\mathcal{P}(B)$) and whose *structure* will be induced by that of ${}^4\mathfrak{F}$ via the world-view functions $w_m : {}^4F \longrightarrow \mathcal{P}(B)$ belonging to the observers. Cf. the geometry chapter 4. In the *simplest* cases of special relativity, space-time will be *isomorphic* with our coordinate-system ${}^4\mathfrak{F}$. However, in order to be prepared for generalizations coming in the more advanced chapters of the present work, we need to treat space-time[53] as a structure strictly *different* from ${}^4\mathfrak{F}$.

* * *

We are ready now to define formally our frame-language. In this chapter we introduce a relatively rich language because we want to use this language throughout the present work. At the beginning, and especially throughout Chapters 2,3, we could have used a much simpler language, e.g. the one introduced in [16]. More specifically, in Chapter 2 we will not really need G, E , Ib introduced in Def.2.1.2 below.

[52] Misner & Thorne & Wheeler [192, p.6] (cf. Figure 1.2 therein) uses basically the same notion of an event as we do. They also give there detailed intuitive motivation for this definition of an event. For completeness, we note the following. In AMN [18, §6.9 ("On what we learned (so far) about choosing our first-order language for relativity")] we arrive at a more abstract, more sophisticated notion of an "event" cf. item (402) on p.1210 and the explanation following it. The intuition behind that notion, however, is basically the same as the present one.

[53] Space-time will be a structure $\langle Mn, \dots\rangle$ with $Mn \subseteq \mathcal{P}(B)$ the set of "observable" events. (In this connection, we note that e.g. Friedman [91, p.32, lines 4–5] defines space-time as "the set of ... all actual and possible events".)

Definition 2.1.2 (frame-language of relativity theory)
Let B, Q and G denote three sorts called *bodies*, *quantities* and *lines*, respectively. Let a natural number $n > 1$ be fixed.[54] Intuitively, n will be the dimension of our "space-time".

We are defining a first-order language with sorts[55] B, Q, G by first defining its models, as follows. $\mathfrak{M}$ is a *model (of dimension n) of this language* iff

$$\mathfrak{M} = \langle B^{\mathfrak{M}}, F^{\mathfrak{M}}, G^{\mathfrak{M}};\ Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, Ib^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, \leq^{\mathfrak{M}}, \mathrm{E}^{\mathfrak{M}}, W^{\mathfrak{M}} \rangle ,$$

also denoted as

$$\mathfrak{M} = \langle B, F, G;\ Obs, Ph, Ib, +, \cdot, \leq, \mathrm{E}, W \rangle$$

for brevity[56], where:

- B is a nonempty set, it is $\mathfrak{M}$'s universe of sort B. B is called the set of *bodies* (of $\mathfrak{M}$).

- F is $\mathfrak{M}$'s universe of sort Q. Intuitively, F serves both to be our "*time scale*" and "*space scale*". Relations $+, \cdot, \leq$ are of sort Q, hence $\langle F, +, \cdot, \leq \rangle$ forms a structure. We will assume that $\mathfrak{F} := \langle F, +, \cdot, \leq \rangle$ is a linearly ordered field.[57] That is, the following set $\underline{\mathbf{Ax}_{\mathrm{OF}}}$ of axioms is satisfied by $\langle F, +, \cdot, \leq \rangle$.

  $\mathbf{F} := \langle F, +, \cdot \rangle$ is a field[58]
  $\langle F, \leq \rangle$ is a linear order, and for every $a, c \in F$,
  $a \leq c \ \Rightarrow\ (\forall d \in F)(a + d \leq c + d)$ and
  $(a \leq c \text{ and } d > 0) \ \Rightarrow\ (d \cdot a \leq d \cdot c)$ hold.

  0 and 1 denote the usual zero and unit elements of the field. Further, for every $a \in F$, $|a|$ denotes the absolute value of a, that is,
  $|a| \stackrel{\text{def}}{=} \max\{a, -a\}$ (where "$-$" is the usual group theoretic inverse operation determined by $+$).

  We will denote the ordered field $\langle F, +, \cdot, \leq \rangle$ by $\mathfrak{F}$ and its field reduct $\langle F, +, \cdot \rangle$ by $\mathbf{F}$. Often we write $\mathfrak{F}^{\mathfrak{M}}$ for $\mathfrak{F}$ ($\mathbf{F}^{\mathfrak{M}}$ for $\mathbf{F}$) when we want to indicate explicitly that we look at $\mathfrak{F}$ ($\mathbf{F}$) as the "quantity part" of $\mathfrak{M}$. $\mathfrak{F}^{\mathfrak{M}}$ is called the *ordered field reduct* of $\mathfrak{M}$, following the

[54] We will be interested only in the case $n \in \{2, 3, 4\}$, but we give definitions and lemmas for arbitrary n if this does not cost any extra effort.

[55] Many-sorted logic is known to be reducible to one-sorted logic the following way (cf. Monk [194], Enderton [82]): One uses the union $\mathrm{B} \cup \mathrm{Q} \cup \mathrm{G}$ of the universes of the sorts of the many-sorted model as the universe of our new one-sorted model and one calls B, Q, G unary predicates.

[56] As is usual in logic, B, F, G, Obs etc. are *symbols* (sort symbols and relation symbols) of the language of $\mathfrak{M}$ while $B^{\mathfrak{M}}, \dots, Obs^{\mathfrak{M}}$ etc. are *objects denoted* by these symbols according to the model $\mathfrak{M}$. *If* and where there is no danger of confusion, we will identify the symbols with the objects they denote (hence we write B for $B^{\mathfrak{M}}$ etc).

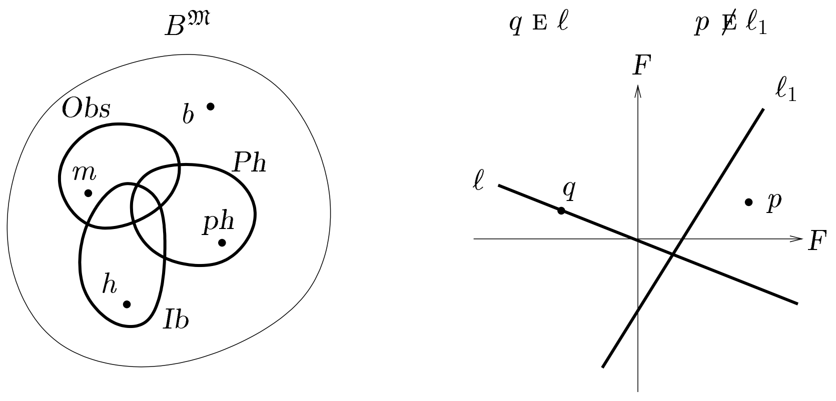
[57] This is why the universe of sort Q of $\mathfrak{M}$ is denoted by $F^{\mathfrak{M}}$ instead of $Q^{\mathfrak{M}}$. Occasionally we may refer to sort Q as sort F or as the field-sort of $\mathfrak{M}$. (Since in standard mathematical practice Q often denotes the field of rationals, there is a potential danger for ambiguity here for which we apologize to the reader. Anyway, we will *not* use Q to denote the rationals.)

[58] For completeness, we recall here the definition of a field. $\langle F, +, \cdot \rangle$ is called a *field* iff
$\langle F, + \rangle$ is a commutative group, we let 0 denote its neutral element;
$\langle F \setminus \{0\}, \cdot \rangle$ is a commutative group, we let 1 denote its neutral element;
$\cdot$ distributes over $+$, that is, $a \cdot (c + d) = a \cdot c + a \cdot d$ holds for every $a, c, d \in F$.
Sometimes we think of a field as a structure $\mathbf{F} = \langle F, +, \cdot, -, 0, 1 \rangle$, we hope this will cause no confusion. (We omitted 0, 1 and "$-$" from the original definition because they are first-order definable from $+$ and "$\cdot$". One thing that can be slightly influenced by this omission is the set of homomorphisms between two fields.)

standard notation and terminology of many-sorted model theory. We note that every linearly ordered field is infinite. Fields form an abstract (axiomatic[59]) approximation of the field of real numbers; one can work with fields in most of the cases as if they were the field of real numbers.

- G is a nonempty set, it is $\mathfrak{M}$'s universe of sort G. G is called (the set of) *lines* (or geometry, but geometry will be used in Chapter 4 in a slightly different and more comprehensive sense).[60] Intuitively, lines represent motion (in the form of "life-lines") of inertial bodies.

- $Obs, Ph, Ib \subseteq B$ are unary relations (of sort B). Their names are: set of *observers*, set of *photons*, and set of *inertial bodies*, respectively. See the left-hand side of Figure 3.



Bodies of $\mathfrak{M}$ Quantities and lines of $\mathfrak{M}$

Figure 3: Bodies, quantities and lines of an arbitrary model $\mathfrak{M}$.

- $\mathrm{E} \subseteq {}^nF \times G$ is an $(n+1)$-ary relation of sort $\langle \mathrm{Q}, \ldots, \mathrm{Q}, \mathrm{G}\rangle$. Intuitively, for $p = \langle p_0, \ldots, p_{n-1}\rangle \in {}^nF$ and $\ell \in G$, $\mathrm{E}\,(p_0, \ldots, p_{n-1}, \ell)$ expresses that the point $p \in {}^nF$ is on the line ℓ. If p and ℓ are as above, we abbreviate $\mathrm{E}\,(p_0, \ldots, p_{n-1}, \ell)$ by $p \mathrel{\mathrm{E}} \ell$. We postulate axiom $\mathbf{Ax}_{\mathrm{G}}$ below, called the axiom of *extensionality* of lines.

$$\mathbf{Ax}_{\mathrm{G}} \quad (\forall \ell_1, \ell_2 \in G)\Big((\forall p \in {}^nF)(p \mathrel{\mathrm{E}} \ell_1 \;\Leftrightarrow\; p \mathrel{\mathrm{E}} \ell_2) \;\Rightarrow\; \ell_1 = \ell_2\Big).$$

Here we note that the axiom of extensionality allows us to identify $\ell \in \mathrm{G}$ with a subset of ${}^nF$. (See the right-hand side of Figure 3.) Indeed, *we will identify* ℓ with the set $\{\, p \in {}^nF \,:\, p \mathrel{\mathrm{E}} \ell \,\}$ (which is sometimes called the extension of ℓ). By this identification we may assume that $G \subseteq \mathcal{P}({}^nF)$ and E is the real "element-of" relation, $\in$. We will do this from now on, cf. Convention 2.1.3 (p.10).[61]

[59]We mean finitely axiomatizable in FOL.
[60]So the acronym G refers to geometry, but to avoid misunderstandings in Chapter 4, we pronounce it simply as "lines".
[61]This is a standard technique for handling higher-order objects of a logic.

- Let $p = \langle p_0, \dots, p_{n-1}\rangle \in {}^nF$. Then, p_0 is called the *time component* of p, while $\langle p_1, \dots, p_{n-1}\rangle$ is the *space component* of p.[62] Often we write p_t, p_x, p_y, p_z for p_0, p_1, p_2, p_3 respectively.

  ${}^nF$ is called the *coordinate-system* of $\mathfrak{M}$. We refer to p as a *point* or a (space-time) *location*. $\langle p_1, \dots, p_{n-1}\rangle$ is a *(space) location*. We will use the word location ambiguously.

- $W \subseteq B \times {}^nF \times B$, that is, W is an $n+2$-ary relation of sort $\langle \mathrm{B}, \underbrace{\mathrm{Q}, \dots, \mathrm{Q}}_{n\text{-times}}, \mathrm{B}\rangle$.

  W is called the *world-view relation* (of $\mathfrak{M}$). The most important part of our model is this relation. Intuitively, for n=4, $W(m, t, x, y, z, b)$ means that the observer m "observes" or "sees"[63] the body b at time t at (space) location $\langle x, y, z\rangle$. From the $(n+2)$-ary relation W and arbitrary observer $m \in Obs$ we define the *world-view function* $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ as follows:

  $$w_m(p) \stackrel{\text{def}}{=} \{\, b \in B \;:\; W(m, p, b)\,\} \qquad \text{for every } p \in {}^nF\,,$$

  see Figure 4.

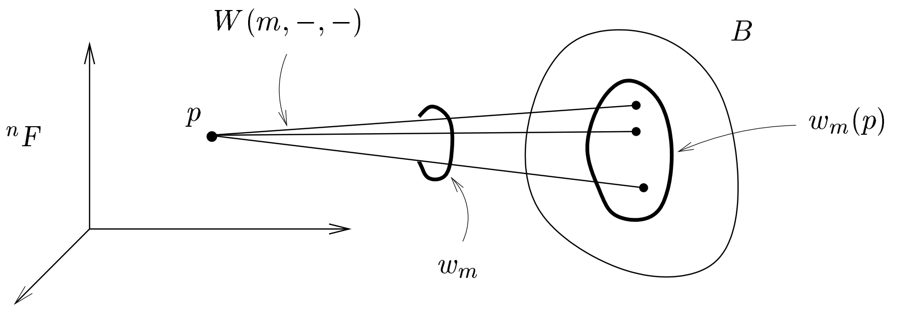


Figure 4: The world-view function w_m.

For $p \in {}^nF$, we call the set $w_m(p)$ of bodies the *event* "happening" at location p as seen by m.[64] Intuitively, w_m defines the "subjective reality" of m. That is, w_m tells us how observer m "arranges" the events (elements of $\mathcal{P}(B)$) in the coordinate-system ${}^nF$; in other words, w_m tells us how m *"coordinatizes"* the set of events $\mathcal{P}(B)$. See Figure 5.

In the literature sometimes ${}^nF$ is called space-time, and sometimes the set of events $\mathcal{P}(B)$ is called space-time. The reason for calling $\mathcal{P}(B)$ space-time is that ${}^nF$ is only

[62] It is important to emphasize here that ${}^nF$ is only the coordinate system of $\mathfrak{M}$ as opposed to being say "space-time" itself of $\mathfrak{M}$. Space-time will not be one of our primitive (i.e. basic) concepts, instead, it will be a derived "theoretical" concept and it will appear e.g. in §4. For the observational/theoretical duality (in the sense of Friedman [91] or Reichenbach [218]) cf. §1, §4.1 herein and AMN [18, §1.1 on p.11].

[63] We want to emphasize that here "*observing*" or "*seeing*" has nothing to do with the intuitive notion of observing in the form of measurement, or with the everyday notion of seeing via photons. In the present text, "observer" and "observing" are technical expressions which we use for historical reasons. Our "observing" is really a kind of coordinatizing, i.e. when we say that observer m observes event e at coordinates t, x, y, z, we mean only to say that m associates coordinates t, x, y, z to event e. (As opposed to "real observing", this is a very abstract act only.) By the word "observer" we mean what is sometimes called *frame of reference* or "system of reference" (or coordinate-system), cf. footnote 51 (p.4) and Remark 2.2.5 (p.25).

[64] Two or more bodies occupying the same space at the same time might contradict the physical intuition. However, presently we abstract from the sizes of the bodies and therefore we permit two or more bodies to be at the same place at the same time. We also note the following. The reader may ask "why is an event a set of bodies". Motivation for this definition of an event can be found e.g. in Misner-Thorne-Wheeler [192, p.6], and Friedman [91, p.31] starting with line 9 therein.

a coordinate-system (consisting of labels), using which observers coordinatize the set of events $\mathcal{P}(B)$.[65] On the long run it will be more fruitful to use the word space-time for the thing which is being coordinatized, that is for $\mathcal{P}(B)$.[66] We will see more reasons for calling the set of events space-time in the geometry chapter §4, pp.127–364. The sets $B^{\mathfrak{M}}, F^{\mathfrak{M}}, G^{\mathfrak{M}}$ are also called the <u>*universes*</u> of $\mathfrak{M}$ (of sorts B, Q, G respectively).

Summing it up: The similarity type of our first-order language consists of

– the sort symbols B, Q, G;

– the unary relation symbols Obs, Ph, Ib (most often, their interpretations in models are denoted by Obs, Ph, Ib as well);

– the symbols $+, \cdot, \leq$ of the ordered field $\mathfrak{F}$ (the neutral elements 0 and 1 of + and $\cdot$, respectively, and "$-$" will also be treated as basic symbols);

– the (n+1)-ary relation symbol E, which we will systematically replace by the set theoretic "$\in$" (cf. Convention 2.1.3);

– the (n+2)-ary relation symbol W. Further:

The reduct $\langle B, Obs, Ph, Ib\rangle$ of $\mathfrak{M}$ is purely of sort B (body);

$\mathfrak{F} = \langle F, +, \cdot, \leq\rangle = \langle \mathbf{F}, \leq\rangle$ is purely of sort Q (quantities);

G is the universe of sort G (lines), and there are no relation or function symbols which would be purely of sort G.

E (which we will replace by $\in$) acts between sorts Q and G, while W involves B and Q.

Variables ranging over the universes B, F, G of $\mathfrak{M}$ are most often chosen as follows. For arbitrary $i \in \omega$,

$b, b_i, h, h_i, k, k_i, m, m_i, ph, ph_i \in B$;
$a, a_i, c, c_i, d, d_i, t, t_i, x, x_i, y, y_i, z, z_i, \varepsilon, \lambda, \eta \in F$;
$\ell, \ell_i \in G$.

Let us recall that
$\mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} =$
$\{$the axioms postulating that $\mathfrak{F}$ is a linearly ordered field, axiom of extensionality$\}$.
Now the <u>*frame-language*</u> of relativity theory of dimension n is defined to be the 3-sorted first-order language built up from the above symbols the usual way. A model $\mathfrak{M} = \langle B, F, G; Obs, Ph, Ib, +, \cdot, \leq, \mathrm{E}, W\rangle$ is called a <u>*frame model*</u> (of relativity theory, of dimension n) iff

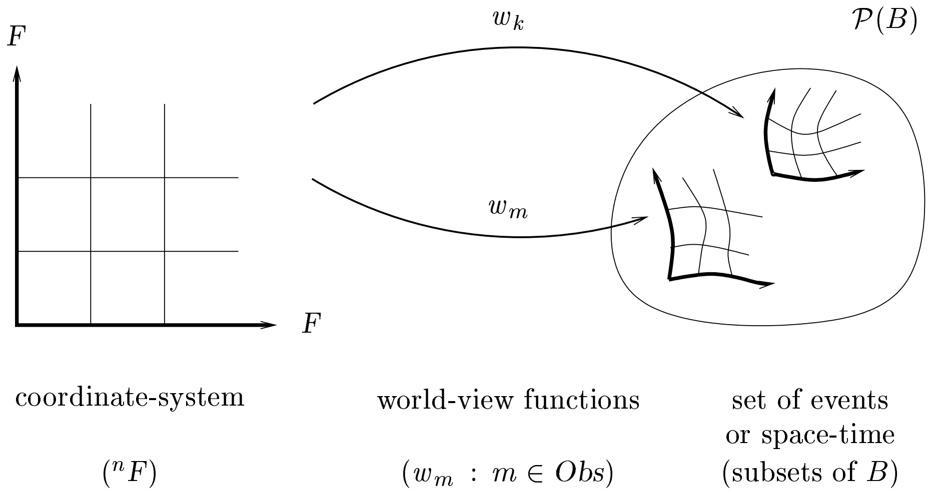
$$\mathfrak{M} \models \mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,b) \to Obs(m)\}\,.$$ [67]

---

[65] A location $p \in {}^nF$ functions only as an "address" or "label" used by an observer m in labeling those events which exist for m.

[66] To help the reader's intuition we note that the world-view function w_m connects up the coordinate-system ${}^nF$ with the set of events $\mathcal{P}(B)$. Therefore if for someone it were easier to imagine ${}^nF$ as space-time, he/she can use the world-view function w_m to translate his/her intuition for viewing set of events as space-time.

[67] We use the standard convention from logic that an axiom $\varphi(x)$ automatically means its universal closure $\forall x\, \varphi(x)$. Throughout we write p for $p_0, \ldots, p_{n-1}$, hence $W(m,p,b)$ abbreviates $W(m, p_0 \ldots, p_{n-1}, b)$.



The heart of our model is W, which is represented by functions $w_m : {}^nF \to \mathcal{P}(B)$ for each $m \in Obs$.

Figure 5: This is a useful way for visualizing a model $\mathfrak{M}$.

FM denotes the class of all frame models and FF denotes the set of frame formulas.[68] We call $\mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,b) \to Obs(m)\}$ the <u>*frame theory of special relativity theory*</u> (or <u>*frame theory*</u> for short). By $\models^{\mathrm{OFG}}$ we denote semantical consequence within our present frame theory $\mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,b) \to Obs(m)\}$. That is, for two sets Σ and Γ of formulas in our frame language,

$$\Sigma \models^{\mathrm{OFG}} \Gamma \iff (\forall \mathfrak{M} \in \mathsf{FM})(\mathfrak{M} \models \Sigma \Rightarrow \mathfrak{M} \models \Gamma)\,.$$

Also we define

$$\mathsf{Mod}_{\mathrm{OFG}}(\Sigma) \stackrel{\mathrm{def}}{=} \mathsf{FM} \cap \mathsf{Mod}(\Sigma)\,.$$

For brevity, throughout this work, we will write $\mathsf{Mod}(\Sigma)$ for $\mathsf{Mod}_{\mathrm{OFG}}(\Sigma)$. We hope that this causes no confusion, since we never want to talk about models (of type of our frame language) in which $\mathbf{Ax}_{\mathrm{OF}}$, $\mathbf{Ax}_{\mathrm{G}}$, or $(W(m,p,b) \to Obs(m))$ would fail.

Similarly, throughout we denote $\models^{\mathrm{OFG}}$ simply by $\models$, and we will never use $\models$ in its original purely logical sense <u>*in the context*</u> of our frame language (to avoid misunderstanding). Of course, when talking about structures or formulas of a different similarity type like $\mathfrak{F}$, we use "$\models$" in its usual logical sense.

END OF DEFINITION 2.1.2 (FRAME LANGUAGE).

$\triangleleft$

CONVENTION 2.1.3 (1) As we indicated on p.7, below the definition of $\mathbf{Ax}_{\mathrm{G}}$, we will identify our E with the set theoretic membership relation "$\in$". As it was indicated there, this causes no loss of generality because every frame model $\mathfrak{M}$ is isomorphic to a frame model $\mathfrak{N}$

[68]i.e. the formulas in our frame language

such that $\mathrm{E}^{\mathfrak{N}}$ coincides with the set theoretic "$\in$". Therefore throughout the rest of this work a frame model is of the form

$$\mathfrak{M} = \langle B, F, G;\ Obs, Ph, Ib, +, \cdot, \leq, \in, W \rangle .$$

Throughout, we use the semicolon ";" to separate the sorts of a model from its relations and functions, as in the above equality. Often, we will use the more concise notation

$$\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle .$$

If we want to indicate that a universe (or sort) like B or a relation like W comes from a particular model $\mathfrak{M}$, we use the superscript $B^{\mathfrak{M}}, W^{\mathfrak{M}}$ respectively. This is why on p.6 we wrote $\mathfrak{M} = \langle B^{\mathfrak{M}}, \ldots, Obs^{\mathfrak{M}}, \ldots, W^{\mathfrak{M}} \rangle$. However, if $\mathfrak{M}$ is understood from context, we will usually omit the superscript. All this ($B^{\mathfrak{M}}$ etc.) is standard notation from model theory and universal algebra, cf. e.g. Hodges [130], Monk [194], Grätzer [108], McKenzie&McNulty&Taylor [189], [120], Barwise-Feferman [45, p.27]. (As an exception, Chang-Keisler [60] uses lower indices like $B_{\mathfrak{M}}$ instead of $B^{\mathfrak{M}}$. But the general style and notational philosophy remains the same in [60], too, as adopted here.)

(2) As is customary in (parts of) universal algebra and model theory, structures are denoted by German capitals like $\mathfrak{A}, \mathfrak{B}, \ldots, \mathfrak{M}$. If $\mathfrak{A}$ is a <u>one-sorted</u> structure, its universe is denoted by the corresponding latin capital A. Sometimes $Uv(\mathfrak{A})$ <u>also</u> denotes the universe of $\mathfrak{A}$. Hence $A = Uv(\mathfrak{A})$, $B = Uv(\mathfrak{B})$, etc. assuming $\mathfrak{A}$, $\mathfrak{B}$ are one sorted. This kind of convention will be extended to many-sorted structures in §4.3 on p.219. Till then, we will not need the many-sorted version of convention $A = Uv(\mathfrak{A})$ because (before Chapter 4) we will deal with only one kind of many-sorted structures, namely, ones like $\mathfrak{M}$.

(3) Throughout, n denotes a natural number with $n > 1$. Hence e.g. "for all n" means "for all $n > 1$". Thus if in a theorem n is not specified, the theorem is claimed to hold for all $n > 1$. $\lhd$

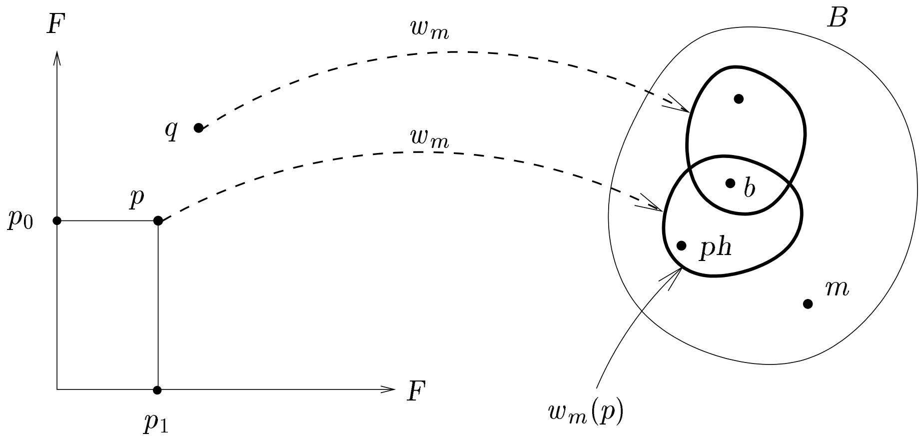
As we said, intuitively, n is the dimension of our space-time. If $n = 2$, we have one time-dimension, and one space-dimension, i.e. space is one-dimensional. If $n = 3$, space is two-dimensional, and $n = 4$ represents our usual 4-dimensional space-time, i.e. space is three-dimensional. In the case of $n = 2$ it is rather easy to illustrate things, so we will often use $n = 2$ in our drawings. When $n = 3$, one still can illustrate ideas by drawings quite well. Many ideas can be better seen in the case $n = 2$ and work completely analogously for arbitrary n. Some statements, however, are true for $n = 2$ and not true for $n = 3, 4$. In these cases we will emphasize that $n = 2, n = 3$, or $n = 4$. (Sometimes [but not frequently], the cases of 3 and 4 behave differently. In such cases, of course, one emphasizes this difference. But most of the time, for understanding the key ideas, we will concentrate on the case of $n = 3$.)[69] There is another reason why it may be useful to allow the dimension of space-time to vary. Later we devise models in which not all observers coordinatize events with the same dimensional coordinate-system ${}^nF$. E.g. we can allow that most of the observers coordinatize events in 4-dimension, while some special (e.g. faster-than-light) observers coordinatize events with 2-dimensional coordinate-system only.[70]

[69] Sometimes it is worth contemplating why the proofs are different for different dimensions.

[70] Cf. the section on faster-than-light observers in dimension 2 (§2.7) and its references to AMN [19]. Such models were also presented at seminars in Rényi Institute of Mathematics in 1999.

Figures 3 - 6 illustrate the structure of an arbitrary model $\mathfrak{M}$ (of dimension 2) in the sense of Definition 2.1.2. Consider the coordinate-system in the right-hand side of Figure 3 (or in the left-hand sides of Figures 4, 5, 6). Intuitively, the first (vertical) axis is the *time scale* while the second (horizontal) axis represents *space*. The straight lines ℓ and ℓ_1 represent "lines" in Figure 3. The world-view relation W, which is the heart of our model, is illustrated in Figures 4–6. W is represented by the system of world-view functions $\langle\, w_m \,:\, m \in Obs \,\rangle$, cf. Figure 5. In Figure 6, $w_m(p) = \{b, ph\}$ means that m "sees" at time p_0 at location p_1 two bodies: b and ph. I.e, $W(m, p, b)$, $W(m, p, ph)$ are true, while e.g. $W(m, p, m)$ is not true.

For the time being we do not have a structure on the set $\mathcal{P}(B)$ of events, which we also call space-time. In the geometry chapter §4 we will put some structure on our space-time, too. Sometimes, the structure in Figure 5 is mathematically modeled by a so-called manifold.[71]



The world-view relation W and world-view functions w_m.
m "sees" at time p_0 at location p_1 two bodies: b and ph.

Figure 6: Second drawing of the world-view function w_m.

We will use the following notation. For $Obs(b), Ph(b), Ib(b)$ we often write $b \in Obs$, $b \in Ph$, $b \in Ib$, respectively. Moreover, we will reserve the variables m, m_i, k, k_i to denote observers; we reserve ph, ph_i for photons; finally we use the symbols p, q, r, s to denote elements of ${}^nF$. Thus we have[72]

$$m, m_i, k, k_i \in Obs;$$
$$ph, ph_i \in Ph;$$
$$p, q, r, s \in {}^nF.$$

Using the terminology of vector spaces, elements of ${}^nF$ will often be referred to as *vectors*. As we mentioned, we use the convention from logic that

$\varphi(m)$ when used as an axiom, means $(\forall m \in Obs)\varphi(m)$.

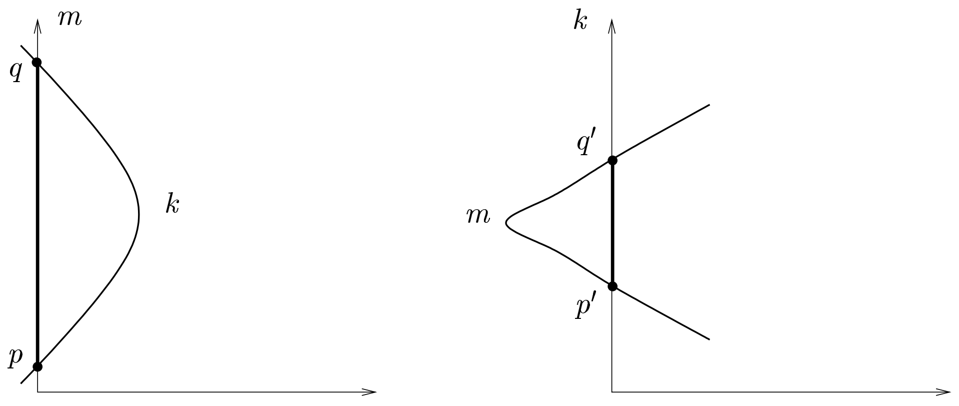
[71] The manifold structure is not particularly relevant at the present point, but it will be relevant in later developments.

[72] Sometimes we will deviate from this convention though, for lack of enough letters. E.g. sometimes we will use m, k to denote natural numbers also.

(This is based on our convention above that m ranges over elements of Obs, and not B.)

Consider a frame model $\mathfrak{M}$ and its ordered field reduct $\mathfrak{F}$. We will sometimes impose the condition on our $\mathfrak{M}$ that $\mathfrak{F} = \mathfrak{R}$, the ordered field of real numbers.

We close this section with giving a possible formulation of the so-called "*twin paradox*", as an example of a formula in our frame language.[73] Intuitively, the twin paradox says that if one of two twin brothers leaves the other (accelerating) and returns to him later, the brother who stayed behind will be *older* at the time of their reunion. That is, more time has passed for the "non-moving" brother than for the traveling one.



m: "non-moving" (inertial) brother $\qquad$ k: traveling (accelerated) brother

Figure 7: The "twin paradox".

(TwP) $(\forall m \in Obs \cap Ib)(\forall k \in Obs \setminus Ib)(\forall p, q, p', q' \in {}^4F)$
$\Big(m, k \in w_m(p) \cap w_m(q) \;\wedge\; w_m(p) = w_k(p') \;\wedge\; w_m(q) = w_k(q')\Big) \;\Rightarrow$
$|p_t - q_t| > |p'_t - q'_t|$. [74]

See Figure 7. It is not a coincidence that in Figure 7, the life-line of m as seen by k is more "exotic" than that of k as seen by m. (The acceleration of m is sometimes negative and sometimes positive.) We will discuss the reason for this in §2.8, p.95 herein, and in Chapter 8 of a future edition [19] of AMN [18].

[73] We use natural abbreviations here, as well as later. E.g. we write "$m, k \in w_m(p) \cap w_m(q)$" in place of the longer "$W(m,p,m) \;\wedge\; W(m,q,m) \;\wedge\; W(m,p,k) \;\wedge\; W(m,q,k)$".

[74] The notation p_t, q_t was introduced on p.8. Further, $|p_t - q_t|$ denotes the absolute value of $p_t - q_t$, cf. p.6 for the notation $|a|$ when $a \in F$.

2.2 Basic axioms Basax

Our next task is to postulate axioms in our frame language, expressing parts of our knowledge about physical reality. Our first set of axioms to be proposed shortly will be called **Basax**, and it serves as one possible starting point for axiomatizing *special* relativity theory.

Before presenting **Basax**, we would like to say a few words about its place in the hierarchy of axiom systems which will be studied in the present work.

In the 1998 December version [24] of AMN [18] as well as in section 8.1 of a future edition [19] of AMN [18] we introduce a further (actually a more "advanced") set **Acc** of axioms, in which we will allow accelerated observers, and accordingly, in **Acc** we will modify some of the postulates of **Basax** (e.g. we will modify item 7 below).[75] In section 3 we define variants of the axioms of **Basax**, and variants of **Basax** itself (e.g. **Newbasax**).[76] These new versions will be more "balanced" in a sense, and will make it easier to move towards having accelerated observers, i.e. towards **Acc**. (On the other hand, our first choice, **Basax** has the advantage that its axioms are easy to formulate and understand, so it might be considered as a good starting point.) In later parts we will introduce stronger as well as weaker (than **Basax**) axiom systems. As we indicated in §1.4 and §3.1, (and in more detail in AMN [18, §1.1 (Broad Introduction)]), a plurality of competing axiom systems (or relativity theories) is an essential feature of logical analysis of a theory like relativity. Accordingly, in §3 herein as well as in AMN [18, §3.4.2 and Chapter 4] we introduce several axiom systems for the purposes indicated in AMN [18, §1.1]. One of these purposes is conceptual analysis (started e.g. in Friedman [91] and Rindler [222]) which asks which axiom of relativity is responsible for which conclusion of the theory. Another purpose of this plurality is to study such variants of relativity as e.g. the Reichenbach-Grünbaum version and to compare them with the standard version. Also, we want to "fine-tune" our axiom systems in various regards. A further, but not negligible purpose in studying weaker axiom systems is to prove stronger theorems. For more on the motivation for having a plurality of axiom systems we refer the reader to §1, §3.1, and to AMN [18, §§ 1.1, 3.4.2, Chapter 4]. See also Figures 60 and 223 on pp.126 and 653 (and also AMN [18, Figure 138, p.A-31]). Finally we note that besides weakening (and/or modifying) **Basax** we also study the possibility of making it <u>*stronger*</u> by adding a few new, natural axioms, cf. e.g. §§2.8,3 and AMN [18, §§ 3.8, 3.9]. In AMN [18, §3.8], we also study an extension "**BaCo**+**Ax**($\sqrt{\ }$)" of **Basax**, which completely describes the standard, Minkowskian models of special relativity, cf. §3, p.125.

Before presenting **Basax**, we emphasize that it is only our first and simplest variant of an axiom system for special relativity. Later, we will also have: (i) axiom systems in which accelerated observers are permitted (i.e. informal postulate 2 below will be withdrawn), (ii) systems in which for different observers different events may exist (i.e. postulate 7 below will be withdrawn), (ii) systems in which the speed of light will be not the same for all observers,

[75] **Acc** (and its theory) can be considered as a *first step* in the direction of *experimenting* with the idea of treating general relativity (in **Acc** we will have gravity, event horizons etc) in the framework of first-order logic in a spirit analogous with that of the present work. Cf. e.g. pp.95-98 herein.

[76] It belongs to the spirit of the axiomatic method that we start out with a simple set of axioms (like **Basax**), investigate its properties, prove some theorems from it, and then we use our so obtained experience for modifying this axiom system. After that, we restart the "cycle", i.e. we start investigating the new axiom system etc.

(iv) the Reichenbachian version of relativity where there is even less restriction on the speed of light, (v) systems in which the coordinate-system of an observer may be not the whole of ${}^4F$ but only a subset of ${}^4F$, etc.

Informally, about a model $\mathfrak{M} = \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, G;\, \in, W\rangle$, **Basax** will postulate the following.

> $\mathfrak{F}$ is a linearly ordered field; we can thus define straight lines of the usual, Cartesian geometry over $\mathfrak{F}$, i.e. of ${}^nF$ (which, intuitively, are "*life-lines*" or "*traces*" of the motions of inertial bodies), and we can define angles of straight lines (which represent "*speeds*" of inertial bodies). In this sense of the word we will postulate the following:
>
> 1. G is the set of straight lines of the Cartesian geometry over $\mathfrak{F}$.
> 2. Observers and photons are inertial bodies.
> 3. The "trace" of an inertial body h as seen[77] by any observer m is in G.
> 4. Any observer m sees itself as being at rest in the origin.
> 5. Any observer sees some observer on each "slow" line.[78]
> 6. Each line which could be the life-line of a photon (according to item 8 below) is indeed the life-line of a photon.
> 7. Any two observers see the same events.
> 8. All observers see all photons moving with the same speed.
>
> In items 5 and 6 above by existence we mean only *potential* existence. I.e. when we say that on each slow line there exists an observer, what we mean is that potentially there can exist an observer, but in reality all these potential observers and photons need not be really there. The same applies to the existence of "potential" photons in item 6.[79]

For the formal definition of **Basax**, we need some preparation. We start with recalling some basic notions of linear algebra e.g. from Halmos [114] or Kostrikin-Manin [148] or Hausner [115] or [226] (or any other textbook on linear algebra).

If $p \in {}^nF$ for some set F and $n \in \omega$ then, for any $i < n$, p_i denotes the i-th component (projection) of p. Thus $p = \langle p_0, \ldots, p_i, \ldots, p_{n-1}\rangle = \langle p_i\rangle_{i<n}$.

Recall from any textbook on vector spaces (e.g. [114]) that, to any field $\mathbf{F} = \langle F, +, \cdot\rangle$ and natural number $n \in \omega$, an n-dimensional *vector space* ${}^n\mathbf{F}$ can be associated the following natural way. Defining $+^V : {}^nF \times {}^nF \longrightarrow {}^nF$ by

$$(\forall p, q \in {}^nF)\; p +^V q \stackrel{\text{def}}{=} \langle p_i + q_i\rangle_{i<n}\,,$$

[77] Below, and later on, we will use the word "*see*" as a kind of intuitive (or "animated") way of referring to the act of *observing* via the world-view function, as we already indicated this (cf. footnote 63 on p.8).

[78] A line is called slow if its "speed" (i.e. angle with the time axis) is smaller than that of a photon.

[79] We will return later to clarifying the issue of these potentially existing entities (observers, photons) which exist only potentially but need *not* exist *actually*. This can be made precise e.g. by using first-order modal logic as a framework as will be discussed soon. Cf. Vályi [263].

$\langle {}^nF, +^V\rangle$ turns out to be a commutative group with neutral element

$$\bar{0} \stackrel{\text{def}}{=} \langle 0\rangle_{i<n}$$

and inverse $-^V p = \langle -p_i\rangle_{i<n}$ for any $p \in {}^nF$. With defining "multiplication by scalars" $\cdot^V : F \times {}^nF \longrightarrow {}^nF$ by

$$a \cdot^V p \stackrel{\text{def}}{=} \langle a \cdot p_i\rangle_{i<n} \quad \text{for each } a \in F \text{ and } p \in {}^nF\,,$$

$\langle {}^nF, +^V\rangle$ becomes a vector space over the field $\mathbf{F}$. We denote this vector space by ${}^n\mathbf{F}$. We note that any n–dimensional vector space over $\mathbf{F}$ is isomorphic to ${}^n\mathbf{F}$ (see e.g. Halmos [114]). In universal algebra, there are two ways for making the notion of a vector space like ${}^n\mathbf{F}$ precise. These are the "one-sorted" and the "two-sorted" versions, defined below. The one-sorted version is defined as follows:

$${}^n\mathbf{F}_1 \stackrel{\text{def}}{=} \langle {}^nF, +^V, -^V, \bar{0}, f_a\rangle_{a\in F}$$

with f_a unary and $f_a(p) \stackrel{\text{def}}{=} a \cdot^V p$ for $p \in {}^nF$ and $a \in F$. Cf. Burris-Sankappanavar [53]. The two-sorted version is the structure

$${}^n\mathbf{F}_2 \stackrel{\text{def}}{=} \langle \mathbf{F}, {}^nF; +^V, -^V, \bar{0}, \cdot^V\rangle\,,$$

where the operations $+^V$, $-^V$, $\bar{0}$ are defined on sort ${}^nF$ while $\cdot^V$ is of mixed sort, i.e. $\cdot^V : F \times {}^nF \longrightarrow {}^nF$. Throughout this work, ${}^n\mathbf{F}$ denotes either ${}^n\mathbf{F}_1$ or ${}^n\mathbf{F}_2$ depending on context. Occasionally we will explicitly indicate which one is meant. So whenever ${}^n\mathbf{F}$ shows up, it denotes the n-dimensional vector space over $\mathbf{F}$ without specifying whether we mean the one-sorted or the two-sorted version (the reader is asked to use the context if he wants to decide this).

We note that the notation ${}^n\mathbf{F}$ is slightly ambiguous (from a different point of view too) because ${}^n\mathbf{F}$ can denote the vector space over the field $\mathbf{F}$ but also (by the standard notation of universal algebra) it can denote the n'th direct (or Cartesian) power of the algebraic structure $\mathbf{F}$. This direct power happens to be a ring. Therefore we might talk about the vector space ${}^n\mathbf{F}$ or the ring ${}^n\mathbf{F}$ (they are not the same because they have different operations). If we do not indicate which one is meant then, by *default*, we mean the vector space. I.e. if the symbol ${}^n\mathbf{F}$ appears in the text (without an indication of whether we mean a vector space or a ring) then it *denotes a vector space*. A completely analogous convention applies to $\mathfrak{F}$ in place of $\mathbf{F}$.

As usual, we will often write $p -^V q$ in place of $p +^V (-^V q)$ for simplicity. Further, we will often omit the index V from $\cdot^V$, $+^V$ and $-^V$, and hope that context will always save us from misunderstandings.

CONVENTION 2.2.1

(i) Throughout, $\mathfrak{F}$ $(= \langle \mathbf{F}, \leq\rangle)$ denotes an arbitrary linearly ordered field. However, this is a context sensitive convention in the following sense: If there is a frame model $\mathfrak{M}$ around, then automatically $\mathfrak{F}$ denotes the ordered field reduct of $\mathfrak{M}$. A similar convention applies to the field $\mathbf{F}$, its universe F, coordinate system ${}^nF$, and vector space ${}^n\mathbf{F}$, e.g. if there is an $\mathfrak{F}$ around then automatically F denotes its universe etc. In the other direction if we talk about, say, F then implicitly we assume that there is an $\mathfrak{F}$ in the background etc.

(ii) As we already said in Def.2.1.2, when we work in ${}^n\mathbf{F}$ $(2 \leq n \leq 4)$, to match the physical intuition, we call the 0-th coordinate p_0 of a point $p = \langle p_0, \ldots, p_{n-1}\rangle \in {}^nF$ the *time coordinate* or <u>*time component*</u> of p. Accordingly, when drawing coordinate systems, we call the 0-th axis of it the <u>*time axis*</u> or <u>*$\bar{t}$-axis*</u>. The rest of the coordinates are the <u>*space coordinates*</u> or <u>*space components*</u>. We denote the first four coordinate axes as follows:

$$\begin{aligned}
\bar{t} &\stackrel{\text{def}}{=} F \times {}^{n-1}\{0\} \; (= F \times \{0\} \times \ldots \times \{0\}) \,, \\
\bar{x} &\stackrel{\text{def}}{=} \{0\} \times F \times {}^{n-2}\{0\} \,, \\
\bar{y} &\stackrel{\text{def}}{=} \{0\} \times \{0\} \times F \times {}^{n-3}\{0\} \,, \text{ and} \\
\bar{z} &\stackrel{\text{def}}{=} \{0\} \times \{0\} \times \{0\} \times F \times {}^{n-4}\{0\} \,.
\end{aligned}$$

In general $\bar{x}_i$ denotes the i'th coordinate axis, that is

$$\bar{x}_i \stackrel{\text{def}}{=} {}^{i}\{0\} \times F \times {}^{n-i-1}\{0\} \,.$$

Also, we put

$$\begin{aligned}
p_t &\stackrel{\text{def}}{=} p_0 \,, \\
p_x &\stackrel{\text{def}}{=} p_1 \,, \\
p_y &\stackrel{\text{def}}{=} p_2 \,, \\
p_z &\stackrel{\text{def}}{=} p_3 \,,
\end{aligned}$$

for each $p \in {}^nF$.

(iii) Throughout this work, the dimension n $(\in \omega)$ of our space-time is a parameter of almost all of our concepts. Therefore a possibility for a rigorous presentation would be to indicate n in the name of each concept we introduce, e.g., by putting something like "(n)" after it. But then the text would become too complicated. Therefore we chose omitting the "(n)"-s except when this would lead to misunderstanding or when we want to emphasize the presence of n.

But sometimes we will define or state things for one particular n only (e.g., for just $n = 2$). In these cases we will indicate this fact by putting the particular number, in parenthesis, after the name of the concept involved. For example, we will formulate an axiom **Ax1**, where n will be a parameter of **Ax1**. Then the instance of **Ax1** for the case $n = 2$ will be denoted by **Ax1**(2).

Throughout this work $n > 1$. Therefore, we will not mention this explicitly.

We will treat some other parameters likewise. E.g., we will sometimes state things for a collection of models from FM such that all $\mathfrak{M} \in \mathsf{FM}$ share the same ordered field $\mathfrak{F}$ as their "quantity part". Then we will denote this collection by $\mathsf{FM}(\mathfrak{F})$.

In cases when we will need more than one parameter we will list them in parentheses, separated by commas. For example,

$$\mathsf{FM}(3, \mathfrak{R}) = \left\{ \mathfrak{M} \in \mathsf{FM}(3) \; : \; \mathfrak{F}^{\mathfrak{M}} = \mathfrak{R} \right\} \,.$$

That is, $\mathfrak{M} \in \mathsf{FM}(3, \mathfrak{R})$ iff $\mathfrak{M}$ is of dimension 3 and the quantity part of $\mathfrak{M}$ is the ordered field $\mathfrak{R}$ of real numbers.

$\lhd$

Besides our frame language introduced in section 2.1, we will also use the language of the vector space ${}^n\mathbf{F}_2$ (as an extension of our frame language) for expressing ideas concisely. (E.g., for $r, s \in {}^nF$ we may mention the vectors $r + s$ or $3 \cdot r$.) We are allowed to do this since the

${}^n\mathbf{F}_2$ formulas are translatable to our frame language. As a first example of this and for the other natural abbreviations we will use, we introduce our first axiom **Ax1** both as a formula in a concise style translatable to our frame language[80] and, equivalently, as a (longer) formula written purely in the frame language.

The set of straight lines of ${}^n\mathbf{F}$ in the usual Euclidean sense is denoted by $\mathsf{Eucl} := \mathsf{Eucl}(n, \mathfrak{F}) := \mathsf{Eucl}(n, \mathbf{F})$, that is,

$$\ell \in \mathsf{Eucl}(n, \mathbf{F}) \stackrel{\text{def}}{\Longleftrightarrow} (\exists r, s \in {}^nF)\Big(s \neq \bar{0} \;\wedge\; \ell = \{\, r + a \cdot s \;:\; a \in F\,\}\Big).\ ^{81}$$

Ax1 in a concise language:
$G = \mathsf{Eucl}(n, \mathbf{F})$.

Ax1 in the frame language of relativity theory:

Ax1' $(\forall r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1} \in F)$
$\Big(\{s_0, \ldots, s_{n-1}\} \neq \{0\} \;\Rightarrow\; (\exists \ell \in G)(\forall p_0, \ldots, p_{n-1} \in F)(\; \mathrm{E}\ (p_0, \ldots, p_{n-1}, \ell) \Leftrightarrow (\exists a \in F)\ \bigwedge_{i<n} p_i = r_i + a \cdot s_i)\Big)$
and
$(\forall \ell \in G)(\exists r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1} \in F)$
$\Big(\{s_0, \ldots, s_{n-1}\} \neq \{0\} \;\wedge\; (\forall p_0, \ldots, p_{n-1} \in F)(\; \mathrm{E}\ (p_0, \ldots, p_{n-1}, \ell) \Leftrightarrow (\exists a \in F)\ \bigwedge_{i<n} p_i = r_i + a \cdot s_i)\Big).$

Here we emphasize that **Ax1** is designed to serve the purposes of special relativity only. In later parts when dealing with more general theories of relativity, **Ax1** will be changed.

If $\ell \in \mathsf{Eucl}(n, \mathbf{F})$, then we can consider the angle between ℓ and the time axis. By $ang^2(\ell)$ we denote the square of the tangent of the angle between ℓ and the time axis.[82] Thus, for $\ell = \{\, r + a \cdot s \;:\; a \in F\,\} \in \mathsf{Eucl}(n, \mathbf{F})$,

$$\begin{array}{lcll} ang^2(\ell) & \stackrel{\text{def}}{=} & \dfrac{s_1^2 + s_2^2 + \ldots + s_{n-1}^2}{s_0^2} & \text{if } s_0 \neq 0, \text{ and} \\ ang^2(\ell) & \stackrel{\text{def}}{=} & \infty & \text{if } s_0 = 0.^{83} \end{array}$$

(It will cause no problem that infinity ∞ is not an element of F.) Thus $0 \leq ang^2(\ell) \leq \infty$. $ang^2(\ell) = 0$ means that ℓ is vertical, $ang^2(\ell) = 1$ intuitively means that the angle between ℓ and the time axis is 45^o, and $ang^2(\ell) = \infty$ means that ℓ is horizontal. The definition of $ang^2(\ell)$ is illustrated in Figure 8.

[80] I.e. in **Ax1** we use convenient abbreviations reducible to our frame language.

[81] Note that after this definition the formula $\ell \in \mathsf{Eucl}(n, \mathbf{F})$ counts as a formula of our frame language. Namely, it abbreviates the following formula of our frame language:
$\ell \in G$ and $(\exists r, s \in {}^nF)\Big(s \neq \bar{0} \;\wedge\; (\forall p \in {}^nF)[p \in \ell \;\Leftrightarrow\; (\exists a \in F)\, p = r + a \cdot s]\Big)$.

[82] We consider the square of the tangent (instead of the tangent itself) of this angle because, in general, we do not assume that square-roots exist in $\mathbf{F}$.

[83] We use ∞ to denote "infinite" in the usual sense. In more detail, ∞ is a new constant symbol not in the language of $\mathfrak{M}$ and we use it as a *new*, greatest element added to the structure $\mathfrak{F}$. We will remain informal about ∞ because we will use it only in such formulas from which it can be easily eliminated. Cf. also p.535 of AMN [18] about the notation ∞.

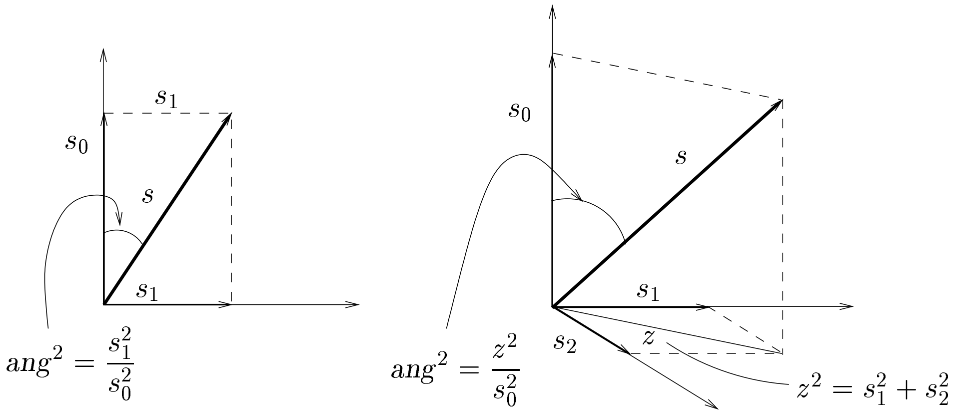


Figure 8: Angle of a line.

Definition 2.2.2 (life-line (or trace), speed)
Let $\mathfrak{M}$ be a frame model as in Definition 2.1.2. Let $m \in Obs$ and $b \in B$ be arbitrary but fixed. Recall from Definition 2.1.2 that the world-view function $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ of m was defined as follows:

$$w_m(p) = \{\, b \in B \;:\; \langle m, p, b\rangle \in W \,\} \qquad \text{for every } p \in {}^nF\,.$$

(i) By the *life-line (or trace) of b as seen by m* (or life-line (or trace) of b by the world-view of m) we mean the set

$tr_m(b) \stackrel{\text{def}}{=} \{\, p \in {}^nF \;:\; b \in w_m(p) \,\} \;=\{\, p \in {}^nF \;:\; W(m,p,b) \,\}.$

(ii) If $tr_m(b) \in \mathsf{Eucl}(n, \mathbf{F})$, then by the *speed of b as seen by m* we mean

$v_m(b) \stackrel{\text{def}}{=} ang^2(tr_m(b))$,

cf. Figure 9.

The formula $v_m(b) = a$ will abbreviate that

$tr_m(b) \in \mathsf{Eucl}(n, \mathbf{F})$ and $ang^2(tr_m(b)) = a$.

◁

In Figure 9, the line $tr_m(b)$ illustrates the life-line of a body b (in case $n = 2$). The acronym "tr" stands for "*trace*". If $tr_m(b) = \{\, \langle t, x, y, z\rangle \;:\; t \in F \,\}$, then m always, at each time instance $t \in F$, sees b at location $\langle x, y, z\rangle$, i.e. m sees the body b at rest at location $\langle x, y, z\rangle$. Thus, $tr_m(b)$ is a vertical line (a line parallel with the time axis), i.e. $v_m(b) = 0$, means that "b is at rest, as seen by m". Similarly, the bigger $v_m(b)$ is, the more "speed" b is moving with, as seen by m, cf. Figure 9.

As we said $v_m(b)$ is called the *speed* of b as seen by m. To be more precise it is the square of the usual speed (since we used ang^2 instead of ang). The reason for using the square of quantities (in place of the original quantities) is that we do not want to assume that square-roots exist in $\mathbf{F}$. So, speed is a scalar (i.e. element of F). As opposed to speed, the *velocity* $\vec{v}_m(b)$ of b as seen by m is an $(n-1)$-vector, i.e. $\vec{v}_m(b) \in {}^{n-1}F$, defined as

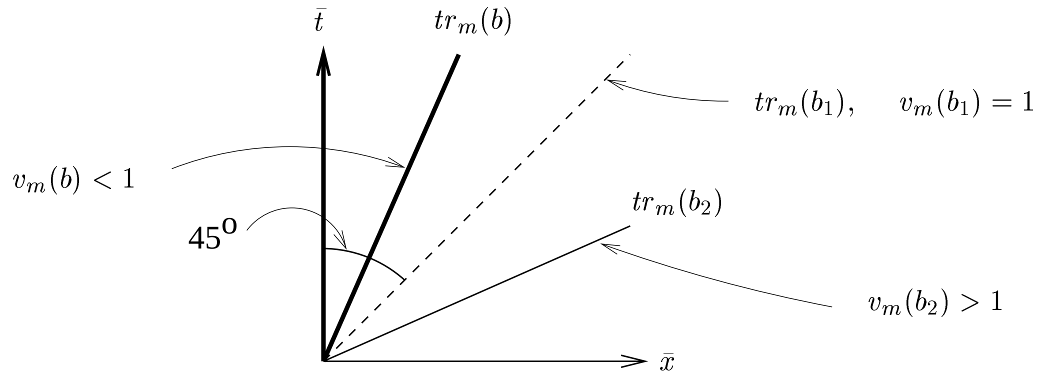


Figure 9: Traces and speeds.

follows (cf. Figures 10, 11). Let $\mathfrak{M}$ be a frame model. Let $m \in Obs$ and $b \in B$ such that $tr_m(b) = \ell = \{ r + a \cdot s \ : \ a \in F \} \in \mathsf{Eucl}$, for some r and $s \neq \bar{0}$. Assume that $s_0 > 0$. Then

$$\vec{v}_m(b) \stackrel{\text{def}}{=} \vec{v}_m(\ell) \stackrel{\text{def}}{=} \langle s_1/s_0, \ldots, s_{n-1}/s_0 \rangle \ (= \langle s_1, \ldots, s_{n-1} \rangle / s_0).$$

If $s_0 = 0$ then

$$\vec{v}_m(b) \stackrel{\text{def}}{=} \{ a \cdot s : a \in F \} .$$[84]

If $s_0 = 0$, then $\vec{v}_m(b)$ is infinite (i.e. $\vec{v}_m(b) = \infty$), therefore we cannot represent $\vec{v}_m(b)$ as a finite vector. Therefore, the information content of $\vec{v}_m(b) = \ell$ (where ℓ is in the space part of ${}^nF$) remains that b is moving in direction ℓ with infinite speed both "forwards" and backwards". We note that the speed $v_m(b)$ is the (square of) distance covered by b in unit time; while the velocity $\vec{v}_m(b)$ is the vector representing the change of location which happened in unit time, see Figure 10, assuming $v_m(b) \neq \infty$. For more on the distinction between speed and velocity cf. e.g. Gardner [95, p.7].

We are ready to postulate axioms **Ax2**–**Ax6**.

Ax2 $Obs \cup Ph \subseteq Ib$.

That is, observers are inertial bodies; and so are photons.

Ax3 $(\forall h \in Ib)(\forall m \in Obs) \Big(tr_m(h) \in G \Big)$.

That is, the life-line of any inertial body h as seen by any observer m must be a "line".

Ax4 $(\forall m \in Obs) \Big(tr_m(m) = \bar{t} \quad (= F \times {}^{n-1}\{0\}) \Big)$.

Ax4 states that the life-line $tr_m(m)$ of an observer as seen by itself is the 0-th axis (the time axis). Thus **Ax4** says that each observer sees itself to be a body at rest (not moving) at (space) location $\langle 0, \ldots, 0 \rangle$. In particular, $v_m(m) = 0$. This is one of the basic

[84] We note that the $s_0 = 0$ case of this definition in <u>*not*</u> very important. If $s_0 = 0$ (and $s \neq \langle 0, \ldots, 0 \rangle$), then the velocity $\vec{v}_m(b)$ is infinite. The direction of this infinite velocity is characterized by the space-like line $\{a \cdot s : a \in F\}$.

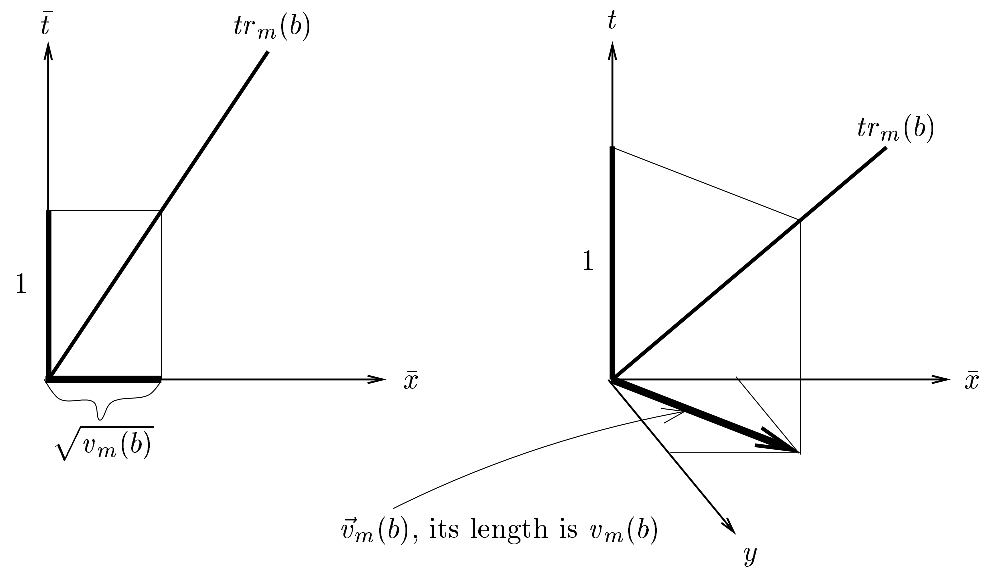


Figure 10: Speed and velocity.

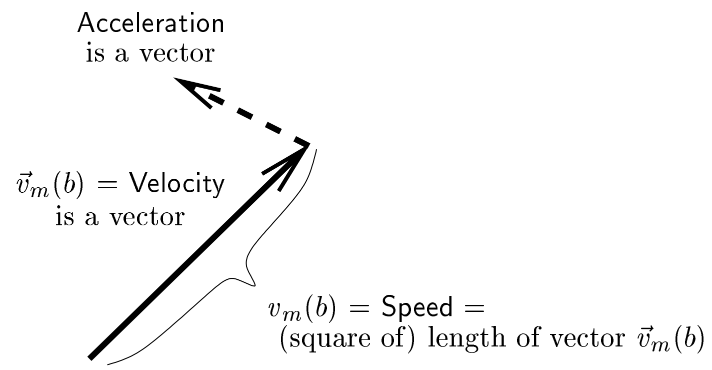


Figure 11: Velocity, speed, and acceleration represented purely in space (the time dimension is suppressed). (Of this figure, acceleration is relevant only in Chapter 8 of a future edition [19] of AMN [18].)

axioms of relativity theory. This was a "relativistic axiom" already before Einstein.[85] It expresses that each inertial observer can "think" that he is at rest and all other bodies are moving. The first step towards general relativity theory will be that we will extend **Ax4** to accelerated observers, too[86]: then even accelerated observers can "think" that they are at rest (and then, in a poetic language, gravity will come into the picture to explain certain strange behavior of other bodies).[87]

Ax5 $(\forall m \in Obs)(\forall \ell \in G)\ \Big(ang^2(\ell) < 1 \ \Rightarrow\ (\exists k \in Obs)\, \ell = tr_m(k)$ and

$$ang^2(\ell) = 1 \ \Rightarrow\ (\exists ph \in Ph)\, \ell = tr_m(ph)\Big).$$

Ax5 makes sense only in the presence of **Ax1** (because $ang^2(\ell)$ is not defined otherwise). Then it states that we have the tools for (performing) thought-experiments: on any appropriate straight line *we can assume* there is an observer; and the same for photons.[88] Later we will weaken the first part of **Ax5** to say that there is a positive c such that in every direction for every positive $\lambda < c$ there is an observer going in that direction and with speed λ. (I.e., $(\forall m)(\exists c > 0)(\forall \ell)[ang^2(\ell) < c \Rightarrow (\exists k \in Obs)\ell = tr_m(k)]$. Cf. $\mathbf{Ax(5nop)^{-+}}$ in the list of axioms (p.A-19) and in AMN [18, Chapter 5, pp.761, 763].) This weaker form of the axiom is sufficient for many purposes.

Ax6 $(\forall m, k \in Obs)\ \Big(Rng(w_m) = Rng(w_k) \Big).$

Ax6 states that all observers see the same set of events. I.e. whenever an observer m sees a set E of bodies at some time point t and space location s, any other observer k must see the same set E of bodies at *some* time point t' and space location s'. In still other words, the same events "exist" or "are available" for all observers. **Ax6** is quite strong. In particular, it will not be true in our theory of accelerated observers (or in general relativity).[89] Later we will weaken **Ax6** to $\mathbf{Ax6_{00}}$ such that the new version will be true for our accelerated observers, too. The new version $\mathbf{Ax6_{00}}$ will say that if m sees an event E on the trace of the observer k, then k itself sees this event E. Cf. §3.

[85] Sometimes a stronger form of this is referred to as *Galileo's relativity principle*. Galileo's relativity principle says a bit more than just **Ax4**. Cf. e.g. Geroch [96], pp.32-39, in particular §3 entitled "The Galilean View". Cf. also Einstein's Special Principle of Relativity (SPR) in §2.8.3 and footnote 185 on p.84.

[86] According to e.g. Friedman [91, p.5], general relativity begins with the study of accelerated observers (or accelerated reference frames), at least when they are treated "equivalently" with inertial reference frames. In this sense, Chapter 8 of a future edition [19] of AMN [18] deals with the (first steps of the) generalization of our (logic-based) method from special relativity to general relativity.

[87] Cf. e.g. p.98 and Figure 47 in the discussion of the twin paradox in §2.8. On an intuitive level, a generalization of **Ax4** called Einstein's Special Principle of Relativity (SPR) states that the "laws of nature" are the same for all inertial observers (or inertial reference frames), cf. §2.8.3, pp.84–87. According to Einstein [80], roughly, the General Principle of Relativity (GPR) (on which, according to Einstein [80], general relativity is, partially, based) generalizes a refinement of this (SPR) to arbitrary, e.g. to accelerated, observers. A price of this generalization is that Einstein had to put restrictions on which statements count as laws of nature, and which do not. Hence the concept of a law of nature we use in formulating SPR^+ in §2.8.3 is not suitable (not refined enough) for the purposes of GPR. (In GPR a key point is that each observer may imagine that he is not moving and it is the rest of the universe which moves, accelerates etc.; and that the "laws of nature" are the same for all observers, cf. §2.8.3.)

[88] In a future edition [19] of AMN [18] we will see a (first-order) modal logic refinement (or variant) of our axioms (and formalism) in which **Ax5** sounds "less radical" (that is, sounds more convincing intuitively). The modal version of **Ax5** avoids making space-time "overcrowded" with observers and photons. Cf. [263].

[89] One reason for this is, very roughly, that if observer k accelerates (in m's world) so fast that its clock will never reach 12 o'clock as seen by m, then the "event" seen by k at 12 o'clock (or after 12) will not be "seen" by m. Cf. e.g. Etesi-Németi [84] or Hogarth [132] for more realistic settings with similar effects.

Our last axiom in the present section is the most distinguished one in relativity theory:[90]

AxE $(\forall m \in Obs)(\forall ph \in Ph)\ v_m(ph) = 1.$

AxE ("*Einstein's axiom*") states that the speed of a photon ph, as seen by any observer m, is always 1. In **Basax**, we choose the "speed of light" to be 1. This is a rather ad-hoc decision, the important part of **AxE** is that all observers see all photons as having the same speed. Later, e.g. in Chapter 4 of AMN [18], we weaken **AxE** in several ways.[91] We will see that *already* most of these weak forms of **AxE** will be enough for proving the majority of the important consequences of **Basax**. In particular, we will see that the weaker postulates saying that in each direction there is a photon going forwards and that "photons do not race with one another like bullets do" in place of **AxE** are already sufficient (together with the other axioms, of course) to prove most of the interesting theorems of special relativity theory. Cf. Chapter 3.

Definition 2.2.3 (Basax) We define

$$\mathbf{Basax} \stackrel{\text{def}}{=} \{\, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{Ax6}, \mathbf{AxE} \,\},$$

where the axioms **Ax1**–**Ax6**, **AxE** were defined above.

Here is a summary of the axioms in **Basax**:

Ax1 $G = \mathsf{Eucl}(n, \mathbf{F}).$

Ax2 $Obs \cup Ph \subseteq Ib.$

Ax3 $(\forall h \in Ib)(\forall m \in Obs)\ \Big(tr_m(h) \in G\Big).$

Ax4 $(\forall m \in Obs)\ \Big(tr_m(m) = \bar{t}\Big).$

Ax5 $(\forall m \in Obs)(\forall \ell \in G)\ \Big(ang^2(\ell) < 1 \;\Rightarrow\; (\exists k \in Obs)\, \ell = tr_m(k)$ and

$$ang^2(\ell) = 1 \;\Rightarrow\; (\exists ph \in Ph)\, \ell = tr_m(ph)\Big).$$

Ax6 $(\forall k, m \in Obs)\ \Big(Rng(w_m) = Rng(w_k)\Big).$

AxE $(\forall m \in Obs)(\forall ph \in Ph)\ v_m(ph) = 1.$ $\lhd$

It follows from **Ax2**,**Ax3** that the trace of any observer is a line. We give this conclusion a name:

(geod) $(\forall m, k \in Obs)\ tr_m(k) \in G.$

[90] One could refer to e.g. the Michelson-Morley experiment for motivation, but instead of doing that, we refer to the introduction of Friedman [91].

[91] One of these says that each observer m sees all photons with the same speed, another one is the Reichenbach-Grünbaum version of **AxE** etc. Cf. Chapter 3 here and Chapter 4 of AMN [18]. Moreover, following an idea of Gyula Dávid [68], in Chapter 5 of AMN [18] we see a variant of **Basax** which (proves most of usual relativity and) does not need **AxE** at all.

Statement **(geod)** together with **Ax1** imply that $tr_m(k)$ is a *Euclidean* straight line. Later, **Ax1** will be generalized so that G will be a more general geometry-like structure, e.g. G might consist of the geodesics of some structure. Beginning with Chapter 8 of a future edition [19] of AMN [18] where we will have accelerated observers too, **(geod)** will be restricted to inertial observers.

The world-view function w_m can be recovered from the family of traces of all bodies (from $\langle tr_m(b) : b \in B \rangle$), and the world-view-relation W can be recovered from all the world-view functions (from $\langle w_m : m \in Obs \rangle$). Thus we can "represent" the function w_m by the world-view of m, which is just the indexed family $\langle tr_m(b) : b \in B \rangle$, and which, in turn, we represent by drawing the traces of bodies that we are interested in. See Figure 12.

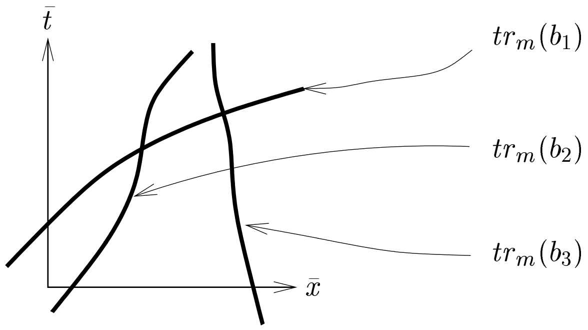


Figure 12: World-view of m.

Assuming **Basax**, we can (and will often) draw the world-view of an observer m as shown in Figure 13. In this figure and in similar pictures, most often we simply write the name of a body h instead of writing out the long expression $tr_m(h)$, when indicating the life-line of h (as seen by m).

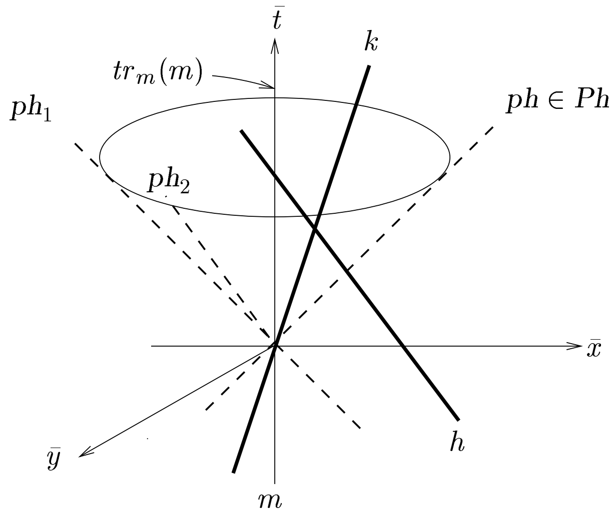


Figure 13: The world-view of an observer m in a model of **Basax**.

We will sometimes use the following.

FACT 2.2.4

(i) *Assume* **Basax**. *Let $h \in Ib$ be an* inertial body *with $v_m(h) \neq \infty$. Then $tr_m(h) : F \longrightarrow {}^{n-1}F$ is a function everywhere defined on F, where we think of F as the time axis $\bar{t}$ and of ${}^{n-1}F$ as "space".*[92]

(ii) *Statement (i) above remains true if we replace* **Basax** *with* $\{$**Ax1**, **Ax3**$\}$.

∎

By the *space part* S of ${}^nF$ we understand the subspace $S \stackrel{\text{def}}{=} \{ \langle 0, q_1, \ldots, q_{n-1} \rangle : q \in {}^nF \}$ $(= \{q \in {}^nF : q_0 = 0\})$. Throughout chapter 2 we will *identify* S with ${}^{n-1}F$ to simplify notation. In later chapters we do *not* identify S with ${}^{n-1}F$. By a *space-vector* we understand an element of ${}^{n-1}F$.

Remark 2.2.5 (Terminology: Observers, reference frames, "slim observers", "fat observers") We call the (sometimes partial[93]) function $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ the *world-view* function of observer m.

(i) Some authors call w_m the *reference frame* of observer m, cf. e.g. d'Inverno [73]. We could have used that word instead of world-view function, it is only a historical accident that we chose the other name.

(ii) Some authors eliminate "observers" and talk *only* about reference frames (i.e. world-view functions) w's (with $w : {}^nF \longrightarrow \mathcal{P}(B)$), instead. This is absolutely justified, because given a world-view function $w : {}^nF \longrightarrow \mathcal{P}(B)$ we can *recover* an observer, call it m, from w such that, after some modifications, basically w will be the world-view function of m. In more detail: We "create" a *new body* m by *postulating* that the set of events in which m is present should be $w[\bar{t}]$. Next, we *expand* all the world-view functions of our model with this new m. With this all properties of m *as a body* are defined. Now, we *raise* m to the *rank* of an observer by postulating that the world-view function w_m of m is *defined* to be w. This construction shows that a reference frame w completely determines an observer m such that m's world-view function is w. The above illustrates that *if we wish* we could forget the observers m and talk about reference frames w instead. Then instead of a set *Obs*, another set *Rfm* of reference frames would be given as one of our primitives. (We could let $Rfm := \{ w_m : m \in Obs \}$.) The above train of thought shows that our approach and the "only reference frames" approach are equivalent (inter-definable) and it is not important whether we start out with observers (*Obs*) or reference frames (*Rfm*) in our basic vocabulary.

(iii) Our observers are "slim" in the respect that their life-lines (or traces) are thin curves in ${}^nF$. This again is not important, it is again *only* a *choice of words*: Namely, we could identify observer m with its world-view function w_m, and then it would cease to be "slim" in the above sense. In passing, we also note that instead of a single body m, we could have used as an observer *m together with a set K* of bodies [slim observers] such that $(\forall k \in K)$ $(tr_m(k)$ would

[92] More precisely, we can regard the relation $tr_m(h) \subseteq {}^nF$ as a function $tr_m(h) : F \longrightarrow {}^{n-1}F$ by identifying $F \times {}^{n-1}F$ with ${}^nF$.

[93] w_m will become a partial function in §3, in AMN [18, §4.9] and in a future edition [19] of AMN [18] e.g. in the chapter on accelerated observers, Chapter 8 of AMN [19].

be parallel with $\bar{t}$).[94] But since the final mathematical effects would remain[95] more or less the same[96] (via interdefinability), we decided to stick with an observer being a single body $m \in B$ and whenever we would need a "fat observer" like K above, we will simply recover it from the reference frame (i.e. world-view) w_m of m.

$\triangleleft$

Remark 2.2.6 Throughout, we will use the standard practice from logic of introducing new relation and function symbols by defining them, and then treating them as if they were symbols of our original language. E.g. we defined the function w_m and then we used it in our axioms (as if it was part of our language). We believe that translating the so enriched language back to the original first-order language is straightforward (and therefore it is better not to include it). For such translating algorithms see e.g. Monk [194, pp. 206–210] or Bell-Machover [46, p.97]. For this translation see also our chapter 4.3 on definability theory, in particular subsection 4.3.3 and Convention 2.3.10 on p.31.

$\triangleleft$

[94] In this case we could think of an element k of K as a "partner" of m representing a time-like coordinate-line for m. Then $m \in K$ could be called the "central partner" in $\langle m, K\rangle$. Such an observer $\langle m, K\rangle$ could be visualized as a cloud of "particles" floating in space and each particle having a clock. Etc.

[95] at least from the point of view of questions investigated in this work

[96] As we said, on the long run we allow $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ to be a partial function, i.e. $Dom(w_m) \subsetneq {}^nF$ is allowed.

2.3 Some properties of Basax, world-view transformation

In this section we introduce the notion of world-view transformations. We discuss some simple consequences of our basic axioms – to get a feel for them – and then we investigate those functions that occur as world-view transformations in models of **Basax**. We close this section with listing some basic properties of **Basax** as a logical theory (like consistency, independence, categoricity).

Definition 2.3.1 (world-view transformation)
Given $m, k \in Obs$, we define the *world-view transformation* f_{mk} as follows:

$$\mathsf{f}_{mk} \stackrel{\text{def}}{=} w_m \circ w_k^{-1} .$$

$\lhd$

We note that w_k^{-1} is a relation, hence the composition $w_m \circ w_k^{-1}$ is again a relation, cf. the definition of composition on p.1. Thus $\mathsf{f}_{mk} \subseteq {}^nF \times {}^nF$ and

$$\mathsf{f}_{mk} = \{ \langle p, q\rangle \in {}^nF \times {}^nF \ : \ w_m(p) = w_k(q) \} ,$$

see Figure 14. Thus f_{mk} is a binary relation on the coordinate system ${}^nF$; two points are f_{mk}-related when m and k see the same "events" at those points. See also Figure 15.

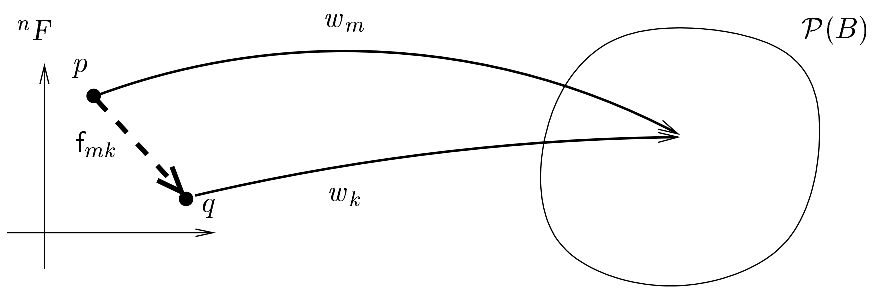


Figure 14: The world-view transformation.

The name "world-view *transformation*" suggests that f_{mk} is a function, i.e. to any $p \in {}^nF$ there is at most one q such that p is f_{mk}-related to q. This indeed will be the case in models of e.g. **Basax**, see Prop.2.3.3(v).[97] In arbitrary frame models, f_{mk} can be an arbitrary binary relation.[98] As we said, in models of **Basax**, f_{mk} cannot be an arbitrary binary relation, e.g. it has to be a function. Towards the end of this section we *characterize* those functions that occur in models of **Basax**(2) as world-view transformations cf. Thm.2.3.12, and also there we give some hints for the $n > 2$ case.

Figure 15 illustrates the world-view transformation f_{mk} for the 2-dimensional case. We drew the picture under the assumption that $\mathsf{f}_{mk} : {}^2F \longrightarrow {}^2F$, and we indicated two copies of

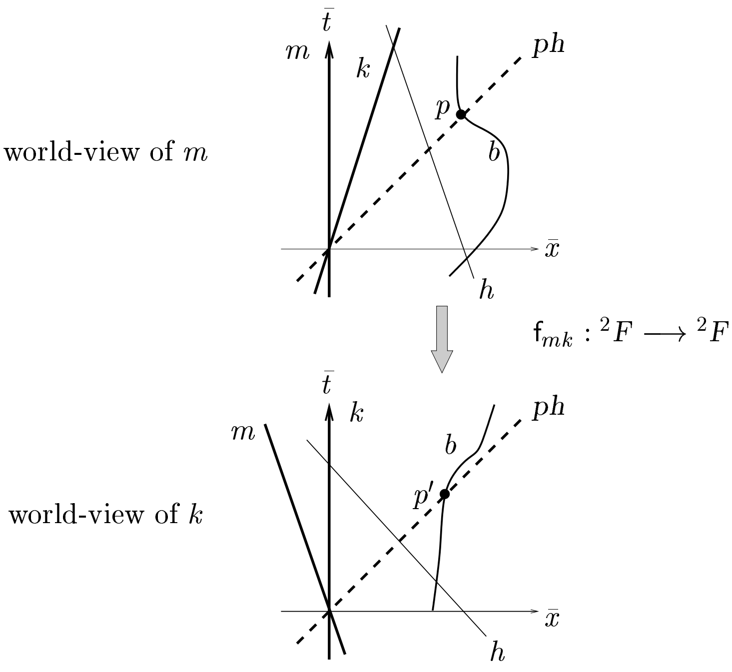


Figure 15: World-view transformation. The event when "m, k and ph are together" happens at $\bar{0}$ both for m and for k, hence $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. The event when ph and b are together happens at p for m and at p' for k; thus $\mathsf{f}_{mk}(p) = p'$.

${}^2F$, the usual coordinate system way. The world-view of m is illustrated in the top coordinate system, and the world-view of k is in the bottom coordinate system (we did not represent in the picture all traces and all points of the world-views).

As a warm-up we begin with simple statements about our axiom system **Basax**. Let us recall that $\mathsf{Eucl} = \mathsf{Eucl}(n, \mathfrak{F})$ is the set of straight lines defined on p.18.

Notation 2.3.2 We define the sets of n-dimensional *slow-lines* SlowEucl and *photon-lines* PhtEucl over an ordered field $\mathfrak{F}$ as follows.

$$\begin{aligned}\mathsf{SlowEucl} &\stackrel{\text{def}}{=} \mathsf{SlowEucl}(n,\mathfrak{F}) \stackrel{\text{def}}{=} \{\, \ell \in \mathsf{Eucl}(n,\mathfrak{F}) \; : \; ang^2(\ell) < 1 \,\} ,\\ \mathsf{PhtEucl} &\stackrel{\text{def}}{=} \mathsf{PhtEucl}(n,\mathfrak{F}) \stackrel{\text{def}}{=} \{\, \ell \in \mathsf{Eucl}(n,\mathfrak{F}) \; : \; ang^2(\ell) = 1 \,\} . \quad \triangleleft\end{aligned}$$

In connection with Prop.2.3.3(x) below, let us recall from p.2, that Id is the identity function on ${}^nF$.

PROPOSITION 2.3.3 *Let $\mathfrak{M}$ be a frame model of* **Basax**. *Then the following are true for all $m, k, h \in Obs$, $ph \in Ph$ and $b \in B$.*

(i) $Obs \cap Ph = \emptyset$, *i.e. no photon can be an observer.*

(ii) $tr_m(k) \neq tr_m(ph)$, *i.e. no observer can travel together with a photon.*

[97] f_{mk} will be a *partial function* in all of the axiom systems, besides **Basax**, studied in the present work.

[98] By this we mean that for any ordered field $\mathfrak{F}$ and a binary relation $R \subseteq {}^nF \times {}^nF$, there are a frame model $\mathfrak{M}$ and two observers m, k in $\mathfrak{M}$ such that $R = \mathsf{f}_{mk}$.

(iii) $v_m(k) \neq 1$, *i.e. the speed of an observer is never* 1.

(iv) *The world-view function* w_m *is an injection (i.e. one-one). That is, no observer "sees" the same event at two different space-time locations.*

(v) *The world-view transformation* f_{mk} *is a bijection (i.e. one-one, defined on* ${}^nF$ *and onto* ${}^nF$*).*

(vi) $w_m = \mathsf{f}_{mk} \circ w_k$. *I.e. we get the world-view of* m *from that of* k *by "applying* f_{mk}*" to it;* f_{mk} *is the "conversion" between* m*'s and* k*'s world-views.*

(vii) f_{mk} *takes the trace of a body as seen by* m *to the trace of the body as seen by* k*, i.e.* $\mathsf{f}_{mk}[tr_m(b)] = tr_k(b)$.

(viii) f_{mk} *takes slow-lines to straight lines, i.e. if* $\ell \in \mathsf{SlowEucl}$*,* $\mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$.

(ix) f_{mk} *takes photon-lines to photon-lines, i.e. if* $\ell \in \mathsf{PhtEucl}$*,* $\mathsf{f}_{mk}[\ell] \in \mathsf{PhtEucl}$.

(x) $\mathsf{f}_{mm} = \mathrm{Id}$, $\mathsf{f}_{mk} = \mathsf{f}_{km}^{-1}$, *and* $\mathsf{f}_{mk} = \mathsf{f}_{mh} \circ \mathsf{f}_{hk}$.

All of the statements in Proposition 2.3.3 can be expressed with (first-order) formulas in our frame-language. We note that none of (i)-(ix) in Prop.2.3.3 is true without assuming (at least part of) **Basax**. We invite the reader to construct frame models in which these statements fail. We will prove the items in Prop.2.3.3 one-by-one, so that we can single out the axioms we need for proving them.

Claim 2.3.4 $\{\mathbf{Ax4}, \mathbf{AxE}\} \models \mathit{Obs} \cap \mathit{Ph} = \emptyset$.

Proof: Assume that $m \in \mathit{Obs} \cap \mathit{Ph}$. Look at $v_m(m)$. By **Ax4** we have that $v_m(m) = 0$, and by **AxE** and $m \in \mathit{Ph}$ we have that $v_m(m) = 1$. Since in all fields 0 and 1 are different elements, we reached a contradiction. ∎

Claim 2.3.5 $\{\mathbf{Ax4}, \mathbf{Ax6}, \mathbf{AxE}\} \models tr_m(k) \neq tr_m(ph)$.

Proof: Assume that $tr_m(k) = tr_m(ph)$. Then $tr_k(k) = \bar{t}$ and $v_k(ph) = 1$ by **Ax4** and **AxE**; in this connection note that $v_k(ph) = 1$ implies that $tr_k(ph) \in \mathsf{Eucl}$ by the convention on p.19. Thus $tr_k(k) \neq tr_k(ph)$. Then k sees an event in which k is present but ph is not present (namely, such is $w_k(p)$ for any $p \in tr_k(k) \setminus tr_k(ph)$). However, m does not see such an event by $tr_m(k) = tr_m(ph)$. This contradicts **Ax6**, proving the proposition. See Figure 16. ∎

Claim 2.3.6 $\{\mathbf{Ax1}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{Ax6}, \mathbf{AxE}\} \models v_m(k) \neq 1$.

Proof: Assume that $v_m(k) = 1$ for some $m, k \in \mathit{Obs}$. Then $ang^2(tr_m(k)) = 1$, thus by **Ax5**, $tr_m(k) = tr_m(ph)$ for some $ph \in \mathit{Ph}$. This contradicts Claim 2.3.5. ∎

Claim 2.3.7 $\{\mathbf{Ax1}, \mathbf{Ax5}\} \models (\forall m \in \mathit{Obs})(w_m$ is an injection$)$.

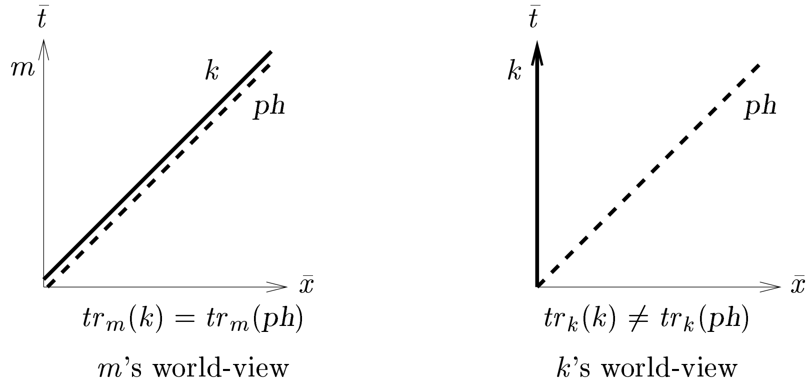


Figure 16: An observer cannot travel together with a photon.

Proof: Let $m \in Obs$ and assume that $p, q \in {}^nF$, $p \neq q$. Then, by **Ax1** and by the properties of $\mathsf{Eucl}(n, \mathbf{F})$, $(\exists \ell \in G)(p \in \ell \ \wedge \ q \notin \ell \ \wedge \ ang^2(\ell) < 1)$. By **Ax5**, $(\exists k \in Obs)\ell = tr_m(k)$. For such a k, $k \in w_m(p)$ but $k \notin w_m(q)$. ∎

Claim 2.3.8

(i) $\{\mathbf{Ax1}, \mathbf{Ax5}, \mathbf{Ax6}\} \models (\mathsf{f}_{mk}$ is a bijection $\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF)$.

(ii) $\{\mathbf{Ax1}, \mathbf{Ax5}\} \models (\mathsf{f}_{mk}$ is a (possibly) partial one-to-one function).

(iii) $\{\mathbf{Ax1}, \mathbf{Ax5}, \mathbf{Ax6}\} \models (\mathsf{f}_{mm} = \mathrm{Id}, \mathsf{f}_{mk} = \mathsf{f}_{km}^{-1}, \mathsf{f}_{mk} = \mathsf{f}_{mh} \circ \mathsf{f}_{hk})$.

Proof: That f_{mk} is one-to-one follows from Claim 2.3.7. That f_{mk} is defined everywhere and is onto ${}^nF$ follows from **Ax6**. $\mathsf{f}_{mm} = \mathrm{Id}$, $\mathsf{f}_{mk} = \mathsf{f}_{km}^{-1}$ and $\mathsf{f}_{mk} \supseteq \mathsf{f}_{mh} \circ \mathsf{f}_{hk}$ follow from the definition of the world-view transformation relations. Assume **Ax1**, **Ax5**, **Ax6**, let $p \in {}^nF$, and $\mathsf{f}_{mk}(p) = q$, i.e. $w_m(p) = w_k(q)$. By **Ax6** there is $p' \in {}^nF$ such that $w_m(p) = w_h(p')$. Now $\mathsf{f}_{mh}(p) = p'$ by $w_m(p) = w_h(p')$ and $\mathsf{f}_{hk}(p') = q$ by $w_h(p') = w_m(p) = w_k(q)$. Thus $\mathsf{f}_{mk}(p) = \mathsf{f}_{hk}(\mathsf{f}_{mh}(p))$. ∎

Remark 2.3.9 By Claim 2.3.8 we have that if the set $Wtm \stackrel{\text{def}}{=} Wtm^{\mathfrak{M}} \stackrel{\text{def}}{=} \{\mathsf{f}_{mk} : m, k \in Obs^{\mathfrak{M}}\}$ of the world-view transformations is closed under composition $\circ$, $\langle Wtm, \circ, {}^{-1}, \mathrm{Id}\rangle$ forms a group (under assuming **Ax1**,**Ax5**,**Ax6**). In AMN [18, Def.3.6.11, p.269] we define a class GM of models of **Basax**, such that for some $\mathfrak{M} \in \mathsf{GM}$ we have that Wtm is not closed under composition.[99] However, in §2.8 we introduce a "symmetry axiom" **Ax□1** and we see in AMN [18] that if $\mathfrak{M} \models \mathbf{Basax} \cup \{\mathbf{Ax\Box 1}\}$, $\langle Wtm^{\mathfrak{M}}, \circ, {}^{-1}, \mathrm{Id}\rangle$ is a group. ◁

The proof of Prop.2.3.3(viii) consists of noting that every slow-line is the trace of some observer k_1 as seen by m, and that $tr_k(k_1)$ is a straight line. Similarly, the proof of Prop.2.3.3(ix) consists of noting that every photon-line is a trace of some photon ph_1 as seen by m (by **Ax5**), and that $tr_k(ph_1)$ is a photon-line again (by **AxE**). The proofs of Proposition 2.3.3 (vi), (vii) are similar to those of Proposition 2.3.3 (i)–(v), (x). We leave them to the reader. By this, Proposition 2.3.3 has been proved. ∎

[99] For more on models $\mathfrak{M}$ of **Basax** in which Wtm is not a group cf. section 3.10 of AMN [18]. Cf. also [261].

By Claim 2.3.8(ii), in most of the situations we will investigate, f_{mk} will be a *function*. This will remain so, even when we will study refinements of our axiom system **Basax**, or even when we will omit some or most of our axioms, f_{mk} will be at least a *partial function* ${}^nF \supseteq Dom(\mathsf{f}_{mk}) \xrightarrow{\mathsf{f}_{mk}} {}^nF$. Therefore, we would like to use the standard notation $\mathsf{f}_{mk}(p)$ when $p \in {}^nF$ as if f_{mk} were a (partial) *function* symbol. But then (since in our original frame-language f_{mk} is only a relation symbol) we have to define a translation mechanism ensuring that the formulas involving notation like $\mathsf{f}_{mk}(p)$ remain formulas of our frame language. To ensure this we make the following convention.

CONVENTION 2.3.10 We introduced f_{mk} as a binary relation symbol (in the extended version of our frame-language). Since in models of **Basax** it is a function (cf. Prop.2.3.3(v)), we will also use f_{mk} as if it were a unary function symbol. There is a well known practice of doing this; a precise translation algorithm can be found e.g. in Monk [194, pp. 206–210] or Bell-Machover [46, p.97] ("Elimination of function symbols"). However, *later* we want to treat theories where f_{mk}'s will be *only partial* functions. Therefore, instead of the algorithms for translating total functions given e.g. in Monk [194], we want to use a slightly more general translation algorithm suitable for handling partial functions as well, see e.g. Andréka-Németi [29]. This translation is quite intuitive: whenever we write "$\mathsf{f}_{mk}(p)$" we mean "f_{mk} is defined on p, i.e. there is a unique q such that $\langle p,q\rangle \in \mathsf{f}_{mk}$, and $\mathsf{f}_{mk}(p)$ denotes this unique q".

In more detail: Let τ, $\sigma(p)$ be terms and R be a relation symbol like "$=$" or "$\leq$" in our frame language (expanded, for convenience, with the language of the vector space[100] ${}^n\mathbf{F}_2$). Let us recall that p, q are variable symbols ranging over ${}^nF$. Then an atomic formula of the "shape" $\mathsf{f}_{mk}(p) = \tau$ means

$$\exists! q\ (\langle p,q\rangle \in \mathsf{f}_{mk}) \quad \wedge \quad \exists q\ (\langle p,q\rangle \in \mathsf{f}_{mk}\ \wedge\ q = \tau),$$

where q is a new variable and "$\exists!$" means "there is a unique". That is, the new formula says, f_{mk} is defined on the argument p and is a function on $\{p\}$ and $\mathsf{f}_{mk}(p) = \tau$.[101]

Similar convention applies to more general atomic formulas like $R(\mathsf{f}_{mk}(p), \tau)$ or $\sigma\Big(\mathsf{f}_{mk}(p)\Big) = \tau$. In both cases the new formula begins with $\exists! q(\langle p,q\rangle \in \mathsf{f}_{mk})$. E.g. the translated version of the second formula is

$$\exists! q\ (\langle p,q\rangle \in \mathsf{f}_{mk}) \quad \wedge \quad \exists q\ (\langle p,q\rangle \in \mathsf{f}_{mk}\ \wedge\ \sigma(q) = \tau)\,,$$

where q does not occur in τ or $\sigma(p)$.

Let Tr denote the "translation function" which we are in the process of defining, which is defined on formulas, and which eliminates *function-symbol* style occurrences of the f_{mk}'s. So far we described how to translate atomic formulas, call them φ_i, possibly containing f_{mk}'s as function symbols to new formulas $Tr(\varphi_i)$ in which f_{mk}'s do not occur as function symbols (and hence $Tr(\varphi_i)$ is truly in our frame language). Now, if we want to translate a *complex formula*, call it ψ, the same way (i.e. eliminate using f_{mk}'s as functions), first we translate all the atomic formulas φ_i occurring in ψ, and then we put together the translations exactly as ψ was put together. E.g. $Tr(\varphi \wedge \psi) = Tr(\varphi) \wedge Tr(\psi)$, $Tr(\neg\varphi) = \neg Tr(\varphi)$, $Tr(\exists x\varphi) = \exists x(Tr(\varphi))$.

$\lhd$

Now, we turn to characterizing the world-view transformations in models of **Basax**(2). Figures 17 and 18 illustrate these transformations, and give perhaps a hint for why we will call

[100] As we already said, ${}^n\mathbf{F}_2$ formulas are translatable to our frame language.

[101] The first subformula $\exists! q\ \langle p,q\rangle \in \mathsf{f}_{mk}$ means, simply, that $\mathsf{f}_{mk}(p)$ is uniquely defined.

such transformations later "rhombus transformations". Their relationship with the literature (Lorentz transformations, Poincaré transformations) is discussed in §2.9.[102] In Figure 18 the world-view transformation f_{mk} is illustrated in such a way that the world-views of both k and m are drawn in the same copy of ${}^2F$. I.e. k's coordinate system is drawn into m's world-view, cf. also Figure 17.

Before giving the characterization (of the world-view transformations), we cite a theorem from the next chapter. (Cf. Thm.3.2.6 on p.110.)

THEOREM 2.3.11 *Assume* **Basax**. *Let* $m, k \in Obs$. *Then* f_{mk} *takes straight lines to straight lines, that is,* $(\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$.

We **prove** the above theorem in the form of the more general Thm.3.2.6 (p.110). ∎

Throughout, by a *transformation* f (of ${}^n\mathbf{F}$) we mean a function $\mathsf{f} : {}^nF \longrightarrow {}^nF$.[103] By a *photon-preserving* transformation f (of ${}^n\mathbf{F}$) we mean a bijective transformation such that both f and f^{-1} take photon-lines to photon-lines. Further, by a *collineation* f (of ${}^n\mathbf{F}$) we mean a transformation (of ${}^n\mathbf{F}$) which takes straight lines to straight lines, i.e. which preserves Eucl. A *homomorphism* between two structures is defined the natural way. Intuitively, it is a structure preserving map between the universes of the structures involved. A detailed definition is given in Convention 4.3.1 (p.220). We note that *structure* means both (universal) algebra, model and any combination of the two (like our frame-models $\mathfrak{M}$). We recall from the standard literature of algebra that by a *linear transformation* of a vector space ${}^n\mathbf{F}$ we understand a homomorphism of the one-sorted vector space ${}^n\mathbf{F}_1$ onto itself, cf. e.g. Halmos [114]. (The homomorphisms of the two-sorted vector space ${}^n\mathbf{F}_2$ into itself are something else, cf. Remark 2.3.13.) An *automorphism* of a structure is an injective and surjective homomorphism of that structure into itself such that the inverse of the map is a homomorphism, too (cf. AMN [18, p.160] for more detail).

THEOREM 2.3.12 (Characterization of world-view transformations in Basax(2).) *Let* $\mathfrak{F} = \langle \mathbf{F}, \leq \rangle$ *be any ordered field, and* $\mathsf{f} : {}^2F \longrightarrow {}^2F$.

1. *Assume first that* $\mathbf{F}$ *has no (nontrivial) automorphisms*[104] *and* $\mathsf{f}(\bar{0}) = \bar{0}$. *Then (i)–(iii) below are equivalent.*

   (i) f *is a world-view transformation in some model of* **Basax**(2) *whose ordered field reduct is* $\mathfrak{F}$.[105]

   (ii) f *is like on Figures 17 and 18, i.e.* f *is a bijective linear transformation of the vector-space* ${}^2\mathbf{F}$ *such that* $\mathsf{f}[\bar{t}]$ *and* $\mathsf{f}[\bar{x}]$ *are mirror images of each other w.r.t. a photon-line passing through* $\bar{0}$. *Moreover, the vectors* $\mathsf{f}(\langle 1, 0\rangle)$ *and* $\mathsf{f}(\langle 0, 1\rangle)$ *are of the same length.*[106] *Cf. Figure 19.*

---

[102] Using that terminology, world-view transformations in models of **Basax** are exactly the Poincaré transformations composed with expansions and with functions induced by field-automorphisms. Cf. Theorem 2.9.4.

[103] Often we write *mapping* or *map* instead of transformation, e.g. photon-preserving mapping or linear mapping.

[104] Let us note that the property of $\mathfrak{M}$ that "$\mathbf{F}^{\mathfrak{M}}$ has no (nontrivial) automorphisms" cannot be expressed by a set of (first-order) formulas in our frame-language, since this property is not preserved under taking ultrapowers. We also note that the field of reals (real numbers) and the field of rational numbers enjoy this property.

[105] I.e. $(\exists \mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(2)))\,[(\exists m, k \in Obs)\ \mathsf{f} = \mathsf{f}_{km}$ and $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{F}]$.

[106] We use $p_0^2 + p_1^2$ for the length of $p \in {}^2F$. (We do not take square roots because no axiom ensures their existence *yet*.)

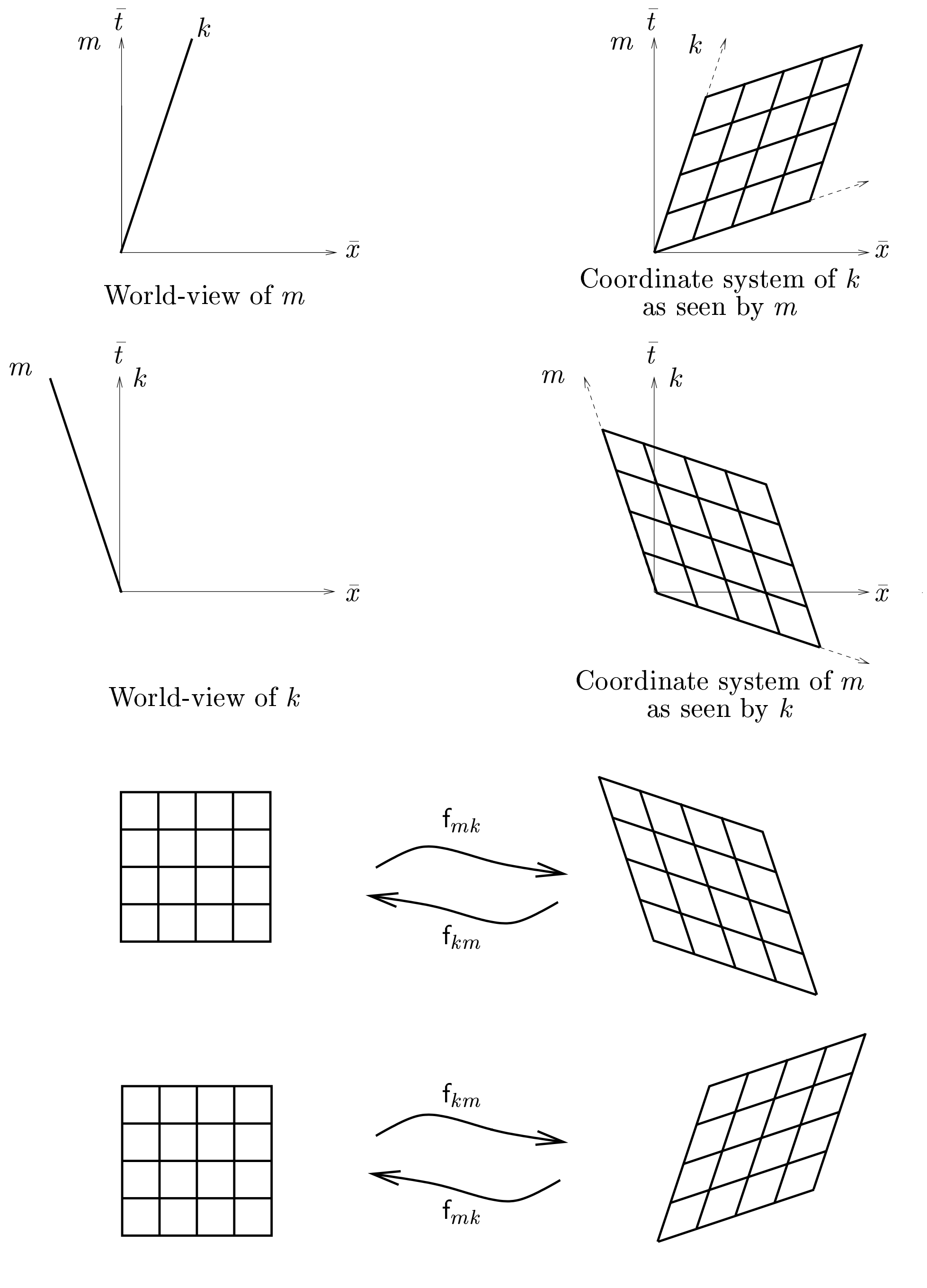


Figure 17: World-view transformation in two space-time dimensions assuming **Basax**.

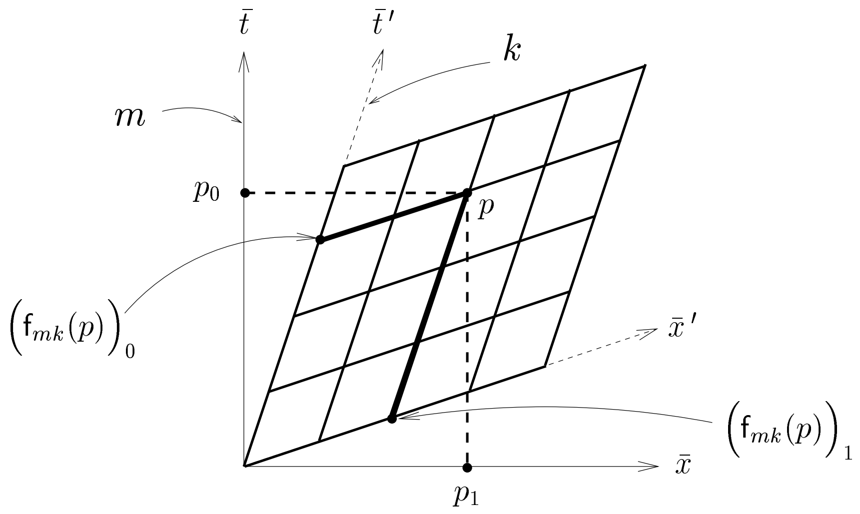


Figure 18: Two-dimensional word-view transformation in **Basax**(2).

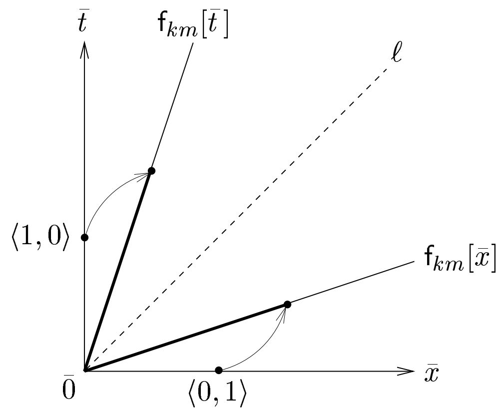


Figure 19: Two-dimensional world-view transformation in **Basax**(2).

(iii) f *is a photon-preserving bijective* <u>collineation</u> *(i.e.* f *is bijective, takes straight lines and photon-lines to straight lines and photon-lines respectively).*

2. *In the more general case when* $\mathbf{F}$ *is permitted to have (nontrivial) automorphisms, we still have that (i), (iii) above are equivalent (with each other and) with both (ii)' and (ii)** *below:*

(ii)' $\mathsf{f} = \gamma \circ \mathsf{g}$ *where* g *is like* f *was in (ii) above and there is an automorphism* $\varphi : \mathbf{F} \longrightarrow \mathbf{F}$ *of* $\mathbf{F}$ *such that* $\gamma(p) = \langle \varphi(p_0), \varphi(p_1) \rangle$ *for all* $p \in F^2$. [107]

(ii)* f *is a bijective collineation such that* $\mathsf{f}(\langle 1, 0 \rangle)$ *and* $\mathsf{f}(\langle 0, 1 \rangle)$ *are mirror images of each other w.r.t. a photon-line passing through* $\bar{0}$. *I.e.* f *is like on Figures 17–19.*

3. *If in 2 above we drop the assumption* $\mathsf{f}(\bar{0}) = \bar{0}$, *(ii)' and (ii)** *have to be changed to (ii)" and (ii)***, *respectively, below.*

(ii)" f *is a composition of a function* f' *which is like in (ii)' and a translation, i.e.* $\mathsf{f} = \mathsf{f}' \circ \tau$ *where* f' *is exactly like* f *was in (ii)' and* $\tau : {}^2F \longrightarrow {}^2F$ *is a translation*[108].

(ii)** f *is a bijective collineation such that* $\mathsf{f}(\langle 1, 0 \rangle)$ *and* $\mathsf{f}(\langle 0, 1 \rangle)$ *are mirror images of each other w.r.t. a photon-line passing through* $\mathsf{f}(\bar{0})$.

Before proving Thm.2.3.12, we include the following two remarks.

Remark 2.3.13 The bijective collineations of ${}^n\mathbf{F}$ came up in the above theorem (and they will keep on coming up later, too). Therefore, we note that the $\bar{0}$-preserving <u>*bijective collineations*</u> are exactly the <u>*automorphisms*</u> of the two-sorted version ${}^n\mathbf{F}_2$ of the vector space ${}^n\mathbf{F}$.

Another characterization (of the bijective collineations preserving $\bar{0}$) is that they are exactly the maps obtainable as a composition of a bijective linear transformation (i.e. an automorphism of the one-sorted version ${}^n\mathbf{F}_1$ of the vector-space) and a map induced by an automorphism of the field $\mathbf{F}$. Cf. Lemma 3.1.6 on p.163 of AMN [18].

$\lhd$

Remark 2.3.14 The above theorem (characterizing the f_{mk}'s) involves <u>*field*</u> <u>*automorphisms*</u>. Intuitive (as well as mathematical) discussion of field automorphisms with examples, pictures, and their roles in **Basax** models, in collineations and in the world-view transformations (the f_{mk}'s) will be discussed in a separate item in Chapter 3 of a future edition [19] of AMN [18]. We note that a partial version of the just promised dicussion (of field automorphisms etc.) can be found in the 1997 October 27 version of [25], pp. 25–26. The just promised discussion will include e.g. the following: (i) In any **Basax** model $\mathfrak{M}$, if $\mathfrak{F}^{\mathfrak{M}}$ is Archimedean[109] and Euclidean (for "Euclidean" see p. 55), the f_{mk}'s are affine transformations[110].[111] (ii) There are **Basax**(2) models with Archimedean ordered field reducts containing non-betweenness preserving hence

[107] To help the reader's intuition we note that $\gamma \circ \mathsf{g}$ on the points with rational coordinates, e.g. $p = \langle 1, 1 \rangle$, is the same as g. (Let us recall that, for any $\mathfrak{F}$ the rational numbers can be considered as elements of $\mathfrak{F}$.)

[108] A translation is a map of the form $\langle p + q \ : \ p \in {}^nF \rangle$, where $q \in {}^nF$ is fixed.

[109] $\mathfrak{F}$ is <u>*Archimedean*</u> iff to each positive $x \in F$ there is a natural number $\varrho \in \omega$ which is larger than x, i.e. $\varrho > x$. (We note that for every ordered field the set ω of the natural numbers can be considered as a subset of the ordered field, or in more careful wording ω is embeddable into the ordered field in a natural way.) For brevity, by <u>*"Archimedean field"*</u> we mean "Archimedean ordered field". We further note that $\mathfrak{F}$ is <u>*Archimedean*</u> iff it is embeddable into (i.e. isomorphic to a subfield of) $\mathfrak{R}$.

[110] Affine transformations are linear transformations composed with translations, as we will discuss this in §2.9.

[111] For undefined terminology the reader is referred to the Index.

non-continuous and not affine world-view transformations.[112] (iii) If, to **Basax**(2), we add the axiom that the f_{mk}'s are betweenness preserving, we will obtain a strictly stronger and natural version (of **Basax**(2)). (For $n > 2$, **Basax** implies that the f_{mk}'s are betweenness preserving, cf. Prop.4.5.4 on p.289 of AMN [18]). (iv) We guess that in **Basax** models the assumption that the f_{mk}'s are betweenness preserving implies that they are continuous, but we did not check this. (v) There are **Basax** models with Euclidean ordered field reducts in which some of the f_{mk}'s are not affine, for every $n \geq 2$. (vi) There are **Basax** models with Euclidean ordered field reducts where some of the f_{mk}'s are continuous collineations which are still not affine transformations. This means that if we add to **Basax** continuity of the f_{mk}'s as an extra axiom, we still cannot force all the f_{mk}'s to be affine. (vii) If $n > 2$ and $\mathfrak{F}$ is a reduct of a **Basax** model, all the automorphisms of $\mathbf{F}$ are order preserving, i.e. using a standard notation of universal algebra $Aut(\mathbf{F}) = Aut(\mathfrak{F})$, cf. Corollary 6.7.12 on p.1142 of AMN [18].

$\triangleleft$

Proof of Thm.2.3.12: The main idea of the proof is illustrated in Figure 20.

Assume that $\mathbf{F}$ has no (nontrivial) automorphisms and $\mathsf{f}(\bar{0}) = \bar{0}$.

(i) $\Rightarrow$ (iii): f is a bijection and photon-preserving by Prop.2.3.3(v),(ix); and f is a collineation by Thm.2.3.11.

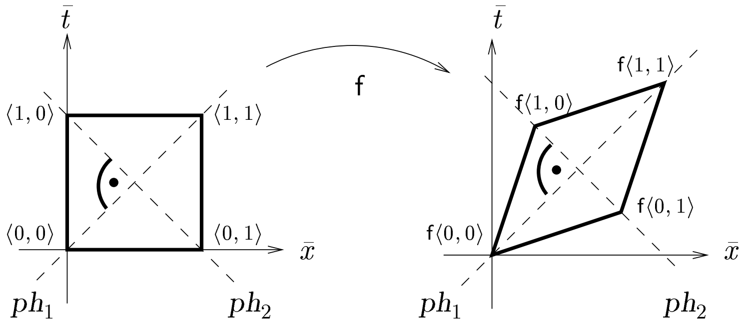


Figure 20: This is the main idea of the proof of Thm.2.3.12.

(iii) $\Rightarrow$ (ii): Since $\mathbf{F}$ has no (nontrivial) automorphisms and $\mathsf{f}(\bar{0}) = \bar{0}$, a bijective collineation is a linear transformation, cf. Remark 2.3.13. If we do not assume that $\mathbf{F}$ has no (nontrivial) automorphisms – but we still assume $\mathsf{f}(\bar{0}) = \bar{0}$ – , f is like in (ii)', i.e. f is a composition of a linear transformation with a map coming from a field automorphism, cf. Remark 2.3.13. If we do not assume $\mathsf{f}(\bar{0}) = \bar{0}$ either, f is like in (ii)", i.e. we have to compose with a translation also. The main idea of the rest of the proof is illustrated in Figure 20.

For any two distinct points $p, q \in {}^2F$, $\overline{pq}$ denotes the Euclidean line containing both p and q.

Consider the two photon-lines (in Figure 20) illustrated on the left-hand copy of ${}^2F$, they are $\overline{\langle 0,0\rangle\langle 1,1\rangle}$ and $\overline{\langle 1,0\rangle\langle 0,1\rangle}$. These two photon-lines are taken to $\overline{\mathsf{f}\langle 0,0\rangle\mathsf{f}\langle 1,1\rangle}$ and $\overline{\mathsf{f}\langle 1,0\rangle\mathsf{f}\langle 0,1\rangle}$.[113]

These last two are photon-lines because f is photon-line preserving. They cannot be parallel, because the original two photon-lines are not parallel. Thus they have to be orthogonal (in the usual Euclidean sense) to each other because we are in two dimensions. The two pairs of lines

[112] We note that "affine $\Rightarrow$ continuous $\Rightarrow$ betweenness preserving" (for f_{mk}'s of **Basax** models if $\mathfrak{F} = \mathfrak{R}$).
[113] Sometimes we write $\mathsf{f}p$ for $\mathsf{f}(p)$ like $\mathsf{f}\langle 0,0\rangle$ for $\mathsf{f}(\langle 0,0\rangle)$.

$\overline{\mathsf{f}\langle 0,0\rangle\mathsf{f}\langle 1,0\rangle}$, $\overline{\mathsf{f}\langle 0,1\rangle\mathsf{f}\langle 1,1\rangle}$ and $\overline{\mathsf{f}\langle 0,0\rangle\mathsf{f}\langle 0,1\rangle}$, $\overline{\mathsf{f}\langle 1,0\rangle\mathsf{f}\langle 1,1\rangle}$ are parallel because the original lines are so. Thus the square with vertices $\langle 0,0\rangle$, $\langle 1,0\rangle$, $\langle 0,1\rangle$, $\langle 1,1\rangle$ is taken to the parallelogram with vertices $\mathsf{f}\langle 0,0\rangle$, $\mathsf{f}\langle 1,0\rangle$, $\mathsf{f}\langle 0,1\rangle$, $\mathsf{f}\langle 1,1\rangle$. The latter parallelogram is indeed a rhombus, because its diagonals are orthogonal. This implies (ii).
(ii) $\Rightarrow$ (i): We prove this as AMN [18, Thm.2.4.2]. ∎

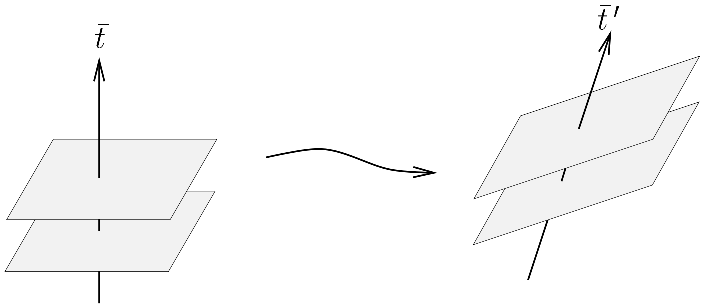


Figure 21: World-view transformation in three space-time dimensions, cf. Figures 17 and 18.

A large part of Thm.2.3.12 remains true in higher dimensions (i.e. for **Basax**(n) in place of **Basax**(2)), e.g., under a mild extra assumption[114] on $\mathfrak{F}$, (i) and (iii) remain equivalent, cf. Thm.2.9.4 on p.103 and Thm.3.6.16 on p.273 of AMN [18]. We now generalize the kind of transformations described in (ii) (of Thm.2.3.12) to arbitrary dimensions $n \geq 2$; we will call such transformations "rhombus transformations". Cf. Figures 21, 22.

In two dimensions, the trace of an observer, as everything else, is in the plane of the time-axis and the $\bar{x}$-axis. In higher dimensions this is not so. Below we will single out a special case in higher dimensions that resembles the 2-dimensional case, and we will call it "standard configuration".

Notation 2.3.15

1. For every $i \in n$, $1_i \in {}^nF$ denotes the <u>*unit vector*</u> pointing in direction of the i'th coordinate axis $\bar{x}_i$, that is,

$$1_i \stackrel{\text{def}}{=} \langle \underbrace{0,\ldots,0}_{i}, 1, \underbrace{0,\ldots,0}_{(n-i-1)} \rangle \,.$$

   Usually, we will write

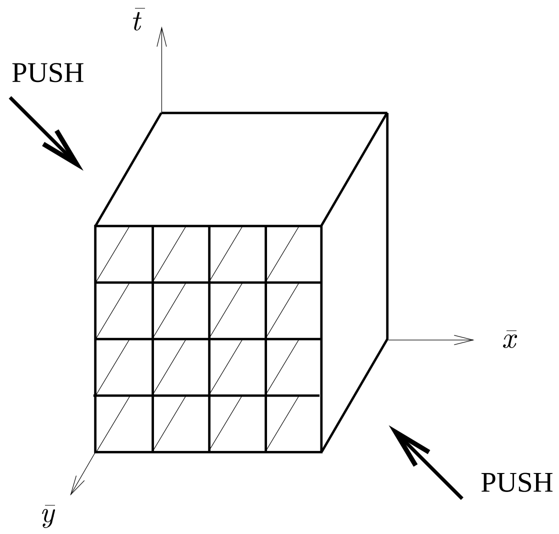
$$1_t, 1_x, 1_y, 1_z \quad \text{for} \quad 1_0, 1_1, 1_2, 1_3, \quad \text{respectively.}$$

2. Let $j \leq n$. We say that P is a <u>*j-dimensional plane*</u> iff there is a j-dimensional subspace[115] $\mathbf{W}$ of ${}^n\mathbf{F}$ and a vector $p \in {}^nF$ such that $P = W + p$, where $W + p \stackrel{\text{def}}{=} \{\, w + p \,:\, w \in W \,\}$.[116]

[114] This extra assumption is that the square roots of positive elements exist in $\mathfrak{F}$ (i.e. that $\mathfrak{F}$ is Euclidean).

[115] Let us recall from the literature that by a <u>*subspace*</u> of the vector space ${}^n\mathbf{F}$ we understand a subalgebra (in the universal algebraic sense, cf. Conv.3.1.2) $\mathbf{W} \subseteq {}^n\mathbf{F}_1$ of the one-sorted vector space ${}^n\mathbf{F}_1$. Further a one-sorted vector space $\mathbf{W}$ is <u>*j-dimensional*</u> iff there is a j-element minimal generator system $G \subseteq W$, i.e. G generates $\mathbf{W}$ but no proper subset of G generates $\mathbf{W}$. (G <u>*generates*</u> $\mathbf{W}$ if no proper subalgebra of $\mathbf{W}$ contains G.)

[116] We use the universal algebraic convention that $\mathbf{W}$ denotes the algebra (vector space) and W denotes its universe. (We also note that by a plane one understands a set of form $W + p$, where $\mathbf{W}$ is 2-dimensional.)



Preparation for drawing 3-dimensional f_{mk}.

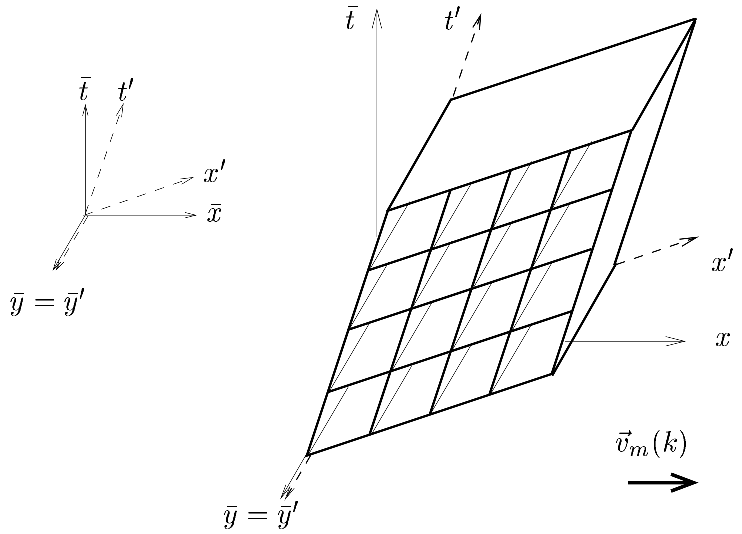


Figure 22: 3-dimensional world-view transformation f_{mk} in "standard" configuration, cf. Figures 16, 18. For the notion of standard configuration cf. Def.2.3.16 and Figures 23, 24.

By a *plane* we understand a 2-dimensional plane.
By a *hyper-plane* we understand an $n-1$-dimensional plane.

3. Let $\ell_1, \ell_2 \in \mathsf{Eucl}$.

   (i) We say that ℓ_1 and ℓ_2 are in the same plane if there is a 2-dimensional plane P such that $\ell_1, \ell_2 \subseteq P$.[117]

   (ii) If there is a unique 2–dimensional plane P such that $\ell_1, \ell_2 \subseteq P$, we denote this unique P by
   $$\mathsf{Plane}(\ell_1, \ell_2)\,.$$
   E.g. $\mathsf{Plane}(\bar{t}, \bar{x}) = F \times F \times {}^{n-2}\{0\}$ and
   $\mathsf{Plane}(\bar{t}, \bar{y}) = F \times \{0\} \times F \times {}^{n-3}\{0\}$.

   (iii) We say that ℓ_1 and ℓ_2 are *parallel*, in symbols $\ell_1 \parallel \ell_2$, iff ℓ_1 and ℓ_2 are in the same plane and $\ell_1 \cap \ell_2 = \emptyset$ or $\ell_1 = \ell_2$.

   (iv) Whenever we write $\ell \parallel \ell'$ and we do not indicate what kinds of objects ℓ and ℓ' are, then the symbol $\ell \parallel \ell'$ abbreviates the formula ($\ell \parallel \ell'$ and $\ell, \ell' \in \mathsf{Eucl}$).(This will be slightly different in the geometry chapter, §4.) $\lhd$

We are ready to define standard configuration. We will write about the intuitive meaning of standard configuration after the definition.

Definition 2.3.16 (Standard configuration)

(i) Let $\mathfrak{M}$ be a frame model. Let $m, k \in Obs$. We say that m and k are in *standard* configuration if
$$\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t}, \bar{x})] = \mathsf{Plane}(\bar{t}, \bar{x}) \quad \text{and} \quad (\forall 1 < i \in n)(\exists 0 < \lambda \in F)\mathsf{f}_{mk}(1_i) = \lambda \cdot 1_i.$$

(ii) We say that m and k are in *strict standard* configuration if in addition to the above we have $\mathsf{f}_{mk}(1_x)_x > 0$.

See Figures 23, 24. Cf. also Figure 22.

$\lhd$

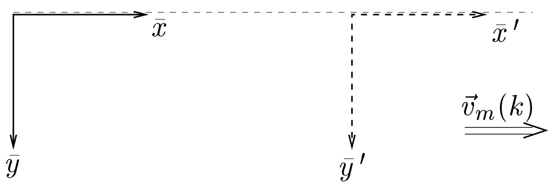


Figure 23: Standard configuration. Here $\bar{x}$ and $\bar{y}$ are space axes of m while $\bar{x}\,'$ and $\bar{y}\,'$ are space axes of k. The spatial coordinate system $\{\bar{x}\,', \bar{t}\,'\}$ of k is moving relative to that of m.

The next proposition says that m and k are in standard configuration iff they meet at $\bar{0}$, they see each other moving in direction 1_x (forwards or backwards), and they see each other's unit-vectors other than $\bar{t}, \bar{x}$ as perhaps shrinking or growing but pointing in the same direction.

[117]The standard geometry literature uses the expression "ℓ_1 and ℓ_2 are coplanar" for this.

PROPOSITION 2.3.17 *Assume* **Ax1** – **Ax5**. *Then* m *and* k *are in standard configuration iff (i)-(iv) below hold.*

(i) $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$

(ii) $tr_m(k), tr_k(m) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$

(iii) *If* $v_m(k) = 0$, $\mathsf{f}_{mk}[\bar{x}] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$

(iv) *Let* $1 < i \in n$. *Then* $\mathsf{f}_{mk}(\underbrace{0, \ldots, 0}_{i}, 1, 0, \ldots, 0) = \langle \underbrace{0, \ldots, 0}_{i}, \lambda, 0, \ldots, 0 \rangle$ *for some* $0 < \lambda \in F$.

∎

We note that **Ax3** and **Ax5** in the above Proposition 2.3.17 can be replaced with their much weaker forms $\mathbf{Ax3_0}$ and $\mathbf{Ax(5nop)^{-+}}$ i.e. with $(\forall k \in Obs)[tr_m(k) = \emptyset$ or $tr_m(k) \in G]$ and with $(\exists c > 0)(\forall \ell)[ang^2(\ell) < c \Rightarrow (\exists k \in Obs)\ell = tr_m(k)]$ respectively, <u>where</u> the axiom $\mathbf{Ax3_0}$ is defined in §3 and $\mathbf{Ax(5nop)^{-+}}$ is defined in AMN [18, Chapter 5, p.761], see the list of axioms. Thus in later parts when we deal with weaker axiom systems, (i)-(iv) in Proposition 2.3.17 will still give an equivalent definition of standard configurations (because the weaker axioms that we mentioned will be included in all our weak axiom systems).

We note that being in standard configuration is a symmetric relation, i.e. if m and k are in standard configuration, k and m are also in standard configuration. Very often it simplifies the discussion if we assume that m and k are in standard configuration. (Sometimes, in intuitive discussions we may assume that m and k are in standard configuration without explicitly mentioning this.)

The reader is invited to contemplate Figures 17–22. They all represent cases of a natural kind of transformations $\mathsf{f} : {}^nF \longrightarrow {}^nF$ which we will call rhombus transformations, their set will be denoted by *Rhomb*, cf. Def.2.3.18 below. They are generalizations of the kind of functions occurring in Thm.2.3.12(ii); they will be strongly related to what we will call Lorentz transformations in standard configuration, cf. Thm.2.9.7 on p.104.

Now we turn to a common generalization of the transformations illustrated in Figures 17–22.

Definition 2.3.18 (Rhombus transformation, *Rhomb*)
Assume $\mathfrak{F}$ is an ordered field and $n \geq 2$.
By a <u>*rhombus transformation*</u> (of ${}^n\mathfrak{F}$)[118] we understand a bijective linear transformation $\mathsf{f} : {}^nF \longrightarrow {}^nF$ of the vector space ${}^n\mathbf{F}$ satisfying (i)-(iii) below.

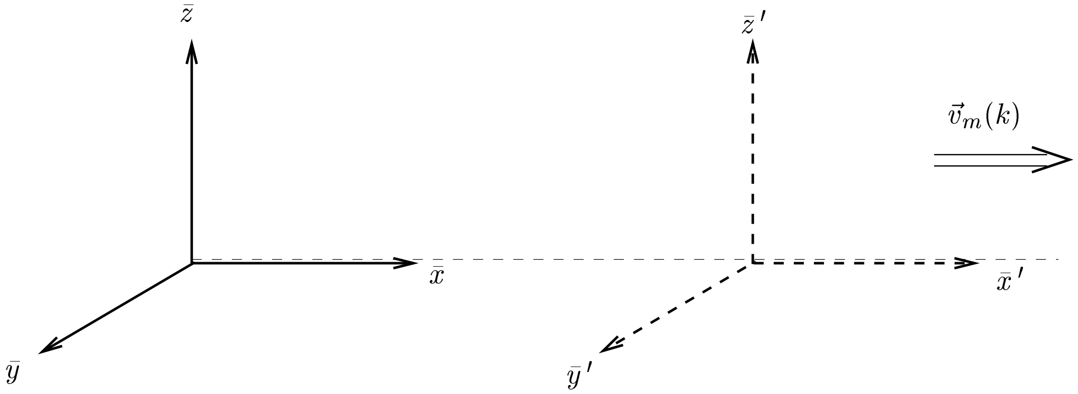
(i) $\mathsf{f}(1_t)$ and $\mathsf{f}(1_x)$ are both in $\mathsf{Plane}(\bar{t}, \bar{x})$ and are mirror images of each other w.r.t. a photon-line ℓ with $\bar{0} \in \ell \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.[119]

(ii) $(\forall 1 < i \in n)$ $(\mathsf{f}(1_i) = \lambda \cdot 1_i$, for some $0 < \lambda \in F)$.[120]

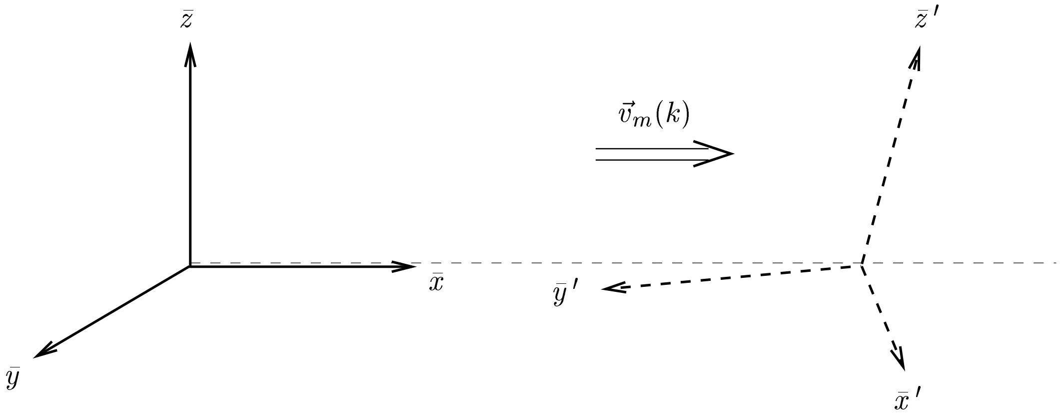
[118] Occasionally we mention this symbol ${}^n\mathfrak{F}$. Since $\mathfrak{F}$ is an algebraic structure, so is its Cartesian power ${}^n\mathfrak{F}$ (which happens to be a partially ordered ring). However, in this work, we think of ${}^n\mathfrak{F}$ as a <u>*partially ordered*</u> vector space $\langle {}^n\mathbf{F}, \leq \rangle$ where the partial ordering $\leq$ of ${}^n\mathfrak{F}$ is induced by $\leq^{\mathfrak{F}}$ of $\mathfrak{F}$ in the usual, "Cartesian power" style. (In particular, the coordinate axes like $\bar{t}$ are linearly ordered by this partial order $\leq$ of ${}^n\mathfrak{F}$.)

[119] This mirror image part means that if $\mathsf{f}(1_t) = \langle p_0, p_1, 0, \ldots, 0 \rangle$ then either $\mathsf{f}(1_x) = \langle p_1, p_0, 0, \ldots, 0 \rangle$ or $\mathsf{f}(1_x) = \langle -p_1, -p_0, 0, \ldots, 0 \rangle$.

[120] For completeness we note that more on the choice of λ can be found in §§ 3.2, 3.5 of AMN [18]. However, we emphasize that the above definition makes sense (i.e. is complete) without any further discussion of the choice of λ.



Standard configuration



A nonstandard configuration, which in "animated" form is drawn below.
The picture shows a spaceship flying in the indicated "nonstandard" direction.

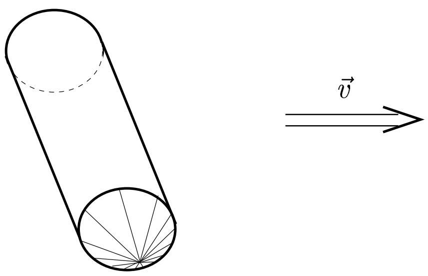


Figure 24: A standard, and a nonstandard configuration.

(iii) f preserves the set of photon-lines, i.e. $(\forall \ell \in \mathsf{PhtEucl})\ \mathsf{f}[\ell] \in \mathsf{PhtEucl}$.

Condition (iii) is needed only if $n > 2$. The role of (iii) is to regulate the choice of λ in (ii). $Rhomb = Rhomb(n, \mathfrak{F})$ denotes the set of rhombus transformations of ${}^n\mathfrak{F}$.

$\lhd$

We note that rhombus transformations will play a central role in proving that **Basax**(n) is consistent, cf. §2.4 and AMN [18, §3.5].

Remark 2.3.19 Assume that square roots of positive elements of $\mathfrak{F}$ exist, that is $(\forall 0 < x \in F)(\exists y \in F)\ x = y^2$. Assume $n > 2$. In chapter 3 of AMN [18] we see that[121] for any slow-line ℓ with $\bar{0} \in \ell \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ there is a rhombus transformation taking $\bar{t}$ to ℓ. The idea will be that first we choose $\mathsf{f}(1_t)$ and $\mathsf{f}(1_x)$ so that they are mirror-images of each other like in (i) of Def.2.3.18, and $\mathsf{f}(1_t)$ is on ℓ. $\mathsf{Plane}(\bar{t}, \bar{x}_i)$ is the plane determined by $\bar{t}$ and $\bar{x}_i$ in an analogous way as $\mathsf{Plane}(\bar{t}, \bar{x})$ was defined. Then for every $i \in n$, $i > 1$ there is a unique λ making (ii) (of Def.2.3.18) true so that photon-lines in $\mathsf{Plane}(\bar{t}, \bar{x}_i)$ are mapped to photon-lines. These now fix our linear transformation f. Finally, we have to check that (iii) of Def.2.3.18 is satisfied, i.e. that *every* photon-line is mapped (by f) to a photon-line, and not only those in $\mathsf{Plane}(\bar{t}, \bar{x}_i)$.

$\lhd$

We note that if for observers m and k we have $\mathsf{f}_{km} \in Rhomb$, m and k are in standard configuration.

In §2.9 we will recall from the literature the so-called Lorentz transformations. A special case of the latter will be called Lorentz transformations in standard configuration. The elements of the above introduced *Rhomb* will turn out to be generalizations of Lorentz transformations in standard configuration, cf. Thm.2.9.7 on p.104. At this point we would like to suggest that the reader go through Figures 17–22 and compare them with the definition of *Rhomb*.

Connections between the world-view transformations f_{mk} and Lorentz transformations will be discussed in §2.9. It will turn out that for establishing these connections it is enough to assume **Basax**. Roughly speaking, these connections will say that every f_{mk} is a composition of a *Lorentz transformation*, an *"expansion"*, and a map induced by a *field automorphism*.

* * *

Now we turn to listing some (logical) properties of **Basax** as a first-order theory.

According to our Convention 2.2.1(ii), **Basax**(2) denotes **Basax** in the 2-dimensional case. Next, in section 2.4, we will see that **Basax**(2) is *consistent*, that is, there exist frame models satisfying **Basax**(2). In AMN [18, sections 3.2, 3.5] we see that **Basax**(3) is also consistent, and that generally, **Basax**(n) is consistent for all $n \geq 3$ (cf. AMN [18, Definition 3.5.5, Thm.3.5.6]).

The next two properties "count as logical" in the sense that the above property (consistency) concerns the existence of models while the next two properties concern existence of special kinds of models (namely, models with faster than light observers, and models with special ordered field reducts).

We will see, in section 2.4, that there are models of **Basax**(2) in which there *are* observers moving *faster than light*, while if $n > 2$ then there are no such models of **Basax**(n) (i.e.

[121] Cf. §3.2 of AMN [18], cf. also Lemma 3.8.46 of AMN [18].

for $n > 2$, $\mathbf{Basax}(n) \models (\forall m, k \in Obs) v_m(k) < 1$, see Thm.3.2.13 (p.118), while $\mathbf{Basax}(2) \not\models (\forall m, k \in Obs) v_m(k) < 1)$.

We will see that *every* linearly ordered field is the ordered field reduct of some model of **Basax**(2), while the ordered field reducts of **Basax**(3) are exactly the Euclidean ordered fields (i.e. those in which square roots of positive elements exist). For $n > 3$, we do not know exactly which ordered fields occur as ordered field reducts of **Basax**(n) models, but we know that all Euclidean ordered fields do occur.

An axiom system **Th** is called *independent* if no axiom of **Th** follows from the rest of **Th**, i.e. if $\mathbf{Th} \setminus \{\mathbf{Ax}\} \not\models \mathbf{Ax}$ for all $\mathbf{Ax} \in \mathbf{Th}$. **Basax**($n$) is independent for every $n > 1$.[122] We omit the proof of this, but cf. [16]. To make this independence statement about **Basax** precise we have to make the formulation of **Ax5** a little-bit more careful. Namely, we have to replace the subformula $ang^2(\ell) < 1$ with the formula $(\ell \in \mathsf{Eucl} \quad \wedge \quad ang^2(\ell) < 1)$; similarly for the subformula $ang^2(\ell) = 1$.

We now list some further *logical* properties of **Basax**.[123] We already stated that **Basax** is consistent and independent. We classify the models of **Basax**, and we see that there are continuum many non-elementarily equivalent models of **Basax** (such that they have the same ordered field reduct $\mathfrak{F}$), cf. Thm.3.8.18 of AMN [18]. Hence, **Basax** is not complete (even if we add the theory $\mathsf{Th}(\mathfrak{F})$, for any choice of $\mathfrak{F}$); **Basax** is non-categorical in any cardinality even if we fix the reduct $\mathfrak{F}$ (it has non-isomorphic models [with a common ordered field reduct] of the same cardinality, for each infinite cardinality), cf. Thm.3.8.18 of AMN [18]. We prove that the theory generated by **Basax**, i.e. the set of first-order consequences of **Basax**, is undecidable cf. Chapter 7 of AMN [19]. This also proves that **Basax** is not complete (hence not categorical in any cardinality, since its models are infinite), because **Basax** is finite. We define some natural axioms, call them **Axnob** [124] and **Axisb** and we show that $\mathbf{Basax} \cup \{\mathbf{Axnob}\}$ is complete[125] (cf. §3.8 of AMN [18]), while $\mathbf{Basax} \cup \{\mathbf{Axisb}\}$ is hereditarily undecidable, thus no finite extension of it can be complete, cf. AMN [16] and Chapter 7 of a future edition [19] of AMN [18].[126] Moreover, the conclusion of Gödel's second incompleteness theorem also applies to $\mathbf{Basax} \cup \{\mathbf{Axisb}\}$.

Definability issues related to **Basax** and its variants will be discussed in §4.6. In more detail, in §4.5 we will see that **Basax** admits a nice "duality theory" acting between models of **Basax** and certain geometries.[127] This duality theory involves, among others, "representation theorems" (in the Tarskian sense[128]). So in a sense **Basax** admits a kind of "geometrization"[129]. Studying this duality theory will lead us (in §4.6) to *definability* properties of **Basax** (and its geometric counterpart) in the sense of the chapter of model theory called *definability theory*.

[122] Let **Basax**$'$ be the axiom system obtained from **Basax** by replacing **Ax2** and **Ax3** with a single axiom $(\forall h \in Obs \cup Ph)\ tr_m(h) \in G$. Then **Basax**$'(2)$ is independent, **Basax**$'(3)$ is not independent, and we do not know whether **Basax**$'(n)$ is independent for $n > 3$. These properties of **Basax**$'$ are proved in [16], taken together with Thm.3.6.17 of AMN [18].

[123] For the notions from logic used below (like categorical theory, complete theory, theory generated by a set of axioms etc.) we refer the reader to §3.8 of AMN [18] and to AMN [16].

[124] To be precise, we note that **Axnob** is only a schema of axioms (as opposed to being a single axiom).

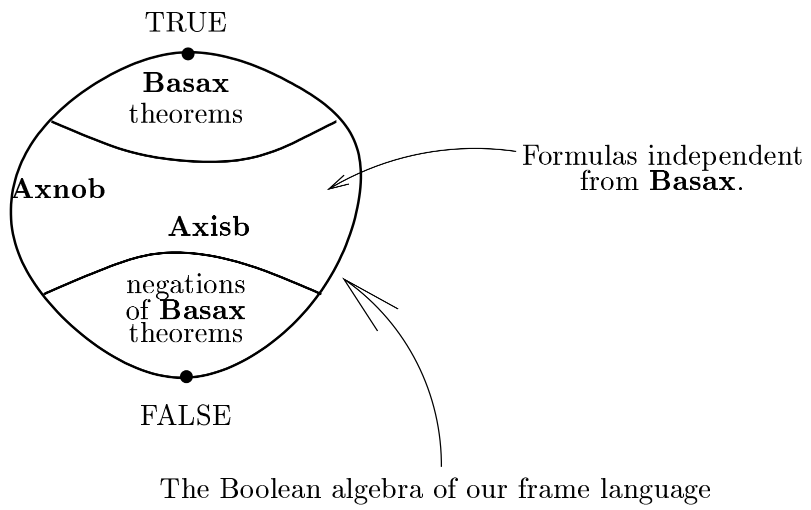
[125] and also *categorical* in *some* natural sense made precise in §3.8 of AMN [18]

[126] The name **Axnob** refers to the fact that this axiom says, among others, that there are no accelerated bodies. On the other hand **Axisb** refers to the fact that the key part of this axiom says that there do exist accelerated bodies.

[127] For this we first add a few natural axioms to **Basax**, and then we find that this duality theory works already for "fragments" and variants of **Basax**.

[128] bringing together Tarski's approach to geometry and his approach to algebraic logic

[129] This is analogous with "algebraization" of logics in Tarskian algebraic logic.



Summing it up:

- **Basax** is *independent.*[130]
- **Basax** is *consistent* (cf. §2.4 and AMN [18, §§ 3.2, 3.5]).
- **Basax** has *many non-elementarily equivalent* models (even if we add to it $Th(\mathfrak{R})$), cf. Thm.3.8.18 of AMN [18].
- We give a classification of the models of **Basax** in §3.6 of AMN [18].
- The first-order theory T(**Basax**) generated by **Basax** is *undecidable* hence not complete, cf. AMN [16] and Chapter 7 of a future edition [19] of AMN [18].
- Adding an extra axiom-schema can make T(**Basax**) *complete* hence *decidable*, since **Basax** is finite, cf. §3.8 of AMN [18] and AMN [16].
- Adding a different extra axiom can make T(**Basax**) *hereditarily undecidable* hence hereditarily not complete. The conclusions of *Gödel's incompleteness theorems* apply to the so extended version of **Basax**. Cf. AMN [16] and Chapter 7 of AMN [19].
- Adding an extra axiom-schema makes **Basax** equivalent with the standard, "textbook version" of "Einsteinian" special relativity, cf. §§ 2.8, 2.9, 3, 4.2, 4.5 and AMN [18, §3.8].
- Other distinguished versions like the Reichenbach-Grünbaum version of relativity can (and will) be formalized in first-order logic (and compared with the Einsteinian version) by appropriately modifying **Basax** (cf. §3 and §4.5 herein and AMN [18, §§3.4.2, 4.4] and the section [on "Reichenbachian relativity"] of Chapter 4 (§4.5) of AMN [18]). Cf. also Szabó [244, 243] in connection with these versions of relativity theory, which the present author extensively studied in AMN [18, §4.5] by the methods of the present work.

The above 9 items about **Basax** are proved in AMN [16] and [18] (cf. e.g. §3.8 therein).

On p.100 we introduce an extension **Specrel** of **Basax**. It would be interesting to check if all the 9 items above remain true for **Specrel** in place of **Basax**. (Items 2,3,5-8 remain true for **Specrel**. We did not yet found the time for thinking about the rest.)

[130] However, **Basax**(n) + **Ax**$(\sqrt{\ })$ is not independent for $n > 2$, while **Basax**(2) + **Ax**$(\sqrt{\ })$ is independent. Formal statement and proof is in AMN [16]. Cf. also footnote 122 on p.43.

2.4 Models for Basax in two dimensions

In this section we show that **Basax**(2) is consistent, via defining a frame model $\mathfrak{M}$ and showing that $\mathfrak{M} \models$ **Basax**(2). We will also give a model of **Basax**(2), in which there are faster than light observers.

First, let us have some *intuitive* considerations on why **Basax**(2) is consistent. (Later we will give a formal proof.) The main reason why **Basax**(2) is consistent is the following:

($\star$) for each slow-line ℓ there is a photon-preserving bijective collineation taking $\bar{t}$ to ℓ.

The reader is invited to study Figures 17–22 (pp. 33–38) to convince himself that ($\star$) is true, and then use ($\star$) the following way to show that **Basax**(2) is consistent.

(I) Assume we are given a "partial model"

$$\mathfrak{M} = \langle (B;\ \{m_0\}, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle\,,$$

which satisfies all the axioms in **Basax** *except* for the observer-part of **Ax5**. Let us use the notation **Ax5** = **Ax5(Obs)** $\wedge$ **Ax5(Ph)**. Then

$$\mathfrak{M} \models (\mathbf{Ax1}\text{–}\mathbf{Ax4}, \mathbf{Ax5(Ph)}, \mathbf{Ax6}, \mathbf{AxE})\,.$$

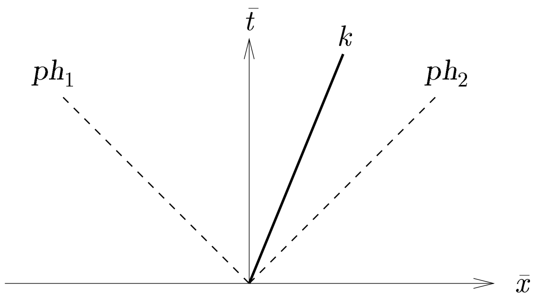
Assume further $\mathfrak{F} = \mathfrak{R}$, and that

$$(\forall \ell \in G)(\exists b \in Ib)\ \ell = tr_{m_0}(b)\,.$$

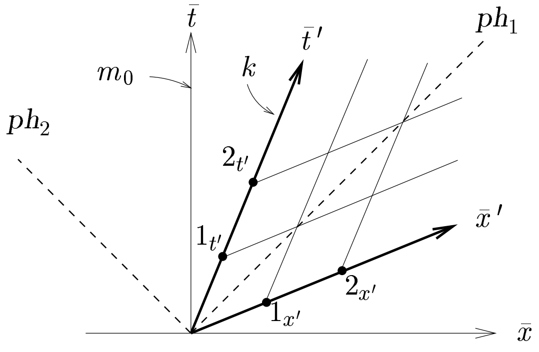
Constructing such a partial model is easy, and is left to the reader.

(II) Next, we would like to add new observers to $\mathfrak{M}$ so that eventually **Ax5(Obs)** would become true without destroying validity of the other axioms (hence **Basax** would become true).

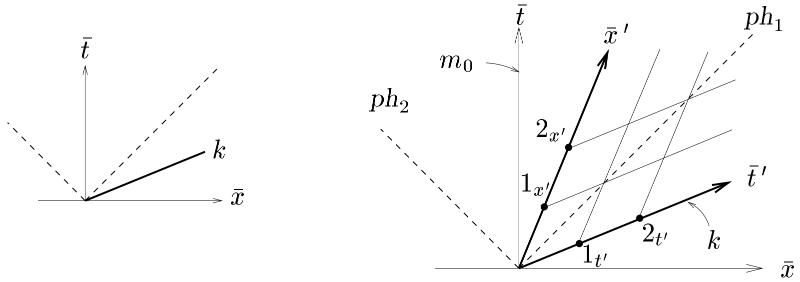
Clearly, in $\mathfrak{M}$ we do have a world-view function $w_{m_0} : {}^2\mathrm{R} \longrightarrow \mathcal{P}(B)$, to begin with. From this world-view function we will construct world-views for new observers. Let us pick randomly $k \in Ib$ such that $v_{m_0}(k) \neq 1$. Now, we would like to raise k to the level of being an observer. Assume m_0 sees this:



Our task is to choose the world-view of k such that, among other things, **AxE** remains valid, i.e. that k observes all photons moving with speed 1. Following Figures 17–22, let us choose k's world-view like this (the figure shows k's coordinate system as seen by observer m_0):



where $\bar{t}\,'$ and $\bar{x}\,'$ are the time-axis and $\bar{x}$-axis, respectively, of k. We note that if k moves faster than light relative to m_0 (i.e. if $v_{m_0}(k) > 1$) then k's coordinate system (as seen by m_0) is like in the following picture:



Now clearly, k will observe photon ph_1 moving with speed 1 and the same applies to ph_2. Then one can check that for these two particular observers m_0, k our axiom **AxE** holds, i.e. both k and m_0 will observe all photons moving with speed 1. One can check that for the extended model

$$\mathfrak{M}' := \langle (B;\ \{m_0, k\}, Ph, Ib), \mathfrak{F}, G;\ \in, W^+ \rangle$$

we have **Ax1**–**Ax4**, **Ax5(Ph)**, **Ax6**, **AxE** still valid. Here, W^+ denotes the extension of W with the world-view function w_k of the <u>new</u> observer k.

To complete the "intuitive" proof, one does the above extension <u>not only</u> with a single $k \in Ib$ but with the class $K = \{\, k \in Ib \ :\ v_{m_0}(k) \neq 1 \,\}$ of all potential candidates for being an observer. This will make **Ax5(Obs)** true. We note that the condition $\bar{0} \in tr_m(k)$ was <u>not</u> needed in our construction of w_k.

In passing we note that in the above constructed model faster than light observers exist. It is easy to modify the construction in such a way that faster than light observers will not exist in the modified model. This modification begins with adding to statement ($\star$) above that the photon-preserving bijective collineation in question takes slow-lines to slow-lines. The rest of the modifications are straightforward, we leave them to the reader.

<u>*END of Intuitive Idea of Proof.*</u>

Let us turn to giving a detailed construction.

Let P be a function that with each $\ell \in \mathsf{Eucl}(2, \mathfrak{R})$ associates a pair of two distinct points lying on ℓ. We will denote $P(\ell)$ by $\langle o_\ell, t_\ell \rangle$. To each such function P, we will define two frame models, $\mathfrak{M}_0^P$ and $\mathfrak{M}_1^P$. These two frame models will be very similar in spirit, but in $\mathfrak{M}_0^P$ we have as few observers as possible, while in $\mathfrak{M}_1^P$ there will be an observer on each line (with angle $\neq 1$, cf. Prop.2.3.3(iii)).

First we define $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P \stackrel{\text{def}}{=} \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle$, where

$\mathfrak{F} \stackrel{\text{def}}{=} \mathfrak{R}$, the ordered field of real numbers,

$G \stackrel{\text{def}}{=} \mathsf{Eucl}(2,\mathfrak{R})$, the set of straight lines over $\mathfrak{R}$,

$Obs \stackrel{\text{def}}{=} \{\, \ell \in \mathsf{Eucl}(2,\mathfrak{R}) \ :\ ang^2(\ell) < 1 \,\}$,

$Ph \stackrel{\text{def}}{=} \{\, \ell \in \mathsf{Eucl}(2,\mathfrak{R}) \ :\ ang^2(\ell) = 1 \,\}$,

$B \stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} Obs \cup Ph = \{\, \ell \in \mathsf{Eucl}(2,\mathfrak{R}) \ :\ ang^2(\ell) \le 1 \,\}$.

By the above, **Ax1** and **Ax2** are true in $\mathfrak{M}$. It remains to define W. Let

$$m_0 \stackrel{\text{def}}{=} \bar{t} \stackrel{\text{def}}{=} \mathrm{R} \times \{0\}.$$

First we will define $w_{m_0} : {}^2\mathrm{R} \longrightarrow \mathcal{P}(B)$ and $\mathsf{f}_{km_0} : {}^2\mathrm{R} \longrightarrow {}^2\mathrm{R}$ for all $k \in \mathsf{Eucl}(2,\mathfrak{R}), ang^2(k) \neq 1, k \neq m_0$. To define w_{m_0}, let $p \in {}^2\mathrm{R}$. Then

$$w_{m_0}(p) \stackrel{\text{def}}{=} \{\, \ell \in B \ :\ p \in \ell \,\}\,.$$

By this we have that for all $\ell \in B$,

$$tr_{m_0}(\ell) = \ell,$$

in particular, $tr_{m_0}(m_0) = m_0$. Thus **Ax3**, **Ax4**, **Ax5**, **AxE** are satisfied when m is replaced in them by m_0. See Figure 25.

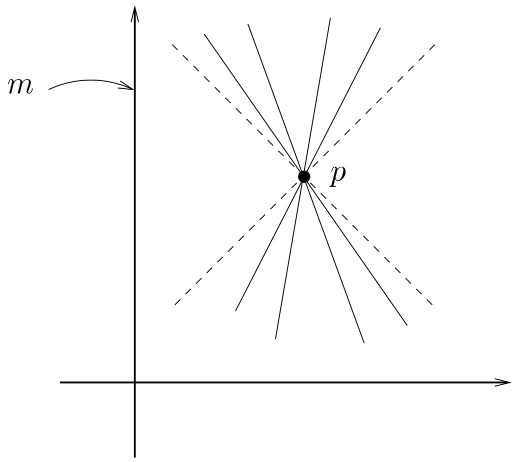


Figure 25: $w_{m_0}(p)$ in $\mathfrak{M}_0^P$.

Let $k \in \mathsf{Eucl}(2,\mathfrak{R}), k \neq m_0, ang^2(k) \neq 1$ be arbitrary. We are going to define f_{km_0}. In the following, we will write f for f_{km_0}.

For any two distinct points $p, q \in {}^nF$, $\overline{pq}$ denotes the Euclidean line containing both p and q.

Sometimes we write (x, y) for the ordered pair $\langle x, y\rangle$. We apologize to the reader for this inconsistency.

Recall that two distinct points, o_k and t_k are given to us by the parameter P of the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P$. First we define the point s_k as the mirror image of t_k w.r.t. the line ℓ_k such that $o_k \in \ell_k$ and ℓ_k is parallel to the line $\overline{(0,0)(1,1)}$. See Figure 26.

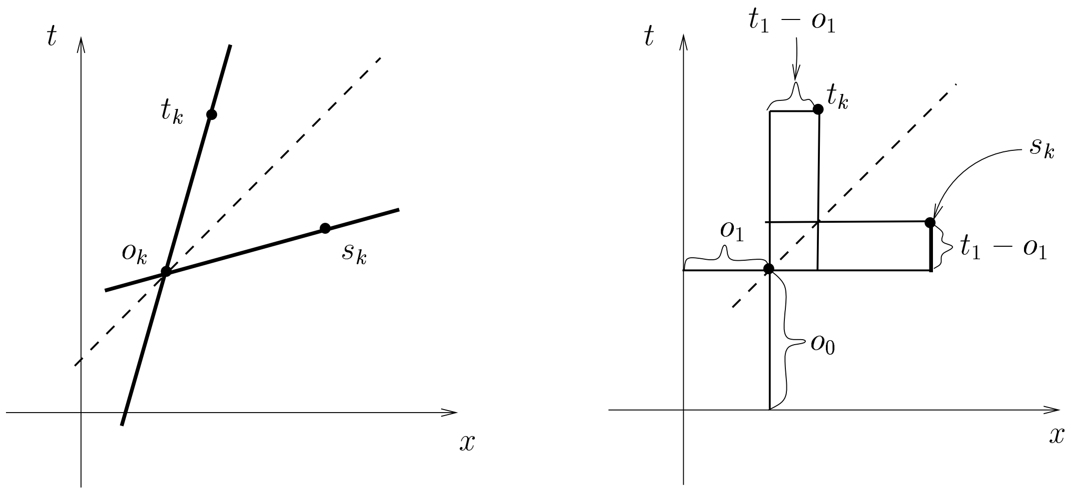


Figure 26: The definition of the point s_k.

In more detail: Let $o_k = (o_0, o_1), t_k = (t_0, t_1)$. We define

$$s_k \stackrel{\text{def}}{=} (o_0 + (t_1 - o_1), o_1 + (t_0 - o_0)).$$

By $ang^2(k) \neq 1$ we have that $s_k \neq t_k$, moreover, $s_k \neq a \cdot t_k$ for all $a \in \mathrm{R}$.

We will define $\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{km_0} : {}^2\mathrm{R} \longrightarrow {}^2\mathrm{R}$ to be the affine transformation[131] that takes $(0,0), (1,0), (0,1)$ to o_k, t_k, s_k respectively. See Figure 27.

In more detail,

$$\mathsf{f}_{km_0}(a, d) \stackrel{\text{def}}{=} a \cdot (t_k - o_k) + d \cdot (s_k - o_k) + o_k.$$

(Here we used that t_k, s_k, o_k are also vectors.) See Figure 28.

Intuitively, take a point $p = (a, d)$ in ${}^2\mathrm{R}$, and let $\mathsf{f}_{km_0}(a, d) = (a', d')$. Then a', d' are the coordinates of p in the coordinate system with basis $\{(1, 0), (0, 1)\}$, while a, d are the coordinates of p in the coordinate system with basis $\{(t_k - o_k), (s_k - o_k)\}$, see Figure 28.

By this, f_{km_0} is defined for all $k \in \mathsf{Eucl}(2, \mathfrak{R}), k \neq m_0, ang^2(k) \neq 1$. We now define

[131] For the definition of an affine transformation see §2.9. We will not need the definition here.

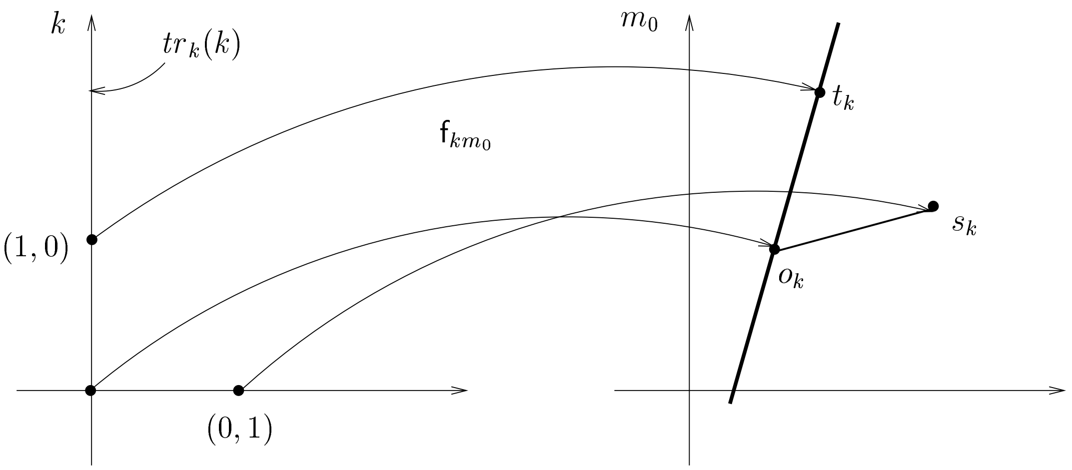


Figure 27: The world-view transformation f_{km_0}.

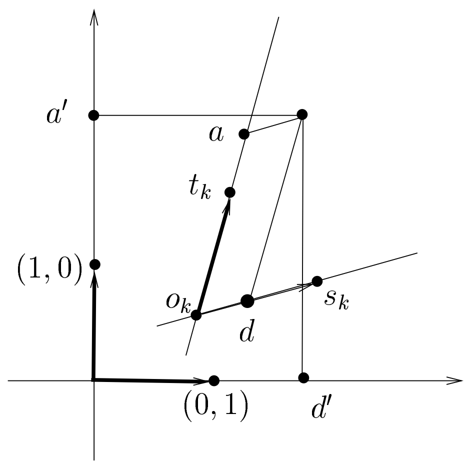


Figure 28: The world-view transformation f_{km_0}.

$w_k \stackrel{\text{def}}{=} \mathsf{f}_{km_0} \circ w_{m_0}$ for all $k \in Obs \setminus \{m_0\}$, and

$W \stackrel{\text{def}}{=} \{ \langle m, p, b\rangle \ : \ m \in Obs, b \in w_m(p) \}$.

By this, the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P \stackrel{\text{def}}{=} \langle B, \dots, W\rangle$ has been defined. $\mathfrak{M}_0^P$ is a frame model.

THEOREM 2.4.1 $\mathfrak{M}_0^P \models \mathbf{Basax}(2)$. ∎

Now we define the other model $\mathfrak{M}_1^P$. The definition of $\mathfrak{M}_1^P$ is completely analogous to that of $\mathfrak{M}_0^P$, the only difference is that we allow all lines (with angle $\neq 1$) to be observers. In detail: let

$Obs_1 \stackrel{\text{def}}{=} \{ \ell \in \mathsf{Eucl}(2, \mathfrak{R}) \ : \ ang^2(\ell) \neq 1 \}$,

$B_1 \stackrel{\text{def}}{=} Ib_1 \stackrel{\text{def}}{=} Obs_1 \cup Ph = \mathsf{Eucl}(2, \mathfrak{R})$.

Then $m_0 \in Obs \subseteq Obs_1$. We define

$w'_{m_0}(p) \stackrel{\text{def}}{=} \{ \ell \in B_1 \ : \ p \in \ell \}$,

$w'_k \stackrel{\text{def}}{=} \mathsf{f}_{km_0} \circ w'_{m_0}$,

$W' \stackrel{\text{def}}{=} \{ \langle m, p, b\rangle \ : \ b \in w'_m(p) \}$,

$\mathfrak{M}_1^P \stackrel{\text{def}}{=} \langle (B_1;\ Obs_1, Ph_1, Ib_1), \mathfrak{R}, G;\ \in, W'\rangle$.

THEOREM 2.4.2 $\mathfrak{M}_1^P \models \mathbf{Basax}(2)$. ∎

COROLLARY 2.4.3 $\mathbf{Basax}(2) \not\models$ *"there are no faster than light observers"*.[132] ∎

In Figure 29, $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ represent possible models of $\mathbf{Basax}(2)$. There $\bar{t}, \bar{t}', \bar{t}''$ are the time axes of observers $k, k', k'' \in Obs$.

Consider e.g. the picture representing $\mathfrak{M}_1$ (first picture of Figure 29). What the picture *really* represents is the *world-view of a particular observer k* and also how k sees k', k'' etc. In the picture $\bar{t}', \bar{t}''$ represent the life-lines of observers k', k''. Further $1_{t'} = \mathsf{f}_{k'k}(1_t)$, $1_{t''} = \mathsf{f}_{k''k}(1_t)$ and $1_{x'} = \mathsf{f}_{k'k}(1_x)$ etc. Intuitively, $1_{t'}$ is the time-unit vector of k' as seen by k, while $1_{x'}$ is the x-unit vector of k' as seen by k. We do *not* claim that the world-view of observer k' would be similar. Actually it is not. The *only* thing we claim is that there is an observer k of $\mathfrak{M}_1$ whose world-view is as represented in the picture. The same convention applies to the pictures representing $\mathfrak{M}_2$ and $\mathfrak{M}_3$.

Figure 29 represents possible choices for the parameter P of the model $\mathfrak{M}^P$ introduced on p.46. Recall that $\langle o_\ell, t_\ell\rangle = P(\ell)$. In Figure 29 $\ell \in \{\bar{t}, \bar{t}', \bar{t}'',$ etc.$\}$. In the figure we chose $o_\ell := \bar{0}$, further $t_\ell := 1_t$ etc.

[132] In Thm.3.2.13 (p.118) we will see that this corollary does not generalize to $n > 2$.

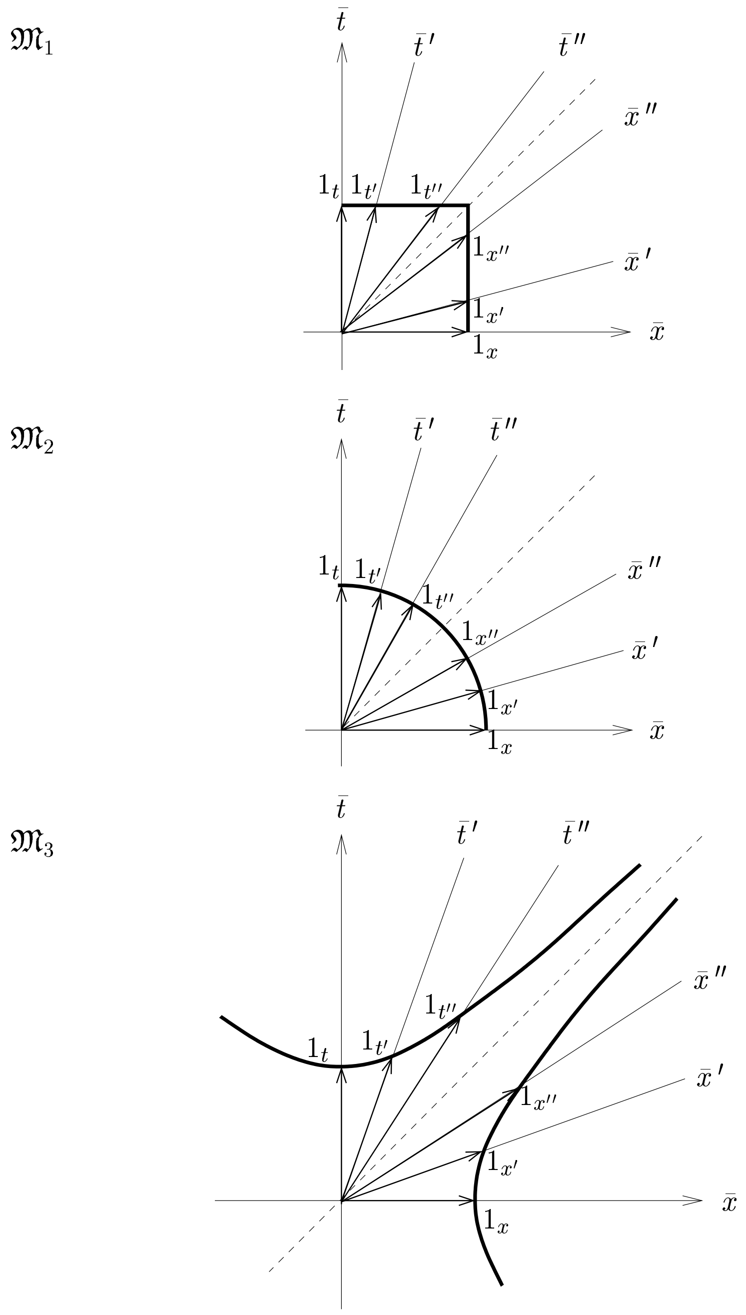


Figure 29: Possible models of **Basax**. Possible choices for the parameter P (on p.46): $o_\ell = \bar{0}$ and t_ℓ are represented in the picture, for slow-lines ℓ going through $\bar{0}$.

Let us turn now to the *ideas* we wanted to represent in these pictures.

In the third picture, the curves connecting $1_t, 1_{t'}, 1_{t''}$ etc. are hyperbolas. In §2.8 we will introduce a symmetry axiom called **Ax(symm)**. We note that $\mathfrak{M}_3 \models$ **Ax(symm)**, while $\mathfrak{M}_i \not\models$ **Ax(symm)** for $i < 3$. Roughly speaking **Ax(symm)** says that "I see you the same way as you see me". Thus in $\mathfrak{M}_3$ all observers see the other observers' unit vectors as m sees it (while as we mentioned, in $\mathfrak{M}_1$ and in $\mathfrak{M}_2$ this is not so). We also note that $\mathfrak{M}_3$ corresponds to the *usual* (or classically standard) so-called Minkowskian models of relativity, while $\mathfrak{M}_1, \mathfrak{M}_2$ are "non-Minkowskian" (for the definition of a Minkowskian model see Definition 3.8.42 on p.331 of AMN [18]).

A common feature of $\mathfrak{M}_1$–$\mathfrak{M}_3$ in Figure 29 is that, for m fixed,

$$v_m(k) \longmapsto |\mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0})|$$

is (i) a function (of $v_m(k) \in F$) and this function is (ii) continuous. These properties will re-emerge as potential axioms in a future edition [19] of AMN [18]. Although these properties do *not* follow from **Basax**, we will not put too much emphasis on studying models which do not satisfy (i) or (ii). *Analogous* properties show up in §4.4 of AMN [18] as potential axioms.

We will return to the pictures in Figure 29 in §4.6. A complete classification of the isomorphism classes of **Basax**(2) models is given in Madarász [162]. It turns out that there are only finitely many non-isomorphic models under fixing some natural parameters like $\mathfrak{F}$, cardinality of the model etc.

Minkowski-circles, Minkowski-spheres.

Assume $n = 2$. The drawings in Figure 29 are called Minkowski-circles.[133] They are often useful in representing models by simple drawings.

Definition 2.4.4 Minkowski-sphere Let $n \geq 2$, $\mathfrak{M}$ be a frame model, and $m \in Obs$. Then the *Minkowski-sphere* MS around m is defined as

$$\mathsf{MS} \stackrel{\text{def}}{=} \mathsf{MS}(\mathfrak{M}, m) \stackrel{\text{def}}{=} \{p : (\exists k \in Obs)(\exists i < n)(\mathsf{f}_{mk}(\bar{0}) = \bar{0} \text{ and } p \in \{\mathsf{f}_{mk}[\{1_i, -1_i\}]\})\}.$$

◁

For very nice models (e.g. the ones studied in §2.8) MS forms a kind of surface such that one can imagine that this surface is a boundary[134] of a connected region like the inside of a ball (or a cube, or something like these). This is indeed the case with the three models in Figure 29 (p.51). In two dimensions, instead of "spheres" we speak of *Minkowski-circles*. What we said above about the Minkowski-spheres in n dimensions, sounds like the following for $n = 2$. In nice 2-dimensional models, MS as defined above looks like a nice curve (like a circle, or a square etc) such that one can imagine that MS is the boundary of a connected subset of the plane like the circle is the boundary of a "disc". This is the case in all three drawings in Figure 29. Classically, in standard relativity theory, only the figure associated with $\mathfrak{M}_3$ was called

[133]Instead of Minkowski-circles, we should call these sets of points relativistic circles because only a small fraction of the models of our relativity theories (studied in this work) are Minkowski models as defined in AMN [18] p.331 (Def.3.8.42) and p.726 (Def.5.0.65). This distinction in terminology ("Minkowskian" versus "relativistic") is carried through in Chapter 4 herein systematically, cf. e.g. Minkowskian geometry versus relativistic geometry (p.188, 146), Minkowskian orthogonality versus relativistic orthogonality $\perp_r$, Minkowskian distance g_μ versus relativistic distance g (p.145) etc. All the same, for reasons of tradition, we make an exception here in our terminology.

[134]if we disregard the points on the life-lines of photons crossing the origin.

a Minkowski-circle. (The reason for this is that only $\mathfrak{M}_3$ satisfies the symmetry axiom to be introduced in §2.8.) However, here we generalize this concept to arbitrary frame models. As we said, in nice models, $\mathsf{MS}(\mathfrak{M}, m)$ looks like a curve surrounding (or forming the boundary of) some connected area. However, in many less "well behaved" models $\mathsf{MS}(\mathfrak{M}, m)$ is just a set of points and does not even form a curve. Later we will introduce an axiom called **Ax(||)**. Typically, if **Ax(||)** fails, then MS tends to become more like a random set of points than a curve. With this, we stop the discussion of Minkowski-spheres and Minkowski-circles, but from time to time they will serve us as pleasant devices for visualizing certain nice, well behaved models.

In passing we note that in the case of $n = 2$, it is more often the case that $\mathsf{MS}(\mathfrak{M}, m)$ is like a curve surrounding a well defined area, while if $n > 2$ then this is more rare.[135] We base this latter statement on the following. Basically, if $n > 2$, and $(\forall m \in Obs)[\mathsf{MS}(\mathfrak{M}, m)$ *is a surface surrounding a connected and well defined area*], then the extra axiom **Ax(symm)** to be defined later is true in our model $\mathfrak{M}$, and then the Minkowski-sphere becomes practically the same what is called such in the classical literature (cf. e.g. "Minkowski-metric" in Friedman [91]). On the other hand, for $n = 2$ this is far from being true as is illustrated e.g. by Figure 29.

[135] We mean this with "surface" in place of "curve", of course.

2.5 The three "paradigmatic" theorems of relativity

What the average layperson usually knows about relativity is that

(I) moving clocks slow down,

(II) moving spaceships shrink (cf. Figure 30), and

(III) moving clocks get out of synchronism, i.e. the clock in the nose of the spaceship is late (shows less time) when compared with the clock in the rear, see Figure 31.

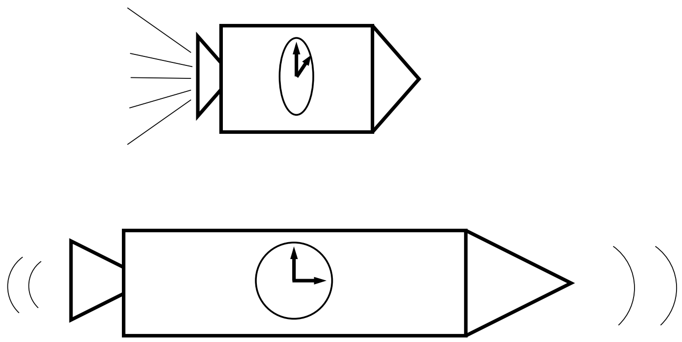


Figure 30: Moving clocks slow down and moving spaceships shrink.

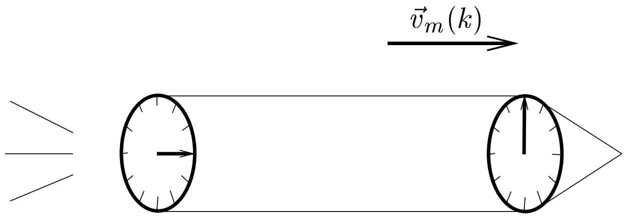


Figure 31: Moving clocks get out of synchronism.

In all of (I)–(III) above the spaceship is represented by an observer k, "we" who look at the spaceship are represented by observer m, and all of (I)–(III) are understood in the world-view of m. Below we formalize (I)–(III) as our *"paradigmatic" theorems*.[136] We will prove them from **Basax**. In Chapter 4 of AMN [18] when investigating weaker (or subtler) versions of **Basax** (e.g. the Reichenbachian version with non-standard simultaneities) we systematically re-visit our paradigmatic theorems to see if they are still true. It turns out that these paradigmatic theorems can be proved from surprisingly weak axioms. Cf. Chapter 4 of AMN [18], and especially section §4.8 of AMN [18] which is devoted to paradigmatic effects. In §2.8 we will

[136] In passing we note that the official names for effects (I) and (II) are "time dilation" and "length contraction" cf. d'Inverno [73, §§3.3, 3.4].

see that our paradigmatic theorems (I)-(III) hold in a stronger and simpler form in the stronger axiom system **Basax** + **Ax(symm)**.

Our next axiom, **Ax**($\sqrt{\ }$), is of a technical nature. Namely, sometimes we will need to assume that square roots of positive (greater than 0) elements exist in the ordered field reduct $\mathfrak{F}$ of the frame model $\mathfrak{M}$ we are speaking about.

Ax($\sqrt{\ }$) $\quad (\forall 0 < x \in F)(\exists y \in F)\ y^2 = x.$

If $\mathfrak{F} \models \mathbf{Ax}(\sqrt{\ })$ then we say that $\mathfrak{F}$ is *Euclidean*. Clearly, $\mathfrak{R} \models \mathbf{Ax}(\sqrt{\ })$. For any $0 < x \in F$, $\sqrt{x}$ denotes that positive y for which $y^2 = x$. For brevity, by a *Euclidean field* we mean a Euclidean ordered field.

CONVENTION 2.5.1 1. Throughout this section we assume $Obs^{\mathfrak{M}} \neq \emptyset$.

2. Let **Th** be a set of formulas of our frame language. Let $\mathbf{Ax}_1$, $\mathbf{Ax}_2$ be further formulas. Then
$$\mathbf{Th} + \mathbf{Ax}_1 + \mathbf{Ax}_2 \quad \text{denotes} \quad \mathbf{Th} \cup \{\mathbf{Ax}_1, \mathbf{Ax}_2\}.$$
Similar convention applies to other combinations like $\mathbf{Th} + \mathbf{Ax}_1$. (This notational convention is taken from axiomatic set theory.)

◁

The intuitive meaning of Thm.2.5.2 below is the following. Item (i) of the theorem states that observer m thinks that k's clocks are late at time-instance 1. As a generalization of this, (ii) says the same for many time instances $\lambda \in F$ namely, for those λ's which are not "infinitely big" or "infinitely small".

THEOREM 2.5.2 (Clocks slow down.)
Assume **Basax** + **Ax**($\sqrt{\ }$). *Then (i)–(iii) below hold.*

(i) *There are observers m and k such that m "thinks" that k's clocks run slow; formally:*
$$(\exists m, k \in Obs)\ |\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > 1\,,$$
see Figure 32. Moreover;

(ii) *There are observers m and k such that m "thinks" at each "finite" time instance λ that k's clocks run slow; formally:*
$(\exists m, k \in Obs)$
$(\forall \lambda \in F)\ \Big((\exists 0 < j \in \omega)\ 1/j < |\lambda| < j \quad \Rightarrow \quad |\mathsf{f}_{km}(\lambda \cdot 1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > |\lambda|\Big)$.[137]

(iii) *Assume $m, k \in Obs$ and $0 \neq v_m(k) < 1$. Then either m thinks that k's clocks run slow* or *k thinks that m's clocks run slow (cf. Figure 33); formally:*
$$(\exists m', k' \in \{m, k\})\ |\mathsf{f}_{k'm'}(1_t)_t - \mathsf{f}_{k'm'}(\bar{0})_t| > 1.$$

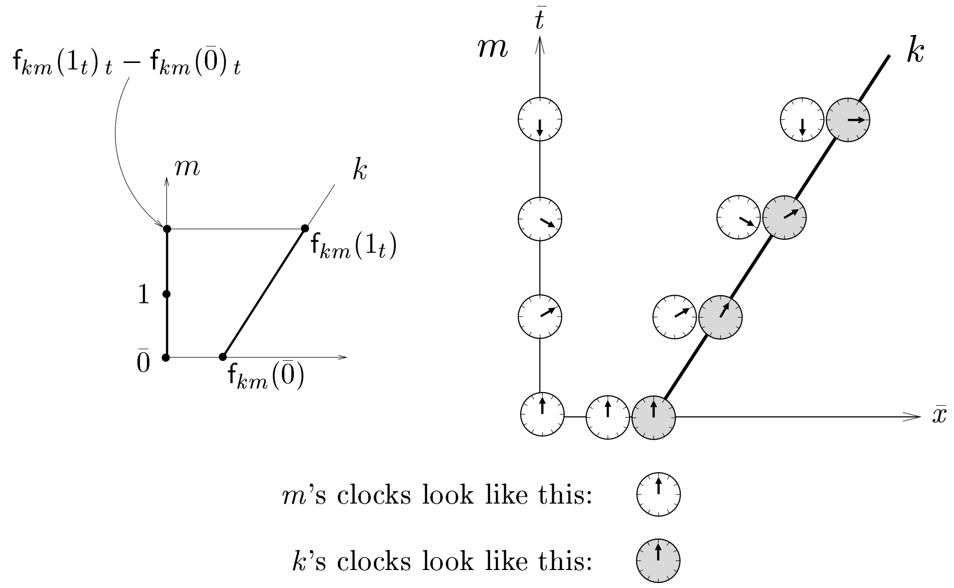


Figure 32: m thinks that k's clocks run slow.

On the proof: The main idea of the proof of (iii) is illustrated in Figure 33. (i) is a corollary of (iii). (ii) can be proved from Thm.2.5.3(i) below as follows:
By Thm.2.5.3(i) below, there are $m, k \in \mathit{Obs}$ such that

$$|\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > 2\,. \tag{2}$$

We have that every automorphism of $\mathbf{F}$ is order preserving, i.e. every automorphism of $\mathbf{F}$ is an automorphism of $\mathfrak{F}$ since $\mathfrak{F}$ is Euclidean. So, by Prop.3.1.4 (p.162) of AMN [18], we have that

$$(\forall \lambda \in F)\ \Big(|\mathsf{f}_{km}(\lambda \cdot 1_t)_t - \mathsf{f}_{km}(\bar{0})_t| \;=\; \varphi(|\lambda|) \cdot |\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t|\Big), \tag{3}$$
$$\text{for some automorphism } \varphi \text{ of } \mathfrak{F}.$$

For every automorphism φ of $\mathfrak{F}$ we have

$$(\forall \lambda \in F)\ \Big((\exists 0 < j \in \omega)\ \frac{1}{j} < |\lambda| < j \;\Rightarrow\; \varphi(|\lambda|) > \frac{|\lambda|}{2}\Big), \tag{4}$$

because of the following. Let $\lambda \in F$ such that $(\exists j \in \omega)\ 1/j < |\lambda| < j$. Between $|\lambda|/2$ and $|\lambda|$ there is a rational number, say x. Let such an x be fixed. Every automorphism (of $\mathfrak{F}$) is the identity function on the rational numbers. Therefore by $|\lambda|/2 < x < |\lambda|$, we have $|\lambda|/2 < x = \varphi(x) < \varphi(|\lambda|)$. So (4) holds. Let $\lambda \in F$ such that there is $0 < j \in \omega$ with $1/j < \lambda < j$. Then

$$\begin{array}{rcll} |\mathsf{f}_{km}(\lambda \cdot 1_t)_t - \mathsf{f}_{km}(\bar{0})| & = & \varphi(|\lambda|) \cdot |\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| & \text{by (3)} \\ & > & 2\,\varphi(|\lambda|) & \text{by (2)} \\ & > & |\lambda| & \text{by (4).} \end{array}$$

This completes the proof of (ii). ∎

[137] We note that for every ordered field the set ω of the natural numbers can be considered as a subset of the ordered field, or in more careful wording ω is embeddable into the ordered field in a natural way. Further we note that if $\mathfrak{F}$ is Archimedian (cf. footnote 109 on p.35) then (ii) above is true in the following simpler form:
(ii)' $(\exists m, k \in \mathit{Obs})(\forall p \in \bar{t}\,)\ |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(\bar{0})_t| > |p_t|$.
An analogous remark applies to Thm.2.5.3(ii).

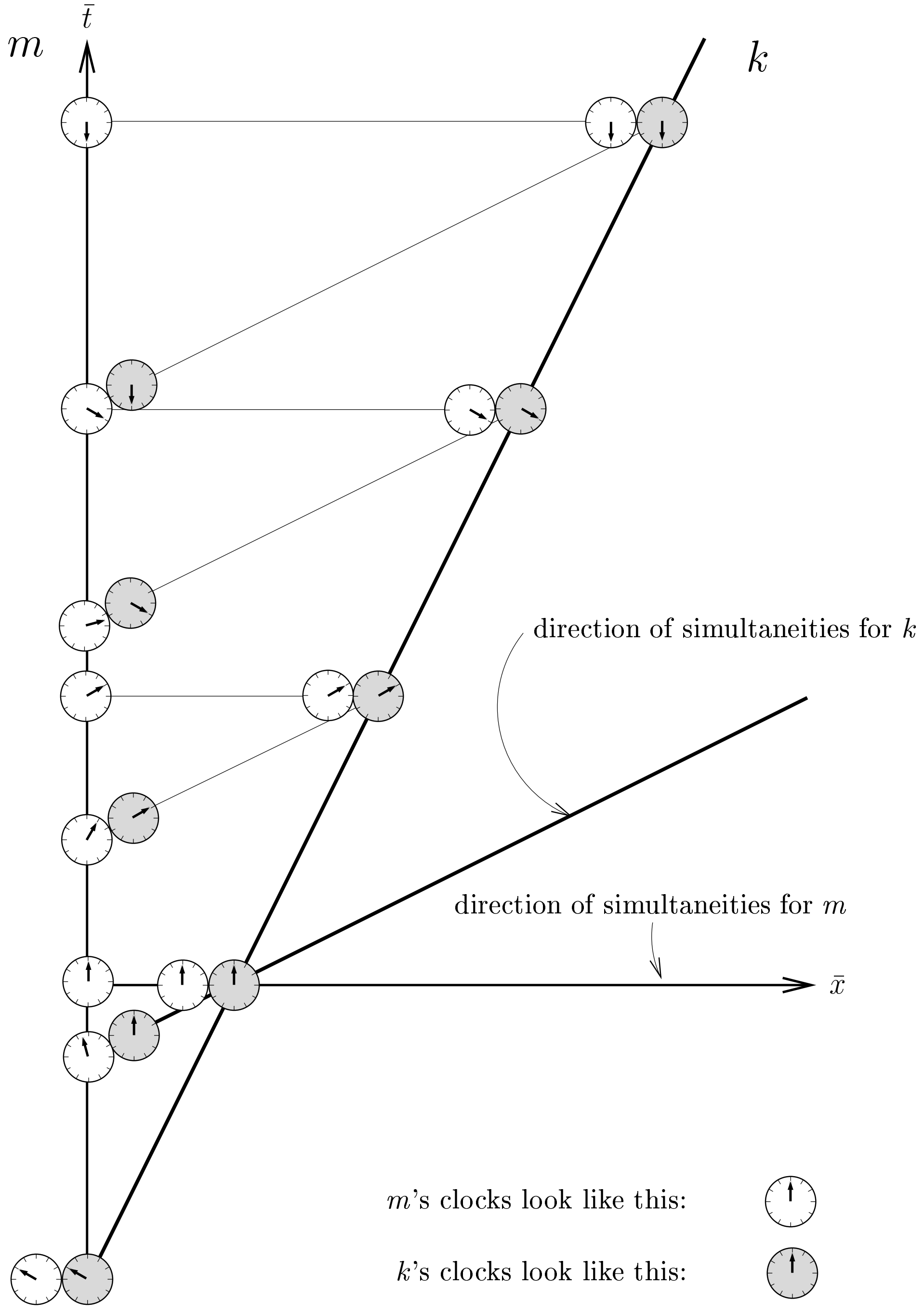


Figure 33: Assume that for m, k's clocks do not run slow. Then k will think that m's clocks run slow.

THEOREM 2.5.3 (Clocks can run very slow.)
Assume **Basax** + **Ax**($\sqrt{\ }$). *Let* $\varrho \in \omega$ *be arbitrary.*

(i) *There are observers* m, k *such that* m *thinks that* k*'s clocks run more than* ϱ*-times more slowly than* m*'s; formally:*

$$(\exists m, k \in Obs)\ |\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > \varrho\,.$$

Moreover:

(ii) $(\exists m, k \in Obs)$

$$(\forall \lambda \in F)\ \Big((\exists 0 < j \in \omega)\ 1/j < |\lambda| < j\ \Rightarrow\ |\mathsf{f}_{km}(\lambda \cdot 1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > \varrho \cdot |\lambda|\Big)\,.$$

On the proof: We include Figure 34 as a hint for the idea of the proof of (i). (ii) follows from (i) similarly as item (ii) of Thm.2.5.2 did. ∎

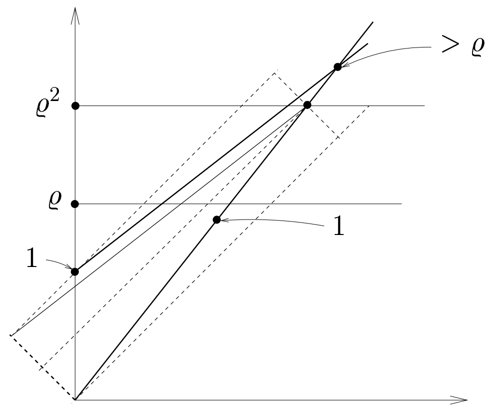


Figure 34: Hint for the idea of proof of Thm.2.5.3(i).

<u>*Let us turn to clocks getting out of synchronism*</u> ("effect" (III) on our "paradigmatic" list). First we need some definitions.

Definition 2.5.4 Let $\mathfrak{M}$ be a frame model. Events $e, e_1 \in \mathcal{P}(B)$ are said to be <u>*simultaneous*</u> for observer $m \in Obs$ iff

$$e, e_1 \in Rng(w_m) \quad \wedge \quad (\forall p \in w_m^{-1}(e))(\forall q \in w_m^{-1}(e_1))\ p_t = q_t\,.\ ^{138}$$

◁

[138] To improve readability we write $w_m^{-1}(e)$ instead of $w_m^{-1}[\{e\}]$, where $m \in Obs$ and $e \in \mathcal{P}(B)$.

THEOREM 2.5.5 (Clocks get out of synchronism.)
Assume **Basax**. *Let $m, k \in Obs$ be such that $v_m(k) \neq 0$. Then (i) and (ii) below hold.*

(i) *There are events $e, e_1 \in \mathcal{P}(B)$ which are simultaneous for m, but are* <u>*not*</u> *simultaneous for k.*

(ii) *Assume that k moves in direction $\bar{x}$ as seen by m, formally: $tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. Then*

$$(\forall p, q \in {}^nF)\ \Big((p_t = q_t \quad \wedge \quad p_x \neq q_x) \quad \Rightarrow \quad \mathsf{f}_{mk}(p)_t \neq \mathsf{f}_{mk}(q)_t\Big),$$

cf. Figure 35. I.e. if m thinks that $e = w_m(p)$ and $e_1 = w_m(q)$ are simultaneous but their x-coordinates are different, then k will think that e and e_1 are not simultaneous.

Intuitively, let us imagine that k is traveling on a spaceship and is being observed by m. Then m will think that clocks in the nose and the rear of k's spaceship are not synchronous, cf. Figures 31, 37. (They do not show the same time.) ∎

We note that Thm.2.5.5 can be refined in the style of Thm.2.5.7 below.

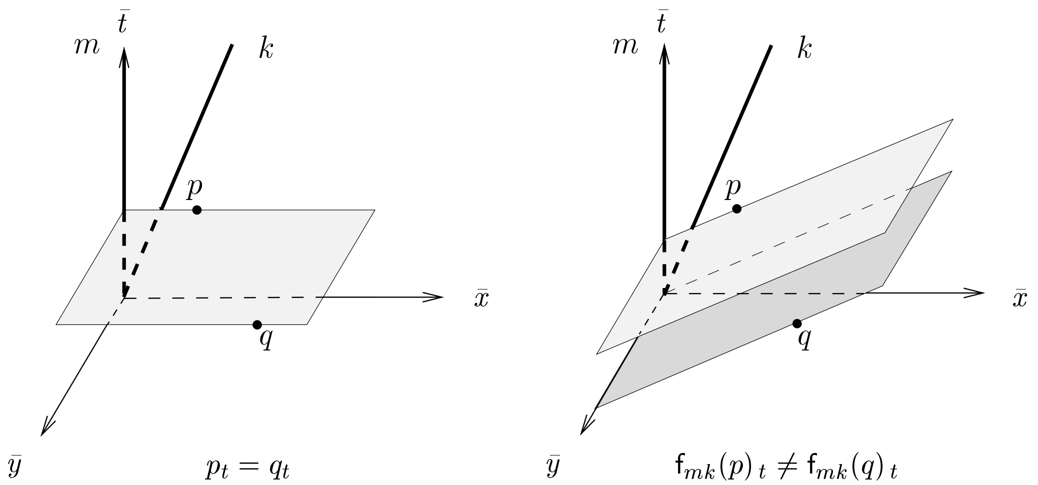


Figure 35: Events $w_m(p)$ and $w_m(q)$ are simultaneous for observer m, but they are not simultaneous for observer k.

THEOREM 2.5.6 (Clocks do not get out of synchronism in direction orthogonal to movement.)
From the point of view of synchronism, "nothing" happens in the spatial direction orthogonal to the direction of movement (cf. Figure 36);[139] *formally: Assume* **Basax**. *Let $m, k \in Obs$. Assume m sees that k does not move in direction $\bar{y}$, i.e. $(\forall p, q \in tr_m(k))\ p_y = q_y$. Then,*

$$(\forall p, q \in {}^nF)\ \Big((\forall i \in n)(i \neq 2 \ \Rightarrow\ p_i = q_i) \quad \Rightarrow \quad \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t\Big).$$

In particular

$$p, q \in \bar{y} \qquad \Rightarrow \qquad \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t .$$

That is, simultaneous events observed by m as separated only in a direction $\bar{y}$ orthogonal to the direction of movement <u>*remain*</u> *simultaneous for the moving observer k.* ∎

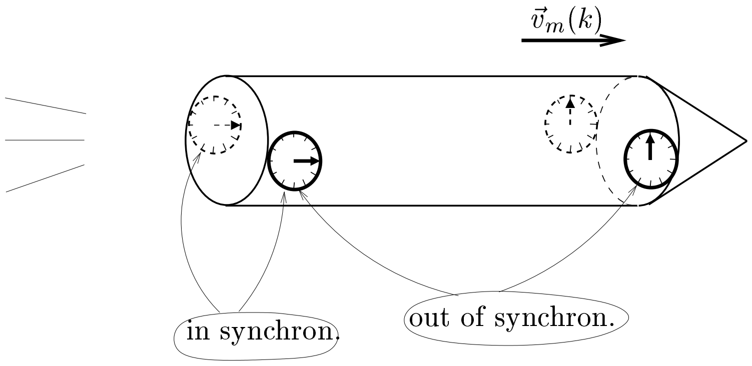


Figure 36: Clocks do not get out of synchronism orthogonal to movement. Imagine the little clocks glued to the hull of the spaceship.

Let us return to discussing the "out of synchronism" effects in Thm.2.5.5. Throughout the rest of this section (in §2.5) **Basax + Ax($\sqrt{\ }$)** *is assumed* (unless otherwise specified), therefore we do not indicate this.

Let $\mathfrak{M}$ be a frame model, and let $m, k \in Obs$ such that $tr_m(k) \in \mathsf{Eucl}$. Let us recall that the velocity of k as seen by m is denoted by $\vec{v}_m(k)$, cf. p.20, and it is a "space vector", i.e. an element of ${}^{n-1}F$.

In connection with the next two theorems we note the following. Since we assumed **Basax**, for every $m, k \in Obs$, $tr_m(k)$ can be considered as a function $tr_m(k) : F \longrightarrow {}^{n-1}F$, if $v_m(k) \neq \infty$; therefore $tr_m(k)(0)$ is well defined.[140]

THEOREM 2.5.7 (The clock in the nose of the spaceship is late.)
Let $m, k, k_1 \in Obs$. Assume $tr_k(k_1)$ is parallel with $\bar{t}$,[141] $0 \neq v_m(k) < 1$, and that time passes forwards for k as seen by m, formally: $(\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t) > 0$. Intuitively, k represents the rear of the "spaceship", while k_1 represents the nose of the "spaceship". Assume further that this "spaceship" is moving forwards as seen by m;[142] formally:

$$\Big(tr_m(k_1)(0) - tr_m(k)(0)\Big) = \lambda \cdot \vec{v}_m(k) \qquad \textit{for some positive } \lambda \in F.$$

(i) *Then m thinks that the clock in the nose of the spaceship is late w.r.t. the clock in the rear of the spaceship (see Figure 37); formally:*

$$(\forall p \in tr_m(k))(\forall q \in tr_m(k_1)) \ \Big(p_t = q_t \ \Rightarrow \ \mathsf{f}_{mk}(p)_t > \mathsf{f}_{mk}(q)_t\Big).$$

(ii) *Let $m, k, k_1 \in Obs$ satisfy all the conditions of the present theorem. Assume further that the length of the "spaceship" as seen by k is 1. For simplicity we formalize this condition as $1_x \in tr_k(k_1)$. Then*

$$(\forall p \in tr_m(k))(\forall q \in tr_m(k_1)) \ \Big(p_t = q_t \ \Rightarrow \ (\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t) < 1\Big).$$

[139] More precisely, if two clocks are separated ***only*** in a spatial direction which is orthogonal to the direction of movement then they do ***not*** get out of synchronism.

[140] Cf. Fact 2.2.4.

[141] This means that k_1 is in rest w.r.t. k, i.e. we use the relationship parallelism between lines in the sense of Euclidean geometry.

[142] Intuitively, this means that m sees the spaceship moving in the direction of its nose.

Intuitively, this says that assuming the length of the spaceship as seen by k is 1 then the difference between the two clock readings (in the rear and in the nose) as seen by m is always smaller than 1.

(iii) *For any $\lambda \in F$ with $0 < \lambda < 1$, there are m, k, k_1 satisfying all the conditions of the present theorem, including the condition in (ii) saying that the length of the "spaceship" as seen by k is 1, such that*

$$(\forall p \in tr_m(k))(\forall q \in tr_m(k_1)) \Big(p_t = q_t \;\Rightarrow\; (\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t) > \lambda \Big).$$

Intuitively, the asynchronism between the two clock readings can get arbitrarily close to 1. ∎

Item (iii) of the above theorem describes how much "asynchronism" we can obtain as a relativistic effect.

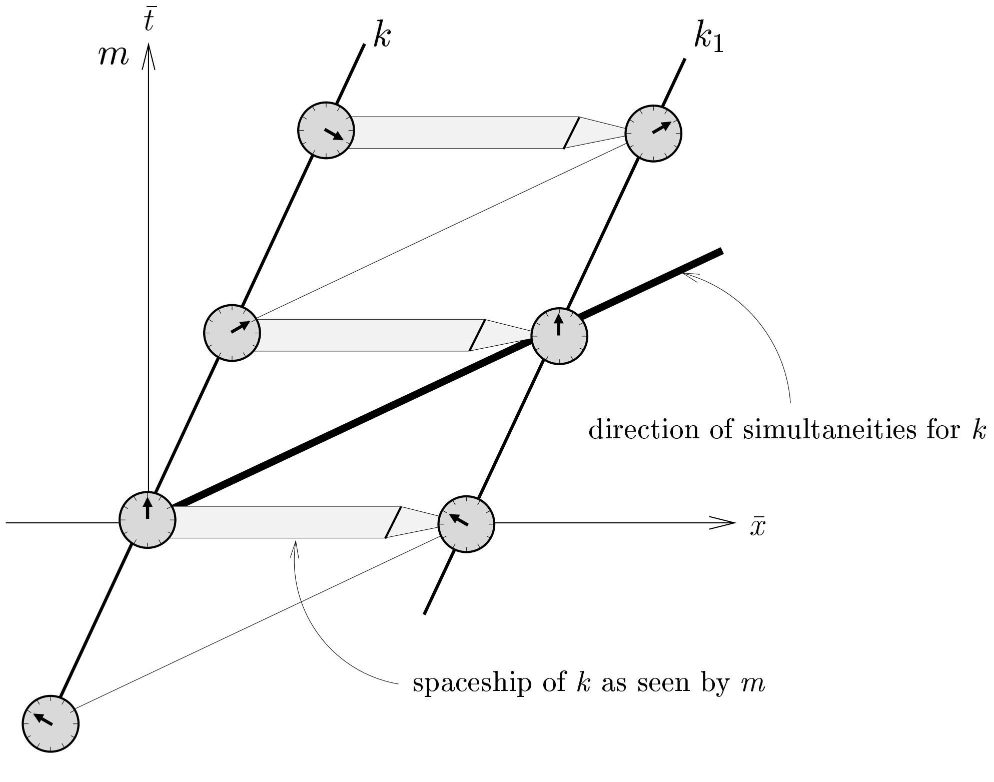


Figure 37: Clock (of k) in the nose of the spaceship is late w.r.t. the clock in the rear, when viewed by m. (The length of this spaceship is more than 1 as seen by k.)

<u>*Next, we turn to discussing how meter-rods shrink,*</u> i.e. to paradigmatic effect (II). (Strangely enough, one has to represent meter-rods and their shrinking slightly differently than it was the case with clocks.)

Notation 2.5.8 Assume $\mathfrak{F}$ is Euclidean, i.e. that $\mathfrak{F} \models \mathbf{Ax}(\sqrt{\ })$. Let $p \in {}^nF$. Then $|p|$ denotes the <u>*Euclidean length*</u> of vector p, i.e.

$$|p| \stackrel{\text{def}}{=} \sqrt{p_0^2 + p_1^2 + \ldots + p_{n-1}^2}\,.$$

◁

THEOREM 2.5.9 (Spaceships shrink.)
There are observers $m, k, k_1 \in Obs$ with $\bar{0} \in tr_m(k)$, and with $tr_k(k_1)$ parallel to $\bar{t}$,[143] *such that for $p := tr_m(k_1)(0)$ and $q := tr_k(k_1)(0)$, we have $|p| < |q|$.*[144]

The last statement $|p| < |q|$ can be interpreted as saying that m thinks that the purely spatial distance between observers k and k_1 is shorter than it is observed by k, see Figure 38.

Intuitively, if k represents the rear of the "spaceship" and k_1 represents the nose of the "spaceship", then this spaceship is shorter for m than for k. ∎

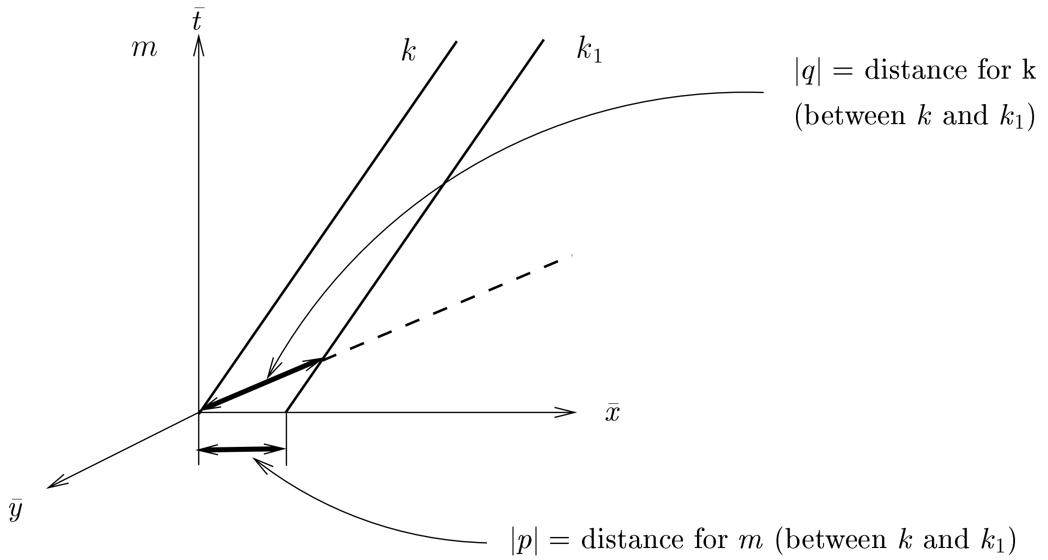


Figure 38: Illustration for Thm.2.5.9.

Remark 2.5.10 An improved version of Thm.2.5.9 could be formulated analogously to Thm.2.5.3 saying that meter-rods can get arbitrarily short (if they are parallel with the direction of movement).

◁

Remark 2.5.11 Analogously with Thm.2.5.6, we could formulate a theorem saying that meter-rods orthogonal to the direction of movement do <u>*not*</u> get shorter (at least not as a consequence of relativistic effects). But for this we would need extra conditions formulated in §2.8 "A symmetry axiom". Cf. Item 2.8.12 (p.133).

◁

[143] This means that k_1 is in rest w.r.t. k, i.e. we use the relationship parallelism between lines in the sense of Euclidean geometry.

[144] Here again $|p|$ is the Euclidean length of $p \in {}^{n-1}F$.

Remark 2.5.12 Throughout the present remark we assume that the ordered field reduct $\mathfrak{F}$ of our model $\mathfrak{M}$ has no nontrivial automorphisms.[145]

(i) Thm.2.5.9 above could be interpreted and modified intuitively as follows: There are observers m and k such that m sees k moving in direction $\bar{x}$, and those meter-rods of k which are pointing in direction $\bar{x}$ as seen by m, are shorter when observed by m than as observed by k. In short: m thinks that k's meter-rods pointing in direction $\bar{x}$, as seen by m, shrink. We will use this intuitive language in the rest of the remark without formalizing it. The reader is invited to formalize it.

(ii) Assume m sees k moving in direction $\bar{x}$ more slowly than light and with nonzero speed.

(a) Let us concentrate on meter-rods pointing in direction $\bar{x}$ as seen by m.[146] Let us call them x-meter-rods. Then either m will think that k's x-meter-rods shrink or k will think that m's x-meter-rods shrink or both.[147]

(b) Let us concentrate on meter-rods pointing in direction $\bar{y}$ as seen by m.[148]
Let us call them y-meter-rods. Then if m thinks that k's y-meter-rods shrink, then k will think that m's y-meter-rods grow.

(iii) Let $m, k \in Obs$. Assume $v_m(k) < 1$. Then one of them thinks that all meter-rods of the other shrink or remain unchanged. Those meter-rods shrink the most which point in the direction of movement. Further, those can remain unchanged which are orthogonal to the direction of movement.

(iv) Let us return to clocks getting out of synchronism in connection with item (iii), cf. Theorems 2.5.5–2.5.7. Let $m, k \in Obs$. Consider pairs of clocks which are synchronous for k and the distance between two clocks in a pair is 1 for k. Then that pair will get out of synchronism most the connecting line of which is parallel to the direction of motion of k (as seen by m). $\triangleleft$

The following theorem says that on a moving spaceship (i) either clocks slow down or meter rods (pointing in the direction of movement) shrink (or both, of course) and (ii) clocks in the rear and the nose of ship get out of synchronism.

THEOREM 2.5.13 (Clocks slow down or meter rods shrink.) *Assume* **Basax** + **Ax**($\sqrt{\ }$). *Let $m, k \in Obs$, $0 < v_m(k) < 1$. Then (i), (ii) below hold.*

(i) *Either the clocks of k run slow or meter rods of k parallel with $\vec{v}_m(k)$ shrink (as seen by m).*

(ii) *Clocks in the nose and rear of the ship of k get out of synchronism.* ∎

Remark 2.5.14 If we omit the condition **Ax**($\sqrt{\ }$) from Theorem 2.5.13 above, then the theorem remains basically true but the formulation gets more complicated, cf. e.g. the formulation of Theorem 2.5.7(iii). Further, in Theorem 2.5.13, **Ax**($\sqrt{\ }$) can be replaced by the weaker assumption that f_{mk} is *betweenness-preserving*[149]. $\triangleleft$

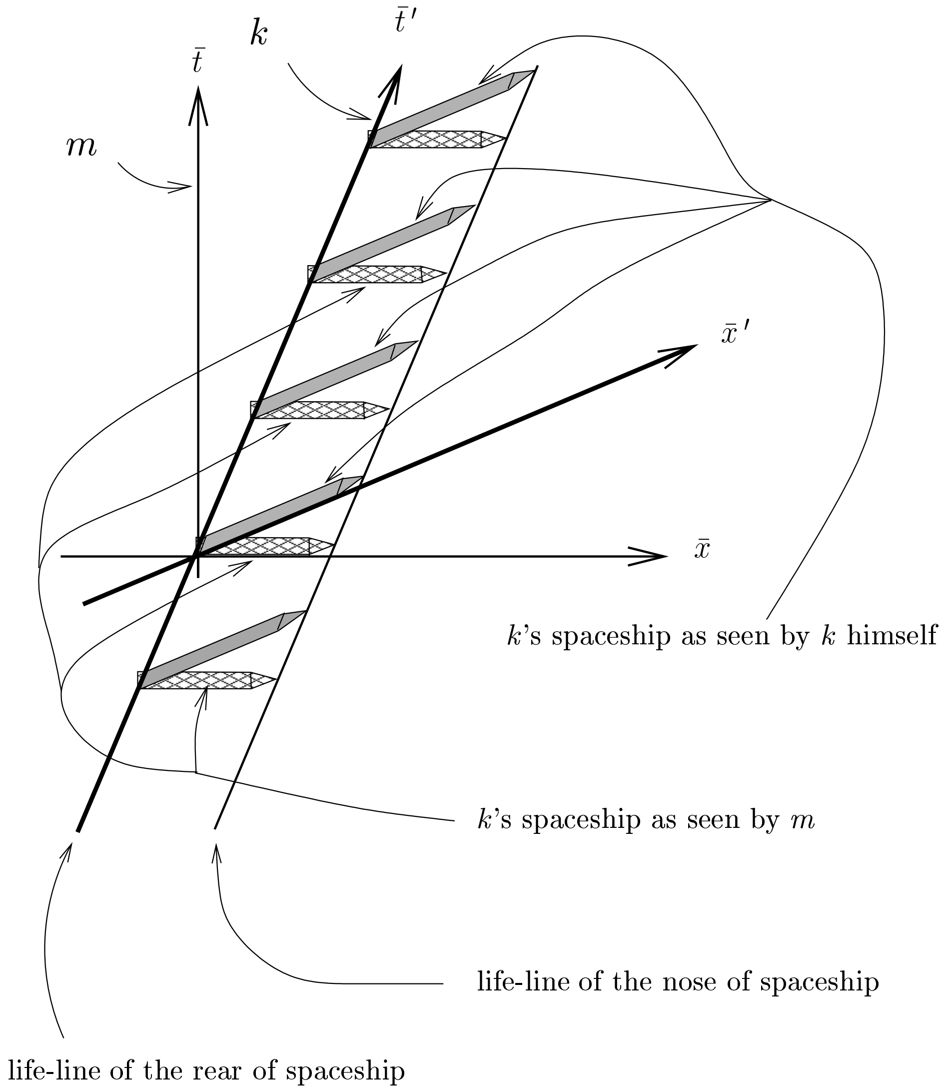
[145] This assumption can be eliminated on the expense of restricting discussion to meter-rods of rational length as seen by that observer whose meter-rods they are.
[146] I.e. even if the meter-rod is k's one we check whether m sees it pointing in direction $\bar{x}$.
[147] Here the emphasis is on that it is consistent with **Basax** that both m and k think that the other's x-meter-rods shrink.
[148] I.e. even if the meter-rod is k's one we check whether m sees it pointing in direction $\bar{y}$.
[149] The ternary relation Betw of betweenness is defined on p.140. Intuitively, for $p, q, r \in {}^nF$ the relation $\mathsf{Betw}(p, q, r)$ holds if p, q, r are collinear and q is between p and r.

The relativistic effects (I)–(III) discussed so far will lead e.g. to the famous *twin paradox*. However, for that we will need a symmetry axiom **Ax(symm)** discussed in section "A symmetry axiom" (§2.8). So, we will return to the twin paradox in §2.8, cf. Thm.2.8.27 (p.92). An even more satisfactory discussion of this paradox can be given by looking at accelerated observers, hence we will return to the "twin" in Chapter 8 ("Accelerated observers") of a future edition [19] of AMN [18] again.

We will see in §2.8 how **Ax(symm)** (introduced in §2.8) influences the paradigmatic effects (I)–(III). E.g. we will see that these effects (for example, the effect of clocks slowing down) admit a simpler and stronger formulation in **Basax** + **Ax(symm)** than in pure **Basax**. Cf. Items 2.8.7–2.8.12.

In AMN [18, §4.8] we investigate how paradigmatic effects (I)–(III) (discussed in items 2.5.2– 2.5.11) change if we use axiom systems more subtle than **Basax**. Many of these axiom systems are introduced in Chapter 3 herein.

Our next figure illustrates the meter-rod shrinking effect, i.e. items 2.5.9–2.5.12. To be more intuitive, we draw spaceships instead of meter-rods. The figure represents how observers m and k see k's spaceship.



Here $\bar{t}\,'$ and $\bar{x}\,'$ denote the respective coordinate axes of observer k.

2.6 Are all three paradigmatic effects necessary?

A central question that motivates our interest in the models of **Basax** is the following:

($\star$) Are all three paradigmatic effects discussed in section 2.5 necessary consequences of special relativity? If not, are they independent of one another?

It is question ($\star$) that triggers our interest in the class $\mathsf{Mod}(\mathbf{Basax})$, i.e. in the question how different the models of **Basax** can be from each other. [150]

Turning to the question itself, the following idea naturally comes to one's mind. Are all the paradigmatic effects, items (I) to (III) on page 54, necessary consequences of **Basax**? If not, in which combinations can they occur? Cf. also the pictures on p. 51.

First, concerning effect (III) (moving clocks get out of synchronism), the answer is simple. It *is* a necessary consequence of **Basax**, by Theorem 2.5.7. That is, whenever $v_m(k) > 0$, m thinks that the clocks in the nose and in the rear of k's spaceship are out of synchronism (provided, of course, that k thinks they are synchronized). This is so in every model of **Basax** and for every $m, k \in Obs$.

On the other hand, to answer the question as far as the other effects are concerned, we must pose it more precisely. Let us fix an observer m_0 in a model $\mathfrak{M}$ of **Basax**. We will think in m_0's world-view. For example, "k moves" means that k moves relative to m_0. Now, we will seek for the answers to our question in a systematic manner (cf. items 1-3 below).

1. As we have already pointed out, paradigmatic effect (III) must be true in m_0's world-view.

2. Clocks do *not* necessarily slow down on moving spaceships (i.e. effect (I) is not necessary). More formally, there are a model $\mathfrak{M} \models \mathbf{Basax}$ and an observer $m_0 \in Obs^{\mathfrak{M}}$ such that

   $(\forall k \in Obs)[m_0$ thinks that k's clocks tick with exactly the same rate as his clocks$]$. [151]

   Such a model $\mathfrak{M}$ (with a distinguished m_0) is represented in Figure 29 (p. 51) under the name $\mathfrak{M}_1$. That model is 2-dimensional, but it can be extended to 3 or 4 dimensions, too. However, this generalization from $n = 2$ to $n \geq 3$ is not completely straightforward. We do not go into the details here, but the key idea is described in AMN [18, §3.2]. We should mention one difference between the cases $n = 2$ and $n = 3$. In the case $n = 2$ we have a so-called Minkowski-sphere around the origin which, assuming $\mathsf{f}_{m_0k}(\bar{0}) = \bar{0}$, tells us for each k how long its unit-vectors are (i.e. where $\mathsf{f}_{m_0k}(1_i)$ is). This sphere works uniformly for all choices of $k \in Obs$ (assuming $\mathsf{f}_{m_0k}(\bar{0}) = \bar{0}$). By contrast, in the case of $n = 3$ all we know is that all the points $\mathsf{f}_{m_0k}(1_i)$ are in a horizontal plane as depicted in Figure 39. However, after 1_t^k is determined by this plane for each choice of k, we still have to fix the rest of k's unit vectors as it is done in AMN [18, §3.2]. (Where it is shown that choosing 1_t^k *arbitrarily*, the other unit vectors can be fixed so that the axioms of **Basax** are validated.)

[150] In AMN [18], this interest will lead us to the investigations in §3.6 (Models of **Basax**), as well as to the study of non-elementarily equivalent models of **Basax** in e.g. Theorem 3.8.18(ii) on p. 303 of AMN [18]. Cf. also AMN [18, Remark 3.6.15 on p. 271].

[151] We emphasize again that m_0 thinks that *all observers* have clocks running with the correct rate.

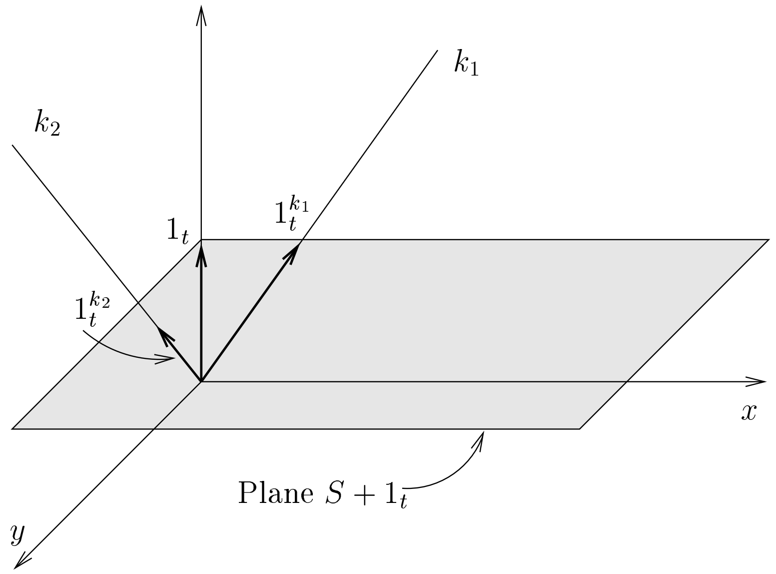


Figure 39: Illustration for item 2.

In the second part of this section, when discussing the independence of the particular paradigmatic effects, we will concentrate on the case $n = 2$; but all the results can be generalized to $n \geq 3$, analogously to the generalization indicated in item 2 above. We invite the interested reader to figure out what the answers look like for $n = 3$ first, and later to all $n \geq 3$.

Let us return to discussing what happens if m_0 thinks that all moving clocks tick with the correct rate (i.e., no clocks slow down or run fast). The present answer to the question ($\star$) (namely, that effect (I) is not necessary) applies if we are allowed to fix $\mathfrak{M}$ *and* m_0. However, in the same model $\mathfrak{M}$ there will be an observer m_1 who thinks that moving clocks *do* slow down. Indeed, if $v_{m_0}(k) > 0$, then k will think that m_0's clocks do slow down (and they slow down more than would be necessary if we did not force k's clocks not to slow down for m_0; this will be implicitly seen in §2.8).

3. Similarly to item 2, moving spaceships *need not* shrink. That is, there are $\mathfrak{M} \models$ **Basax** and $m_0 \in Obs^{\mathfrak{M}}$ such that in m_0's world-view moving spaceships do not shrink. Formally,

$$(\forall k \in Obs)[m_0 \text{ thinks that } k\text{'s meter-rods are of the correct length}].$$

(The reader is invited to formalize this statement in our frame language.) The model proving this claim is remotely similar to $\mathfrak{M}_3$ in Figure 29 (p. 51), but the functions that are parts of the Minkowski circle must grow faster in the model. The reader is invited to construct (and draw) such a model (based on the world-view of some m_0 in which *no* spaceship shrinks). For this exercise it might be useful to consult Figure 38 on p. 62 proving that spaceships "usually" do shrink. On the other hand, see picture 40.

By items 2 and 3 above we received permissive answers to our question concerning the removability (or changeability) of the paradigmatic effects. The second part of our question asked how *independent* effects (I)-(III) are of one another.

The paradigmatic effects (I)-(III) are *not* independent. We have already seen that effect (III) (violation of synchronism) is necessary. Further, assume $v_{m_0}(k) > 0$, and consider m_0's

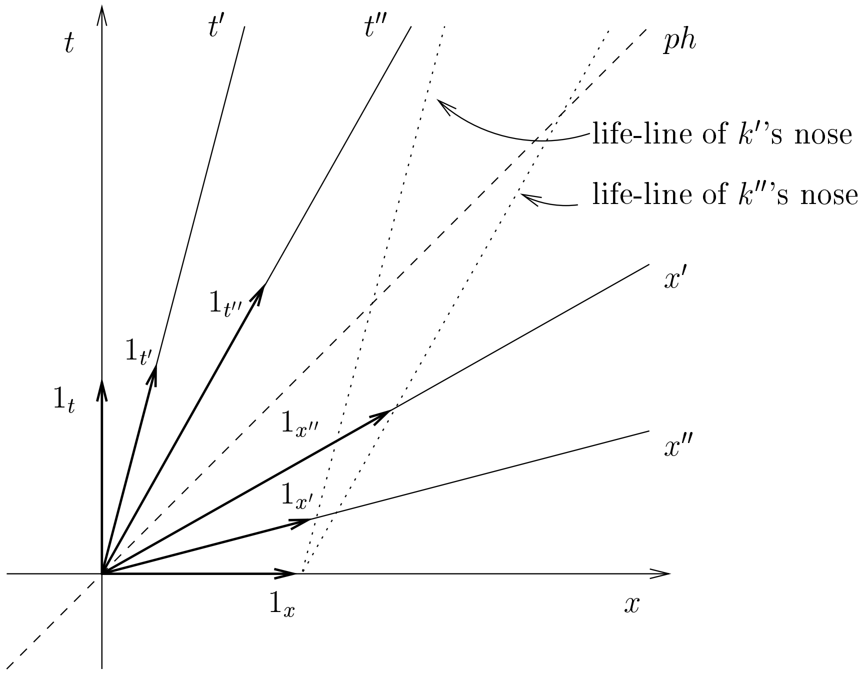


Figure 40: Illustration for item 3.

world-view. The following holds:

$$(k\text{'s clocks do } \textit{not} \text{ slow down}) \Rightarrow (k\text{'s spaceship shrinks}).$$

Similarly,

$$(k\text{'s spaceship does } \textit{not} \text{ shrink}) \Rightarrow (k\text{'s clocks slow down}).$$

These statements are stated in Theorem 2.5.13. Actually in Thm. 2.5.13, $\mathbf{Ax}(\sqrt{\ })$ was assumed, but we conjecture that it is not needed here, because we are stating only that

($\star$) (k's clocks always show the correct time) $\Rightarrow$ (one of k's meter rods[152] shrinks),

that is,

$$(\forall p \in \bar{t})p_t = \mathsf{f}_{km}(p)_t \Rightarrow (\text{one of } k\text{'s meter rods shrinks}).$$

It seems to us that **Basax** $\models (\star)$, but we have not checked this claim carefully.

To sum up: On the one hand, we can get rid of effect (I), but *then* we must have (II) and (III).[153] On the other hand, we can get rid of effect (II), but *then* we must have (I) and (III). So, for a possible observer m_0 in a possible model $\mathfrak{M} \models$ **Basax**, the following combinations can be realized:

(A) All three effects (I), (II) and (III) are experienced by m_0.

(B) Effects (I) and (III) prevail, but (II) does not.

(C) Effect (I) is not present, but (II) and (III) are.

[152] Namely, the one pointing in the direction of k's movement.

[153] Moreover, we pay for not having (I) by having (II) to a higher extent.

There are no other possibilities. Thus we have at least three essentially different classes of models for **Basax**,[154] and this fact triggers our interest in asking how many, and what sorts of, *non-elementarily equivalent* models **Basax** has. The reason why we talk about non-elementarily equivalent models is that this expression means that the models in question are not only different (i.e., non-isomorphic), but they are actually distinguishable by a formula in our frame language like (the formalized versions of) (A), (B) and (C) are. Actually, the above mentioned three classes of models are distinguishable by thought experiments too, which might be a stronger notion of distinguishability (than the one using formulas). It would be interesting to see how many classes of models of **Basax** are distinguishable by means of thought experiments, but for this purpose we would need to define which formulas of our frame language count as thought experiments. We do not deal with this issue here.[155]

Later, in §2.8, we will introduce a natural axiom $\mathbf{Ax}(\|)$ saying that observers not moving relative to each other see the world *essentially* the same way. We mention this because the presently discussed issue concerning the connection between the paradigmatic effects is even more interesting in $\mathbf{Basax} + \mathbf{Ax}(\|)$ than in pure **Basax**. Therefore we mention that the answer to our question remains exactly the same, i.e. cases (A) to (C) are all the possibilities, for $\mathbf{Basax} + \mathbf{Ax}(\|)$ too. Actually, most of those axioms to be introduced that we will call *auxiliary axioms* (cf. (3) in the list of axioms here or AMN [18, §3.8]) leave the answer to the present question *unchanged* (i.e., (A), (B), (C) remain possible). For example, for $\mathbf{Basax} + \mathbf{Ax}(\surd) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\uparrow) + \mathbf{Ax(ext)} + \mathbf{Ax(rc)}$ the situation is the same as outlined above for **Basax**.[156] We could even add the continuity axiom **Ax(cont)** from §5 of AMN [18] to the axioms without changing this result. However, the symmetry axioms, e.g. **Ax(symm)** to be introduced soon, in §2.8, will change this picture.

A more systematic discussion of the paradigmatic effects, their interdependence, their dependence on the axioms of our relativity theory etc. is found in AMN [18, §4.8(pp.635–704)].

[154] Say, $\mathfrak{M}'$ is such that all observers are of type (A), $\mathfrak{M}''$ is such that it has both (A) and (B) type observers but none of type (C), and $\mathfrak{M}'''$ has observers of type (A) and (C), but none of type (B). It takes some time to check that $\mathfrak{M}''$ and $\mathfrak{M}'''$ exist, but they do. We omit the proof of this claim.

[155] Cf. also Remark 3.6.15 of AMN [18] for further philosophical reasons for studying non-elementarily equivalent models of **Basax**.

[156] The mentioned extra axioms can be found in the list of axioms here and in AMN [18, §3.8].

2.7 Faster than light in two dimensions

In §2.4, Cor.2.4.3 we saw that if $n = 2$ then faster than light (FTL) observers are possible, more formally, the existence of FTL observers is consistent with **Basax**(2). For completeness we note that **Basax**(n) with $n > 2$ excludes FTL observers, by Theorem 3.2.13 in Chapter 3 herein. However, there are refinements of **Basax** which allow FTL observers for $n > 2$ too. Cf. e.g. the axiom system **Relphax** and Thm.3.4.22 (p.223) in AMN [18] (or the version of **Basax** mentioned in AMN [18] where FTL-observers use only 2 dimensions for coordinatizing the set of events). Further, in **Basax**(n), n arbitrary, we still can have interesting FTL effects with the only difference that, in $m\,\mathrm{FTL}\,k$, k is not a fully fledged observer but only a *body having an "inner clock"*. Such FTL bodies are consistent with **Basax**(n) and practically all of its variants.[157] Bodies with inner clocks are discussed in the manuscript Madarász-Németi [175] and will be more extensively discussed in AMN [19]. Their theory is extremely similar to "**Basax**(2) + *FTL observers exist*". The 2-dimensional "parts" of these theories are fairly close to **Basax**(2) which adds extra motivation for our looking into FTL in **Basax**(2).

Let $k, m \in Obs$. We call k FTL w.r.t. m iff $v_m(k) > 1$. We also write $k\,\mathrm{FTL}\,m$ for this. We call k STL w.r.t. m [158] iff $v_m(k) < 1$. We also write $k\,\mathrm{STL}\,m$ for this. (Warning: The definitions of STL and FTL will be refined in §2.8.5 (p.91), but in the present section we need not worry about that.[159]) In AMN [18, §2.7], assuming **Basax** + $\mathbf{Ax}(\sqrt{\ })$, we prove that STL is an equivalence relation.

In Thm.2.7.1 and in Figure 41 below we will see that it is impossible to have symmetry when considering the directions of "flows of time" (i.e. considering whether the observed clocks run forwards or backwards).

THEOREM 2.7.1 *Assume* **Basax**(n) + $\mathbf{Ax}(\sqrt{\ })$.[160] *Let $m, k \in Obs$. Assume k moves FTL relative to m, i.e. $v_m(k) \geq 1$. Then the following hold. If m thinks that k's clock runs forwards then k will think that m's clock runs backwards. However, if m thinks that k's clock runs backwards then k will think that m's clock runs forwards.*[161] *Summing it up, m and k see each other's clocks differently. Formally:*

$$\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t > 0 \quad \Leftrightarrow \quad \mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t < 0\,.$$

On the proof: By Prop.2.3.3(iii), we may assume $v_m(k) > 1$. Then for $n = 2$, the idea of the proof is illustrated in Figure 41. For $n > 2$, one either checks that the idea represented in Figure 41 works; or equivalently one may use the no FTL theorems in §3. ∎

---

[157] In the physics literature, FTL bodies are called *tachyons*, cf. Gott [106], [107], Feinberg [85], d'Inverno[73, pp.22,23,51,228], Ricami [221], Gibbs [97], Rembielinski [219], cf. also the references in §1.3(iv) on p.xii. We should also mention ideas (on tachyons) of physicists S. Tanaka, O. M. P. Bilaniuk, V. K. Despande, E. C. G. Sudarshan from the 1960's. In the just discussed literature, the formulation of some interesting questions assume that tachyons have inner clocks. Actually, for formulating these interesting questions (e.g. FTL allows sending messages to the past) we have to assume that these bodies have inner clocks.

[158] slower than light as seen by m

[159] In **Basax** the two definitions are equivalent.

[160] We note that a variant of this theorem remains true without $\mathbf{Ax}(\sqrt{\ })$, i.e. in pure **Basax**.

[161] Sometimes we quote this theorem as if it stated "... k's clocks run backwards ...". In these quotations we have in mind the clock in the rear of k's spaceship together with the clock in the nose of the spaceship etc. and that is why we write in the plural k's clocks instead of just k's clock as the theorem above says.

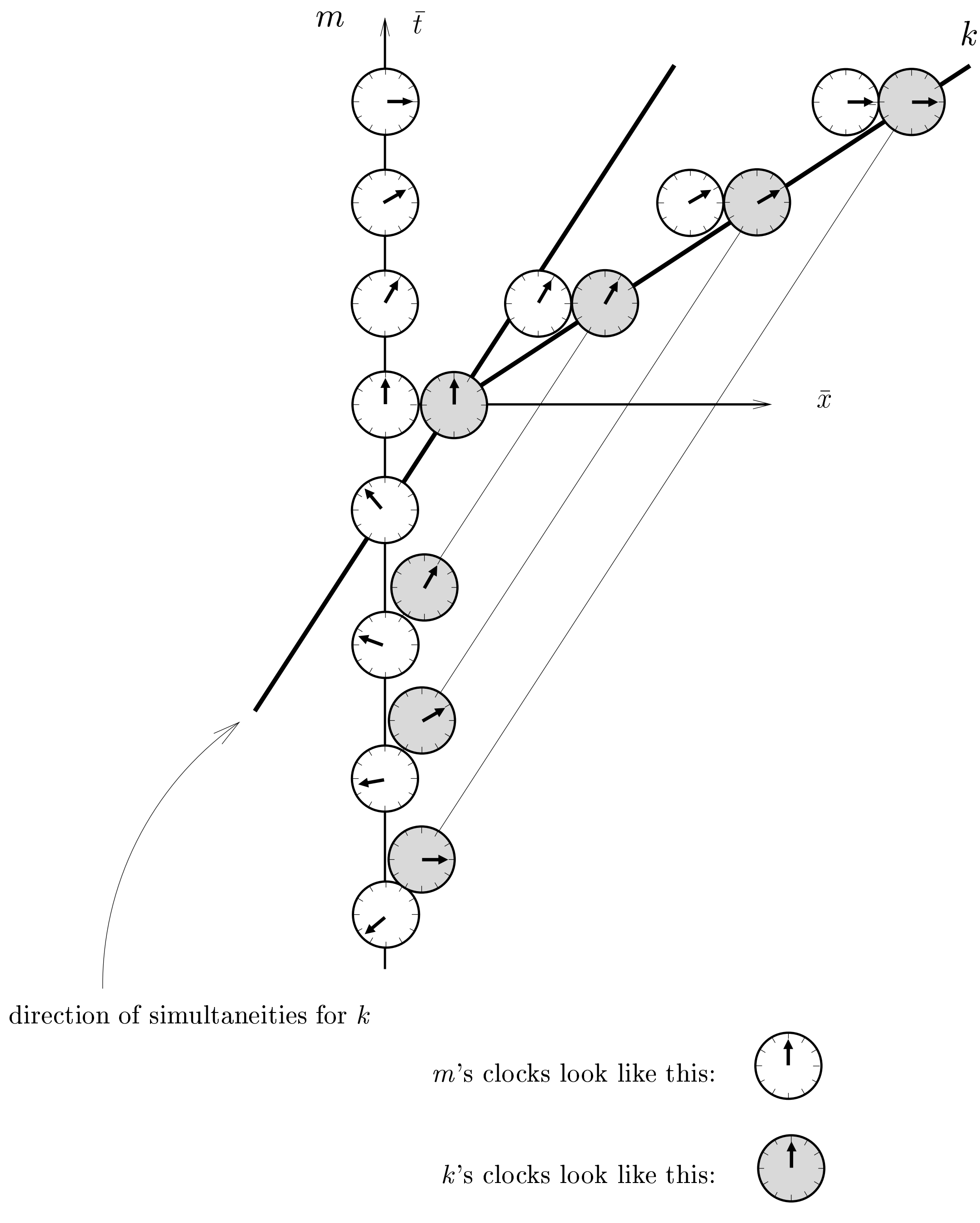


Figure 41: Assume that m thinks that k's clock runs forwards. Then k will think that m's clock runs backwards.

The above theorem already indicates that FTL leads to unusual temporal behavior. Putting it more boldly, the theorem can be interpreted as pointing in the direction that FTL seems to cause phenomena with a "time-travel-ish" flavor. More in this direction is proved in AMN [18, §2.7].

It was known[162] that the existence of FTL observers has some exotic, even "fantastic" consequences, e.g. messages can be sent to the past.[163] In AMN [18, §2.7] and AMN [19, §2.7] we prove even more "exotic" consequences of FTL, e.g. if FTL is combined with the theory of accelerated observers as outlined in AMN et al. [26] (in a suitable way), then it makes "actual time-travel" to the past possible. Motivated by these challenging possibilities, in AMN [18, §2.7], AMN [19, §2.7] the present author provides a careful, logically precise and easily understandable analysis of FTL in **Basax**(2).[164] The exotic possibilities motivate us to apply a wide range of the possibilities offered by the methodology of logical analysis,[165] e.g. we ask ourselves exactly what kinds of arrangements in FTL-models of **Basax**(2) [166] are logically possible, exactly how the details look like, how an FTL observer observes (coordinatizes) the "STL world", what the visual effects are, e.g. how does an FTL observer visually (via photons) see the STL world, etc. The point is that after proving that (in **Basax**(2)) the possibility of sending messages to the past does *not* lead to logical contradictions, our intuition (which "feels" that there should be a contradiction here) leads to several natural questions like "if this is possible, then how do these or those details look like". E.g. how does the "normal" STL observer m see in detail the clocks of an FTL observer k who is moving "so fast" that in some sense he is moving "backwards in time" as observed by m. After answering this, we ask how this FTL observer k sees the very same things, and in particular, how k sees the STL observer's clocks. Among other things, it turns out that there cannot be "perfect symmetry" between these two observers (AMN [18] Thm.2.7.5).

---

[162] Cf. e.g. Gott [107, pp.126-129], Friedman [91, pp.161-162] and the relevant references in these two books.

[163] Actually, this is why e.g. Reichenbach [218] excluded FTL by adding an extra axiom to relativity. Namely, many people at that time thought that the above mentioned consequences lead to logical paradoxes, cf. e.g. Friedman [91, p.162, discussing axiom (P3)], or Rindler [222, p.36, line 15]. Already in 1949 Gödel showed that they *do not* lead to logical paradoxes. Cf. footnote A on p.199 of Gödel [99]. Leading logician David Lewis devoted a whole paper to proving and explaining in detail that FTL particles (with inner clocks) and their "time-travel-like" consequences do *not* lead to logical problems and that from the logical point of view such things and in particular closed temporal loops are possible. Cf. Lewis [157]. Cf. also Gott [107, e.g. pp.16-20], Earman [77, p.170, pp.193-194 or the entire chap.6], Novikov [207], Yourgrau [270], Gödel [100, Fig.1 on p.286, p.228, and the quotation from Weyl's 1918 paper on p.228]. Actually, relativity theorist Hermann Weyl seems to be the first (1918) to notice that General Relativity is consistent with "time travel", i.e. with closed temporal loops. About the "anti-time travel" attitudes Earman [77, p.54 line 10] writes: "... such an attitude is no longer as popular as it once was." Physicist Newman [206, p.982] writes (in General Relativity and Gravitation Vol 21) "It has become customary to claim that closed time-like curves render a space-time physically unreasonable. Certainly, if the universe does contain closed time-like curves, a revision of fundamental premises of physics, and philosophy, may be necessary. However, to dismiss this, and other forms of causality violation, out of hand is reminiscent of the dogmatism regarding singularities prior to the singularity theorems." Cf. also Tipler [260, p.442 lines 8-6 bottom up], Friedman-Morris et al. [90], the works of Novikov, Everett, J. Friedman, Grant, Headrick, Holst&Matschull, Li&Gott, Ori, Pickover, Simon in the bibliography of Gott [107]. Cf. also Gott [107, p.129 lines 8-1 bottom up, p.130 last 8 lines]. For more on this cf. §1.3(iv), p.xii. Summing it up: If we do not add an extra axiom to general relativity saying that closed time-like curves are prohibited, then it is not clear why we should add an axiom to special relativity saying that faster than light causal signals are prohibited. The motivation for both axioms is the same: to exclude causal loops in an a priori manner. But if we are not willing to do this a priori (pre-)judgment in general relativity, why should we do it in the special theory?

[164] and in AMN [19, §2.7] the present author provides an analysis of FTL in the accelerated observer version of some of our more flexible (than **Basax**) theories. Actually, these theories are the localized versions **Loc**(Th) in Chapter 3 herein.

[165] cf. the introductions of AMN [18] and of Chapter 3 herein

[166] i.e. models of **Basax**(2) + *there are FTL observers*

Since in our theories like **Basax** we have chosen a level of abstraction in which mass and energy are disregarded, i.e. since we are in relativistic kinematics (as opposed to relativistic dynamics), the question comes up naturally whether in a suitably flexible version[167] of **Basax**(2) enriched with accelerated observers, it might be consistent to accelerate an STL observer up to an FTL speed (relative some fixed inertial $m_0 \in Obs$, of course). If yes, then what does the so "fantastically" accelerated observer observe (coordinatization-wise), what does he see (via photons) etc. In particular, how does he observe/see the STL world in detail. These latter questions are discussed in AMN [19] but in less detail than the previous questions not involving acceleration.[168]

Someone might object that discussing FTL in two dimensions is not so relevant to understanding our world because the "real world" is 4-dimensional. The answers to this objection are in (i), (ii) below. (i) This part of the answer coincides with what we said in the introduction to the present sub-section about FTL in the case of $n > 2$. Namely, there are refinements of **Basax**(4) allowing either FTL or some interesting consequences of FTL in **Basax**(2), and these refinements are analogous with **Basax**(2), in various ways. (ii) The really exotic predictions of FTL (like time-travel to the past) are consistent[169] with Einstein's general theory of relativity, as was proved probably first by Gödel[170] whose result was essentially strenghtened by Ozsváth & Schücking [209], cf. also Thorne [259], Gott [107], Deutsch [71] and the references at the very beginning of the Introduction to this work. Now, after knowing that these predictions are possible in the general theory, it might be of some intellectual interest to see how these predictions are proved in detail (and how they are realized in detail) in a tangible, transparent and purely logical form, in a simplified version of relativity theory e.g. in **Basax**(2). Of course, the proofs in **Basax**(2) cannot be easily[171] generalized to e.g. Gödel's rotating universe (or to Kerr-type rotating black holes), but they still can provide us with some degree of intellectual insight in connection with how and why these exotic things are logically possible at all in a relativity theoretic frame of mind. The above considerations motivated the present author to elaborate AMN [18, §2.7] in detail and to illustrate it with a large number of pictures (and intuitive explanations). For lack of space, here we do not recall more from AMN [18, §2.7] or AMN [19, §2.7].[172]

We will return to *FTL* in two dimensions in connection with Einstein's Special Principle of Relativity in §2.8.3 (p.84) after discussing our symmetry principles corresponding to Einstein's SPR.

[167] e.g. **Loc**(**Newbasax**(2)) in Chapter 3 herein

[168] This investigation in AMN [19] answers questions of L. Pólos at ILLC of University of Amsterdam. It also answers questions of participants of the research course at University of Amsterdam (1998) for which the first version of AMN [18] served as course material.

[169] Warning: by saying "is consistent" we do not mean "is implied by" and especially we do not mean to claim "is true in the real world" (though we are not excluding the latter, either).

[170] Cf. Figure 134 on p.365 for an illustration of Gödel's model for Einstein's theory, with closed time-like curves.

[171] or even directly

[172] For ideas related to the ones mentioned in the present sub-section we refer to Andai [5], and to Andai's writings connected to tachyon theory. At the same time we would like to express our special thanks to Attila Andai for helpful and thorough discussions of the present sub-section §2.7.

2.8 Some symmetry axioms and the twin paradox

In order to discuss the twin paradox, we will need some kind of symmetry axioms.[173] In this section we study the possibility of adding certain symmetry axioms to **Basax**. An example of a symmetry axiom is **Ax(symm)** to be introduced soon. We consider **Ax(symm)** as a possible formalization of (an instance or a fragment of) *Einstein's Special Principle of Relativity (SPR)*, cf. Friedman [91, p.149] principle (R) therein. Roughly speaking, SPR states that from a certain point of view all inertial observers are equivalent, while our symmetry axioms will say that "the way two observers see each other 'is the same"'. In §2.8.3 we will discuss the relationship between SPR and our symmetry axiom **Ax(symm)**.

Axiom **Ax(symm)** below is an "optional" postulate; sometimes we add it to **Basax** (or other theories of special relativity introduced later in this study) and sometimes we do not. Its usage is somewhat analogous with the Axiom of Choice (AC) in set theory, where people are interested in set theory *both* without AC and with AC. Moreover, **Ax(symm)** is of a *different nature* than the other axioms introduced up to this point. It expresses a sort of methodological (or aesthetics-motivated) principle: by making all observers similar (in a certain sense) we commit ourselves to describing the world as simply as possible. In this respect **Ax(symm)** will serve as a kind of "Occam's razor" in our analysis. To distinguish aesthetics-motivated axioms like our symmetry principles (e.g. **Ax(symm)**) from experiments-motivated ones (e.g. **AxE**), statements like our **Ax(symm)** are often called *principles of parsimony* (i.e. principles of economy of explanation in conformity with Occam's razor), cf. e.g. Friedman [91, p.29 line 23]. So, what we call in the present work symmetry axioms[174] all belong to the kind of axioms called principles of parsimony. For more on the special nature of **Ax(symm)** in connection with Occam's razor etc. we refer to §2.8.4 on page 90 herein and to item 4.2.18 (p.464) of AMN [18]. Principles of parsimony (their historical-methodological background, their role herein) are further discussed in §4.1.

We provide deeper discussions of symmetry principles in AMN [18, §§ 3.9, 4.2, 4.7].

In the present section we include a relatively brief discussion of **Ax(symm)** and its effects on theorems (or phenomena) already studied in the preceding parts. E.g. we will discuss how **Ax(symm)** influences paradigmatic effects (I)–(III) discussed in §2.5. We will see that these effects (for example the effect of clocks slowing down) admit a simpler and stronger formulation in **Basax** + **Ax(symm)** than in pure **Basax**, and all three paradigmatic effects are necessary if we assume **Ax(symm)**. After this, and motivated by these theorems, we introduce some other axioms and show that they are equivalent to **Ax(symm)**. After this we investigate in what sense **Ax(symm)** can be considered as a special case of Einstein's SPR. We then briefly investigate what **Ax(symm)** says about the physical world. After this we show that, (in the presence of **Basax** + **Ax**($\sqrt{\ }$)), **Ax(symm)** implies the twin paradox, more precisely, an approximated version of the twin paradox. We then investigate the twin paradox a little. We conclude this section with introducing one of our central axiom systems, **Specrel**.

First, we postulate a natural symmetry principle **Ax(symm$_0$)**, and then an auxiliary axiom **Ax(eqtime)**. **Ax(symm)** will be defined to be
Ax(symm$_0$) + **Ax(eqtime)**.

[173] This is so because we will approximate the accelerated twin by several inertial observers, and thus we need a kind of "similar behavior" of these inertial observers.

[174] Cf. besides the present section AMN [18, §§ 3.9, 4.2, 4.7].

Ax(symm₀) $(\forall m, k \in Obs)(\exists m', k' \in Obs)$
$$\Big(tr_m(m') = tr_k(k') = \bar{t} \quad \wedge \quad \mathsf{f}_{mk} = \mathsf{f}_{k'm'}\Big).$$

Let us see what $\mathbf{Ax(symm_0)}$ says intuitively, and why we claim that $\mathbf{Ax(symm_0)}$ is a natural symmetry postulate about "how the world behaves". Assume m, k are two observers. We would like to state that observers m and k are "equivalent" in some sense. A natural thing to say in this direction would be saying that "as I see you so do you see me". That is

$(\star)$ as m sees k so does k see m.

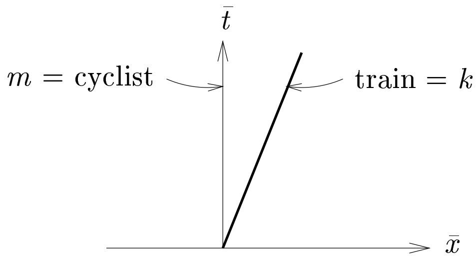
But formally this would mean saying that $\mathsf{f}_{mk} = \mathsf{f}_{km}$ which is a too strong statement, e.g. because k may be "looking in the wrong direction". If the bicyclist sees the train moving forwards, the train inhabitants may see the bicycle moving backwards. Cf. the next sequence of pictures. However, this can be easily mended; instead of $(\star)$ we state the following more subtle version $(\star\star)$.

$(\star\star)$ As m sees k so does some sister k' of k see some brother m' of m.

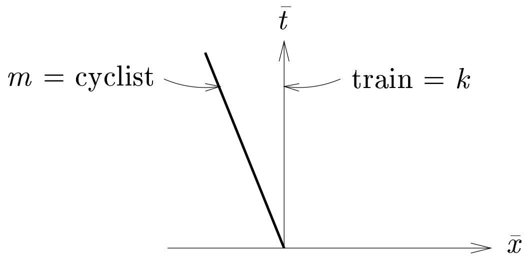
Here saying that k' is a sister of k means that $tr_k(k') = \bar{t}$, i.e. they have the same life-line.[175] Indeed it is exactly the formalized version of $(\star\star)$ what is stated in axiom $\mathbf{Ax(symm_0)}$.

Perhaps a more natural form of $\mathbf{Ax(symm_0)}$ would state the existence of brothers m' and k' such that m' and k' see each other exactly the same way, i.e. $\mathsf{f}_{m'k'} = \mathsf{f}_{k'm'}$. If there are no FTL observers (e.g. if $n > 2$), then this is an equivalent form of $\mathbf{Ax(symm_0)}$, see §3.9 of AMN [18]. However, this more natural form excludes FTL observers (see Theorem 2.7.5 in AMN [18] or Thm.2.7.1 herein), this is why se stated $\mathbf{Ax(symm_0)}$ in its present form. Cf. Theorem 2.8.2.

As an illustration for why k' and m' are needed in $\mathbf{Ax(symm_0)}$ we include the following sequence of pictures. (Throughout the discussion of these pictures we assume **Basax**(2).)



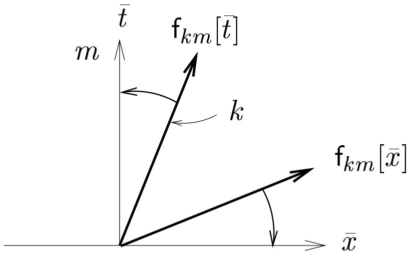
world-view of m = cyclist



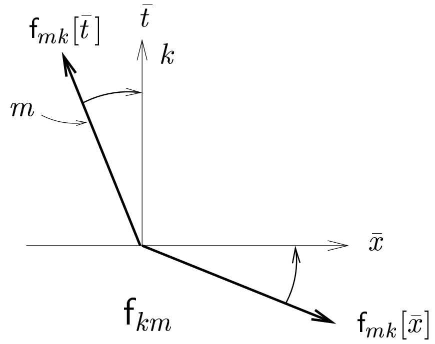
world-view of k = train

The above represents one possible configuration of the cyclist and the train: The cyclist sees the train moving forwards in the positive $\bar{x}$-direction, while the train people see the cyclist moving backwards in the negative $\bar{x}$-direction. For this configuration the world-view transformations are represented in the following picture.

[175] By quantifying over observers having the same life-line (like our quantifiers $\exists k'$, $\exists m'$) we sort of abstracted from "the directions in which our observers are looking" and this is exactly what we needed.

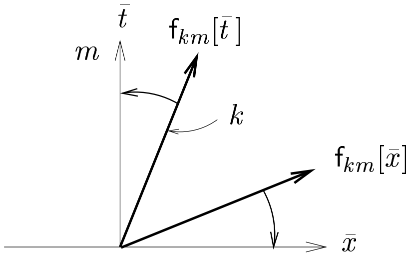


f_{mk}

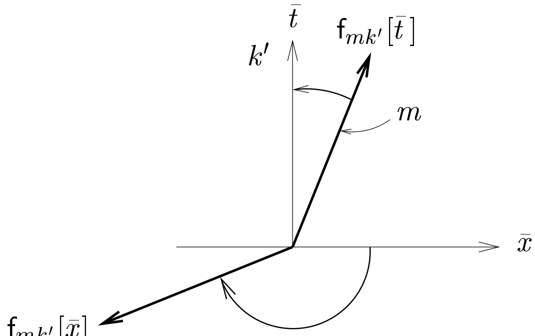


f_{km}

The above are the world-view transformations corresponding to the configuration represented in the previous picture (the train and the cyclist look in the same direction). Obviously $\mathsf{f}_{mk} \neq \mathsf{f}_{km}$. Let us try to mend this by turning the train around such that the train people will be looking backwards, which in our formalism means that we choose a sister k' of k as illustrated in the picture below.

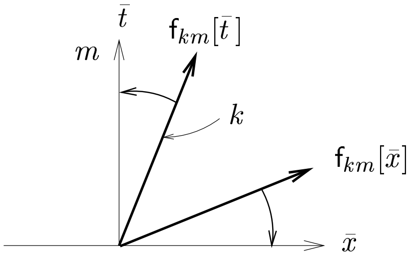


f_{mk}

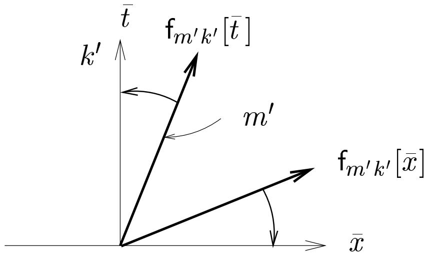


$\mathsf{f}_{k'm}$

Well, $\mathsf{f}_{mk} = \mathsf{f}_{k'm}$ is still not satisfied (but we made a step forwards, look at $\mathsf{f}_{km}[\bar{t}]$ and $\mathsf{f}_{mk'}[\bar{t}]$). Let us turn the cyclist around, too. This means that we take a brother m' of m as illustrated in the picture below.



f_{mk}



$\mathsf{f}_{k'm'}$

This picture shows that now we have a chance for $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$ being true since the lines which are mapped to the axes $\bar{t}$ and $\bar{x}$ are mapped to the right places. (In connection with this we note that $v_m(k) = v_{k'}(m')$ follows from Thm.2.8.5 on p.78 which says that under some mild assumptions, $v_m(k) = v_k(m)$. This implies that in our present case $tr_m(k) = tr_{k'}(m')$.) But in addition to this, we need that f_{mk} and $\mathsf{f}_{k'm'}$ agree on these lines pointwise, and not just that they take these lines to the same sets. The fact that it is possible to <u>*arrange pointwise agreement*</u>, too (i.e. that $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$) at least in <u>*some*</u> model of **Basax** will be seen in Theorems 2.8.1-2.8.2 below (and in more detail in §3.8.2 of AMN [18]).

Next, we introduce an auxiliary axiom **Ax(eqtime)**. We call **Ax(eqtime)** _axiom of "equi-time"_ because it says that time passes with the same rate for "observer brothers" m and m'.

Ax(eqtime) $(\forall m, m' \in Obs)$

$$\Big(tr_m(m') = \bar{t} \quad \Rightarrow \quad (\forall p, q \in \bar{t})\ |p - q| = |\mathsf{f}_{mm'}(p) - \mathsf{f}_{mm'}(q)|\Big).$$

Concerning **Ax(eqtime)** we note that this is a very natural and convincing axiom, it only says that if two observers _do not move_ relative to each other (moreover they are at the same place) then their clocks have the same rate. In other words this means that our paradigmatic effect (I) [176] does _not_ show up in the absence of motion. (This is a natural assumption which has always been assumed beginning with ancient Greeks, then by Galileo and Newton and of course by Einstein.)

Let us turn to defining **Ax(symm)**.

$$\mathbf{Ax(symm)} :\stackrel{\text{def}}{=} \mathbf{Ax(symm_0)} + \mathbf{Ax(eqtime)}\,.$$

Let us see first if studying **Basax+Ax(symm)** makes sense at all. We already stated on p.44 that **Basax** is consistent, cf. also §3.5 ("Simple models for **Basax**") of AMN [18].

THEOREM 2.8.1 **Basax**(n) + **Ax(symm)** *is consistent, for all* $n > 1$. ∎

The next theorem says that, despite of Thm.2.7.1 (p.70) which says that perfect symmetry is ruled out by FTL observers, **Ax(symm)** does *not* exclude the existence of FTL observers. (Thm.2.8.25 will state a stronger result in this direction.) Then the next Thm.2.8.3 describes a consequence of **Ax(symm)** in situations when FTL observers are present.

THEOREM 2.8.2

(i) (**Basax**(2) + **Ax(symm)** + **"∃ FTL observers"**) *is consistent i.e.*

(ii) *there is* $\mathfrak{M} \in \mathsf{Mod}$(**Basax**(2) + **Ax(symm)**) *such that in* $\mathfrak{M}$ *there are FTL observers.*

∎

THEOREM 2.8.3 *Assume* **Basax** + **Ax(symm)** + *"∃FTL observers". Then there is* $m \in Obs$ *such that*

($\star$) *m has a brother m' such that m''s clocks run backwards as seen by m.*[177]

Moreover, let the above m be fixed. Then for any observer k which is not FTL w.r.t. m, statement ($\star$) holds, i.e. k has a brother k' etc. Intuitively this means that at least "half of" the observers have a "counter-clock-wise brother".

On the proof: The idea of the proof can be reconstructed from Figure 41 (p.71). ∎

The next theorem states that in models of **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$), no field-automorphisms are involved in the world-view transformations.

[176] Moving clocks slow down.

[177] I.e. $tr_m(m') = \bar{t}$ and $\mathsf{f}_{mm'}(1_t)_t < \mathsf{f}_{mm'}(\bar{0})_t$.

THEOREM 2.8.4 *Assume* $\mathfrak{M} \models (\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ }))$. *Let* $m, k \in Obs$. *Assume* $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. *Then* f_{mk} *is a bijective* *linear* *transformation of the vector space* ${}^n\mathbf{F}$ *preserving the set of photon-lines.* ∎

It is interesting to compare the above theorem about f_{mk}'s with Thm.2.3.12(ii)' on p.35. Namely, in the above theorem we did not need mentioning field automorphisms while we did need them in Thm.2.3.12. There is a similar contrast between the above theorem and Thm.3.1.4 on p.162 of AMN [18]. In connection with the above theorem we note that, under assuming $\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ })$, the world-view transformations f_{mk} are so-called Poincaré transformations; and those world-view transformations which preserve $\bar{0}$ are Lorentz transformations, cf. Thm.2.9.6 on p.104.

So far we investigated a symmetry property of the the kind "as I see you so do you see me". The following theorem shows that a simple property of this kind also follows from **Ax(symm)**. Compare Theorem 2.7.3 on p.111 of AMN [18] which says that there are $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(2))$ and $m, k \in Obs^{\mathfrak{M}}$ such that $v_m(k) < 1$ while $v_k(m) > 1$.

THEOREM 2.8.5 $\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ }) \models v_m(k) = v_k(m)$. ∎

Remark 2.8.6 (Ax(symm) and Minkowski-circles) Consider the possible models $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ of $\mathbf{Basax}(2)$ represented in Figure 29 (p.51). As we said, of these only $\mathfrak{M}_3$ is a model of **Ax(symm)**. In particular **Ax(symm)** fails both in $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Therefore **Basax** + **Ax(symm)** has radically fewer kinds of models than **Basax** does.

Moreover, **Basax** + **Ax(symm)** implies that the Minkowski-sphere can look like only as in the case of $\mathfrak{M}_3$ in Figure 29. This follows from Thm.2.9.6(ii) (p.104). We note that $\mathbf{Basax} + \mathbf{Ax(symm_0)}$ is not sufficient for this.[178]

Further, under some natural assumptions[179], the following three statements are equivalent for models $\mathfrak{M}$ of **Basax**:

(i) **Ax(symm)** holds in $\mathfrak{M}$.

(ii) All observers have the same Minkowski-spheres, i.e.
$(\forall m, k \in Obs)\mathsf{MS}(\mathfrak{M}, m) = \mathsf{MS}(\mathfrak{M}, k)$.

(iii) The Minkowski-sphere of any observer is like $\mathfrak{M}_3$ in Figure 29, i.e.

$$\mathsf{MS}(\mathfrak{M}, m) = \{p \in {}^nF : g^2_\mu(p, \bar{0}) = 1\},$$

where $g^2_\mu(p, \bar{0})$ will be introduced on p.101.

A more detailed formulation of this statement is in AMN [19].

◁

[178] This was noted by Gergely Székely and Ramon Horváth.
[179] e.g. $\mathbf{Ax}(\|) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ })$ to be introduced soon, or that $\mathfrak{F}$ has no nontrivial automorphism

2.8.1 Ax(symm) and the paradigmatic effects

Let us turn to seeing how **Ax(symm)** simplifies the "picture" of special relativity, e.g. what it "says" about our paradigmatic effects (I)–(III) (p.54).

THEOREM 2.8.7 (Clocks slow down.)
Assume **Basax** + **Ax(symm)** + **Ax($\sqrt{\ }$)**. *Let $m, k \in Obs$, with $0 < v_m(k) \le 1$. Then:*

(i) m <u>thinks</u> that k's <u>clocks run slow</u>, i.e.

$$|\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > 1\,; \ \textit{moreover}$$

(ii) $(\forall 0 \neq \lambda \in F)\ |\mathsf{f}_{km}(\lambda \cdot 1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > |\lambda|$.

∎

In connection with the above theorem cf. Theorem 2.5.2. The novelty in Theorem 2.8.7 is that it says that all observers' clocks slow down in the presence of **Ax(symm)**, while without **Ax(symm)** we only know that some clocks slow down. This also means that both m thinks that k's clocks slow down and k thinks that m's clocks slow down. This is counterintuitive to the thinking we got used to in our Newtonian world where if k thinks that m's clocks run slow, then m will think that k's clocks run fast. That both can think that the other's clocks run slow is possible because they do not perceive the same events as simultaneous, i.e. because of paradigmatic effect (III). In connection with this see Figure 49 on p.104.

Analogous statement can be made about paradigmatic effect (II), i.e. about shrinking of meter-rods, cf. the following theorem. In connection with the next theorem we note the following: If we assume **Basax**, then for every $m, k \in Obs$, such that $v_m(k) \neq \infty$, $tr_m(k)$ can be considered as a function $tr_m(k) : F \longrightarrow {}^{n-1}F$, therefore $tr_m(k)(0)$ is well defined (cf. Fact 2.2.4).

THEOREM 2.8.8 (Meter-rods shrink.)
Assume **Basax** + **Ax(symm)** + **Ax($\sqrt{\ }$)**. *Let $m, k \in Obs$, with $0 < v_m(k) \le 1$. Then (i) and (ii) below hold.*

(i) *Meter-rods of k parallel with the direction of movement of k shrink when observed by m. I.e. m will think that k's meter-rods are shorter than k thinks, formally:*

For simplicity assume that $\bar{0} \in tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. Let $k_1 \in Obs$ with $\bar{t} \neq tr_k(k_1) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ such that $tr_k(k_1)$ is parallel with $\bar{t}$.[180] *Let $p := tr_m(k_1)(0)$ and $q := tr_k(k_1)(0)$.*

Then $|p| < |q|$. Cf. Figure 38 on p.62.

(ii) *Those meter rods of k which are not orthogonal to the direction of movement shrink when observed by m. Formally:*

For simplicity assume again that $\bar{0} \in tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. Let $k_1 \in Obs$ such that $tr_k(k_1)$ is parallel with $\bar{t}$ and $tr_k(k_1)(0)_0 \neq 0$.

Then $|p| < |q|$.

∎

[180] Intuitively k and k_1 together represent a meter-rod of k.

In connection with the above theorem cf. Items 2.5.9–2.5.12 (pp.62–63).

THEOREM 2.8.9 (Clocks slow down exactly the same way.)
Assume **Basax** + **Ax(symm)**. *Assume* $m, k \in Obs$. *Then* m *sees* k*'s clocks slowing down to exactly the same degree as* k *sees* m*'s clocks doing the same; formally:*

(i) $|\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| = |\mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t|$, *moreover:*

(ii) $(\forall p \in \bar{t})\ |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(\bar{0})_t| = |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(\bar{0})_t|$.

∎

Remark 2.8.10 An analogous statement can be made about the effect of shrinking meter-rods as follows.
Assume **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$). Then (i) and (ii) below hold.

(i) Assume further $m, k \in Obs$ are in standard configuration. Let us concentrate on meter-rods *parallel with the direction of movement*. Then m will see k's meter-rods shrink exactly with the same ratio as k sees m's meter-rods shrink.

(ii) Assume k moves in direction $\bar{x}$ when observed by m (i.e. $tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$). Let us concentrate on meter-rods which are parallel with direction $\bar{x}$ when observed by m. I.e. even if the meter-rod is k's one we check whether m sees it parallel with the plane $\mathsf{Plane}(\bar{t}, \bar{x})$ determined by axes $\bar{t}$ and $\bar{x}$.

Then m will see k's meter-rods shrinking with the same ratio as k sees m's meter-rods. ◁

Roughly, the following theorem implies that meter-rods orthogonal to the direction of movement do not shrink or grow, assuming **Basax** + **Ax(symm)**, cf. Corollary 2.8.12.

THEOREM 2.8.11 *Assume* **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$). *Let* $m, k \in Obs$. *Let* e, e_1 *be two events which are simultaneous for both* m *and* k. *Then the spatial distance between* e *and* e_1 *is the same for* m *as for* k*; formally:*

$$(\forall p, q \in {}^nF)\ [\,(p_t = q_t\ \wedge\ \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \quad\Rightarrow\quad |p-q| = |\mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q)|\,]\,.$$

∎

The following is a corollary of Thm.2.8.11 above and Thm.2.5.6 (p.59) which says that (under assuming **Basax**) if two clocks are separated <u>*only*</u> in a spatial direction which is orthogonal to the direction of movement, then they do not get out of synchronism.

COROLLARY 2.8.12 (Meter-rods orthogonal to movement do not shrink.)
Assume **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$). Then meter-rods orthogonal to the direction of movement do <u>*not*</u> get shorter. A detailed formal statement of this is in AMN [18, Cor.2.8.12 (p.133)]. ∎

2.8.2 Equivalent forms of Ax(symm)

In this part we show that certain forms of the paradigmatic effects are actually equivalent to **Ax(symm)** (in **Basax** under mild conditions). Different equivalent forms of **Ax(symm)** are given in §3.9 of AMN [18].

Thm.2.8.9 motivates the axiom $\mathbf{Ax(syt_0)}$ below. We note that the "name" $\mathbf{Ax(syt_0)}$ intends to refer to "symmetry of time". Intuitively, $\mathbf{Ax(syt_0)}$ says that

"as I see your clocks slowing down (because of your speed relative to me) so do you see my clocks (because of my speed relative to you) slowing down".

In the formulation of $\mathbf{Ax(syt_0)}$ below the assumption $tr_m(k) \neq \emptyset$ is superfluous at the present point, because $\mathbf{Basax} \models tr_m(k) \neq \emptyset$. However in later sections this assumption will become useful.[181]

$\mathbf{Ax(syt_0)}$ $$(\forall m, k \in Obs)\Big(tr_m(k) \neq \emptyset \quad \Rightarrow \\ (\forall p \in \bar{t}\,)\ |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(\bar{0})_t| = |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(\bar{0})_t|\Big).$$

In terms of the just defined $\mathbf{Ax(syt_0)}$, Thm.2.8.9 says that

$$\mathbf{Basax} \models \mathbf{Ax(symm)} \to \mathbf{Ax(syt_0)}.$$

In Thm.2.8.13 below we will see that, under mild assumptions, the implication holds in the other direction too, i.e. $\mathbf{Ax(syt_0)}$ is an equivalent form of **Ax(symm)** (in the presence of **Basax**). To formulate Thm.2.8.13 we introduce auxiliary axioms $\mathbf{Ax(}Triv\mathbf{)}$ and $\mathbf{Ax(}Triv_t\mathbf{)}$. First we introduce the notion of an isometry and the set *Triv* of trivial transformations.

Triv denotes the set of all mappings of ${}^nF$ into itself which preserve Euclidean distance, take $\bar{t}$ to a line parallel with it, and so that the order of points does not change on $\bar{t}$. Formally: Let $\mathfrak{F}$ be an ordered field. Then $f : {}^nF \longrightarrow {}^nF$ is said to be an <u>*isometry*</u> iff it preserves the square of Euclidean distances, i.e. $(\forall p, q \in {}^nF)$

$$(p_0 - q_0)^2 + (p_1 - q_1)^2 + \ldots + (p_{n-1} - q_{n-1})^2 = \\ (f(p)_0 - f(q)_0)^2 + (f(p)_1 - f(q)_1)^2 + \ldots + (f(p)_{n-1} - f(q)_{n-1})^2,$$

cf. also Def.3.9.3 on p.349 of AMN [18]. Let ${}^{({}^nF)}\,{}^nF$ denote the set of all functions mapping ${}^nF$ into itself. Then

$$Triv \stackrel{\text{def}}{=} Triv(n, \mathfrak{F}) \stackrel{\text{def}}{=} \left\{ f \in {}^{({}^nF)}\,{}^nF \ : \ f \text{ is an isometry, } f[\bar{t}] \parallel \bar{t},\ f(1_t)_t - f(\bar{0})_t > 0 \right\}.$$

As we explain in §3.5 of AMN [18] in more detail, the transformations in *Triv* involve *no "relativistic effects"*, one could say that they are very non-relativistic or, so to speak, trivial. To illustrate this, assume $f(\bar{0}) = \bar{0}$. Then $f \in Triv$ if and only if f is linear, it is identity on $\bar{t}$ and f maps the space-part S to itself (i.e. $f[S] = S$) so that it preserves Euclidean distance on S.

[181] We would like to remind the reader that we mentioned that when generalizing our axioms towards general relativity, **Ax6** and **Ax3** will be weakened and therefore $tr_m(k) = \emptyset$ will be possible for some choices of $m, k \in Obs$.

Ax(*Triv*) $(\forall m \in Obs)(\forall f \in Triv)(\exists k \in Obs)\ \mathsf{f}_{mk} = f$.

> **Ax(*Triv*)** says that every observer can "re-coordinatize" his world-view by any trivial transformation. As is explained in §3.9 of AMN [18], **Ax(*Triv*)** is first-order, because each isometry is an affine transformation, and so quantifying over elements of *Triv* can be replaced by quantifying over elements of $\mathfrak{F}$.
>
> **Ax(*Triv$_t$*)** below is a weaker form of **Ax(*Triv*)**.

Ax(*Triv$_t$*) $(\forall m \in Obs)(\forall f \in Triv)\ \Big(f[\bar{t}] = \bar{t} \quad \Rightarrow \quad (\exists k \in Obs)\ \mathsf{f}_{mk} = f\Big)$.

THEOREM 2.8.13 *Assume $n > 2$. Then*

$$\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t) \ \models\ \mathbf{Ax(symm)} \leftrightarrow \mathbf{Ax(syt_0)}.$$

∎

We consider **Ax(*Triv*)** and **Ax(*Triv$_t$*)** as some of our auxiliary axioms.[182] A similar auxiliary axiom is **Ax(||)** to be introduced below.

Ax(||) $(\forall m, k \in Obs)\Big(tr_m(k) \parallel \bar{t} \quad \Rightarrow \quad (\mathsf{f}_{mk} \text{ is an isometry})\Big)$.

> Intuitively, assuming **Ax4**, axiom **Ax(||)** says that if you do not move relative to me then we will agree on which events are simultaneous, which occurred at the same place and we agree on both spatial distances and temporal distances between events. Hence **Ax(||)+Ax4** implies that none of the paradigmatic effects shows up in the absence of motion. **Ax(||)** is a stronger version of **Ax(eqtime)**. In passing we note that later (in Chapter 4 on p.145) we will introduce an axiom called **Ax(eqm)** which (under mild assumptions) will be a stronger version of **Ax(||)**.

The proposition below says that, assuming **Basax**+**Ax(*Triv*)**, the auxiliary axioms **Ax(||)** and **Ax(eqtime)** are equivalent.

PROPOSITION 2.8.14 $\mathbf{Basax} + \mathbf{Ax}(Triv) \models \mathbf{Ax}(\|) \leftrightarrow \mathbf{Ax(eqtime)}$. ∎

The following proposition says that, assuming **Basax**+**Ax**($\sqrt{\ }$), both **Ax(syt$_0$)** and **Ax(symm)** imply **Ax(||)**.

[182] The ***axioms*** we call ***auxiliary*** are of a status that we assume them without any hesitation whenever we need them. I.e. we consider them as true in the "real world" and we omit them from some of our theories only to make these theories look prettier. To be on the safe side, we note that **Ax(*Triv*)** and **Ax(||)** will "not survive" the transition from special to general relativity. They both will need some refining already in our chapter on accelerated observers. The following form **Ax(*Triv$_{t0}$*)** of **Ax(*Triv$_t$*)** will remain "true": $(\forall f \in Triv)[f(\bar{0}) = \bar{0} \Rightarrow (\exists k \in Obs)\mathsf{f}_{mk} = f]$. In the case when we will allow only uniformly accelerated observers, also **Ax(*Triv$_t$*)** will remain "true". Here "true" means "usable" or consistent with our intentions.

PROPOSITION 2.8.15

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)} \models \mathbf{Ax}(\|)$.

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(symm)} \models \mathbf{Ax}(\|)$.

∎

Since $\mathbf{Ax}(\|)$ is a stronger form of **Ax(eqtime)**, the above proposition implies that

$$\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)} \models \mathbf{Ax(eqtime)}.$$

Thm.2.8.11 motivates the following potential axiom, which we call the *axiom of "equi-space"*.

Ax(eqspace) $(\forall m, k \in Obs)(\forall p, q \in {}^nF)$
$$\Big((p_t = q_t \ \wedge \ \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \quad \Rightarrow \quad |p - q| = |\mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q)| \Big).$$

Intuitively, **Ax(eqspace)** says that if two events are simultaneous both for m and k, then the spatial distance between those two events is the same for m as for k. Theorem 2.8.16 below explains why we consider **Ax(eqspace)** as one of our symmetry axioms.

In terms of the just defined **Ax(eqspace)**, Thm.2.8.11 says that

$$(\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ })) \models \mathbf{Ax(eqspace)}\,.$$

In the next theorem we will see that (under assuming $n > 2$ and $\mathbf{Ax}(\mathit{Triv}_t)$) the implication holds in the other direction, too.

THEOREM 2.8.16 *Assume $n > 2$. Then*

$$(\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)) \models \mathbf{Ax(symm)} \leftrightarrow \mathbf{Ax(eqspace)}\,.$$

∎

Next we introduce another natural symmetry axiom. Intuitively, **Ax(speedtime)** below says that the rate with which moving clocks slow down depends only on the relative velocity $\vec{v}_m(k)$ with which one observer sees the other moving. Roughly, the idea is the following. The relativistic effects are caused by relative motion (of k relative to m). Motion is completely determined[183] by velocity $\vec{v}_m(k)$ (of k relative to m). Therefore one concludes that relativistic effects (involving f_{mk}) should be determined by $\vec{v}_m(k)$. (At least if we disregard acceleration). For technical reasons, the axiom is formulated in terms of speeds instead of velocities. Interestingly, we will see that this axiom turns out to be one of the symmetry axioms equivalent to **Ax(symm)** and $\mathbf{Ax(syt_0)}$, cf. Thm.2.8.17.

Ax(speedtime) $(\forall m, k, m', k' \in Obs)\Big(v_m(k) = v_{m'}(k') \quad \Rightarrow$
$$(\forall p \in \bar{t})\ |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(\bar{0})_t| = |\mathsf{f}_{m'k'}(p)_t - \mathsf{f}_{m'k'}(\bar{0})_t|\Big).$$

[183] If we disregard acceleration.

Ax(speedtime) also turns out to be equivalent to an instance of (or fragment of) Einstein's SPR, under some assumptions.[184]

The theorem below says that (under mild assumptions) the symmetry axioms introduced in this section are equivalent to each other. For similar equivalence theorems we refer the reader to §§ 3.9, 4.7 of AMN [18].

THEOREM 2.8.17 *Assume $n > 2$. Then (i) and (ii) below hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t) \models$

$$\mathbf{Ax(symm)} \leftrightarrow \mathbf{Ax(speedtime)} \leftrightarrow \mathbf{Ax(syt_0)} \leftrightarrow \mathbf{Ax(eqspace)};$$

where the "transitive notation" $\psi_1 \leftrightarrow \psi_2 \leftrightarrow \psi_3$ intends to abbreviate $(\psi_1 \leftrightarrow \psi_2)$ & $(\psi_2 \leftrightarrow \psi_3)$. Similarly for the case when we have four formulas say $\psi_1, \ldots, \psi_4$.

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax(speedtime)} \leftrightarrow \mathbf{Ax(syt_0)} \leftrightarrow \mathbf{Ax(eqspace)}$.

∎

2.8.3 Model theoretic characterization of Einstein's SPR; connections with our symmetry principles

Roughly speaking, Einstein's SPR states that — from a certain point of view — all inertial observers are equivalent. Then we only have to specify the relevant aspect from which inertial observers are all equivalent. The usual stipulation is that no <u>law of nature</u> distinguishes any observer from the others.[185] We will allow concrete numbers (i.e. elements of sort quantities Q, i.e. F) occur in a law of nature, but we will not allow individual names to occur in a law of nature. More formally: Let $\mathfrak{M}$ be a frame model, and let $\overline{\mathfrak{M}}$ be the model obtained from $\mathfrak{M}$ by expanding it with the elements of $F^{\mathfrak{M}}$ as new distinguished constants denoting themselves. We want to formalize the statement:

(†) The same laws of nature are true for all inertial observers.

To analyze the meaning of (†), we will speak about *<u>potential</u>* laws of nature. We use the expression *<u>law-like statement</u>* as a synonym for potential law of nature.[186] Now, (†) means

[184] In some respects, **Ax(speedtime)** would have a clearer intuitive meaning if we wrote $\vec{v}_m(k) = \vec{v}_{m'}(k')$ in place of $v_m(k) = v_{m'}(k')$ in its assumption part.

[185] It is interesting that we call this principle Einstein's SPR while it is due to Galileo Galilei (1632), cf. Taylor-Wheeler [257, Chap.III, first 5 pages] and/or Galileo [93]. The principle of relativity is not the novel part of the theory of relativity. The principle was formulated in 1632, what the (special) theory of relativity does is making Galileo's principle *consistent* with the facts of physics discovered much later than Galileo's time, e.g. with the laws of electrodynamics or with the outcome of the Michelson-Morley experiment, cf. Einstein [80, §§7,5]. (According to [80], general relativity generalizes (a suitably refined form of) this principle even further, to not necessarily inertial observers. To this end, among other things, Einstein had to refine the answer to the question of what statements count as laws of nature. The new principle is called General Principle of Relativity (GPR), cf. e.g. Einstein[p.66 and §28]Einst and Friedman[e.g. pp.6,379]Fr83.)

[186] A large part of the literature uses the expression "law-like statement", cf. e.g. Huoranszki [136, Chap.II.'Laws of nature' (e.g. p.66, line 8)], Cambridge Dictionary of Philosophy [35], Hempel-Oppenheim [119, §6]. We use the expressions "law-like statement" and "law-like formulas" as synonyms. (Cf. also "law-like generalization" in Cambridge Dictionary of Philosophy.) For more on law-like formulas i.e. potential laws of nature we refer to §4.1, p.131 and to AMN [18, §6.6.8, §6.1 and pp.777-778].

If $m, k \in Obs \cap Ib$, then the same law-like statements are true for m and k.

In the present work we adopt the convention that law-like statements are *formulas* in the language of $\overline{\mathfrak{M}}$.[187] So, a law-like statement is a formula φ. But, if we want to be able to formulate the claim that "φ is *true for observer m iff it is true for observer k*", then φ must have a free variable, say m, of sort B (which can be evaluated to various observers). We indicate this by writing $\varphi(m)$ for φ.[188] Then the claim "law-like statement $\varphi(m)$ is true for observer k" is formalizable as "$\overline{\mathfrak{M}} \models \varphi[k]$".[189]

Now, we say that *Einstein's SPR holds in* $\mathfrak{M}$ if

($\star$) Whenever a formula $\varphi(m')$ in the language of $\overline{\mathfrak{M}}$ qualifies as a potential law of nature, and $m, k \in Obs^{\mathfrak{M}}$, then $(\overline{\mathfrak{M}} \models \varphi[m] \iff \overline{\mathfrak{M}} \models \varphi[k])$.

The above formulation of ($\star$) is based on our assumption that a potential law of nature for $\mathfrak{M}$, in other words a *law-like statement* about $\mathfrak{M}$, is a formula φ in the language of $\overline{\mathfrak{M}}$ and moreover, it has exactly one free variable of sort B. Such a law of nature φ is true in the reference frame of m, or equivalently in the world-view of m, if $\overline{\mathfrak{M}} \models \varphi[m]$. Which formulae are then potential laws of nature? This notion has not been formalized yet to a satisfactory way in the literature but there is an agreement that no individual names of observers can occur in a law of nature.[190] Thus, if for all law-like formulas $\varphi(m')$ in the language of $\overline{\mathfrak{M}}$ with no free variables other than m' we have

$$\overline{\mathfrak{M}} \models \varphi[m] \quad \text{iff} \quad \overline{\mathfrak{M}} \models \varphi[k],$$

then we can conclude that the same potential laws of nature are true in the reference frames of m and k. But the latter means that the laws of nature are the same in the reference frames of m and k. If we postulate this for all $m, k \in Obs^{\mathfrak{M}} \cap Ib^{\mathfrak{M}}$, we obtain exactly Principle (R) on p. 149 of Friedman [91] which in turn is the usual formulation of Einstein's SPR (as is explained in Friedman [91, pp.150-153]).

We will write $\mathfrak{M} \models$ *Einstein's SPR* when ($\star$) above holds.

Let $\langle \overline{\mathfrak{M}}, m \rangle$ denote the expansion of $\overline{\mathfrak{M}}$ with distinguished constant $m \in Obs^{\mathfrak{M}}$.

The phrase "$\mathfrak{M} \models$ *Einstein's SPR*" does not have a definite meaning yet because we did not specify which formulas in the language of $\overline{\mathfrak{M}}$ count as law-like statements. But if

($\star\star$) $\mathsf{Th}(\langle \overline{\mathfrak{M}}, m \rangle) = \mathsf{Th}(\langle \overline{\mathfrak{M}}, k \rangle)$ for all $m, k \in Obs \cap Ib$,

[187] We do not claim that all formulas of $\overline{\mathfrak{M}}$ are law-like statements, we only claim the other direction that "statements" are formulas (in the language of $\overline{\mathfrak{M}}$). E.g. Hempel-Oppenheim [119] follows basically the same convention. The subject matter of law-like formulas or law-like statements, in basically the present spirit, has a quite extensive literature, cf. e.g. Goodman [104], Hempel [118], Nagel [199], besides the already quoted works, to mention only a few. Very roughly, the general schema is *"law-like formulas"* $\subseteq \mathsf{FF}$, or instead of FF one may use an expansion of the "language" FF (e.g. the language of $\overline{\mathfrak{M}}$).

[188] The symbols φ and $\varphi(m)$ denote the same mathematical object, namely, the formula φ; the only role of the "(m)" part is to emphasize that m is a free variable of φ. Cf. the notation $\mathfrak{M} \models \psi[\overline{a}]$ on p.231 (in §4.3) herein or Chang-Keisler [60] for more on this notation.

[189] If we are given $\varphi(m)$ and $k \in B^{\mathfrak{M}}$, then $\mathfrak{M} \models \varphi[k]$ means that φ is true in $\mathfrak{M}$ if the variable m is evaluated to the element k. (Roughly, $\varphi[k]$ is the "formula" obtained from $\varphi(m)$ by substituting the element k of our model $\mathfrak{M}$ into the place of the free variable m in $\varphi(m)$.)

[190] Cf. e.g. Hempel-Oppenheim [119] about the problem of characterizing law-like statements/formulas among all statements/formulas.

then we can be sure that Einstein's SPR holds in $\mathfrak{M}$, in spite of the fact that we did not specify which formulas are law-like. In fact, $(\star\star)$ corresponds to the choice that all formulas with one free variable of sort B in the language of $\overline{\mathfrak{M}}$ count as law-like.[191] Next we define a set SPR^+ of formulas in the frame language of $\mathfrak{M}$ such that SPR^+ expresses $(\star\star)$, i.e. $\mathfrak{M} \models SPR^+$ iff $(\star\star)$ holds for $\mathfrak{M}$. Thus SPR^+ will be a very strong form of Einstein's SPR.

Definition 2.8.18 The subset $SPR^+ \subseteq \mathsf{FF}$ of our frame-language FF is defined as follows. Let m, k be distinct variables of sort B.

$$\begin{aligned} SPR^+ \stackrel{\text{def}}{=} \{ & Obs(m) \wedge Obs(k) \to [\varphi(m) \leftrightarrow \varphi(k)] \quad : \\ & \varphi(m) \in \mathsf{FF} \text{ is a formula containing } \underline{\textit{no free}} \text{ variables of sort } B \text{ or } G \\ & \text{other than } m\}. \end{aligned}$$

We call SPR^+ the <u>*strong version of Einstein's SPR*</u>. $\lhd$

Perhaps more intuitively we could write

$$\begin{aligned} SPR^+ \stackrel{\text{def}}{=} \{ & \varphi(m, r_1, \ldots, r_\ell) \leftrightarrow \varphi(k, r_1, \ldots, r_\ell) \quad : \quad \varphi(m, \ldots, r_\ell) \text{ is a formula} \\ & \text{containing no free variables other than } m, r_1, \ldots, r_\ell, \text{ further } r_1, \ldots, r_\ell \\ & \textit{are of sort } F \textit{(or Quantities), while } m, k \textit{ are of "sort" } Obs\}. \end{aligned}$$

Now, $\mathfrak{M} \models SPR^+$ implies that $\mathfrak{M} \models$ *Einstein's SPR*, by what we wrote above. Intuitively,

$$SPR^+ \models \textit{Einstein's SPR.}$$

We will return to discussing law-like formulas and Einstein's SPR in §4.1.

Notation 2.8.19 $Aut(\mathfrak{M})$ denotes the automorphism group of the model or structure $\mathfrak{M}$.[192] Cf. also p. 146, Def. 4.2.3(II), Convention 4.2.4.

The following is a model theoretic or <u>*algebraic characterization of SPR^+*</u>. The reader not familiar with the model theoretic/algebraic notions involved may safely skip the next result.

[191] This is not as "crazy" as it may seem at first sight. A formula $\varphi(m)$ in the language of $\overline{\mathfrak{M}}$ containing exactly one free variable, m, of sort B does satisfy the conditions of Cambridge Dictionary of Philosophy as well as Hempel's conditions since it does not depend on <u>*individual*</u> parameters (unlike a formula $\varphi(m, k)$), nor does it contain an individual name (because there is no constant symbol of sort B in it). The reason why we allow constants of sort F in law-like formulas $\varphi(m)$ is that unlike $k \in B$, a "number" $r \in F$ is not an individual physical entity but a result of mathematical abstraction. This is why in physical laws real numbers as "natural constants" (e.g. π or Plank distance in terms of radius of the hydrogen atom, or the speed of light c) are allowed to occur despite of the fact that individual names (like our Sun) are not. Cf. "Lawlike-generalization" in [35] or Hempel-Oppenheim [119]. All the same, to be on the safe side, we do <u>*not*</u> identify Einstein's SPR with $(\star\star)$. Similarly, we do <u>*not*</u> claim that all formulas in the language of $\overline{\mathfrak{M}}$ containing exactly one free variable of sort B are law-like. We note that the above choice of law-like formulas is *not* suitable as a basis of Einstein's GPR because it allows too many law-like formulas. (But here, we are discussing SPR only.)

[192] I.e. $Aut(\mathfrak{M})$ consists of all the isomorphisms $h : \mathfrak{M} \longrightarrow \mathfrak{M}$ of $\mathfrak{M}$ onto $\mathfrak{M}$, c.f. e.g. Def. 4.2.3(II) p. 146, or AMN [18, p. 298] for the well-known notion of an isomorphism.

THEOREM 2.8.20 *Let $\mathfrak{M}$ be a frame model. Then conditions (i), (ii) and (iii) below are equivalent.*

(i) $\mathfrak{M} \models SPR^+$.

(ii) There is an elementary extension $\mathfrak{M}^+$ of $\mathfrak{M}$ such that $(\forall m, k \in Obs^{\mathfrak{M}} \cap Ib^{\mathfrak{M}})(\exists h \in Aut(\mathfrak{M}^+))[h(m) = k$ and $h \restriction F^{\mathfrak{M}} \subseteq \mathrm{Id}]$.[193]

(iii) There is an ultrapower $\mathfrak{M}^+ \stackrel{\text{def}}{=} {}^I\mathfrak{M}/F$ (for some I and ultrafilter F) of $\mathfrak{M}$ such that if $\delta : \mathfrak{M} \rightarrowtail \mathfrak{M}^+$ is the usual diagonal embedding induced by the ultrapower construction, then $(\forall m, k \in Obs^{\mathfrak{M}^+} \cap Ib^{\mathfrak{M}^+})(\exists h \in Aut(\mathfrak{M}^+))[h(m) = k$ and $\delta \circ h \restriction F = \delta \restriction F]$.

Proof: Let $\overline{\mathfrak{M}}$ denote the expansion of $\mathfrak{M}$ with all the elements of F as distinguished constants denoting themselves. Clearly, $\mathfrak{M} \models SPR^+$ iff $\overline{\mathfrak{M}} \models SPR^+$. So we may work with $\overline{\mathfrak{M}}$ instead of $\mathfrak{M}$. Now, direction $(iii) \Rightarrow (i)$ follows by Łos' Lemma. Direction $(i) \Rightarrow (iii)$ is based on an ω-long iterated application of the Keisler-Shelah isomorphic ultrapowers theorem and is similar to analogous (Keisler-Shelah-based) proofs in Chapter 4 and in AMN [18], c.f. e.g. items 6.7.4-6.7.8 on pp. 1138-1140 of AMN [18]. The equivalence $(iii) \Leftrightarrow (ii)$ can be proved by using results in Chang & Keisler [60]. To save space, we omit the details which are available from the author. ∎

Justification of using Ax(symm) as an instance of Einstein's SPR

The following symmetry axiom (**Ax□1**) is a more literal instance of <u>*Einstein's SPR (Special Principle of Relativity)*</u> than our **Ax(symm)**.

Ax□1 $(\forall m, k, m' \in Obs)(\exists k' \in Obs)\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$.

Cf. the part beginning with p.350 of AMN [18] for more on **Ax□1** and its discussion, relationships etc. In Proposition 2.8.21 below, we will illustrate that the above considerations can shed some light on why we feel that **Ax□1** can be regarded as kind of an instance (in some sense) of Einstein's SPR. In the proof we show that **Ax□1** can be interpreted as saying that a certain law-like statement holds for m iff it holds for k.

PROPOSITION 2.8.21 *Assume $\mathfrak{M}$ is such that all the f_{mk}'s are definable (with using members of $F^{\mathfrak{M}}$ as parameters) in the first-order-logic language*[194] *of $\overline{\mathfrak{M}}$.*[195] *Then,*

$$\mathfrak{M} \models SPR^+ \;\Rightarrow\; \mathfrak{M} \models \mathbf{Ax\Box 1}.$$

Proof: Assume $\mathfrak{M} \models SPR^+$. Let $m, k, m' \in Obs \cap Ib$. Since f_{mk} is definable in $\overline{\mathfrak{M}}$, there is a formula $\psi(p,q)$ in the language of $\overline{\mathfrak{M}}$ such that

$$(\forall p \in {}^nF)\left(\mathsf{f}_{mk}(p) = q \quad \text{iff} \quad \overline{\mathfrak{M}} \models \psi(p,q)\right).$$

Then

$$\langle \overline{\mathfrak{M}}, m\rangle \models (\exists k \in Obs)(\forall p, q \in {}^nF)\left[\mathsf{f}_{mk}(p) = q \leftrightarrow \psi(p,q)\right].$$

[193] I.e. h is the identity function on $F^{\mathfrak{M}}$.

[194] This kind of definability is discussed in detail in §4.3.

[195] For definability of all the f_{mk}'s it is enough (but not necessary) to assume that all the f_{mk}'s are affine transformations.

Let $\varphi(m)$ denote the formula $(\exists k \dots q)]$ on the right hand side of $\models$. It is a law of nature (in the sense belonging to SPR^+) in the reference frame of m. Hence $\langle\overline{\mathfrak{M}}, m'\rangle \models \varphi(m')$ must also hold by SPR^+. But in $\varphi(m')$ we may replace all occurrences of the *bound* variable k by k'. This means that there is k' such that $\mathsf{f}_{m'k'} = \mathsf{f}_{mk}$. This proves $\mathfrak{M} \models$ **Ax□1**. ∎

In the form of Prop. 2.8.21 above we have seen a justification of using **Ax□1** as an instance of Einstein's SPR. This in turn, by Thm. 2.8.22 below, justifies our using $\mathbf{Ax(symm_0)}$ too as an instance of Einstein's SPR.

THEOREM 2.8.22 $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax\Box 1} \models \mathbf{Ax(symm_0)}$.

The **proof** for the case $n > 2$ is in AMN [18, Thm.3.9.31(i), p.380]. The proof for $n = 2$ is available from the author, cf. Madarász [168]. ∎

COROLLARY 2.8.23 Assume $\mathfrak{M}$ is such that all the f_{mk}'s are definable in the first-order-logic language of $\overline{\mathfrak{M}}$. Then

$$\mathfrak{M} \models SPR^+ + \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) \;\Rightarrow\; \mathfrak{M} \models \mathbf{Ax(symm_0)}. \quad ∎$$

On the role of Ax(eqtime): The reason why we use **Ax(symm)** instead of $\mathbf{Ax(symm_0)}$ as an instance of Einstein's SPR is the following. $\mathbf{Ax(symm)} = \mathbf{Ax(symm_0)} + \mathbf{Ax(eqtime)}$. Assuming **Basax**, **Ax(eqtime)** excludes <u>only</u> the so-called "ants and elephants" situations in our relativistic models $\mathfrak{M}$. The "ants and elephants" situation refers to the possibility of having $m, k \in Obs$ with $tr_m(k) = \bar{t}$ such that m thinks that k's clocks do not tick with the right rate.[196] So typically m can be a small observer like an ant and k a big observer like an elephant. While we do not want to exclude such situations "once and for all", one feels that studying these situations can be done independently of relativity. Hence, when we assume a simplifying condition like Einstein's SPR then it seems reasonable to also exclude the (more or less irrelevant) complications caused by the "ants and elephants" arrangements.[197] We found that we can make the present work shorter if we use $\mathbf{Ax(symm_0)}$ together with the other symmetry principle **Ax(eqtime)** <u>unless</u> something interesting can be gained by using $\mathbf{Ax(symm_0)}$ by itself.

$\mathbf{Ax(symm_0)}$ is not only an instance of Einstein's SPR, but it is also a <u>*symmetry principle*</u>. **Ax(eqtime)** is another, relatively mild, symmetry principle. However, **Ax(eqtime)** does not seem to be an instance of Einstein's SPR, as the following theorem indicates.

THEOREM 2.8.24 *There is a model* $\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + SPR^+$ *such that in* $\mathfrak{M}$ *the* f_{mk}*'s are definable, yet* $\mathfrak{M} \not\models \mathbf{Ax(eqtime)}$*. Intuitively,*

$$SPR^+ + \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) \;\not\Rightarrow\; \mathbf{Ax(eqtime)},$$

even if we assume that all the f_{mk}*'s are definable.* ∎

Theorem 2.8.24 implies that

[196]This has nothing to do with synchronism, instead this involves the "speed of time" (slow, fast etc). The intuitive content of **Ax(eqtime)** is about choosing our units of measurement. Namely, basically, it says that different observers choose the same units of measurement, in some sense. Cf. §2.8.4 on p.90.

[197]Motivation for doing this from the literature comes soon.

(*) $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathit{Einstein's\ SPR} \not\models \mathbf{Ax(eqtime)}$.[198]

In many of Einstein's derivations, when he uses as axiom the constancy of speed of light together with SPR, he is basically using **Basax** + SPR (parts of **Basax** remain tacit in his formulation). In such derivations Einstein uses **Ax(eqtime)** as a *tacit assumption*.[199] This is an extra reason for us to add **Ax(eqtime)** as an *explicit* assumption to $\mathbf{Ax(symm_0)}$.[200]

In this connection, we also note the following. Taylor and Wheeler [257] at the end of §I.4 (around Figure 12) mention a symmetry principle which we formalized as $\mathbf{Ax\triangle 3}$ in AMN [18, §3.9.6 ("Further ideas") on p. 406]. Intuitively, $\mathbf{Ax\triangle 3}$ says that meter-rods orthogonal to the direction of movement do not shrink or grow.[201] To save space, we refer the reader to AMN [18, §3.9.6] for the formal presentation of $\mathbf{Ax\triangle 3}$. Taylor and Wheeler mention $\mathbf{Ax\triangle 3}$ in connection with Einstein's SPR. (Cf. Remark 2.8.26 below.) We prove in AMN [19] that, under reasonable conditions, **Ax(eqtime)** and $\mathbf{Ax\triangle 3}$ are equivalent, namely,

(**) $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(symm_0)} \models (\mathbf{Ax(eqtime)} \leftrightarrow \mathbf{Ax\triangle 3})$,

if $n > 2$. Therefore by Cor.2.8.23 and Thm.2.8.24 we have

(***) $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathit{Einstein's\ SPR} \not\models \mathbf{Ax\triangle 3}$.

This together with Taylor and Wheeler [257, §I.4] also adds motivation to our including **Ax(eqtime)** into **Ax(symm)**.

Finally, we turn to the connection between Einstein's SPR and existence of faster than light observers.

THEOREM 2.8.25

(i) **Basax**(2)+ *Einstein's SPR is consistent with the existence of FTL observers. Moreover,*

(ii) $\mathbf{Basax}(2) + SPR^+ + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax\square 1} + \mathbf{Ax(symm)} \not\models \nexists\ \mathit{FTL\ observers}$.

Proof: The proof goes by checking that the model $\mathfrak{M}$ constructed in the proof of Thm. 3.9.8(iii) on p. 352 in AMN [18] satisfies SPR^+ too. Since $\mathfrak{M}$ contains FTL observers, we are done. ∎

Remark 2.8.26 The above result is in contrast with an argument of Einstein (from 1916) in Einstein [80], where from SPR and the constancy of speed of light he derives, *without* using the assumption $n > 2$, that no body can move FTL. In our opinion, the reason for this unjustified conclusion was that on pp.126-127 Einstein seems to conclude from SPR that m sees k the same way as k sees m, i.e. roughly $\mathsf{f}_{mk} = \mathsf{f}_{km}$ (in standard configuration). (He uses letters K and K'

[198] Of course, (*) holds only in the presently chosen framework of "pure relativity theory". If the size of a hydrogen atom or e.g. the Planck length would be expressible in our frame language (cf. p. 6 or the Index), then the strength of Einstein's SPR could increase and **Ax(eqtime)** could become derivable from SPR^+. This is possible because SPR is not a single formula but a *schema* of formulas (c.f. §2.8.4 below), hence if we have a richer language, then SPR involves these richer formulas, too. For more on this see §2.8.4.

[199] Examples are his derivation of Lorenz transformations in Einstein [80, Appendix], and e.g. [79].

[200] Recall that one of the purposes of applying mathematical logic to relativity is to replace tacit (i.e. implicit) assumptions by explicit axioms. In other words, it is to make tacit assumptions explicit (so that even the "non-initiated" can securely follow what is going on and exactly why). Cf. [18, §1.1] and Matolcsi [187].

[201] E.g. if m and k are in standard configuration (k moving in direction $\bar{x}$), then meter-rods of k parallel with the $\bar{y}$ axis of k are of the same length when observed by m and k. Cf. Cor.2.8.12.

for what we denote by m and k.) But this does not follow from SPR because the statement that John ($=m$) sees Mary ($=k$) in a certain way (i.e. as described by a fixed function f_{mk}), from the point of view of John, is not a law-like statement. To see this more clearly, let us observe that this statement is of the form "m sees Mary as described by f", where m is a free variable. Now, e.g. Hempel-Oppenheim [119] write that individual names like Mary should not occur in law-like statements. (Many authors agree with this, cf. e.g. the recent paper Hintikka-Halonen [128, §7, in particular e.g. item (2) on p.649].) So, the law-like statement that remains from the above is "for m there is k' such that m sees k' moving as described by the function f". But this formula is of the pattern of our **Ax□1** and does not imply "[*m and k are in standard configuration and* $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$] $\Rightarrow$ $\mathsf{f}_{mk} = \mathsf{f}_{km}$" which seems to be the pattern [80] uses on pp.126-127. To see that [80] does not use $n > 2$ in this particular argument, we note that on pp.124-127 n is not mentioned and only coordinates x and t are mentioned, no mention is made of y or z. But the Lorentz-transform (item (8) on p.128) is immediate by what is on p.127. Actually, after having derived (8a) on p.128, Einstein explicitly writes that next he *generalizes* the result to the case where coordinates y and z are also taken into account. A similar remark applies to the derivation of **Ax△3** from *SPR + tacit assumptions* in [257, p.37 in the Hungarian edition].

In connection with the above we mention that in §3 we prove that there are no faster than light observers if $n > 2$ and if we assume some very mild conditions (see Thm.3.2.13). It is interesting to note that our conditions involve no instance of Einstein's SPR and no version of the constancy of the speed of light. ◁

2.8.4 Is Ax(symm) objective or subjective?

Instead of **Ax(symm)** let us discuss its corollary **Ax(syt$_{00}$)** formulated below, because this simplifies the discussion. However, the whole discussion extends to **Ax(symm)** too.

Ax(syt$_{00}$) $(\forall m, k \in Obs)\ [\mathsf{f}_{mk}(\bar{0}) = \bar{0} \ \Rightarrow\ |\mathsf{f}_{mk}(1_t)_t| = |\mathsf{f}_{km}(1_t)_t|]$.

> That is, "as I see your clocks slowing down (as a consequence of your motion relative to me) so will you see my clocks slowing down (as a consequence of my motion relative to you)".

Meditating over the meaning of **Ax(syt$_{00}$)** leads to the following question. **Ax(syt$_{00}$)** can be made true (or false) by choosing the *units of measurement k uses.* (The same applies to **Ax(symm)**). But choosing units of measurement is something subjective. Assume that m lives on the Earth while k lives in a spaceship from another galaxy. They can see each other all right, but how can they compare their meter-rods (or their clocks), i.e. how can they agree on using the same units of measurement. Suppose, they are in radio communication. If they cannot compare their units of measurement via radio communication, perhaps there is no thought-experiment for them to check whether **Ax(syt$_{00}$)** is true, which could render this axiom either meaningless or to be a matter of agreement for convenience. In other words, **Ax(syt$_{00}$)** would become kind of subjective (i.e. something that does not say too much about what the world is really like, but instead it is about how we choose to describe the world).

The following intuitive argument says that this danger is not present i.e. that **Ax(syt$_{00}$)** and **Ax(symm)** are objective, i.e. they are <u>*checkable*</u> by some <u>*thought experiment*</u>. This goes as follows. Electrons and hydrogen atoms are the same in all parts of our Universe, according to the best of our knowledge.[202] Observers m and k can agree through their radio contact that they will use the hydrogen atom for defining their units of measurement (both for space and for time). I.e. they can agree to use the *same* units of measurement. After this, it is only a matter of patience to work out a thought experiment for checking whether **Ax(syt₀₀)** holds for m and k. A similar argument applies to **Ax(symm)** in place of **Ax(syt₀₀)**. Therefore, we can conclude that **Ax(syt₀₀)** and **Ax(symm)** are meaningful (objective) axioms about what the world is like (and not only "linguistic toys" like, say, absolute time).

The above considerations (using hydrogen atoms for matching units of measurement) comes up in Chapter 4 of AMN [18] where we look into axiom systems weaker than **Basax** and the question comes up whether the difference between the weak system[203] and **Basax** is testable by thought experiments (i.e. objective) or not.

2.8.5 The twin paradox

The twin paradox **(TwP)** was formulated on p.13. However, that formulation cannot be used in **Basax** because it uses non-inertial (i.e. accelerated) observers. Below we will introduce a variant of **(TwP)** in which we will simulate an accelerated observer by several inertial ones. To formulate (our present version[204] of) the twin paradox, we will need the binary relation STL of being slower than light between observers to be recalled from AMN [18, Thm.2.7.2]. Let $\mathfrak{M}$ be a frame model and $m, k \in Obs^{\mathfrak{M}}$.

Intuitively, $m\ STL\ k$ intends to mean that m thinks that k is moving more slowly than light (relative to m, of course). (E.g. in the case of the **Basax**(2) model $\mathfrak{M}_1^P$ with FTL observers, on p.50, STL is an equivalence relation on $Obs^{\mathfrak{M}_1^P}$ with exactly two equivalence classes.)

At the present point of this work, we could use the following simple potential definition (*) for STL, because we are assuming **Basax** (hence **AxE**).

(*) $m\ STL\ k \iff v_m(k) < 1.$

This definition would work because at this point we are assuming **AxE**, and **AxE** postulates that the speed of light is 1 (for every observer, in every direction). However, in Chapter 3 we will start to study more flexible, more general axiom systems (i.e. theories) than **Basax**. In these, **AxE** in its present form will not always be assumed, hence we need a more subtle definition of STL than (*) above.

Intuitively, our definition will say that $m\ STL\ k$ holds iff m sees no photon ph moving forwards in the same direction as k does such that $v_m(ph) \le v_m(k)$ would be the case. Formally:

$$m\ STL\ k \overset{\text{def}}{\iff} (\forall ph \in Ph)$$

$$\Big[\textit{if } \vec{v}_m(k) \textit{ and } \vec{v}_m(ph) \textit{ are vectors pointing in the same direction, then } \vec{v}_m(k) \textit{ is shorter than } \vec{v}_m(ph) \Big].^{205}$$

[202] At this point we acknowledge that we brought a new axiom into our picture of the world. But we think that should be all right as far as one acknowledges it. (I.e. what we are saying about **Ax(syt₀₀)** is not based on pure logic only.) In this connection cf. also Feynman [86, p.90 ("Symmetry in physical law")] and Galilei [93].

[203] in which for different observers the speed of light might be different

[204] In reality this is only an "approximation" of the original paradox, and this is called "clock paradox" in d'Inverno [73]p.24.

[205] The complete formalization of the above definition needs some case distinctions (and some common sense),

In Theorems 2.7.2 and 3.4.1 (pp.110, 203) of AMN [18] we saw that STL is an equivalence relation on the set of observers Obs (assuming $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }))$.

Let us turn to formulating our present version of the twin paradox. Although the formula below might look long at first sight, its intuitive content is simple cf. Figure 42. The key idea is the following. Originally in **(TwP)** we had two twin brothers m and k. Of these, m was inertial while k was accelerated. As we already said, since now (in the present section) we do <u>*not*</u> have accelerated observers, we will have to <u>*simulate*</u> (or approximate) brother (i.e. observer) k by two "auxiliary" inertial observers k_1 and k_2.

We will return to discussing the role of STL in **Ax(TwP)**, soon.

It is easier to digest the rather simple and natural meaning of **Ax(TwP)** below if one looks <u>first</u> at Figure 42. (We were careful to use the same letters in the figure and in the formula below.)

Ax(TwP) $(\forall m, k_1, k_2 \in Obs)(\forall p, q, r \in {}^nF)$

$$\Big([\, m\ STL\ k_1 \quad \wedge \quad m\ STL\ k_2 \quad \wedge \quad p_t < q_t < r_t \quad \wedge$$
$$\{p\} = tr_m(m) \cap tr_m(k_1) \ \wedge\ \{q\} = tr_m(k_1) \cap tr_m(k_2) \ \wedge\ \{r\} = tr_m(m) \cap tr_m(k_2)\,] \ \Rightarrow$$
$$|p_t - r_t| > |\mathsf{f}_{mk_1}(p)_t - \mathsf{f}_{mk_1}(q)_t| + |\mathsf{f}_{mk_2}(q)_t - \mathsf{f}_{mk_2}(r)_t| \Big),$$

see Figure 42.

THEOREM 2.8.27 $(\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ })) \models \mathbf{Ax(TwP)}$. ∎

The condition "*STL*" can be replaced in **Ax(TwP)** by the following perhaps more natural condition: All three observers m, k_1, k_2 think that event $w_m(q)$ was "temporally between" events $w_m(p)$ and $w_m(r)$. We leave the complete formalization of this version of **Ax(TwP)** to the reader. For more on the role of "*STL*" in **Ax(TwP)** we refer to AMN [18, Remark 2.8.19 (p.140)].

Figure 42 shows how the inertial brother, m, observes his accelerated twin brother, k. Let us see how the accelerated brother, k, observes his inertial twin brother, m. Below (and when looking at Figures 42-47) it is important to keep in mind that "m observes k" means that m represents k's life-line in m's coordinate system. Hence "observing" means "coordinatizing" and not visually seeing via photons. Hence "observing" does not involve any visual effect like the doppler effect. As we said before, this convention applies throughout the present work.

As a contrast to Thm.2.8.27 we note that the converse of Thm.2.8.27 is not true. Namely,

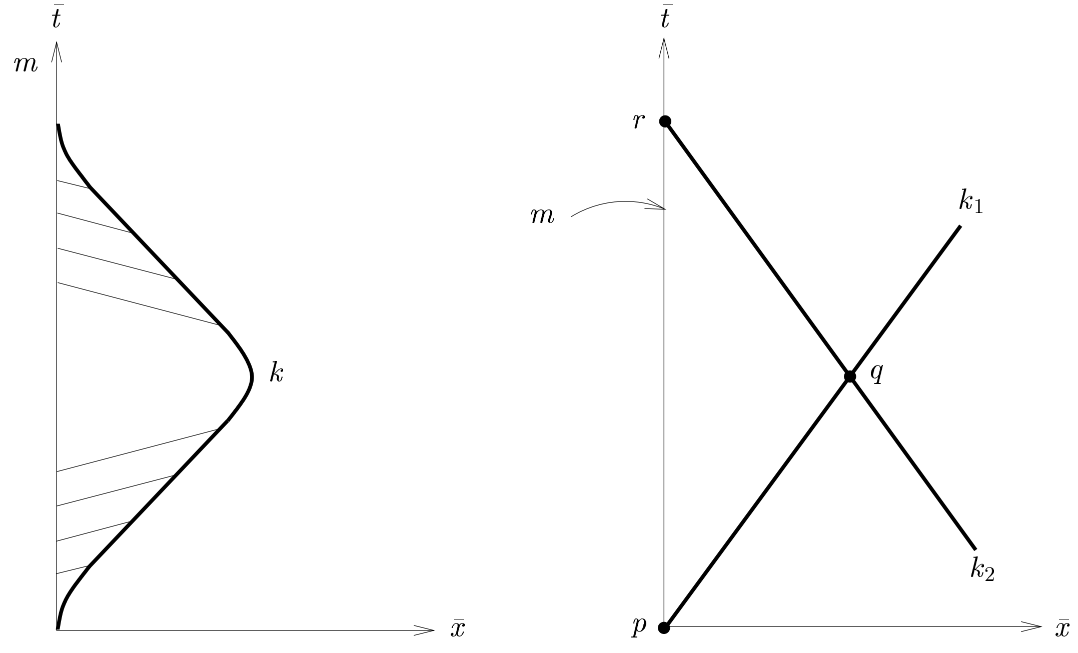
THEOREM 2.8.28 $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)} \not\models \mathbf{Ax(symm)}$.[206]

In Chapter 4 of AMN [18] we consider **Ax(TwP)** as a symmetry principle. In this context the above results can be interpreted as saying that **Ax(symm)** is a strictly stronger symmetry principle than **Ax(TwP)**, in some sense.

because in the formal definition of the vectors $\vec{v}_m(k)$ we represented the infinitely long vectors in a different spirit than the finite ones (cf. pp.19-20 for $v_m(k)$ and $\vec{v}_m(k)$). An equivalent definition of $m\ STL\ k$ says that $(\forall ph \in Ph)$[*if m sees both k and ph moving forwards in the same direction, then* $v_m(k) < v_m(ph)$], where the formalizations of the expressions (like "direction" etc) used in this formulation are found in Chapter 3, Def.3.2.4 (p.108) herein. Cf. also the definition of STL in AMN [18, p.460].

Else: we did not check how many of our axioms are needed for proving that STL is an equivalence relation on Obs, but we note that $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$ is sufficient for this.

[206] This is a result of Gergely Székely (ELTE University), solving a problem posed by Judit Madarász. Cf. Székely [246].



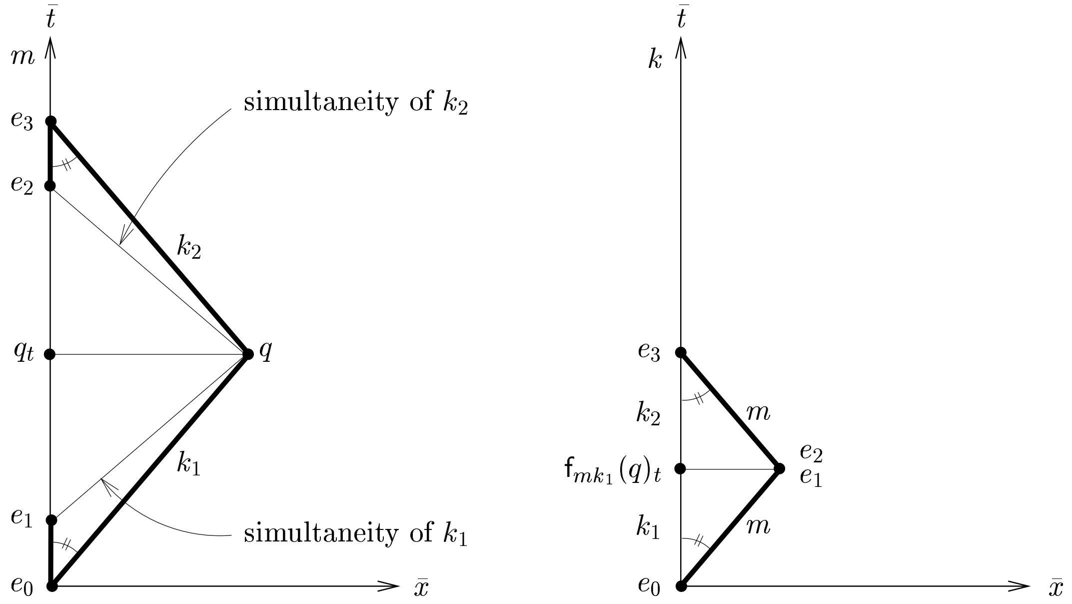
This is what we approximate. $|p_t - r_t| > |\mathsf{f}_{mk_1}(p)_t - \mathsf{f}_{mk_1}(q)_t| + |\mathsf{f}_{mk_2}(q)_t - \mathsf{f}_{mk_2}(r)_t|$

Figure 42: Twin paradox. (In this figure we choose the speed of light to be 2 instead of 1 for better representation of the effects we want to illustrate.) The slanted lines in the left-hand picture represent simultaneities of observer k.

In **Ax(TwP)** we approximated k by two inertial observers, k_1 and k_2. We can imagine that k travels with k_1 until k_1 meets k_2, when k "jumps over" to k_2's spaceship. We then put together k's world-view from k_1's and k_2's such that k's world-view agrees with k_1's world-view until they meet k_2, and from that time on k's world-view agrees with k_2's world-view. We also assume that the clocks of k_1 and k_2 are such that they show the same time at their encounter (i.e. we assume that $\mathsf{f}_{mk_1}(q) = \mathsf{f}_{mk_2}(q)$ where $\{q\} = tr_m(k_1) \cap tr_m(k_2)$).

From now on we assume **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$). We will use properties of the world-view transformations in models of **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$) that we proved in this section, see e.g. Theorems 2.8.7–2.8.8.

Figure 43 shows how k observes m when k is approximated by k_1 and k_2 as in Figure 42. Recall that in Figure 42, $\vec{v}_m(k_1) = -\vec{v}_m(k_2)$ and k_1 and k_2 meet at q, i.e. $tr_m(k_1) \cap tr_m(k_2) = \{q\}$, and $p = \bar{0}$. I.e., according to Figure 42, m observes k receding with speed v until time q_t, when k turns back and begins to approach with the same speed v. As illustrated in Figure 43, k will observe m to recede with the *same* speed v until time $\mathsf{f}_{mk_1}(q)_t$, when m turns back (as observed by k) and begins to approach with speed v. This is very similar to how m observes k, except that m, as observed by k, turns back sooner than k does so as observed by m, because by paradigmatic effect (I) (moving clocks slow down) we have that $\mathsf{f}_{mk_1}(q)_t < q_t$. I.e., m needs less time for the journey as k observes it than k needs for the journey as m observes it (this is the twin paradox). This also implies that the *distance* m covered according to k is *less than* the distance k covered according to m.



This is how m sees k when k is approximated by k_1 and k_2.

This is how k sees m when k is approximated by k_1 and k_2.

Figure 43: Twin paradox approximated by two inertial observers of the same speed.

Let us analyze further (from a different point of view) how k observes m. Assume that when k departs, m is standing there waving goodbye, then goes home, has breakfast, and then comes back to the departing spot again to meet his brother k. Now, k will observe m waving goodbye and starting to go home in slow motion (i.e. all of m's processes are slower than usual), then before m reached home, according to k's world-view, suddenly he is already coming back again (in slow motion) to meet him at the departing spot. In turn, m will observe his twin brother in slow motion all the time, and he will observe all events that happened to k on his journey.

In more technical terminology, using Figures 42 and 43: $e_0 = w_m(\bar{0})$ is the event of k's departing, and $w_m(q)$ is the event of k's turning back on his journey. Let event e_1 in m's life be simultaneous with $w_m(q)$ according to k_1 (i.e. $m \in e_1$ and $\mathsf{f}_{mk_1}(q)_t = w_{k_1}^{-1}(e_1)_t$). See Figure 43. Similarly, let event e_2 in m's life be simultaneous with $w_m(q)$ according to k_2. We can see that e_2 happens much later in m's life than e_1 and that k does not observe the events in m's life that happen between e_1 and e_2. In our story, e_1 is an event in m's life when he is on his way home after waving goodbye to his twin, and e_2 is an event in m's life when he is already on his way back to meet his twin brother upon his return. k observes m in slow motion because k observes that m's clock slows down, and so for k more time passes between the events e_0 and e_1 than for m.[207]

On the other hand, as we said, m will observe his twin brother in slow motion all the time, and he will observe all events that happened to k on his journey. What is the reason for this strong asymmetry between the twins? The reason is that m is an inertial observer while k

[207] This is one of our paradigmatic effects, the **Ax(symm)** version of "moving clocks slow down", cf. Thm. 2.8.7.

is not; k's world-view is put together from the world-views of two different inertial observers, and at the "pasting point" (i.e. at the event when k turns back) there are strange effects, e.g. a large part of m's life-line gets "cut out" (k observes m suddenly at a much later point in m's life). As a side-effect of approximating k by only two inertial observers, "at point q" k experiences infinite acceleration (which in turn naturally causes funny effects). Soon we will approximate k by more and more inertial observers. Then the "irrelevant" parts of the funny effects will gradually fade away while the "relevant" parts of the effects will stay with us (cf. e.g. Figure 46).

If we approximate k by two inertial observers differently than in Figure 42, e.g. if k comes back more slowly than he was traveling outwards, k will observe m at the turning point suddenly placed at a bigger distance, as in Figure 44. But if k_1 and k_2 have the same speed, this "instantaneous displacement" will not occur.

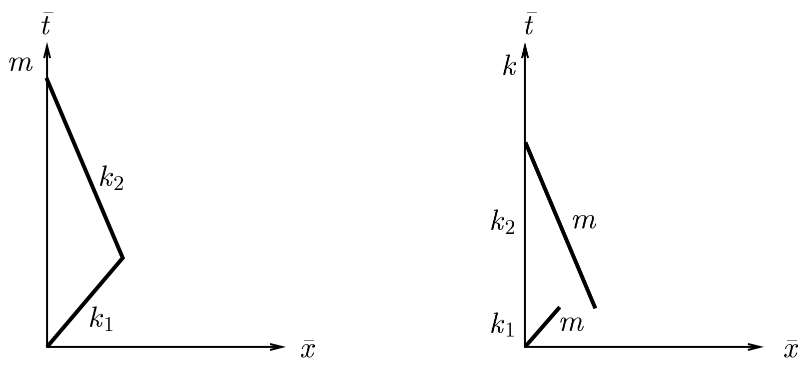


Figure 44: Twin paradox approximated by two inertial observers of different speeds.

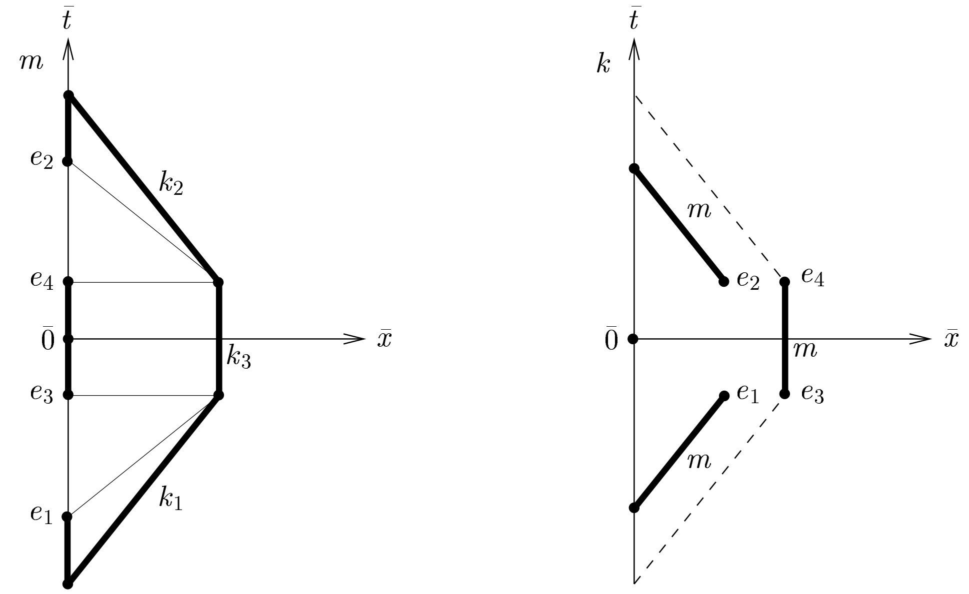
Let us see what the above scenario looks like if we *refine* our approximation of the accelerated twin, i.e. if we approximate the accelerated twin by more and more inertial observers. From now on we assume that the life-line of k is symmetric in the sense that his motion outwards is exactly of the same kind as his motion inwards, i.e. k's life-line is symmetric w.r.t. the horizontal line containing q. Further, we assume that both m's and k's clocks show 0 at the turning point of k's life-line.[208] Also, for simplicity, we assume $\mathfrak{F} = \mathfrak{R}$.

Figure 45 shows m's and k's world-views when k is approximated by three inertial observers, and when k is approximated by five inertial observers. We can see that as we approximate k by more and more inertial observers, the intervals in m's life-line that k will not observe become shorter and shorter, and eventually k will observe all events in m's life-line. Similarly, m's life-line will eventually become a continuous curve as k observes it (i.e. the displacements at the "pasting points" will eventually disappear).[209] (The word "eventually" here means "at the limit of this approximating process". We approximate in a way that the difference of speeds of the consecutive inertial observers approaches 0 and we choose the "pasting points" appropriately.)

Figure 46 shows the "limit" of this approximating process. We concentrated on "smoothing out" the turning point in k's life-line, and we disregarded the initial and last segment of k's

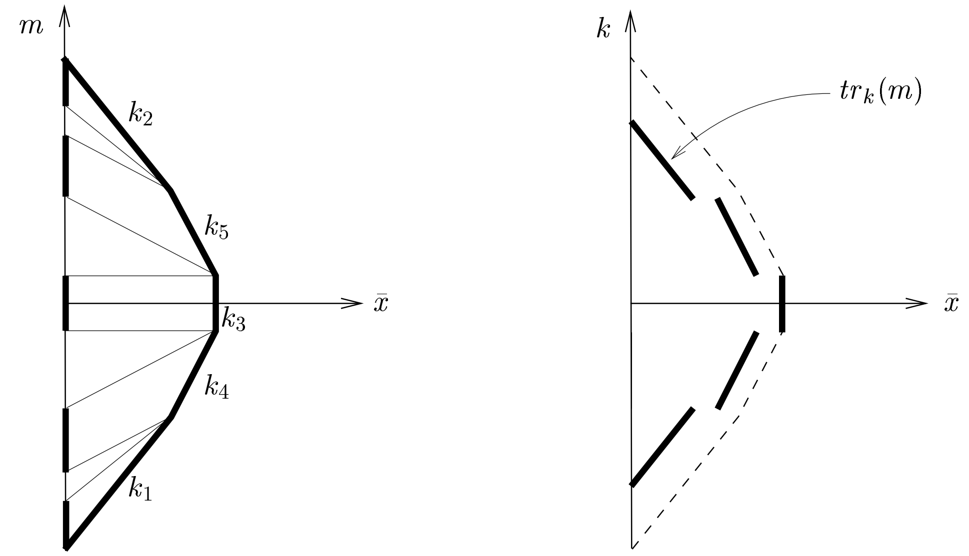
[208] in order to get simpler drawings

[209] This is so because the extent of the paradigmatic effects increases with speed, they do not occur at speed 0, and the extent they occur to continuously depends on speed of movement. See Theorem 2.9.6 in §2.9.



The traveling twin as approximated by three inertial observers

This is how m sees k This is how k sees m



The traveling twin as approximated by five inertial observers

Figure 45: The twin paradox approximated by more and more inertial observers.

acceleration and deceleration (cf. the left-hand side of Figure 42).[210] Thus k goes outwards with constant speed v for a certain amount of time, then gradually (smoothly) he decelerates until he is momentarily at rest with respect to m, then he continues decelerating which means that he turns back and begins to gain speed[211] until he attains speed v again, and then he stops decelerating and approaches m with constant speed v until he reaches m. Cf. the left-hand side of Figure 46. This is how m observes k. Let us turn to how the accelerated twin k observes his inertial brother m. In Figure 46 we can see that k observes m first receding with constant speed v, then m accelerates (increases (!) his speed), then m begins to decelerate till m is momentarily at rest w.r.t. k, and then m reverses this process. Thus the two life-lines are not alike: k's life-line, as observed by m, is "convex" in the sense that k's movement is uniform, it keeps decelerating. At the same time, m's life-line as k observes it is *both "convex and concave"*. This has to be so because of the following: $tr_m(k)$ and $tr_k(m)$ are both continuous (because in physics all movements are continuous), their initial and last segments are straight and parallel (because $v_m(k) = v_k(m)$ in the inertial parts of the journey), these segments are closer in m's life-line than in k's one (because for k less time has passed between departing and meeting, i.e. between e_0 and e_3, than for m), while the "width" of both life-lines are the same (because at the turning point k and m are at rest with respect to each other, so they see each other being at the same spatial distance). See Figure 46.

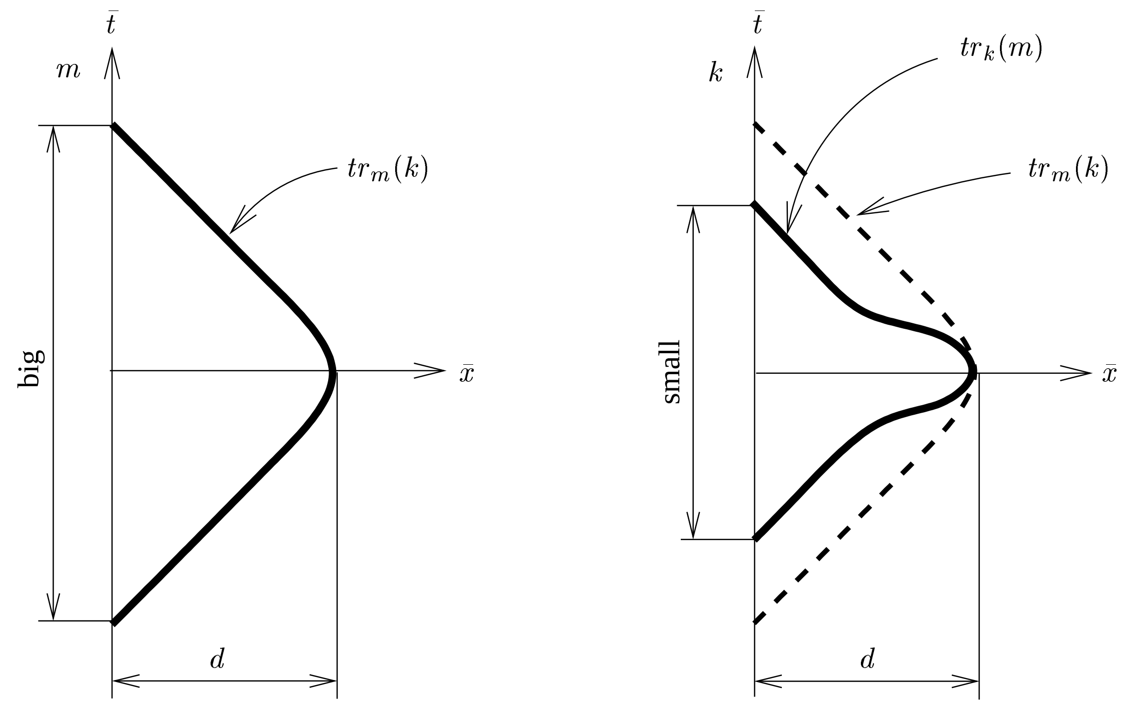


Figure 46: The inertial brother's life-line is different from that of the accelerated one.

[210] For formulating and discussing the Twin Paradox, we do not need to assume that before event e_0 (or after e_3) the two twins k and m are at relative rest. Instead, we may assume that they simply meet at e_0 (moving with relative speed v). This way we can get rid of the initial (and final) acceleration without losing anything essential. The acceleration "around" q, however, is essential, it cannot be "argued away" in the just used spirit.

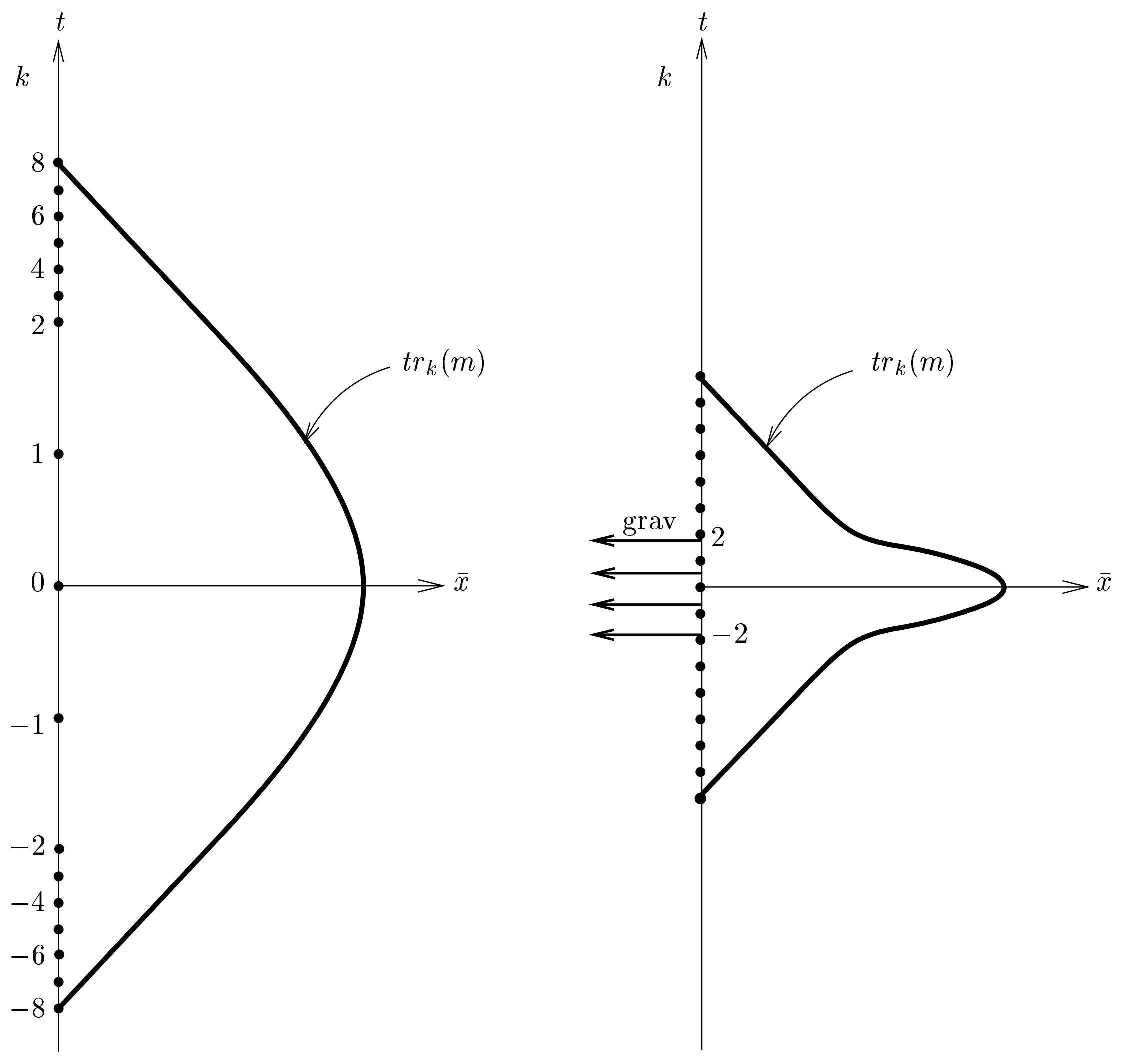
[211] This is so because k's velocity changes gradually from $\vec{v}$ to $-\vec{v}$. So in terms of velocity, k's velocity is constantly decreasing. In terms of speed, this implies losing speed gradually from v to 0, and then gaining speed gradually from 0 to v again.

We will return to the twin paradox in the chapter on accelerated observers of AMN [19] and in [26], where we will begin to study gravity, too. Jumping ahead for a short while, let us see how k will "explain" m's strange movement (life-line) by using his knowledge about gravity. This explanation serves also to explain how the "laws of physics" can be the same for m and for k despite of the fact that each observes his brother as behaving rather differently. The reader does not have to understand the explanation which comes below, since it uses (i) Einstein's equivalence principle (of acceleration and gravity) and some of the effects of gravity which we will prove in the chapter on accelerated observers of AMN[19], [26], namely, that (ii) gravity causes clocks to run slow relative to clocks far away from the "source of" gravity, and (iii) in some sense gravity does not affect processes which take place sufficiently far away from the source of gravity. Therefore we advise the reader to read the explanation below as a "fairy tale" (which, in turn, will become easily understandable after studying the basic parts of the theory of accelerated observers in AMN [19], [26]).

The accelerated brother k thinks that he is at rest and m is moving away from him with speed v. When m is already at a distance, a gravitational field appears in k's world-view where k stands. To remain motionless despite of this strange gravitational field (which appeared "out of nowhere" so to speak), k starts up the engine of his spaceship to balance the effect of gravity. (In contrast, m thinks that k started his engines in order to decelerate.) This gravity slows down k's clock, and this explains why, for k, m appears to accelerate when gravity appears. See Figure 47. After a while, since this gravity "pulls" m towards k, m begins to decelerate till it comes to a momentary rest w.r.t. k, then turns back and begins to "fall back" towards k with increasing speed. When gravity disappears (then k stops the engine in order to stay motionless), m first slows down[212], then reaches speed v and continues to approach k with constant speed v. (In passing we note that the reader might have the impression that for k sometimes m moves faster than light. However, this is not the case, because as a side-effect of *gravity* in k's coordinate system, at places far away from k the speed of light becomes greater than usual. This will be seen in the chapter on accelerated observers.)

In the above we used the expression "k observes" in place of the expression "k sees", because we wanted to emphasize that we meant everything according to k's coordinate system, and not according to how k actually "sees" via photons. Let us briefly turn to the visual effects, i.e. let us see how m and k visually see the journey via photons. See Figure 48. Again, we will find that the two brothers see the journey differently. The inertial brother will see k such that k travels outwards (with clocks slowed down) for a long time and then he approaches (with fast running clocks) for a very short time. On the other hand, k will see that his inertial brother m travels outwards (with slow clocks) for about the first half of the time needed for the whole experiment (i.e. until event $w_m(q)$ which is when m thinks that k turns around), and from that time on m approaches (with fast clocks). Thus for k, m's outwards and inwards parts of the journey (as k sees via photons) lasted approximately for the same time, while m will see (via photons) that k's journey outwards lasted much longer than k's journey backwards. If k decelerates only for a short time around its turning point, then this difference of ratio of outward and inward trips as k and m see them via photons will remain.

[212] because of the already mentioned effect of gravity on k's clocks



k knows that his clock slows down
in the interval of "acceleration"
(i.e. in the time interval around $t = 0$)

the same drawing (as on the left)
but taking the readings of
k's clock seriously

Figure 47: When k starts up his engine, k's clock slows down, and thus m's movement seems to speed up (m seems to accelerate).

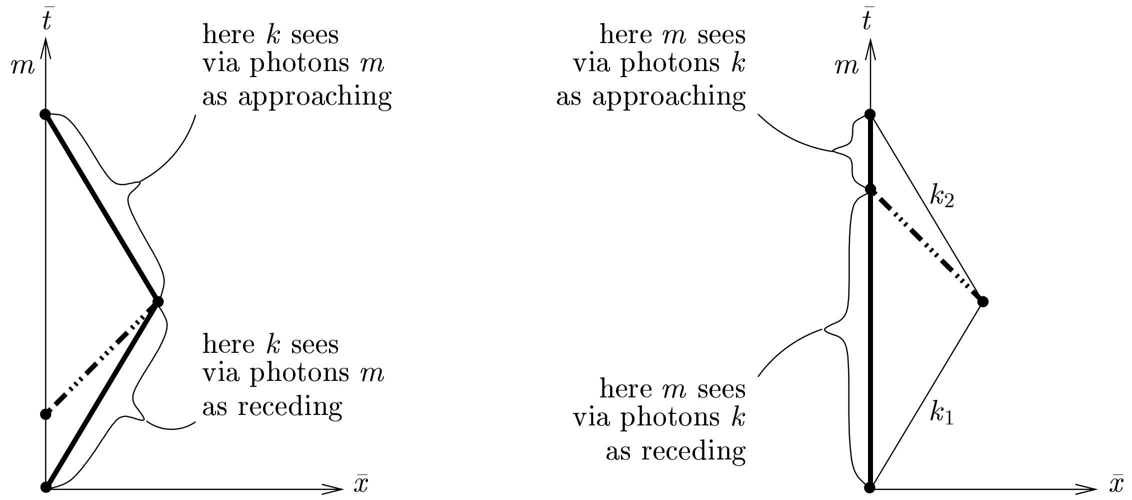


Figure 48: The two brothers' visual observations of each other's journey are also different.

2.8.5 Our central axiom system Specrel.

We conclude this part with introducing one of our central axiom systems for special relativity. Theorem 2.9.6 in §2.9 (p.104) states that the world-view transformations in models of **Basax+Ax(symm)** are all so-called Poincaré transformations, i.e. world-view transformations that occur in the standard models of special relativity. Thus, in models of **Basax+Ax(symm)** all the usual formulas for coordinate-transformations, used in the physics books, are valid. Therefore, models of **Basax + Ax(symm)** are very close to the standard Minkowskian models. In proofs we will often need the auxiliary axioms **Ax(**_Triv_**)** and **Ax(||)**. Though **Ax(||)** follows from **Basax + Ax(symm)**, for later weaker versions of **Basax** this will not be so, e.g. **Bax + Ax(symm)** $\not\models$ **Ax(||)**, here **Bax** is an axiom system introduced in §3. Therefore we define one of our stronger kind[213] of special relativity theories as follows.

Definition 2.8.29

$$\mathbf{Ax(symm)}^{\dagger} \stackrel{\text{def}}{=} \mathbf{Ax(symm)} + \mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax(\|)}$$

$$\mathbf{Specrel} \stackrel{\text{def}}{=} \mathbf{Basax} + \mathbf{Ax(symm)}^{\dagger}.$$

$\lhd$

Specrel is a first-order-logic theory of special relativity that is basically equivalent to the standard version of special relativity theory. The only omissions (missing from **Specrel**) are some auxiliary axioms that we almost never use, see the definitions of **BaCo** and[214] Minkowski model in AMN [18, §3.8, pp.298,331]. For more on this see AMN [18, §3.8].

[213] As indicated in the introduction, in the present work we will have stronger axiomatic versions as well as weaker axiomatic versions of (special) relativity. At each point, we will choose between the stronger and weaker versions depending on our purposes at that point, cf. e.g. items II, III, V in §1.1 of AMN [18].

[214] **BaCo** is a complete axiomatization of (what we consider as) usual special relativity, see Chapter 3 herein, and AMN [18].

2.9 Connections with standard Lorentz and Poincaré transformations

In order to compare our results with the literature, we recall some standard concepts from the literature.

Definition 2.9.1 Assume $\mathfrak{F} = \langle \mathbf{F}, \leq \rangle$ is an ordered field and $n \geq 2$.

1. $Linb = Linb(n, \mathfrak{F})$ denotes the set of <u>bi</u>jective <u>lin</u>ear transformations of ${}^n\mathbf{F}$.

2. Let $p \in {}^nF$. Then $\tau_p : {}^nF \longrightarrow {}^nF$ denotes the <u>*translation*</u> by vector p, defined as follows:
$$\tau_p \stackrel{\text{def}}{=} \langle\, q + p \ :\ q \in {}^nF \,\rangle \ .$$
$Tran = Tran(n, \mathfrak{F})$ denotes the set of translations of ${}^n\mathbf{F}$, i.e.
$Tran \stackrel{\text{def}}{=} \{\, \tau_p \ :\ p \in {}^nF \,\}$.

3. A function $\mathsf{f} : {}^nF \longrightarrow {}^nF$ is called an <u>*affine transformation*</u> of ${}^n\mathbf{F}$ iff it is a composition of a bijective linear transformation and a translation, i.e.
$\mathsf{f} = \mathsf{g} \circ \tau_p, \quad$ for some $\mathsf{g} \in Linb(n, \mathfrak{F})$ and $p \in {}^nF$.
$Aftr = Aftr(n, \mathfrak{F})$ denotes the set of affine transformations of ${}^n\mathbf{F}$.

4. Let $p, q \in {}^nF$. Then the square of their <u>*Minkowski-distance*</u> $g^2_\mu(p, q)$ is defined as follows:
$$g^2_\mu(p, q) \stackrel{\text{def}}{=} \left|(q_0 - p_0)^2 - \left(\sum\nolimits_{0<i\in n}(q_i - p_i)^2\right)\right| \ .$$
We note that $g^2_\mu : {}^nF \times {}^nF \longrightarrow F$.

5. By a <u>*Lorentz transformation*</u> of ${}^n\mathbf{F}$ we understand $\mathsf{f} \in Linb$ such that f preserves the square of Minkowski-distance, that is,
$$(\star) \qquad (\forall p, q \in {}^nF)\ g^2_\mu(p, q) = g^2_\mu(\mathsf{f}(p), \mathsf{f}(q))\,.$$
$Lor = Lor(n, \mathfrak{F})$ denotes the set of Lorentz transformations of ${}^n\mathbf{F}$.

6. By a <u>*standard Lorentz transformation*</u>[215] we understand a Lorentz transformation f such that
$$\mathsf{f}[\bar{t}], \mathsf{f}[\bar{x}\,] \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \quad \text{and} \quad (\forall 1 < i \in n)\ \mathsf{f}(1_i) = 1_i\,.$$
$SLor = SLor(n, \mathfrak{F})$ denotes the set of standard Lorentz transformations of ${}^n\mathbf{F}$.

7. By a <u>*Poincaré transformation*</u> of ${}^n\mathbf{F}$ we understand $\mathsf{f} \in Aftr$ such that f preserves the square of Minkowski-distance, that is, $(\star)$ in item 5 holds for f.
$Poi = Poi(n, \mathfrak{F})$ denotes the set of Poincaré transformations.[216]

[215] Or equivalently a Lorentz transformation in standard configuration.
[216] An equivalent definition says that a Poincaré transformation is a composition of a Lorentz transformation and a translation, i.e. is of the form $lor \circ \tau_p, \quad$ for some $lor \in Lor$ and $p \in {}^nF$.

8. A bijective linear transformation f of ${}^n\mathfrak{F}$ is called an *expansion*[217] iff

$$(\exists 0 < \lambda \in F) \quad \mathsf{f} = \langle\, \lambda \cdot p \,:\, p \in {}^nF \,\rangle \ .$$

   $Exp = Exp(n, \mathfrak{F})$ denotes the set of expansions.

◁

CONVENTION 2.9.2 For better readability, the elements of Exp and Lor will often be denoted by exp and lor, respectively. Similarly for $Linb$, $SLor$, Poi, $Rhomb$ etc.

◁

For completeness, we note that our distinguished sets of transformations are contained in each other in the following way:

$$\begin{array}{ccccccc} SLor & \subset & Lor & \subset & Linb & \supset & Exp \\ & & \cap & & \cap & & \\ Tran & \subset & Poi & \subset & Aftr\,, & & \end{array}$$

where $\begin{array}{c} A \\ \cap \\ B \end{array}$, $B \supset A$, etc. all denote that A is a proper subset of B.

Thm. 2.9.4 below is a kind of *characterization* of the world-view transformations f_{mk} in $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$. Intuitively, it says (assuming $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$) that a world-view transformation f_{mk} is always a composition of a Poincaré transformation, an expansion, and a map $\widetilde{\varphi}$ induced by an automorphism φ of the ordered field $\mathfrak{F}$ (cf. Notation 2.9.3 below for $\widetilde{\varphi}$). Moreover all such compositions are world-view transformations (of some **Basax** model), if we assume that $\mathfrak{F}$ is Euclidean. To formulate this theorem we need Notation 2.9.3 below.

Notation 2.9.3

- $Aut(\mathfrak{F})$ denotes the set of automorphisms of the ordered field $\mathfrak{F}$. For any algebraic structure or model $\mathfrak{A}$, $Aut(\mathfrak{A})$ is defined similarly (i.e. is the set of automorphisms of $\mathfrak{A}$).

- $\widetilde{\varphi}$ denotes the function induced by other function φ the following way. Assume $\varphi : F \longrightarrow F$. Then the induced function $\widetilde{\varphi} : {}^nF \longrightarrow {}^nF$ is defined the natural way, i.e.

$$\widetilde{\varphi}(p) \stackrel{\text{def}}{=} \langle \varphi(p_0), \varphi(p_1), \ldots, \varphi(p_{n-1}) \rangle\,, \quad \text{for every } p \in {}^nF.$$

◁

[217] We note that the official name for an expansion is a *transformation of similitude*. (Coxeter [62] uses the word *dilation* while Burke [51] calls it an expansion. Sometimes it is also called *homothetic transformation*). For reasons of convenience we restricted the notion of an expansion for multiplying with positive λ's only.

THEOREM 2.9.4 (Characterization of the world-view transformations in models of Basax) *Assume* **Basax** + **Ax**($\sqrt{\ }$). *Let* $m, k \in Obs$. *Then:*

(i) $\mathsf{f}_{mk} = poi \circ exp \circ \widetilde{\varphi}$, *for some* $poi \in Poi$, $exp \in Exp$ *and* $\varphi \in Aut(\mathfrak{F})$.

(ii) *Assume in addition that* $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. *Then* $\mathsf{f}_{mk} = lor \circ exp \circ \widetilde{\varphi}$, *for some* $lor \in Lor$, $exp \in Exp$ *and* $\varphi \in Aut(\mathfrak{F})$.

(iii) *Let* $\mathfrak{F}$ *be a fixed Euclidean ordered field. Assume* f *is a composition of a Poincaré transformation, an expansion, and a map* $\widetilde{\varphi}$, *for some* $\varphi \in Aut(\mathfrak{F})$. *Then there is a* **Basax** *model* $\mathfrak{N}$ *with ordered field reduct* $\mathfrak{F}$ *such that* $\mathsf{f} = \mathsf{f}_{m'k'}$, *for some* $m', k' \in Obs^{\mathfrak{N}}$.

∎

Remark 2.9.5 In connection with Thm.2.9.4 above the following are natural questions. Let $\mathfrak{F} = \langle \mathbf{F}, \leq \rangle$ be an arbitrary ordered field. The questions:

(i) Under what conditions on $\mathfrak{F}$ is $\mathfrak{F}$ the field-reduct $\mathfrak{F}^{\mathfrak{M}}$ of some **Basax** model $\mathfrak{M}$?

(ii) Which automorphisms $\varphi \in Aut(\mathbf{F})$ of $\mathbf{F}$ can occur in some **Basax** model in the style of Thm.2.9.4 (i.e. for which φ are there an $\mathfrak{M} \models$ **Basax** and an f_{mk} such that $\mathsf{f}_{mk} = poi \circ exp \circ \widetilde{\varphi}$ for some poi and exp as in Thm.2.9.4)?

In this connection we state the following.

(1) If $n = 2$, then <u>all</u> ordered fields $\mathfrak{F}$ and all $\varphi \in Aut(\mathbf{F})$ can come from some **Basax** model, by Thm.2.3.12 on p.32.

(2) For arbitrary n, all Euclidean $\mathfrak{F}$ and all $\varphi \in Aut(\mathbf{F})$ can occur in some **Basax** model. (This follows from (iii) of Thm.2.9.4 above.)

(3) Further information related to questions (i),(ii) above can be found in Thm.6.7.10, Cor.6.7.12 and the discussion below Cor.6.7.12 in AMN [18]. Cf. also AMN [18] §3.5 and items 3.1.4, 3.1.6 therein.

In connection with the above, we conjecture that for any ordered field $\mathfrak{F}$, if $\mathfrak{F} = \mathfrak{F}^{\mathfrak{M}}$ for some $\mathfrak{M} \models$ **Basax**, then all $\varphi \in Aut(\mathfrak{F})$ come from f_{mk}'s of some $\mathfrak{M}' \in \mathsf{Mod}(\textbf{Basax})$. ◁

Let us recall that the symmetry axiom **Ax(symm)** was introduced in §2.8 on p.77.

The following theorem says that, under assuming **Basax**+**Ax(symm)**+**Ax**($\sqrt{\ }$), a world-view transformation f_{mk} is a Poincaré transformation. Moreover all Poincaré transformations over a Euclidean $\mathfrak{F}$ are world-view transformations in some **Basax** + **Ax(symm)** model. That is, **Ax(symm)** implies that expansions and automorphisms are not needed in the above characterization of world-view transformations.

THEOREM 2.9.6 (Characterization of the world-view transformations in models of Basax + Ax(symm)) *Assume* **Basax + Ax(symm) + Ax($\sqrt{\ }$)**. *Let $m, k \in Obs$. Then (i)–(iii) below hold.*

(i) $\mathsf{f}_{mk} \in Poi$.

(ii) *Assume in addition that* $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. *Then* $\mathsf{f}_{mk} \in Lor$.

(iii) *Let $\mathfrak{F}$ be a fixed Euclidean ordered field. Let $\mathsf{f} \in Poi(n, \mathfrak{F})$. Then there is a* **Basax + Ax(symm)** *model $\mathfrak{N}$ whose ordered field reduct is $\mathfrak{F}$ such that $\mathsf{f} = \mathsf{f}_{m'k'}$, for some $m', k' \in Obs^{\mathfrak{N}}$.*

The **proof** is given in §3.8 of AMN [18]. Here we show the idea of proof of **(ii)** in the case of $n = 2$. Assume that $\mathfrak{M} \models$ **Basax + Ax(symm) + Ax($\sqrt{\ }$)**, $m, k \in Obs$ and $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. Then each of m and k thinks that the other's clock is slow (by Thm.2.8.7), and moreover the rate of slowing down is the same for both of them (see Thm.2.8.9). Figure 49 shows how this is possible. By using this figure, it is not difficult to show that the unique place where $e = \mathsf{f}_{mk}(1_t)$ can be is such that the Minkowski-distance between $\bar{0}$ and e is 1. ∎

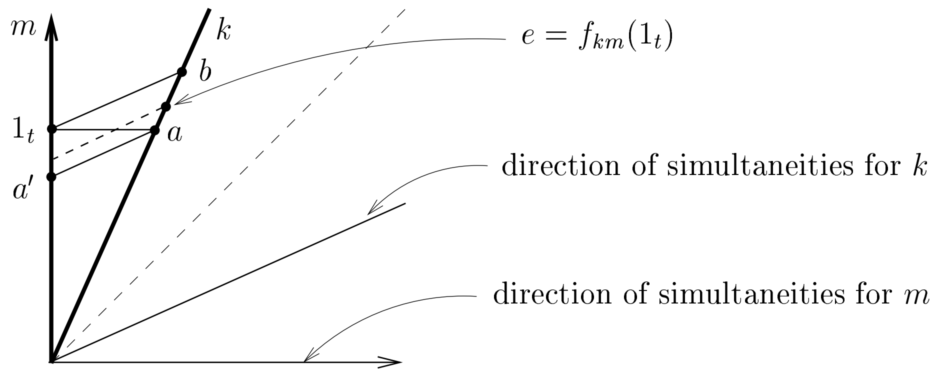


Figure 49: Both m and k think that the other's clock slows down iff $f_{mk}(1_t)$ is in between a and b. The rates of slowing down will be equal at a unique point. This unique point is closer to a than to b, and a geometrical construction for it is given in Chapter 3 of AMN [18]. The Minkowski-distance between $\bar{0}$ and e is 1.

Recall that the set $Rhomb(n, \mathfrak{F})$ of rhombus transformations was defined in Def.2.3.18 (p.40). The following theorem says that, under some mild assumptions, rhombus transformations are compositions of standard Lorentz transformations and expansions. Moreover all such compositions are rhombus transformations.

THEOREM 2.9.7

(i) *Assume $\mathfrak{F}$ is Euclidean, i.e. that $\mathfrak{F} \models$* **Ax($\sqrt{\ }$)**. *Then*

$$Rhomb(n, \mathfrak{F}) = \{\, slor \circ exp \ :\ slor \in SLor \quad and \quad exp \in Exp \,\}.$$

(ii) *Assume $n > 2$. Then*

$$Rhomb(n, \mathfrak{F}) = \{\, slor \circ exp \ :\ slor \in SLor \quad and \quad exp \in Exp \,\}.$$

(iii) $$Rhomb(n, \mathfrak{F}) \supseteq \{\, slor \circ exp \ :\ slor \in SLor \quad and \quad exp \in Exp \,\}.$$ ∎

3 More general, more flexible axiom systems (than Basax, or Specrel)

3.1 Introduction

Below we introduce refinements of **Basax** (and **Specrel**). Roughly, this means that we will study axiom systems for relativity which are weaker (or more subtle) than **Basax**. One aspect of this is that we increase the so-called *"lego" character*[218] or logical decomposability of **Basax**. This means (among other things) that we take one axiom of **Basax**, decompose it into a set of weaker axioms and then replace the original axiom in **Basax** by this set of weaker axioms. The resulting axiom system is equivalent to **Basax** but it is built up from a greater number of smaller pieces in a more flexible way. Then we can experiment with removing one or more of these refined axioms from (the new version of) **Basax** and then ask ourselves what happens to the so obtained weaker (hence more general) theory, which predictions of relativity remain provable, how these new weaker theories are related to each other and to the literature of relativity, etc.

In other words, we take a strong and important theory, **Specrel**, and decompose it (or analyze it) into a lattice of weaker, subtler and more flexible theories such that the supremum of the lattice remains **Specrel**. Then by analyzing the lattice (or hierarchy) of these subtheories of **Specrel** we gain more insight into **Specrel** (and related questions). One could say that this way we analyze the fine-resolution structure of **Specrel** and its possible variants.

The above outlined fine-resolution analysis of important theories is not new. It has been done to axiomatic Set Theory, Peano's Arithmetic (cf. e.g. Hájek-Pudlák [113]). What is known as reverse mathematics[219] is also an example of the fine-resolution analysis which we intend to carry through for relativity herein. A similar fine-resolution analysis of relativity was initiated in Friedman [91, §IV.6.] to which we refer from [18, §4.4] as "conceptual analysis".

At this point someone may ask "*why* should we *study weaker and weaker* subtheories of **Specrel**?" The answer is manyfold: Much of the answer is scattered through [18], Friedman [91] and the literature of mathematical logic; but without aiming for completeness, in items 1 - 6 below we collect some of the motivation for this.

1. If we prove an interesting prediction (i.e. theorem) of **Specrel** like the Twin Paradox or the nonexistence of FTL observers from a weaker subtheory of **Specrel**, then we obtain a *stronger*, more informative *theorem*. Formally, assume **Specrel** $\models Th_{weak} \not\models$ **Specrel** and $Th_{weak} \vdash \varphi$ where φ is an interesting prediction. Then $Th_{weak} \vdash \varphi$ is a stronger theorem than **Specrel** $\vdash \varphi$.

[218] By speaking about "lego" character we mean to invoke the spirit of the toy world known as lego where almost arbitrarily complex "models" of buildings, windmills, cars, spaceships etc. can be constructed by the child using a small number of types of building blocks. One of the analogies with our logic-based relativity theories is that if the child wants to make a small modification on the structure (say bridge or house) he built, he can do so by removing only a small number of pieces and replacing them by others (in a pattern different from the original one). In this analogy, the emphasis is on flexibility, decomposability, and "fine-resolution" structure.

[219] Cf. e.g. Andréka-Kurucz-Németi [14], Simpson [235], Friedman-Simpson-Smith [89].

2. $Th_{weak} \vdash \varphi$ has many further advantages over **Specrel** $\vdash \varphi$. E.g. proving $Th_{weak} \vdash \varphi$ might guarantee for us that the prediction φ of relativity will *survive transition* from the theory **Specrel** to a different (desirable) theory of relativity in the future. This transition might have various motivations, e.g. (i) a desire to generalize[220] **Specrel**, or (ii) a desire to compare **Specrel** with alternative theories of relativity, like Reichenbach's, or Lorentz's (cf. Szabó [244] and AMN [18, §4.5]), or (iii) wanting to know "why" φ is predicted by **Specrel**, i.e. which axiom (or axioms) of **Specrel** is responsible for φ [221], wanting to find a version of relativity which is (strong enough in some sense) but which does not predict φ.

3. Studying the hierarchy (or lattice) of weak sub-theories of **Specrel** (together with some naturally related theories) helps us to address the so-called *why-type questions* outlined in [18, §1.1 items (X), (III), (V)].

4. Studying the lattice of subtheories of (**Specrel** + some relevant other theories)[222] might help us in finding a common generalization of, say, the Newtonian theory and the Einstenian theory, which may help us to advance in the direction of *clarifying* the so-called incommesureability issues proposed by certain followers of Kuhn (cf. [149]) belonging to the direction known as *scientific relativism.*[223] So this can be viewed as a contribution to the (logic-based) philosophy of science. Actually, we did use the hierarchy of subtheories in this direction in [18, §4.1, p.423].

5. As it was explained in Reichenbach's pioneering book [218], definability theory is of fundamental importance for relativity theory. Now, by studying the weak subtheories of **Specrel**, we can make the definability results and duality results for relativity formulated in our next chapter stronger, more substantial and more invariant under possible changes in the theory.

6. Studying weak subtheories makes our understanding of the theory in question more flexible, and it *prepares the ground for generalizations* (in various directions and motivated by various reasons). In other words, this makes the theory more "fine-tuneable" with much more freedom of movement.[224]

With this we stop listing motivations for weak subtheories of **Specrel**, and we turn to introducing and discussing them.

3.2 The axiom systems

We will proceed in a bottom-up fashion: first we will introduce a quite weak axiom system **Pax** for relativity, and then we will introduce new systems by adding axioms to **Pax**. We note that **Pax** is *not* the weakest important version we will discuss, since to every theory like **Pax** or **Basax** we will introduce its partial (or bounded domain) version **Loc**(**Pax**), **Loc**(**Basax**) etc. For a theory *Th* its partial domain version **Loc**(*Th*) will be important because the

[220] e.g. in the direction towards relativity with accelerated observers and eventually general relativity

[221] To make this answer useful, first we need to replace the "few strong axioms" version of **Specrel** by a "many weak, well balanced axioms" version as indicated way above.

[222] See e.g. the lattices in Figures 60,138 (pp.126,A-31) herein and in AMN [18, p.653].

[223] Cf. the key-word "theory-laden" on p.797 of [35].

[224] By contrast: if our theory is given in the form of a differential equation, (roughly) the only freedom of movement in fine-tuning is changing the values of the constants in the equation [e.g. in the case of Einstein's equations, one can "play" with the value of the cosmological constant Λ].

$Th \mapsto \mathbf{Loc}(Th)$ generalization is a typical step needed for moving towards general relativity. For brevity, we will not recall all the detail of our work on $\mathbf{Loc}(Th)$ here, we refer the reader to AMN [18, §4.9] for more (see also p.122 herein).

CONVENTION 3.2.1 When there is no danger of confusion, we will use the word "theory", "axiom system" (and sometimes "set of formulas") interchangeably. When we call an axiom system a theory then we mean the theory generated by the axiom system. $\lhd$

After this little detour (to partial relativity) let us turn to *introducing* **Pax**.

We obtain **Pax** from **Basax** by "attacking" two axioms: **Ax6** and **AxE** and then adjusting the rest (to the change).

The **AxE** *part:* We will throw away all of the axioms mentioning photons (except for **Ax2**).[225] Because of this, we will have to reformulate **Ax5** because we still want to postulate that motion is possible. The new version of **Ax5** will be denoted as $\mathbf{Ax5_{Obs}}^{--}$, and will be defined soon. The subscript $_{Obs}$ refers to the fact that this is that part of **Ax5** in which we talk about motion of observers (as opposed to motion of photons).

The **Ax6** *part:* **Ax6** postulates that all events seen[226] by some observer are seen by all observers. The motivation for essentially weakening this axiom comes from general relativity. E.g. if one observer is inside a big, slowly rotating black hole and the other is outside, far away and remains outside, then there will be events observed by one of them but not observed by the other. Therefore we will replace **Ax6** by the weaker axioms $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$. $\mathbf{Ax6_{01}}$ says that $Dom(\mathsf{f}_{mk})$ is an open subset of ${}^nF$. $\mathbf{Ax6_{00}}$ says, intuitively, that if observer m "sees" observer k participate in an event, then k is not allowed to deny that that event happened at all. Soon we will present the formal definitions of these axioms.[227] Similarly to the case of **AxE**, after weakening **Ax6** we have to check whether the remaining axioms need adjustment to the change. Indeed, **Ax3** needs to be replaced by $\mathbf{Ax3_0}$, where the latter says, roughly, that *if* observer m sees an inertial body b then the life-line of b as seen by m is a straight line.

Summing it up, we obtain **Pax** from **Basax** by first drastically weakening the axioms **AxE**, **Ax6**[228] and then adjusting axioms **Ax1**-**Ax5** to the change.[229]

CONVENTION 3.2.2 (On *terminology:*) As we emphasized in §2, we call ${}^nF$ the *coordinate-system* of our model $\mathfrak{M}$ and not *space-time*. Space-time of $\mathfrak{M}$ will be introduced in §4 and it will be something else; namely, a structure $\langle Mn, \dots \rangle$ whose universe Mn is a subset of $\mathcal{P}(B)$. Cf. Item 4.2.6 in §4. Cf. also Matolcsi [187, §II.1.2, p.151]. Despite of this, occasionally we use the word "space-time" for ${}^nF$, for reasons of convenience. Namely, ${}^nF$ contains a time-axis and $n-1$ space axes. Therefore it is handy to speak about the *space-part* $S = \{0\} \times {}^{n-1}F$, the *time-part* $\bar{t}$ $(= F \times {}^{n-1}\{0\})$ of ${}^nF$, and to call the rest of ${}^nF$ space-time part (since it involves both space and time coordinates). We hope, this will cause no confusion, and that the reader will remember that we do not intend to regard ${}^nF$ as space-time.

---

[225] We could throw away the photon-part of **Ax2** too without any consequence, but to save space we did not go into that here.

[226] i.e. "observed" or coordinatized

[227] In passing we note that in the partial versions $\mathbf{Loc}(Th)$ of our theories Th, $\mathbf{Ax6_{01}}$ will be further weakened, cf. AMN [18, §4.9].

[228] Actually, we throw away **AxE**.

[229] The purpose of the adjustment is threefold: (i) It might happen that $(\mathbf{Ax1} - \mathbf{Ax5} + \mathbf{Ax6_{00}} + \mathbf{Ax6_{01}}) \models \mathbf{Ax6}$. We want to aviod this since we really want to make **Ax6** weaker. (ii) We want the adjusted versions of **Ax1**-**Ax5** make sense (i.e. represent the original intuition) *after* **AxE**, **Ax6** have been weakened. (iii) We want the adjusted versions of **Ax1**-**Ax5** to be consistent with the spirit of the change we made (i.e. of the weakening of **AxE**, **Ax6** we did).

Sometimes, ${}^nF$ is called "*relative space-time*" because the observer "splits" space-time to a space-part and a time-part as in ${}^nF$. Cf. Matolcsi [187, e.g. bottom of p.154, p.165, and §II.1.7].

◁

Below, we will define two functions *time* and *space* such that for any point p in our coordinate-system $time(p)$ and $space(p)$ are the time coordinate and the space "coordinate" of p, respectively.

Definition 3.2.3 We define functions $time : {}^nF \longrightarrow F$ and $space : {}^nF \longrightarrow {}^{n-1}F$ as follows.

$$(\forall p \in {}^nF)(time(p) \stackrel{\text{def}}{:=} p_0 \quad \wedge \quad space(p) \stackrel{\text{def}}{:=} \langle p_1, p_2, \dots, p_{n-1}\rangle).$$

◁

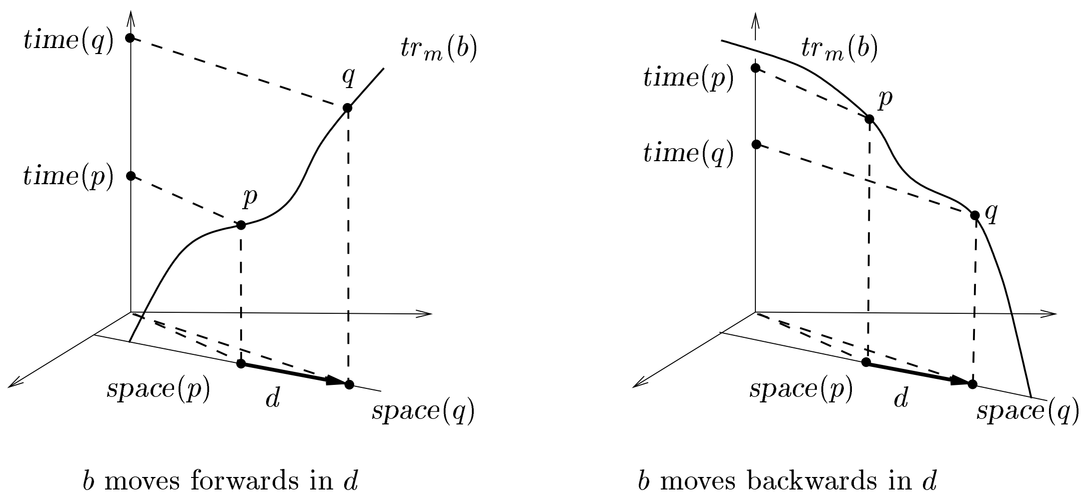


Figure 50: Illustration for Def.3.2.4.

Definition 3.2.4 (direction, moving forwards, backwards)

(i) By a *spatial direction* or simply by a *direction* we understand a *space*-vector $d \in {}^{n-1}F$, with $d \neq \bar{0}$.

(ii) $\mathsf{directions} \stackrel{\text{def}}{:=} \{d \in {}^{n-1}F \;:\; d \neq \bar{0}\}$.

(iii) Let $\mathfrak{M}$ be a frame model. Then body b is said to *move in direction* d (as seen by observer m) iff $|tr_m(b)| \geq 2$ and

$$(\forall p, q \in tr_m(b))(\exists \lambda \in F)(space(q) - space(p) = \lambda \cdot d).$$

Body b is said to move *forwards* in direction d (as seen by observer m) iff

$\Big([b \text{ moves in direction } d] \quad \text{and}$
$[(\forall p, q \in tr_m(b))(\exists 0 \le \lambda \in F)$
$\quad (time(p) < time(q) \ \Rightarrow\ space(q) - space(p) = \lambda \cdot d)]\Big)$, see Figure 50.

Body b is said to move *backwards* in direction d (as seen by observer m) iff

$\Big([b \text{ moves in direction } d] \quad \text{and}$
$[(\forall p, q \in tr_m(b))(\exists 0 \le \lambda \in F)$
$\quad (time(p) > time(q) \ \Rightarrow\ space(q) - space(p) = \lambda \cdot d)]\Big)$, see Figure 50.

More generally, let ℓ be a straight line. We say that *ℓ moves in direction d* iff

$$(\forall p, q \in \ell)(\exists \lambda \in F)(space(q) - space(p) = \lambda \cdot d).$$

(iv) When $d \in S, d \neq \bar{0}$, we say that body b moves in direction d (forwards, backwards), if b moves in direction $space(d)$ (forwards, backwards).

(v) We extend the notion of being parallel to directions, and in more general, to space-vectors as follows. If $d, d_1 \in {}^{n-1}F$, then $d \parallel d_1$ denotes that $d = \lambda \cdot d_1$ or $d_1 = \lambda \cdot d$ for some $\lambda \in F$. $\lhd$

Now we are ready to define **Pax**.

$$\mathbf{Pax} \overset{\text{def}}{=} \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Obs}}^{--}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}\},$$

where the new axioms $\mathbf{Ax3_0}$–$\mathbf{Ax6_{01}}$ are defined as follows.

$\mathbf{Ax3_0}$ $(\forall h \in Ib)\ (tr_m(h) \in G \cup \{\emptyset\} \ \wedge\ (\exists k \in Obs) tr_k(h) \neq \emptyset)$.[230]

That is, the life-line of any inertial body h as seen by any observer m must be a line or the empty-set, and there is an observer k such that the life-line of h for k is not the empty-set. $\mathbf{Ax3_0}$ differs from **Ax3** in that the life-line of an inertial body seen by an observer can be the empty-set.

$\mathbf{Ax5_{Obs}}^{--}$ $(\forall m \in Obs)(\forall d \in \mathsf{directions})(\forall p \in {}^nF)(\exists \lambda \in {}^+F)$
$(\forall q \in {}^nF)\Big[space(p) - space(q) = \delta \cdot d$ for some $\delta \in F \ \Rightarrow\ (\forall 0 \le \varepsilon < \lambda)$
$(\exists k \in Obs)(k$ moves forwards in direction d with speed ε and $q \in tr_m(k))\Big]$.

Intuitively, $\mathbf{Ax5_{Obs}}^{--}$ says that for each direction d there is a λ such that through any point there are observers moving forwards in direction d with all speeds smaller than λ. $\mathbf{Ax5_{Obs}}^{--}$ allows that these λ's be different for points of different planes parallel with $\bar{t}$.

[230] The part "$(\exists k \in Obs) tr_k(h) \neq \emptyset$" of $\mathbf{Ax3_0}$ is needed for reasons of convenience only: If we had omitted this part from $\mathbf{Ax3_0}$ then e.g. the formulation of Thm.3.3.12 on p.196 of AMN [18] would have been more complicated than it is in its present form. We note that our no FTL theorem (Thm.3.2.13 on p.118) remains true if we omit this condition from $\mathbf{Ax3_0}$.

Ax6$_{00}$ $(\forall m, k \in Obs)\ w_m[tr_m(k)] \subseteq Rng(w_k)$.

Intuitively, observer k sees all those events which are seen by another observer m on k's life-line. Even more intuitively, if someone sees k participating in an event then k should not be allowed to deny that that event happened at all.

Ax6$_{01}$ $Dom(\mathsf{f}_{mk})$ is an open[231] subset of ${}^nF$, for all $m, k \in Obs$.

Intuitive discussion of **Pax**.

As we said, **Pax** is not the weakest system we consider, because (i) the partial versions of our theories e.g. **Loc**(**Basax**) are not stronger-or-equivalent than **Pax** (i.e. **Loc**(**Basax**) $\not\models$ **Pax**) and (ii) in theories of accelerated observers (cf. e.g. AMN et al. [25], [26]) and in general relativity we will have to replace Euclidean lines by geodesics in **Ax1** (where geodesics are discussed in §4.7 p.350.). For the rest of this intuitive discussion let us <u>*ignore*</u> facts (i), (ii) above.

Pax is an extremely weak theory. One could say that **Pax** is a common fragment of the "organizational" or "book-keeping" parts[232] of both **Basax** and Newtonian kinematics.[233] **Pax** says nothing about photons or about anything related to electrodynamics. Further, **Pax** does not involve any symmetry principles. **Pax** says only a little more than saying that observers use ${}^nF$ for coordinatizing events, the life-line of inertial bodies as seen by inertial observers are straight lines and a few more things of this kind. Therefore we cannot expect **Pax** to prove any of the exotic predictions of relativity, in particular any of the paradigmatic effects discussed in §2.5. Despite of this, there are some useful and interesting things which can be proved from **Pax**. An example is the statement that all non-empty world-view transformations (i.e. the non-empty f_{mk}'s) are collineations.

Notation 3.2.5 Let $m \overset{\odot}{\to} b$ denote that m sees b, i.e.

$$m \overset{\odot}{\to} b \overset{\text{def}}{\Longleftrightarrow} tr_m(b) \neq \emptyset.$$

THEOREM 3.2.6 *Assume* **Pax**. *Then all non-empty* f_{mk}*'s are bijective collineations of* ${}^nF$. *More formally,*

$$\mathbf{Pax} \models \left(m \overset{\odot}{\to} k \Rightarrow [\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF \textit{ is a bijective collineation}]\right).$$

Proof: In the proof below we will use geometrical properties of $\mathsf{Eucl}(n, \mathfrak{F})$, like e.g. "for any two distinct points there is a unique line containing these points". We will use only such geometrical properties of $\mathsf{Eucl}(n, \mathfrak{F})$ which are provable in Tarski's generalization[234] of Euclidean geometry. These properties can be easily verified either by deriving them from Tarski's axioms (synthetic approach to geometry), or by checking that they hold for $\mathsf{Eucl}(n, \mathfrak{F})$ for any ordered field $\mathfrak{F}$, directly.

Assume $\mathfrak{M} \models$ **Pax**, $m, k \in Obs$ and $m \overset{\odot}{\to} k$. If $p \in Dom(\mathsf{f}_{mk})$, then we say that <u>*$k$ sees p*</u>.

Claim 3.2.7 Assume $v_m(m') = 0$. Then if k sees at least one point on $tr_m(m')$, then there is at most one point on $tr_m(m')$ which k does not see.

[231] in the usual sense. Cf. Notation 4.2.32(iii) on p.177.
[232] This "only book-keeping" feature is discussed in AMN [18, Remark 4.5.29 on p.596].
[233] Cf. AMN [18, §4.1, p.423] for Newtonian kinematics.
[234] which in turn is motivated by Hilbert's second-order logic axiomatization of Euclidean geometry

Proof of Claim 3.2.7: Assume that $p \in tr_m(m') \cap Dom(\mathsf{f}_{mk})$. Let P be a plane parallel with $\bar{t}$ which contains $tr_m(m')$. We will "work" in P. See Figure 51. Let S be a neighborhood of p such that $p \in S \subseteq Dom(\mathsf{f}_{mk})$. Such a neighborhood exists by $\mathbf{Ax6_{01}}$. Let d be a direction such that all straight lines lying in P move in direction d. Let λ belong to p and d according to $\mathbf{Ax5_{Obs}}^{--}$. I.e. from all points $q \in P$ and for all $\varepsilon < \lambda$ there are observers moving through q forwards in direction d with speed ε.

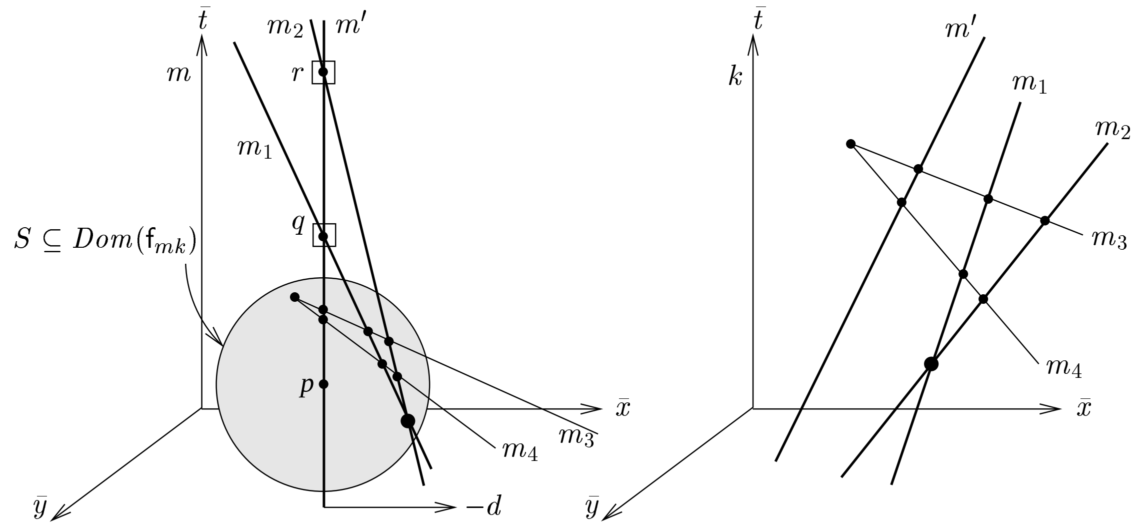


Figure 51: Illustration for the proof of Claim 3.2.7.

Assume that $q, r \in tr_m(m')$, $q \neq r$ such that k sees neither q nor r. Let us choose observers $m_1, \ldots, m_4$ according to Figure 51. This is possible by $\mathbf{Ax5_{Obs}}^{--}$: we choose these observers so that they all go forwards in direction d and their speeds are sufficiently small. The important thing is that all the indicated meeting points are inside S, i.e. k sees all these meeting points (except for q, r which k does not see). Then k sees all the observers $m', m_1, \ldots, m_4$ with those meeting points which are inside S. Thus in k's world-view, the traces of the observers $m', m_1, \ldots, m_4$ are all in one plane, i.e. they are coplanar. On the other hand, we will show that $q \notin Dom(\mathsf{f}_{mk})$ implies that in k's world-view, m_1 does not meet m'. Indeed, assume that m_1 and m' meet in k's world-view, say in point s. Then by $\mathbf{Ax6_{00}}$, m' sees the event $w_k(s)$, say $w_k(s) = w_{m'}(s_1)$. Also by $\mathbf{Ax6_{00}}$, m' sees the event $w_m(q)$, say $w_m(q) = w_{m'}(s_2)$. But both m' and m_1 are present in both events $w_{m'}(s_1)$ and $w_{m'}(s_2)$, so s_1 must equal s_2, since the traces of m' and m_1 meet only in one point, since they are different in m''s world-view (also by $\mathbf{Ax6_{00}}$, since e.g. in m's world-view there is an event on m''s trace in which m_1 is not present). Since $s_1 = s_2$, we then have $w_m(q) = w_k(s_1)$, which contradicts our assumption $q \notin Dom(\mathsf{f}_{mk})$. Similarly, in k's world-view, m_2 does not meet m' because $r \notin Dom(\mathsf{f}_{mk})$. Thus, in k's world-view both $tr_k(m_1)$ and $tr_k(m_2)$ are parallel with $tr_k(m')$, though m_1 and m_2 meet. This contradicts the fact that in a plane to each line ℓ and point u there is only one line parallel to ℓ which goes through u. This finishes the proof of Claim 3.2.7. ∎

We say that an observer $m_1 \in Obs$ is <u>*slow*</u> if, in m's world-view, m_1 moves forwards in direction d in a plane with less speed than a λ belonging to this plane according to $\mathbf{Ax5_{Obs}}^{--}$.

Claim 3.2.8 Assume that m_1 is a slow observer. If k sees a point on $tr_m(m_1)$, then k sees all points on $tr_m(m_1)$.

Proof of Claim 3.2.8: Let $p \in tr_m(m_1) \cap Dom(\mathsf{f}_{mk})$ and let S be a neighbourhood of p such that $S \subseteq Dom(\mathsf{f}_{mk}) \cap Dom(\mathsf{f}_{mm_1})$. Such a neighbourhood exists by $\mathbf{Ax6_{01}}$. Let $q \in tr_m(m_1)$ be arbitrary. See Figure 52.

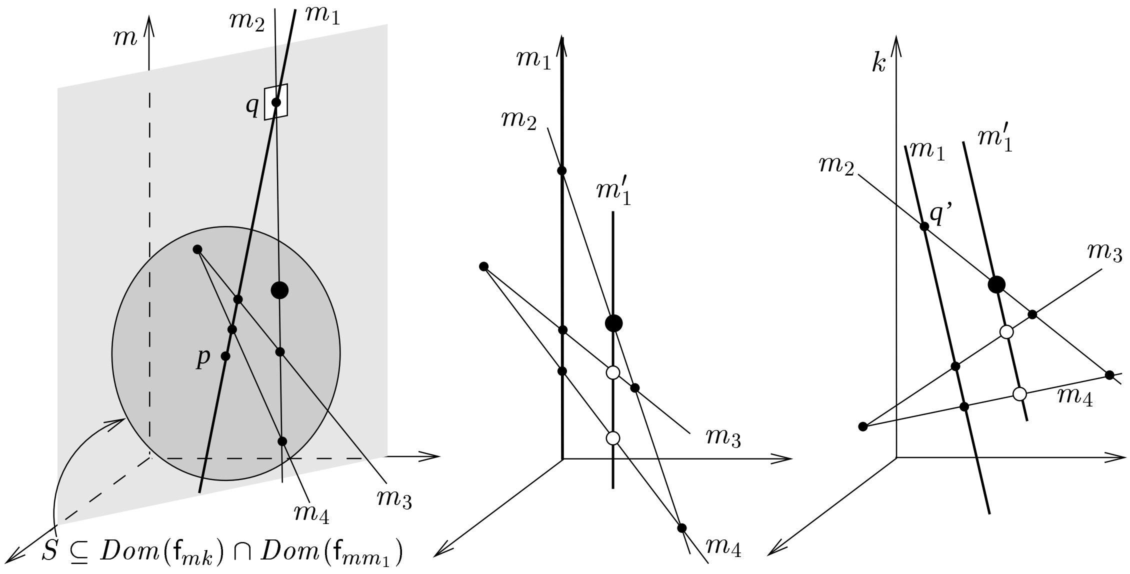


Figure 52: Illustration for the proof of Claim 3.2.8.

Let m_2, m_3, m_4 be as in Figure 52: m_2 meets m_1 at q, and m_3, m_4 meet m_1 and m_2 and each other inside S. Further, $m_1, \ldots, m_4$ have different traces in m's world-view. Such observers exist by $\mathbf{Ax5_{Obs}}^{--}$ as in the proof of the previous claim. Let r be a point on $tr_m(m_2)$ inside S, but different from the meeting points with m_3, m_4. (This is the fat point in Figure 52.)

Let us move into the world-view of m_1. m_1 sees all the meeting points that are inside S, because $S \subseteq Dom(\mathsf{f}_{mm_1})$. Thus, the traces of m_3, m_4, m_1, m_2 are all in one plane in m_1's world-view. Let m_1' be an observer such that $tr_{m_1}(m_1')$ is a straight line parallel with $\bar{t}$ in this plane, which goes through $r' \stackrel{\text{def}}{=} \mathsf{f}_{mm_1}(r)$. Such an observer exists by $\mathbf{Ax5_{Obs}}^{--}$.

Let us move now into the world-view of k. By $r \in S \subseteq Dom(\mathsf{f}_{mm_1}) \cap Dom(\mathsf{f}_{mk})$, k sees the event on m_1's trace which is at r'. In m_1's world-view, the meeting points of m_3 and m_4 with m_1' are different from each other and from r', because the traces of m_1', m_3, m_4 are all different in m_1's world-view. Therefore, by Claim 3.2.7, k does not see at most one of these points, and thus k sees one of the meeting points of m_1' with m_3 or m_4. Therefore, the trace of m_1' in k's world-view is also coplanar with the traces of $m_1, \ldots, m_4$. Also, m_1' and m_1 cannot meet in k's world-view by $\mathbf{Ax6_{00}}$, because they do not meet in m_1's world-view. Now, in k's world-view, the traces of m_2 and m_1' meet, they are coplanar with the trace of m_1, and m_1 and m_1' do not meet. Thus m_2 must meet m_1 (as before, because on a plane through a point there is only one straight line parallel with a given one). Let us assume that m_2 and m_1 meet in k's world-view at q'.

It remains to show that the event e in m's world-view at q is the same as the event e' in k's world-view at q'. To show this we will go back to m_1's world-view. By $\mathbf{Ax6_{00}}$, m_1 sees both events e, e' and he must see them on his own life-line, because $m_1 \in e \cap e'$. On the other hand, also $m_2 \in e \cap e'$, and on m_1's life-line there is only one point where m_2 is present, namely, in the meeting point of m_1 with m_2, because the traces of m_1 and m_2 are different (e.g. by

Ax6$_{00}$, because these traces are different in m's world-view). Thus $e = e_1$ and this finishes the proof of Claim 3.2.8. ∎

Claim 3.2.9 If $Dom(\mathsf{f}_{mk}) \neq \emptyset$, then $Dom(\mathsf{f}_{mk}) = {}^nF$.

Proof of Claim 3.2.9: We are in the world-view of m. We will connect any two points of ${}^nF$ with traces of slow observers. Let $p, q \in {}^nF$ be arbitrary. See Figure 53.

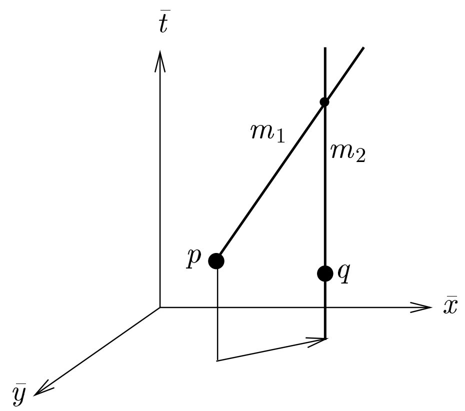


Figure 53: Illustration for the proof of Claim 3.2.9.

If $space(p) = space(q)$, then there is an observer with speed 0 whose trace connects p and q. Assume therefore $space(p) \neq space(q)$, and let $d = space(q) - space(p)$. Then $d \in \mathsf{directions}$. Let m_1 be any slow observer moving forwards in direction d with nonzero speed, and through p, and let m_2 be another observer which is at rest at point $space(q)$. Such observers exists by $\mathbf{Ax5_{Obs}}^{--}$. Then m_1 and m_2 will meet, say in point r. Then by Claim 3.2.8, $p \in Dom(\mathsf{f}_{mk})$ implies $q \in Dom(\mathsf{f}_{mk})$. This finishes the proof of Claim 3.2.9. ∎

From here on the proof is basically the same as the proof of Theorem 3.1.1 in AMN [18]. One of the changes we make is that we replace "slow lines" by "traces of slow observers". For completeness, we briefly include the rest of the proof.

Assume that $m, k \in Obs$, $m \xrightarrow{\odot} k$.

First we show that f_{mk} takes midpoints on the trace of an observer to midpoints, see Figure 54. Let $m_1 \in Obs$ and let $p, q, r \in tr_m(m_1)$ be such that q is the midpoint of p and r. In m's world-view, let us choose slow observers $m_2, \ldots, m_6$ as in Figure 54: the traces of $m_1, \ldots, m_5$ are coplanar, m_2 and m_3 do not meet, and similarly m_4 and m_5 do not meet, m_2 and m_5 meet at r, m_3 and m_4 meet at p, m_6 and m_1 meet at q and m_2, m_6, m_4 all meet in one point, and m_3, m_6, m_5 all meet in one point. Informally, $m_1, \ldots, m_6$ form a paralelogram as in Figure 54. Since f_{mk} is everywhere defined by Claim 3.2.9, k also sees all these meeting points. Thus, in k's world-view $m_1, \ldots, m_6$ also form a paralelogram. Since the diagonals of a paralelogram bisect each other, $\mathsf{f}_{mk}(q)$ is the midpoint of $\mathsf{f}_{mk}(p)$ and $\mathsf{f}_{mk}(r)$.

Next we show that f_{mk} takes collinear points to collinear ones. Assume that $\ell \in \mathsf{Eucl}$ and $p, q, r \in \ell$. In m's world-view we choose slow observers $m_1, \ldots, m_4$ as in Figure 55. From here on the proof is practically the same as on pages 169-170 of AMN [18]. Also the proof of f_{mk} being an injection is the same as that of Claim 2.3.7 on p.29 herein, because in that proof we used only the consequence of **Ax5** that through each point there move at least two different observers, and $\mathbf{Ax5_{Obs}}^{--}$ also implies this fact. By this, Theorem 3.2.6 has been proved. ∎

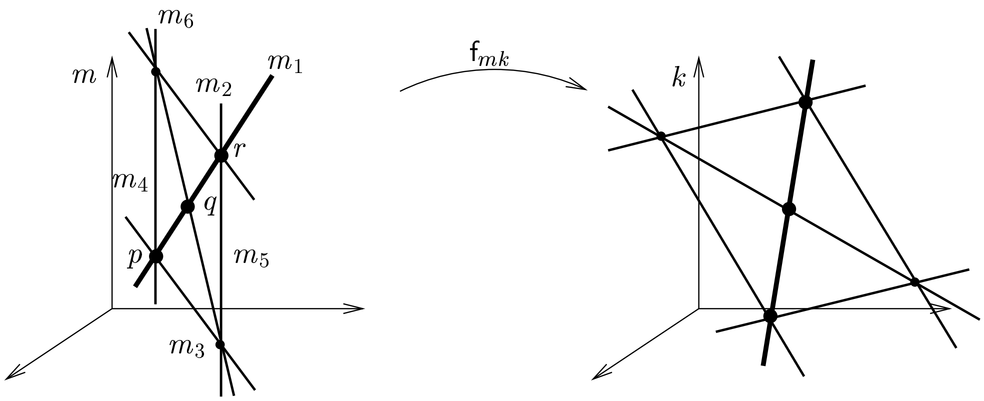


Figure 54: f_{mk} takes midpoints to midpoints

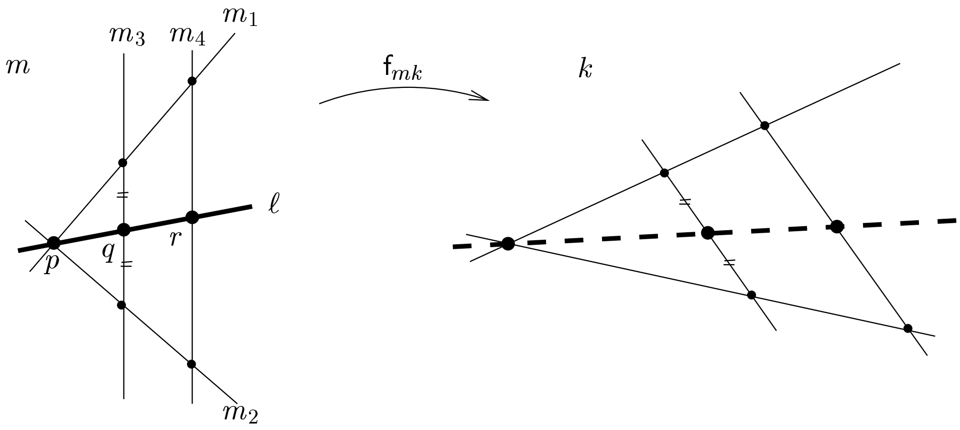


Figure 55: f_{mk} takes collinear points to collinear ones

It remains a future research task to see which axioms of **Pax** are really needed for the conclusion of the above theorem. Clearly, $\mathbf{Ax5_{Obs}}^{--}$ cannot be dropped and none of **Ax1**-$\mathbf{Ax3_0}$ can be dropped. We did not check the rest.

In connection with the above theorem we note that most presentations of special relativity assume its conclusion as an axiom and rely on this axiom rather heavily while building up special relativity (cf. e.g. Einstein [80, (1921)], Nagy [200, p.233, lines 28-30], or Friedman [91, p.139 lines 1-3]).[235] The above theorem shows that this axiom is superfluous since it follows from a very small fraction of the rest of the usual axioms (or postulates) of special relativity, which are always assumed in the usual presentations.

Next we introduce stronger axiom systems.

We will obtain $\mathbf{Bax}^-$ from **Pax** by adding rather weak assumptions on photons to it.[236] Intuitively, they will say two things: (i) Photons do not behave like bullets fired from guns in that their speed does not depend on the velocity of their sources. (ii) If an observer is sitting in his inertial spaceship (curtains on the windows drawn), and he points his flashlight in any direction d, then the flashlight can emit photons moving forwards (and not, say, backwards) in direction d and moving with nonzero speed. We note that the speed of the photon may be infinite. First we define $\mathbf{Bax}_0^-$ and after that some considerations will lead to $\mathbf{Bax}^-$ itself.

$$\mathbf{Bax}_0^- \stackrel{\text{def}}{=} \mathbf{Pax} \cup \{\mathbf{Ax5_{Ph}}, \mathbf{AxP1}, \mathbf{AxE_{01}}\},$$

where the new axioms are defined as follows.

AxP1 Intuitively, starting out from one point p of space-time, in every direction (forwards) there is at most one "speed of light" (i.e. photon-trace), formally:

$(\forall m \in Obs)(\forall ph_1, ph_2 \in Ph)(\forall d \in \mathsf{directions})$[237]

$\Big((ph_1$ and ph_2 are moving forwards in direction d as seen by m and

$tr_m(ph_1) \cap tr_m(ph_2) \neq \emptyset) \;\Rightarrow\; tr_m(ph_1) = tr_m(ph_2)\Big).$

$\mathbf{Ax5_{Ph}}$ Intuitively, from any point p of space-time in any direction there is a photon moving forwards in that direction, formally:

$$(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})(\exists ph \in Ph)$$
$$[p \in tr_m(ph) \;\wedge\; (ph \text{ is moving forwards in direction } d \text{ as seen by } m)].$$

$\mathbf{AxE_{01}}$ $v_m(ph) \neq 0$.

Having defined $\mathbf{Bax}_0^-$, let us briefly return to its intuitive content. $\mathbf{Bax}_0^-$ postulates that there is such a thing as the speed of light but this speed may be different for different observers, different in different directions and may be different at different points (of the observer's coordinate system). Therefore we can have a three variable function which represents the speed of light observed by m at point $p \in {}^nF$ in direction $d \in \mathsf{directions}$.

[235] Often this axiom is formulated by saying that the f_{mk}'s are affine transformations. Suppes [241] writes about this: "Every physics textbook on relativity makes a linearity assumption at the minimum." He also writes: "It is philosophically and empirically interesting that the Lorentz transformations can be derived without any extraneous assumptions of continuity or differentiability."

[236] In this work we do not explain the origin of the acronyms **Pax**, $\mathbf{Bax}^-$ etc. which we inherited from AMN [18].

[237] Let us recall that directions are (nonzero) space-vectors, i.e. $\mathsf{directions} = {}^{n-1}F \setminus \{\bar{0}\}$, cf. p.108.

Definition 3.2.10 Let $\mathfrak{M} \models \mathbf{Bax}_0^-$. Then the function

$$c : Obs \times {}^nF \times \mathsf{directions} \longrightarrow F^\infty$$

is defined as outlined above.[238] The formal definition is in AMN [18, p.535]. $\triangleleft$

Thus, $c(m, p, d)$ is the speed of light in m's world-view, at point p and direction d. We will write $c_m(p, d)$ in place of $c(m, p, d)$.

By the <u>*light-cone* $\mathsf{Cone}_{m,p}$ *starting at* p *as seen by* m</u> we understand

$$\mathsf{Cone}_{m,p} := \bigcup \{tr_m(ph) \ : \ ph \in Ph \ \& \ p \in tr_m(ph)\}.$$

By the above, $\mathbf{Bax}_0^-$ ensures that the light-cones defined above have some minimal "cone-like" property in some intuitive sense. (E.g. for any line ℓ containing p it becomes meaningful to ask whether ℓ is inside the light-cone $\mathsf{Cone}_{m,p}$.) Light-cones play an important role both in special and in general relativity. In models of $\mathbf{Bax}_0^-$, each observer m to each point $p \in {}^nF$ can associate a light-cone with tip at p and which can be arbitrarily deformed, cf. Figure 56.

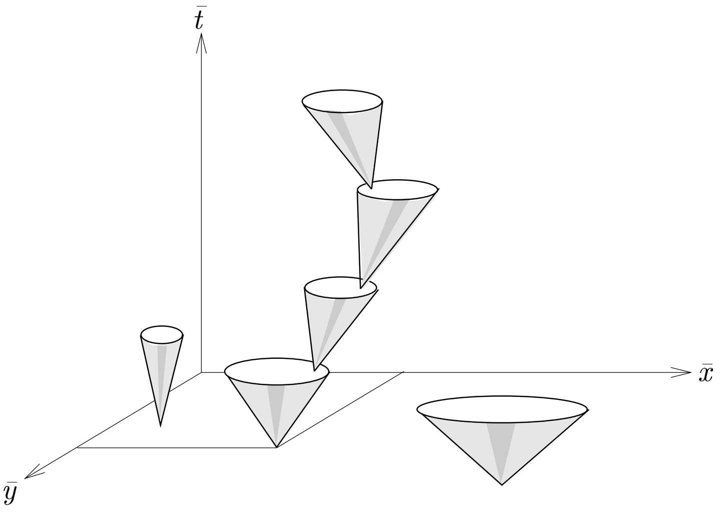


Figure 56: With every point p of ${}^nF$, we associate a so-called light-cone.

By saying that the light-cone may be arbitrarily deformed we mean to say that, say when $n = 3$, its intersection with a horizontal plane need not be a circle or even an ellipse. However, it can be regarded as a set of points which forms the "boundary" of a set of internal points, containing the point above p. AMN [18] contains a detailed discussion of the possible shapes of light-cones in models of $\mathbf{Bax}_0^-$, cf. e.g. pp. 473-517 (especially p.505) and pp. 507-508 therein. The feature of $\mathbf{Bax}_0^-$ that light-cones can be irregular etc. is an important one and its applications are discussed in detail in AMN [18, §§4.3, 4.4, 4.5]. In particular, we note that this feature is strongly used in discussions of general relativity (cf. e.g. D'Inverno [73], or Penrose [211][239]).

If we meditate over $\mathbf{Bax}_0^-$, an axiom of parsimony[240] suggests itself in a natural way. In **Pax** we had a "natural constant" (a kind of speed limit) which we called λ in $\mathbf{Ax5_{Obs}}^{--}$.[241] Namely,

[238] Recall from AMN [18, p.535] that $F^\infty = F \dot\cup \{\infty\}$.

[239] or cf. the discussion of the conformal structure in Ehlers-Pirani-Schield [78]

[240] a simplifying assumption

[241] Actually, instead of "speed limit", we should call λ something like a "speed permit", since it says that certain speeds are permitted (and not prohibited).

for each $m \in Obs$, $p \in {}^nF$, and $d \in \mathsf{directions}$ we postulated the existence of a $\lambda_{m,p,d} = \lambda$ such that all speeds below $\lambda_{m,p,d}$ can be "realized". Now we have two such constants, $\lambda_{m,p,d}$ and $c_m(p,d)$. In AMN [18, Figure 257 (p.762)], these two "data" are represented as two cones, a light-cone and a so-called "observer-cone".[242] The simplifying assumption that comes to one's mind at this point is that let us assume that these two constants coincide because we have no evidence for their being different, further because there seems to be no theoretical insight by keeping them different.[243] So the new axiom says $\lambda_{m,p,d} = c_m(p,d)$ for all m, p and d. Formally

$\mathbf{Ax5_{Obs}}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})c_m(p,d) = \lambda_{m,p,d}$.

Equivalently, $\mathbf{Ax5_{Obs}}$ says

$\mathbf{Ax5_{Obs}}$ $(\forall \rho < c_m(p,d))\Big[\rho \in {}^+F \Longrightarrow (\exists k \in Obs)$
m *observes* k *moving in direction* d *forwards,* $v_m(k) = \rho$, *and* $p \in tr_m(k)\Big]$.

Now, we can define $\mathbf{Bax}^-$.

$$\mathbf{Bax}^- \stackrel{\text{def}}{=} \mathbf{Bax}_0^- + \mathbf{Ax5_{Obs}}.$$

About the above axiom $\mathbf{Ax5_{Obs}}$ we note that it can be motivated both (i) as an axiom of aesthetics (i.e. of parsimony), and (ii) as an experimental axiom.[244] In the present we shall concentrate on the simpler $\mathbf{Bax}^-$ instead of $\mathbf{Bax}_0^-$. It remains an interesting future research task to see how much of our results about $\mathbf{Bax}^-$ generalize to $\mathbf{Bax}_0^-$. In particular, it would be interesting to see whether

$$(\mathbf{Bax}_0^- + c_m(p,d) < \infty) \vdash \nexists\mathrm{FTL}_{Obs}$$

is true, where the right-hand side is defined in Notation 3.2.12 below.[245]

Notation 3.2.11 $Th^{\oplus} \stackrel{\text{def}}{=} Th +$ *the speed of light is finite*, formally

$$Th^{\oplus} \stackrel{\text{def}}{=} Th + (v_m(ph) \neq \infty),$$

for any axiom system $Th \subseteq \mathsf{FF}$ in our frame language. Note that $v_m(ph) \neq \infty$ means that $v_m(ph)$ *exists* $\Rightarrow v_m(ph) < \infty$. $\lhd$

Now we are ready to formulate our no-FTL theorem stating that in models of $\mathbf{Bax}^{-\oplus}$ no observer can move faster than light. First we formalize the statement "no observer can move faster than light", and will denote it as $\nexists\mathrm{FTL}_{Obs}$.

[242] AMN [18] uses $\mathbf{Bax}^-_{\mathbf{nobs}}$ in place of our $\mathbf{Bax}_0^-$. However, our $\mathbf{Bax}_0^-$ is only very slightly weaker than $\mathbf{Bax}^-_{\mathbf{nobs}}$ in AMN [18], and here we may safely ignore the difference between the two.

[243] Later, if at some point of investigation we would find some use for distinguishing these constants, then we will withdraw our present simplifying assumption.

[244] Since (indirect) experimental evidence points in the direction that there does not seem to be a speed limit below the speed of light.

[245] The question remains interesting for $\mathbf{Bax}^-_{\mathbf{nobs}}$ introduced in AMN [18], too. Actually, it remains interesting even for ($\mathbf{Bax}^-_{\mathbf{nobs}}$ + *the* f_{mk}*'s are collineations*).

Notation 3.2.12 $\nexists$FTL*Obs* $\overset{\text{def}}{\Longleftrightarrow}$

[*m sees k and ph moving forwards in the same direction* $\Rightarrow v_m(k) < v_m(ph)$].

$\lhd$

In passing we note the following. If $dir_m(k)$ denotes the direction in which m sees k moving, then $\nexists$FTL*Obs* implies $[p \in tr_m(k) \Rightarrow v_m(k) < c_m(p, dir_m(k))]$. I.e. life-lines of observers stay inside the light-cones.

THEOREM 3.2.13 *Assume* $\mathbf{Ax}(\sqrt{\ })$ *and* $n > 2$. *Then*

(i) $\mathbf{Bax}^{-\oplus}$ $\models$ *"there are no faster than light observers", formally*
$\mathbf{Bax}^{-\oplus}$ $\models$ $\nexists$*FTLObs*, *and therefore*

(ii) $\mathbf{Bax}^{-\oplus}$ $\models$ *"there is a speed limit for moving observers, in some sense".*

(iii) $\mathbf{Bax}^{-\oplus}$ $\models$ *"velocities of observers do not add up the usual Newtonian way".*

Before proving the theorem we indicate that in certain ways it seems to be close to being "strongest possible". Namely, it becomes false if we replace $\mathbf{Bax}^{-\oplus}$ by $\mathbf{Bax}^{-}$ in it, cf. Thm.3.2.14 below. Further, in Thm.4.8.12 of AMN [18, p.651] we proved that if we replace $\mathbf{Bax}^{-}$ by its slightly different variant $\mathbf{Bax}(\mathbf{P1})$ then[246] our no-FTL theorem becomes false, i.e. there are FTL observers in some models of $\mathbf{Bax}(\mathbf{P1})^{\oplus}$. Cf. also Thm.4.4.14 (p.545) therein.

THEOREM 3.2.14 $\mathbf{Bax}^{-} + \mathbf{Ax}(\sqrt{\ }) \not\models \nexists$*FTLObs*, *for any n.*

Proof: This is proved as Thm.4.3.25 in AMN [18, p.500]. To save space we do not recall the proof. ∎

Proof of our no-FTL theorem 3.2.13: We give the proof for (i). The rest, (ii) and (iii), follow from (i). We will prove that if m sees an FTL observer, then m sees also a photon with infinite speed. Assume that k is an FTL observer in m's world-view, i.e. k's speed is greater than the speed of a photon ph going in the same direction d as k.

Let $p \in tr_m(k)$ and let ph' be a photon going through p and moving forwards in direction d. Such a photon exists by $\mathbf{Ax5_{Ph}}$, and then $tr_m(ph')$ is a straight line by $\mathbf{Ax3_0}$. Thm.4.3.17 on p.488 in AMN [18] states that $c_m(p,d)$ does not depend on p (but it may depend on m and d). Therefore, $v_m(ph') = v_m(ph)$, since both ph' and ph move in direction d.

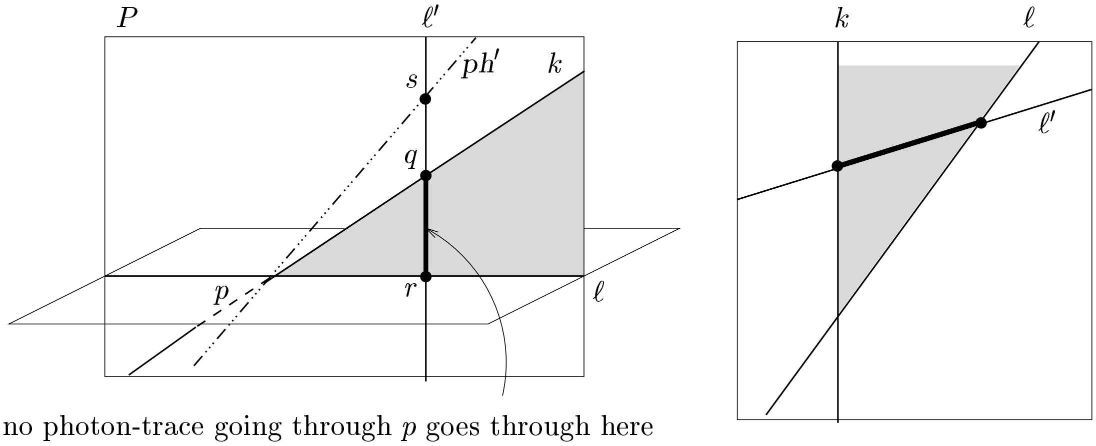
Let P denote the (2-dimensional) plane containing $tr_m(k)$ and parallel with $\bar{t}$.[247] Then P will contain $tr_m(ph')$ as well, because ph' goes in the same direction as k does. Let ℓ be the line in P which goes through p and for which $ang^2(\ell) = \infty$.[248] We are going to show that ℓ is the trace of an observer (in m's world-view).

Let $\ell' \subseteq P$ be a line parallel with $\bar{t}$ and not going through p. Let q and r denote the intersection points of ℓ' with $tr_m(k)$ and ℓ, respectively. There are such intersection points, because $v_m(ph') \neq 0$ by $\mathbf{AxE_{01}}$ and $v_m(k) \neq 0$ since k is a faster-than-light observer in m's world-view. See Figure 57. By $v_m(ph') = v_m(ph) < v_m(k)$ we have that the intersection

[246] $\mathbf{Bax}(\mathbf{P1})$ can be found in AMN [18, p.544] where it is investigated and motivated intuitively to a reasonable degree.

[247] Recall that $p \in tr_m(k) \in \mathsf{Eucl}(n, \mathbf{F})$ and k moves in direction $d \in {}^{n-1}F$. Then $P = \{p + a \cdot \langle \lambda, d_0, \ldots, d_{n-2} \rangle : a, \lambda \in F\}$.

[248] $\ell = \{p + \lambda \cdot \langle 0, d_0, \ldots, d_{n-2} \rangle : \lambda \in F\}$.



The plane P as seen by m

The plane P as seen by k

Figure 57: Illustration for the proof of Thm.3.2.13.

point s of $tr_m(ph')$ with ℓ' is not between q and r, where "s is between q and r" means that $s = q + \lambda \cdot (r - q)$ for some λ such that $0 < \lambda < 1$. [249] Cf. the definition of $\mathsf{Betw}(q, s, r)$ on p.140. By **AxP1**, there is only one photon-trace through p and going in direction d. Thus, there is no photon-trace in m's world-view going through p and intersecting ℓ' between q and r.

By $\mathbf{Bax}^- \models \mathbf{Pax}$ and Thm.3.2.6 we have that f_{mk} is a bijective collineation. By $\mathbf{Ax}(\sqrt{\ })$ then we have that f_{mk} is order-preserving.[250] This means that the image $\mathsf{f}_{mk}(u)$ of any point u between q and r lies between $\mathsf{f}_{mk}(q)$ and $\mathsf{f}_{mk}(r)$; and the same holds for f_{km}. Thus, since there is no photon-trace in m's world-view between $tr_m(k)$ and ℓ,

(*) there is no photon-trace in k's world-view between $\mathsf{f}_{mk}[tr_m(k)]$ and $\mathsf{f}_{mk}[\ell]$,

either. By $\mathbf{Ax5_{Ph}}$, there is a photon ph_1 going through $\mathsf{f}_{mk}(p)$ in the same direction as $\mathsf{f}_{mk}[\ell]$. By (*) then $ang^2(\ell) < ang^2(tr_k(ph_1)) = v_k(ph_1)$. By $\mathbf{Ax5_{Obs}}$ then there is an observer k' whose trace is $\mathsf{f}_{mk}[\ell]$. Then $tr_m(k') = \ell$. So far we have seen that ℓ is the trace of an observer k' in m's world-view.

We now use $n \geq 3$. Let P_1 be a 2-dimensional plane containing ℓ and such that for each line ℓ'' in P_1 we have $ang^2(\ell'') = \infty$. There is such a plane P_1 by $n \geq 3$.[251] See Figure 58.

Let P_1' be the image of P_1 under $\mathsf{f}_{mk'}$, i.e. $P_1' = \mathsf{f}_{mk'}[P_1]$. Then P_1' is a plane because $\mathsf{f}_{mk'}$ is a collineation (by Thm.3.2.6); and $\bar{t} = tr_{k'}(k') \subseteq P_1'$ by $tr_m(k') \subseteq P_1$. By $\mathbf{Ax5_{Ph}}$, in k''s world-view there is a photon ph'' which goes through $\mathsf{f}_{mk'}(p)$ and lies in P_1', i.e. $tr_{k'}(ph'') \subseteq P_1'$. But then $tr_m(ph'') \subseteq P_1$ which means that $v_m(ph'') = \infty$. ∎

[249] This is so because of the following. Since $tr_m(k)$ goes through p and in direction d, there is $\mu_k \in F$ such that $tr_m(k) = \{p + a \cdot \langle \mu_k, d_0, \ldots, d_{n-2}\rangle : a \in F\}$, and similarly, $tr_m(ph') = \{p + a \cdot \langle \mu_p, d_0, \ldots, d_{n-2}\rangle : a \in F\}$, for some μ_p. By $v_m(ph') < v_m(k)$ we have $\mu_p > \mu_k$. Let us choose ℓ' to be $\ell' = \{p + a \cdot \bar{d} : a \in F\}$, where $\bar{d} = \langle 0, d_0, \ldots, d_{n-2}\rangle$. Then $r = p + \bar{d}$, $q = p + \bar{d} + \langle \mu_k, 0, \ldots, 0\rangle$, and $s = p + \bar{d} + \langle \mu_p, 0, \ldots, 0\rangle$. Clearly, s is not between q and r.

[250] It is known that a bijective collineation is an affine mapping composed with a mapping $\widetilde{\varphi}$ induced by an automorphism φ of $\mathbf{F}$, cf. e.g. Lemma 3.1.6 in AMN [18]. A bijective affine mapping preserves betweenness. If square-roots exist in $\mathbf{F}$, then φ preserves order, and hence $\widetilde{\varphi}$ also preserves betweenness. Thus every bijective collineation preserves the betweenness relation.

[251] E.g. $P_1 = \{p + a \cdot \langle 0, \lambda, d_1, \ldots, d_{n-2}\rangle : a, \lambda \in F\}$ is such a plane.

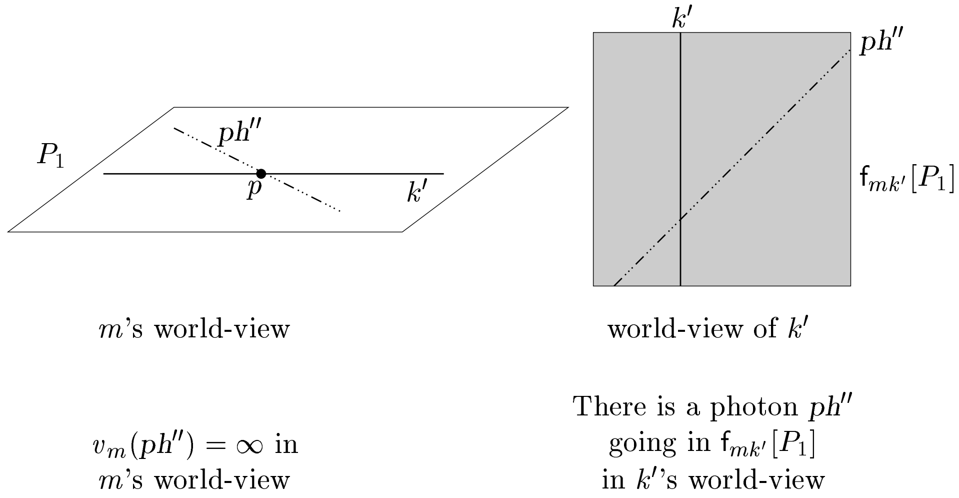


Figure 58: Illustration for the proof of Thm.3.2.13.

Discussion of our no-FTL theorem 3.2.13: The $\nexists$FTL*Obs* theorem is one of the exotic predictions of relativity, and part of the goals of the investigations in AMN [18] is to find an answer to the question "why" it is being predicted. As we explained earlier, this "why"-type question means finding the explicit or tacit assumption which is responsible for the prediction in question. The $\nexists$FTL*Obs* theorem is certainly not true in Newtonian mechanics. Most relativity books, even Reichenbach's carefully axiomatic [218], assume $\nexists$FTL*Obs* as an axiom. Our theorem above shows that assuming $\nexists$FTL*Obs* as an axiom is superfluous because it is a logical consequence of a small subset of the remaining axioms which are always assumed in all variants of special relativity. (In particular, these axioms, i.e. $\mathbf{Bax}^{-\oplus}$, are assumed in Reichenbach's book.) Further, the motivation usually given for assuming $\nexists$FTL*Obs* as an axiom (cf. e.g. Rakić [215, p.11]) are not really convincing after having studied e.g. Gödel's rotating universe which is a model of Einstein's equations for general relativity, cf. e.g. [270]. Namely, the standard argument says that FTL observers would lead to closed time-like loops which in turn are claimed to lead to *logical* paradoxes. But clearly, Gödel's universe contains closed time-like loops[252] and by its existence as a mathematical model of Einstein's axioms shows that no logical paradoxes are present. Our theorem shows that one does not need these arguments about alleged logical paradoxes[253] since the $\nexists$FTL*Obs* claim follows from the simple axioms in $\mathbf{Bax}^{-\oplus}$. Further, in AMN [18, §4.9] we show that even in the localized version $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$ of $\mathbf{Bax}^{-\oplus}$ one can prove the conclusion $\nexists$FTL*Obs* of our no-FTL theorem, cf. Thm.3.2.15 below. The importance of this generalization to $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$ is that the generalization $\mathbf{Bax}^{-\oplus} \mapsto \mathbf{Loc}(\mathbf{Bax}^{-\oplus})$ is a generalization in the direction of the theory of accelerated observers[254] and eventually of general relativity.

So we can safely conclude that $\nexists$FTL*Obs* is not needed as an axiom because it can be proved from other axioms which are assumed anyway and which are much easier to accept intuitively as axioms (i.e. which are much more convincing as axioms). But this is not the end of the story. Let us return to asking why $\nexists$FTL*Obs* is predicted, i.e. which axiom is responsible for it.

[252] Cf. Figure 134 on p.365.

[253] Leading logician David Lewis devoted a separate paper to showing that there is nothing paradoxical about time travel, cf. footnote 163 on p.72 herein. For a similar statement see footnote "a" in Gödel [99, p.199].

[254] Cf. e.g. AMN et al. [25], AMN et al. [26].

A possible first reaction to the question why we believe in $\nexists$FTL$_{Obs}$ is to say that the cause is the Michelson-Morley experiment which, summarized as **AxE**, implies $\nexists$FTL$_{Obs}$ for which conclusion only the innocent looking **Pax** is needed. However, proving $\mathbf{Bax}^{-\oplus} \models \nexists$FTL$_{Obs}$ is only the beginning of the analysis of "why $\nexists$FTL$_{Obs}$" conducted in AMN [18]. Namely, in AMN [18] two possible alternative theories are pointed out in which **AxE** and $n > 2$ are assumed but in which $\nexists$FTL$_{Obs}$ fails. One of them is **Relphax** on p.223 in §3.4.2 of AMN [18]. The other is based on noticing the tacit assumption in **Pax** that all observers use ${}^nF$ with the <u>same</u> number n for coordinatizing space-time. A way out of this might be if e.g. m coordinatizes space-time with using ${}^4F$ while a fast moving observer k uses only ${}^2F$ for coordinatizing the events he experiences. The details of this second alternative have not yet been elaborated, but it seems to yield a possible modification of **Basax** (with the full power of **AxE** assumed) which might allow FTL observers. Cf. Madarász-Németi [172, 173, 174] in this connection. We note that Gyula Dávid[255] reached similar conclusions (in the direction where n is not the same for all observers).

Let us turn to theories <u>*stronger*</u> than $\mathbf{Bax}^-$ (or $\mathbf{Bax}^{-\oplus}$). **Bax** is obtained from $\mathbf{Bax}^-$ by adding a kind of isotropy and homogenity as follows.

$$\mathbf{Bax} \stackrel{\text{def}}{=} \mathbf{Bax}^- + (c_m(p,d) = c_m(p',d')),$$

where $p, p' \in {}^nF$ and $d, d' \in \mathsf{directions}$ are universally quantified (in accordance with the usual convention concerning free variables).

A word of caution: In AMN [18], **Bax** is formulated slightly differently. We <u>*claim*</u> that **Bax** as formulated above is equivalent to **Bax** as formulated in AMN [18]. We omit the simple proof.

Bax-models are similar to disjoint unions (in some sense[256]) of **Basax** models with the only difference that the speed of light might be different for different observers. Therefore in **Bax** the notation $c_m = r \stackrel{\text{def}}{\Longleftrightarrow} (\exists p, d) c_m(p,d) = r$ is useful and natural. Namely, c_m is the speed of light for observer m. Let us notice that $c_m = \infty$ is still allowed.

A further natural axiom says that the speed of light is the same for everybody:

$$\mathbf{Flxbasax} \stackrel{\text{def}}{=} \mathbf{Bax} + (\forall m, k \in Obs) c_m = c_k.$$

Flxbasax is an important and natural theory in many ways. E.g. **Flxbasax** + **Ax6** is the natural common generalization of Newtonian kinematics and the relativity theory **Basax**. Similarly,

$$\mathbf{Flxbasax} + \mathbf{Ax6} + \mathbf{Ax(symm)}$$

is a <u>*natural common generalization*</u> of Newtonian kinematics and **Specrel**. Cf. AMN [18, §4.1] for the formalization of Newtonian kinematics we have in mind and Figure 123 on p.429 therein for the common generalization ideas mentioned above. These common generalization ideas are elaborated in greater detail and precision in the rest of §4.1 therein. We do not recall the details for lack of space.

The models of **Flxbasax** + **Ax6** can be obtained from **Basax** models by changing the speed of light which remains constant for the whole model. I.e. if $\mathfrak{M} \models \mathbf{Flxbasax} + \mathbf{Ax6}$ then $\mathfrak{M}$ can be obtained from some $\mathfrak{M}' \models \mathbf{Basax}$ by replacing $c = 1$ in $\mathfrak{M}'$ with $c = r$ for some

[255] Dept. of Gen. Physics, Eötvös Univ. Budapest.

[256] This sense is made precise in AMN [18, Thm.3.3.12 and Figure 65 (p.195)].

positive $r \in F^\infty$. Here for an **Flxbasax** model $\mathfrak{M}$, the constant c denotes the speed of light for all observers in $\mathfrak{M}$.

Very roughly speaking, **Flxbasax** models are nothing but disjoint unions of **Flxbasax** + **Ax6** models. We refer to Thm.3.3.12 (p.196) and Figure 65 (p.195) of AMN [18] for a precise formulation of the above claim.[257]

In many respects $(\mathbf{Flxbasax} + \mathbf{Ax6})^\oplus$ seems to be a more satisfactory formalization of special relativity without symmetry (than **Basax**).

$$\mathbf{Newbasax} \stackrel{\text{def}}{=} \mathbf{Flxbasax} + c = 1.$$

To our minds, there seems to be no essential difference between **Newbasax** and $\mathbf{Flxbasax}^\oplus$.

$$\mathbf{Newbasax} + \mathbf{Ax6} \mathrel{=\!\!|\!\!\models} \mathbf{Basax}.$$

Similarly to the case of **Flxbasax**, **Newbasax**-models are, basically, the same thing as disjoint unions of **Basax** models, in some sense. This claim is made precise in AMN [18, Thm.3.3.12 (p.196) and Figure 65 (p.195)].

It is interesting to notice that for $\mathbf{Bax} \subseteq Th \subseteq \mathbf{Basax}$ the difference of Th with **Ax6** or with $\mathbf{Ax6_{00}} + \mathbf{Ax6_{01}}$ can be semantically characterized by taking disjoint unions of models in the sense described in AMN [18, Thm.3.3.12 (p.196)]. It would be interesting to see whether analogous reasoning works in the case of $\mathbf{Bax}^-$ or **Pax**. Certainly it does not work for the localized theories $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$ or $\mathbf{Loc}(\mathbf{Basax})$ described in AMN [18, §4.9].

So far we have described our most important theories $\mathbf{Pax}, \ldots, \mathbf{Specrel}$ with two important omissions. These are the localized versions $\mathbf{Loc}(Th)$ and the Reichenbachian versions $\mathbf{Reich}(Th)$ of $Th \in \{\mathbf{Pax}, \ldots, \mathbf{Specrel}\}$. Recall that we already introduced an operator $Th \mapsto Th^\oplus$ which to any one Th of our theories produces a new one, $Th^\oplus$. Now, $\mathbf{Loc}(-)$ and $\mathbf{Reich}(-)$ will be similar operators.

The idea of $\mathbf{Loc}(-)$ is the following. Our theories studied so far assume that the observers use the whole of ${}^nF$ for coordinatizing events. I.e. $w_m : {}^nF \longrightarrow \mathcal{P}(B)$. Motivated by accelerated observers and general relativity, we want to replace this by a more modest version where observers use only subsets of ${}^nF$ for coordinatizing events. It is reasonable to assume that these subsets are open and connected, but we do not want to assume that they are the whole of ${}^nF$. So, with any $m \in Obs$ we want to associate a domain $Do^-(m) \subseteq {}^nF$, and we want m to use only this domain for coordinatizing events, i.e. we want something like $w_m : Do^-(m) \longrightarrow \mathcal{P}(B)$. To avoid inconsistent notation, instead of changing w_m we introduce a new world-view function $w_m^- \stackrel{\text{def}}{=} \{\langle p, e\rangle \in w_m : e \neq \emptyset\}$. Then the domain of m will be $Dom(w_m^-) \subseteq {}^nF$.

Interestingly, $\mathbf{Pax} \models Dom(w_m^-) = {}^nF$. So, we ask ourselves, which are the axioms which "pump up" the domains to be so big. One of them is **Ax4** saying $tr_m(m) = \bar{t}$. Clearly, $tr_m(m) \subseteq Dom(w_m^-)$, so **Ax4** forces $Dom(w_m^-)$ to be infinitely long (time-wise). A similar culprit is **Ax3**, which forces $Dom(w_m^-) \supseteq tr_m(b) \in G$ to be infinite again, but now not only along the time-axis. This forces infinity of $Dom(w_m^-)$ along the photon-lines too, which in **Basax** are in 45° relative to $\bar{t}$. $\mathbf{Ax3_0}$ is not better either.

To define $\mathbf{Loc}(Th)$ we do the following. We replace **Ax4** by its relativization to the domain of w_m^-, i.e. with $tr_m(m) = \bar{t} \cap Dom(w_m^-)$. This latter formula is $\mathbf{Ax4}^{\text{par}}$. Similarly, $\mathbf{Ax3}^{\text{par}}$ says $tr_m(b) = h \cap Dom(w_m^-)$ for some $h \in G$. The rest of the localization of a theory is analogous, we look at those axioms which can "pump up" the size of $Dom(w_m^-)$ and then reformulate

[257] In the quoted theorem **Newbasax** is used instead of **Flxbasax** but this distinction will be clarified soon.

them by "relativizing" to $Dom(w_m^-)$ as we did in the case of **Ax4**, **Ax3** ensuring that they retain their original meaning but in such a way that they no longer "pump up" the size of domains. We leave the details to the reader, but we note that they can be found in AMN [18, §4.9]; see also the List of axioms herein.

After this process of relativizing the axioms, we look at the undesirable side-effects of the changes we made. Then we add axioms to eliminate these side-effects. An example of such an axiom is $\mathbf{Ax(syBw)}^{\mathrm{par}}$, which says that the f_{mk}'s are betweenness preserving both backwards and forwards. For a detailed definition of the axioms and notation used in Thm.3.2.15 below we refer to the List of axioms, p.A-27.

THEOREM 3.2.15 *Assume $n > 2$.*
$\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(syBw)}^{\mathrm{par}} \models$ *"there are no FTL observers".*[258]

Proof: We indicate only some of the ideas involved. First of all, we note that the proof method of Thm.3.2.13 presented way above is relevant here, too. Assume the hypotheses. Assume there is an FTL observer. This means that there are $m, k \in Obs$ and a photon ph_1 such that m thinks that ph_1 and k are moving in the same direction and k is faster than ph_1. Further, ph_1 and k meet. Figure 59 shows the idea of deriving from this the conclusion that $v_m(k') = \infty$ for some observer k'. From this we derive a contradiction. This last part of the proof uses a lemma of geometric nature which the present author proved as Lemma 4.9.16 in AMN [18]. For lack of space we do not recall that lemma or the remaining part of the proof. For a detailed proof of the present theorem we refer to AMN [18, §4.9, Thm.4.9.14 (pp.687–693)]. ∎

We stated Theorem 3.2.15 – though we do not have space for its precise formulation and proof – because we think it is an important generalization of Thm.3.2.13, so we regard it as a result of the present work which has to be mentioned.

Besides $\mathbf{Loc}(-)$, we promised a third operator $\mathbf{Reich}(-)$ called Reichenbachization of our theories. Reichenbach started out from the idea that the Michelson-Morley experiment measured only the two-way speed of light and not the one-way speed of light. (The two-way speed is characterized by the time needed to go from a source to a mirror *and* come back.) As it is summarized e.g. in Szabó [244], ever since Reichenbach discovered this, nobody could come up even with a thought experiment which would determine the one-way speed of light. Therefore, using Occam's razor, followers of Reichenbach suggested to treat the one-way speed of light as "unknowable" (analogously to the aether) and replace throughout the axioms of our theory the one-way speed of light by the two-way speed of light. E.g. instead of saying that the speed of light is the same in all directions (e.g. as in **Bax**) we should say that the two-way speed of light is the same in all directions. If we start out with **Basax**, then the resulting theory is called $\mathbf{Reich}(\mathbf{Basax})$. More generally, if we start out with $Th \in \{\mathbf{Bax}^-, \ldots, \mathbf{Basax}\}$, then the resulting theory is denoted as $\mathbf{Reich}(Th)$. We hope that the idea is clear enough, the detailed mathematical definition of $\mathbf{Reich}(Th)$ is in AMN [18, §4.5 (pp. 553-600)]. Also the properties of $\mathbf{Reich}(-)$ are studied there, e.g. there we prove a transfer theorem by which the models of $\mathbf{Reich}(Th)$ are obtainable from $\mathsf{Mod}(Th)$ by a simple model theoretic construction called relativization. We will occasionally use $\mathbf{Reich}(-)$ and the ideas behind it in our next chapter on Geometry. So the reader is asked to contemplate $\mathbf{Reich}(-)$ a little bit at this point.

[258] This is understood only locally, in the following sense. We call k *locally FTL* w.r.t. m iff $(\exists ph \in Ph)[tr_m(ph) \cap tr_m(k) \neq \emptyset$ *and m sees k and ph move forwards in the same direction and* $v_m(k) \geq v_m(ph)]$. Now, the conclusion of our theorem says $(\forall m, k \in Obs)$ *k is not locally FTL w.r.t. m.*

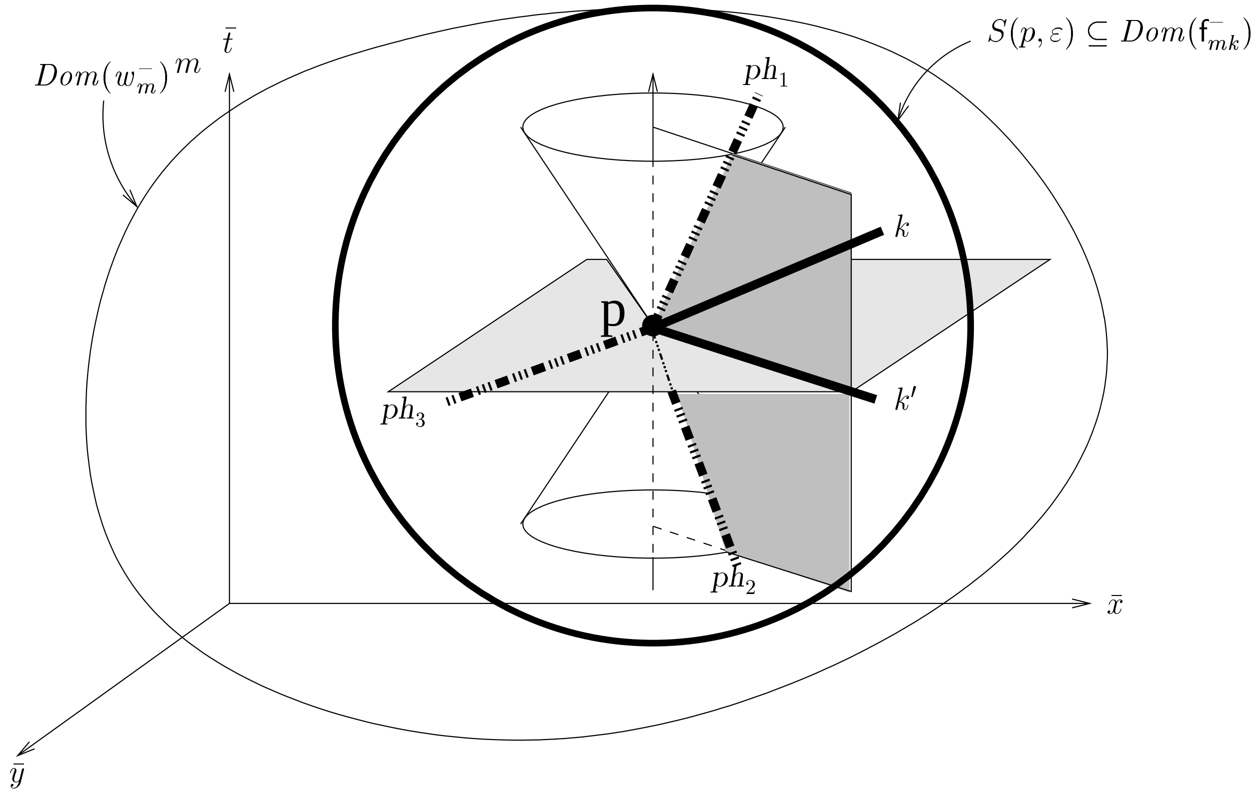


Figure 59: Illustration of the main idea of the proof of Thm. 3.2.15.

Finally, we introduce three axioms, **Ax♡**, **Ax(ext)**, **Ax(↑↑)** and an axiom system **BaCo**. These will come up in the rest of this work, occasionally.

Ax♡ says, intuitively, that we want to deal with "only the heart" of relativity theory in the sense of p.vii herein and of AMN [18, item (IV) (p.8) of the Introduction]. To achieve this effect, this axiom excludes all bodies except photons and observers. Formally

(**Ax♡**) $B = Obs \cup Ph.$ [259]

Ax(ext) is analogous to the axiom of extensionality of set theory. It says, roughly, that if all "properties" of two things (of sort B) coincide, then they must be the same.[260]

$$\mathbf{Ax(ext)} \quad (\forall m, k \in Obs)\,[w_m = w_k \;\Rightarrow\; m = k] \quad \wedge$$
$$(\forall b, b_1 \in B \setminus Obs)(\forall m \in Obs)\,[tr_m(b) = tr_m(b_1) \;\Rightarrow\; b = b_1].$$

Ax(↑↑) can be interpreted as saying (something like) that there is such a thing as the "general direction of the flow of time"[261]. More concretely, it says that every observer sees the clock of every other observer running forward. This will be formalized on p.176.

The name **BaCo** refers to the fact that this is an extension of **Basax** aiming for completeness.

$$\mathbf{BaCo} \stackrel{\text{def}}{=} \mathbf{Specrel} + \mathbf{Ax♡} + \mathbf{Ax(ext)} + \mathbf{Ax(\uparrow\uparrow)}.$$ [262]

We note that **BaCo** $\models$ **Specrel** + SPR$^+$ + *"all the other symmetry axioms discussed in this work"*.[263] Let $\mathfrak{F}$ be an arbitrary Euclidean field. Then (i) **BaCo** + $\mathsf{Th}(\mathfrak{F})$ is a complete theory (deciding all the formulas of our frame language). Further, (ii) **BaCo** + $\mathsf{Th}(\mathfrak{F})$ is categorical over $\mathfrak{F}$ in the following sense: **BaCo** admits exactly one model $\mathfrak{M}$ with $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{F}$, up to isomorphism. Further, this $\mathfrak{M}$ is *the* standard Minkowski model over $\mathfrak{F}$. Claims (i),(ii) above are proved in AMN [18]. These are also theorems in AMN [16].

In Figure 60 we summarize the definitions of the theories introduced in this chapter.

[259] $(\forall b \in B) Obs(b) \vee Ph(b)$

[260] This can be related to Leibniz's principle of indistinguishables.

[261] This principle is violated, at least in a sense, e.g. by Gödel's model for general relativity represented on Figure 134 (p.365) at the end of our section of geodesics. Cf. also §2.7 ("FTL") for violation of **Ax(↑↑)**.

[262] This definition of **BaCo** is equivalent to the one given in AMN [18].

[263] This is proved in AMN [18].

BaCo

simplifying axioms ← $\begin{cases} B = Obs \cup Ph & (\mathbf{Ax\heartsuit}) \\ \text{extensionality} & (\mathbf{Ax(ext)}) \\ \text{time flows forwards} & (\mathbf{Ax(\uparrow\uparrow)}) \end{cases}$

Specrel

← $\begin{cases} \text{Symmetry axiom} & (\mathbf{Ax(symm_0)}) \\ \text{co-moving observers measure time the same way} & (\mathbf{Ax(\|)}) \\ \text{observers can “rotate” their coordinate system} & (\mathbf{Ax(}\mathit{Triv}\mathbf{)}) \end{cases}$

Basax

← $\begin{cases} \text{Each observer can see the same events} & (\mathbf{Ax6}) \\ \text{(or each observer can see any other observer, } m \overset{\odot}{\to} k\text{)} \end{cases}$

Newbasax

← $c_m(p, d) = 1$ (**AxE**)

Flxbasax

“observers see the world the same way” (Specrel, Basax, Newbasax, Flxbasax)

← $\begin{cases} \text{The speed of light is the same} \\ \text{for any observer} \end{cases}$ $c_m(p, d) = c_{m'}(p, d)$

Bax

isotropy ← $\begin{cases} \text{The speed of light is the same} \\ \text{in each direction} \end{cases}$ $c_m(p, d) = c_m(p, d')$

Bax$^-$

light-cones exist ← $\begin{cases} \begin{cases} \text{Photons do not race with eachother} & (\mathbf{AxP1}) \\ \text{photons move forwards} \end{cases} & (\mathbf{Ax5_{Ph}} + \mathbf{AxE_{01}}) \\ \text{speed of light } c_m(p, d) \text{ exists} \end{cases}$

organizational axioms **Pax**

Figure 60: The hierarchy (lattice) of the axiom systems defined so far. We start out from the bottom (**Pax**) and obtain the next axiom system via adding the axioms listed on the right hand side and assigned to the vertical edge by an arrow. E.g. **Bax**$^-$ = **Pax** + (**Ax5**$_{\mathbf{Ph}}$ + **AxE**$_{\mathbf{01}}$) + **AxP1**.

4 Observer-independent geometry

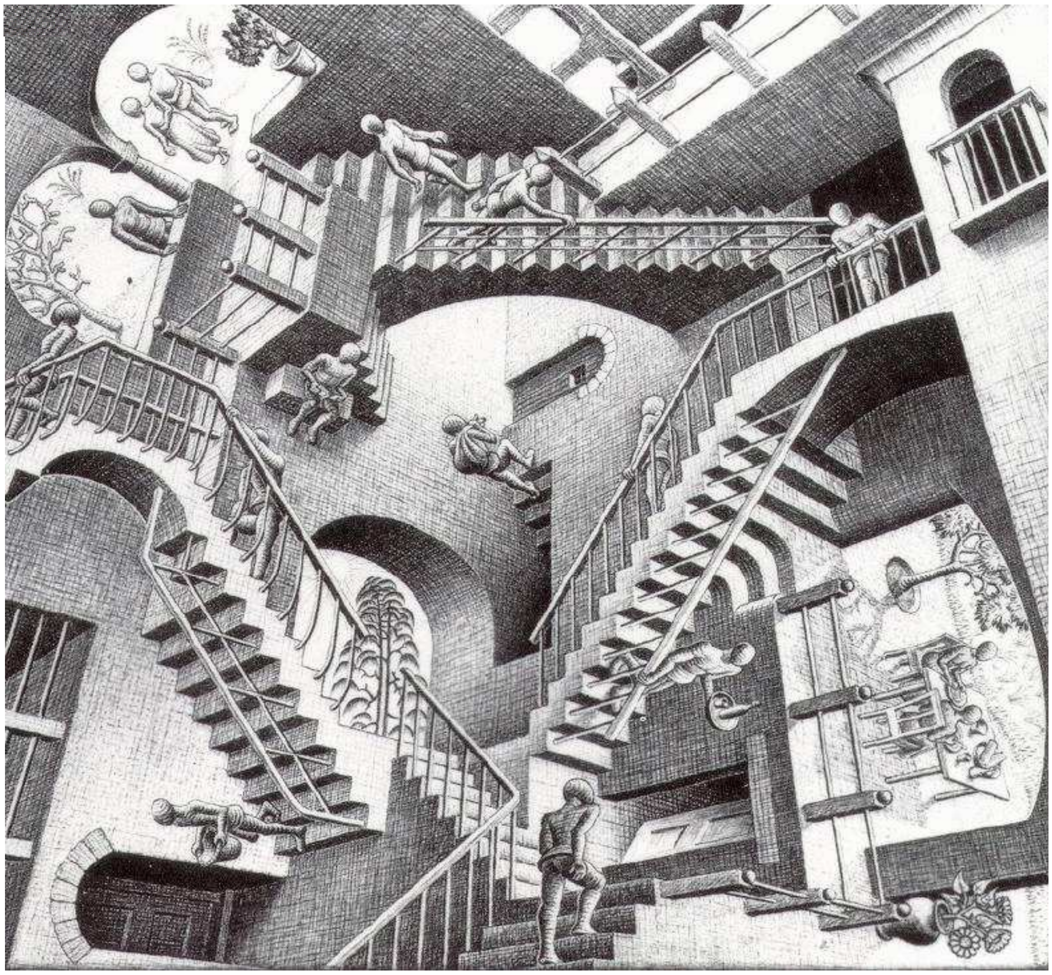


Figure 61: Escher's litograph *Relativity*.

A note to the reader. In this work, but especially in the introductory sections like §4.1, the footnotes contain more important information than the main text. Therefore, especially during reading §4.1 we recommend that the reader pay more attention to the footnotes than to the main text. We know that this is a highly unusual arrangement, but the logic of the situation forced this arrangement upon us. Namely, this way a reader who has little time can read the main text without looking at the footnotes and he will get a coherent story about what we want to say. But this is *at the expense* of not learning about the relevant ideas of Kant, Mach, Einstein, Reichenbach, Gödel etc. which, in our opinion, make the subject interesting. If the reader wants to read about these ideas (and about why we do things the way we do) he can find them in the footnotes. We are afraid, if we lifted these interesting ideas from the footnotes into the main text, the main text would lose its coherence (too much detours, rumblings etc). A pleasant way of reading §4.1 might be the following: First read through the main text only (to get a general impression) and then read the main text together with the footnotes paying more attention to the footnotes than to the main text. The same philosophy applies to the whole of this work (Chapters 1 - 4) and not only to §4.1, but to a less pronounced extent.

4.1 Introduction (to the present chapter on geometries)

In this chapter we will see how observer-independent structures can be found in our frame models of relativity theory, i.e. we will show that there is an *observer-independent* "geometric" structure $\mathfrak{G}_{\mathfrak{M}}$ inside every model $\mathfrak{M}$ of our frame language. We will consider the observer independent geometric structures $\mathfrak{G}_{\mathfrak{M}}$ (associated with "observer-oriented" models $\mathfrak{M}$) as representing so-called "theoretical" concepts, while we will consider the original $\mathfrak{M}$'s as representing so-called more "observational" concepts. Here, the expressions "observational" and "theoretical" are technical terms explained and used in the relativity books Reichenbach [218], Friedman [91][264], cf. also §1.1(IX) on p.11 of AMN [18] for a brief explanation and motivation for the observational/theoretical distinction.

The key idea is that in some situations or at some level of the development of our scientific theories, certain concepts can be considered more observational while others can be regarded as being more on the theoretical side. For a more careful description of this distinction (and its justification etc.) we refer to Reichenbach [218]. We are aware of the fact that the observational/theoretical distinction is not absolute[265], it may change during the development of our scientific theories, etc. but, as Friedman [91] writes on p.4 and on p.31, if we are aware of its limitations and its "tentativeness", it can be used rather fruitfully.[266]

Next we discuss the role of theoretical/observational concepts in scientific theories. The methodology and ideas we are going to sketch below originate from an intensive and fruitful interaction between the originators of relativity theory, e.g. Einstein on the one part, and the logical positivists[267], e.g. Reichenbach, on the other. The key ideas go back to Kant[268] (1724-1804), Leibniz (1646–1716) and Occam (1295–1349).[269] Cf. e.g. Friedman [91, §I (pp.3-31)]. (These ideas are also strongly related to ontology, i.e. to the field of research studying the

[264] The words "observational" etc. come from Friedman [91]. Reichenbach used other expressions with basically the same meaning. E.g. he writes about theory formation: "... it is advantageous to approach the axiomatization in a different fashion. It is possible to start with the observable facts and to end with the abstract conceptualization", cf. [218, pp. 4–5]. Later he writes "... start an axiomatization with so-called empirical facts", also "... this investigation starts with elementary facts as axioms...", cf. [218, p.8]. Cf. also p.174 in Loose [158].

[265] e.g. a concept which is observational in one situation may appear as theoretical in another situation

[266] Actually, Friedman [91, p.24, first 30 lines] writes that the birth of the modern form of the observational/theoretical distinction can be credited to Einstein's fundamental 1916 paper [81, p.117].

[267] We are neither supporting nor attacking positivism, we simply want to use those of their ideas which proved useful while avoiding their mistakes e.g. oversimplification.

[268] For the (positive) role of Kant cf. e.g. Friedman [91] p.7 lines 8–25 and p.18 lines 14–20.

[269] E.g. we mention Leibniz's principle of identity of indistinguishable concepts, and what became popularly known as Occam's razor, c.f. e.g. Friedman [91], Hodges [130, pp. 9, 21]. Roughly, Occam's razor says: do not assume the existence of unnecessary theoretical entities. In passing we note that Leibniz's principle appears as axiom C_7 in algebraic logic (called there Leibniz rule) cf. Henkin-Monk-Tarski [120, Part I, p.172] and Andréka et al. [31]. C_7 is the algebraic counterpart of an axiom of first-order logic. (William Occam was a 14-th century English logician. His razor is usually summarized as "Do not assume the existence of more entities than you have to".)

question of which of our theoretical entities exist and in what sense they exist.)[270]

The methodology (of the above origin) is the following:[272] Assume we want to study a part (or aspect) of the physical world. Then first we build models, like $\mathfrak{M}$ in the present work, which involve observational concepts only. I.e. we try to keep the "ingredients" of $\mathfrak{M}$ to be on the observational side as much as possible. Then we study $\mathfrak{M}$ and develop a theory Th in the language of $\mathfrak{M}$ with $\mathfrak{M} \in \mathsf{Mod}(Th)$. After having studied Th and $\mathsf{Mod}(Th)$ long enough, we begin to see what kind of new, theoretical, concepts would be useful for understanding Th, $\mathfrak{M}$ etc. even better. The methodology of introducing such new theoretical concepts is the following.

By the principles of parsimony[273] (i.e. refinements of Occam's razor), we require the new, theoretical concepts to be definable by means of first-order logic, over $\mathfrak{M}$ (or more generally over $\mathsf{Mod}(Th)$). Cf. §4.3 way below for the theory of definability. (The importance of definability is emphasized in the relativity book Reichenbach [218] e.g. on p.3, pp.7-13.) Then we expand our observational model $\mathfrak{M}$ with the defined concepts obtaining something like $\mathfrak{M}^+ = \langle \mathfrak{M}, \text{defined concepts} \rangle$ in the hope that the theory of $\mathfrak{M}^+$ will be more "streamlined", more elegant and more illuminating (than that of $\mathfrak{M}$) in various ways. Indeed, in the present chapter, we will define a streamlined theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ over the model $\mathfrak{M}$, and we will call $\mathfrak{G}_{\mathfrak{M}}$ the "observer independent geometry" associated with $\mathfrak{M}$.[274] First we will identify what desirable theoretical entities we would like to put into $\mathfrak{G}_{\mathfrak{M}}$, and then comes the "hard work" of checking whether these new entities are indeed first-order logic definable over $\mathfrak{M}$, cf. §§ 4.2.2, 4.3, and Theorems 4.3.22–4.3.25 (p.244) for the definability investigations, while considerations on what should go into $\mathfrak{G}_{\mathfrak{M}}$ are in §4.2.3 (but cf. also §§ 4.2.1–4.2.5).

Having defined, over $\mathfrak{M}$, our structure $\mathfrak{G}_{\mathfrak{M}}$ of theoretical entities, we expand our observational structure $\mathfrak{M}$ with these theoretical entities obtaining the richer structure $\mathfrak{M}^+ = \langle \mathfrak{M}, \mathfrak{G}_{\mathfrak{M}} \rangle$. Our theoretically enriched structure $\mathfrak{M}^+$ corresponds to the structure $\mathcal{A}$ in Friedman [91, p.236] while our observational $\mathfrak{M}$ corresponds to the sub-reduct $\mathcal{B}$ of $\mathcal{A}$ on the same page in [91]. In the language of our enriched structure $\mathfrak{M}^+$ we have both theoretical and observational concepts, so we could go on indefinitely studying the theory of our $\mathfrak{M}^+$. However, this is not what we do, because, so to speak, we become greedy to improve our language and our concepts. Namely, *if we are lucky*, we will find that (not only $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{M}$ but) also our observational structure $\mathfrak{M}$ is definable over the theoretical $\mathfrak{G}_{\mathfrak{M}}$. *If* this is the case, we may forget our original observational structure $\mathfrak{M}$, and may stick with the more streamlined

[270] It is of interest to note how much philosophy influenced the development of relativity. E.g. Mach's philosophy influenced Einstein in developing general relativity, cf. e.g. Barbour [40] or Friedman [91]. Further, Gödel proved very interesting things about the so obtained general relativity. Gödel's main motivation came from Kant's philosophy, he wanted to justify Kant's views on the nature of time.[271] Gödel's results lead up to one of the most exciting parts of modern relativity, namely, to the theory of rotating black holes (closed time-like curves, i.e. "time travel"), at least in some sense. Further, (in a different direction) Gödel's results show that Einstein's equations do not imply Mach's principle, after all (for seeing this in full form one uses Ozsvath's and Schücking's 1969 paper [209]), cf. also Friedman [91, pp. 209–211]. This does not prove that Mach's principle would not be true, instead it proves only that it is not implied by Einstein's axioms for general relativity. Cf. e.g. Gödel's collected works [99, pp. 189–217], and [100, pp. 202–289], and Dawson [70]. Cf. also footnote 281 on p.132. See Figure 134 (p.365) for a visual representation of Gödel's model satisfying Einstein's equations but not Mach's principle. This model usually is called Gödel's rotating universe.

[271] In this connection we recommend that the reader read footnote 477 on p.214.

[272] We present it, here, only in a simplified form.

[273] "principle of parsimony"= "economy of explanation", cf. footnote 269

[274] At the beginning of this chapter it will not be very obvious why we think that $\mathfrak{G}_{\mathfrak{M}}$ is much more streamlined than $\mathfrak{M}$, but around the end of this chapter, in §4.6, we will see that $\mathfrak{G}_{\mathfrak{M}}$ admits rather streamlined reformulations. Cf. e.g. Theorems 4.6.2,4.6.3 (pp.346,347) and AMN [18, Theorems 6.7.20 (p.1157), 6.7.30 (p.1164), 6.7.37 (p.1166)].

and elegant theoretical structure $\mathfrak{G}_{\mathfrak{M}}$.[275] (The reason for this is that if $\mathfrak{M}$ is definable over $\mathfrak{G}_{\mathfrak{M}}$, in a rather concrete sense $\mathfrak{M}$ is "present" [or available] in $\mathfrak{G}_{\mathfrak{M}}$, e.g. all questions about $\mathfrak{M}$ can be translated into questions about $\mathfrak{G}_{\mathfrak{M}}$.) Very probably, if we are permitted to concentrate on $\mathfrak{G}_{\mathfrak{M}}$ and to forget about $\mathfrak{M}$, our investigations of the theory of $\mathfrak{G}_{\mathfrak{M}}$ will be more efficient, we will be able to reach deeper insights in a shorter time etc. Motivated by these considerations, in the present work we will prove various results to the effect that the observational "world" $\mathfrak{M}$ is indeed first-order-logic-definable over the theoretical world $\mathfrak{G}_{\mathfrak{M}}$. This will be one of the main themes of §4.5.

We will extend these definability results from individual models to axiomatizable classes of models. E.g. if $\mathsf{Mod}(Th)$ is an axiomatizable class of observational models, we will write $\mathsf{Ge}(Th)$ for the corresponding class of theoretical models, i.e. the corresponding class of geometries. Then we will prove that $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$, and in the other direction, $\mathsf{Mod}(Th)$, too, is definable over $\mathsf{Ge}(Th)$.

Actually, we will do more than this in two respects:

(i) We will prove that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are <u>definitionally equivalent</u>[276] which in some sense means that they are different "linguistic representations" of the same theory. (Cf. Thm.4.3.38 on p.261.[277])

(ii) We will also elaborate a so-called <u>duality</u> theory between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ which is analogous with the various duality theories (adjoint situations, etc.) playing important roles all over mathematics.[278] We will make the connections with several distinguished duality theories explicit in Appendix A (§§ A.1–A.3), cf. also pp. 293–296, pp. A-15–A-18. This subject is discussed in more detail in AMN [18, §§6.6.5–6.6.7], Remark 6.6.4 (p.1014) and on pp.1096–1107 therein, and also in Madarász et al. [171]. One of the uses of these duality theories is that they establish strong connections between seemingly distant parts of mathematics, and they help us to solve problems in one area by using the methods of another, completely different, area (where the solution to this particular problem might be drastically easier). The above quoted parts[279] make the unity between the author's earlier papers, e.g. Madarász [161, 165, 164, 166, 170, 163], Madarász et al. [177, 176, 23, 20] and the present work explicit.

In §4.6 we will use the methods of definability theory for streamlining our "theoretical structure" $\mathfrak{G}_{\mathfrak{M}}$ in the spirit outlined way above. Since definability theory plays such a central role in our investigations (as well as in other parts of relativity, cf. e.g. Reichenbach [218], Friedman [91]), we devoted §4.3 to recalling and further elaborating this theory.

<u>*Potential laws of nature, characterization of symmetry principles:*</u>

Our theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ can also be used in identifying potential laws of nature and in characterization of symmetry principles, as follows. Recall from the introduction of §3.9 of AMN [18] and from §2.8 herein that some of our axioms like **Ax(symm)** were called symmetry principles and were regarded as special instances of Einstein's SPR. In §2.8 and in AMN [18, §3.9] we experimented with giving logical or model-theoretic characterizations for

[275] In the above sentence we want to refer to a kind of "tension" which regards $\mathfrak{M}$ as being too close "to the original thing being modeled", detail-oriented or "mosaic-like" or coordinate-systems-oriented while $\mathfrak{G}_{\mathfrak{M}}$ is regarded to be more "whole-oriented" or more "essence-oriented". Cf. item (7) on p.185.

[276] Cf. §4.3 (p.255) for definitional equivalence.

[277] Cf. also Remark 4.3.37 on p.258 and the intuitive text above that remark.

[278] This duality theory works under weaker conditions needed for (i) above. (Note that definitional equivalence between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ automatically implies a very strong form of duality. [Actually what we will call weak definitional equivalence is sufficient for this.] However, duality in general does not imply definitional equivalence.)

[279] e.g. Appendix A herein

Einstein's SPR and for symmetry principles. Cf. Thm.2.8.20 and the first theorem in §3.9 of AMN [18] which is based on Def.3.8.2 (p.298) of AMN [18]. The intuitive idea was, roughly, that Einstein's SPR says that inertial observers cannot be distinguished from each other by laws of nature. (An equivalent formulation says that the same laws of nature hold for m and k if m, k are inertial observers.) So if $\varphi(x)$ is a potential law of nature with x ranging over observers, then $\mathfrak{M} \models$ "SPR" $\Rightarrow$ $[\,\mathfrak{M} \models \varphi(m) \leftrightarrow \varphi(k),$ for all inertial observers m and k of $\mathfrak{M}\,]$. Moreover, $\mathfrak{M} \models$ "SPR" iff we have $[(\forall m, k \in Obs \cap Ib)\mathfrak{M} \models \varphi(m) \leftrightarrow \varphi(k),$ for all potential laws $\varphi(m)]$. The problem with carrying this programme through was that we did not know which formulas of our frame language $Fm(\mathfrak{M})$ count as potential laws of nature and which formulas are of an "accidental" (or contingent) character (e.g. making some random statement about the state of affairs on the life-line of m, say, at the event where m sees the origin). For the distinction between "accidental" statements and potential laws cf. e.g. the entry "lawlike generalization" in the Cambridge Dictionary of Philosophy [35]. So, the problem was to provide a logical or model theoretic distinction between those formulas in $Fm(\mathfrak{M})$ which are regarded as potential laws from those formulas which count only as potential "accidental facts"[280]. We can use our theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ for distinguishing those elements of $Fm(\mathfrak{M})$ which are closer to being potential laws, and also we can use $\mathfrak{G}_{\mathfrak{M}}$ for characterizing Einstein's SPR and **Ax(symm)** in a model theoretical way. We will not go into this issue in the present work, but we investigated this on pp. 74–91 herein and in §6.1, §6.6.8 in AMN [18]. However, cf. AMN [18, §6.2.8] for related (but different) results.

Let us return to explaining in what sense we regard $\mathfrak{G}_{\mathfrak{M}}$ as an observer independent geometry (sitting in $\mathfrak{M}$). Originally, in the Newtonian world-view, there was a common "outside reality" for all observers. In our models $\mathfrak{M}$, each observer has a "kind of private world", namely, his world-view (determined by $w_m : {}^nF \longrightarrow \mathcal{P}(B)$). The f_{mk} transformations tell us how these worlds are connected. However, they do not tell us which of these worlds is the "real one". Moreover, by Einstein's SPR these worlds are of equal status. (Of course, one can live with this arrangement forever, there is nothing wrong with it.) The question comes up naturally: Can one find a single "monolithic" (or "fundamental") reality behind all these "pluralistic" personal worlds? If the answer is yes, we could call this "monolithic" reality the outside reality (behind our experience). All the personalized worlds (i.e. the world-views) could be regarded as different projections of this single outside reality. This situation is analogous with those described by descriptive geometry, where we have a spatial body (or figure) which has a "front-view", a "side-view", etc., i.e. which can be viewed from all possible angles or directions. That spatial figure corresponds to our observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$ while the views (or projections) of that body from possible directions correspond to the "personalized" world-views of our observers $m \in Obs^{\mathfrak{M}}$. Cf. Fig.62 for descriptive geometry put into analogy with the connections between our single observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ and the many personalized worlds, i.e. the w_m's in $\mathfrak{M}$.

Our observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$ is intended to serve as such a monolithic outside reality. Indeed, in our duality-theory section (§4.5) we will see that the different personalized world-views (of form $w_m : {}^nF \longrightarrow \mathcal{P}(B)$) can be recovered from the single geometry $\mathfrak{G}_{\mathfrak{M}}$, cf. e.g. the definition of functor $\mathcal{M}$ on p.310.[281]

[280] Like, "the number of non-inertial bodies present at the origin is smaller than that at coordinates $\langle 1, 0, 0, 0\rangle$".

[281] There seems to be an analogy here with Kantian philosophy: Namely, $\mathfrak{G}_{\mathfrak{M}}$ corresponds to the outside world in itself ("Ding an sich") and each observer creates his "own" world of phenomena via perceiving (in the Kantian sense) the outside world, where Kant emphasizes that each observer contributes to the creation of the world of phenomena (and not only the outside world contributes), cf. Kant [145, 146]. In our case, the contribution of observer m is his *coordinatization* of $\mathfrak{G}_{\mathfrak{M}}$. Cf. Friedman [91, pp.286-287], Reichenbach [217].

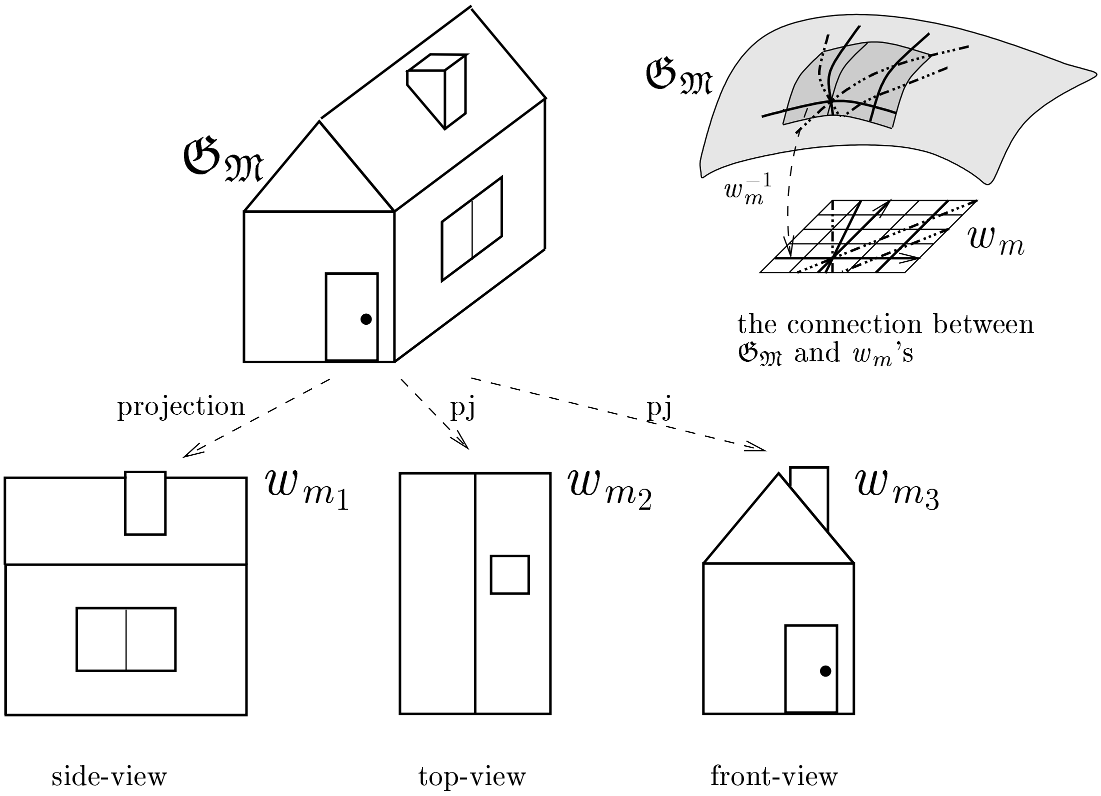


Figure 62: Descriptive geometry put into analogy with the connection between the unique $\mathfrak{G}_{\mathfrak{M}}$ and the many world-views w_m in $\mathfrak{M}$.

For further introductory thoughts on why we "celebrate" the observer independent character of our geometries $\mathfrak{G}_{\mathfrak{M}}$ we refer to items (1)–(11) on pp. 185–186. (That is in sub-section "On the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$".)

On the contents of some of the sections (in this chapter). §4.2 contains the definition of the observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$. §4.3 contains the basics of definability theory we will need. §4.5 contains our duality theories between observation-oriented models $\mathfrak{M}$ and observer independent geometries $\mathfrak{G}_{\mathfrak{M}}$. More precisely, the duality theories act between axiomatizable classes of frame models and of geometries. §4.6 studies interdefinability of the ingredients (sorts, relations, functions etc) of $\mathfrak{G}_{\mathfrak{M}}$, and thereby it aims at simplifying and streamlining $\mathfrak{G}_{\mathfrak{M}}$ as a mathematical structure. §4.7 defines and discusses geodesics of $\mathfrak{G}_{\mathfrak{M}}$. Geodesics play an essential role in the theory of accelerated observers (to be discussed in AMN [19], [26])[282] and in the theory of general relativity. Some mathematical results/methods treated in AMN [18] as a part of the Geometry chapter are moved to the Appendix in the present work to make

In passing we also note that Kant's philosophy of science was continued by the logical positivists, e.g. Carnap [58], Reichenbach [217]. Logical positivism began as a neo-Kantian movement whose central preoccupation was the content/form distinction where the "content" is supplied by the outside world while the "form" is supplied by the observer's mind (e.g. by his logic). [Here, phenomenon = (content + form).] In Carnap's works, the "form" part or the part supplied by the mind is logic. (In this respect, our present approach is positively related to those of Carnap and Reichenbach.) Reichenbach emphasizes that the content part, e.g. the basic definitions of the concepts of a theory do change during the evolution (or development) of the theory in question. In agreement with Reichenbach, we think that this is in agreement with the modern view of Kant-oriented philosophy of science. Cf. also footnote 270 on p.130.

[282] cf. also the "Accelerated observers" chapter of [24]

the main body shorter (cf. Appendix A).

The figure representing Gödel's rotating universe (proving e.g. that Einstein's equations do not imply Mach's principle), mentioned on p.130, is postponed to the section on geodesics (Figure 134, p.365) because the notion of a geodesic is essential for understanding the picture.

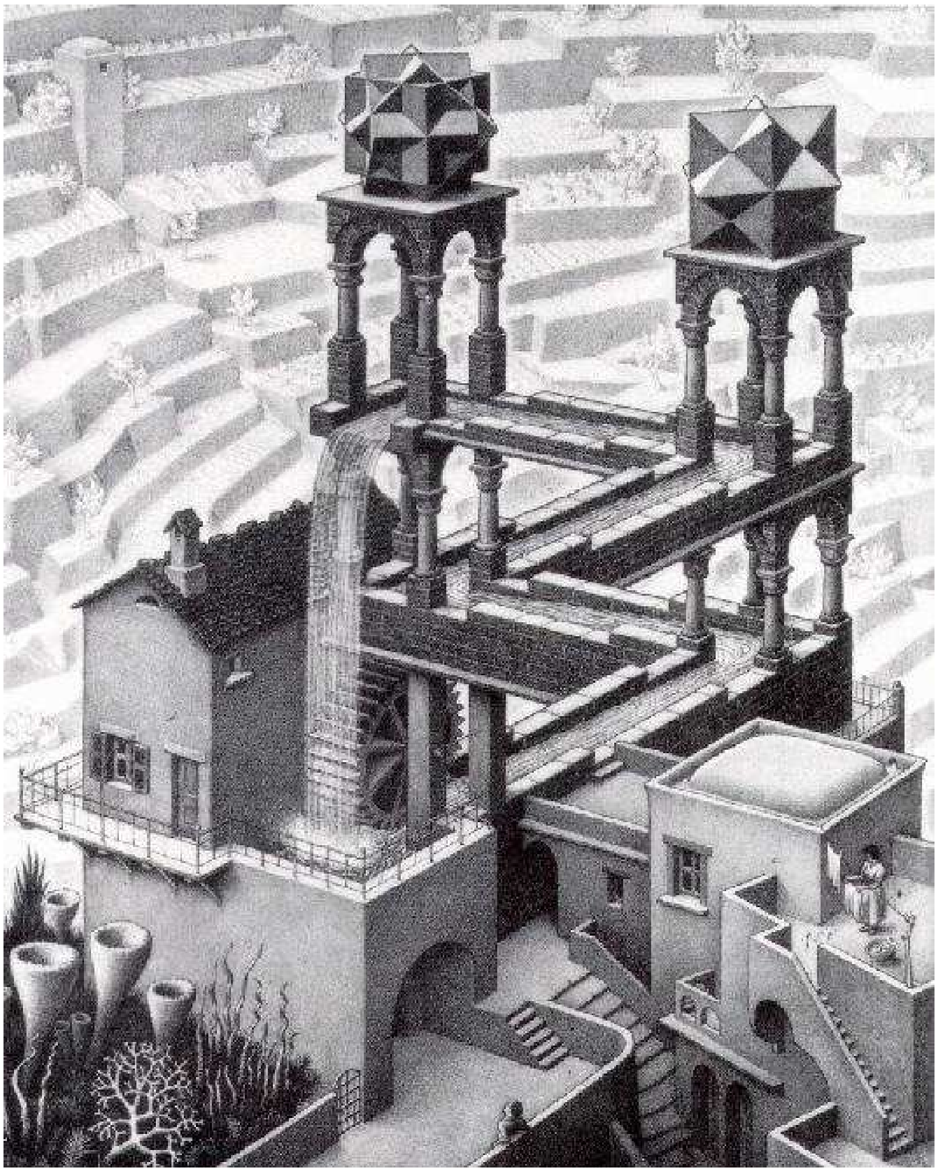


Figure 63: Paving the road towards general relativity by gluing together locally well behaved geometries yielding something globally strange, cf. §4.2.5 for gluing geometries. This Escher picture shows a "paradigm for general relativity" which locally behaves like special relativity: In the picture locally everything is normal, globally it is like time travel via a rotating black hole, cf. Thorne [259] or O'Neil [208] for the latter. Cf. also Fig.61 on p.127.

4.2 Basic concepts

In this section we show how observer-independent structures can be found in our frame models of relativity theory, i.e. we will show that there is an *observer-independent* "geometric" structure $\mathfrak{G}_{\mathfrak{M}}$ inside every model $\mathfrak{M}$ of our frame language. We will also define the Reichenbachian version $\mathfrak{G}^R_{\mathfrak{M}}$ of the geometric structure corresponding to a frame model $\mathfrak{M}$ of relativity theory.[283]

Conventions, clarifying possible ambiguities: The symbol $\perp$ for orthogonality will be used in the present work in an ambiguous way. Sometimes it denotes Euclidean orthogonality (as defined in §3.1 of AMN [18]) and sometimes it denotes *relativistic orthogonality* as will be the case in the middle of Definition 4.2.3 below. We hope that context will help. If somewhere we want to emphasize the difference then we will write $\perp_e$ and $\perp_r$ (for the Euclidean and the relativistic version, respectively). We note that the so-called Minkowskian orthogonality is a special case of our relativistic orthogonality $\perp_r$. A further source of ambiguity is the following (issue about where exactly our geometry lives). For a second let $Mn := \mathcal{P}(B)$, where B is the set of bodies for our model $\mathfrak{M}$. (Later we will slightly change this convention but that is beside the point now.) In §2 we had *"Lines"* $\subseteq \mathcal{P}({}^nF)$ while in the present chapter we will have *"Lines"* $\subseteq \mathcal{P}(Mn)$. That is, now lines are understood on the set Mn of events, while at the beginning (when we defined frame models) lines were understood on the vector-space ${}^n\mathbf{F}$. We hope, context will help.

For a class K of models $\mathbf{I}\mathsf{K}$ denotes the class of isomorphic copies of members of K.

Warning 4.2.1 The word algebra is used in 3 different senses, both here and in the literature. These are:

(i) Algebra is a branch of mathematics.

(ii) An algebra[284] is a structure $\mathfrak{A} = \langle A; f_i\rangle_{i\in I}$ in the sense of universal algebra.[285]

(iii) An algebra over a field $\mathbf{F}$ is a vector space over $\mathbf{F}$ with an extra binary operation "$\cdot$" as indicated in footnote 1105, p.1101 (§6.6.6, sub-title "On … omnipresence …" item (2)) of AMN [18].[286] $\lhd$

4.2.1 Definition of the observer-independent (or relativistic) geometry $\mathfrak{G}_{\mathfrak{M}}$

The reader may find that $\mathfrak{G}_{\mathfrak{M}}$ defined below has too many components; however there is no need to worry, our theory will not be as complicated as suggested by the number of these components as it will be explained in §4.2.6 (Some reducts …). The reader is asked not to be disturbed by the complexity (or size) of the geometry $\mathfrak{G}_{\mathfrak{M}}$. We include here the whole of $\mathfrak{G}_{\mathfrak{M}}$

[283] Cf. §4.5 of AMN [18] for what we call the Reichenbachian approach to relativity.
[284] or equivalently an algebraic structure
[285] Here A is an arbitrary set and $f_i : {}^nA \longrightarrow A$ is arbitrary too (for some $n \in \omega$).
[286] The literature often writes simply "an algebra" for an algebra over a field.

only for completeness: We will almost never study the whole $\mathfrak{G}_{\mathfrak{M}}$. Most of the time, we will study simpler geometries, e.g. $\mathbf{G}_{\mathfrak{M}}$ defined in the fifth line of Def.4.2.3 or some even simpler variant of this simpler geometry like $\langle Mn, L;\ \in, eq\rangle$ or $\langle Mn, L;\ \in, \perp\rangle$ or the streamlined time-like metric structure $\langle Mn, \mathbf{F_1};\ g^{\prec}\rangle$ on p.346 (§4.6.1). Cf. also the beginning of §4.2.6 (p.215).

We would like to emphasize that we want to treat relativistic geometries as <u>abstract structures</u>. An abstract structure is determined only up to isomorphism.[287] Therefore it is important to emphasize that relativistic geometries are defined up to isomorphism only, cf. Def.4.2.3(III) (p.146). That is, any isomorphic copy of the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ counts as "the geometric counterpart" of the frame model $\mathfrak{M}$ (where recall, that $\mathfrak{G}_{\mathfrak{M}}$ is the observer-independent geometry associated with $\mathfrak{M}$). In still other words this means that when studying $\mathfrak{G}_{\mathfrak{M}}$ we will concentrate on its <u>*isomorphism invariant*</u> properties only (as is usual in the structuralist branches of mathematics like algebra). The reason why this is important is explained in Remark 4.2.5 (p.149). Treating $\mathfrak{G}_{\mathfrak{M}}$ as <u>abstract</u> structure will make some of our results, e.g. the duality theory, stronger. The fact that we treat isomorphic geometries as identical is important for the philosophy of the present chapter, cf. Remark 4.2.5 (p.149).

More motivation for the definition below will come in §4.2.3 ("On the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$"), we would like to, particularly, emphasize Remark 4.2.42 on p.186.

Notation 4.2.2 $^{+}F \stackrel{\text{def}}{=} \{\, x \in F \,:\, x > 0 \,\}$. I.e. ^{+}F is the set of positive elements of $\mathfrak{F}$.

Definition 4.2.3
(Observer-independent, relativistic geometry and related definitions)
Let $\mathfrak{M}$ be a frame model.

(I) The <u>*observer-independent geometry*</u> $\mathfrak{G}_{\mathfrak{M}}$ is a <u>*three-sorted*</u> structure to be defined below.[289] But cf. also the improved geometry $\mathfrak{G}^{*}_{\mathfrak{M}}$ in §4.5.5 (p.332). (The "simplified" geometries $\mathbf{G}_{\mathfrak{M}}$ and $\mathsf{G}_{\mathfrak{M}}$ will be only two-sorted.)

$$\mathfrak{G}_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq, g, \mathcal{T}\,\rangle, \text{ and}$$

$\mathbf{G}_{\mathfrak{M}}$ is the $(g, L^S, \mathcal{T})$-free reduct

$$\mathbf{G}_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ L^T, L^{Ph}, \in, \prec, Bw, \perp, eq\rangle$$

[287]By an <u>*abstract structure*</u> we understand a class K of structures such that $(\forall \mathfrak{A} \in \mathsf{K})\, \mathsf{K} = \mathbf{I}\{\mathfrak{A}\}$. Similarly an <u>*abstract class of structures*</u> is one which is closed under isomorphisms. As a contrast, a concrete class is usually not closed under $\mathbf{I}$. An example of the abstract/concrete distinction is provided by Stone duality on pp. 1015, 1019 of AMN [18]. The class BA of Boolean algebras is an abstract class (since $\mathsf{BA} = \mathbf{I}\mathsf{BA}$). The class BSA of Boolean set algebras, i.e. algebras whose operations are the real, set theoretic $\cup, \cap, -$ is a <u>concrete</u> class of structures because if we know the universe A of an algebra $\mathfrak{A} \in \mathsf{BSA}$ then from A the rest of $\mathfrak{A}$ is recoverable.[288] Accordingly $\mathsf{BSA} \neq \mathbf{I}\mathsf{BSA}$ $(= \mathsf{BA})$. Stone's representation theorem says that every member of the abstract class BA is representable by (i.e. is isomorphic to) a member of the concrete class BSA. Cf. also p.147 and Remark 6.6.87 ("On representation theorems ...") on p.1106 in AMN [18]. E.g. on p.147 $Geom(Th)$ will be a concrete class while $\mathsf{Ge}(Th) = \mathbf{I}Geom(Th)$ will be an abstract class. The theorems later saying that for an axiomatizable class $\mathsf{Mog}(TH)$ of geometries $\mathsf{Mog}(TH) = \mathbf{I}Geom(Th)$ are typical <u>representation theorems</u>. Cf. e.g. items 4.5.57, A.1.7, A.1.10, A.1.11 (pp. 328, A-4, A-5, A-6). Though these items are not exactly of the desired form "$\mathsf{Mog}(TH) = \mathbf{I}Geom(Th)$" they (and the "tools" scattered around them) can be used for obtaining theorems of the desired form. Cf. for more on "concrete", "abstract", "axiomatic-abstract" classes and their connections with representation theorems in AMN [18, Remark 6.6.87] and also Németi [204].

[288]For the notion of concrete classes of algebras and for the importance of the concrete/abstract distinction cf. Németi [204].

[289]Later, in Remark 4.2.5 we will define the geometric counterpart of the model $\mathfrak{M}$ to be $\mathbf{I}\mathfrak{G}_{\mathfrak{M}}$.

of $\mathfrak{G}_{\mathfrak{M}}$, and $\mathsf{G}_{\mathfrak{M}}$ is the $(L^T, L^{Ph}, \prec, Bw, eq)$-free reduct

$$\mathsf{G}_{\mathfrak{M}} \stackrel{\text{def}}{:=} \langle Mn, L; \in, \perp\rangle$$

of $\mathbf{G}_{\mathfrak{M}}$; where:

1. The universes (or sorts) of $\mathfrak{G}_{\mathfrak{M}}$ are Mn, F (= universe of the structure $\mathbf{F_1}$) and L, while the rest are the relations of $\mathfrak{G}_{\mathfrak{M}}$.[290]

2. $Mn \stackrel{\text{def}}{:=} \bigcup\{\, Rng(w_m) \;:\; m \in Obs\,\}$ $(\subseteq \mathcal{P}(B))$. Intuitively Mn is the set of all *events* in our relativistic model $\mathfrak{M}$. Mn is the *set of points* of our geometry $\mathfrak{G}_{\mathfrak{M}}$. We also call Mn *space-time*, cf. Convention 4.2.6 (p.149). The acronym Mn abbreviates the word *manifold*.[291]

3. $\mathbf{F_1} \stackrel{\text{def}}{:=} \langle F; 0, 1, +, \leq\rangle$ is the ordered group reduct $\langle F; 0, +, \leq\rangle$ of the ordered field $\mathfrak{F}^{\mathfrak{M}}$ expanded with the constant 1, where 0 and 1 are the usual zero and one of the field $\mathfrak{F}^{\mathfrak{M}}$.

4. $$\begin{aligned} L^T &\stackrel{\text{def}}{:=} \{\,\{\, e \in Mn \;:\; m \in e\,\} \;:\; m \in Obs \cap Ib\,\}.^{292} \\ L^{Ph} &\stackrel{\text{def}}{:=} \{\,\{\, e \in Mn \;:\; ph \in e\,\} \;:\; ph \in Ph\,\}. \end{aligned}$$

   I.e. L^T, called the set of *time-like lines*, is the set of life-lines of *inertial observers*, and similarly L^{Ph}, called the set of *photon-like lines*, is the set of life-lines of *photons*. Here, life-lines are understood as subsets of Mn $(\subseteq \mathcal{P}(B))$, while in earlier parts of this work they were understood as subsets of ${}^nF$.

---

[290] The statuses of all the relations $\in$, $\prec$ etc. should be clear with the possible exception of the topology $\mathcal{T}$. We can declare that $\mathcal{T}$ is a so-called second-order relation on Mn. Equivalently, we could declare that $\mathcal{T}$ is the 4^{th} *sort* (or universe) of $\mathfrak{G}_{\mathfrak{M}}$, and use the set theoretic membership relation $\in_{Mn,\mathcal{T}} \subseteq Mn \times \mathcal{T}$ to connect $\mathcal{T}$ with the remaining sorts. Cf. §4.3 for more detail on this.

[291] We do not need the manifold structure on Mn yet. So, the reader may safely skip the following. Mn is only the universe of a manifold $\mathbf{Mn}$. Assume Mn comes from $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax}(\sqrt{\ }))$. Then $\mathbf{Mn} := \langle Mn, \mathfrak{F};\, w_m\rangle_{m\in Obs}$. (Here the w_m's are called the maps and $\{\, w_m \;:\; m \in Obs\,\}$ is called the atlas of $\mathbf{Mn}$.) This structure looks like a manifold except that $\mathfrak{F}$ may be different from $\mathfrak{R}$ and the topology induced on Mn by the coordinatizations $\{\, w_m \;:\; m \in Obs\,\}$ may be of an uncountable base. In a *generalized manifold* we allow the base set to be uncountable but otherwise we do require all the remaining usual properties. So if $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ and $\mathfrak{M}$ satisfies some natural conditions then $\mathbf{Mn}$ is a generalized manifold. In later generalizations to accelerated observers and towards general relativity we will have to generalize $\mathbf{Mn}$ further, e.g. there will be maps much more general than world-views of inertial observers. (Cf. also footnote 96 on p.26 where we indicated generalizations such that w_m becomes a partial function $w_m : {}^nF \stackrel{\circ}{\longrightarrow} Mn$, i.e. w_m coordinatizes a subset of Mn with only a subset of ${}^nF$. Cf. also Fig.5 on p.10, together with the sentence on p.12 containing [reference to] footnote 71. Cf. also footnote 198 on p.188 of AMN [18] and Fig.64 on p.191 of AMN [18]. This is of course only a first step in the direction of generalization we are discussing.)

[292] In our theories studied so far we always assumed $Obs \subseteq Ib$. The latter is implied by $\mathbf{Ax2}$. Therefore instead of inertial observers i.e. members of $Obs \cap Ib$ we usually talked about simply observers, Obs only, for simplicity (since we knew that $Obs = Obs \cap Ib$ was the case). However, later when studying accelerated observers and other generalizations towards general relativity we will need to pay special attention to $Obs \cap Ib$, since the Obs-part of $\mathbf{Ax2}$ (i.e. $Obs = Obs \cap Ib$) will not be assumed any more. This is why at the present point we start to pay attention to the distinction between Obs and $Obs \cap Ib$. (In some sense, in general relativity $Obs \cap Ib$ will be a kind of "backbone" of our theory.) Cf. in connection with these ideas Remark 4.2.42 on p.186.

L^S consists of the <u>*space-like lines*</u> of $\mathfrak{M}$ defined as follows:

$$L^S :\stackrel{\text{def}}{=} \{\, w_m[\bar{x}_i] \ : \ m \in Obs \cap Ib,\ 0 < i \in n \,\}.$$

I.e. L^S consists of the w_m-images of the spatial coordinate axes (like $\bar{x},\ \bar{y},\ \bar{z}$) of ${}^nF$ for inertial m's.

We note that, assuming $\mathbf{Ax4} + \mathbf{Ax6_{00}}$,

$$L^T = \{\, w_m[\bar{t}] \ : \ m \in Obs \cap Ib \,\},$$

i.e. L^T consists of the w_m-images of the time axis for inertial m's.[293]

The set L of all <u>*lines*</u> of $\mathfrak{G}_{\mathfrak{M}}$ is defined as

$$L :\stackrel{\text{def}}{=} L^T \cup L^{Ph} \cup L^S.$$

Cf. Figure 90 on p.210 for the spirit of working in Mn instead of ${}^nF$ and for the connections of the two.

5. $\in$ is the set theoretic membership relation between Mn and L.[294] In other words, $\in$ is the usual <u>*incidence relation*</u> of our geometry $\langle Mn, \ldots, L; \ldots \rangle$.

6. We define the binary relation, called <u>*causality pre-ordering*</u>,[295] $\prec$ on Mn as follows. Let $e, e_1 \in Mn$. Intuitively, $e \prec e_1$ holds if there is an inertial observer who is present both in e and e_1 (i.e. e and e_1 are on his life-line) and sees that event e precedes event e_1 in time; formally:

$$e \prec e_1$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs \cap Ib)\ (m \in e \cap e_1 \ \land\ (\exists p \in w_m^{-1}(e))(\exists q \in w_m^{-1}(e_1))\ p_t < q_t).^{296}$$

7. The <u>*relation*</u> $Bw \subseteq Mn \times Mn \times Mn$ <u>*of betweenness*</u> is a ternary relation defined as follows: Let $e, e_1, e_2 \in Mn$. Intuitively, $Bw(e, e_1, e_2)$ holds if there is an inertial observer who thinks that event e_1 is between events e and e_2; formally:

$$\begin{aligned} Bw(e, e_1, e_2) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad & \Big((\exists m \in Obs \cap Ib)(\exists p, q, r \in {}^nF) \\ & [w_m(p) = e \ \land\ w_m(r) = e_1 \ \land\ w_m(q) = e_2 \ \land\ \mathsf{Betw}(p, r, q)] \Big), \end{aligned}$$

[293] The difference between the style of definitions of L^T and L^S is connected to the fact (emphasized e.g. by Reichenbach) that in relativity theory L^S is somewhat less tangible than L^T, cf. §4.5 of AMN [18] (and the definition of "Reichenbachian" geometries on p.147 herein).

[294] In our many-sorted approach we encounter several situations where members of one sort U_1 act as sets of members of another sort, say U_2. In such situations we use the set theoretical symbol "$\in$" as the relation connecting U_2 and U_1, i.e. "$\in$" $\subseteq U_2 \times U_1$. We can add the names of the sorts involved as indices of $\in$ like $\in_{Mn,L}$ but for simplicity we often omit these indices.

[295] The word "causality" in "causality pre-ordering" here is used <u>only</u> because we want to be consistent with the literature. We emphasize that with this word we do <u>not</u> mean to imply that we would have a theory of real causality around at this point. Cf. AMN [18, Remark 6.7.22 on p.1158].

[296] Under very mild assumptions on $\mathfrak{M}$, $\prec$ becomes a so-called irreflexive pre-ordering, i.e. $\prec \cup \mathrm{Id}$ is a pre-ordering, i.e. is transitive and reflexive. Note that $e \prec e_1 \ \Rightarrow\ e, e_1 \in \ell \in L^T$, for some ℓ. We defined $\prec$ in the "existential" style. The universal version $\prec^u$ of $\prec$ is defined as follows. $(\exists m \in Obs \cap Ib)\, m \in e \cap e_1 \ \land\ (\forall m \in Obs \cap Ib)[m \in e \cap e_1 \ \Rightarrow\ (\exists p \in w_m^{-1}(e))(\exists q \in w_m^{-1}(e_1))\ p_t < q_t]$. Under mild assumptions $\prec^u$ is a (strict) partial ordering, moreover $\prec^u$ is the antisymmetric part of $\prec$ (i.e. $x \prec^u y \ \Leftrightarrow\ [x \prec y \ \land\ y \not\prec x]$).

where, we recall from §3, p.119 that $\mathsf{Betw}(p, r, q)$ means that p, r, q are collinear points of ${}^nF$ and r is strictly in between p and q, formally: $r \neq p, q$ and $r = p + \lambda \cdot (q - p)$ for some $0 < \lambda < 1$.

8. In analogy with our notation "$\mathfrak{G}_{\mathfrak{M}}$", if we want to indicate that Mn or L comes from $\mathfrak{G}_{\mathfrak{M}}$ then we will write $Mn_{\mathfrak{M}}$, $L_{\mathfrak{M}}$ etc.[297]

9. Next we define the derived relation of parallelism in our observer-independent geometries $\mathfrak{G} = \langle Mn, L; \in, Bw \rangle \cong \langle Mn_{\mathfrak{M}}, L_{\mathfrak{M}}; \in, Bw_{\mathfrak{M}} \rangle$, where "$\cong$" is the usual relation of isomorphism between structures (cf. e.g. Conventions 3.1.2, 3.8.4 of AMN [18]). Let $\ell, \ell_1 \in L$. Intuitively, ℓ and ℓ_1 are <u>*$\mathfrak{G}$-parallel*</u> iff each inertial observer who sees them thinks, they are parallel; formally:

$$\ell \parallel_{\mathfrak{G}} \ell_1$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\forall a, b, c \in Mn)\Big((\, Bw(a, b, c) \ \wedge\ a \in \ell \ \wedge\ a \notin \ell_1 \ \wedge\ c \in \ell_1\,) \ \Rightarrow$$
$$[\, (\exists d \in \ell)(\exists e \in \ell_1)(\, Bw(d, b, e) \ \wedge\ d \neq a \ \wedge$$
$$(\nexists f \in \ell_1)(Bw(a, d, f) \ \vee\ Bw(a, f, d) \ \vee\ Bw(d, a, f)\,)\,]\,\Big)^{298}$$

see Figure 64.

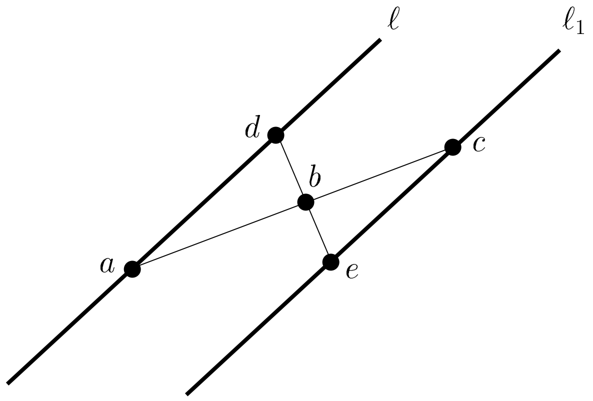


Figure 64: $\ell \parallel_{\mathfrak{G}} \ell_1$.

10. For the definition of (relativistic) <u>*orthogonality*</u> $\perp = \perp_r$ we need first an auxiliary definition.[299]

    Alternative (shorter) definitions of relativistic orthogonality ($\perp_r$) are available in Definitions 4.2.10, 4.2.17 (pp. 156, 161) below, cf. Remark 4.2.9, too.

[297] Formally, $Mn^{\mathfrak{G}_{\mathfrak{M}}}$ would be the standard model theoretic notation. However, it is too complicated.

[298] The formal definition became so long because we have to take into account lines which are present in several windows, for "windows" cf. the intuitive text above Thm.3.3.12 on p.196 of AMN [18] (think of photon-like lines). Recall from §3 that **Newbasax** models are disjoint unions of **Basax** models, roughly speaking. These **Basax** models are informally referred to as "windows" in [18].

[299] We would like to mention that on p.161 we will give an alternative definition ($\perp_r^\omega$) for relativistic orthogonality which is just as natural as the present one and is shorter. The only disadvantage of $\perp_r^\omega$ is that it "works" only for $n > 2$.

Ordinals denotes the class of ordinal numbers in the usual set theoretic sense, cf. e.g. Handbook of Mathematical Logic [44].

The definition of convergence given below agrees with what one would intuitively expect, cf. Figure 65.

Definition: Let $\alpha \in$ Ordinals. Let $\mathcal{S} \in {}^{\alpha}L$ (i.e. $\mathcal{S}$ is an α-sequence of lines) and $\ell \in L$. Then we say that $\mathcal{S}$ *converges to* ℓ iff

$$(\exists p \in Mn)\Big[(\forall i \in \alpha)(p \in \mathcal{S}(i) \cap \ell) \;\wedge$$
$$(\exists \ell' \in L)\Big(p \notin \ell' \;\wedge\; (\exists q \in {}^{\alpha}Mn)(\forall i \in \alpha)\;[\,q(i) \in \mathcal{S}(i) \cap \ell' \;\wedge$$
$$(q \text{ converges to some } q^+ \in \ell' \cap \ell \text{ w.r.t. } Bw)\;{}^{300}]\Big)\Big], \text{ see Figure 65.}$$

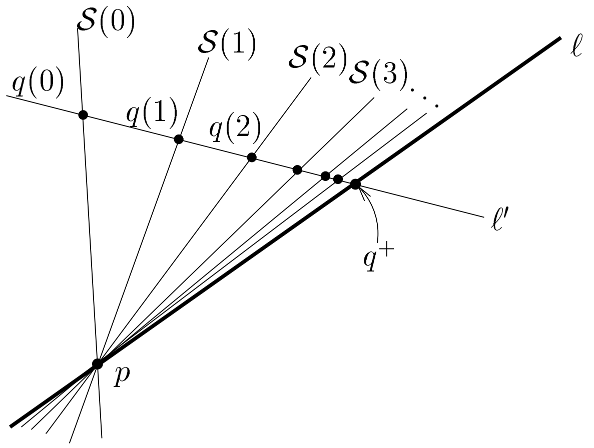


Figure 65: $\mathcal{S} \in {}^{\alpha}L$ converges to $\ell \in L$.

First we define *basic orthogonality* $\perp_0 \subseteq L \times L$. Intuitively, two lines are $\perp_0$-orthogonal if there is an inertial observer who thinks that these two lines coincide with two distinct coordinate axes; formally: Let $\ell, \ell' \in L$. Then

$$\ell \perp_0 \ell' \;\stackrel{\text{def}}{\Longleftrightarrow}\; \Big((\exists m \in Obs \cap Ib)(\exists i, j \in n)(i \neq j \;\wedge\; \ell = w_m[\bar{x}_i] \;\wedge\; \ell' = w_m[\bar{x}_j])\Big),$$

see Figure 66.

The relation of *relativistic orthogonality* $\perp = \perp_r$ is defined to be the smallest subset of $L \times L$ containing $\perp_0$ and closed under taking limits and parallelism, i.e. $\perp_r$ is the smallest subset of $L \times L$ having properties (i)–(iii) below.

(i) $\perp_0 \subseteq \perp_r$, i.e. $\ell \perp_0 \ell' \;\Rightarrow\; \ell \perp_r \ell'$.

(ii) $\perp_r$ is closed under taking limits, i.e.

[300] I.e., $(\exists q^+ \in \ell \cap \ell')(\forall a, b \in \ell')\,[Bw(a, q^+, b) \;\Rightarrow\; (\exists \beta \in \alpha)(\forall i \in (\alpha \setminus \beta))\, Bw(a, q(i), b)]$.

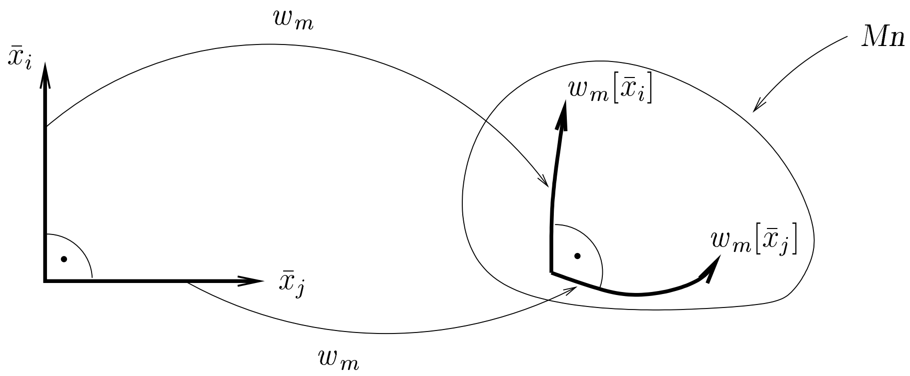


Figure 66: Illustration for the definition of $\perp_0$.

$\Big((\exists\alpha \in \mathsf{Ordinals})(\exists\mathcal{S},\mathcal{S}' \in {}^{\alpha}L)(\forall i \in \alpha)(\mathcal{S}(i) \perp_r \mathcal{S}'(i) \quad \wedge$
$\mathcal{S}$ and $\mathcal{S}'$ converge to ℓ and ℓ' respectively$)\Big) \quad \Rightarrow \quad \ell \perp_r \ell'$,

see Figure 67.[301] We note that this property (i.e. that $\perp_r$ is closed under taking limits) can be formulated in the first-order language of the structure $\langle Mn, L; \in, Bw, \perp_r\rangle$, cf. axiom $\mathbf{L_{10}}$ on p.331.

(iii) $\perp_r$ is closed under parallelism, i.e.
$(\ell \perp_r \ell_1 \ \wedge\ \ell' \parallel_{\mathfrak{G}} \ell \ \wedge\ \ell_1' \parallel_{\mathfrak{G}} \ell_1) \ \Rightarrow\ \ell' \perp_r \ell_1'$.

In connection with Figure 67, it might be useful to have a look also at Remark 4.2.7 (pp.149–152) and Figure 69 in that remark.

We refer to Remark 4.2.7 (p.149) at the end of §4.2.1 for intuitive motivation (and considerations) for our using limits in the definition of $\perp_r$. That remark might also help in improving our intuitive picture of $\perp_r$ (and perhaps other parts of $\mathfrak{G}_{\mathfrak{M}}$).

11. The *relation* $eq \subseteq {}^4Mn$ *of equidistance* is a 4-ary relation defined as follows. Intuitively, $eq(a,b,c,d)$ will mean that segments $\langle a,b\rangle$ and $\langle c,d\rangle$ are of equal length (in some sense).[302] First we define the relation eq_0 of basic equidistance. Let $e, e_1, e_2, e_3 \in Mn$. Then

[301] Figure 67 is understood in the world-view of an observer, under assuming $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$. The 8 pictures represent all the possibilities as new "orthogonal pairs" ($\perp_r$-pairs) can be generated by "old orthogonal pairs" ($\perp_0$-pairs) by taking limits as described above. For that possible reader who wants to see the "intuitive counterparts" of these pictures in, say, $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$ models we suggest concentrating on figures (a), (b), (c). We note that we do not claim that all these 8 possibilities are realized in, say, **Basax** models. (Though, in passing we note that, (a), (b), (c), (d), (f), (h) do occur, and we did not check with the rest.)

[302] Intuitive and historical motivation for including eq in our geometries is that eq corresponds to "compass" in the traditional "ruler and compass" conception of geometry going back for a very long time, cf. e.g. Lánczos [151, p.48, lines 5-11, p.25 Postulate 3 (of Euclid)]. (This is so because by using eq we can define circles.)

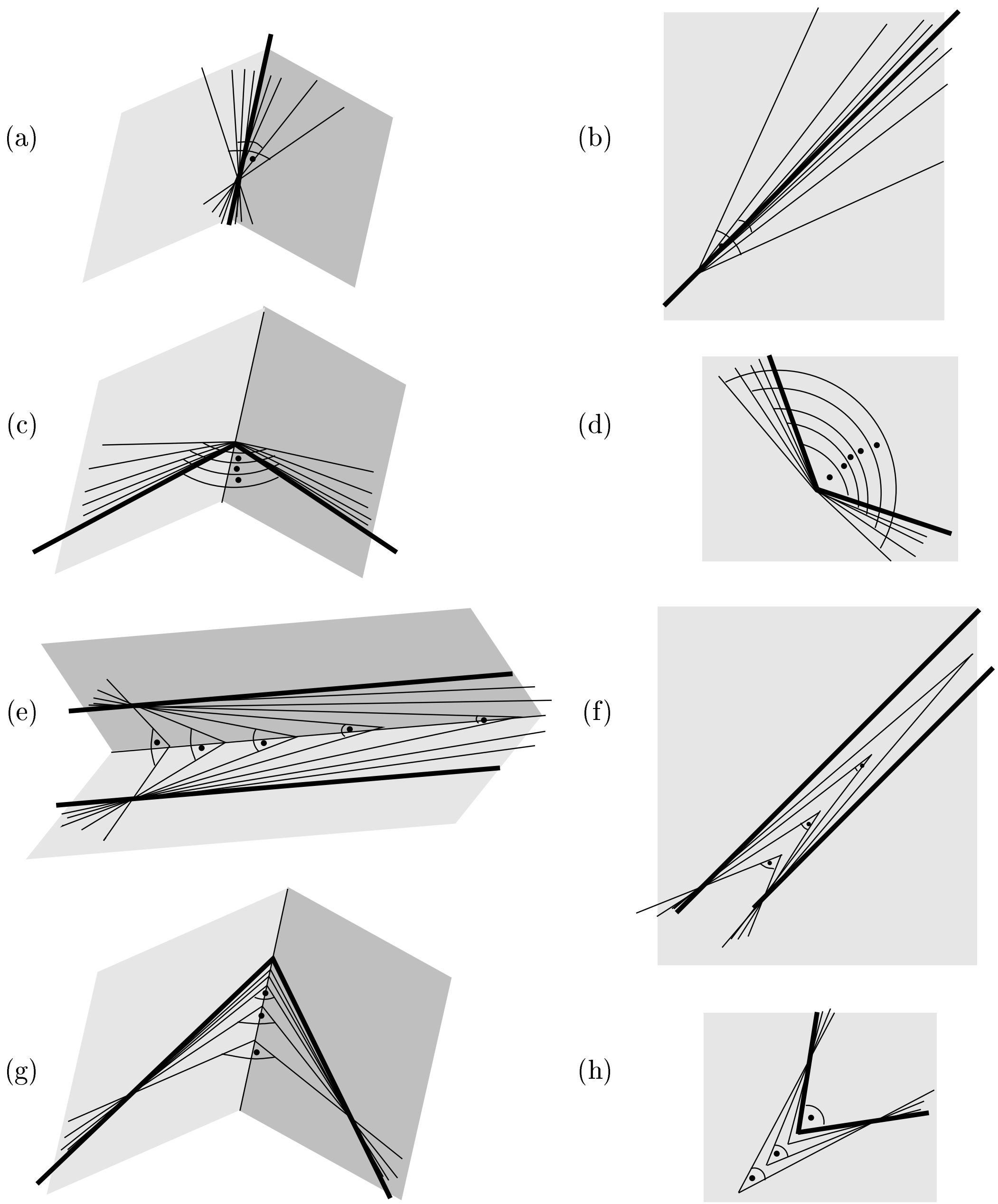


Figure 67: "Taking the closure of $\perp_0$ under limits".

$$eq_0(e, e_1, e_2, e_3)$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs \cap Ib)(\exists i, j \in n)(\exists p, q \in \bar{x}_i)(\exists r, s \in \bar{x}_j)$$
$$\Big(|p - q| = |r - s| \ \wedge \ w_m(p) = e \ \wedge \ w_m(q) = e_1 \ \wedge \ w_m(r) = e_2 \ \wedge \ w_m(s) = e_3\Big).$$[303]

Intuitively, segments $\langle e, e_1\rangle$ and $\langle e_2, e_3\rangle$ are eq_0-related if there is an inertial observer m who "thinks" that the distance between e and e_1 is the same as the distance between e_2 and e_3 (and sees the segments $\langle e, e_1\rangle$ and $\langle e_2, e_3\rangle$ on some [perhaps different] coordinate axes).

Now we define eq to be the transitive closure of eq_0 understood as a binary relation between pairs of points (cf. Figure 68); in more detail: First for every $i \in \omega$ we define eq_{i+1} as follows.

$$eq_{i+1} :\stackrel{\text{def}}{=} \{ \langle a, b, c, d\rangle \in {}^4Mn \ : \ (\exists e, f \in Mn) \ \langle a, b, e, f\rangle, \langle e, f, c, d\rangle \in eq_i \}.$$[304]

Now,

$$eq :\stackrel{\text{def}}{=} \bigcup \{ eq_i \ : \ i \in \omega \}, \quad \text{see Figure 68.}$$

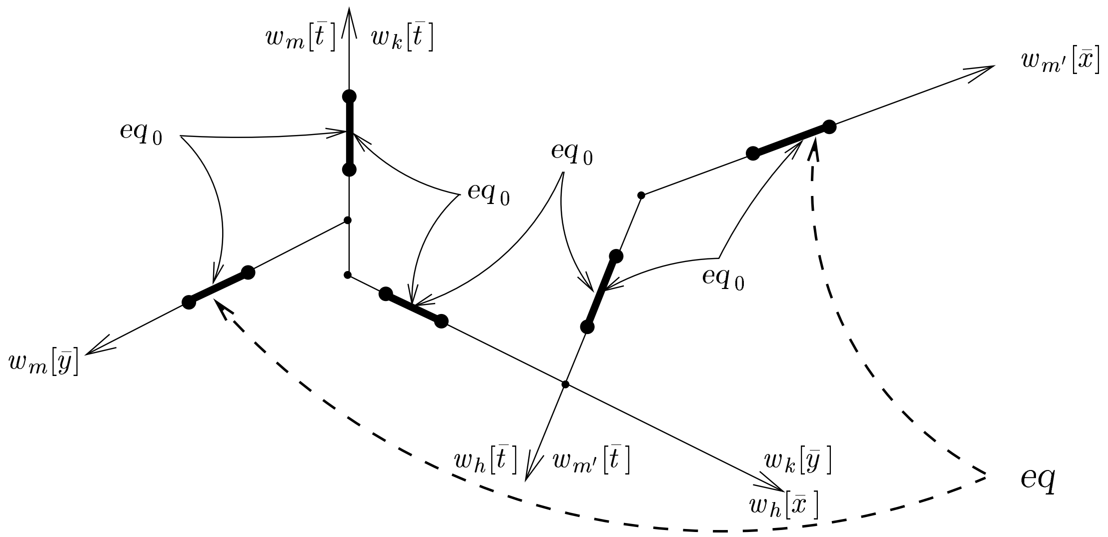


Figure 68: eq is defined to be the transitive closure of eq_0.

Instead of $eq(a, b, c, d)$ sometimes we write $\langle a, b\rangle \ eq \ \langle c, d\rangle$. Similarly for eq_0, eq_1, etc.

We note that eq is an equivalence relation (when understood on pairs of points) on the set $\{ \langle a, b\rangle \in Mn \times Mn \ : \ (\exists m \in Obs \cap Ib)(\exists i \in n) \, a, b \in w_m[\bar{x}_i] \}$.

[303] We could "improve" the definition of eq_0 by adding $eq_0(e, e_1, e, e_1)$. This would perhaps simplify some of our upcoming statements, but we did not explore this. Further, analogously to the definition of $\perp_r$, if we closed eq_0 under taking limits then perhaps the new eq_0 would behave better (i.e. if e, e_1, e_2, e_3 are on a photon-like line then $eq_0(e, e_1, e_2, e_3)$ would be the case).

[304] We note that $(\forall i \in \omega) \ eq_i \subseteq eq_{i+1}$ since eq_0 is "reflexive", i.e. $[\langle a, b, c, d\rangle \in eq_0 \ \Rightarrow \ \langle a, b, a, b\rangle \in eq_0]$.

12. $g : Mn \times Mn \overset{\circ}{\longrightarrow} F$ is a partial function defined as follows. Let $e, e_1 \in Mn$.

Intuitively, the distance between events e and e_1 as measured by an inertial observer, call it m, is λ (where $0 \leq \lambda \in F$) iff m sees both e and e_1 happening on the same coordinate axis $\bar{x}_i$ with coordinate distance λ. Further, the distance between events e and e_1 as measured by a photon, call it ph, is λ iff $\lambda = 0$ and ph is present both in e end e_1.[305] Now,

$$g(e, e_1) :\overset{\text{def}}{=} \min\{\, \lambda \in F \;:\; (\exists h \;\in\; (Obs \cap Ib) \cup Ph\,)$$
$$\text{(the distance between } e \text{ and } e_1 \text{ as measured by } h \text{ is } \lambda\,)\,\};^{306}$$

Formally:

$$g(e, e_1) :\overset{\text{def}}{=} \min\{\, \lambda \in F \;:\; (\exists ph \in Ph)\,[\, ph \in e \cap e_1 \;\wedge\; \lambda = 0\,] \quad \text{or}$$
$$(\exists m \in Obs \cap Ib)(\exists i \in n)(\exists p, q \in \bar{x}_i)\,[w_m(p) = e \;\wedge\; w_m(q) = e_1 \;\wedge\; \lambda = |p - q|\,]\,\},$$

if this min[308] exists, *otherwise* $g(e, e_1)$ is undefined.

Under mild assumptions, the "min" part of the definition of $g(e, e_1)$ can be omitted. (More precisely, the essential occurrence of "min" could be omitted.[309]) An example of such sufficient assumptions is the axiom of equi-measure **Ax(eqm)** below. Intuitively, **Ax(eqm)** says that all inertial observers agree on distances (which they can measure).

Ax(eqm) $(\forall m, k \in Obs \cap Ib)(\forall i, j \in n)(\forall p, q \in \bar{x}_i)(\forall p', q' \in \bar{x}_j)$
$$\Big([w_m(p) = w_k(p') \;\wedge\; w_m(q) = w_k(q')] \;\Rightarrow\; |p - q| = |p' - q'|\Big).^{310}$$

Connections between **Ax(eqm)** and the rest of our axioms are proved by the present author in §6.2.7 of AMN [18]. Among other things, she proves that **Ax(eqm)** is equivalent to the other symmetry axioms (most of which were introduced and discussed in §2.8 herein, cf. also the list of axioms herein[311]), under some conditions.

Let us note that $g(e, e_1)$ can easily become undefined, since either (i) there may exist no inertial observer m who sees e and e_1 on the same coordinate axis and no photon ph who is present both in e and e_1 or (ii) there may exist an infinity of inertial observers who measure smaller and smaller distances between e and e_1.

We will call g the *pseudo-metric*[312] of $\mathfrak{G}_{\mathfrak{M}}$ because it remotely does resemble a metric

[305] We could have achieved the "photons measure zero distance" effect by first using inertial observers only and then closing the concept of a distance under taking limits like we did in the definition of $\perp_r$ (from $\perp_0$).

[306] It is important in the definition of g that we required "$h \in Ib$" i.e. we used only inertial observers in measuring distances, because of the twin paradox cf. §2.8.5 (p.139). Namely, by the twin paradox to time-like separated events e, e_1 we can have accelerated observers who see e and e_1 closer and closer[307] (and therefore $g(e, e_1)$ would not be defined etc).

[307] This closeness would be not a property of e and e_1 instead it would only represent the extent of acceleration of the "measuring observer".

[308] As usual, $\min H$ denotes the minimal element (or smallest element) of the set H taken in the ordered set $\langle F, \leq \rangle$. Note that $\min H$ need not exist (even if $\mathfrak{F}$ is complete).

[309] I.e. before trying to remove min we would reformulate the definition of g according to the following pattern. $g(e, e_1) = 0$ if e, e_1 are on a photon-like line, otherwise $g(e, e_1) = \min\{\, \lambda \in F \;:\; (\exists m \in Obs \cap Ib) \ldots\}$.

[310] Cf. footnote 306 on p.145.

[311] Einstein's SPR^+ is also a symmetry principle, cf. p.84.

[312] In the relativity book Rindler [222, p.62 footnote 1] the expression *"pseudometric"* is used the same way as we use it here. For completeness we note that several other relativity works use a slightly different terminology. Namely, our $g : Mn \times Mn \overset{\circ}{\longrightarrow} F$ is a *variant* of what, in certain relativity works, is called a Lorentzian metric cf. e.g. Naber [198, p.83, line 8] or Wald [269, p.23, line 20] or Hawking-Ellis [116] (where a "metric" is really a bilinear function on a vector-space, like our ${}^n\mathbf{F}$; however this difference does not effect what is important for the present work).

(of a geometry) and because the elements of L will turn out to be so-called "geodesics" (cf. Def.4.7.2 on p.351) w.r.t. g, under very mild assumptions (e.g. $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqm)}$). It is important to note that a pseudo-metric g is usually *not* a metric because e.g. the "triangle inequality axiom of metrics" fails for g.[313]

13. $\mathcal{T}$ is the *topology*[314] on Mn determined by pseudo-metric g. In more detail:

    Let $e \in Mn$, $\varepsilon \in {}^+F$. The *ε-neighborhood* of e is defined as

    $$S(e,\varepsilon) :\stackrel{\text{def}}{=} \{ e_1 \in Mn \ : \ g(e,e_1) < \varepsilon \}.$$ [315]

    See Figures 123–126 (pp. 343–345) for how such neighborhoods can look like (there we use the word g-circle instead of neighborhood). Cf. also Fig.29 on p.51.

    Now, the topology $\mathcal{T} \subseteq \mathcal{P}(Mn)$ is the one generated by[316]

    $$T_0 :\stackrel{\text{def}}{=} \left\{ S(e,\varepsilon) \ : \ e \in Mn, \ \varepsilon \in {}^+F \right\},$$

    i.e. T_0 is a *subbase*[317] for the topology $\mathcal{T}$.[318]

    Alternative definitions for the topology part of $\mathfrak{G}_{\mathfrak{M}}$ are available in Definition 4.2.30 (p.175).

(II) Structures with the same similarity type[319] as that of $\mathfrak{G}_{\mathfrak{M}}$ are called structures *similar* to $\mathfrak{G}_{\mathfrak{M}}$. By an *isomorphism* between $\mathfrak{G}_{\mathfrak{M}}$ and $\mathfrak{G}_{\mathfrak{N}}$ we understand an isomorphism in the usual sense which is a homeomorphism[320] w.r.t. the topologies $\mathcal{T}_{\mathfrak{M}}$ and $\mathcal{T}_{\mathfrak{N}}$.[321] Since $\mathfrak{G}_{\mathfrak{M}}$ is a *three-sorted* structure (with sorts Mn, F and L) an isomorphism is a usual three sorted isomorphism, i.e. it consists of three functions, one defined on Mn, one on F, and one on L, cf. end of Convention 4.3.1 on p.220. The definition of an isomorphism for structures similar to $\mathfrak{G}_{\mathfrak{M}}$, is the same, but as we will see in Convention 4.2.4 (p.148) the membership relations $\in$ of our structures similar to $\mathfrak{G}_{\mathfrak{M}}$ always have to coincide with the standard, set theoretic membership relation.[322]

(III) By a *relativistic geometry* we understand an *isomorphic copy* of $\mathfrak{G}_{\mathfrak{M}}$, for some frame model $\mathfrak{M}$.[323]

---

[313] Under very mild assumptions on $\mathfrak{M}$, our g does satisfy the axioms $g(a,a) = 0$ and $g(a,b) = g(b,a)$ but it does not satisfy the remaining axioms usually required from metrics cf. e.g. James & James [138, p.232]. One of the axioms which fail for g is the triangle inequality $g(a,b)+g(b,c) \geq g(a,c)$, another one is $g(a,b) = 0 \Rightarrow a = b$.

[314] i.e. $\langle Mn, \mathcal{T} \rangle$ forms a topological space in the usual sense, cf. p.198 for a definition

[315] Note that, by our convention on equations involving partial functions, $g(e,e_1) < \varepsilon \Rightarrow (\, g(e,e_1)$ is defined$\,)$. Cf. Convention 2.3.10 on p.31.

[316] Where, "topology generated by T_0" means taking finite intersections first, and then infinite unions as usual. So $\mathcal{T} := \{ \bigcup Y \ : \ Y \subseteq \{ \bigcap X \ : \ X \text{ is a finite subset of } T_0 \} \}$.

[317] By a *subbase* for a topological space $\mathbf{X} = \langle X, \mathcal{O} \rangle$ we understand a set $\mathcal{H} \subseteq \mathcal{O}$ such that $\mathcal{H}$ generates $\mathcal{O}$ by finite intersections and infinite unions i.e. $\mathcal{O} := \{ \bigcup Y \ : \ Y \subseteq \{ \bigcap H \ : \ H \text{ is a finite subset of } \mathcal{H} \} \}$.

[318] In the "standard" literature the members of $\mathcal{T}$ are called the open sets of the topology $\mathcal{T}$. Cf. p.198.

[319] Recall that similarity type = vocabulary.

[320] We note that a homeomorphism between topologies is what the category theorist would call an isomorphism.

[321] Because of the presence of g, the homeomorphism condition is automatically satisfied, but in reducts from which g has been omitted this condition will become nontrivial.

[322] When looking at structures similar to $\mathfrak{G}_{\mathfrak{M}}$ we always assume that they satisfy the axiom of extensionality for $\in$.

[323] Therefore a relativistic geometry is nothing but the observer-independent geometry of some model $\mathfrak{M}$.

Let *Th* be a set of formulas in our frame language for relativity theory. Then the classes of relativistic geometries $Geom(Th)$ and $\mathsf{Ge}(Th)$ associated with *Th* are defined as follows.

Recall that for a class K of models $\mathbf{I}\mathsf{K}$ denotes the class of isomorphic copies of members of K.

Now,

$$\begin{aligned} Geom(Th) &\stackrel{\text{def}}{:=} \{\,\mathfrak{G}_{\mathfrak{M}} : \mathfrak{M} \in \mathsf{Mod}(Th)\,\}, \text{ and} \\ \mathsf{Ge}(Th) &\stackrel{\text{def}}{:=} \mathbf{I}Geom(Th), \text{ i.e.} \\ \mathsf{Ge}(Th) &= \{\,\mathfrak{G} : (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}\,\}. \end{aligned}$$

We will use the just introduced notation $\mathsf{Ge}(Th)$ in the spirit of Convention 4.2.4 (p.148) below.

In the terminology of algebraic logic $Geom(Th)$ is a *concrete class* while $\mathsf{Ge}(Th)$ is an *abstract class* of structures. The distinction between the two becomes important in duality theories (cf. §4.5 way below) and "representation theorems". Cf. footnote 287 on p.137, AMN [18, Remark 6.6.87 (p.1106)] and e.g. Andréka-Németi-Sain [31, the Remark below Def.42].

By a *Th geometry* we understand a member of $\mathsf{Ge}(Th)$. E.g. we will talk about **Basax** geometries. In the same spirit when in a theorem we discuss relativistic geometries then by writing "assume *Th*" we mean that the geometries in question are in $\mathsf{Ge}(Th)$.

(IV) We define the relations $\equiv^T, \equiv^{Ph}, \equiv^S \subseteq Mn \times Mn$ as follows.

$$\begin{aligned} e \equiv^T e_1 &\stackrel{\text{def}}{\iff} (\exists \ell \in L^T) \quad e, e_1 \in \ell. \\ e \equiv^{Ph} e_1 &\stackrel{\text{def}}{\iff} (\exists \ell \in L^{Ph}) \quad e, e_1 \in \ell. \\ e \equiv^S e_1 &\stackrel{\text{def}}{\iff} (\exists \ell \in L^S) \quad e, e_1 \in \ell. \end{aligned}$$

Intuitively: $e \equiv^T e_1$, that is e and e_1 are *time-like separated*, iff there is an inertial observer m which is present both in e and e_1. $e \equiv^{Ph} e_1$, that is e and e_1 are *photon-like separated*[324] iff there is a photon ph which is present both in e and e_1. Further, e and e_1 are called *space-like separated* iff $(\exists m \in Obs \cap Ib)$ m thinks that e and e_1 are simultaneous.[325]

(V) Let $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Then the relation $\|_{\mathfrak{G}}$ of parallelism in $\mathfrak{G}$ is defined in item (I).9 above (p.140).

(VI) We define the *Reichenbachian version* of the geometric structure corresponding to $\mathfrak{M}$ as follows:[326]

$$\mathfrak{G}^R_{\mathfrak{M}} \stackrel{\text{def}}{:=} \langle Mn, \mathbf{F_1}, L^R;\ L^T, L^{Ph}, \in, \prec, Bw, g^R, \mathcal{T}^R \rangle,$$

where $Mn, \mathbf{F_1}, L^T, L^{Ph}, \prec, Bw$ are as defined in item (I), $\in$ is the set theoretic membership relation between Mn and $L^R := L^T \cup L^{Ph}$ and $g^R, \mathcal{T}^R$ are defined in items 1 and 2 below.

1. $g^R(e, e_1) \stackrel{\text{def}}{:=} \min\{\,\lambda \in F : (\exists ph \in Ph)\,[\,ph \in e \cap e_1 \ \wedge\ \lambda = 0]$ or
$(\exists m \in Obs \cap Ib)(\exists p, q \in \bar{t}\,)\,[w_m(p) = e \ \wedge\ w_m(q) = e_1 \ \wedge\ \lambda = |p - q|\,]\,\}$,
if this min exists, *otherwise* $g^R(e, e_1)$ is undefined.

[324] In the literature this is often called *null-separated*. The word null comes from the fact that (if $e \neq e_1$ then) $e \equiv^{\mathrm{Ph}} e_1 \ \Leftrightarrow\ g(e, e_1) = 0$.
[325] The connection between $\equiv^S$ and space-like separateness is discussed in Prop.6.2.56(ii), p.858 of AMN [18].
[326] Cf. §4.5 of AMN [18] for motivation.

2. $\mathcal{T}^R$ is the topology on Mn determined by the pseudo-metric g^R. In more detail: Let $e \in Mn$ and $\varepsilon \in {}^+F$. Then

$$S^R(e,\varepsilon) :\stackrel{\text{def}}{=} \{\, e_1 \in Mn \,:\, g^R(e,e_1) < \varepsilon,\ e \neq e_1 \,\}.$$

Now, the topology $\mathcal{T}^R \subseteq \mathcal{P}(Mn)$ is the one generated by

$$T_0^R :\stackrel{\text{def}}{=} \{\, S^R(e,\varepsilon) \,:\, e \in Mn,\ \varepsilon \in {}^+F \,\},$$

i.e. T_0^R is a subbase for the topology $\mathcal{T}^R$.[327]

We note that a somewhat richer, improved version of the geometry $\mathfrak{G}_{\mathfrak{M}}$ will be defined in §4.5.5 on p.332, it will be denoted as $\mathfrak{G}^*_{\mathfrak{M}}$.

END OF DEFINITION OF $\mathfrak{G}_{\mathfrak{M}}$ AND RELATED DEFINITIONS.

$\lhd$

A discussion of the *intuitive meaning* of (parts of) $\mathfrak{G}_{\mathfrak{M}}$ will be given on pp. 182–185. Connections with the literature will be discussed beginning with p.215 (§4.2.6).[328]

The following convention is only a matter of convenience and does not have far reaching consequences. It is motivated by the fact that the axiom of extensionality holds in $\mathfrak{G}_{\mathfrak{M}}$ (for $\in$ connecting Mn and L), therefore it holds in any isomorphic copy $\mathfrak{G}'_{\mathfrak{M}}$ of $\mathfrak{G}_{\mathfrak{M}}$. Therefore we do not lose generality if we assume that "$\in$" is the real set theoretic membership in $\mathfrak{G}'_{\mathfrak{M}}$, too. To make $\in$ the real one in $\mathfrak{G}'_{\mathfrak{M}}$ the only change we have to make is renaming the lines. Cf. Convention 2.1.3 (p.10), footnote 294 on p.139 and the text below $\mathbf{Ax}_{\mathrm{G}}$ on p.7.

CONVENTION 4.2.4 Let $\mathfrak{G} := \langle Mn, \ldots, L; \ldots, \in, \ldots\rangle \in Geom(\emptyset)$. By an isomorphic copy $\mathfrak{G}'$ of $\mathfrak{G}$ we understand an isomorphic copy in the usual sense as it was explained in item (II) of Def.4.2.3, but with the *restriction* that $\in'$ of $\mathfrak{G}'$ is the real, set theoretic membership relation,[329] cf. Convention 2.1.3 on p.10.

The definition of $\mathsf{Ge}(Th)$ is understood accordingly. Hence $\mathsf{Ge}(Th) = \{\, \mathfrak{G}' \,:\, (\exists \mathfrak{G} \in Geom(Th))\, \mathfrak{G} \cong \mathfrak{G}'$ and $\in'$ of $\mathfrak{G}'$ is the real set theoretic membership relation$\,\}$. Throughout we understand isomorphism closedness of classes of structures in this sense. In this chapter we concentrate on isomorphism closed classes of structures (with the above restriction on $\in$). It is important to emphasize that isomorphism closed classes of models are more important for us than the rest. We also emphasize that the restriction on $\in$ does not contradict our philosophy of concentrating on isomorphism closed classes. In particular we consider $\mathsf{Mod}(Th)$

[327] We note that under some assumptions the topology $\mathcal{T}^R$ agrees with the usual Euclidean topology. E.g. $\mathbf{Reich(Basax)} + \mathbf{R(sym)} + \mathbf{Ax(}\mathit{Triv}\mathbf{)}$ is enough for this. Further we note that the alternative topologies $\mathcal{T}', \mathcal{T}''$ (basically equivalent to $\mathcal{T}$) defined in Def.4.2.30 (p.175) (see Fig.81, p.176) can be used here in the Reichenbachian approach too as alternative possibilities for defining $\mathfrak{G}^R_{\mathfrak{M}}$.

[328] In passing, we note that Busemann [55] obtains very attractive results by using a geometric structure similar to the following version $\mathfrak{G}^B_{\mathfrak{M}}$ of our $\mathfrak{G}^R_{\mathfrak{M}}$. $\mathfrak{G}^B_{\mathfrak{M}} := \langle Mn, \mathbf{F_1}, L^T, L^{Ph}; \in, \prec, g^R, \mathcal{T}^R\rangle$. More precisely, but still very roughly speaking, Busemann uses only (a version of) the $\mathbf{G}^B_{\mathfrak{M}} := \langle Mn, \mathbf{F_1}; \prec, g^R, \mathcal{T}^R\rangle$ reduct of $\mathfrak{G}^B_{\mathfrak{M}}$ and recovers L^T as geodesics in the sense of §4.7 way below. (L^{Ph} is definable in Busemann's structures $\mathfrak{G}^B_{\mathfrak{M}}$.) Then he introduces the local version of $\mathbf{G}^B_{\mathfrak{M}}$ with which he obtains very attractive insights into the problem of obtaining transparent axiomatizations of (aspects of) general relativity. Cf. §4.6.1 (p.346).

[329] We note that this does not cause loss of generality.

and $\mathsf{Ge}(Th)$ as being closed under isomorphisms. I.e. we consider $\mathsf{Mod}(Th) = \mathbf{I}\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th) = \mathbf{I}\mathsf{Ge}(Th)$.

Our conventions concerning the symbol $\in^{\mathfrak{G}}$ (or $\in$ of $\mathfrak{G}$) can be summarized and clarified by postulating that we are working in an extremely weak version of higher-order logic where $\in$ is considered as a logical symbol. This applies to our frame language as well as to our various geometric languages. For more detail on this (i.e. "$\in$" and higher-order logic reduced to many-sorted one etc.) we refer to the Appendix ("Why first-order logic?") of AMN [18]. In order to keep things simple we leave it to the reader to elaborate the logical machinery of treating our various $\in$ symbols as logical symbols. Cf. e.g. $\mathfrak{M}^+$ on p.153 for a case when there are more than one incarnations of the $\in$ symbol in the same language. In such cases one uses $\in$ with appropriate subscripts.

$\lhd$

Remark 4.2.5 Our convention that we regard isomorphic relativistic geometries as *identical* is important for the philosophy of the present chapter. Without this convention it would be easy to cheat. Namely, it would be too easy to "geometrize" relativity (or any other theory, for that matter) so that from the geometry say $\mathfrak{G}^+_{\mathfrak{M}}$ associated with an observational model $\mathfrak{M}$ all properties of the original model $\mathfrak{M}$ would be recoverable. E.g. let $\mathfrak{G}^+_{\mathfrak{M}}$ be obtained from $\mathfrak{G}_{\mathfrak{M}}$ (in Def.4.2.3) by replacing the universe $Mn = Points^{\mathfrak{G}_{\mathfrak{M}}}$ of $\mathfrak{G}_{\mathfrak{M}}$ by $Mn \times \{\mathfrak{M}\}$ in such a way that $\mathfrak{G}^+_{\mathfrak{M}} \cong \mathfrak{G}_{\mathfrak{M}}$ holds. But now, it is a trivial matter to recover $\mathfrak{M}$ from the "concrete" geometry $\mathfrak{G}^+_{\mathfrak{M}}$ since $\mathfrak{M}$ is sitting inside the elements of $\mathfrak{G}^+_{\mathfrak{M}}$ (we can find it if we are willing to "dig" deep enough along the set theoretic membership relation). Actually $\mathfrak{M}$ can be obtained by applying the projection function pj_1 to any element $e \in Points^{\mathfrak{G}^+_{\mathfrak{M}}} (= Mn \times \{\mathfrak{M}\})$. Further[330] $\mathfrak{M} = \bigcup pj_1[Points^{\mathfrak{G}^+_{\mathfrak{M}}}]$, since $\bigcup\{x\} = x$, where pj_1 is the projection function $\langle a, b\rangle \mapsto b$ associating the 1st member of a sequence with the sequence (cf. p.232 for pj_1).

But, if we *define*[331] $\mathbf{I}\mathfrak{G}_{\mathfrak{M}}$ (or equivalently $\mathbf{I}\mathfrak{G}^+_{\mathfrak{M}}$) to be the *geometric counterpart* of $\mathfrak{M}$ [332] then the above trick does not work (for reconstructing $\mathfrak{M}$ from its geometric counterpart in a cheap way). Defining the geometric counterpart this way is the same as defining the geometry of $\mathfrak{M}$ only up to isomorphism.

The presently discussed convention makes some of our theorems in the duality theory section stronger. Moreover, it provides formal justification for Einstein's remark to the effect that it is interesting that relativistic physics can be fully geometrized, cf. e.g. Misner-Thorne-Wheeler [192].[333]

$\lhd$

CONVENTION 4.2.6 Throughout, by the *space-time* of a model $\mathfrak{M}$ we mean either the geometry $\mathfrak{G}_{\mathfrak{M}}$ or a reduct of $\mathfrak{G}_{\mathfrak{M}}$ like e.g. $\mathsf{G}_{\mathfrak{M}}$ in Def.4.2.3.(I) (p.138).

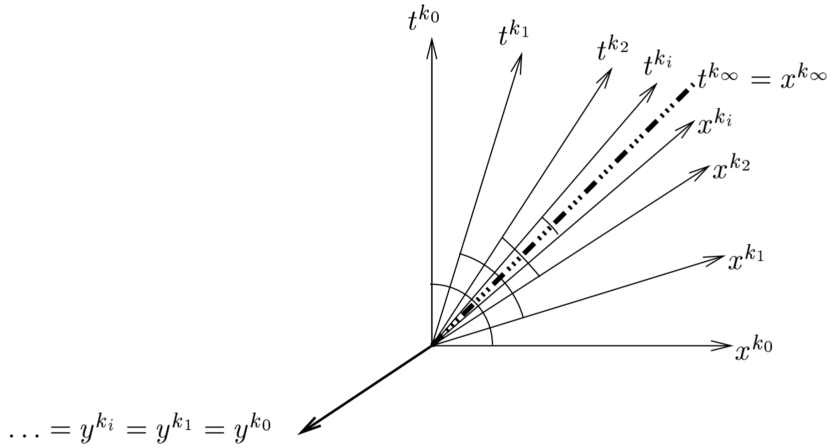
$\lhd$

Remark 4.2.7 (Intuitive motivation for our definition of $\perp_r$)[334]
For simplicity, in the present discussion we are assuming $\mathbf{Bax}^{\oplus}(4) + \mathbf{Ax}(Triv_t) + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ })$ but most of these assumptions are *not* essential (i.e. they could be eliminated on

[330] It is not necessary to understand this formula, for understanding the rest of this work.
[331] For any structure $\mathfrak{G}$, $\mathbf{I}\mathfrak{G} := \mathbf{I}\{\mathfrak{G}\}$ is the class of isomorphic copies of $\mathfrak{G}$.
[332] i.e. we insist on "geometric counterpart of $\mathfrak{M}$" $= \mathbf{I}$"geometric counterpart of $\mathfrak{M}$"
[333] Geometrizability of relativity (or any other theory) would be vacuously true without the condition "geometric counterpart of $\mathfrak{M}$" $= \mathbf{I}$"geometric counterpart of $\mathfrak{M}$".
[334] We note that for the case $n = 2$ Goldblatt [102] defines $\perp_r$ practically the same way as we do, and he provides intuitive motivation which is also similar to ours (e.g. uses limits) cf. Goldblatt [102, p.6 lines 9–6 bottom up and p.8 first 6 lines].

the expense of making the text longer). In particular c_m is the speed of light for m. Also, throughout the present remark we assume $n = 4$ (i.e. we are in four dimensions) but, when it *does not matter*, we often talk as if we were in three dimensions e.g. this is what we do in the pictures.

We would like to base our definition of $\perp_r$ on the intuitive observer-oriented notions in $\mathfrak{M}$. So, what corresponds to orthogonality in $\mathfrak{M}$? Well, two coordinate axes of any observer are considered "orthogonal". So, we would like to say that two lines are $\perp_r$-orthogonal if some observer thinks that they are parallel with two of his coordinate axes. The problem with this is that then no photon-like line will be orthogonal to *any* line because photon-like lines are not parallel with any coordinate axis of any observer. The reason for this is, roughly, that no observer can move with the speed of light, i.e. $v_m(k) \neq c_m$ for any m, k. But this can be circumnavigated because we can have observers whose speed is arbitrarily close to c_m, i.e. we can have a sequence $k_0, k_1, \ldots \in Obs$ with $\lim_{i \mapsto \infty} v_m(k_i) = c_m$.[335] Cf. the picture on p.150.



Let such $m, k_0, k_1, \ldots$ be fixed. Assume

$$\forall i\ (\, m \text{ and } k_i \text{ are in strict standard configuration} \quad \text{and} \quad m \uparrow k_i \,).$$

Now, we are working in the world-view of m. To ensure existence of limits let us work with F^∞ instead of F. We can try to construct an <u>*imaginary observer*</u> k_∞ as the limit of the sequence $k_0, k_1, \ldots, k_i, \ldots$ $(i \in \omega)$ of "real" observers, in some sense.[336] So the intuitive idea is to "define" $k_\infty :=$ "$\lim_{i \mapsto \infty}(k_i)$" and $\mathsf{f}_{mk_\infty} :=$ "$\lim_{i \mapsto \infty}(\mathsf{f}_{mk_i})$". We did *not* define what we mean by $\lim_{i \mapsto \infty}(k_i)$, but we can define at least "parts" of this imaginary observer k_∞ $(= \lim_{i \mapsto \infty}(k_i))$. Cf. the picture on p.150. E.g. we can choose the coordinate axes of k_∞ to be

$$\bar{t}_\infty := \lim_{i \mapsto \infty}(\bar{t}_i)\ (\in L^{Ph})^{337}, \quad \bar{x}_\infty := \lim_{i \mapsto \infty}(\bar{x}_i), \quad \bar{y}_\infty := \lim_{i \mapsto \infty}(\bar{y}_i), \quad \bar{z}_\infty := \lim_{i \mapsto \infty}(\bar{z}_i),$$

where we use the notation $\bar{t}_i = \mathsf{f}_{k_i m}(\bar{t})$ and $1^i_t = \mathsf{f}_{k_i m}(1_t)$, and similarly for $\bar{x}_i, \bar{y}_i, \bar{z}_i$ and for $1^i_x, 1^i_y, 1^i_z$. To ensure existence of the time unit vector 1^∞_t of the imaginary observer k_∞, we define the limit of a growing sequence like $\langle 1, 2, 3, \ldots, i, \ldots \rangle$ of members of F to be ∞. Further for the sake of (nice behavior e.g. convergence of) the unit vectors we assume $\mathbf{Ax(symm)}^\dagger$. However, at the same time we would like to emphasize, that for the present argument about

[335] It is possible that we need sequences longer than ω for this limit to exist but that does not change anything essential.

[336] For a similar train of thought (or construction) cf. AMN [18, Figure 254 on p.749 and §5.1 (pp. 744–750)].

[337] For simplicity we write L^{Ph} for $\{\, tr_m(ph) : ph \in Ph \,\}$, in the present remark.

$\perp_r$ we do *not* need the unit vectors, hence **Ax(symm)**† is not really needed here (we assumed it only for making our "picture prettier"). Then we can define

$$1_t^\infty := \lim_{i\mapsto\infty}(1_t^i), \quad 1_x^\infty := \lim_{i\mapsto\infty}(1_x^i), \quad 1_y^\infty := \lim_{i\mapsto\infty}(1_y^i), \quad 1_z^\infty := \lim_{i\mapsto\infty}(1_z^i).$$

This way we will obtain

$\bar{t}_\infty = \bar{x}_\infty \in L^{Ph}$, $\quad \bar{y}_\infty = \bar{y}, \quad \bar{z}_\infty = \bar{z},$
$1_y^\infty = 1_y, \quad 1_z^\infty = 1_z;$ further
$1_t^\infty = 1_x^\infty =$ "the infinitely long vector pointing in the photon-like direction $\bar{t}_\infty$".
More formally,
$1_t^\infty = 1_x^\infty = \langle\infty, \infty, 0, 0\rangle$.

See Figure 69. In the present remark, *Space*$_k$ denotes the hyper-plane which m "thinks" is k's space. I.e. *Space*$_k$ is the f_{km}-image of the hyper-plane *Space* $\overset{\text{def}}{=} S \subseteq {}^nF$ determined by $x_1, \ldots, x_{n-1}$. I.e. *Space*$_k = \mathsf{f}_{km}[\mathit{Space}]$. As Figure 69 shows, our *imaginary observer* k_∞ has some exotic features. E.g. its space *Space*$_{k_\infty} \subseteq {}^n(F^\infty)$ is a Robb hyper-plane[338], i.e. *Space*$_{k_\infty}$ is a hyper-plane tangent to the light-cone. It contains $\bar{y}$, $\bar{z}$ and a photon-like line $\bar{t}_\infty = \bar{x}_\infty$. Though k_∞ is only an imaginary observer, studying its mathematical structure can give us

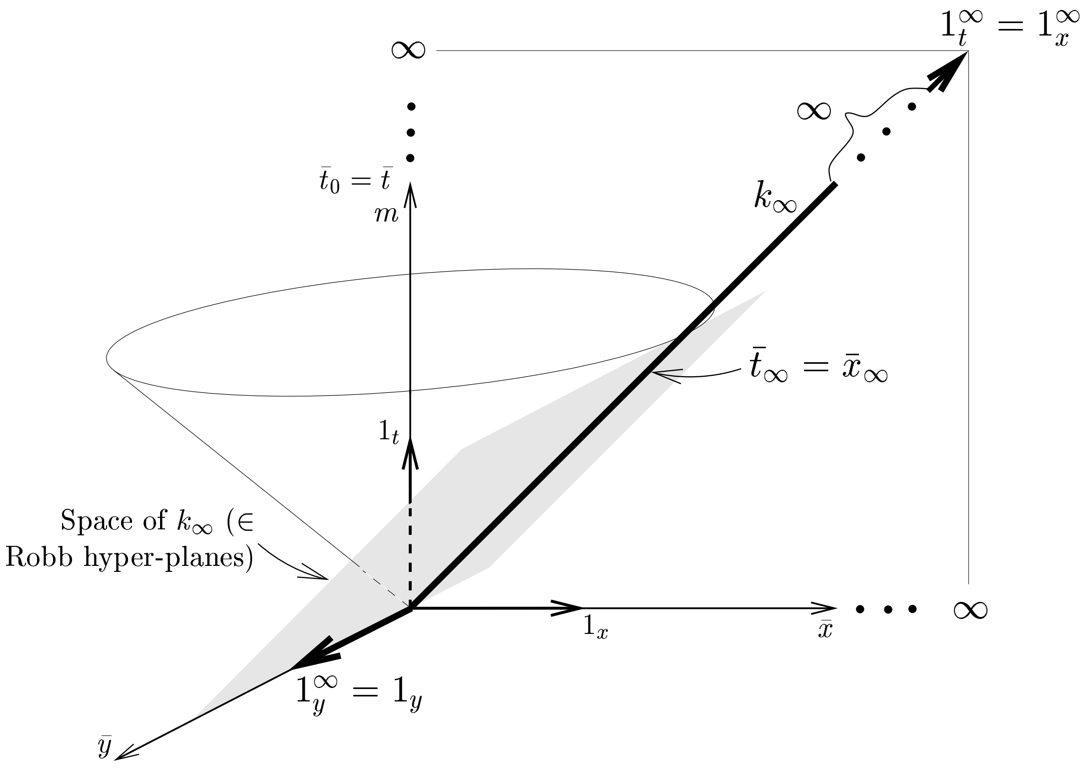


Figure 69: The axes of the imaginary observer k_∞.

insight e.g. to the structure of $\mathfrak{G}_{\mathfrak{M}}$. k_∞ does not satisfy our axiom $\mathbf{Ax6_{00}}$, i.e. k_∞ does not see most of the events m sees, *but* k_∞ does see the events on $\mathsf{Plane}(\bar{y}, \bar{z})$ since 1_y^∞ and 1_z^∞ are finite (and agree with 1_y and 1_z, respectively). In the direction $\langle 1, 1, 0, 0\rangle$ however k_∞ is "blind": of

[338] Cf. e.g. Robb [223] or Goldblatt [102] for the Robb hyper-plane (called in [102] Robb threefold) cf. also p.1163 in AMN [18].

the events on his life-line $\bar{t}_\infty = tr_m(k_\infty)$ he sees only the event at the origin $\bar{0}$ because k_∞'s unit vectors $(1_t^\infty, 1_x^\infty)$ in this direction[339] are too long.

Let us return to relativistic orthogonality $\perp_r$. Our k_∞ thinks that his axes $\bar{t}_\infty$, $\bar{x}_\infty$, $\bar{y}_\infty$, $\bar{z}_\infty$ are orthogonal, therefore according to our philosophy for defining $\perp_r$ it is natural that we wish to have

$$\bar{t}_\infty = \bar{x}_\infty \perp_r \bar{t}_\infty \perp_r \bar{y} \perp_r \bar{z} \qquad \text{etc.}$$

How could we achieve this (e.g. $\bar{t}_\infty \perp_r \bar{y}$) in a natural way? Well, k_∞ was obtained from real observers k_i by taking a limit. Parts of k_∞ (e.g. the coordinate axes of k_∞) were also obtained by the same limit procedure. Therefore, all this suggests that we should close our relativistic orthogonality $\perp_0$ up under taking limits and then probably this will yield for us those orthogonal pairs (like $\bar{t}_\infty \perp_r \bar{t}_\infty$, $\bar{t}_\infty \perp_r \bar{y}$ etc.) which are coordinate axes of imaginary observers which in turn were obtained by a limit procedure analogous to the one with which we obtained k_∞.

In passing, we also note the following. k_∞ thinks that his time axis $\bar{t}_\infty$ is orthogonal to his space, $Space_{k_\infty}$, which in turn is the hyper-plane generated by $\{\bar{x}_\infty = \bar{t}_\infty, \bar{y}, \bar{z}\}$. Hence k_∞ will think that $\bar{t}_\infty$ is orthogonal to *any* line in this hyper-plane.[340] Thus, any photon-like line in a Robb hyper-plane is expected to be $\perp_r$-orthogonal to all lines in that hyper-plane.

Summing it up, on a very-very informal level we could say the following. Of course speed-of-light observers cannot exist. But *if* they existed they would behave like k_∞ does.[341] In claiming this we are relying on the "rule of thumb" that in physics everything is continuous (i.e. is preserved under taking limits). We emphasize that the above train of thought is *not* a precise mathematical argument, and it should not be taken too seriously[342], it only serves to help the intuition about some parts of $\mathfrak{G}_\mathfrak{M}$ (especially about $\perp_r$).

Concerning the above intuitive remark we also refer to Goldblatt [102, middle of page 13] for an analogous argument.

$\triangleleft$

Besides discussing definability issues and alternative definitions, the next sub-section can also serve to improve our intuitive understanding of certain parts of $\mathfrak{G}_\mathfrak{M}$.

4.2.2 On first-order definability of observer-independent geometry over observational concepts; and alternative definitions for $\perp_r$, eq, $\mathcal{T}$

The parts of our observer-independent geometry $\mathfrak{G}_\mathfrak{M}$ can be considered as "theoretical" concepts as opposed to the parts of $\mathfrak{M}$ which in turn can be considered as "observational". Here we use the observational/theoretical distinction as introduced and discussed e.g. in Friedman [91]. The observational/theoretical distinction is known to be relative, hence we are aware of the fact that someone might challenge the observational status of $\mathfrak{M}$, but let us consider observationalness of $\mathfrak{M}$ as a working hypothesis only. There is a long tradition (going back e.g. to Mach,

[339] i.e. in the direction of $\langle 1, 1, 0, 0\rangle$
[340] Cf. Proposition 6.2.51 (p.856) of AMN [18].
[341] Practically the same argument is found in Goldblatt [102, p.8 lines 4–7].
[342] e.g. it uses "rules of thumb" which are *not* axioms in our theories

Carnap) in theoretical physics where people try to restrict attention to such theoretical concepts which are *definable* in terms of observational ones[343], cf. the introduction to the present chapter (§4.1, p.129), cf. also e.g. Friedman [91]. This (among other things) motivates our asking ourselves[344] whether parts of $\mathfrak{G}_{\mathfrak{M}}$ are definable in first-order logic over $\mathfrak{M}$, and more generally whether $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$. Indeed, e.g. in Theorem 4.2.40 (p.182) we will see results in the direction that $\mathsf{Ge}(Th)$ is first-order definable over $\mathsf{Mod}(Th)$, under mild assumptions. Of course, we begin studying definability of $\mathfrak{G}_{\mathfrak{M}}$ (over $\mathfrak{M}$) by discussing definability of parts of $\mathfrak{G}_{\mathfrak{M}}$ over $\mathfrak{M}$. In passing, we also note that the above sketched ideas serve as part of the motivation for our section 4.3 on definability (and for our concern for definability issues throughout the present Chapter 4).

We will use the notion of (first-order logic) *definability* of a new structure say $\mathfrak{N}^+$ in[345] an "old" structure, say, $\mathfrak{N}$. Here $\mathfrak{N}^+$ is an expansion of $\mathfrak{N}$ possibly both with new sorts and new relations. Intuitively, $\mathfrak{N}^+$ looks like

$$\mathfrak{N}^+ = \langle \mathfrak{N}, U_1^{new}, \dots, U_j^{new}; R_1^{new}, \dots, R_l^{new} \rangle$$

where U_i^{new} are new sorts and R_i^{new} are new relations. Such a definition of $\mathfrak{N}^+$ in $\mathfrak{N}$ induces an *interpretation* of the language $Fm(\mathfrak{N}^+)$ of $\mathfrak{N}^+$ in the language $Fm(\mathfrak{N})$ of $\mathfrak{N}$,[346] like

$$\mathrm{intrp} : Fm(\mathfrak{N}^+) \longrightarrow Fm(\mathfrak{N}),$$

cf. Theorems 4.3.27 (p.245) and 4.3.29 (p.247) (in those theorems we will write "*Tr*" instead of "intrp"). In more detail, the basic concepts of "definability theory" (also called "the theory of definability") elaborated for the case of many-sorted first-order logic (i.e. definability of a new sort) will be discussed in §4.3 beginning with p.218 way below.[347]

CONVENTION 4.2.8 By definability we automatically mean explicit definability throughout the present work, cf. §4.3.2. Similarly, first-order logic definability also means explicit definability. The adjective "first-order logic" is there only to emphasize that our explicit definitions of new relations will be formulas of first-order logic as one would expect. Similarly "definition" means explicit definition in the sense of §4.3.2.

$\lhd$

Remark 4.2.9 (On first-order definability of $\mathfrak{G}_{\mathfrak{M}}$ in $\mathfrak{M}$.)
Let $\mathfrak{M}$ be a frame model and let $\mathfrak{G}_{\mathfrak{M}}$ be the observer-independent geometry corresponding to $\mathfrak{M}$ (defined in Def.4.2.3(I) above). We will look into the question whether the ingredients of $\mathfrak{G}_{\mathfrak{M}}$ are definable in $\mathfrak{M}$ using first-order logic (i.e. using our first-order frame language) or, more boldly, whether $\mathfrak{G}_{\mathfrak{M}}$ is first-order definable in $\mathfrak{M}$. In the present remark, for simplicity, instead of $\mathfrak{M}$ we will use its expansion

$$\mathfrak{M}^+ := \langle \mathfrak{M}, Mn, L; \in_{Mn}, \in_L \rangle$$

[343]In our opinion, "definable" should almost always mean definable in some system of logic, e.g. in the logic being used by the "speaker". Therefore, in the present work, definable always means definable in the language of (many-sorted) first-order logic.

[344]cf. p.130

[345]"Definable in" means the same as "definable over".

[346]$Fm(\mathfrak{N})$ is the set of formulas in the "language" of $\mathfrak{N}$, cf. Convention 4.3.26 (p.245).

[347]We will return to defining (but only informally) interpretations on p.263 Fig.96 and on p.1023 of AMN [18]. Cf. also p.251 and footnote 545 (on p.251). [The above $\mathrm{intrp} : Fm(\mathfrak{N}^+) \longrightarrow Fm(\mathfrak{N})$ is only a special kind of interpretations (involving only one of the many possible ways of using this concept).]

where events Mn and lines L are new sorts as defined in items 2, 4 of Def.4.2.3(I) and $\in_{Mn}, \in_L$ are the restrictions of the usual set theoretic membership relation $\in$ to $B \times Mn$ and $Mn \times L$, respectively. We will see in Proposition 4.3.18 (p.240) that

($\star$) $\mathfrak{M}^+$ is first-order definable[348] in $\mathfrak{M}$, moreover this definition is uniform for the whole class FM of frame models.

This justifies our decision of studying definability in $\mathfrak{M}^+$ instead of definability in $\mathfrak{M}$. Whenever below we say that something is definable in $\mathfrak{M}^+$ then by ($\star$) above this *automatically* means that that thing is *also* definable in $\mathfrak{M}$. So, we ask ourselves which parts (ingredients) of the geometry $\mathfrak{G}_{\mathfrak{M}}$ are first-order definable in the expanded frame model $\mathfrak{M}^+$. Let us notice that the definitions of all ingredients of $\mathfrak{G}_{\mathfrak{M}}$, given in Def.4.2.3(I) above, <u>*are*</u> indeed first-order definitions in $\mathfrak{M}^+$ <u>*except*</u> for the relation $\perp_r$ of relativistic orthogonality, the relation eq of equidistance, and the topology $\mathcal{T}$. Therefore, it is sufficient to discuss here definability of $\perp_r$, eq and $\mathcal{T}$ in $\mathfrak{M}^+$. Let us turn to doing this.

On $\perp_r$: The relation $\perp_0$ <u>is</u> first-order definable. This gives us a promising start (for checking definability of $\perp_r$), but disappointingly, the definition of *relativistic orthogonality* $\perp_r$ (item 10 of Def.4.2.3.(I) on p.140) involves closing $\perp_0$ up under taking limits, then closing up under parallelism, and then iterating this two-step procedure arbitrarily many times. Clearly this definition in its present form is not a first-order one. As we indicated, we can translate the step of closing up under limits and the step of closing up under parallelism into our first-order frame language, cf. pp. 142, 331, but it is not completely obvious how to translate iteration into first-order logic. (The iteration comes into the picture when we say that $\perp_r$ is the smallest set with certain properties [this happens above item (i) in the definition of $\perp_r$].)[349] In Definitions 4.2.10, 4.2.17 below we give three alternative definitions for $\perp_r$, which are (i) in the (first-order) language of $\mathfrak{M}^+$, and (ii) they are equivalent to the original definition of $\perp_r$, under some assumptions on $\mathfrak{M}$, e.g. $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax6}$ (cf. Theorems 4.2.11, 4.2.19). Therefore $\perp_r$ *becomes first-order definable* in $\mathfrak{M}^+$, under some assumptions on $\mathfrak{M}$. Another use of exploring alternatives for $\perp_r$ (and proving equivalence) is that we obtain some insights into "how $\perp_r$ works".

On *eq*: The relation eq of equidistance (item 11 of Def.4.2.3.(I) on p.142) was defined to be the transitive closure of the relation eq_0 of basic equidistance, so it uses the set of natural numbers ω. Being a natural number is usually not first-order definable in $\mathfrak{M}^+$. Hence[350] the definition of eq is not a first-order definition in $\mathfrak{M}^+$. Let us recall that for every $i \in \omega \quad eq_i$ was defined to be the "i-long-transitive closure" of eq_0. Let us notice that the definition of each one of our relations eq_i is indeed a first-order definition in $\mathfrak{M}^+$. In Theorems 4.2.21, 4.2.22 we will see that eq_2 coincides with eq under some assumptions on $\mathfrak{M}$, e.g. $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\|) + \mathbf{Ax}(\sqrt{\ })$. Hence *the relation eq of equidistance becomes first-order definable* in $\mathfrak{M}^+$, under some assumptions on $\mathfrak{M}$.[351]

On $\mathcal{T}$: Let us recall that the topology $\mathcal{T}$ was defined from the subbase

$$T_0 = \left\{ S(e, \varepsilon) \ : \ e \in Mn, \ \varepsilon \in {}^+F \right\}.$$

[348] I.e. $\mathfrak{M}^+$ is rigidly definable over $\mathfrak{M}$ in the sense of §4.3.2.

[349] In theory it is possible that one could prove that the above mentioned iteration (of taking limits and parallels) terminates in a bounded finite number of steps, under certain assumptions. If that is the case then the original definition of $\perp_r$ will get translated into our first-order frame language. However, we did not have time to think about this direction. Instead of pursuing this direction (i.e. checking whether iteration stops) we explore alternative definitions for relativistic orthogonality.

[350] Transitive closure is a typical example of (usually) not first-order definable concepts.

[351] If we make no assumptions, eq becomes undefinable in some frame models $\mathfrak{M}$, cf. Thm.4.2.23 (p.168).

We will see in Proposition 4.3.19 (p.242) that

($\star\star$) the subbase T_0 for $\mathcal{T}$ is first-order definable[352] in $\mathfrak{M}$, and that this definition is uniform for the whole class FM of frame models.

By ($\star\star$), we consider the topology $\mathcal{T}$ as first-order definable in $\mathfrak{M}$, however *we do not discuss* here which basic concepts of topology are first-order definable, e.g. we do not discuss whether the set of open subsets of $\mathcal{T}$ (i.e. $\mathcal{T}$ itself) is first-order definable.

($\star\star\star$) To be honest, we should call $\mathcal{T}$ first-order definable only if a *base*[353] , say T, for $\mathcal{T}$ is first-order definable. (This is so because then standard notions of topology, e.g. continuity would become expressible by using T which in turn is definable.) To pursue this direction we should investigate the question, under what conditions (axioms) does definability of a subbase T_0 imply definability of a base T. However, in the present work we do not want to investigate this direction. Therefore (perhaps slightly misleadingly) we call $\mathcal{T}$ definable if a subbase T_0 for $\mathcal{T}$ is definable. Investigating the question of under what assumptions is a base T for $\mathcal{T}$ definable (over FM) remains a task for future research. For a similar notion of explicit definability of a topology $\mathcal{T}$ we refer to the model theory book Barwise-Feferman [45, p.567, lines 5-8, §3.3 (Definability) of Chap.XV].

We will introduce alternative versions $\mathcal{T}'$ and $\mathcal{T}''$ for the definition of the topology part of our observer-independent geometry in Definition 4.2.30 (p.175). From the point of view of first-order definability over $\mathfrak{M}$, $\mathcal{T}''$ will behave just as nicely as $\mathcal{T}$ does (cf. Prop.4.3.20, p.243) while to ensure nice behavior of $\mathcal{T}'$ we will assume $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$ (cf. Prop.4.3.21, p.243).

As a corollary of ($\star$), ($\star\star$) above and Theorems 4.2.11, 4.2.19, 4.2.21, 4.2.22 below we obtain that under reasonably mild assumptions Th on our models $\mathfrak{M} \in \mathsf{Mod}(Th)$, the *geometry* $\mathfrak{G}_{\mathfrak{M}}$ is *definable in first-order logic* in the structure $\mathfrak{M}$. Moreover this definition is uniform for the whole class $\mathsf{Mod}(Th)$, see Theorems 4.3.22 (p.244) and 4.3.24, cf. also Theorems 4.3.25 and 4.2.40. Hence $\mathsf{Ge}(Th)$ is uniformly definable over $\mathsf{Mod}(Th)$.

As we have already said, more on definability theory can be found in §4.3 way below.

$\lhd$

In §4.3 we will give a precise definition of what we mean by first-order definability of $\mathfrak{G}_{\mathfrak{M}}$ in $\mathfrak{M}$ (or over $\mathfrak{M}$). After §4.3 we will prove that $\mathfrak{G}_{\mathfrak{M}}$ is indeed first-order definable over $\mathfrak{M}$, under some mild conditions.

In Definition 4.2.10 below we give two alternative definitions $\perp_r'$ and $\perp_r''$ for the relation $\perp_r$ of relativistic orthogonality. The advantages of the definitions of $\perp_r'$ and $\perp_r''$ over the definition of $\perp_r$ will be that (i) they will be first-order definitions (in the expanded frame models $\mathfrak{M}^+$ defined in Remark 4.2.9, p.153) and (ii) they will be easier to understand. However, we consider the definitions of $\perp_r'$ and $\perp_r''$ less natural than that of $\perp_r$, because they (i.e. the definitions of $\perp_r'$ and $\perp_r''$) use case-distinctions, i.e. they distinguish photon-like lines from the rest of the lines, cf. items (iii) and (iv)' of Def.4.2.10 below. In Definition 4.2.17 (p.161) way below we give two further alternative definitions $\perp_r'''$ and $\perp_r^\omega$ (for relativistic orthogonality) which we consider just *as natural as* the definition of $\perp_r$ is. The definition of $\perp_r'''$ will be a first-order one (in the expanded frame models $\mathfrak{M}^+$). In Theorems 4.2.11, 4.2.18 and 4.2.19 below we will see

[352] More precisely, T_0 together with "$\in$-relation" acting between Mn and T_0 are first-order definable, where definable here means rigidly definable in the sense of §4.3.2.

[353] A set $T \subseteq \mathcal{T}$ is called a base for topology $\mathcal{T}$ iff each member (i.e. "open set") of $\mathcal{T}$ can be obtained as a (possibly infinite) union of sets from T.

that, under some assumptions, all versions of relativistic orthogonality, i.e. $\perp_r, \perp'_r, \perp''_r, \perp'''_r, \perp^\omega_r$, coincide, cf. Corollary 4.2.20. These theorems imply that $\perp_r$ is first-order definable (in the expanded frame models $\mathfrak{M}^+$ mentioned above), under certain conditions.

Definition 4.2.10 (Alternatives $\perp'_r$, $\perp''_r$ for relativistic orthogonality $\perp_r$)
Let $\mathfrak{M}$ be a frame model. L, L^{Ph} and the relation of parallelism ($\|_{\mathfrak{G}}$) on L are defined in items 4, 9 of Def.4.2.3.(I) (pp. 138–140). We define $\perp'_r \subseteq L \times L$ and $\perp''_r \subseteq L \times L$ as follows.

Intuitively, two lines are $\perp'_r$-orthogonal if they are parallel photon-like lines or there is an inertial observer who thinks that one of the lines coincides with a coordinate axis, call it $\bar{x}_i$, and the other line lies in the subspace determined by two (possibly coinciding) coordinate axes different from $\bar{x}_i$, see the left-hand side of Figure 70. Formally: Let $\ell, \ell' \in L$. Then

$$\ell \perp'_r \ell' \quad \stackrel{\text{def}}{\iff} \quad \text{(one of (i)–(iii) below holds),} \quad \text{cf. Figure 70.}$$

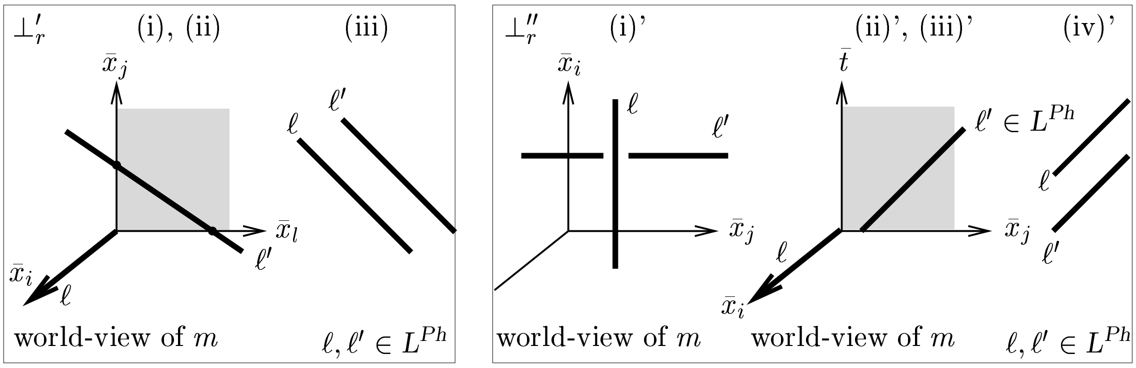


Figure 70: Illustration for the definitions of $\perp'_r$ and $\perp''_r$.

In the formula in item (i) below, if $j = l$ then $\mathsf{Plane}(\bar{x}_j, \bar{x}_l)$ denotes the coordinate axis $\bar{x}_j$ (i.e. $\mathsf{Plane}(\bar{x}_j, \bar{x}_j) = \bar{x}_j$, as one would expect).

(i) $(\exists m \in Obs \cap Ib)(\exists i, j, l \in n)$
$$\Big(j \neq i \neq l \quad \wedge \quad \ell = w_m[\bar{x}_i] \quad \wedge \quad \ell' \subseteq w_m[\mathsf{Plane}(\bar{x}_j, \bar{x}_l)] \Big).$$

(ii) The same as (i) but with ℓ, ℓ' interchanged.

(iii) $(\ell, \ell' \in L^{Ph} \quad \wedge \quad \ell \,\|_{\mathfrak{G}}\, \ell')$.

Now we turn to defining $\perp''_r$. Intuitively, two lines are $\perp''_r$-orthogonal if they are parallel photon-like lines or there is an inertial observer, call it m, who thinks that the two lines are parallel with two different coordinate axes or m thinks that one of the lines coincides with a spatial coordinate axis, call it $\bar{x}_i$, and the other line is the trace of a photon and this photon moves in the (spatial) direction determined by a spatial coordinate axis different from $\bar{x}_i$, see the right-hand side of Figure 70. Formally: Let $\ell, \ell' \in L$.

$$\ell \perp''_r \ell' \quad \stackrel{\text{def}}{\iff} \quad \text{(one of (i)'–(iv)' below holds),} \quad \text{cf. Figure 70.}$$

(i)’ $(\exists m \in Obs \cap Ib)(\exists i, j \in n)$
$$\Big(i \neq j \quad \wedge \quad w_m[\bar{x}_i], w_m[\bar{x}_j] \in L \text{ [354]} \quad \wedge \quad \ell \,\|_{\mathfrak{G}}\, w_m[\bar{x}_i] \quad \wedge \quad \ell' \,\|_{\mathfrak{G}}\, w_m[\bar{x}_j] \Big).$$

[354] We note that, assuming $\mathbf{Ax4} + \mathbf{Ax6_{00}}$, $(\forall m \in Obs)(\forall i \in n)\ w_m[\bar{x}_i] \in L$.

(ii)' $\ell' \in L^{Ph} \quad \wedge \quad (\exists m \in Obs \cap Ib)(\exists i, j \in n)$
$\Big(0 \neq i \neq j \neq 0 \ \wedge\ \ell = w_m[\bar{x}_i] \ \wedge\ \ell' \subseteq w_m[\mathsf{Plane}(\bar{t}, \bar{x}_j)]\Big).$

(iii)' The same as (ii)' but with ℓ, ℓ' interchanged.

(iv)' $(\ell, \ell' \in L^{Ph} \quad \wedge \quad \ell \parallel_{\mathfrak{G}} \ell')$.

$\lhd$

For stating the next theorem we introduce two new axioms. The first one is called axiom of disjoint windows (**Ax(diswind)**) formulated below. We note that there is a model of **Newbasax** in which **Ax(diswind)** fails (see Figure 98 on p.275 or Figure 91 on p.211).

Ax(diswind) $(\forall m, k \in Obs \cap Ib)\ [(m \overset{\odot}{\to} ph \ \wedge\ k \overset{\odot}{\to} ph) \ \Rightarrow\ m \overset{\odot}{\to} k]$.

The intuitive meaning of **Ax(diswind)** is the following. In models of $\mathbf{Bax}^-$ the visibility relation $\overset{\odot}{\to}$ is an equivalence relation on the set of (inertial) observers,[355] cf. Theorem 3.2.6 (p.110) and the intuitive text above Theorem 3.3.12 (p.196) of AMN [18]. The "windows" correspond to the equivalence classes of $\overset{\odot}{\to}$. Now, **Ax(diswind)** says that there is no photon connecting the windows. Cf. also Figure 91 on p.211 for the intuitive idea of a window. Very roughly, one could say that the window of an observer m is that part of space-time which "unquestionably exists" for m.[356]

The second new axiom is the auxiliary axiom $\mathbf{Ax}(\mathit{Triv}_t)^-$ which is a weakened version of $\mathbf{Ax}(\mathit{Triv}_t)$ (p.82). The advantage of the axiom $\mathbf{Ax}(\mathit{Triv}_t)^-$ over $\mathbf{Ax}(\mathit{Triv}_t)$ is that $\mathbf{Ax}(\mathit{Triv}_t)^-$ can survive the transition from special relativity to general relativity, while $\mathbf{Ax}(\mathit{Triv}_t)$ might probably not survive this transition. Recall from p.82, that $\mathbf{Ax}(\mathit{Triv}_t)$ postulates the existence of certain very simple f_{mk} transformations *not involving motion* (or even changing the time axis $\bar{t}$). Recall that

$$\mathit{Triv} = \{\, f \;:\; f \text{ is an isometry of } {}^n\mathbf{F} \text{ and } f(1_t) - f(\bar{0}) = 1_t \,\}.$$

Our new, weaker axiom will say less than $\mathbf{Ax}(\mathit{Triv}_t)$, namely, it will prescribe only what the required f_{mk}'s do with the spatial coordinate axes, i.e. what they do with *spatial directions* $\bar{x}, \bar{y}, \ldots$ but it will not prescribe what they do with e.g. the *lengths* of the unit vectors $1_t, 1_x, 1_y, \ldots$.

$\mathbf{Ax}(\mathit{Triv}_t)^-$ $(\forall m \in Obs)(\forall f \in \mathit{Triv})\,[\, f[\bar{t}\,] = \bar{t} \quad \Rightarrow$
$(\exists k \in Obs)(\forall i \in n)\ (\,\mathsf{f}_{km}[\bar{x}_i] = f[\bar{x}_i] \quad \wedge \quad m \uparrow k\,)\,]$.

That is, assume we are given an observer m and a *Triv* transformation f that leaves the time-axis fixed. Then m has a brother, call it k, such that m thinks that (i) the coordinate axes of k are the f-images of the original coordinate axes $\bar{x}_i$, and (ii) the clock of k runs forwards.

Intuitive motivation explaining *why* we will need $\mathbf{Ax}(\mathit{Triv}_t)^-$ often can be found on p.182 (beginning of §4.2.3).

[355] We note that in $\mathbf{Bax}^-$ all the observers are inertial ones.

[356] One could also say that a window is such a part of Mn which can be obtained in the form $Rng(w_m)$, for some m (i.e. which can be "coordinatized" by m).

THEOREM 4.2.11 *$\perp_r$ coincides with both $\perp'_r$ and $\perp''_r$,* [357] *therefore $\perp_r$ is first-order definable*[358]*, assuming* $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax(diswind)} + \mathbf{Ax}(\sqrt{\ })$. *(The assumption is needed for all parts of the statement, e.g. for $\perp_r = \perp'_r$, of course.)*

For the axiom **Compl** and the axioms in Figure 71 we refer the reader to the List of axioms (on p.A-19).

$$\begin{array}{rll}
 & \mathfrak{M} \models & \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)} \\
\text{changing the units of measurement for time} & \big\downarrow & \\
 & \mathfrak{M}_1 \models & \boxed{\mathbf{Newbasax}} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)} \\
\text{getting rid of field automorphisms from the } \mathsf{f}_{mk}\text{'s} & \big\downarrow & \\
 & \mathfrak{M}_2 \models & \boxed{\mathsf{f}_{mk} \in \mathit{Aftr}} + \\
 & & \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)} \\
\text{changing the units of measurement} & \big\downarrow & \\
 & \mathfrak{M}_3 \models & \boxed{\mathbf{Ax(syt_0)}} + \\
 & & \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)} \\
\text{changing the direction of flow of time for some observers} & \big\downarrow & \\
 & \mathfrak{M}_4 \models & \mathbf{Ax(syt_0)} + \boxed{\mathbf{Ax}(\uparrow) + \mathbf{Ax5}^+} + \\
 & & \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)} \\
\text{throwing away superfluous bodies} & \big\downarrow & \\
 & \mathfrak{M}_5 \models & \mathbf{Ax(syt_0)} + \mathbf{Ax}(\uparrow) + \mathbf{Ax5}^+ + \boxed{\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}} + \\
 & & \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}
\end{array}$$

Figure 71: Illustration for the proof of Thm.4.2.11. By gradually changing $\mathfrak{M}$ we arrive at $\mathfrak{M}_5$. For explanation cf. footnote 359.

Proof: Assume $n > 2$. A sketch of the idea of the proof is illustrated in Figure 71.

Let $\mathfrak{M}, \mathfrak{M}_1, \ldots, \mathfrak{M}_5$ be as in the figure. We start out with $\mathfrak{M}$ and by gradually changing it we arrive at $\mathfrak{M}_5$.[359] What is invariant during this process is that the

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_0, \perp_r, \perp'_r, \perp''_r \rangle$$

[357] Recall that $\perp_r$ is defined in Def.4.2.3 (p.140) and $\perp'_r$ and $\perp''_r$ are defined in Definition 4.2.10 above.

[358] we mean, definable over $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, of course. First one defines $\mathfrak{M}^+ = \langle \mathfrak{M}, Mn, L; \in \rangle$ over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then $\perp_r$ over $\mathfrak{M}^+$. (The point is that for defining $\perp_r$ first we need to have lines.)

[359] See Figure 71. Step $\mathfrak{M} \mapsto \mathfrak{M}_1$ goes exactly as step $\mathfrak{M} \mapsto \mathfrak{N}$ in the proof of item 6.2.89 on p.895 of AMN [18]. Step $\mathfrak{M}_1 \mapsto \mathfrak{M}_2$ goes as follows: Assume $\mathfrak{M}_1 = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle$. Let $m_0 \in Obs$

generalized reducts of geometries corresponding to all these models[360] are isomorphic (actually with the exception of $\mathfrak{M}_5$ these reducts are identical). It can be proved[361] that

$$\mathfrak{M}_5 \models \mathbf{Newbasax} + \mathbf{Compl} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}.$$

Therefore $\mathfrak{M}_5$ is a disjoint union[362] of models of $\mathbf{BaCo}+\mathbf{Ax}(\sqrt{\ })$, i.e. it is a disjoint union of Minkowski models.[363]

Since the above indicated "geometry reducts" of $\mathfrak{M}$ and $\mathfrak{M}_5$ are isomorphic, to prove the theorem, it is enough to prove its conclusion for Minkowski models. I.e. it is enough to prove that in Minkowski models $\bot_r$, $\bot_r'$, $\bot_r''$ coincide. We leave checking this to the reader; but we note that a generalized version of this is proved as Claim 6.2.11 in AMN [18] on p.815.

Assume $n = 2$. Then we use the first part of Figure 71 involving $\mathfrak{M}, \ldots, \mathfrak{M}_3$. For this part the proof is the same as in the $n > 2$ case (e.g. we use the same geometry reduct). It is not hard to prove that $\mathfrak{M}_3$ is a disjoint union of models of $(\mathbf{Basax}+\mathbf{Ax(syt)}+\mathbf{Ax}(\mathit{Triv}_t)+\mathbf{Ax}(\sqrt{\ }))$, cf. footnote 361. Then it is enough to prove the conclusion for $(\mathbf{Basax}+\mathbf{Ax(syt)}+\mathbf{Ax}(\mathit{Triv}_t)+\mathbf{Ax}(\sqrt{\ }))$. Since $n = 2$ this is not too hard. We leave this step to the reader; but we note that a generalized version of this is proved as Claim 6.2.11 in AMN [18] on p.815. We note that there is a different, more detailed proof in AMN [18, p.815]. ∎

Before defining the third and the fourth versions $\bot_r'''$ and $\bot_r^\omega$ of our relativistic orthogonality relation we need the definition of the "plane generated by a set of points $H \subseteq Mn$". To explain certain technicalities in this definition, we include Proposition 4.2.14 below. To improve readability, we will use the following abbreviations.

Notation 4.2.12 Let $\mathfrak{G}$ be a relativistic geometry.

(i) We define the binary relation $\sim$ of <u>*connectedness*</u> on points Mn as follows.[364] Let $e, e_1 \in Mn$. Then

$$e \sim e_1 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \Big(e = e_1 \ \vee \ (\exists e_2 \in Mn)\, Bw(e, e_1, e_2)\Big).$$

(ii) Let $a, b, c \in Mn$. Then

$$\begin{aligned} \mathit{coll}(a,b,c) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad & \Big(a \sim b \sim c \sim a \quad \wedge \\ & (\,Bw(a,b,c) \ \vee \ Bw(a,c,b) \ \vee \ Bw(b,a,c) \ \vee \ a=b \ \vee \ b=c \ \vee \ a=c\,)\Big). \end{aligned}$$

"$\mathit{coll}(a,b,c)$" abbreviates "a, b, c are collinear".

be arbitrary, but fixed. For every $k \in Obs$ let $\varphi_k \in Aut(\mathbf{F})$ be such that $\mathsf{f}_{m_0 k} = f \circ \widetilde{\varphi}_k$, for some $f \in Aftr$. Such φ_k's exist by Thm.3.2.6 (p.110) (and e.g. Lemma 3.1.6 (p.163) of AMN [18]). Let $W' := \{\, \langle k,p,b\rangle \in Obs \times {}^nF \times B \,:\, W(k, \widetilde{\varphi}_k(p), b)\,\}$ and $\mathfrak{M}_2 := \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, G;\, \in, W'\,\rangle$. For the step $\mathfrak{M}_2 \mapsto \mathfrak{M}_3$ we refer the reader to Figures 96, 97 (p.324) of AMN [18] and to the intuitive model construction on pp. 322–325 of AMN [18]. In step $\mathfrak{M}_3 \mapsto \mathfrak{M}_4$ we change the direction of flow of time for some observers so that $\mathbf{Ax}(\uparrow) + \mathbf{Ax5}^+$ becomes true, and in step $\mathfrak{M}_4 \mapsto \mathfrak{M}_5$ we throw away some bodies so that $\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}$ becomes true.

[360] $\mathfrak{M}, \ldots, \mathfrak{M}_5$

[361] by Thm.3.3.12 (p.196) of AMN [18], by Prop.2.8.15 (p.83), by noticing that $\mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}_t)^- \models \mathbf{Ax}(\mathit{Triv}_t)$ and by Thm.2.8.17 (p.84)

[362] For disjoint unions of models cf. pp. 196-197.

[363] Cf. Def.3.8.42 (p.331) of AMN [18] for Minkowski models.

[364] We note that the present notion of connectedness is a completely different thing than the topological notion of connectedness.

Warning: The "real" collinearity relation of our relativistic geometries $\mathfrak{G}$ (to be denoted as *Col*) will be defined later by using the set L of lines and (e.g. in $\mathsf{Ge}(\mathbf{Bax}^-)$) it will not necessarily coincide with the recently defined *coll*. However, the two collinearity relations (*coll* and *Col*) will coincide in the geometries of models of $\mathbf{Bax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. More generally, $coll \subseteq Col$, assuming $\mathbf{Pax} + \mathbf{Ax(diswind)}$, cf. Item 4.5.36 on p.308.

$\lhd$

Remark 4.2.13 Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$. Then two events $e, e_1 \in Mn_{\mathfrak{M}}$ are connected iff there is an observer who sees both of them. I.e.

$$e \sim e_1 \quad \Longleftrightarrow \quad (\exists m \in Obs)\, e, e_1 \in Rng(w_m).$$ [365]

$\lhd$

PROPOSITION 4.2.14 *Assume* $\mathbf{Bax}^-$. *Then* $(\forall a, b, c \in Mn)$

$$coll(a,b,c)$$
$$\Updownarrow$$
$$(\textit{ there is an observer who sees that events } a,b,c \textit{ are collinear })$$ [366]
$$\Updownarrow$$
(each observer who sees events a, b, c "thinks" that they are collinear and some observer sees all of a, b, c)[367].

Proof: The proposition follows by Thm.3.2.6 (p.110) (and by the definition of Bw). ∎

We note that the directions "$\Uparrow$" in the above proposition hold for any frame model (i.e. the assumption $\mathbf{Bax}^-$ is not needed for these directions).

In Def.4.2.15 below, the first definition we give for $Plane(H)$ is short, but is not in the first-order language of our geometry $\langle Mn_{\mathfrak{M}};\ Bw_{\mathfrak{M}}\rangle$. This is why, still in Def.4.2.15 we continue discussing alternative definitions for $Plane(H)$. A similar remark applies to Def.4.2.17 (the definition of $\perp_r^\omega$).

Definition 4.2.15 Let $\mathfrak{M}$ be a frame model. Let $\langle Mn;\ Bw\rangle \cong \langle Mn_{\mathfrak{M}};\ Bw_{\mathfrak{M}}\rangle$.[368]
Let $H \subseteq Mn$.

(i) By $Plane(H)$ we denote the "plane generated by H", i.e. $Plane(H)$ is the smallest subset of Mn having properties 1 and 2 below.[369]

1. $H \subseteq Plane(H)$.
2. $(\, a, b \in Plane(H) \ \wedge\ coll(a,b,c)\,) \ \Rightarrow\ c \in Plane(H)$.

---

[365] This holds by the definitions of Bw and $\sim$.
[366] Formally: $(\exists m)\,[\, a,b,c \in Rng(w_m) \ \wedge\ (w_m^{-1}(a),\ w_m^{-1}(b),\ w_m^{-1}(c)$ are collinear$)\,]$.
[367] Formally: $(\forall m)\,[\, a,b,c \in Rng(w_m) \ \Rightarrow\ (w_m^{-1}(a),\ w_m^{-1}(b),\ w_m^{-1}(c)$ are collinear$\,)\,] \wedge (\exists m)\, a,b,c \in Rng(w_m)$.
[368] $Mn_{\mathfrak{M}}, Bw_{\mathfrak{M}}$ are defined in items 2, 7 of Definition 4.2.3.(I).
[369] What we denote by $Plane(H)$, is usually denoted as $Span(H)$, in the literature.

As we already said, the above definition of $Plane(H)$ is not formulated in first-order logic. (In passing we note, that actually, it is in second-order logic.) Next we prepare for making the definition of $Plane(H)$ a first-order logic one (under some mild assumptions). An equivalent definition for $Plane(H)$ is the following. First, for every $i \in \omega$ we define $Plane^i(H)$ as follows.

$$\begin{aligned} Plane^0(H) &\stackrel{\text{def}}{:=} H, \\ Plane^{i+1}(H) &\stackrel{\text{def}}{:=} \{ c \in Mn : (\exists a, b \in Plane^i(H))\ coll(a,b,c) \}. \end{aligned}$$

We note that

$$Plane^i(H) \subseteq Plane^{i+1}(H), \quad \text{for any } i \in \omega.$$

Now we observe, that

$$Plane(H) = \bigcup \{ Plane^i(H) : i \in \omega \}.$$

(ii) Below we introduce the "first-order version" $Plane'(H)$ of $Plane(H)$ which will be defined in the first-order language of the structure $\langle Mn;\ H, Bw\rangle$. Let us recall that $n > 1$ is the dimension of our space-time. We define

$$Plane'(H) \stackrel{\text{def}}{:=} Plane^n(H).$$

We note that $Plane'(H) = Plane(H)$, assuming $\mathfrak{M} \models \mathbf{Bax}^-$, cf. Prop.4.2.16.

(iii) We write $Plane(\ell_1, \ldots, \ell_i)$ for $Plane(\ell_1 \cup \ldots \cup \ell_i)$, where $\ell_1, \ldots, \ell_i \in L$.

$\lhd$

PROPOSITION 4.2.16 *Assume* $\mathbf{Bax}^-$. *Then* $Plane(H) = Plane'(H)$.

On the proof: A proof can be obtained by items 1g (p.209) and 3b (p.213) of Proposition 4.2.64 way below, cf. also Prop.4.2.14 (p.160). ∎

Now, we are ready for defining our third and fourth versions $\perp_r'''$ and $\perp_r^\omega$ of relativistic orthogonality.

Definition 4.2.17 (Alternatives $\perp_r'''$, $\perp_r^\omega$ for relativistic orthogonality $\perp_r$) Let $\mathfrak{M}$ be a frame model. Mn, L, $\in$, Bw, $\|_{\mathfrak{G}}$, and the basic relation $\perp_0 \subseteq L \times L$ of orthogonality are defined in Def.4.2.3.(I). In the present definition we define two alternatives $\perp_r^\omega$ and $\perp_r'''$ for the relativistic orthogonality. The definition of $\perp_r'''$ will be a first-order one over $\langle Mn, L;\ \in, Bw\rangle$[370] while that of $\perp_r^\omega$ will not be such.

(i) $\perp_r^\omega$ is defined to be the smallest subset of $L \times L$ having properties 1–4 below.

1. $\perp_0 \subseteq \perp_r^\omega$, i.e. $\ell \perp_0 \ell' \Rightarrow \ell \perp_r^\omega \ell'$.
2. $\perp_r^\omega$ is a symmetric relation, i.e. $\ell \perp_r^\omega \ell' \Rightarrow \ell' \perp_r^\omega \ell$. [371]
3. If lines ℓ, ℓ_1, ℓ_2 concur at point e, with $\ell_1 \neq \ell_2$ and ℓ is $\perp_r^\omega$-orthogonal to both ℓ_1 and ℓ_2, then ℓ is $\perp_r^\omega$-orthogonal to every line through e in the plane determined by ℓ_1 and ℓ_2, see Figure 72;[372] formally: Let $e \in Mn$ and $\ell, \ell_1, \ell_2, \ell' \in L$. Then

[370] Recall that the relation of parallelism $\|_{\mathfrak{G}}$ was first-order defined over $\langle Mn, L;\ \in, Bw\rangle$.
[371] We note that in Goldblatt [102, p.115] this is an axiom for a metric affine space called OS1.
[372] We note that in Goldblatt [102, p.115] this is an axiom for a metric affine space called OS4.

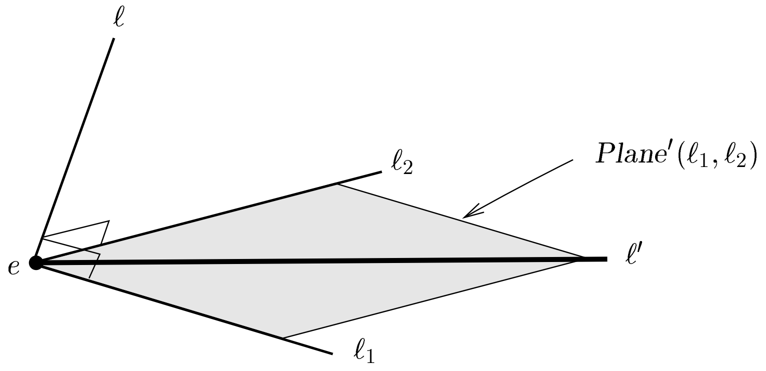


Figure 72: In the Figure, the little "diamonds" around point e indicate that lines ℓ and ℓ_i are orthogonal, for $i \in \{1, 2\}$.

$$[\, e \in \ell \cap \ell_1 \cap \ell_2 \cap \ell' \quad \wedge \quad \ell_1 \neq \ell_2 \quad \wedge$$
$$\ell \perp_r^\omega \ell_1 \quad \wedge \quad \ell \perp_r^\omega \ell_2 \quad \wedge \quad \ell' \subseteq Plane'(\ell_1, \ell_2)\,] \qquad \Rightarrow \qquad \ell \perp_r^\omega \ell'.$$

In such situations we may also say that ℓ is $\perp_r^\omega$-orthogonal to the plane $Plane'(\ell_1, \ell_2)$.

4. $\perp_r^\omega$ is closed under parallelism, i.e.

$$(\, \ell \perp_r^\omega \ell_1 \ \wedge\ \ell_1 \parallel_{\mathfrak{G}} \ell_2 \,) \quad \Rightarrow \quad \ell \perp_r^\omega \ell_2.$$ [373]

Next we prepare for making the definition of $\perp_r^\omega$ a first-order logic one (under some assumptions). An equivalent definition for $\perp_r^\omega$ is the following. First, for every $i \in \omega$ we define $\perp_r^i \subseteq L \times L$ as follows. For easier readability, we note that the formulas ψ_2^i, ψ_3^i, ψ_4^i below correspond to "taking the closure of $\perp_r^i$ in one step[374]" to properties 2, 3, 4 above, respectively.

$$\begin{aligned}
\perp_r^0 &\overset{\text{def}}{:=} \perp_0, \\
\perp_r^{i+1} &\overset{\text{def}}{:=} \{\, \langle \ell, \ell' \rangle \in L \times L \ :\ \psi_2^i \vee \psi_3^i \vee \psi_4^i \,\}, \quad \text{where} \\
\psi_2^i &:= \ell' \perp_r^i \ell, \\
\psi_3^i &:= (\exists \ell_1, \ell_2)(\exists e)\, [e \in \ell \cap \ell_1 \cap \ell_2 \cap \ell' \ \wedge\ \ell_1 \neq \ell_2 \ \wedge \\
&\qquad \ell \perp_r^i \ell_1 \ \wedge\ \ell \perp_r^i \ell_2 \ \wedge\ \ell' \subseteq Plane'(\ell_1, \ell_2)\,], \\
\psi_4^i &:= (\exists \ell_1)\, (\ell \perp_r^i \ell_1 \ \wedge\ \ell_1 \parallel_{\mathfrak{G}} \ell' \,).
\end{aligned}$$

We note that $\perp_r^i \subseteq \perp_r^{i+1}$, for all $i \in \omega$.

Now we observe, that

$$\perp_r^\omega = \bigcup \{\, \perp_r^i \ :\ i \in \omega \,\}\,.$$

Let us notice that, for every $i \in \omega$, $\perp_r^i$ is a first-order definition in $\langle Mn, L;\, \in, Bw, \perp_0 \rangle$.

[373] This property is called axiom OS5 (for metric affine space) in Goldblatt [102, p.116].

[374] i.e. to "making one step only in the process of taking the closure".

(ii) $\perp_r''' \stackrel{\text{def}}{:=} \perp_r^4$. So the definition of $\perp_r'''$ is a first-order definition in $\langle Mn, L;\ \in, Bw, \perp_0\rangle$. Therefore the definition of $\perp_r'''$ is a first-order one in the expanded frame model $\mathfrak{M}^+$ defined in Remark 4.2.9 on p.153.

$\lhd$

THEOREM 4.2.18 *Assume* $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. *Then* $\perp_r^{\omega} = \perp_r'''$*, therefore* $\perp_r^{\omega}$ *is first-order definable*[375].

Proof: The proof for the case $n = 2$ is easy. Namely, if $n = 2$ then both $\perp_r^{\omega}$ and $\perp_r'''$ coincide with $\perp_0$. For the case $n > 2$, the theorem follows by the proof of Theorem 4.2.19 below. Namely, in the proof of Thm.4.2.19 we will prove (a) $\perp_r' \subseteq \perp_r'''$ and (b) 1–4 of Def.4.2.17 hold for $\perp_r'$, i.e. 1–4 hold when $\perp_r^{\omega}$ is replaced by $\perp_r'$ in them. Since $\perp_r''' \subseteq \perp_r^{\omega}$ and $\perp_r^{\omega}$ is the smallest subset of $L \times L$ having properties 1–4, (a) and (b) imply that $\perp_r' = \perp_r''' = \perp_r^{\omega}$. ∎

THEOREM 4.2.19 *Assume* $n > 2$ *and* $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. *Then* $\perp_r$ *and* $\perp_r'''$ *coincide, therefore* $\perp_r$ *is first-order definable*[375].

The proof will be given below item 4.2.20.

The following is an immediate corollary of Theorems 4.2.11 (p.158), 4.2.18 and 4.2.19.

COROLLARY 4.2.20 Assume $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. Then $\perp_r$, $\perp_r'$, $\perp_r''$, $\perp_r'''$, $\perp_r^{\omega}$ coincide (and are definable[375]). ∎

Proof of Thm.4.2.19: Assume $n > 2$.

Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. By Thm.4.2.11 (which says that $\perp_r, \perp_r', \perp_r''$ coincide) it is enough to prove (a) and (b) below, as it was shown in the proof of Thm.4.2.18.

(a) $\perp_r' \subseteq \perp_r'''$, i.e. $\ell \perp_r' \ell' \ \Rightarrow\ \ell \perp_r''' \ell'$.

(b) $\perp_r'$ has the properties 1–4 in Def.4.2.17, i.e. 1–4 in Def.4.2.17 hold when $\perp_r^{\omega}$ is replaced by $\perp_r'$ in them.

Let $\mathfrak{N}$ be a model of **Newbasax** obtained from $\mathfrak{M}$ by changing the units of measurement for time, i.e. $\mathfrak{N}$ is obtained from $\mathfrak{M}$ exactly the same way as in the proof of item 6.2.89 on p.896 of AMN [18]. The generalized geometry reducts

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r', \perp_r'''\rangle$$

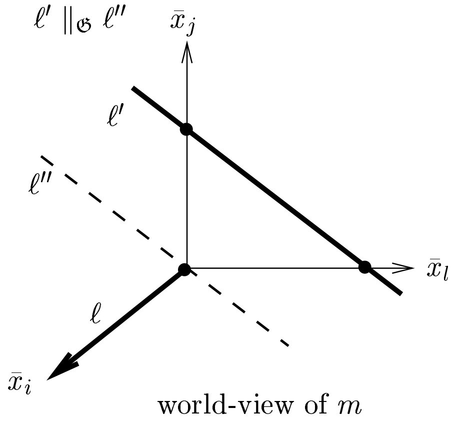
of $\mathfrak{M}$ and $\mathfrak{N}$ coincide. Further,

$$\mathfrak{N} \models \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}.$$

Therefore $\mathfrak{N}$ is a photon-disjoint union[376] of models of $(\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$. Since the above indicated "geometry reducts" of $\mathfrak{M}$ and $\mathfrak{N}$ coincide, it is enough to prove (a) and (b) for $(\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$-models. Now, we turn to proving these two statements.

Proof of (a): Recall that $\perp_r^i$ $(i \in \omega)$ is the "i-long closure" of $\perp_0$ to properties 2, 3, 4 in Def.4.2.17 and that $\perp_r''' = \perp_r^4$, cf. Def.4.2.17 (p.161).

Assume $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$. Let $\ell, \ell' \in L$ be such that $\ell \perp_r' \ell'$, see the left-hand side of Figure 70 (p.156). Then one of (i)–(iii) in the definition of $\perp_r'$ on p.156 hold for



$\ell \perp_r^0 w_m[\bar{x}_l]$, $\ell \perp_r^0 w_m[\bar{x}_j]$ by 1 in Def.4.2.17

$\ell \perp_r^1 \ell''$ by 3 in Def.4.2.17 [378]

$\ell \perp_r^2 \ell'$ by 4 in Def.4.2.17 [379]

Figure 73: In case (i) $\ell \perp_r^2 \ell'$.

ℓ, ℓ'. By $\mathbf{Ax}(\mathit{Triv}_t)^-$,[377] in cases (i) and (ii) $\ell \perp_r''' \ell'$ holds, cf. Figure 70. Actually, in case (i) $\ell \perp_r^2 \ell'$ and in case (ii) $\ell \perp_r^3 \ell'$, see Figure 73. So it remains to prove (a) for case (iii).

Assume (iii) holds for ℓ, ℓ', i.e. $\ell, \ell' \in L^{Ph}$ and $\ell \parallel_{\mathfrak{G}} \ell'$, cf. Figure 70. To prove $\ell \perp_r''' \ell'$ it is enough to prove $\ell \perp_r^3 \ell$ because $\ell \perp_r^3 \ell$ and $\ell \parallel_{\mathfrak{G}} \ell'$ imply $\ell \perp_r^4 \ell'$, i.e. $\ell \perp_r''' \ell'$.

Now, we turn to proving $\ell \perp_r^3 \ell$. There is $m \in \mathit{Obs}$ and $ph \in \mathit{Ph}$ such that

$$w_m[\mathit{tr}_m(ph)] = \ell.$$ [380]

Fix such m and ph. Without loss of generality we can assume that $\bar{0} \in \mathit{tr}_m(ph) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ because of $\mathbf{Ax}(\mathit{Triv}_t)^-$ and $\mathbf{Ax5}$. Throughout the remaining part of the proof of (a) the reader is advised to consult Figure 74. We are in the world-view of m. Let P be the plane determined by $\bar{y}$ and ph, i.e.

$$P := \mathsf{Plane}(\bar{y}, \mathit{tr}_m(ph)),$$

cf. the upper picture in Figure 74. Let $k \in \mathit{Obs}$ be such that m sees that k passes through $\bar{0}$ with nonzero speed and lies in $\mathsf{Plane}(\bar{t}, \bar{y})$, i.e. $\bar{0} \in \mathit{tr}_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{y})$ and $v_m(k) \neq 0$. Such a k exists by $\mathbf{Ax5}$. Without loss of generality we can assume that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ because of $\mathbf{Ax}(\mathit{Triv}_t)^-$. Let

$$\bar{y}_k := \mathsf{f}_{km}[S] \cap P,$$

i.e. in the world-view of m $\bar{y}_k$ is the intersection of k's space part with plane P. Clearly, $\bar{y}_k \in \mathsf{Eucl}$ and $\bar{y}_k, \bar{y}, \mathit{tr}_m(ph)$ are pairwise distinct, since k lies in $\mathsf{Plane}(\bar{t}, \bar{y})$, is of nonzero speed as seen by m and since in the direction of movement clocks get out of synchronism. Without loss of generality, by $\mathbf{Ax}(\mathit{Triv}_t)^-$, we can assume that the $\bar{y}$-axis of k as seen by m is y_k, formally

$$\mathsf{f}_{km}[\bar{y}] = \bar{y}_k.$$

Let us switch over from the world-view of m to the world-view of k. We claim that k sees ph moving in the spatial direction orthogonal to $\bar{y}$ (in the Euclidean sense). To prove this

[375] we mean, definable over $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, of course. First one defines L over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then $\perp_r$ over $\mathfrak{M}$ and L.

[376] For disjoint and photon-disjoint union of models cf. item 1 on p.196.

[377] and some basic properties of **Basax**

[378] and by 5b of Prop.4.2.64

[379] and by 5a of Prop.4.2.64

[380] Cf. items 1c and 2a of Prop.4.2.64 (p.208).

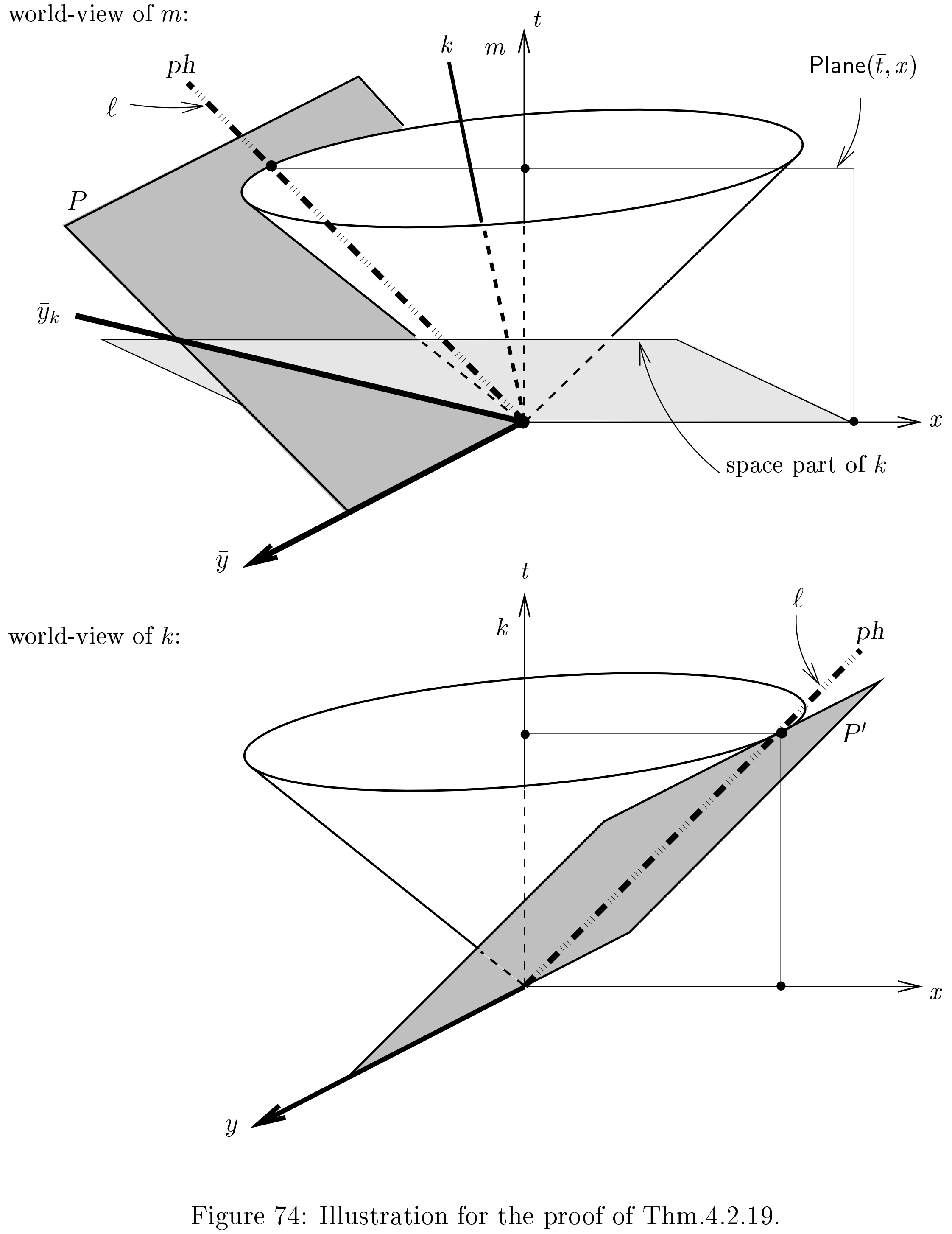


Figure 74: Illustration for the proof of Thm.4.2.19.

claim, let P' be the f_{mk} image of P, cf. Figure 74. Then $\bar{y} \subseteq P'$. Since f_{mk} takes $\mathsf{LightCone}(\bar{0})$, P, $tr_m(ph)$ to $\mathsf{LightCone}(\bar{0})$, P', $tr_k(ph)$, respectively and since $\mathsf{LightCone}(\bar{0}) \cap P = tr_m(ph)$ we get that

$$\mathsf{LightCone}(\bar{0}) \cap P' = tr_k(ph).$$

This and $\bar{y} \subseteq P'$ imply that $\bar{y} \perp_e tr_k(ph)$, proving our claim.

Then, by $\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^-$, we can assume that k sees ph in $\mathsf{Plane}(\bar{t}, \bar{x})$, i.e. $tr_k(ph) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. Then

$$w_m[\bar{y}] \perp_r^1 \ell \quad \text{and} \quad w_k[\bar{y}] \perp_r^1 \ell,$$

see Figure 74. By this, by $w_m[\bar{y}_k] = w_k[\bar{y}]$ and by $\bar{y}, \bar{y}_k \subseteq P$, we have

$$(*) \qquad \ell \perp_r^2 w_m[\bar{y}] \quad \text{and} \quad \ell \perp_r^2 w_m[\bar{y}_k] \quad \text{and} \quad w_m[\bar{y}], w_m[\bar{y}_k] \subseteq w_m[P].$$

See the upper picture in Figure 74. By item 5b of Prop.4.2.64 (p.213), we have

$$\mathit{Plane}'(w_m[\bar{y}], w_m[\bar{y}_k]) = w_m[P].$$

This, $(*)$ and $\ell \subseteq w_m[P]$ imply $\ell \perp_r^3 \ell$, which completes the proof of (a).

Proof of (b): Assume $\mathbf{Basax}+\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^-+\mathbf{Ax}(\sqrt{\ })$. It is easy to check that $\perp_r'$ has properties 1, 2, 4 in Def.4.2.17. So it remains to prove that $\perp_r'$ has property 3 in Def.4.2.17. To prove this we will use Minkowskian orthogonality $\perp_\mu \subseteq \mathsf{Eucl} \times \mathsf{Eucl}$ which will be introduced in Def.4.2.44 (p.189). Now, by (I)–(II) below and item 5b of Prop.4.2.64, it can be checked that $\perp_r'$ has property 3 in Def.4.2.17; where (I) holds by item (d) in the proof of Claim 6.2.11 (p.816) of AMN [18] and by the def. of $\perp_r'$, and (II) can be checked by the definition of Minkowskian orthogonality.

(I) Let $\ell, \ell' \in L$. Then $\quad \ell \perp_r' \ell' \quad \Longleftrightarrow \quad (\forall m)(w_m^{-1}[\ell] \perp_\mu w_m^{-1}[\ell'])$.

(II) Minkowskian orthogonality has property 3 in Def.4.2.17, i.e. if lines ℓ, ℓ_1, ℓ_2 ($\in \mathsf{Eucl}$) concur at point p ($\in {}^nF$), with $\ell_1 \neq \ell_2$ and ℓ is Minkowski-orthogonal to both ℓ_1 and ℓ_2, then ℓ is Minkowski-orthogonal to every line through p in $\mathsf{Plane}(\ell_1, \ell_2)$, cf. Figure 72.

At this point Thm.4.2.19 is fully proved. ∎

Let us recall that eq is a 4-ary relation on the set of points Mn of an observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ and was defined in item 11 of Def.4.2.3(I) (p.142). Further, eq was defined to be the transitive closure of the relation eq_0 which was first-order logic defined (in the expanded frame-model $\mathfrak{M}^+$ defined in Remark 4.2.9 on p.153); and eq_i was defined to be the "i-long-transitive closure" of eq_0. As we have already said in Remark 4.2.9, each one of eq_i's is first-order defined (in $\mathfrak{M}^+$).[381]

The next two theorems (4.2.21 and 4.2.22) say that eq is first-order definable in $\mathfrak{M}^+$ under certain conditions.

THEOREM 4.2.21 *Assume* $\mathbf{Basax}+\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^-+\mathbf{Ax}(\sqrt{\ })$. *Then $eq_2 = eq$, therefore eq is first-order definable*[382].

[381] First-order definable is the same as first-order logic definable (which in turn is the same as definable, at least in the present work).

[382] we mean, definable over $\mathsf{Mod}(\mathbf{Basax}+\ldots)$, of course. First one defines Mn over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then eq over $\mathfrak{M}$ and Mn.

A **proof** is given in AMN [18, §6.2.6, on p.906].

To formulate our next theorem, we introduce a weakened version $\mathbf{Ax(\|)}^-$ of $\mathbf{Ax(\|)}$.

$\mathbf{Ax(\|)}^-$ $(\forall m, k \in Obs \cap Ib)$
$[\, tr_m(k) = \bar{t} \;\Rightarrow\; (\mathsf{f}_{mk} = h \circ I, \quad \text{for some expansion } h \text{ and isometry } I)\,]$.[383]

Assuming **Bax**, $\mathbf{Ax(\|)}^-$ is equivalent to the following: If two observers, say m and k, have the same life-line (i.e. $tr_m(k) = \bar{t}$) then they agree on the speed of light (i.e. $c_m = c_k$) and the world-view transformation f_{mk} is an affine transformation, i.e. there is no field automorphism involved in f_{mk} (cf. Fact 4.7.7 of AMN [18]).

The essential feature of $\mathbf{Ax(\|)}^-$ is that it does not exlude the "ant and the elephant version of relativity" mentioned on p.88 herein and in Remark 4.2.1 of AMN [18], while $\mathbf{Ax(\|)}$ does.

Let

$$Th^{+-} := \mathbf{Bax}^{\oplus} + \mathbf{Ax(\|)}^- + \mathbf{Ax(}Triv_t\mathbf{)}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}.$$

This theory Th^{+-} will play an essential role in the following theorems and propositions: Thm's 4.2.22 (p.168), 4.2.40(iii) (p.182), 4.3.38 (p.261); and 6.6.114 (p.1130) and Prop's 6.2.88 (p.895), 6.2.92 (p.901) of AMN [18]. Because of this, we point out a few intuitive and helpful properties of Th^{+-} (which eventually will be proved as parts of various later theorems). We collect these properties in items 1–4 below. In 1–4 below $n > 2$ is assumed.

1. The reduct
$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r \rangle$$
of $\mathsf{Ge}(Th^{+-})$ is a disjoint union[384] of (the similar reducts of) Minkowskian geometries[385].

2. eq behaves well in Th^{+-}, in the following sense. Whenever a, b, c in Fig.75 exist then d also exists. Further the arrangement in Fig.76 cannot happen. Formal statements of

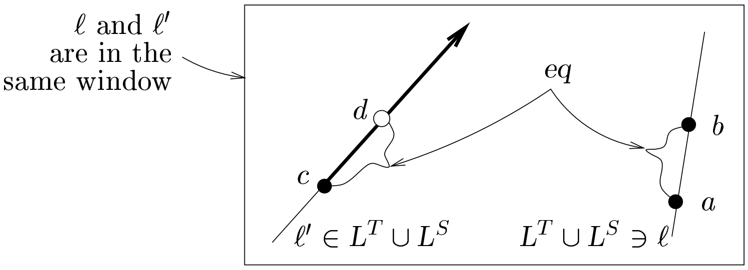


Figure 75: $(\forall a, b, c)(\exists d$ as in the figure).

these are in Prop's 6.2.88 (p.895), 6.2.92 (p.901) of AMN [18].

3. The space-like hyper-planes of the $\langle Mn, L;\ \in, Bw, \perp_r, eq\rangle$ reducts[386] of the elements of $\mathsf{Ge}(Th^{+-})$ are Euclidean geometries, assuming **Ax(eqtime)**, cf. Thm.6.6.114 (p.1130) of AMN [18].

---

[383] Though $\mathbf{Ax(\|)}^-$ is not a first-order formula in its present form, it can be easily reformulated in the first-order frame language, cf. p.82.

[384] Cf. pp. 198, 200 for disjoint union of geometries.

[385] Cf. Def.4.2.44 (p.189) for Minkowskian geometries.

[386] We will call these reducts Goldblatt-Tarski reducts or $GT_{\mathfrak{M}}$'s on p.215.

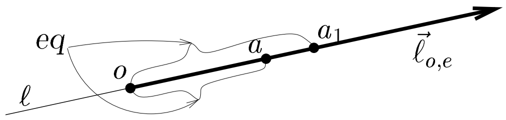


Figure 76: This cannot happen.

4. This theory Th^{+-}, despite of having all the nice properties in items 1–3 above, is not too strong e.g. we will see that even a strengthened version of Th^{+-} does not imply **Flxbasax**, i.e.

$$Th^{+-} \;+\; \text{“extra axioms”} \quad \not\models \mathbf{Flxbasax}$$

cf. AMN [18, Prop.6.2.101 (p.912) and the intuitive text below it on p.912].

THEOREM 4.2.22 *Assume $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^{-} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ })$. Then $eq_2 = eq$, therefore eq is first-order definable*[387].

A **proof** is given in §6.2.6 on p.906 of AMN [18].

In connection with the theorem below, cf. Proposition 6.2.96 on p.907 of AMN [18].

THEOREM 4.2.23

(i) *Theorem 4.2.21 does not generalize from* **Basax** *to* $\mathbf{Bax}^{\oplus}$ *(and the assumption* $\mathbf{Ax}(\|)^{-}$ *cannot be omitted from Thm.4.2.22). Moreover:*

For any $n > 1$, there is $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }))$ such that eq is not first-order definable in the expanded frame model $\mathfrak{M}^{+} := \langle \mathfrak{M};\, Mn_{\mathfrak{M}}, \in \rangle$.

(ii) *Theorem 4.2.22 does not generalize to $n = 2$. Moreover:*

There is $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }))$ such that eq is not first-order definable in the expanded frame model $\mathfrak{M}^{+} := \langle \mathfrak{M};\, Mn_{\mathfrak{M}}, \in \rangle$.

Proof:
Outline of the proof: We choose $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }))$ (for the case of (ii) $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }))$ such that $\mathfrak{M}$ has properties (a)–(c) formulated below.

(a) $\mathfrak{F}^{\mathfrak{M}}$ is a real-closed field.

(b) $\langle \mathfrak{M};\, Mn, \in \rangle$ is first-order definable (in the sense of §4.3.2) over $\mathfrak{F}^{\mathfrak{M}}$.

(c) The subset $\{\, 2^i \,:\, i \in \mathrm{Z} \,\}$ [388] of $F^{\mathfrak{M}}$ is first-order definable over $\langle \mathfrak{M};\, Mn, \in, eq \rangle$.

Since $\langle \mathfrak{M};\, Mn, \in \rangle$ is definable over $\mathfrak{F}^{\mathfrak{M}}$, a subset A of $F^{\mathfrak{M}}$ is definable over $\langle \mathfrak{M}; Mn, \in \rangle$ iff it is definable over $\mathfrak{F}^{\mathfrak{M}}$ (cf. Thm.4.3.27, p.245). If eq was definable over $\langle \mathfrak{M}; Mn, \in \rangle$ then by property (c) the set $\{\, 2^i \,:\, i \in \mathrm{Z} \,\}$ would be definable over $\mathfrak{F}^{\mathfrak{M}}$. We will prove that the set

[387] we mean, definable over $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, of course. First one defines Mn over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then eq over $\mathfrak{M}$ and Mn.

[388] Recall that Z denotes the set of all integers.

$\{\, 2^i \,:\, i \in \mathrm{Z} \,\}$ is not definable over $\mathfrak{F}^{\mathfrak{M}}$ as a corollary of Lemma 4.2.27 way below. Hence, eq is not definable over $\langle \mathfrak{M};\, Mn, \in \rangle$.

Details of the proof:

Case of (i): Let $\mathfrak{F}$ be a real-closed field. Let $\mathfrak{M}$ be the frame-model over $\mathfrak{F}$ obtained from the Minkowski model[389] $\mathfrak{M}^M_{\mathfrak{F}}$ as follows. Intuitively, for each observer m of $\mathfrak{M}^M_{\mathfrak{F}}$ we include a new observer k such that clock of k runs twice more slowly than that of m and in all other properties m and k agree (i.e. $w_m(p) = w_k(p_0/2, p_1, \ldots, p_{n-1})$, for all $p \in {}^nF$). The speed of light for new observers is 2^2, while the speed of light for the old observers is 1. Formally, $\mathfrak{M}$ is defined over

$$\begin{aligned}
\mathfrak{M}^M_{\mathfrak{F}} &= \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, G;\, \in, W \rangle \qquad \text{as follows:}\\
\mathfrak{M} &\stackrel{\text{def}}{:=} \langle (B';\, Obs', Ph', Ib'), \mathfrak{F}, G;\, \in, W' \rangle, \quad \text{where}\\
Obs' &\stackrel{\text{def}}{:=} Obs \times \{1, 2\},\\
Ph' &\stackrel{\text{def}}{:=} Ph \times \{1, 2\},^{390}\\
B' &\stackrel{\text{def}}{:=} Ib' \stackrel{\text{def}}{:=} Obs' \cup Ph',\\
W' &\stackrel{\text{def}}{:=} \Big\{ \Big\langle \langle m, i\rangle, p, \langle b, j\rangle \Big\rangle \in Obs' \times {}^nF \times B' \,:\, W(m, ip_0, p_1, \ldots, p_{n-1}, b) \Big\}.
\end{aligned}$$

We note that the speed of light for observers of the form $\langle m, 1\rangle$ is 1 while for observers of the form $\langle m, 2\rangle$ is 2^2.

It can be checked that $\mathfrak{M} \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ })$.

Further, it can be checked that the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ is first-order definable over $\mathfrak{F}$ in the sense of §4.3.2. *Hint:* The observers of $\mathfrak{M}^M_{\mathfrak{F}}$ can be identified with special Poincaré transformations of ${}^n\mathbf{F}$, namely, with elements of PT^M (cf. Prop.3.8.63 on p.346 of AMN [18] and Def.'s 3.8.38, 3.8.42 of AMN [18]). Since all these are affine transformations, they can be represented by matrices together with a vector. But a matrix together with a vector can be identified with a sequence (of length $n \cdot n + n$) of elements of $\mathfrak{F}$. The rest of defining $\mathfrak{M}^M_{\mathfrak{F}}$ over $\mathfrak{F}$ goes in the style of §4.3.2 using the "concrete construction" given for $\mathfrak{M}^M_{\mathfrak{F}}$ in Def.3.8.38 (p.325) of AMN [18] and Def.3.8.42 (p.331) of AMN [18].

Since $\mathfrak{M}$ was first-order defined (in the sense of §4.3.2) over $\mathfrak{M}^M_{\mathfrak{F}}$, we conclude that $\mathfrak{M}$ is first-order definable over $\mathfrak{F}$. Therefore, by Prop.4.3.18 (p.240), $\langle \mathfrak{M};\, Mn, \in\rangle$ is first-order definable over $\mathfrak{F}$.

By these, $\mathfrak{M}$ has properties (a) and (b) (formulated on p.168). Next we turn to proving that $\mathfrak{M}$ has property (c).

Let

$$H \stackrel{\text{def}}{:=} \left\{\, x \in {}^+F : (\exists m \in Obs')(c_m = 1 \;\wedge\; \langle w_m(\bar{0}), w_m(1_t)\rangle \; eq \; \langle w_m(\bar{0}), w_m(x \cdot 1_t)\rangle) \,\right\}.$$

Claim **4.2.24** $H = \{\, 2^i \,:\, i \in \mathrm{Z} \,\}$.

Proof: The proof of $\{\, 2^i \,:\, i \in \mathrm{Z} \,\} \subseteq H$ is depicted in Figure 77. In the figure $m, k \in Obs'$ are such that the speed of light for m is 1, while the speed of light for k is 2^2, m and k are "brothers" in the sense that $m = \langle h, 1\rangle$ and $k = \langle h, 2\rangle$, for some $h \in Obs$.

The proof of $H \subseteq \{\, 2^i \,:\, i \in \mathrm{Z} \,\}$ goes as follows. We will use the Minkowski distance $g_\mu : {}^nF \times {}^nF \longrightarrow F$ which is defined in Definition 4.2.44 (p.189), cf. also Def.2.9.1 (p.101). It can be easily checked, e.g. by the proof of Claim 6.2.84 (p.892) of AMN [18], that

[389] cf. Def.3.8.42 on p.331 of AMN [18] for Minkowski models

[390] We defined Ph' as $Ph \times \{1, 2\}$ only for technical reason.

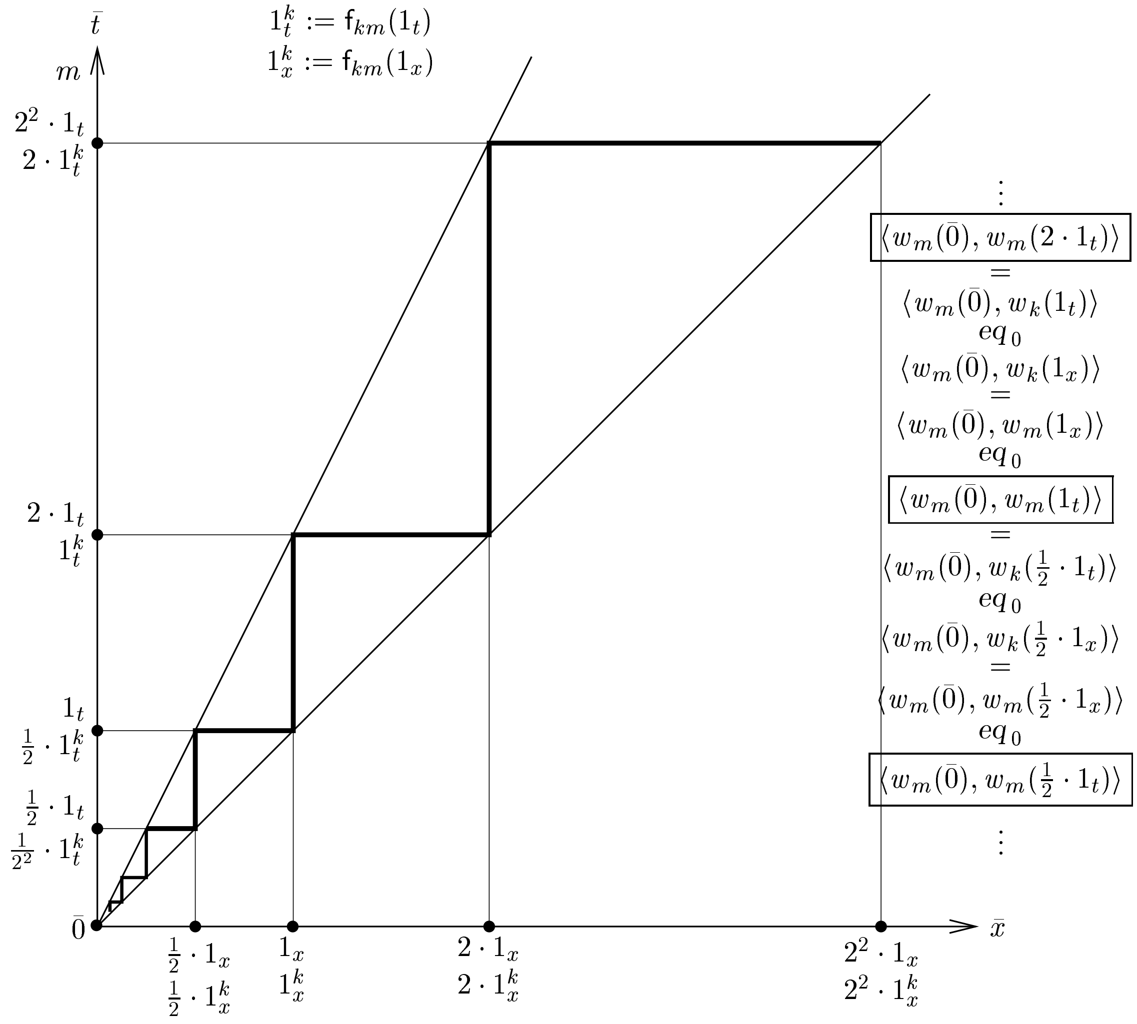


Figure 77: Proof of $\{\, 2^i \,:\, i \in \mathrm{Z} \,\} \subseteq H$. The right-hand side column illustrates the computational part of why m thinks that $\langle \bar{0}, 1_t \rangle$ is "eq-related" to $\langle \bar{0}, 2^i \cdot 1_t \rangle$ (which means $2^i \in H$ by the definition of H).

$$(\forall m \in Obs')(\forall p, q, r, s \in {}^nF) \Big((\, c_m = 1 \;\wedge\; \langle w_m(p), w_m(q)\rangle \; eq_0 \; \langle w_m(r), w_m(s)\rangle \,) \;\Rightarrow$$
$$(\, g_\mu(p,q) = 2^i g_\mu(r,s), \text{ for some } i \in \{-1,0,1\}\,)\Big).$$

Since eq was defined to be the transitive closure of eq_0, the above implies that

$$(\forall m \in Obs')(\forall p, q, r, s \in {}^nF) \Big((\, c_m = 1 \;\wedge\; \langle w_m(p), w_m(q)\rangle \; eq \; \langle w_m(r), w_m(s)\rangle \,) \;\Rightarrow$$
$$(\, g_\mu(p,q) = 2^i g_\mu(r,s), \text{ for some } i \in \mathrm{Z}\,)\Big).$$

By this, it can be easily checked that $H \subseteq \{\, 2^i \;:\; i \in \mathrm{Z}\,\}$ indeed holds.

QED (Claim 4.2.24)

By Claim 4.2.24 (and by the definition of H), we have that property (c) holds for $\mathfrak{M}$. To complete the proof for item (i), it remains to prove that the subset $\{\, 2^i \;:\; i \in \mathrm{Z}\}$ of F is not first-order definable over $\mathfrak{F}$. This will be an immediate corollary of Lemma 4.2.27 way below.

<u>*Case of (ii):*</u> The proof of item (ii) is similar to that of (i). We will construct a model $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(||) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\;}))$ such that $\mathfrak{M}$ has properties (a)–(c) formulated on p.168. Let $\mathfrak{F}$ be a real-closed field. Let $\mathfrak{M}$ be a model over $\mathfrak{F}$ obtained from the 2-dimensional Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ as follows. Intuitively, for each observer m of $\mathfrak{M}^M_{\mathfrak{F}}$ we include a new observer k such that

$$\mathsf{f}_{km}(1_x) = 1_t \quad \text{and} \quad \mathsf{f}_{km}(1_t) = 2 \cdot 1_x, \quad \text{see Figure 78.}$$

The speed of light for new observers is 2^2 while for the old ones it is 1. Further, the new

$$1^k_t := \mathsf{f}_{km}(1_t)$$
$$1^k_x := \mathsf{f}_{km}(1_x)$$

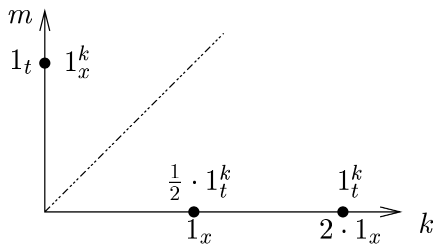


Figure 78: The picture represents the world-view of observer m.

observers are FTL observers relative to the old ones. Formally, $\mathfrak{M}$ is defined over $\mathfrak{M}^M_{\mathfrak{F}} = \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, G;\, \in, W\rangle$ as follows:

$$\begin{aligned}
\mathfrak{M} &\overset{\text{def}}{:=} \langle (B';\, Obs', Ph', Ib'), \mathfrak{F}, G;\, \in, W'\,\rangle, \quad \text{where}\\
Obs' &\overset{\text{def}}{:=} Obs \times \{1,2\},\\
Ph' &\overset{\text{def}}{:=} Ph \times \{1,2\},\\
B' &\overset{\text{def}}{:=} Ib' \overset{\text{def}}{:=} Obs' \cup Ph',\\
W' &\overset{\text{def}}{:=} \Big\{ \Big\langle \langle m,i\rangle, p_0, p_1, \langle b,j\rangle \Big\rangle \in Obs' \times F \times F \times B' \;:\; W(m, p_1, ip_0, b) \Big\}.
\end{aligned}$$

We note that the speed of light for observers of the form $\langle m,1\rangle$ is 1 while for observers of the form $\langle m,2\rangle$ is 2^2.

It can be checked that $\mathfrak{M} \models \mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ })$. The rest of the proof goes similarly to the proof given for item (i), i.e. we define H exactly the same way as in the proof of item (i); it can be proved that H coincides with $\{\, 2^i \,:\, i \in \mathrm{Z} \,\}$, etc. We omit the relatively easy details.

To complete the proof, it remains to prove that $\{\, 2^i \,:\, i \in \mathrm{Z} \,\}$ is not definable over $\mathfrak{F}$. A generalized version of this will be proved as Lemma 4.2.27 below. Thus the theorem is proved modulo Lemma 4.2.27. ∎

For stating Lemma 4.2.27 we need a convention and a definition.

CONVENTION 4.2.25 From now on, $\mathbb{Q}$ denotes the ordered field of rational numbers. Throughout we identify $\mathbb{Q}$ with its universe. $\mathbb{Q}$ is embeddable in a natural way into every ordered field $\mathfrak{F}$. When discussing an ordered field $\mathfrak{F}$ we will pretend that $\mathbb{Q}$ is a subfield of $\mathfrak{F}$. I.e. we identify $\mathbb{Q}$ with its unique isomorphic copy sitting inside $\mathfrak{F}$.

By an <u>*algebraic*</u> element of $\mathfrak{F}$ we understand an element which is algebraic over $\mathbb{Q}$.[391]

$\lhd$

Definition 4.2.26 Let $\mathfrak{F}$ be an ordered field. Let $H \subseteq F$. We call H <u>*gapy*</u> in $\mathfrak{F}$ iff

$$\Big(H \neq \emptyset \quad \text{and} \quad (\forall \text{ algebraic } a \in H)(\exists b, c \in F)(a < b < c \ \wedge\ b \notin H \ \wedge\ c \in H) \Big),$$

see Figure 79.

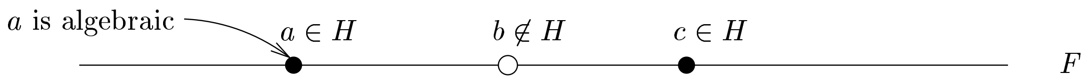


Figure 79: $H \subseteq F$ is gapy in $\mathfrak{F}$ iff it is nonempty and
$(\forall$ algebraic $a \in H)(\exists b, c$ as in the figure$)$.

$\lhd$

<u>*Examples:*</u> Z, ω and $\{\, 2^i \,:\, i \in \omega \,\}$ are gapy subsets in $\mathfrak{F}$, for any ordered field $\mathfrak{F}$.

LEMMA 4.2.27 *Assume $\mathfrak{F}$ is a real-closed field. Then <u>no</u> gapy subset $H \subseteq F$ in $\mathfrak{F}$ is definable over $\mathfrak{F}$.*

<u>*Proof:*</u> Assume $\mathfrak{F}$ is a real-closed field. Throughout the proof we will use the following fact from field theory.

Fact **4.2.28** Let $p(x)$ be a unary term in the language of $\mathfrak{F}$ extended with the unary operation symbol "$-$". Then (i) and (ii) below hold.

(i) Assume that $p(x) = 0$ is a nontrivial[392] equation. Then this equation has only finitely many solutions. Further, the solutions of $p(x) = 0$ in $\mathfrak{F}$ are algebraic elements of $\mathfrak{F}$.

[391] For completeness we recall that an element of $\mathfrak{F}$ is algebraic over $\mathbb{Q}$ iff it is a root of a nonzero polynomial with coefficients in $\mathbb{Q}$. (A root of a polynomial $p(x)$ is the same as a solution of the equation $p(x) = 0$.)

[392] $p(x) = 0$ is called trivial in $\mathfrak{F}$ iff $\mathfrak{F} \models p(x) = 0$.

(ii) The intermediate value theorem holds for the function defined by $p(x)$, i.e. if $p(a){\cdot}p(b) < 0$ then $p(c) = 0$ for some c strictly between a and b.[393]

Proof: Assume $p(x)$ is as above. Item (i) follows by the fact that $p(x)$ is a nonzero polynomial with coefficients in Z. Hence $p(x)$ has finitely many roots[394] and the roots of $p(x)$ are algebraic over $\mathbb{Q}$. For item (ii) cf. [130, Fact 8.4.5, p.386].

QED (Fact 4.2.28)

Now we turn to proving Lemma 4.2.27. The proof goes by contradiction. Assume that $H \subseteq F$ is gapy in $\mathfrak{F}$ and that H is definable over $\mathfrak{F}$. Then there is a first-order formula $\varphi(x)$ in the language of $\mathfrak{F}$ such that $H = \{\, a \in F \, : \, \mathfrak{F} \models \varphi[a] \,\}$. By Tarski's elimination of quantifiers Theorem for real-closed fields, i.e. by Thm.8.4.4 on p.385 of [130] and line 9 on p.376 of [130], $\varphi(x)$ is equivalent in $\mathfrak{F}$ to a *quantifier free* formula $\psi(x)$, i.e. $\mathfrak{F} \models \forall x(\varphi(x) \leftrightarrow \psi(x))$. Then $\psi(x)$ defines H, i.e.

$$H = \{\, a \in F \, : \, \mathfrak{F} \models \psi[a] \,\}\,.$$

Since ψ is quantifier free, it is a Boolean combination of atomic formulas. It is not hard to see that ψ is equivalent to a *disjunction* of formulas of the form[395]

$$(+) \qquad p_0(x) = 0 \,\wedge\, \ldots \,\wedge\, p_{k-1}(x) = 0 \,\wedge\, q_0(x) > 0 \,\wedge\, \ldots \,\wedge\, q_{m-1}(x) > 0,$$

where $m, k \in \omega$ and $p_i(x)$, $q_j(x)$ $(i \in k,\ j \in m)$ are unary terms in the language of $\mathfrak{F}$ extended with the operation symbol "$-$". *Warning:* here we include the unary operation "$-$" in the language of $\mathfrak{F}$. But then H is a finite union of sets definable by formulas of the form (+). Then one of these sets must be gapy in $\mathfrak{F}$ since H is gapy in $\mathfrak{F}$.[396] Therefore there is $H' \subseteq H$ such that H' is gapy in $\mathfrak{F}$ and H' is definable by a formula of the form (+). We may assume that this formula is exactly the one displayed in (+).

If one of the $(p_i(x) = 0)$'s is a nontrivial equation, then it has only finitely many solutions in $\mathfrak{F}$ and these solutions are algebraic elements of $\mathfrak{F}$ (by Fact 4.2.28(i)), hence H' is a finite set of algebraic elements of $\mathfrak{F}$ which contradicts the fact that H' is gapy in $\mathfrak{F}$. Therefore we may assume $k = 0$. Thus

$$(*) \qquad H' = \{\, a \in F \, : \, q_0(a) > 0 \,\wedge\, \ldots \,\wedge\, q_{m-1}(a) > 0 \,\}\,.$$

We may assume that none of the $(q_i(x) = 0)$'s is trivial. Therefore the set

$$Sol :\stackrel{\text{def}}{=} \{\, d \in F \, : \, (\exists i \in m) q_i(d) = 0\}$$

(of solutions) is finite by Fact 4.2.28(i).

Claim **4.2.29** $(\forall \text{ algebraic } a \in H')(\exists b, c \in F)$

$$\Big((c \text{ is algebraic}) \,\wedge\, a < b < c \,\wedge\, b \notin H' \,\wedge\, c \in H'\Big).$$

---

[393] This can be memorized by e.g. thinking of the Bolzano Theorem from elementary calculus.

[394] That each nonzero polynomial in $\mathfrak{F}$ has only finitely many roots is a well known property of ordered fields.

[395] using facts like $\tau(x) < \sigma(x) \Leftrightarrow \sigma(x) - \tau(x) > 0$, or $\neg(\tau < \sigma) \Leftrightarrow (\tau = \sigma \vee \sigma < \tau)$ etc.

[396] In more detail: $H = \bigcup_{i \in n} H_i$ for some $n \in \omega$ and each H_i is definable by a formula of the form (+). Then one of the H_i's is gapy in $\mathfrak{F}$ because of the following. Assume that none of the H_i's is gapy in $\mathfrak{F}$. Without loss of generality we can assume that each H_i is nonempty. Then, for all $i \in n$

$$(\exists \text{ algebraic } a_i \in H_i)(\{\, y \, : \, y > a_i \,\} \subseteq H_i \,\vee\, \{\, y \, : \, y > a_i \,\} \subseteq F \setminus H_i).$$

But then, for $a := \max\{\, a_i \, : \, i \in n \,\}$ we have that $a \in H$ and a is algebraic, further

$$\{\, y \, : \, y > a \,\} \subseteq H \,\vee\, \{\, y \, : \, y > a \,\} \subseteq F \setminus H.$$

This contradicts our assumption that H is gapy in $\mathfrak{F}$. Therefore one of the H_i's is gapy in $\mathfrak{F}$.

Proof: Let $a \in H'$ be such that a is an algebraic element of $\mathfrak{F}$. We have to prove that there are $b, c \in F$ such that $a < b < c$, $b \notin H'$, $c \in H'$ and c is algebraic. Let $b, c' \in F$ be such that $a < b < c'$, $b \notin H'$ and $c' \in H'$. Since H' is gapy in $\mathfrak{F}$ such b and c' exist. To prove the claim it is enough to prove that there is an algebraic $c \in H$ such that $b < c$. Clearly,

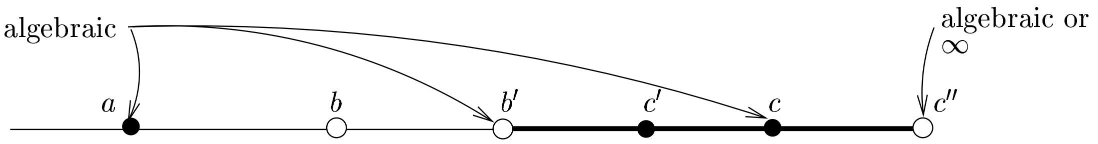


Figure 80:

$$q_i(c') > 0, \quad \text{for all } i \in m$$

by $(*)$ and by $c' \in H'$. See Figure 80. Further, by $b \notin H'$ and $(*)$, there is $j \in m$ such that $q_j(b) \leq 0$. Let such a j be fixed. Thus, by Fact 4.2.28(ii) (and by $b < c'$), there is $d \in F$ such that $b \leq d < c'$ and $q_j(d) = 0$. Therefore the set $\{ d \in Sol \,:\, d < c' \}$ is nonempty (and is finite), and

$$b \leq \max \{ d \in Sol \,:\, d < c' \} .$$

Let

$$b' \stackrel{\text{def}}{:=} \max \{ d \in Sol \,:\, d < c' \} .$$

Let

$$c'' \stackrel{\text{def}}{:=} \begin{cases} \min \{ d \in Sol \,:\, d > c' \} & \text{if } (\exists d \in Sol) d > c' \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, $b \leq b' < c' < c''$ and none of the equations $q_0(x) = 0, \ldots, q_{m-1}(x) = 0$ has a solution in the open interval $(b', c'') := \{ d \in F \,:\, b' < d < c'' \}$, cf. Figure 80 (recall that $q_i(c') > 0$, for all $i \in \omega$). By this, by Fact 4.2.28(ii), by $(*)$ and by $c' \in H'$, we conclude that $(b', c'') \subseteq H'$. Further (by Fact 4.2.28(i)) b' is an algebraic element of $\mathfrak{F}$ and c'' is either an algebraic element of $\mathfrak{F}$ or is ∞. Thus there is an algebraic element c of $\mathfrak{F}$ such that $c \in (b', c'') \subseteq H'$. For this choice of c we have $b < c$, $c \in H'$ and c is an algebraic element of $\mathfrak{F}$.

QED (Claim 4.2.29)

Let $a^i, b^i \in F$ $(i \in \omega)$ be such that for all $i \in \omega$, a^i is an algebraic element of $\mathfrak{F}$, $a^i \in H'$, $b^i \notin H'$, and

$$a^i < b^i < a^{i+1} < b^{i+1}.$$

By Claim 4.2.29, such a^i's and b^i's exist. By $(*)$,[397] there are $j \in m$ and an infinite subset I of ω such that

$$(\forall i \in I) \, (\, q_j(b^i) \leq 0 \quad \wedge \quad q_j(a^i) > 0 \,).$$

Let such j and I be fixed. Let $h : \omega \rightarrowtail\!\!\!\twoheadrightarrow I$ be an order preserving bijection. Then clearly,

$$(\forall i \in \omega)(\, q_j(a^{h(i)}) > 0 \;\wedge\; q_j(b^{h(i)}) \leq 0 \;\wedge\; a^{h(i)} < b^{h(i)} < a^{h(i+1)} < b^{h(i+1)} \,).$$

Thus, by Fact 4.2.28(ii), for every $i \in \omega$ there is $c^{h(i)} \in F$ such that $a^{h(i)} < c^{h(i)} \leq b^{h(i)}$ and $q_j(c^{h(i)}) = 0$. By the above we conclude that the equation $q_j(x) = 0$ has infinitely many solutions, and this contradicts item (i) of Fact 4.2.28. ∎

[397] and by $a^i \in H'$, $b^i \notin H'$

At this point all parts of the proof of Thm.4.2.23 have been taken care of.

One of the reasons for looking at the alternative notions like $\perp_r'$, $\perp_r''$, eq_2 is that they can behave better from the point of view of definability issues. (There are of course other reasons, too, for experimenting with alternative concepts.) Similarly, we will look at alternative definitions of the topology part $\mathcal{T}$ of our geometries. Namely, $\mathcal{T}'$ will be based on Bw while $\mathcal{T}''$ will be based on causality $\prec$.

Definition 4.2.30 (Alternatives $\mathcal{T}'$, $\mathcal{T}''$ for topology $\mathcal{T}$)
Assume $n > 1$. Let $\mathfrak{M}$ be a frame model of n dimensions. $Mn, Bw, \prec$ are defined in items 2, 7, 6 of Def.4.2.3(I). We define the topologies $\mathcal{T}'$ and $\mathcal{T}''$ on Mn in items (i) and (ii) below, respectively.

(i) Intuitively, first by using Bw we define interiors of simplexes,[398] cf. the left-hand side of Figure 81. Then by using these (as a subbase) we define the topology $\mathcal{T}'$ on Mn the natural way, formally:

For every $H \subseteq Mn$ the <u>*convex hull*</u> $Ch(H)$ of H is the smallest subset of Mn having properties 1 and 2 below.[399]

1. $H \subseteq Ch(H)$.
2. $(\, a, b \in Ch(H) \ \wedge \ Bw(a, c, b)\,) \quad \Rightarrow \quad c \in Ch(H)$.

We define the collection $\mathsf{simplexes} \subseteq \mathcal{P}(Mn)$ as follows.

$$\mathsf{simplexes} :\stackrel{\text{def}}{=} \{\, H \subseteq Mn \ : \ |H| = n+1, \ (\exists m \in Obs)\, Plane(H) = Rng(w_m)\,\}\,.$$

Let $H \in \mathsf{simplexes}$. Then, intuitively, the neighborhood $S'(H)$ is defined to be the "interior" of the convex hull $Ch(H)$ of H; formally:

$$S'(H) :\stackrel{\text{def}}{=} Ch(H) \setminus \bigcup_{e \in H} Plane(H \setminus \{e\}),$$

see the left-hand side of Figure 81. Now, the topology $\mathcal{T}' \subseteq \mathcal{P}(Mn)$ is the one generated by T_0' below, i.e. T_0' is a subbase for $\mathcal{T}'$.

$$T_0' :\stackrel{\text{def}}{=} \{\, S'(H) \ : \ H \in \mathsf{simplexes}\,\}\,.$$

We note that, assuming $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$, T_0' is a *base* for $\mathcal{T}'$, cf. Figure 81 (and the proof of Thm.4.2.33).

(ii) For every $a, b \in Mn$ with $a \prec b$ we define the neighborhood

$$S''(a, b) :\stackrel{\text{def}}{=} \{c \in Mn \ : \ a \prec c \prec b\,\}\,,$$

see the right-hand side of Figure 81. Now, the topology $\mathcal{T}'' \subseteq \mathcal{P}(Mn)$ is the one generated by T_0'' below, i.e. T_0'' is a subbase for $\mathcal{T}''$.

$$T_0'' :\stackrel{\text{def}}{=} \{\, S''(a, b) \ : \ a, b \in Mn, \ a \prec b\,\}\,.$$

[398] We note that if $n = 2$ then the simplexes are the triangles and if $n = 3$ then the simplexes are the tetrahedra.

[399] The usual notation in the literature is "$co(H)$" for our $Ch(H)$.

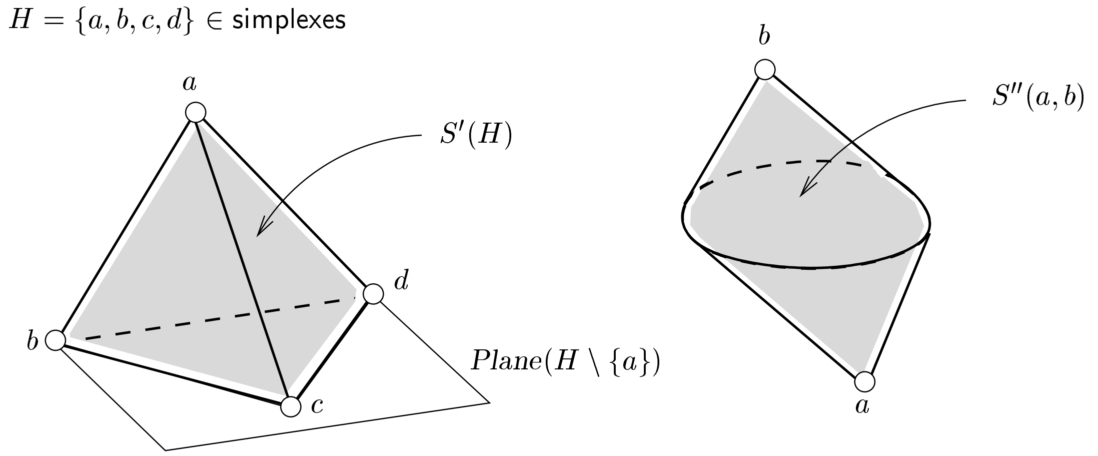


Figure 81: In the figure $n = 3$.

We note that, assuming $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\surd\,)$ and $[\,(\forall m \in Obs)\,($ m thinks that there is an upper bound for the speed of light $)$[400] <u>or</u> $\mathfrak{F} = \mathfrak{R}]$, T''_0 is a *base* for $\mathcal{T}''$, where $\mathbf{Ax}(\uparrow\uparrow_0)$ is defined below, cf. Figure 81 (and the proof of Thm.4.2.33).

$\triangleleft$

Theorems 4.2.33, 4.2.37 and 4.2.38 below say that topologies $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ coincide, under some assumptions. For stating these theorems we introduce weakened versions $\mathbf{Ax}(\uparrow\uparrow_0)$ and $\mathbf{Ax}(\uparrow\uparrow_{00})$ of our axiom $\mathbf{Ax}(\uparrow\uparrow)$ saying that each observer sees any other observer's time flow forwards. Recall that $m \overset{\odot}{\to} k$ means that $tr_m(k) \neq \emptyset$ and m STL k means that m sees k moving more slowly than light. $m \uparrow k$ denotes that <u>*m sees k's clock running forwards*</u>, i.e.

$$m \uparrow k \overset{\text{def}}{\Longleftrightarrow} \mathsf{f}_{km}(1_t)_t > \mathsf{f}_{km}(\bar{0})_t.$$

Note that by Convention 2.3.10 (p.31), $m \uparrow k$ implies that f_{km} is defined on 1_t and thus $m \uparrow k$ implies $m \overset{\odot}{\to} k$. Similarly, m STL k implies $m \overset{\odot}{\to} k$.

$\mathbf{Ax}(\uparrow\uparrow)$ $(\forall m, k \in Obs) m \uparrow k$.

$\mathbf{Ax}(\uparrow\uparrow_0)$ $(\forall m, k \in Obs)\,(m \overset{\odot}{\to} k \;\to\; m \uparrow k)$.

Intuitively, if m sees k then k's clock runs forwards as seen by m.

$\mathbf{Ax}(\uparrow\uparrow_{00})$ $(\forall m, k \in Obs)\,(m\ STL\ k \;\to\; m \uparrow k)$.

Intuitively, if m sees k moving more slowly than light then k's clock runs forwards as seen by m.

The reason for introducing $\mathbf{Ax}(\uparrow\uparrow_0)$ is that $\mathbf{Ax}(\uparrow\uparrow)$ blurs the distinction between **Basax** and **Newbasax**.[401] The reason for introducing $\mathbf{Ax}(\uparrow\uparrow_{00})$ is that $\mathbf{Ax}(\uparrow\uparrow_0)$, together with $\mathbf{Bax}^- + \mathbf{Ax}(\surd\,)$, excludes FTL observers already in two dimensions, cf. Prop.4.2.31 below.

[400] formally: $(\exists \lambda \in F)(\forall d \in \text{directions})\, \mathsf{c}_m(d) < \lambda$.

[401] $\mathbf{Newbasax} + \mathbf{Ax}(\uparrow\uparrow) \models \mathbf{Basax}$ because $m \overset{\odot}{\to} k$ implies that f_{km} is defined on 1_t (by Convention 2.3.10), and thus $m \uparrow k$ implies $m \overset{\odot}{\to} k$.

On the other hand, $\mathbf{Ax}(\uparrow\uparrow_{00})$ does not exclude FTL observers, and hence we can use $\mathbf{Ax}(\uparrow\uparrow_{00})$ in theories in which we want to allow FTL observers. We will see that the issue of the existence of FTL observers is relevant to our relativistic geometries. E.g. if there are FTL observers, then the sets of time-like and space-like lines are not disjoint.

PROPOSITION 4.2.31 (i) $\mathbf{Bax}^- + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ }) \models$ **"$\nexists$ FTL observers".**

(ii) $\mathbf{Bax}^- + \mathbf{Ax}(\uparrow\uparrow_{00}) + \mathbf{Ax}(\sqrt{\ }) \not\models$ **"$\nexists$ FTL observers".**

Proof: <u>**(i):**</u> In the proof of Thm.3.2.13 (p.118) we proved that $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) \models (\exists k)m \text{ FTL } k \;\rightarrow\; (\exists k)v_m(k) = \infty$. If $v_m(k) = \infty$, then $m \uparrow k$ does not hold in spite of $m \overset{\odot}{\to} k$. This proves (i). <u>**(ii):**</u> The parameters of the model constructed in the proof of Thm.4.3.25 (p.500) of AMN [18] can be chosen so that $\mathbf{Ax}(\uparrow\uparrow_0)$ holds in it. This proves (ii). ∎

Notation 4.2.32 Assume $\mathfrak{F}$ is an ordered field.

(i) Let $p \in {}^nF$. Then the square $\|p\|$ of the Euclidean length of the vector p is defined as follows.[402]

$$\|p\| \stackrel{\text{def}}{=} p_0^2 + p_1^2 + \ldots + p_{n-1}^2 .$$

(ii) Let $p \in {}^nF$ and $\varepsilon \in {}^+F$. Then by $S(p,\varepsilon)$ we denote the <u>ε-neighborhood of p</u> defined as follows.[403]

$$S(p,\varepsilon) \stackrel{\text{def}}{=} \{\, q \in {}^nF \;:\; \|q-p\| < \varepsilon \,\} .$$

(iii) Let $H \subseteq {}^nF$. We say that H is an <u>open set</u> iff

$$(\forall q \in H)(\exists \varepsilon \in {}^+F)\; S(q,\varepsilon) \subseteq H .$$

The set of open subsets of ${}^nF$ is denoted by $Open = Open(n,\mathfrak{F})$.[404]

◁

THEOREM 4.2.33 *Assume* $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$. *Assume that* $(\forall m \in Obs)$ (*m thinks that there is an upper bound for the speed of light*)[405] <u>*or*</u> $\mathfrak{F} = \mathfrak{R}$. *Then (i) and (ii) below hold.*

(i) *The topologies $\mathcal{T}'$ and $\mathcal{T}''$ coincide.*

(ii) *The topology $\mathcal{T}' = \mathcal{T}''$ is a Euclidean one in the following sense:*

[402] We use the square of the length instead of the length itself because we did not assume that $\mathfrak{F}$ is Euclidean.

[403] In the notation $S(p,\varepsilon)$ the letter S refers to the word "sphere". Further we note that there is a slight danger of confusion because S will denote the space-part of our coordinate-system ${}^nF$. We hope context will help.

[404] The set *Open* is of course not definable (at least not as an entity on its own right) in our frame language, but if we have a definable subset like $Dom(\mathsf{f}_{mk})$ then $Dom(\mathsf{f}_{mk}) \in Open$ counts as a first-order formula of our frame language, i.e. is translatable to a formula of our frame language. A similar remark applies e.g. to Eucl, *Linb*, *Rhomb* etc, and we will not repeat this remark each time we introduce an abbreviation like *Open*, Eucl, *Linb* etc. Summing it up, one could say that *Open*, Eucl etc. are definable only as "predicate symbols" (but not necessarily as individual objects or entities.)

[405] formally: $(\exists \lambda \in F)(\forall d \in \mathsf{directions})\, \mathsf{c}_m(d) < \lambda$, where $\mathsf{c}_m(d)$ is defined on p.A-21.

(a) *For any $m \in Obs$, $\{\, w_m^{-1}[H] \,:\, H \in \mathcal{T}' \,\}$ is the usual Euclidean topology on ${}^nF$, i.e. the one with base $\{\, S(p,\varepsilon) \,:\, p \in {}^nF,\ \varepsilon \in {}^+F \,\}$.*

(b) *$\mathcal{T}'$ is homeomorphic to a sum topology (i.e. a coproduct)*[406] *of usual Euclidean topologies on ${}^nF$.*

Proof: Assume the assumptions of the theorem. By Thm.3.2.6, we have that: the visibility relation $\xrightarrow{\odot}$ is an equivalence relation when restricted to Obs, and if $m \xrightarrow{\odot} k$ then $Rng(w_m) = Rng(w_k)$, otherwise $Rng(w_m) \cap Rng(w_k) = \emptyset$.

Let $O \subseteq Obs$ be a class of representatives for the equivalence relation $\xrightarrow{\odot}$.[407] Then

(∗) Mn is the disjoint union of the family $\langle\, Rng(w_m) \,:\, m \in O \,\rangle$ (and the members of this family are mutually disjoint).

It is easy to check that for every $m \in Obs$

$$(**) \qquad Rng(w_m) \in \mathcal{T}' \qquad \text{and} \qquad Rng(w_m) \in \mathcal{T}'',$$

i.e. $Rng(w_m)$ is an open set w.r.t. both topologies. For every $m \in Obs$, let $\mathcal{T}' \restriction Rng(w_m)$ and $\mathcal{T}'' \restriction Rng(w_m)$ be the subspace topologies of $\mathcal{T}'$ and $\mathcal{T}''$ on $Rng(w_m)$,[408] respectively, i.e.

$$\begin{aligned}
\mathcal{T}' \restriction Rng(w_m) &\stackrel{\text{def}}{:=} \{\, H \cap Rng(w_m) \,:\, H \in \mathcal{T}' \,\} = \{\, H \in \mathcal{T}' \,:\, H \subseteq Rng(w_m) \,\}, \\
\mathcal{T}'' \restriction Rng(w_m) &\stackrel{\text{def}}{:=} \{\, H \cap Rng(w_m) \,:\, H \in \mathcal{T}'' \,\} = \{ H \in \mathcal{T}'' \,:\, H \subseteq Rng(w_m) \,\};
\end{aligned}$$

further let $\mathcal{T}'_m$ and $\mathcal{T}''_m$ be the topologies on the coordinate system ${}^nF$ defined as follows.

$$\begin{aligned}
\mathcal{T}'_m &\stackrel{\text{def}}{:=} \{\, w_m^{-1}[H] \,:\, H \in \mathcal{T}' \,\}. \\
\mathcal{T}''_m &\stackrel{\text{def}}{:=} \{\, w_m^{-1}[H] \,:\, H \in \mathcal{T}'' \,\}.
\end{aligned}$$

It is easy to see that for every $m \in Obs$

(∗∗∗) $w_m : {}^nF \longrightarrow Rng(w_m)$ is a homeomorphism between $\mathcal{T}'_m$ and $\mathcal{T}' \restriction Rng(w_m)$ and between $\mathcal{T}''_m$ and $\mathcal{T}'' \restriction Rng(w_m)$.

To prove item (i) of the theorem, by (∗), (∗∗), (∗∗∗) above it is enough to prove that for each m, $\mathcal{T}'_m$ and $\mathcal{T}''_m$ coincide. This holds by Claim 4.2.34 below.

Claim **4.2.34** Let $m \in Obs$. Then (a) and (b) below hold.

(a) $\mathcal{T}'_m$ is the Euclidean topology on ${}^nF$, i.e. the one with base $\{\, S(p,\varepsilon) \,:\, p \in {}^nF,\ \varepsilon \in {}^+F \,\}$.

(b) $\mathcal{T}''_m$ is the Euclidean topology on ${}^nF$.

<u>*Proof:*</u>

<u>Proof of (a):</u> A set $H \subseteq {}^nF$ is called a <u>*simplex*</u> iff $|H| = n+1$ and for each $p \in H$, $\{\, q - p \,:\, q \in H,\ q \neq p \,\}$ is a basis[409] for the vector space ${}^n\mathbf{F}$, cf. the left-hand side of Figure 81.[410]

---

[406] Cf. p.198 for coproduct of topological spaces. Cf. also Engelking [83] under the name "sum of spaces".

[407] I.e. $(\forall m \in Obs)\, |O \cap m/\xrightarrow{\odot}| = 1$, where $m/\xrightarrow{\odot}$ is the equivalence class of m w.r.t. $\xrightarrow{\odot}$, as usual.

[408] i.e. they are the restrictions to $Rng(w_m)$ of $\mathcal{T}'$ and $\mathcal{T}''$, respectively

[409] i.e. a minimal set of generators

[410] This is practically the same notion as "simplexes" in Def.4.2.30, the only difference being that now we are in ${}^nF$ while there we were in $\langle Mn, \ldots \rangle$.

Clearly, a subbase for $\mathcal{T}'_m$ is

$$T'_m \stackrel{\text{def}}{:=} \{\, w_m^{-1}[H] \;:\; H \in T'_0,\; w_m^{-1}[H] \neq \emptyset \,\};$$

where recall that T'_0 is the subbase of $\mathcal{T}'$. Since the world-view transformations are betweenness preserving collineations[411] it can be checked (by item 1f of Prop.4.2.64) that T'_m consists of the interiors of the convex hulls of the simplexes, where interiors of sets are defined via the Euclidean topology, and convex hulls of sets are defined in Def.4.3.28(iii) (p.509) of AMN [18].

T'_m is a base for the Euclidean topology (on ${}^nF$) because of the following. Let H be an open set of the Euclidean topology. Then for any $p \in H$, there is a "neighborhood" of p in T'_m which is contained in H. Hence H is a union of members of T'_m.

But then, $\mathcal{T}'_m$ is the Euclidean topology on ${}^nF$.

Proof of (b): Let $\prec_m$ be a binary relation on ${}^nF$ defined as follows.

$$\prec_m \;\stackrel{\text{def}}{:=}\; \{\, \langle p, q\rangle \in {}^nF \times {}^nF \;:\; w_m(p) \prec w_m(q) \,\}.$$

For every $p \in {}^nF$, let

$$\mathsf{Future}_p \;\stackrel{\text{def}}{:=}\; \{\, q \in {}^nF \;:\; p \prec_m q \,\},$$

$$\mathsf{Past}_p \;\stackrel{\text{def}}{:=}\; \{\, q \in {}^nF \;:\; q \prec_m p \,\}.$$

Clearly, a subbase for $\mathcal{T}''_m$ is

$$T''_m \;\stackrel{\text{def}}{:=}\; \{\, w_m^{-1}[H] \;:\; H \in T''_0,\; w_m^{-1}[H] \neq \emptyset \,\},$$

where recall that T''_0 is the subbase of $\mathcal{T}''$. It is easy to see that

(16) $$T''_m \;=\; \{\, \mathsf{Future}_p \cap \mathsf{Past}_q \;:\; p, q \in {}^nF,\; p \prec_m q \,\}.$$

By item 1h of Prop.4.2.64 (p.209), we have that

(17) $$p \prec_m q \;\Leftrightarrow\; [\, p_t < q_t \;\wedge\; (\exists k \in Obs)\, p, q \in tr_m(k) \,].$$

There are no FTL observers, by Prop.4.2.31. Thus, by Thm.4.3.29 (p.510) of AMN [18], by (17) and by $\mathbf{Ax5_{Obs}}$, we have that for any $p \in {}^nF$

(18) Future_p is the interior of the convex hull of $\{\, q \in \mathsf{Cone}_{m,p} \;:\; p_t < q_t \,\}$, and

(19) Past_p is the interior of the convex hull of $\{\, q \in \mathsf{Cone}_{m,p} \;:\; p_t > q_t \,\}$;

where interiors of sets are defined via the Euclidean topology, and convex hulls of sets are defined in Def.4.3.28(iii) (p.509) of AMN [18]. By (16), (18), (19) and Thm.4.3.29 (p.510) of AMN [18], we conclude that T''_m is a base for the Euclidean topology (on ${}^nF$), cf. the right-hand side of Figure 81. Hence, $\mathcal{T}''_m$ is the Euclidean topology.

QED (Claim 4.2.34)

By this, item (i) of our theorem is proved. Item (ii) follows by $(*)$, $(**)$, $(***)$ and Claim 4.2.34. Namely, by $(*)$, $(**)$ we have that $\mathcal{T}'$ is the sum topology (i.e. the coproduct) of the family $\langle\, \mathcal{T}' \restriction Rng(w_m) \;:\; m \in O \,\rangle$ which in turn, by $(***)$, is homeomorphic to the sum topology (i.e. the coproduct) of the family $\langle\, \mathcal{T}'_m \;:\; m \in O \,\rangle$; while by Claim 4.2.34 we have that each $\mathcal{T}'_m$ is the Euclidean topology on ${}^nF$. ∎

PROPOSITION 4.2.35 *Assume* $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$. *Then the topology* $\mathcal{T}'$ *is the Euclidean one in the sense of Thm.4.2.33(ii), i.e. it has properties (a) and (b) in the formulation of Thm.4.2.33(ii).*

Moreover T'_0 *is a base for* $\mathcal{T}'$.

Proof: The proposition is a corollary of the proof of Thm.4.2.33. ∎

[411] by Thm.3.2.6 (p.110) and $\mathbf{Ax}(\sqrt{\ })$.

To see when $\mathcal{T}$ coincides with $\mathcal{T}'$ and $\mathcal{T}''$, we will need the axioms introduced below. We have introduced a strong symmetry principle $\mathbf{Ax}(\omega)$ in §3.9 of AMN [18] (cf. the list of axioms here (p.A-19)). Below we introduce four *weak* variants $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax}(\omega)^{00}$, $\mathbf{Ax}(\omega)^\sharp$, $\mathbf{Ax}(\omega)^{\sharp\sharp}$ of $\mathbf{Ax}(\omega)$, where $\mathbf{Ax}(\omega)^0$ and $\mathbf{Ax}(\omega)^{00}$ can be considered as natural *weakened versions* of $\mathbf{Ax}(\omega)$; while $\mathbf{Ax}(\omega)^\sharp$ and $\mathbf{Ax}(\omega)^{\sharp\sharp}$ can be considered as natural *weakened versions* of $\mathbf{Ax}(\omega) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$. I.e.

$$\begin{array}{ccccc} [\mathbf{Ax}(\omega) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })] & > & \mathbf{Ax}(\omega)^\sharp & > & \mathbf{Ax}(\omega)^{\sharp\sharp} \\ \vee & & \vee & & \vee \\ \mathbf{Ax}(\omega) & > & \mathbf{Ax}(\omega)^0 & > & \mathbf{Ax}(\omega)^{00}. \end{array}$$

We will use these axioms in formulating some of our theorems.

Definition 4.2.36 Axioms $\mathbf{Ax\Box 2}$, $\mathbf{Ax\triangle 1}$, $\mathbf{Ax\triangle 2}$ are defined in the list of axioms (p.A-19) herein.

$\mathbf{Ax}(\omega)^0$ is defined to be the <u>disjunction</u> of the following symmetry axioms: $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle 1{+}Ax(eqtime)}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\Box 1{+}Ax(eqtime)}$, $\mathbf{Ax\Box 2}$.[412]

$\mathbf{Ax}(\omega)^\sharp$ is defined to be $\mathbf{Ax}(\omega)^0{+}\mathbf{Ax}(\mathit{Triv}_t)^-{+}\mathbf{Ax}(\sqrt{\ })$.

$\mathbf{Ax}(\omega)^{00}$ is defined to be the <u>disjunction</u> of the following symmetry axioms: $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}{+}\mathbf{Ax}(\mathit{Triv}_t)^-$.[413]

$\mathbf{Ax}(\omega)^{\sharp\sharp}$ is defined to be $\mathbf{Ax}(\omega)^{00} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$.

$\lhd$

For characterizing the strengths of the just introduced axioms we refer to AMN [18, items 6.2.38–6.2.40].

In connection with the following theorem recall that

$$\mathbf{Basax} \models \mathbf{Newbasax} \models \mathbf{Flxbasax}^\oplus.$$

Let Th^+ be the theory

$$\mathbf{Flxbasax}^\oplus + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax(diswind)}$$

which will occur in Thm.4.2.37 below. This theory or its variants with **Basax** or **Newbasax** in place of $\mathbf{Flxbasax}^\oplus$ will often occur in our subsequent theorems. Therefore we note that by our previously mentioned 3 results (items 6.2.38–6.2.40 in AMN [18]), Th^+ is almost equivalent to "official special relativity" with disjoint windows allowed.[414]

[412] We note that, assuming $\mathbf{Flxbasax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$ these symmetry axioms are equivalent to one another, cf. Thm.2.8.17 (p.84) herein, AMN [18, Thm.3.9.11 (p.356), Thm.6.2.98 (p.910)] and [168].

[413] We note that, assuming $n > 2$ and $\mathbf{Flxbasax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$ the symmetry axioms involved in $\mathbf{Ax}(\omega)^0$ and $\mathbf{Ax}(\omega)^{00}$ are equivalent to one another.

[414] By "official special relativity" we refer to **Specrel**.

THEOREM 4.2.37 *Assume* $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\omega)^{\sharp} + \mathbf{Ax(diswind)}$. *Then (i) and (ii) below hold.*

(i) *$\mathcal{T}$ and $\mathcal{T}'$ coincide.*

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ coincide.*[415]

Further the topology $\mathcal{T} = \mathcal{T}' = \mathcal{T}''$ is the Euclidean one in the sense of Thm.4.2.33(ii).

To save space, we omit the proof. It is available from the author. ∎

Since $\mathbf{Ax}(\omega)^{\sharp}$ was designed to be weak, Theorems 4.2.37, 4.2.38 say that $\mathbf{Flxbasax}^{\oplus}$ + (some mild assumptions) suffice for $\mathcal{T} = \mathcal{T}' = \mathcal{T}''$.

The next theorem says that if $n > 2$ then in the above theorem we could use the weaker $\mathbf{Ax}(\omega)^{\sharp\sharp}$ in place of $\mathbf{Ax}(\omega)^{\sharp}$.

THEOREM 4.2.38 *Assume $n > 2$ and* $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax(diswind)}$. *Then (i) and (ii) in Thm.4.2.37 hold.*

The **proof** is available from the author. ∎

Theorems 4.2.11 (p.158), 4.2.21 (p.166), 4.2.22 (p.166), 4.2.33 (p.177) and 4.2.37 motivate the following definition.

Definition 4.2.39
(Alternatives $\mathfrak{G}'_{\mathfrak{M}}$, $\mathfrak{G}''_{\mathfrak{M}}$ and $\mathsf{Ge}'(Th)$, $\mathsf{Ge}''(Th)$ for $\mathfrak{G}_{\mathfrak{M}}$ and $\mathsf{Ge}(Th)$)

(i) Assume $\mathfrak{M}$ is a frame model. Then we define $\mathfrak{G}'_{\mathfrak{M}}$ to be the geometry obtained from $\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \mathbf{F_1}, \ldots\rangle$ by replacing $\perp_r$, eq by $\perp'_r$, eq_2, respectively, i.e.

$$\mathfrak{G}'_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp'_r, eq_2, g, \mathcal{T}\rangle.$$

We define $\mathfrak{G}''_{\mathfrak{M}}$ to be the geometry obtained from $\mathfrak{G}'_{\mathfrak{M}}$ by replacing the topology $\mathcal{T}$ by $\mathcal{T}'$, i.e.

$$\mathfrak{G}''_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp'_r, eq_2, g, \mathcal{T}'\rangle.$$

(ii) Let Th be a set of formulas in our frame language for relativity theory. Then the classes of relativistic geometries $\mathsf{Ge}'(Th)$ and $\mathsf{Ge}''(Th)$ associated with Th are defined as follows.

$$\mathsf{Ge}'(Th) :\stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{G}'_{\mathfrak{M}} : \mathfrak{M} \in \mathsf{Mod}(Th)\},$$
$$\mathsf{Ge}''(Th) :\stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{G}''_{\mathfrak{M}} : \mathfrak{M} \in \mathsf{Mod}(Th)\},$$

where for taking isomorphic copies of our geometries we apply Convention 4.2.4 (i.e. we stick with the "real" membership relation "$\in$"). ◁

Our next theorem says, roughly, that our class $\mathsf{Ge}(Th)$ of relativistic geometries is definable over the corresponding class of observational models.

In Theorem 4.2.40 below instead of definability of the topology part we claim definability of <u>only a subbase</u> for the topology. An exception is item (ii) of Thm.4.2.40, because there a base T'_0 will be definable, too. The content of Thm.4.2.40 below will be presented (discussed etc.) in a greater detail in §4.3 (cf. the proof of Thm.4.2.40).

[415] A physical consequence of Thm.4.2.37 is that for the various definitions of our topology (i.e. $\mathcal{T}, \mathcal{T}', \mathcal{T}''$) the so-called measurable sets remain the same (under the assumptions of the theorem). The reason for this is that the measurable sets are usually derived from the topology. In principle results like this might be relevant for recent theories of physical measurement (where the notion of measurement is related to measurable sets) cf. Attila Andai personal communication. Cf. e.g. Misner-Thorne-Wheeler [192, p.1184 (lower part of the page)]. Cf. also Andai [6, Chap.4, §5] and Pulmanová [214].

THEOREM 4.2.40

(i) *The class* $\mathsf{Ge}'(Th)$ *is uniformly first-order* *__definable__*[416] *over the class* $\mathsf{Mod}(Th)$*, for any set* Th *of formulas in our frame language.*[417]

(ii) $\mathsf{Ge}''(Th)$ *is uniformly first-order* *__definable__* *over* $\mathsf{Mod}(Th)$*, assuming*

$$Th \models \mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }).$$

(iii) $\mathsf{Ge}(Th)$ *is uniformly first-order* *__definable__* *over* $\mathsf{Mod}(Th)$*, assuming* $n > 2$ *and*

$$Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^- + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax(diswind)} + \mathbf{Ax}(\sqrt{\ }).$$

(iv) $\mathsf{Ge}(Th)$ *is uniformly first-order* *__definable__* *over the class* $\mathsf{Mod}(Th)$*, assuming*

$$Th \models \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }).$$

Proof: The theorem is restated and is proved in §4.3 as Theorems 4.3.25 (p.245), 4.3.22 (p.244) and 4.3.24. ∎

We will see that more is true, namely, $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent[418], in symbols

$$\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th),$$

assuming Th is strong enough[419], cf. Thm.4.3.38 (p.261).

__On the conditions of Thm.4.2.40(iii):__ The assumption $n > 2$ cannot be omitted by (the proof of) Thm.4.2.23(ii) (p.168). The assumption $\mathbf{Ax}(\|)^-$ is needed because of (the proof of) Thm.4.2.23(i). Further we conjecture that **Ax(diswind)** cannot be omitted, cf. Conjecture 4.3.23 on p.244 and Fig.93 on p.244.

4.2.3 On the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$

Recall that $\mathbf{Ax}(\mathit{Triv}_t)^-$ is a weakened version of $\mathbf{Ax}(\mathit{Triv}_t)$ and $\mathbf{Ax}(\mathit{Triv})$, and it was introduced on p.157 in the present section. We will need $\mathbf{Ax}(\mathit{Triv}_t)^-$ and $\mathbf{Ax}(\mathit{Triv})$ quite often for the following reason. We defined, roughly speaking, the set L of lines such that something is a line if it "coincides" with a coordinate axis of some inertial observer. Therefore we have rather few lines, i.e. to have enough lines we need $\mathbf{Ax}(\mathit{Triv}_t)^-$. We could have defined lines as sets "parallel" either with the time-axis $\bar{t}$ or with a Euclidean line in the space part S of our space-time for some inertial observer. In that case we would not need $\mathbf{Ax}(\mathit{Triv}_t)^-$ so often. The *only* reason why we did *not* include $\mathbf{Ax}(\mathit{Triv}_t)^-$ into our basic theories like **Basax** or **Basax + Ax(symm)** is that we could derive our main theorems (e.g. no FTL observers, Twin

[416] cf. ($\star$) in Remark 4.2.9 on p.154 (or for greater detail §4.3)
[417] With the exception of §4.3 Th is in our frame language (i.e. Th denotes an arbitrary set of formulas in our frame language).
[418] The notion of definitional equivalence will be discussed in §4.3.
[419] The conditions of Thm.4.2.40(iii) together with $\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)} + \mathbf{Ax(eqtime)}$ are sufficient.

Paradox) *even* without $\mathbf{Ax}(\mathit{Triv}_t)^-$. But whenever we need $\mathbf{Ax}(\mathit{Triv}_t)^-$ for something, we will assume it without a second thought.[420]

We will also need **Ax(eqm)** often, where **Ax(eqm)** was defined on p.145. The reason for our needing **Ax(eqm)** is the following: Without **Ax(eqm)**, g could easily become degenerate because g was defined via "min". Further, failure of **Ax(eqm)** can produce strange things, e.g. $(eq(a,b,c,d) \;\Rightarrow\; g(a,b)=g(c,d))$ can fail even in **Basax** without **Ax(eqm)**. (Connections between **Ax(eqm)** and some earlier introduced axioms are discussed in §6.2.7 of AMN [18].)

Discussion of the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$: Intuitively, the points of $\mathfrak{G}_{\mathfrak{M}}$ are the events. The L^T-lines are the life-lines of inertial observers. The L^{Ph}-lines are the life-lines of photons. Intuitively, one could say that the set of space-like lines L^S consists of the life-lines of the potential faster than light *inertial* bodies (which are called tachions in the literature). However, these bodies need not exist in our model $\mathfrak{M}$. But certainly, if there exists an FTL *inertial* body b in a model $\mathfrak{M}$, then the life-line $\{\, e \in Mn \,:\, b \in e \,\}$ of b is in L^S, under some assumptions on $\mathfrak{M}$, [421] cf. Prop.6.2.55 (p.858) of AMN [18]. Two events are $\equiv^T$-related if there is an inertial observer, whose life-line contains both events. This is equivalent to saying that there is an inertial observer who sees them happening at the same place, under mild assumptions[422], cf. Prop.6.2.56(i) (p.858) of AMN [18]. Two events are $\equiv^{Ph}$-related if they are connected by a photon. Two events are $\equiv^S$-related iff there is an inertial observer who sees them happening at the same time, if we assume $\mathbf{Ax}(\mathit{Triv})+\mathbf{Ax}(\sqrt{\ })$, cf. Prop.6.2.56(ii) (p.858) of AMN [18]. Assuming $\mathbf{Ax}(\mathit{Triv})+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(eqm)}+\mathbf{Ax4}+\mathbf{Ax6_{00}}$, the g-distance $g(e,e_1)$ between two events e, e_1 is (i) the Euclidean distance between them if they are simultaneous for some inertial observer, is (ii) the time elapsed between e, e_1 if they are on the life-line of some inertial observer, is (iii) zero if a photon connects them and is (iv) undefined if no inertial observer can see both of them (under some mild assumptions).

Remark 4.2.41 We have seen in earlier sections that (assuming **Ax1**, **Ax2**, $\mathbf{Ax3_0}$, **Ax4**, $\mathbf{AxE_{01}}$, $\mathbf{Ax6_{00}}$) the irreflexive parts of $\equiv^T$ and $\equiv^{Ph}$ are disjoint because no observer moves with the speed of light,[423] hence $(e \neq e_1 \;\wedge\; e \equiv^T e_1) \;\Rightarrow\; e \not\equiv^{Ph} e_1$.

For completeness, we note that there is a tradition in the literature which codes g, $\equiv^T$, $\equiv^{Ph}$ up into one complex-valued (pseudo-metric) function

$$g^+(e,e_1) = \begin{cases} g(e,e_1) & \text{if } e \equiv^T e_1 \text{ or } g(e,e_1) \text{ is undefined} \\ 0 & \text{if } e \equiv^{Ph} e_1 \\ i \cdot g(e,e_1) & \text{otherwise.} \end{cases}$$

Here $i=\sqrt{-1}$ and $g^+ : Mn \times Mn \longrightarrow \mathbf{C}(\mathfrak{F})$, where $\mathbf{C}(\mathfrak{F}) = \mathfrak{F}(i)$ is the field of complex numbers over $\mathfrak{F}$.

However, in the present work we will not need g^+ because the information carried by g^+ is recoverable from our structure $\langle Mn, \mathbf{F_1};\, g, \equiv^T, \equiv^{Ph}\rangle$.[424]

$\lhd$

[420] Omitting (or weakening) certain axioms of a theory (like $\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp$) of special relativity leads to exciting questions (such an axiom is e.g. **AxE**) but for some other axioms (e.g. "$tr_m(m) = \bar{t}$" or the other axiom $[\forall p\,(p \in \ell \leftrightarrow p \in \ell_1) \to \ell = \ell_1]$) this does not seem to be the case. It is our impression that $\mathbf{Ax}(\mathit{Triv}_t)^-$ might belong to this second kind of axioms (though we did not think much about this, so we may be wrong).

[421] e.g. $\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ })$

[422] e.g. $\mathbf{Ax}(\mathit{Triv})+\mathbf{Ax4}+\mathbf{Ax6_{00}}$ suffices

[423] More precisely, no observer has the same life-line as a photon.

[424] In the relativity book d'Inverno [73, pp. 107-108], our g^+ is called a *Minkowski metric* (and is denoted as η_{ab}). More precisely, the square $(g^+)^2$ of g^+ is called there a Minkowski metric, we guess that this is done there in order to avoid complex numbers. (It is important to note that a Minkowski metric is not a metric [in the usual sense] cf. footnote 313 on p.146.)

Below we continue the discussion of the intuitive meaning of the parts of our geometries. Intuitively, two lines ℓ, ℓ_1 are orthogonal in the relativistic sense (i.e. $\perp_r$-related) if there is an inertial observer m who thinks that they are parallel with two different coordinate axes. There is a slight problem with this intuitive definition because in most of our models $\mathfrak{M}$ no line will be orthogonal to photon-like lines. To help this we introduced a limit construction in our definition of $\perp_r$. We refer to Remark 4.2.7 (pp. 149–152) for intuitive motivation (and cosiderations) for our using limits in the definition of $\perp_r$. If we assume $\mathbf{Bax}^{\oplus}$ and some mild assumptions then our relativistic orthogonality gets very close to the usual Minkowskian orthogonality, cf. Thm.4.2.50 (p.195). On the other hand if we do not assume $\mathbf{Bax}^{\oplus}$, then the relativistic orthogonality $\perp_r$ can behave in quite interesting, unusual ways. E.g. in **NewtK** geometries, two lines are orthogonal iff at least one of them is space-like. Further, there is a $\mathbf{Bax}^{-\oplus}$ geometry with two parallel space-like lines which are $\perp_r$-orthogonal, see Figure 82. (The "meanings" of $\perp_r, L^T, L^S, \equiv^T, \equiv^S, \ldots$ are discussed above and in items 6.2.48–6.2.57, pp.

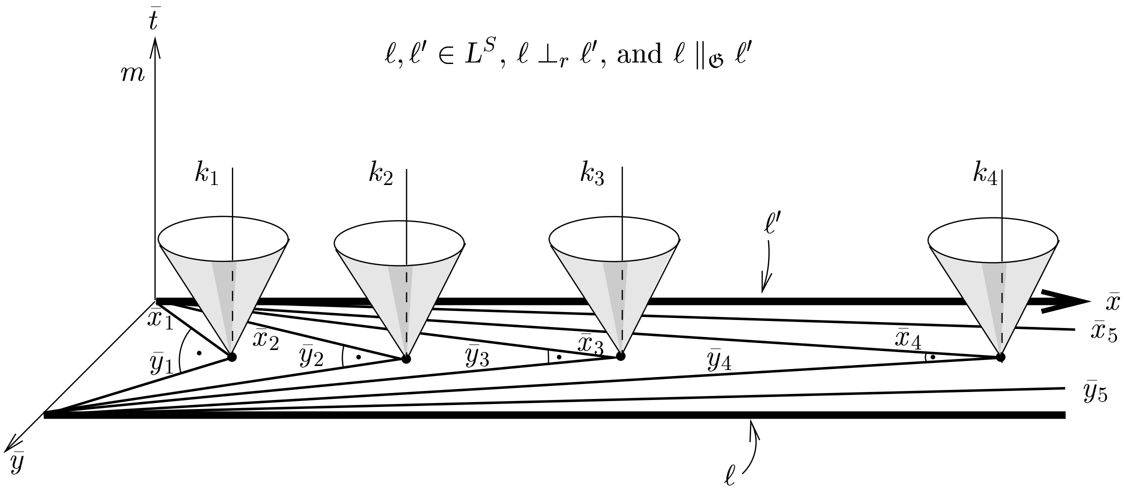


Figure 82: $\mathbf{Bax}^{-\oplus}$ geometry with two parallel space-like lines which are $\perp_r$-orthogonal.

854–858 of AMN [18]. Betweenness Bw and equidistance eq are the usual geometric relations used e.g. by Hilbert [125, 127]. $Bw(a, b, c)$ means that some inertial observer thinks that event b is between events a and c. Intuitively, $eq_0(a, b, c, d)$ means[425] that segments $\langle a, b\rangle$ and $\langle c, d\rangle$ have the same length, for some inertial observer (and this observer sees these segments on coordinate axes). Further, $eq(a, b, c, d)$ means that there is a finite chain of inertial observers such that they together (in a kind of collaboration) think that segments $\langle a, b\rangle$ and $\langle c, d\rangle$ have the same length, see Figure 68 on p.144. Further, $a \prec b$ means that there is an inertial observer who thinks that a happened earlier than b *and* who sees both a and b on his life-line.

The reader may ask what the role of the constant $1 \in \mathbf{F_1}$ is in the geometry $\mathfrak{G}_{\mathfrak{M}}$. Clearly the role of $\mathbf{F_1}$ is to represent the range of g as a special sort (or universe), but for this purpose the additive group $\mathbf{F_0} := \langle F; 0, +, \leq\rangle$ would be sufficient. The answer is the following. Later, in §4.5, we will experiment with reconstructing the "observational-oriented" models $\mathfrak{M}$ from the observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$. The role of the constant 1 is to help us to reconstruct the

[425] Recall that eq was defined as the transitive closure of eq_0. Hence eq_0 can be considered as a kind of "core" of eq.

"units of measurement" or in other words "the size of a hydrogen atom" (cf. p.91) in $\mathfrak{M}$ from $\mathfrak{G}_{\mathfrak{M}}$, at least to some extent (and under some conditions). E.g., under assuming **Ax(eqm)**, we can reconstruct the units of measurement of $\mathfrak{M}$ from $\mathfrak{G}_{\mathfrak{M}}$, cf. e.g. Thm.4.5.11 (p.290). In passing we note that as "patterns" (A)-(E) on p.284 suggest, there will be stronger results of "recoverability" than the just quoted one, in later parts of §4.5.1. We will return to the present subject (the role of "1" etc) in more detail in §4.5. (Cf. e.g. Remark 4.5.51, p.322.) In particular, we will discuss how much of $\mathfrak{M}$ is recoverable from $\mathfrak{G}_{\mathfrak{M}}$ without using 1 in the form of a "duality theory" called in 4.5.4 (p.325) $(\mathcal{G}o, \mathcal{M}o)$-duality.[426]

Summing it up, the role of $1 \in \mathbf{F_1}$ is to help us to recover the units of measurement (in $\mathfrak{M}$) from the geometry $\mathfrak{G}_{\mathfrak{M}}$. (Referring back to the intuitive explanation using hydrogen atoms in §2.8 on p.91 [about justification of **Ax(symm)**], we could say that the constant "1" helps us to remember in $\mathfrak{G}_{\mathfrak{M}}$ what the "size of a hydrogen atom" was in $\mathfrak{M}$.)

Let us turn to discussing why we "celebrate" the observer-independent character of $\mathfrak{G}_{\mathfrak{M}}$. In answering this question we will deliberately mix talking about $\mathfrak{G}_{\mathfrak{M}}$ and its (first-order logic) theory $\mathsf{Th}(\mathfrak{G}_{\mathfrak{M}})$.[427]

(1) Much of what we should say about this was already said in the introduction §4.1 (of this chapter). We will not repeat those thoughts here, the reader is asked to have a look in §4.1.

(2) Clearly $\mathfrak{G}_{\mathfrak{M}}$ is the same to all observers.

(3) By the duality theory to be developed later in §4.5, all the information available in $\mathfrak{M}$ is also available in $\mathfrak{G}_{\mathfrak{M}}$,[428] so we do not lose information when switching to $\mathfrak{G}_{\mathfrak{M}}$.

(4) $\mathfrak{G}_{\mathfrak{M}}$ satisfies certain important, desirable philosophical principles (e.g. the one saying that all our concepts should be definable from observational ones, associated with certain modern refinements[429] of Occam's razor[430]). These principles were already satisfied by $\mathfrak{M}$, and $\mathfrak{G}_{\mathfrak{M}}$ inherits this property from $\mathfrak{M}$ because $\mathfrak{G}_{\mathfrak{M}}$ is first-order-logic-definable over $\mathfrak{M}$ (under some conditions).[431]

(5) Around the end of this chapter, we will see that $\mathfrak{G}_{\mathfrak{M}}$ admits mathematically elegant streamlined versions (cf. e.g. the time-like-metric geometry $\langle Mn, \mathbf{F_1}; g^{\prec}\rangle$ in §4.6.1 p.346 as an example). These streamlined versions of $\mathfrak{G}_{\mathfrak{M}}$ provide us with a simple, mathematically elegant, and transparent picture of the world (which, in many respects, is simpler and more elegant than $\mathfrak{M}$).

(6) $\mathfrak{G}_{\mathfrak{M}}$ provides us with a stepping-stone towards theories admitting accelerated observers and eventually towards general relativity. Cf. e.g. §4.7 on geodesics.

(7) In some sense one feels that $\mathfrak{G}_{\mathfrak{M}}$ represents "deeper" and more essential aspects of the world than $\mathfrak{M}$ does. One could say that the ingredients of $\mathfrak{M}$ are the things one sees on the "surface" of the phenomena or reality being studied while $\mathfrak{G}_{\mathfrak{M}}$ contains ingredients which make these surface phenomena "tick". One could say that $\mathfrak{G}_{\mathfrak{M}}$ contains something that could be regarded as "explanation" for $\mathfrak{M}$ (where explanation is understood in the sense of Friedman [91]). Cf. footnote 275 on p.131.

(8) The various reducts of $\mathfrak{G}_{\mathfrak{M}}$ provide us with aspects of the world that we can contemplate. So for a while we may decide to concentrate on one aspect (represented by one reduct)

[426] Forgetting 1 from $\mathfrak{G}_{\mathfrak{M}}$ is related to what we called on p.88 "ant and elephant version of relativity", and which we plan to outline in some future work. Cf. also AMN [18, Remark 4.2.1 on p.458].

[427] Or more precisely $\mathsf{Th}(\{\, \mathfrak{G}_{\mathfrak{M}} : \mathfrak{M} \models Th_1 \,\})$ for some fixed Th_1.

[428] under some mild conditions

[429] We mean principles proposed by Reichenbach, Carnap, Mach (and also by the logical positivists).

[430] For further desirable philosophical principles satisfied by $\mathfrak{G}_{\mathfrak{M}}$ we refer the reader to the introduction of the present chapter (§4.1).

[431] Cf. Thm.4.2.40 (p.182). (In this respect we do not gain over $\mathfrak{M}$ but we do not lose either.)

and ignore the rest. Then we can experiment with how far we can get by concentrating on this aspect. Later we may concentrate on some other aspect (reduct). Eventually we can compare the results (and try to obtain insight into which aspect is responsible for which effect). In other words this provides us with the machinery of "abstraction".[432] For more on this ("decomposing" the world into reducts etc.) cf. the first 5 lines of §4.5.4 (p.325), p. 342, and p.341 of AMN [18]. (Note that the same kind of "decomposability" is not available in the original structures the $\mathfrak{M}$'s.)

(9) $\mathfrak{G}_{\mathfrak{M}}$ may be helpful in comparing the various observers, seeing their relationships with each other. We feel that this is so because in $\mathfrak{G}_{\mathfrak{M}}$ when, say, we are thinking about e.g. three inertial observers simultaneously, we are not forced to do so from the world-view of some particular observer, instead we can look at our three observers from, so to speak, the "objective" perspective of $\mathfrak{G}_{\mathfrak{M}}$. In contrast, when working in $\mathfrak{M}$, we always have to choose an observer and we have to describe things from his particular perspective. This may make e.g. proofs longer (because we might have to switch perspectives).

(10) For more on why we celebrate the observer-independent character of $\mathfrak{G}_{\mathfrak{M}}$ we refer to the book Matolcsi [187].

(11) For completeness, we note the following: Many of the so-called thought experiments can be translated into the language of $\mathfrak{G}_{\mathfrak{M}}$, and the outcome of the thought experiment can be predicted by knowing $\mathfrak{G}_{\mathfrak{M}}$, cf. "laws of nature" part in AMN [18]. An example is the so-called twin paradox, assuming e.g. $n > 2$ and $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqtime)}$. For the case $\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp}+\mathbf{Ax}(\uparrow\uparrow)$, [433] the importance of $\mathfrak{G}_{\mathfrak{M}}$ is further elaborated in e.g. Misner-Thorne-Wheeler [192, pp. 3–47, 163–175].

The usefulness of $\mathfrak{G}_{\mathfrak{M}}$ will be especially apparent when we turn to discussing non-inertial observers. As an illustration, let us assume that we have a body b whose life-line is not in L^T. Assume we would like to raise b to the level of being an observer. For simplicity, assume $n = 2$. Then b would like to coordinatize the "events" Mn, i.e. we would like to define a function $w_b : {}^2F \longrightarrow Mn$. Using $\mathfrak{G}_{\mathfrak{M}}$, there is a natural way for doing this,[434] cf. e.g. Misner-Thorne-Wheeler [192, pp. 163–175].

At this point we stop listing values of $\mathfrak{G}_{\mathfrak{M}}$.

Remark 4.2.42 (On the philosophy of our using inertial and not necessarily inertial observers in the definition of $\mathfrak{G}_{\mathfrak{M}}$ above.)
Before starting, we note that later we will have so-called windows in $\mathfrak{G}_{\mathfrak{M}}$. Roughly, a window is a part of Mn visible for one observer.

Now, what we want to say about the "philosophy ..." is the following: (i) Everything that is "measured" (e.g. g or $\perp_r$) (by observers of course) is defined via $Obs \cap Ib$. As a contrast; (ii) windows, existence of events (ontology of Mn) are defined via Obs (i.e. all observers).

(iii) Cf. also the definition of $\mathfrak{G}^*_{\mathfrak{M}}$ in §4.5.5 p.332.

$\triangleleft$

We will start discussing the connections with the standard Minkowskian geometry on p.188 in §4.2.4.

[432] Decompose the world into aspects, study the aspects separately and in their interaction and then put the results together.

[433] We note that (for $n > 2$) the members of $\mathsf{Ge}(\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp}+\mathbf{Ax}(\uparrow\uparrow))$ are the Minkowskian geometries, up to isomorphism, cf. Def.4.2.44 (p.189) and Thm.4.2.45 (p.190).

[434] This does not contradict what we will say in §4.5(V) on pp. 332–340 (... Gödel incompleteness) about undefinability of non-inertial bodies. (The reason for this is that these two claims about definability "live" as two different levels of abstraction.)

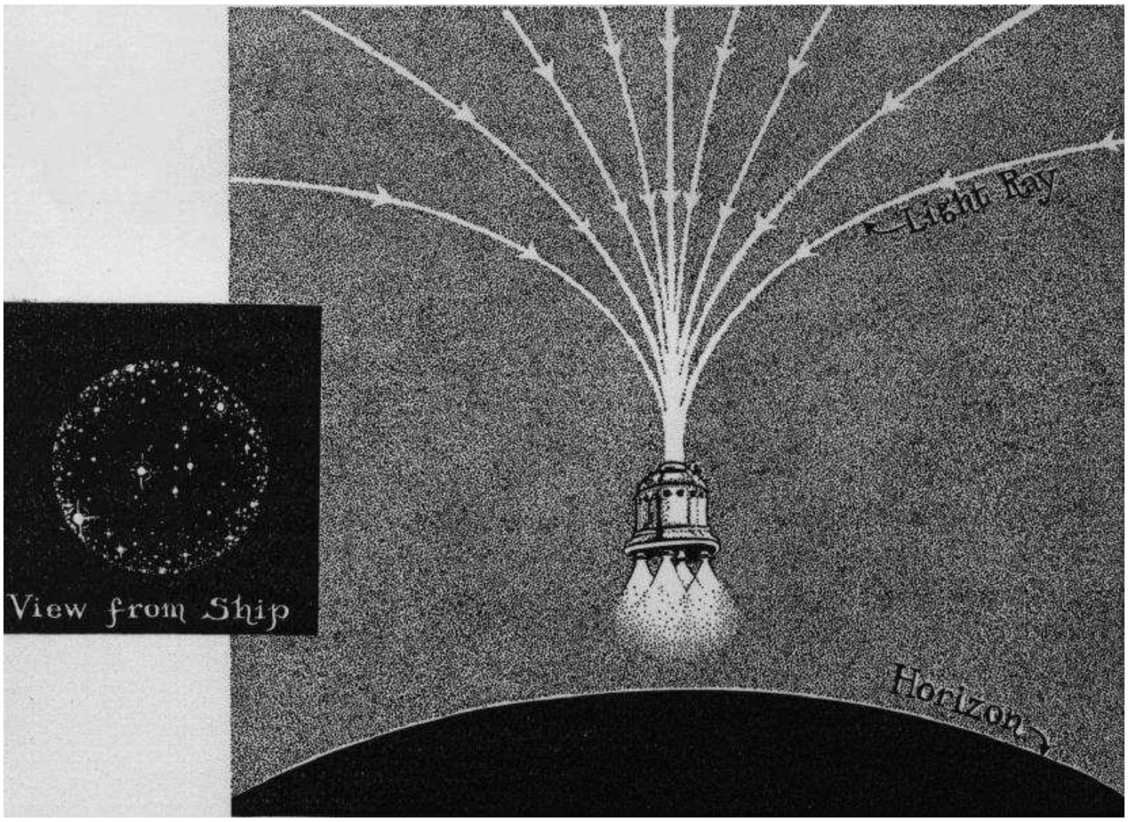


Figure 83: The starship hovering above the black-hole horizon, and the trajectories along which light travels to it from distant galaxies (the light rays). The hole's gravity deflects the light rays downwards ("gravitational lens effect"), causing humans on the starship to see all the light concentrated in a bright, circular spot overhead.

Remark 4.2.43 (On Figure 83 [view from the black hole]) Later, in generalizations towards general relativity, our geometry $\mathfrak{G}$ will be more sophisticated than the present $\mathfrak{G}_{\mathfrak{M}}$. E.g. life-lines of photons (and other inertial bodies) will be so-called geodesics instead of Euclidean lines. Geodesics will be discussed in §4.7.Figure 83 on p.187 represents some spectacular effect caused by geodesics being curved by a black hole.

$\lhd$

The basic properties of L, L^T, L^S, $\equiv^T$, $\equiv^S$, $\perp_r$ etc. and their interconnections are discussed in items 6.2.48–6.2.57 of AMN [18]. In later proofs we will use these basic facts. These propositions establish connections between geometric properties on the Mn-side and similar properties on the ${}^nF$-side. E.g. they reduce orthogonality of two lines in Mn to properties of their pre-images in ${}^nF$.

If the reader wants only a quick overview of the high points of this dissertation and if he is familiar with the standard Minkowskian geometry of special relativity, then he can skip the next three subsections (§4.2.4–4.2.6) and go directly to section §4.3. These three subsections discuss (i) connections with the standard Minkowskian geometry, (ii) characterizations of $\mathsf{Ge}(Th)$ for distinguished choices of Th (and basic properties of our geometries), (iii) streamlined reducts like $\langle Mn, L; \in, \perp\rangle$ of $\mathfrak{G}_{\mathfrak{M}}$ which can be used instead of $\mathfrak{G}_{\mathfrak{M}}$ for most of the purposes in the present chapter, under some conditions. [435]

4.2.4 Connections with the standard Minkowskian geometry

The style of our above definition of $\mathfrak{G}_{\mathfrak{M}}$ followed a certain kind of intuition e.g. (i) events e, e_1 are defined to be spatially separated iff some inertial observer thinks that e and e_1 happened at the same time; and (ii) for events e and e_1 the relation $e \prec e_1$ is defined to hold iff some inertial observer thinks that e precedes e_1 in time (and sees e, e_1 on his life-line); etc. In general, we tried to achieve the effect that, intuitively, some relation holds between given objects iff some inertial observer thinks this is so (sometimes we had to take "min" or limits to complete the picture, but this was the general intuition).

As a contrast, in Definition 4.2.44 below, for every Euclidean $\mathfrak{F}$, we define a geometry on ${}^nF$ in a "computational" style. Following the literature[436], we call this geometry the Minkowskian geometry over $\mathfrak{F}$.

In Thm.4.2.45 below (p.190), we will see that our "intuition-oriented" definition of $\mathfrak{G}_{\mathfrak{M}}$ is equivalent to the standard Minkowskian definition mentioned above, under some assumptions on $\mathfrak{M}$. Further, if $n > 2$, the observer-independent geometries (in our sense[437]) of the Minkowski models (the latter is defined in §3.8 of AMN [18]) will turn out to coincide with the Minkowskian geometries, up to isomorphism, cf. Prop.4.2.48, p.194. (In §4.2.5 we will see that relativistic geometries corresponding to many of our theories can be obtained as "unions" of Minkowskian geometries if we concentrate on a reduct of our geometries, only. Cf. Figures 84, 85, pp. 192, 193.)

[435] In some of the later proofs we use lemmas proved in (ii) but we will refer to these when they are needed.
[436] cf. e.g. Kostrikin-Manin [148], cf. also Goldblatt [102]
[437] in the sense of Def.4.2.3

On terminology: What we call here Minkowskian geometry, is (usually) called in the literature "*Minkowskian spacetime*", cf. e.g. Goldblatt [102] or Schutz [231].[438] But e.g. Goldblatt [102] in its introduction uses the expression "Minkowskian geometry" exactly the same way as we do.

Definition 4.2.44 (Minkowskian geometry)
Assume $\mathfrak{F}$ is Euclidean. Then the n-dimensional *Minkowskian geometry* over $\mathfrak{F}$ is defined as follows.

$$Mink(n, \mathfrak{F}) :\stackrel{\text{def}}{=} Mink(\mathfrak{F}) :\stackrel{\text{def}}{=} \langle {}^nF, \mathbf{F_1}, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, \prec_\mu, Bw_\mu, \perp_\mu, eq_\mu, g_\mu, \mathcal{T}_\mu\rangle;$$

where:

- $\mathbf{F_1} :\stackrel{\text{def}}{=} \langle F;\ 0, 1, +, \leq\rangle$, as defined in Def.4.2.3.
- $L_\mu :\stackrel{\text{def}}{=} \mathsf{Eucl}(n, \mathbf{F}) := \mathsf{Eucl}$.
- $L^T_\mu :\stackrel{\text{def}}{=} \mathsf{SlowEucl}$.
- $L^{Ph}_\mu :\stackrel{\text{def}}{=} \mathsf{PhtEucl}$.
- $L^S_\mu :\stackrel{\text{def}}{=} L_\mu \setminus (L^T_\mu \cup L^{Ph}_\mu)$.
- $\prec_\mu$ is a binary relation on ${}^nF$ defined as follows. Let $p, q \in {}^nF$. Then
$$p \prec_\mu q \quad \stackrel{\text{def}}{\iff} \quad (p_t < q_t \quad \wedge \quad \overline{pq} \in \mathsf{SlowEucl}).$$
- $Bw_\mu = \mathsf{Betw}$.
- The *Minkowskian orthogonality* $\perp_\mu \subseteq L_\mu \times L_\mu$ is defined as follows. Let $\ell, \ell' \in L_\mu$. Then
$$\begin{gathered} \ell \perp_\mu \ell' \\ \stackrel{\text{def}}{\iff} \\ (\forall \text{ distinct } p, q \in \ell)(\forall \text{ distinct } p', q' \in \ell') \\ (p_0 - q_0)(p'_0 - q'_0) - \left(\textstyle\sum_{0<i\in n}(p_i - q_i)(p'_i - q'_i)\right) = 0. \end{gathered}$$
If $\ell \perp_\mu \ell'$ then we say that ℓ and ℓ' are *Minkowski-orthogonal*.
- Let us recall that $g^2_\mu : {}^nF \times {}^nF \longrightarrow F$ is the square of the Minkowski-distance defined in Def.2.9.1.

  We define the *Minkowski distance* $g_\mu : {}^nF \times {}^nF \longrightarrow F$ as follows[439]. Let $p, q \in {}^nF$. Then
$$g_\mu(p, q) :\stackrel{\text{def}}{=} \sqrt{g^2_\mu(p, q)}.^{440}$$

[438] The reason for this is probably the fact that e.g. in Busemann [54, §17] the expression "Minkowskian geometry" is reserved for something else, something not connected to relativity.

[439] exactly as we did above Prop.6.2.38 on p.844 in AMN [18]

[440] In connection with this definition we note that our symbol g^2_μ (introduced on p.101) is ***not*** the square of something denoted by g_μ, but instead it is a basic symbol, like, say γ. Then, g_μ counts as a brand new symbol unrelated to g^2_μ and our definition $g_\mu(\ldots) = \sqrt{g^2_\mu(\ldots)}$ should be understood like $g_\mu(p, q) = \sqrt{\gamma(p, q)}$. (The reason for treating g^2_μ as basic symbol [instead of e.g. g_μ] is explained in footnote 82, p.18.)

- eq_μ is a 4-ary relation on ${}^nF$ defined as follows. Let $p, q, p', q' \in {}^nF$. Then

$$eq_\mu(p,q,p',q') \\ \overset{\text{def}}{\Longleftrightarrow} \\ \Big(g_\mu(p,q) = g_\mu(p',q') \ \wedge \ [\, g_\mu(p,q) = 0 \ \Rightarrow \ (\, p = q \ \wedge \ p' = q'\,)\,]\Big).$$ [441]

- $\mathcal{T}_\mu$ is defined by g_μ as described in item 13 of Def.4.2.3 (p.146).

We will sometimes omit the subscript μ from L_μ etc. because the vocabulary or similarity type of Minkowskian geometries is the same as that of relativistic geometries.

$\lhd$

Assume $\mathfrak{M} \models$ **Basax**. Then for each $m \in Obs$, the bijection $w_m : {}^nF \longrightarrow Mn$ can be used to "copy" the geometry $Mink(\mathfrak{F}^{\mathfrak{M}})$ to Mn (as its new universe, i.e. as its new set of points), yielding a geometry $Mink^m_{\mathfrak{M}}$. However for different observers m, this geometry might be different (though isomorphic), because different observers might copy $Mink(\mathfrak{F}^{\mathfrak{M}})$ *differently* to Mn. Assume further $\mathfrak{M} \models \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$. Then the observers will *agree* on how to copy $Mink(\mathfrak{F}^{\mathfrak{M}})$. Formally,

$$(\forall m, k \in Obs) Mink^m_{\mathfrak{M}} = Mink^k_{\mathfrak{M}},$$

assuming $\mathfrak{M}$ satisfies the mentioned axioms. This is essentially what Thm.4.2.45 below says.[442]

Assume now $\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$. Then we could define a Minkowskian geometry on Mn as follows:

$$Mink_{\mathfrak{M}} := Mink^m_{\mathfrak{M}}$$

for an arbitrary but fixed $m \in Obs$. Our Thm.4.2.45 below says that

$$Mink_{\mathfrak{M}} = \mathfrak{G}_{\mathfrak{M}},$$

assuming $n > 2$. To keep the number of defined symbols in this work relatively small, we will *not* rely on the notation $Mink_{\mathfrak{M}}$ in the rest of this work (at least not without recalling it).

THEOREM 4.2.45 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow))$. *Then (i)–(iii) below hold.*

(i) *Let* $n > 2$. *Then*

$$\mathfrak{G}_{\mathfrak{M}} \cong Mink(\mathfrak{F}^{\mathfrak{M}}),$$

cf. Figures 84, 85.

Moreover, for every $m \in Obs$, $w_m : {}^nF \longrightarrow Mn_{\mathfrak{M}}$ *<u>induces an isomorphism</u> between* $Mink(\mathfrak{F}^{\mathfrak{M}})$ *and* $\mathfrak{G}_{\mathfrak{M}}$ *the natural way.*[443]

[441] We need the subformula "$g_\mu(p,q) = 0 \ \Rightarrow \ \ldots$" only because in our definition of eq by some accident we had the side effect that photon-like separated pairs of points are not eq-related even to themselves, cf. footnote 303 on p.144. Further, because we want to make our definition (of $\mathfrak{G}_{\mathfrak{M}}$) comparable with the Minkowskian definition (i.e. with $Mink(\mathfrak{F})$).

[442] Actually, this idea of somehow identifying ${}^nF$ with Mn via some observer's world-view can be pushed through even in $\mathbf{Bax}^-$, since we have seen that the world-view transformations are line preserving, cf. Def.4.2.61 (p.206) and Prop.4.2.64 (p.208).

[443] Making this precise: Let $m \in Obs$. Let $\widetilde{w_m} : \mathsf{Eucl} \longrightarrow Mn_{\mathfrak{M}}$ be defined by $\widetilde{w_m} : \ell \mapsto w_m[\ell]$. Then $\langle\, w_m,\ \mathrm{Id} \restriction F,\ \widetilde{w_m} \rangle$ is a (three-sorted) isomorphism between $Mink(\mathfrak{F}^{\mathfrak{M}})$ and $\mathfrak{G}_{\mathfrak{M}}$, cf. item (II) of Def.4.2.3 (p.146) for the notion of an isomorphism between geometries.

(ii) *Let $n = 2$. Then the conclusion of (i) remains true with the exception of* eq, *i.e. instead of $\mathfrak{G}_{\mathfrak{M}}$ we have to talk about the* eq*-free reduct of $\mathfrak{G}_{\mathfrak{M}}$. The conclusion of (i) will not remain true if we do not exclude* eq *from our geometries.*

(iii) *The statement in item (i) remains true if we replace the assumption* $\mathbf{Ax}(\omega)^\sharp$ *by* $\mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}$, *where* $\mathbf{Ax}$ *is any one of* $\mathbf{Ax}(\omega)$, $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax}(\omega)^{00}$, $\mathbf{Ax}(\omega)^{\sharp\sharp}$, $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\Box 1} + \mathbf{Ax(eqtime)}$, $\mathbf{Ax\Box 2}$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}$.

The **proof** is available from the author.[444] ∎

The following theorem says that, if $n > 2$, then the $\prec$-free reduct of any $\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp\sharp}$ geometry coincides with the similar reduct of a Minkowskian geometry. In connection with the conditions of Theorems 4.2.45 and 4.2.46 we recall that $\mathbf{Ax}(\omega)^{\sharp\sharp}$ is weaker than $\mathbf{Ax}(\omega)^\sharp$. In Thm.4.2.45 we needed the assumption $\mathbf{Ax}(\omega)^\sharp$ for the $n = 2$ case only; for the $n > 2$ case $\mathbf{Ax}(\omega)^{\sharp\sharp}$ was sufficient.

THEOREM 4.2.46 *Assume $n > 2$. Then (i) and (ii) below hold.*

(i) *Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp\sharp})$. Then the $\prec$-free reduct of $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean $\mathfrak{F}$ such that*

$$(\prec\text{-free reduct of } \mathfrak{G}) \ \cong\ (\prec\text{-free reduct of } Mink(\mathfrak{F})),$$

cf. Figures 84, 85.

(ii) *The statement in item (i) remains true if we replace the assumption* $\mathbf{Ax}(\omega)^{\sharp\sharp}$ *by* $\mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}$, *where* $\mathbf{Ax}$ *is any one of* $\mathbf{Ax}(\omega)$, $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax}(\omega)^{00}$, $\mathbf{Ax}(\omega)^\sharp$, $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\Box 1} + \mathbf{Ax(eqtime)}$, $\mathbf{Ax\Box 2}$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}$.

The **proof** is available from the author. ∎

Roughly, the following proposition says that, assuming $\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$, the world-view transformations f_{mk} are exactly those automorphisms of the observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$ which leave the sort F pointwise fixed, cf. items (iii) and (iv) of the proposition. Let us notice that this means, basically, that the world-view transformations of $\mathfrak{M}$ coincide with the (nice) automorphisms of $\mathfrak{G}_{\mathfrak{M}}$. In connection with the proposition below cf. §6.2.8 in AMN [18].

PROPOSITION 4.2.47 *Assume $\mathfrak{M} \models (\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow))$. Assume $m, k \in Obs$. Then (i)–(iv) below hold.*

(i) *The world-view transformation f_{mk} induces an automorphism of the Minkowskian geometry $Mink(\mathfrak{F}^{\mathfrak{M}})$ the natural way.*[445]

(ii) *For every automorphism α of $Mink(\mathfrak{F}^{\mathfrak{M}})$ which is the identity function on the sort F, there are $m', k' \in Obs^{\mathfrak{M}}$ such that α and $\mathsf{f}_{m'k'}$ coincide on ${}^nF$.*

---

[444] In connection with item (ii) of Thm.4.2.45 cf. the first 8 lines of the proof of Thm.6.2.22 on p.906 in AMN [18].

[445] Making this precise: Let $\widetilde{\mathsf{f}_{mk}} : \mathsf{Eucl} \longrightarrow \mathsf{Eucl}$ be defined by $\widetilde{\mathsf{f}_{mk}} : \ell \mapsto \mathsf{f}_{mk}[\ell]$. Then $\widehat{\mathsf{f}_{mk}} \stackrel{\text{def}}{=} \langle \mathsf{f}_{mk},\ \mathrm{Id} \restriction F,\ \widetilde{\mathsf{f}_{mk}} \rangle$ is a (three-sorted) automorphism of $Mink(\mathfrak{F}^{\mathfrak{M}})$. For the notion "three-sorted automorphism" cf. item (II) of Def.4.2.3 (p.146).

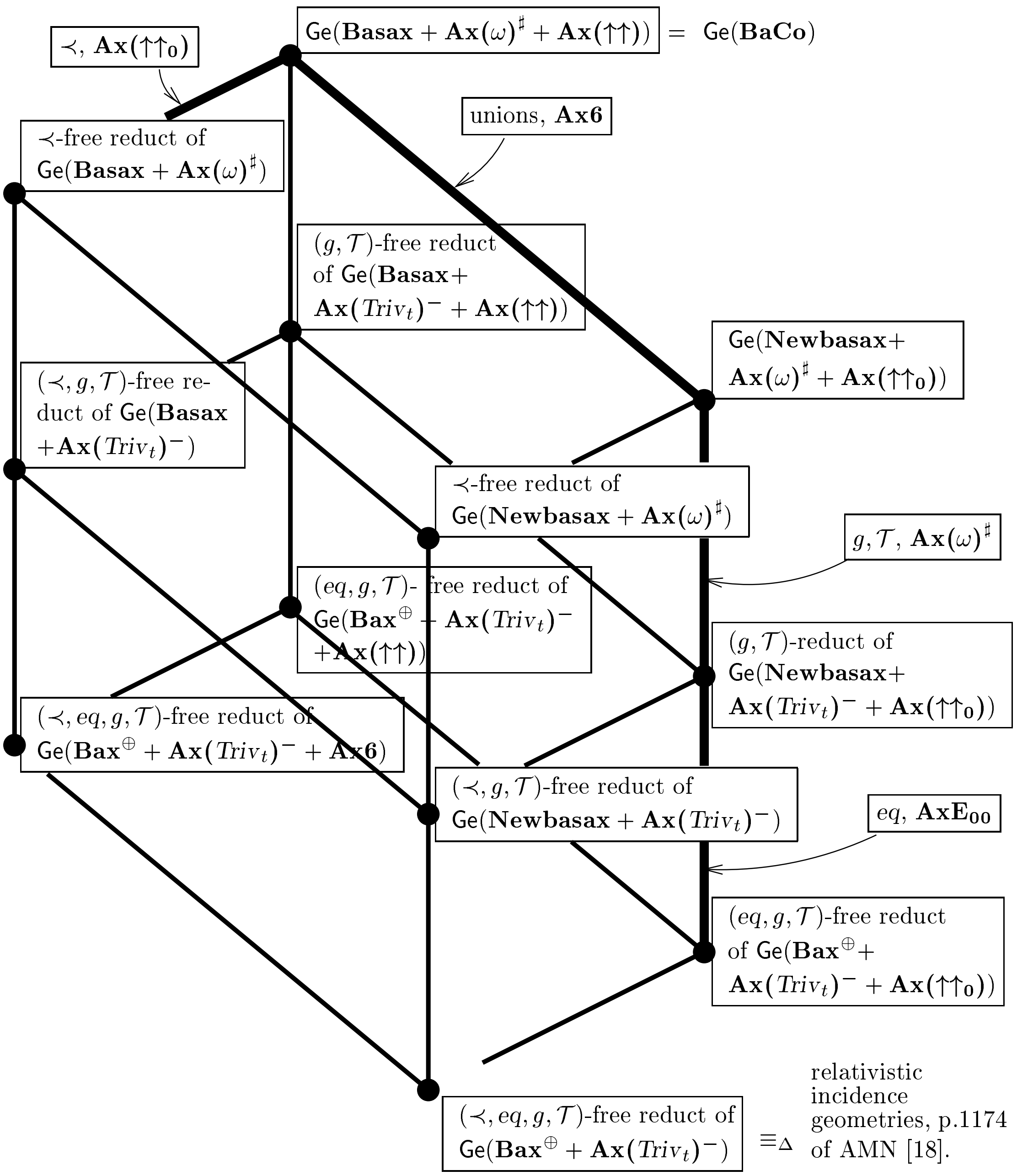


Figure 84: Reducts of geometries agreeing with the corresponding (reducts of) Minkowskian geometries. $\mathbf{Ax}(\sqrt{\ })$ and $n > 2$ are assumed. Nodes are of form $\mathsf{Rd_L}(\mathsf{Ge}(Th))$ determined by the choice of Th and geometric sublanguage L, where the operator $\mathsf{Rd_L}$ is defined on p.205 and intuition etc. about reducts is in Convention 4.3.1. For detailed explanation cf. p.205. Cf. also Fig.85. For $\equiv_\Delta$ cf. p.255.

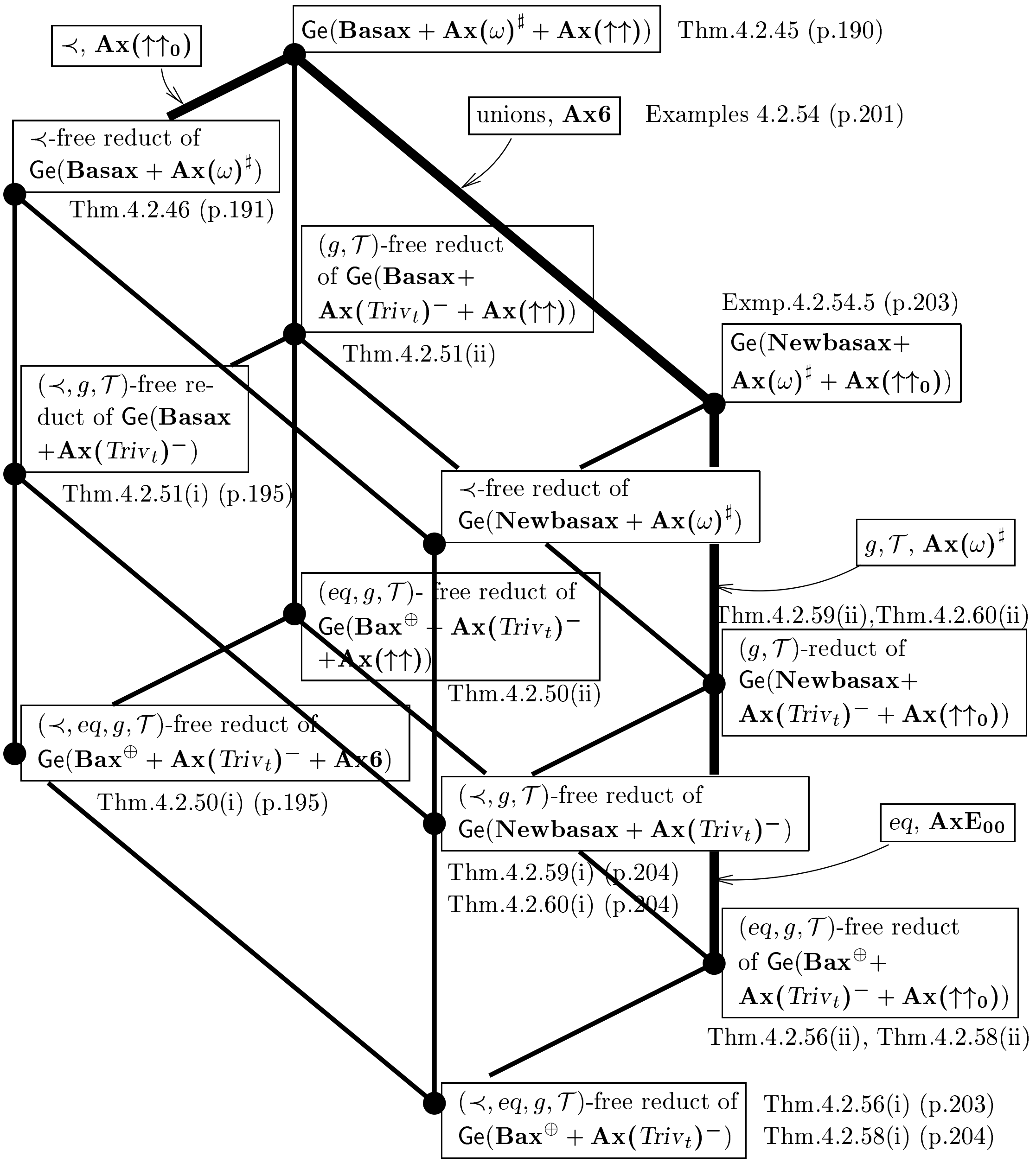


Figure 85: This is Fig.84 enriched with the names of theorems involved.

(iii) $w_m^{-1} \circ w_k$ *induces an automorphism* $\widehat{\mathsf{f}_{mk}}$ *of the geometry* $\mathfrak{G}_{\mathfrak{M}}$*, the natural way.*[446]

(iv) *For every automorphism* α *of* $\mathfrak{G}_{\mathfrak{M}}$ *which is the identity function on the sort* F*, there are* $m', k' \in Obs^{\mathfrak{M}}$ *such that* α *and* $w_{m'}^{-1} \circ w_{k'}$ *coincide on* Mn*. I.e.* $\widehat{\mathsf{f}_{m'k'}}$ *agrees with* α*.*

On the proof: Items (i) and (iii), for the case $n > 2$, are corollaries of Thm.4.2.45. In the case $n = 2$, by Thm.4.2.45, we conclude that items (i) and (iii) hold for the eq-free reducts of the geometries. Checking that f_{mk} and $w_m^{-1} \circ w_k$ are automorphisms of the geometry reducts $\langle {}^2F;\ eq_\mu \rangle$ and $\langle Mn_{\mathfrak{M}};\ eq_{\mathfrak{M}} \rangle$, respectively, is easy and is left to the reader. The proofs of items (ii), (iv) are available from the author. ∎

Items (iii) and (iv) of the above proposition can be summarized, roughly, by saying that $Aut(\mathfrak{G}_{\mathfrak{M}})$ can be identified with the group $\{\, \widehat{\mathsf{f}_{mk}} \ :\ m, k \in Obs \,\}$, which in turn can be identified with $\{\, \mathsf{f}_{mk} \ :\ m, k \in Obs \,\}$. Cf. AMN [18, §6.2.8 (p.913) and p.779]. Items (i) and (ii) say basically the same about $Mink(\mathfrak{F}^{\mathfrak{M}})$ in place of $\mathfrak{G}_{\mathfrak{M}}$.

Let us recall that in Definition 3.8.42 (p.331) of AMN [18], for every Euclidean $\mathfrak{F}$, the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ over $\mathfrak{F}$ was defined. The proposition below says that the observer-independent geometry of the Minkowski model over $\mathfrak{F}$ is the Minkowskian geometry over $\mathfrak{F}$, up to isomorphism.

PROPOSITION 4.2.48 *Assume* $\mathfrak{F}$ *is Euclidean and* $n > 2$*. Then*

$$\mathfrak{G}_{\mathfrak{M}^M_{\mathfrak{F}}} \cong Mink(\mathfrak{F}).$$

Moreover, for every $m \in Obs^{\mathfrak{M}^M_{\mathfrak{F}}}$*,* $w_m : {}^nF \longrightarrow Mn$ *induces an isomorphism between* $Mink(\mathfrak{F})$ *and* $\mathfrak{G}_{\mathfrak{M}^M_{\mathfrak{F}}}$ *the natural way.*[447]

Proof: The proposition follows by Thm.4.2.45. ∎

The following theorem says that in **Basax** models the world-view transformations f_{mk} preserve Minkowskian orthogonality.

THEOREM 4.2.49 *Assume* **Basax***. Let* $\ell, \ell' \in \mathsf{Eucl}$ *and* $m, k \in Obs$*. Then*

$$\ell \perp_\mu \ell' \quad \Rightarrow \quad \mathsf{f}_{mk}[\ell] \perp_\mu \mathsf{f}_{mk}[\ell'].$$

The **proof** is available from the author. ∎

Roughly, the following theorem says that the $(\prec, eq, g, \mathcal{T})$-free reduct of almost any ($\mathbf{Bax}^\oplus$+ **Ax6**)-geometry coincides with the similar reduct of a Minkowskian geometry. Further, the same holds for the $(eq, q, \mathcal{T})$-free reducts of ($\mathbf{Bax}^\oplus$ + **Ax6** + **Ax(↑↑)**)-geometries. Stronger forms of the following theorem, not involving **Ax6**, will be stated in §4.2.5 as Theorems 4.2.56, 4.2.58.

[446] Making this precise: Let $\mathsf{d}_{mk} \stackrel{\text{def}}{=} w_m^{-1} \circ w_k$. Then d_{mk} is a mapping of Mn into itself. Let $\widetilde{\mathsf{d}_{mk}} : L \longrightarrow L$ be defined by $\widetilde{\mathsf{d}_{mk}}(\ell) = \mathsf{d}_{mk}[\ell]$. Now, $\widehat{\mathsf{f}_{mk}} \stackrel{\text{def}}{=} \langle \mathsf{d}_{mk}, \mathrm{Id} \restriction F, \widetilde{\mathsf{d}_{mk}} \rangle$. More detail and intuitive motivation for $\widehat{\mathsf{f}_{mk}}$ is in AMN [18, p.914].

[447] See footnote 443.

THEOREM 4.2.50 *Assume* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\surd\ ) + \mathbf{Ax6})$. *Then (i) and (ii) below hold.*

(i) *Assume* $n > 2$. *Then the* $(\prec, \mathsf{eq}, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean* $\mathfrak{F}$ *such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, Bw_\mu, \perp_\mu\rangle,$$

cf. Figures 84, 85.

(The other direction also holds by Prop.4.2.48.)[448]

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow)$. *Then the* $(\mathsf{eq}, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean* $\mathfrak{F}$ *such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, \prec_\mu, Bw_\mu, \perp_\mu\rangle.$$

(The other direction also holds by Prop.4.2.48.)

Proof: The theorem follows by the first proof given for Thm.4.2.11 (p.158), by Prop.4.2.48 and by Prop.4.2.31 (p.177). ∎

Roughly, the following theorem says that the $(\prec, g, \mathcal{T})$-free reduct of almost any **Basax** geometry $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry. Further, the same holds for $(g, \mathcal{T})$-free reducts of $(\mathbf{Basax} + \mathbf{Ax}(\uparrow\uparrow))$-geometries. Generalizations of the following theorem for **Newbasax** (in place of **Basax**) will be stated in §4.2.5 as Theorems 4.2.59, 4.2.60.

THEOREM 4.2.51 *Assume* $n > 2$ *and* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\surd\ ))$. *Then (i) and (ii) below hold. (Cf. Figures 84, 85.)*

(i) *The* $(\prec, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean* $\mathfrak{F}$ *such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r, eq\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, Bw_\mu, \perp_\mu, eq_\mu\rangle.$$

(The other direction also holds by Prop.4.2.48.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow)$. *Then the* $(g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean* $\mathfrak{F}$ *such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \prec, \perp_r, eq\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, Bw_\mu, \prec_\mu, \perp_\mu, eq_\mu\rangle.$$

(The other direction also holds by Prop.4.2.48.)

On the proof: A proof can be obtained by the proof given for Thm.4.2.11 (p.158), by Prop.4.2.48, by Claim 6.2.84 (p.892) and Prop.6.2.88 (p.895) of AMN [18] and by Prop.4.2.31 (p.177). ∎

[448] I.e. this reduct of any Minkowskian geometry is obtainable as a reduct of a $(\mathbf{Bax}^{\oplus} + \ldots)$-geometry (up to isomorphism of course).

Remark 4.2.52

(i) In **Basax** we know that if we are given a possible life-line ℓ then ℓ completely determines the relation of simultaneity of observers living on ℓ. (By relation of simultaneity we mean a binary relation between events). This generalizes to $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ })$, but it does not generalize e.g. to **Reich**(**Basax**).

(ii) In $\mathbf{Basax}(4) + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ })$ we have the following property. Assume we are given four lines $\ell, \ell_1, \ell_2, \ell_3 \in L$ intersecting at one point and mutually $\perp_r$-orthogonal. Assume exactly one of them is time-like. Then there is an observer whose coordinate axes are exactly these four lines. The other direction is also true: the coordinate axes of any observer behave like $\ell, \ldots, \ell_3$.

This generalizes to $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ })$.

$\triangleleft$

4.2.5 Getting familiar with our geometries; unions of geometries and models

In this section we will analyze how the geometries $\mathfrak{G}_{\mathfrak{M}}$ are "put together" i.e. how one can have a grasp on them. Roughly, we will see that $\mathfrak{G}_{\mathfrak{M}}$ is obtained from the world-views (now regarded as geometries) of inertial observers by gluing them together in some way, cf. Fig.91 (p.211). For more on the intuition behind this (or how these ideas will be implemented) see p.208 above Prop.4.2.64.

As a motivation for studying disjoint union of geometries (and generalizations of this in items 3,4,5 below) we refer the reader to Remark 4.2.66 and Figure 92 on p.212 on the connections with Penrose diagrams from general relativity.

We will use the concept of *disjoint unions of* $\mathbf{Bax}^-$ models as well as *disjoint unions of geometries* similar to our observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$. In both cases we will assume that the field reducts of the structures in question coincide.

1. **Disjoint, generalized disjoint and photon-disjoint unions of models:**

   Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$. Assume $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} = \emptyset$. Then the <u>*disjoint union*</u> is defined as follows.

   $$\mathfrak{M} \,\dot{\cup}\, \mathfrak{N} :\overset{\text{def}}{=} \langle B^{\mathfrak{M}} \cup B^{\mathfrak{N}}, Obs^{Mm} \cup Obs^{\mathfrak{N}}, Ph^{\mathfrak{M}} \cup Ph^{\mathfrak{N}}, Ib^{\mathfrak{M}} \cup Ib^{\mathfrak{N}}, \mathfrak{F}, G, \in, W^{\mathfrak{M}} \cup W^{\mathfrak{N}} \rangle.$$

   For more detail we refer to the definition in the statement of Theorem 3.3.12 (p.196) of AMN [18]. Then

   $$\mathfrak{M} \,\dot{\cup}\, \mathfrak{N} \models \mathbf{Bax}^-.$$

   Actually, $\mathfrak{M} \,\dot{\cup}\, \mathfrak{N}$ to be defined and to be a $\mathbf{Bax}^-$ model, we do not need to assume $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} = \emptyset$ since the "disjointness conditions"[449] in the statement of Thm.3.3.12 of

[449] these conditions were $Obs^{\mathfrak{M}} \cap B^{\mathfrak{N}} = \emptyset$, $Ph^{\mathfrak{M}} \cap B^{\mathfrak{N}} \subseteq Ph^{\mathfrak{N}}$, $Ib^{\mathfrak{M}} \cap B^{\mathfrak{N}} \subseteq Ib^{\mathfrak{N}}$, together with the same conditions but with $\mathfrak{M}$ and $\mathfrak{N}$ interchanged.

AMN [18] are sufficient. This more general notion (using the disjointness conditions) is called <u>*generalized disjoint union*</u> and is denoted by $\mathfrak{M} \cup \mathfrak{N}$.

Instead of only two models, we can form the union of any class K of models (satisfying some disjointness conditions) exactly as we did in Thm.3.3.12 of AMN [18]. In particular let $\mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$. Assume

$$(\forall \text{ distinct } \mathfrak{M}, \mathfrak{N} \in \mathsf{K})\ B^{\mathfrak{M}} \cap B^{\mathfrak{N}} = \emptyset.$$

Then the <u>*disjoint union*</u> $\dot{\bigcup} \mathsf{K}$ of K is defined exactly as in Thm.3.3.12 of AMN [18], i.e.

$$\dot{\bigcup} \mathsf{K} :\stackrel{\text{def}}{=} \Big\langle \bigcup_{\mathfrak{M} \in \mathsf{K}} B^{\mathfrak{M}}, \ldots, \mathfrak{F}, G, \in, \bigcup_{\mathfrak{M} \in \mathsf{K}} W^{\mathfrak{M}} \Big\rangle.$$

Then $\dot{\bigcup} \mathsf{K} \models \mathbf{Bax}^-$. Again (for having $\dot{\bigcup} \mathsf{K} \models \mathbf{Bax}^-$) instead of complete disjointness of $B^{\mathfrak{M}}$ and $B^{\mathfrak{N}}$ it is sufficient to require the milder disjointness conditions (on K) in the formulation of Thm.3.3.12 in AMN [18]. This more general kind of union is again called <u>*generalized disjoint union*</u> (as it was in the case of two models above) and is denoted by $\bigcup \mathsf{K}$.

We note that if $\bigcup \mathsf{K}$ is a generalized disjoint union then

$$(\forall \text{ distinct } \mathfrak{M}, \mathfrak{N} \in \mathsf{K}) Obs^{\mathfrak{M}} \cap Obs^{\mathfrak{N}} = \emptyset,$$

while this does not necessarily hold for Ph in place of Obs.

Generalized disjoint union $\bigcup \mathsf{K}$ is called <u>*photon-disjoint union*</u> iff

$$(\forall \text{ distinct } \mathfrak{M}, \mathfrak{N} \in \mathsf{K})\ Ph^{\mathfrak{M}} \cap Ph^{\mathfrak{N}} = \emptyset.$$

Note that disjoint unions form a special case of photon-disjoint unions, and photon-disjoint unions form a special case of generalized disjoint unions.

2. **Disjoint unions of non-body-disjoint models:** Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$ be such that $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} \neq \emptyset$.[450] The <u>*disjoint union*</u> $\mathfrak{M} \dot{\cup} \mathfrak{N}$ of $\mathfrak{M}$ and $\mathfrak{N}$ is defined as follows. Let $\mathfrak{N}' \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$ be an isomorphic copy of $\mathfrak{N}$ such that (a) and (b) below hold.

   (a) There is an isomorphism between $\mathfrak{N}$ and $\mathfrak{N}'$ which is the identity function on the sort F.

   (b) $\mathfrak{N}'$ is body-disjoint from $\mathfrak{M}$, i.e. $B^{\mathfrak{M}} \cap B^{\mathfrak{N}'} = \emptyset$.

   Now,

   $$\mathfrak{M} \dot{\cup} \mathfrak{N} :\stackrel{\text{def}}{=} \mathfrak{M} \dot{\cup} \mathfrak{N}',$$

   where $\mathfrak{M} \dot{\cup} \mathfrak{N}'$ has already been defined.

   The <u>*disjoint-union*</u> of an arbitrary class $\mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$ of *non-body-disjoint*[451] models is defined analogously to the case of two models and is denoted by $\dot{\bigcup} \mathsf{K}$.

   We note that disjoint unions of (non-body-disjoint) models are determined only up to isomorphism (but this should be no disadvantage, moreover this can be easily avoided if someone wanted to).

[450] The condition $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} \neq \emptyset$ is in principle superfluous but we did not want the present definition of $\mathfrak{M} \dot{\cup} \mathfrak{N}$ overwrite the one in item 1 (approximately previous page).

[451] K is non-body-disjoint if there are distinct $\mathfrak{M}, \mathfrak{N} \in \mathsf{K}$ such that $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} \neq \emptyset$.

3. **Disjoint unions of geometries:**

In the definition of disjoint unions of geometries we will use the following notions from topology.

<u>Topological spaces:</u> By a <u>topological space</u> we understand a pair $\mathbf{X} = \langle X, \mathcal{O}\rangle$ with $\mathcal{O} \subseteq \mathcal{P}(X)$ closed under finite intersections and infinite unions, and such that $\emptyset, X \in \mathcal{O}$. X is the set of <u>points</u> of $\mathbf{X}$ while $\mathcal{O}$ is the set of <u>open sets</u> of $\mathbf{X}$. If $Y \in \mathcal{O}$ then $(X \setminus Y)$ is called a <u>closed set</u>. Hence the closed sets are the complements of the open ones.

<u>Coproduct of topologies:</u> Assume $\mathbf{X}_0 = \langle X_0, \mathcal{O}_0\rangle$ and $\mathbf{X}_1 = \langle X_1, \mathcal{O}_1\rangle$ are disjoint topological spaces, i.e. $X_0 \cap X_1 = \emptyset$. Let us recall from topology that the <u>coproduct</u> (i.e. sum)[452] $\mathbf{X}_0 \coprod \mathbf{X}_1$ of the topological spaces $\mathbf{X}_0$ and $\mathbf{X}_1$ is defined as follows.

$$\begin{aligned} \mathbf{X}_0 \textstyle\coprod \mathbf{X}_1 &\stackrel{\text{def}}{:=} \langle X_0 \cup X_1, \mathcal{O}_0 \textstyle\coprod \mathcal{O}_1\rangle, \quad \text{where} \\ \mathcal{O}_0 \textstyle\coprod \mathcal{O}_1 &\stackrel{\text{def}}{:=} \{\, U_0 \cup U_1 \;:\; U_0 \in \mathcal{O}_0,\ U_1 \in \mathcal{O}_1 \,\}. \end{aligned}$$

Assume $\mathbf{X}_i = \langle X_i, \mathcal{O}_i\rangle$ are topological spaces, for $i \in I$ with fixed set I. Assume that $\mathbf{X}_i$'s are pairwise disjoint, i.e. that $X_i \cap X_j = \emptyset$, for $i \neq j$. Then the <u>coproduct</u> $\coprod_{i\in I} \mathbf{X}_i$ of the family $\langle \mathbf{X}_i \;:\; i \in I\rangle$ is defined as follows.

$$\begin{aligned} \coprod_{i\in I} \mathbf{X}_i &\stackrel{\text{def}}{:=} \langle \bigcup_{i\in I} X_i, \coprod_{i\in I} \mathcal{O}_i\rangle, \quad \text{where} \\ \coprod_{i\in I} \mathcal{O}_i &\stackrel{\text{def}}{:=} \{ \bigcup_{i\in I} U_i \;:\; \langle U_i \;:\; i \in I\rangle \in \mathbf{P}_{i\in I}\mathcal{O}_i \,\}, \end{aligned}$$

where $\mathbf{P}_{i\in I}\mathcal{O}_i$ is the usual Cartesian product of the sets $\mathcal{O}_i$, $i \in I$. ($\mathbf{P}_{i\in I}\mathcal{O}_i$ is the generalization of the direct product $\mathcal{O}_0 \times \mathcal{O}_1$). To help the intuition we note that

$$\coprod_{i\in I} \mathcal{O}_i \;=\; \{ \bigcup_{i\in I} U_i \;:\; (\forall i \in I) U_i \in \mathcal{O}_i \,\}.$$

Note that the "coproduct" $\coprod_{i\in I} \mathcal{O}_i$ of $\mathcal{O}_i$'s has been defined, too.

<u>Disjoint unions of geometries:</u> Disjoint unions of geometries in $\mathsf{Ge}(\emptyset)$ are defined similarly to the case of models (in item 1 above), as follows.

Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots, g_i, \mathcal{T}_i\rangle \in \mathsf{Ge}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$.[453] Assume $Mn_i \cap Mn_j = \emptyset$, for $i \neq j$ $(i, j \in I)$. The <u>*disjoint union*</u> of $\mathfrak{G}_0, \mathfrak{G}_1$ is defined by

$$\mathfrak{G}_0 \,\dot{\cup}\, \mathfrak{G}_1 \stackrel{\text{def}}{:=} \langle Mn_0 \cup Mn_1, \mathbf{F_1}, L_0 \cup L_1; \ldots, g_0 \cup g_1, \mathcal{T}_0 \textstyle\coprod \mathcal{T}_1\rangle.$$

For the general case, the <u>*disjoint union*</u> of the family $\langle \mathfrak{G}_i \;:\; i \in I\rangle$ is

$$\dot{\bigcup_{i\in I}} \mathfrak{G}_i \stackrel{\text{def}}{:=} \Big\langle \bigcup_{i\in I} Mn_i, \mathbf{F_1}, \bigcup_{i\in I} L_i; \ldots, \bigcup_{i\in I} g_i, \coprod_{i\in I} \mathcal{T}_i \Big\rangle.$$

[452] Cf. Engelking [83] under the name "sum of spaces".

[453] I.e. the "field" reducts of $\mathfrak{G}_i$ and $\mathfrak{G}_j$ coincide, for all i, j.

4. **Geometry $\mathfrak{G}^{\perp_0}_{\mathfrak{M}}$ and the class $\mathsf{Ge}^{\perp_0}(Th)$:**

   For every frame model $\mathfrak{M}$ we define $\mathfrak{G}^{\perp_0}_{\mathfrak{M}}$ to be the geometry obtained from $\mathfrak{G}_{\mathfrak{M}}$ by replacing the orthogonality $\perp_r$ by the basic orthogonality $\perp_0$ (cf. p.141 for $\perp_0$). Further, for any set Th of formulas in our frame language we define

$$\mathsf{Ge}^{\perp_0}(Th) :\overset{\text{def}}{=} \{\, \mathfrak{G} \;:\; (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\ \mathfrak{G} \cong \mathfrak{G}^{\perp_0}_{\mathfrak{M}} \,\}\,.$$

<u>*A note to the reader:*</u> At a first reading, the reader may skip item 5 ("Photon-glued ...") below, in such a way that later whenever "photon-glued disjoint unions" are mentioned then the expression "photon-glued ..." should be replaced by "disjoint unions" <u>and</u> **Ax(diswind)** should be added to the assumptions. This is possible because if we assume **Ax(diswind)** then photon-glued disjoint unions become plain disjoint unions. I.e. in the remaining part of this material using photon-glued disjoint unions can be avoided at the expense of assuming **Ax(diswind)**.

5. **Photon-glued disjoint unions of geometries:** In the present item we concentrate on the $\perp_0$-versions of our geometries because of the following. The point is that in the $\perp_0$-versions if two lines are orthogonal then they are in $L^T \cup L^S$. This enables us to define $\perp_0$ in the photon-glued disjoint unions to be the same as it was in the "ordinary" disjoint unions. (If we tried to extend this to $\perp_r$ then we would face the nontrivial task of defining $\perp_r$-orthogonality between the new lines obtained by "gluing" photon-like lines.)

   Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots \rangle \in \mathsf{Ge}^{\perp_0}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$. Assume $Mn_i \cap Mn_j = \emptyset$, for $i \neq j$ $(i, j \in I)$. Then the disjoint union $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ is defined analogously to the case of $\mathsf{Ge}(\emptyset)$ in item 3.

   When forming a disjoint union $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ of geometries ($\mathfrak{G}_i \in \mathsf{Ge}^{\perp_0}(\emptyset)$) sometimes we might want to glue certain photon-like lines together into a single, new, longer photon-like line. The idea is the following. We choose a parameter $H \subseteq L^{Ph} = \bigcup_{i\in I} L_i^{Ph}$ with $|H \cap L_i^{Ph}| \leq 1$ for all $i \in I$. Then

$$\mathrm{Glue}_H\Big(\dot{\bigcup_{i\in I}} \mathfrak{G}_i\Big)$$

   is obtained from $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ by adding the new, "long" line $\bigcup H$ to L^{Ph} and throwing away (all the "old" lines in the set) H from L^{Ph}, and by adjusting $L, g, \mathcal{T}$ to the new set of photon-like lines. In more detail: The new sets of photon-like lines and lines are[454]

$$\begin{aligned} L^{Ph}_{\mathrm{Glue}} &:\overset{\text{def}}{=} (L^{Ph} \setminus H) \cup \{\textstyle\bigcup H\} \\ L_{\mathrm{Glue}} &:\overset{\text{def}}{=} L^{Ph}_{\mathrm{Glue}} \cup L^T \cup L^S, \quad \text{where} \end{aligned}$$

[454] For the "non-set-theory-oriented" reader, we would like to illuminate the intuitive content of the expression $(L^{Ph} \setminus H) \cup \{\bigcup H\}$. Assume $L^{Ph} = \{\, \ell, \{a\}, \{b\} \,\}$ and $H = \{\, \{a\}, \{b\} \,\}$. Then $\bigcup H = \{a, b\}$, $\{\bigcup H\} = \{\, \{a, b\} \,\}$. Hence $(L^{Ph} \setminus H) \cup \{\bigcup H\} = \{\, \ell, \{a, b\} \,\}$. Intuitively, this is what we wanted, we wanted to glue together the photon-like lines $\{a\}$, $\{b\}$ into a single new line $\{a, b\}$, and then to replace the old "short" photon-like lines $\{a\}$, $\{b\}$ by the single new line $\{a, b\}$. Summing it up: $\bigcup H$ is the new long photon-like line obtained by gluing; and H is the <u>set</u> of the old short lines which we want to throw away since they are replaced by their longer version $\bigcup H$. Important: $\bigcup H$ is a line, while H is not. (It is a <u>set</u> of lines.)

$L^T = \bigcup_{i\in I} L_i^T$ and $L^S = \bigcup_{i\in I} L_i^S$; <u>and</u>, letting $Mn = \bigcup_{i\in I} Mn_i$, the pseudo-metric and the topology (of the new geometry) are

$$\begin{array}{lll} g_{\mathrm{Glue}} & \stackrel{\mathrm{def}}{:=} & g \cup \{\, \langle e, e_1, 0\rangle \in Mn \times Mn \times F \;:\; (\exists \ell \in L^{Ph}_{\mathrm{Glue}}) e, e_1 \in \ell \,\}, \\ \mathcal{T}_{\mathrm{Glue}} & & \text{is the topology on } Mn \text{ determined by } g_{\mathrm{Glue}} \end{array}$$

as described in item 13 on p.146. The rest of the ingredients of the new geometry are the same as those of $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$.[455]

We may glue together more than one sequence H of photon-like lines. Namely, let $\mathcal{H} \subseteq \mathcal{P}(L^{Ph})$ be given such that

$$(\forall H \in \mathcal{H})\ (\forall i \in I) |H \cap L_i^{Ph}| \le 1.$$

Now we apply the above outlined gluing procedure for each $H \in \mathcal{H}$. Formally, we obtain

$$\mathrm{Glue}_{\mathcal{H}}\Big(\dot{\bigcup_{i\in I}} \mathfrak{G}_i\Big)$$

which differs from $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ only in L^{Ph}, L, g and $\mathcal{T}$, where the new sets of photon-like lines and lines are

$$\begin{array}{lll} L^{Ph}_{\mathrm{Glue}(\mathcal{H})} & \stackrel{\mathrm{def}}{:=} & (L^{Ph} \setminus \bigcup \mathcal{H}) \cup \{\, \bigcup H \;:\; H \in \mathcal{H} \,\}, \\ L_{\mathrm{Glue}(\mathcal{H})} & \stackrel{\mathrm{def}}{:=} & L^{Ph}_{\mathrm{Glue}(\mathcal{H})} \cup L^T \cup L^S; \end{array}$$

<u>and</u> the pseudo-metric and the topology (of the new geometry) are

$$\begin{array}{lll} g_{\mathrm{Glue}(\mathcal{H})} & \stackrel{\mathrm{def}}{:=} & g \cup \{ \langle e, e_1, 0\rangle \in Mn \times Mn \times F \;:\; (\exists \ell \in L^{Ph}_{\mathrm{Glue}(\mathcal{H})}) e, e_1 \in \ell \,\}, \\ \mathcal{T}_{\mathrm{Glue}(\mathcal{H})} & & \text{is the topology on } Mn \text{ determined by } g_{\mathrm{Glue}(\mathcal{H})}. \end{array}$$

For a representation of this "glued" $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ see Figure 98 (p.275) and the lower picture in Figure 91 (p.211). We call the above defined

$$\mathrm{Glue}_{\mathcal{H}}\Big(\dot{\bigcup_{i\in I}} \mathfrak{G}_i\Big)$$

a <u>*photon-glued disjoint union*</u> of the family $\langle\, \mathfrak{G}_i \;:\; i \in I\,\rangle$ of geometries.

6. **Disjoint and photon-glued disjoint unions of non-disjoint geometries:**

   Disjoint unions of non-disjoint geometries are defined analogously to the case of non-body-disjoint models (in item 2 above), as follows.

   Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots\rangle \in \mathsf{Ge}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$ and assume that $\mathfrak{G}_i$'s are non-disjoint, i.e. $Mn_i \cap Mn_j \neq \emptyset$ for some distinct $i, j \in I$. Let $\mathfrak{G}'_i = \langle Mn'_i, \mathbf{F_1}, L'_i; \ldots\rangle \in \mathsf{Ge}(\emptyset)$, for $i \in I$ be such that (a) and (b) below hold.[456]

[455] Let us notice that $\dot{\bigcup}_{i\in I} \mathfrak{G}_i \models (\,\ell \perp_0 \ell' \;\rightarrow\; \ell, \ell' \in L^T \cup L^S\,)$ by the definition of $\perp_0$.

[456] Concrete construction of the family of geometries $\langle\, \mathfrak{G}'_i \;:\; i \in I\,\rangle$ satisfying (a) and (b): Let $i \in I$. Let $Mn'_i := Mn \times \{i\}$. Let $h_i : Mn \rightarrowtail\!\!\!\twoheadrightarrow Mn'_i$ be the bijection defined by $h_i : e \mapsto \langle e, i\rangle$. Let $h_i^+ = \langle\, h_i,\ \mathrm{Id} \restriction F,\ \widetilde{h}_i\rangle$, where $\widetilde{h}_i : L_i \longrightarrow \{\, h_i[\ell] : \ell \in L_i\,\}$ is defined by $\widetilde{h}_i : \ell \mapsto h_i[\ell]$. Now we define $\mathfrak{G}'_i$ to be the isomorphic copy of $\mathfrak{G}_i$ along h_i^+ (i.e. it is the unique structure for which $h_i^+ : \mathfrak{G}_i \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{G}'_i$ is an isomorphism).

(a) There is an isomorphism beteen $\mathfrak{G}_i$ and $\mathfrak{G}'_i$ which is the identity function on the sort F.

(b) $(\forall$ distinct $i, j \in I)\, Mn'_i \cap Mn'_j = \emptyset$.

Now, the *disjoint union* of the family $\langle\, \mathfrak{G}_i \,:\, i \in I \,\rangle$ is defined to be the disjoint union of the family $\langle \mathfrak{G}'_i \,:\, i \in I \rangle$ (which in turn has already been defined in item 3), and is denoted by $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$.

Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots \rangle \in \mathsf{Ge}^{\perp_0}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$ and assume that $\mathfrak{G}_i$'s are non-disjoint. Let $\mathfrak{G}'_i = \langle Mn'_i, \mathbf{F_1}, L'_i; \ldots \rangle \in \mathsf{Ge}^{\perp_0}(\emptyset)$, for $i \in I$ be such that (a) and (b) above hold. By a *photon-glued disjoint union* of the family $\langle\, \mathfrak{G}_i \,:\, i \in I \,\rangle$ we understand a photon-glued disjoint union of the family $\langle\, \mathfrak{G}'_i \,:\, i \in I \,\rangle$.

We note that disjoint unions and photon-glued disjoint unions of (non-disjoint) geometries are determined only up to isomorphism (but this should be no disadvantage, moreover this can be easily avoided, cf. footnote 456).

Remark 4.2.53 We note that unions *commute* with "geometrization" in the following sense.

Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$. Assume that they satisfy the disjointness conditions[457] in Thm.3.3.12 of AMN [18], i.e. that $\mathfrak{M} \cup \mathfrak{N}$ is a generalized disjoint union.

Then

$$\mathfrak{G}^{\perp_0}_{(\mathfrak{M}\cup\mathfrak{N})} = \text{"a photon-glued disjoint union of } \mathfrak{G}^{\perp_0}_{\mathfrak{M}} \text{ and } \mathfrak{G}^{\perp_0}_{\mathfrak{N}}\text{"}.$$

Intuitively, a *generalized disjoint union* in the "observational world" corresponds to a *photon-glued disjoint union* in the "geometry world", cf. Figure 86.

Assume in addition that $Ph^{\mathfrak{M}} \cap Ph^{\mathfrak{N}} = \emptyset$, i.e. that $\mathfrak{M} \cup \mathfrak{N}$ is a photon-disjoint union. Then

$$\mathfrak{G}_{(\mathfrak{M}\cup\mathfrak{N})} = \mathfrak{G}_{\mathfrak{M}} \,\dot{\cup}\, \mathfrak{G}_{\mathfrak{N}}.$$

Intuitively, a *photon-disjoint union* in the "observational world" corresponds to a *disjoint union* in the "geometry world", cf. Figure 86.

$\lhd$

Examples 4.2.54

1. Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax})$ with $B^{\mathfrak{M}} \cap\ B^{\mathfrak{N}} = \emptyset$. Then

$$\mathfrak{M} \,\dot{\cup}\, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax}).$$

Similarly for any class $\mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax})$. This remains true for generalized disjoint unions of **Basax** models.

$\mathsf{Mod}(\mathbf{Newbasax})$ is the class of all generalized disjoint unions of members of $\mathsf{Mod}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking generalized disjoint unions and contains $\mathsf{Mod}(\mathbf{Basax})$.

$\mathsf{Mod}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$ is the class of all photon-disjoint unions of members of $\mathsf{Mod}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking photon-disjoint unions and contains $\mathsf{Mod}(\mathbf{Basax})$.

[457] cf. footnote 449 on p.196

$$
\begin{tikzcd}
\mathsf{Mod}(Th) \arrow[r, "\substack{\text{closing up under} \\ \underline{\text{generalized}}\text{ disjoint unions}}"] \arrow[d] \arrow[r, phantom, shift right=6ex, "\text{geometrization}"] & \mathsf{Mod}(Th') \arrow[d] \\
\mathsf{Ge}^{\perp_0}(Th) \arrow[r, "\substack{\text{closing up under} \\ \underline{\text{photon-glued}}\text{ disjoint unions}}"'] & \mathsf{Ge}^{\perp_0}(Th')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Mod}(Th) \arrow[r, "\substack{\text{closing up under} \\ \underline{\text{photon}}\text{-disjoint unions}}"] \arrow[d] \arrow[r, phantom, shift right=6ex, "\text{geometrization}"] & \mathsf{Mod}(Th' + \mathbf{Ax(diswind)}) \arrow[d] \\
\mathsf{Ge}(Th) \arrow[r, "\substack{\text{closing up under} \\ \text{disjoint unions}}"'] & \mathsf{Ge}(Th' + \mathbf{Ax(diswind)})
\end{tikzcd}
$$

$$
\begin{aligned}
\langle\, Th,\ Th' \,\rangle \in \{\, &\langle\, \mathbf{Basax}, \mathbf{Newbasax} \,\rangle,\ \langle\, \mathbf{Specrel}, \mathbf{Newbasax} + \mathbf{Ax(symm)}^{\dagger} \,\rangle, \\
&\langle\, \mathbf{Reich}(\mathbf{Basax}), \mathbf{Reich}(\mathbf{Newbasax}) \,\rangle,\ \langle\, \mathbf{Bax}^{-} + \mathbf{Ax6}, \mathbf{Bax}^{-} \,\rangle \,\}
\end{aligned}
$$

Figure 86: Generalized disjoint unions of models correspond to photon-glued disjoint unions of geometries, while photon-disjoint unions of models correspond to disjoint unions of geometries. (Further, the above diagrams commute in the sense of Remark 4.2.53.)

2. The examples in item 1 above show up in the "geometry world" in the following "shape". See Figure 86.

   Let $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathsf{Ge}(\mathbf{Basax})$ with a common "field" reduct. Then

   $$\mathfrak{G}_1 \dot{\cup} \mathfrak{G}_2 \in \mathsf{Ge}(\mathbf{Newbasax}).$$

   Similarly for any family $\langle\, \mathfrak{G}_i \ :\ i \in I \,\rangle$ of **Basax** geometries. This remains true for photon-glued disjoint unions of **Basax** geometries, i.e. the photon-glued disjoint unions of geometries from $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$ are in $\mathsf{Ge}^{\perp_0}(\mathbf{Newbasax})$.

   $\mathsf{Ge}^{\perp_0}(\mathbf{Newbasax})$ is the class of all photon-glued disjoint unions of members of $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking photon-glued disjoint unions and contains $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$.

   $\mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$ is the class of all disjoint unions of members of $\mathsf{Ge}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking disjoint unions and contains $\mathsf{Ge}(\mathbf{Basax})$.

   (If we formed the *non-disjoint* union of two **Basax** geometries say $\mathfrak{G}_0, \mathfrak{G}_1$ then we could obtain a geometry $\mathfrak{G}_0 \cup \mathfrak{G}_1$ which is not even a $\mathbf{Bax}^{-}$ geometry.)

3. Examples similar to those given in items 1 and 2 are illustrated in Figure 86.

4. Let $\mathfrak{G}_0, \mathfrak{G}_1 \in \mathsf{Ge}(\mathbf{Basax})$. Assume they are disjoint. Then in $\mathfrak{G}_0 \dot{\cup} \mathfrak{G}_1$ the parts Mn_0 and Mn_1 are sometimes called windows. Cf. Figure 98 (p.275) and Figure 91 (p.211). Similarly for photon-glued disjoint unions of **Basax** geometries (i.e. $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$-structures).

   More generally in a **Newbasax** geometry, say $\mathfrak{G}$, the maximal "**Basax** subgeometries"[458]

[458] Recall that any **Newbasax** geometry $\mathfrak{G}$ is a photon-glued disjoint union of **Basax** geometries say $\mathfrak{G}_i$'s. These $\mathfrak{G}_i$'s (more precisely the Mn_i's) are called the windows of $\mathfrak{G}$.

are called <u>*windows*</u>. (Here we use the notion of a sub-geometry in an intuitive sense only, but it could be formalized such that all details would match.[459])

In $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^-)$ two points $e, e_1 \in Mn$ are in the same <u>*window*</u> iff they are *connected*, i.e. $e \sim e_1$. If $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$, for some $\mathfrak{M} \models \mathbf{Bax}^-$, then these windows are exactly the $Rng(w_m)$'s, i.e. the subsets of Mn of the form $Rng(w_m)$ (with $m \in Obs$). Cf. Remark 4.2.13 (p.160).

5. Assume $n > 2$. Then every $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax(diswind)})$ is obtainable as a disjoint union of Minkowskian geometries.[460] Further, $\mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax(diswind)})$ is the disjoint unions closure of the class of Minkowskian geometries.

6. $\mathsf{Mod}(\mathbf{Flxbasax})$ is not closed under taking disjoint unions, but disjoint unions of **Flxbasax** models are **Bax** models.

   We did not have time to think about whether $\mathsf{Ge}(\mathbf{Flxbasax})$ is closed under taking disjoint unions but we think it is not closed.

$\triangleleft$

CONVENTION 4.2.55 Besides geometries in $\mathsf{Ge}(\emptyset)$ and in $\mathsf{Ge}^{\perp_0}(\emptyset)$ we will also discuss reducts of these (e.g. $\mathbf{G}_{\mathfrak{M}}$, $\mathsf{G}_{\mathfrak{M}}$, etc.) and also slight variants of $\mathsf{Ge}(\emptyset)$ e.g. $\perp_r'$ or $\perp_r''$ in place of $\perp_r$.

We <u>*extend*</u> the above defined notions of *disjoint unions* and *photon-glued disjoint unions* to these kinds of geometries the natural (and obvious) way. (In the case of generalizing photon-glued disjoint unions we restrict attention to such geometries where relativistic orthogonality is $\perp_0$.)

$\triangleleft$

Now, having disjoint unions etc. at our hands we can state a stronger form of Theorem 4.2.50, not involving **Ax6**. Further, we will generalize Theorem 4.2.51 from **Basax** to **Newbasax**. Roughly, the just quoted theorems say that certain reducts of our geometries agree with the corresponding reducts of Minkowskian geometries, for certain choices of Th. Very roughly the new theorems will say that our relativistic geometries corresponding to many of our theories can be obtained as disjoint (or photon-glued disjoint) unions of Minkowskian geometries if we regard a reduct only.

THEOREM 4.2.56 *Assume* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Then (i) and (ii) below hold. (Cf. Figures 84, 85.)*

(i) *Assume $n > 2$. Then the $(\prec, \mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a disjoint union of the similar reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then the $(\mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a disjoint union of the similar reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

---

[459] One possibility is to add **Ax(diswind)** to **Newbasax**.

[460] This follows by example 1, Remark 4.2.53 (p.201) and Thm.4.2.45 (p.190).

Proof: The theorem follows by Thm.4.2.50 (p.195), Remark 4.2.53 (p.201) and by noticing that each $\mathbf{Bax}^{\oplus} + \mathbf{Ax(diswind)}$ model is a photon-disjoint union of $\mathbf{Bax}^{\oplus} + \mathbf{Ax6}$ models. ∎

Theorem 4.2.58 below is the "photon-glued" version of Theorem 4.2.56 above. For stating this theorem we define the $\perp_0$-versions of Minkowskian geometries.

Definition 4.2.57 Assume $\mathfrak{F}$ is Euclidean. Then the $\underline{\perp_0\textit{-version}}$ $\mathit{Mink}^{\perp_0}(\mathfrak{F})$ $\underline{\textit{of the}}$ $\underline{\text{Minkowskian}}$ $\underline{\text{geometry}}$ $\mathit{Mink}(\mathfrak{F})$ is defined to be the geometry obtained from $\mathit{Mink}(\mathfrak{F})$ by replacing $\perp_\mu$ by $(\perp_0)_\mu$ defined below.

$$(\perp_0)_\mu :\stackrel{\text{def}}{=} \left\{ \langle \ell, \ell' \rangle \in \perp_\mu \ : \ \ell, \ell' \in L^T_\mu \cup L^S_\mu, \ \ell \cap \ell' \neq \emptyset \right\}.$$

◁

THEOREM 4.2.58 *Assume* $\mathfrak{G} \in \mathsf{Ge}^{\perp_0}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$. *Then (i) and (ii) below hold. (Cf. Figures 84, 85.)*

(i) *Assume* $n > 2$. *Then the* $(\prec, \mathrm{eq}, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a photon-glued disjoint union of reducts of* $\perp_0$*-versions of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then the* $(\mathrm{eq}, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a photon-glued disjoint union of reducts of* $\perp_0$*-versions of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

Proof: The theorem follows by Thm.4.2.50 (p.195), Remark 4.2.53 (p.201) and by noticing that each $\mathbf{Bax}^{\oplus}$ model is a generalized disjoint union of $\mathbf{Bax}^{\oplus} + \mathbf{Ax6}$ models. ∎

The following two theorems are generalizations of Theorem 4.2.51 (p.195).

THEOREM 4.2.59 *Assume* $n > 2$ *and* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Then (i) and (ii) below hold. (Cf. Figures 84, 85.)*

(i) *The* $(\prec, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a disjoint union of reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds by item 5 of Examples 4.2.54.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then the* $(g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a disjoint union of reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

Proof: The theorem follows by Thm.4.2.51 (p.195), Remark 4.2.53 (p.201) and by noticing that each $\mathbf{Newbasax} + \mathbf{Ax(diswind)}$ model is a photon-disjoint union of $\mathbf{Basax}$ models. ∎

Theorem 4.2.60 below is the "photon-glued" version of Theorem 4.2.59 above.

THEOREM 4.2.60 *Assume* $\mathfrak{G} \in \mathsf{Ge}^{\perp_0}(\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$ *and* $n > 2$. *Then (i) and (ii) below hold. (Cf. Figures 84, 85.)*

(i) *The $(\prec, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a photon-glued disjoint union of reducts of $\bot_0$-versions of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

(ii) *Assume* $\mathbf{Ax(\uparrow\uparrow_0)}$. *Then the $(g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a photon-glued disjoint union of reducts of $\bot_0$-versions of Minkowskian geometries (up to an isomorphism).*

(The other direction also holds.)

Proof: The theorem follows by Thm.4.2.51 (p.195), Remark 4.2.53 and by Thm.3.3.12 of AMN [18] saying that each **Newbasax** model is a generalized disjoint union of **Basax** models. ∎

Theorems 4.2.56, 4.2.58, 4.2.59, 4.2.60 above are all involved in Figures 84, 85 (pp. 192–193). Here we give an intuitive explanation for these figures.

Notation: For a class K of models, $Voc\mathsf{K}$ denotes the vocabulary of K, cf. p.220. Assume L is a sub-vocabulary of $Voc\mathsf{K}$. Then $\mathsf{Rd}_{\mathsf{L}}(\mathsf{K})$ denotes the class of reducts of members of K to the vocabulary L.[461]

Intuitive explanation for Figures 84, 85: The figures represent reducts of geometries agreeing with the corresponding reducts of (possibly unions of) Minkowskian geometries. Each node (in the figure) is of the form $\mathsf{Rd}_{\mathsf{L}}(\mathsf{Ge}(Th))$ for some relativity theory Th (observational) and subvocabulary L of the vocabulary of our relativistic geometries $\mathfrak{G}_{\mathfrak{M}}$. Hence, each node is characterized by two pieces of data Th and the "geometric reduct" (i.e. the geometric vocabulary) L. $\mathbf{Ax}(\sqrt{\ })$ and $n > 2$ are assumed in the figures. If we disregard the "$\mathsf{Rd}_{\mathsf{L}}\,\mathsf{Ge}$"-part i.e. if we consider the Th-part only then the figure becomes a sublattice of the lattice of our distinguished theories discussed on pp. 451–453 of AMN [18], cf. also Fig.223 on p.653 of AMN [18] and Remark 4.5.14(III) pp. 294–295 herein. If we want to disregard Th, then we get a 6-element lattice of distinguished geometry-reducts of our relativistic geometries $\mathfrak{G}_{\mathfrak{M}}$. At the bottom of this lattice are the $\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \bot_r\rangle$ geometries which are basically the same what we call relativistic incidence geometries $\mathsf{Ge}^{inc}(Th)$ in §6.7.4 (p.1174) of AMN [18]. More precisely $\mathsf{Ge}^{inc}(Th)$ is definitionally equivalent[462] with our "bottom" geometry, with Th as indicated in the figure, assuming **Ax(diswind)**, cf. Thm.6.7.31 (p.1164) of AMN [18]. The top of the lattice represents the whole of $\mathfrak{G}_{\mathfrak{M}}$'s, of course. Besides labelling the nodes, we labelled some of the edges too in Fig.84. The labels on an edge indicate (roughly) the changes that happen when moving along that edge, the same change happens when moving along parallel edges. E.g. the label $\boxed{\text{unions, } \mathbf{Ax6}}$ indicate that, intuitively, we can move from the higher end of that edge to the lower one by taking (possibly photon-glued) disjoint unions of our geometries and dropping **Ax6** from our Th, loosely speaking.

To understand our observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$ (and their connections with the original models $\mathfrak{M}$), below we introduce "observer-dependent" geometries $\mathfrak{G}_m$, for each observer $m \in Obs^{\mathfrak{M}}$. After this we will introduce restrictions $\mathfrak{G} \restriction N$ of geometries to subsets $N \subseteq Mn$ of their set of points.

Our next definition may look, at first sight, somewhat longish, but at second reading it will turn out to be just the natural thing, and it will turn out to be quite useful. E.g. in

[461] Cf. Convention 4.3.1 (p.220) for more familiarity with these notions.
[462] Cf. Def.4.3.33 on p.255 for definitional equivalence.

Prop.4.2.64 we will see that $\mathfrak{G}_{\mathfrak{M}}$ can be obtained from the world-views of observers i.e. from the w_m's by gluing them together (as we planned in the first 2 sentences of §4.2.5). For this, first, the w_m's have to be "geometrized". The geometrized versions of the w_m's will be the $\mathfrak{G}_m$'s defined below.

Definition 4.2.61 Let $\mathfrak{N}$ be a frame model and $\mathfrak{G}_{\mathfrak{N}} = \langle Mn, \mathbf{F_1}, L; \ldots \rangle$ be the geometry corresponding to it. Then using the world-view function w_m each observer m can copy the geometry $\mathfrak{G}_{\mathfrak{N}}$ to his coordinate system ${}^nF$, obtaining the <u>*observer-dependent geometry*</u> $\mathfrak{G}_m$ defined below, cf. Figure 87. Let $m \in Obs$. For every $\ell \in L$, throughout this definition, let

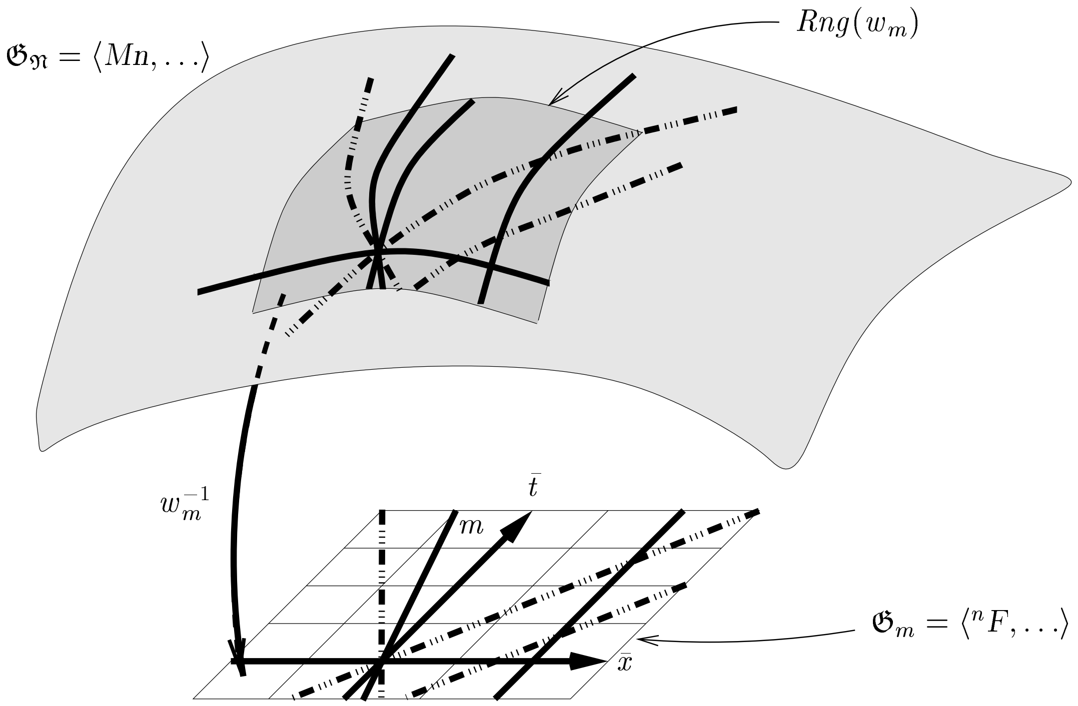


Figure 87: Using the world-view function w_m each observer m can copy the geometry $\mathfrak{G}_{\mathfrak{N}}$ to his coordinate system ${}^nF$.

$$\ell_m :\stackrel{\text{def}}{=} w_m^{-1}[\ell].$$

Now,

$$\mathfrak{G}_m \quad :\stackrel{\text{def}}{=} \quad \langle {}^nF, \mathbf{F_1}, L_m;\ L_m^T, L_m^{Ph}, L_m^S, \in, \prec_m, Bw_m, \perp_m, eq_m, g_m, \mathcal{T}_m \rangle,$$

where

$$\begin{aligned}
L_m &:\stackrel{\text{def}}{=} \{\, \ell_m \,:\, \ell \in L,\ \ell_m \neq \emptyset \,\}, \\
L_m^T &:\stackrel{\text{def}}{=} \{\, \ell_m \,:\, \ell \in L^T,\ \ell_m \neq \emptyset \,\}, \\
L_m^{Ph} &:\stackrel{\text{def}}{=} \{\, \ell_m \,:\, \ell \in L^{Ph},\ \ell_m \neq \emptyset \,\}, \\
L_m^S &:\stackrel{\text{def}}{=} \{\, \ell_m \,:\, \ell \in L^S,\ \ell_m \neq \emptyset \,\},
\end{aligned}$$

$\in$ is the membership relation between ${}^nF$ and L_m,

$$\begin{aligned}
\prec_m &\stackrel{\text{def}}{:=} \{\, \langle p,q\rangle \in {}^nF \times {}^nF \;:\; w_m(p) \prec w_m(q) \,\}\,,\\
Bw_m &\stackrel{\text{def}}{:=} \{\, \langle p,q,r\rangle \in {}^3({}^nF) \;:\; Bw(w_m(p), w_m(q), w_m(r)) \,\}\,,\\
\perp_m &\stackrel{\text{def}}{:=} \{\, \langle \ell_m, \ell'_m\rangle \;:\; \ell \perp_r \ell',\; \ell_m \neq \emptyset,\; \ell'_m \neq \emptyset \}\,,\\
eq_m &\stackrel{\text{def}}{:=} \{\, \langle p,q,r,s\rangle \in {}^4({}^nF) \;:\; eq(w_m(p), w_m(q), w_m(r), w_m(s)) \,\}\,,\\
g_m &\stackrel{\text{def}}{:=} \{\, \langle p,q,\lambda\rangle \in {}^nF \times {}^nF \times F \;:\; g(w_m(p), w_m(q)) = \lambda \,\}\,,\\
\mathcal{T}_m &\stackrel{\text{def}}{:=} \{\, w_m^{-1}[H] \;:\; H \in \mathcal{T} \,\}\,.
\end{aligned}$$

We define $\mathfrak{G}_m^{\perp_0}$ to be the geometry obtained from $\mathfrak{G}_m$ by replacing $\perp_m$ by $(\perp_0)_m$ defined below.

$$(\perp_0)_m \stackrel{\text{def}}{:=} \{\, \langle \ell_m, \ell'_m\rangle \;:\; \ell \perp_0 \ell',\; \ell_m \neq \emptyset,\; \ell'_m \neq \emptyset \}\,,$$

cf. p.141 for the definition of $\perp_0$.

$\lhd$

Definition 4.2.62 Let $\mathfrak{G} = \langle Mn, \mathbf{F_1}, L; \ldots, \mathcal{T}\rangle$ be an observer-independent geometry. Let $N \subseteq Mn$. Then the restrictions $\mathfrak{G} \restriction N$ and $\mathfrak{G} \restriction^+ N$ of $\mathfrak{G}$ to N are defined in (i) and (ii) below, respectively. See Figure 88, cf. also Figure 90.

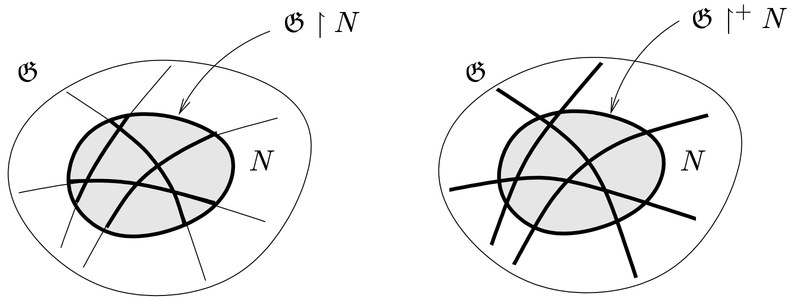


Figure 88: Illustration for Definition 4.2.62.

(i) $\mathfrak{G} \restriction N \stackrel{\text{def}}{:=} \langle N,\ \mathbf{F_1},\ L \restriction N$ [463] $;\ L^T \restriction N,\ L^{Ph} \restriction N,\ L^S \restriction N,\ \in,$
$\prec \restriction N$ [464], $Bw \restriction N,\ \perp_N,\ eq \restriction N,\ g \restriction {}^2N,\ \mathcal{T} \restriction N$ [465]$\rangle$, where

$$\perp_N \stackrel{\text{def}}{:=} \{\, \langle \ell \cap N, \ell' \cap N\rangle \;:\; \ell, \ell' \in L,\ \ell \perp_r \ell' \,\}.$$

(ii) We define $\mathfrak{G} \restriction^+ N$ to be the geometry obtained from $\mathfrak{G} \restriction N$ by replacing $L \restriction N,\ L^T \restriction N,\ L^{Ph} \restriction N,\ L^S \restriction N,\ \perp_N$ by $L_N,\ L^T \cap L_N,\ L^{Ph} \cap L_N,\ L^S \cap L_N,\ \perp_r \restriction L_N$, respectively, where $L_N \stackrel{\text{def}}{:=} \{\, \ell \in L \;:\; \ell \cap N \neq \emptyset \,\}$.

[463] $L \restriction N := \{\, \ell \cap N \;:\; \ell \in L \,\}$. This is the natural restriction of *"Lines"* to $N \subseteq$ *"Points"*. Similarly for the topology $\mathcal{T}$ in place of lines L.

[464] We use the restriction symbol $\restriction$ for relations too the natural way. I.e. $\prec \restriction N := \prec \cap (N \times N)$. Similarly for other relations of perhaps different ranks. (Since functions are special relations our usage of $\restriction$ is ambigous. We hope context will help.)

[465] $\mathcal{T} \restriction N := \{\, H \cap N \;:\; H \in \mathcal{T} \,\}$, cf. footnote 463.

(iii) We extend the definitions of the restrictions $\mathfrak{G} \restriction N$ and $\mathfrak{G} \restriction^+ N$ to similar geometries, e.g. $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$ the natural way. E.g. $\mathfrak{G}_{\mathfrak{N}}^{\perp_0} \restriction N$ is defined the natural way.

$\lhd$

Remark 4.2.63 Let $\mathfrak{G}$ be an observer-independent geometry and $N \in \mathcal{T}$, i.e. $N \subseteq Mn$ is an open set. Then $\mathfrak{G} \restriction^+ N$ is a strong submodel of $\mathfrak{G}$, in symbols $(\mathfrak{G} \restriction^+ N) \subseteq \mathfrak{G}$. $\mathfrak{G} \restriction N$ is not necessarily a submodel of $\mathfrak{G}$; moreover there is $\mathfrak{G}$ and $N \in \mathcal{T}$ such that $\mathfrak{G} \restriction N$ is not isomorphic to any submodel of $\mathfrak{G}$. Such $\mathfrak{G}$ and N are represented in Figure 89 below, cf. also item 2f of Prop.4.2.64 (p.210) and footnote 471 in it.

$\lhd$

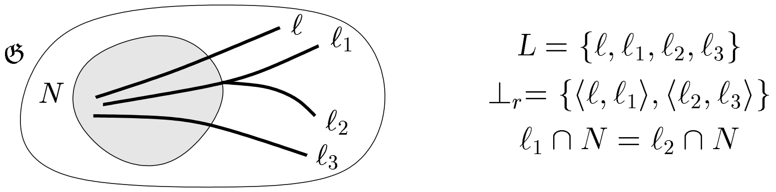


Figure 89: $\mathfrak{G} \restriction N$ is not isomorphic to any submodel of $\mathfrak{G}$.

Item 1 of our next proposition says that, assuming $\mathbf{Bax}^-$, any observer-dependent geometry $\mathfrak{G}_m$ is basically the *familiar picture* which we often called the world-view of observer m; e.g., in $\mathfrak{G}_m$, the set of points is ${}^nF$, L consists of Euclidean lines, L^T consists of the traces (i.e. life-lines) of observers as seen by m, L^{Ph} is the set of life-lines of photons as seen by m, two lines are $\perp_0$-orthogonal iff they are two coordinate axes of some observer as seen by m, etc. For a second, let us call these familiar structures ${}^nF$-geometries. Item 3 says that any observer-independent geometry $\mathfrak{G}_{\mathfrak{N}}$ is a disjoint union of such familiar ${}^nF$-geometries, assuming $\mathbf{Bax}^- + \mathbf{Ax(diswind)}$. Formally

$$\mathfrak{G}_{\mathfrak{N}} = \dot{\bigcup_{m \in O}} \mathfrak{G}_m$$

for some $O \subseteq Obs$. $\mathbf{Ax(diswind)}$ can be omitted if we use photon-glued disjoint unions and $\perp_0$-versions of our geometries. Cf. Figure 91 (p.211).

PROPOSITION 4.2.64 (On Bax$^-$geometries)
Let $\mathfrak{N} \models \mathbf{Bax}^-$. Consider the observer-independent geometry $\mathfrak{G}_{\mathfrak{N}}$. Then 1–5 below hold.

1. *Let $m \in Obs$. Consider the observer-dependent geometry $\mathfrak{G}_m$. Then (a)–(h) below hold.*
   - *(a)* $L_m \subseteq \mathsf{Eucl}$. *Hence,* $(\forall \ell \in L)\; w_m^{-1}[\ell] \in \mathsf{Eucl} \cup \{\emptyset\}$.[466]
   - *(b)* $L_m^T = \{\, tr_m(k) \;:\; k \in Obs,\; m \xrightarrow{\odot} k \,\}$.
   - *(c)* $L_m^{Ph} = \{\, tr_m(ph) \;:\; ph \in Ph,\; m \xrightarrow{\odot} ph \,\}$.
   - *(d)* $L_m^S = \{\, \mathsf{f}_{km}[\bar{x}_i] \;:\; k \in Obs,\; m \xrightarrow{\odot} k,\; 0 < i \in n \,\}$.
   - *(e)* $(\perp_0)_m = \{\, \langle \mathsf{f}_{km}[\bar{x}_i], \mathsf{f}_{km}[\bar{x}_j]\rangle \;:\; k \in Obs,\; m \xrightarrow{\odot} k,\; i \neq j \,\}$.

[466] Cf. Prop.6.2.48 (p.854) in AMN [18].

(f) *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Then* Bw_m *and* Betw *coincide.*

(g) $(\forall$ *distinct* $p, q, r \in {}^nF)$

$$(\, Bw_m(p,q,r) \;\vee\; Bw_m(p,r,q) \;\vee\; Bw_m(q,p,r)\,) \;\Leftrightarrow\; (p, q, r \textit{ are collinear}).$$ [467]

(h) *Assume* $\mathbf{Ax(\uparrow\uparrow_0)} + \mathbf{Ax}(\sqrt{\ })$. *Let* $p, q \in {}^nF$. *Then*

$$p \prec_m q \quad\Longleftrightarrow\quad \Big(p_t < q_t \;\;\wedge\;\; (\exists k \in Obs)\, p, q \in tr_m(k)\Big).$$

2. *Let* $m \in Obs$. *Consider the observer-dependent geometry* $\mathfrak{G}_m$. *Then (a)–(g) below hold.*

(a) *Assume* $\mathbf{Ax6}$. *Then*

$$\mathfrak{G}_m \cong \mathfrak{G}_{\mathfrak{N}} \quad \textit{and} \quad \mathfrak{G}_m^{\perp_0} \cong \mathfrak{G}_{\mathfrak{N}}^{\perp_0}.$$

Actually, the world-view function w_m *induces an isomorphism between* $\mathfrak{G}_m$ *and* $\mathfrak{G}_{\mathfrak{N}}$ *(and between* $\mathfrak{G}_m^{\perp_0}$ *and* $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$*) the natural way.*[468]

(b)

$$\mathfrak{G}_m \cong (\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)) \quad \textit{and} \quad \mathfrak{G}_m^{\perp_0} \cong (\mathfrak{G}_{\mathfrak{N}}^{\perp_0} \restriction Rng(w_m)),$$

see Figure 90. Actually, the world-view function w_m *induces an isomorphism between* $\mathfrak{G}_m$ *and* $\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)$ *the natural way.*[469]

(c) $(\forall \ell \in L^T \cup L^S)\,(\, \ell \cap Rng(w_m) \neq \emptyset \;\Rightarrow\; \ell \subseteq Rng(w_m)\,)$.
Intuitively, time-like and space-like lines do not stick out from the window $Rng(w_m)$, *see Figure 90.*

(d) *Assume* $\mathbf{Ax(diswind)}$. *Then, intuitively, lines do not stick out from the window* $Rng(w_m)$, *formally:*

$$(\forall \ell \in L)\,(\, \ell \cap Rng(w_m) \neq \emptyset \;\Rightarrow\; \ell \subseteq Rng(w_m)\,),$$

cf. Figure 90 (in the figure some photon-like lines do stick out from the window $Rng(w_m)$*).*
Therefore

$$(\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)) = (\mathfrak{G}_{\mathfrak{N}} \restriction^+ Rng(w_m)),$$

cf. Figure 88 (p.207).

(e) *Assume* $\mathbf{Ax(diswind)}$. *Then* $\mathfrak{G}_m$ *is isomorphic to a strong submodel of* $\mathfrak{G}_{\mathfrak{N}}$ *(and* $Rng(w_m) \in \mathcal{T}$*). In more detail:*

$$\mathfrak{G}_m \cong (\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)) = (\mathfrak{G}_{\mathfrak{N}} \restriction^+ Rng(w_m)) \subseteq \mathfrak{G}_{\mathfrak{N}},$$

cf. Remark 4.2.63.
The world-view function w_m *induces an embedding of* $\mathfrak{G}_m$ *into* $\mathfrak{G}_{\mathfrak{N}}$ *the natural way.*[470] *See Figure 90 and the the upper picture in Figure 91.*

---

[467] Cf. Prop.4.2.14 on p.160.

[468] Making this precise: Let $\widetilde{w_m} : L_m \longrightarrow \{\, w_m[\ell] \;:\; \ell \in L_m \,\}$ be defined by $\widetilde{w_m} : \ell \mapsto w_m[\ell]$. Then $Rng(\widetilde{w_m}) = L_{\mathfrak{N}}$ and $\langle w_m,\ \mathrm{Id} \restriction F,\ \widetilde{w_m}\rangle$ is a (three-sorted) isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}}$ (and between $\mathfrak{G}_m^{\perp_0}$ and $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$). Cf. item (II) of Def.4.2.3 (p.146) for isomorphisms between geometries.

[469] Making this precise: Let $\widetilde{w_m}$ be defined as in footnote 468. Then $\langle w_m,\ \mathrm{Id} \restriction F,\ \widetilde{w_m}\rangle$ is an isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)$.

[470] Let $\widetilde{w_m}$ be defined as in footnote 468, p.209. Then

$$Rng(\widetilde{w_m}) = \{\, \ell \in L_{\mathfrak{N}} \;:\; \ell \subseteq Rng(w_m)\,\} = \{\, \ell \in L_{\mathfrak{N}} \;:\; \ell \cap Rng(w_m) \neq \emptyset \,\};$$

and $\langle w_m,\ \mathrm{Id} \restriction F,\ \widetilde{w_m}\rangle$ is an embedding of $\mathfrak{G}_m$ itno $\mathfrak{G}_{\mathfrak{N}}$.

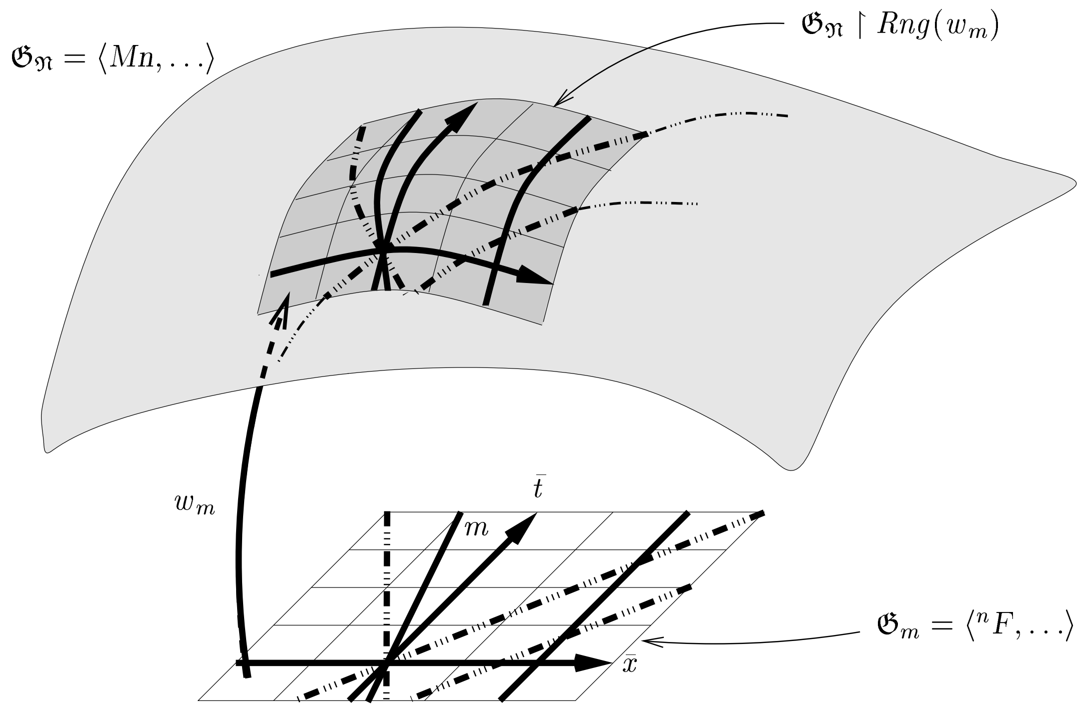


Figure 90: $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)$ are isomorphic.

(f) *The assumption* **Ax(diswind)** *is needed in item (e) above. I.e. there is* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$ *and* $k \in Obs^{\mathfrak{M}}$ *such that* $\mathfrak{G}_k$ *is not isomorphic to any submodel of* $\mathfrak{G}_{\mathfrak{M}}$.[471]

(g) *Assume* $k \in Obs$ *is such that* $m \overset{\odot}{\to} k$. *Then the geometies* $\mathfrak{G}_m$ *and* $\mathfrak{G}_k$ *are isomorphic, i.e.* $\mathfrak{G}_m \cong \mathfrak{G}_k$. *Actually, the world-view transformation* f_{mk} *induces an isomorphism between* $\mathfrak{G}_m$ *and* $\mathfrak{G}_k$ *the natural way.*[472]

3. *By Thm.3.2.6 (and* **Ax4**), $\overset{\odot}{\to}$ *is an equivalence relation when restricted to Obs.*[473] *Let* $O \subseteq Obs$ *be a class of representatives for the equivalence relation* $\overset{\odot}{\to}$.[474] *Then (a) and (b) below hold.*

(a) *Assume* **Ax(diswind)**. *Then* $\mathfrak{G}_{\mathfrak{N}}$ *is the disjoint union of the family* $\langle\, \mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m) \,:\, m \in O \,\rangle$.
Therefore, by item 2b, $\mathfrak{G}_{\mathfrak{N}}$ *is the disjoint union of the family* $\langle\, \mathfrak{G}_m \,:\, m \in O \,\rangle$, *up to isomorphism, i.e.*

$$\mathfrak{G}_{\mathfrak{N}} \;\cong\; \dot{\bigcup_{m \in O}} \mathfrak{G}_m,$$

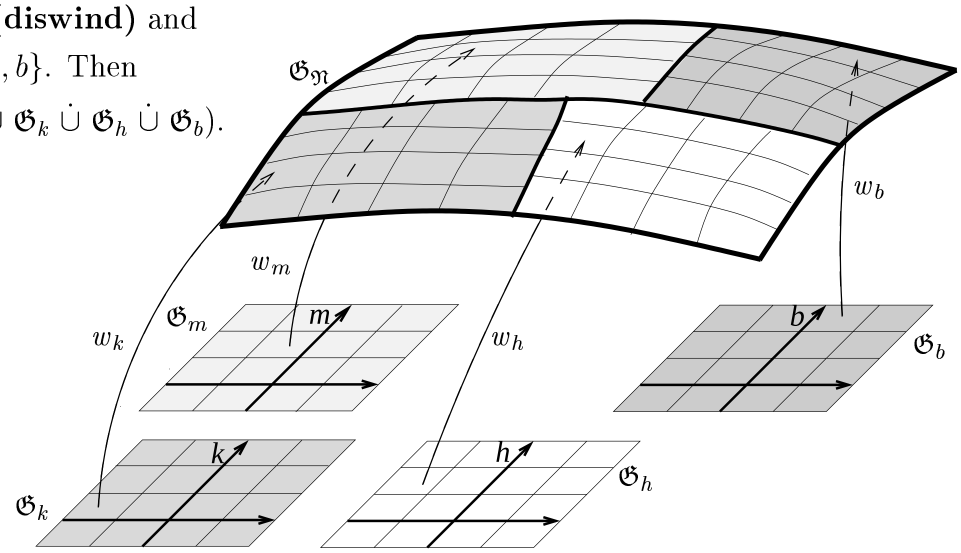
[471] E.g. let $\mathfrak{M}$ be the generalized disjoint union of two **NewtK** models $\mathfrak{M}_1$, $\mathfrak{M}_2$ with $Ph^{\mathfrak{M}_1} = Ph^{\mathfrak{M}_2}$. Then for each observer k, $L_k^{Ph} \cap L_k^S \neq \emptyset$, while $L^{Ph} \cap L^S = \emptyset$. Thus for each k, $\mathfrak{G}_k$ is not ismorphic to any submodel of $\mathfrak{G}_{\mathfrak{M}}$.

[472] Cf. footnote 445 on p.191.

[473] assuming $\mathbf{Bax}^-$ of course

[474] I.e. $(\forall m \in Obs)\, |O \cap m/\overset{\odot}{\to}\,| = 1$, where $m/\overset{\odot}{\to}$ is the equivalence class of m w.r.t. $\overset{\odot}{\to}$, as usual.

Assume **Ax(diswind)** and
$O = \{m, k, h, b\}$. Then
$\mathfrak{G}_{\mathfrak{N}} \cong (\mathfrak{G}_m \dot\cup \mathfrak{G}_k \dot\cup \mathfrak{G}_h \dot\cup \mathfrak{G}_b)$.



Assume $O = \{m, k, h, b\}$. Then
$\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$ is a photon-glued disjoint union of $\mathfrak{G}_m^{\perp_0}, \mathfrak{G}_k^{\perp_0}, \mathfrak{G}_h^{\perp_0}, \mathfrak{G}_b^{\perp_0}$, up to isomorphism:

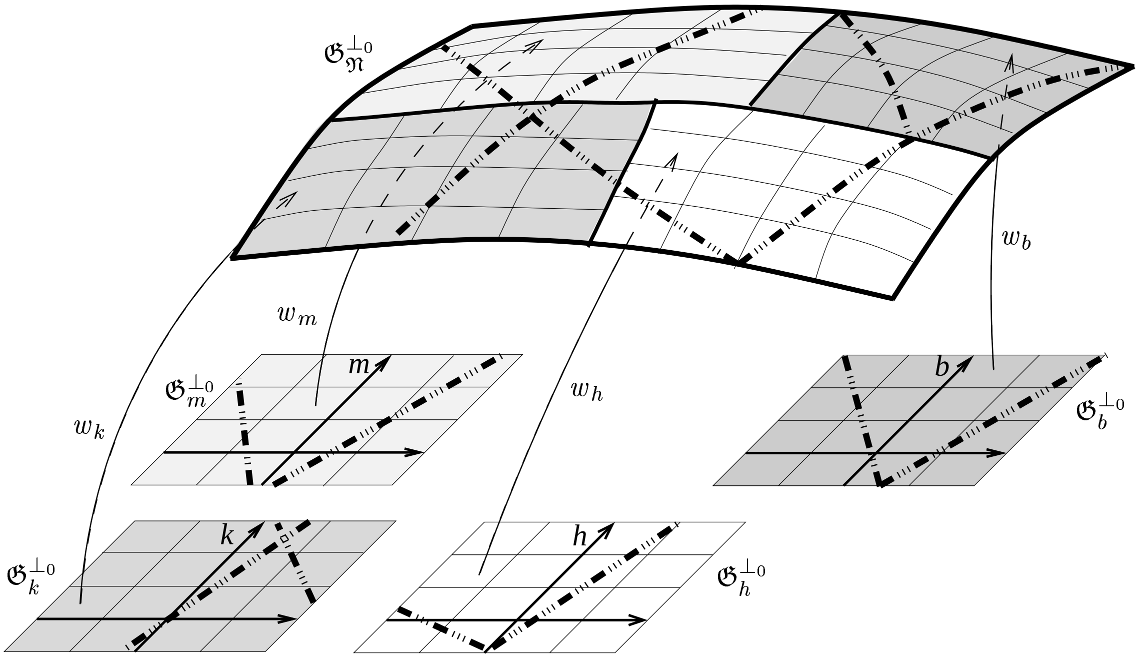


Figure 91: Notice that a "photon-line" splits up to two in the lower picture.

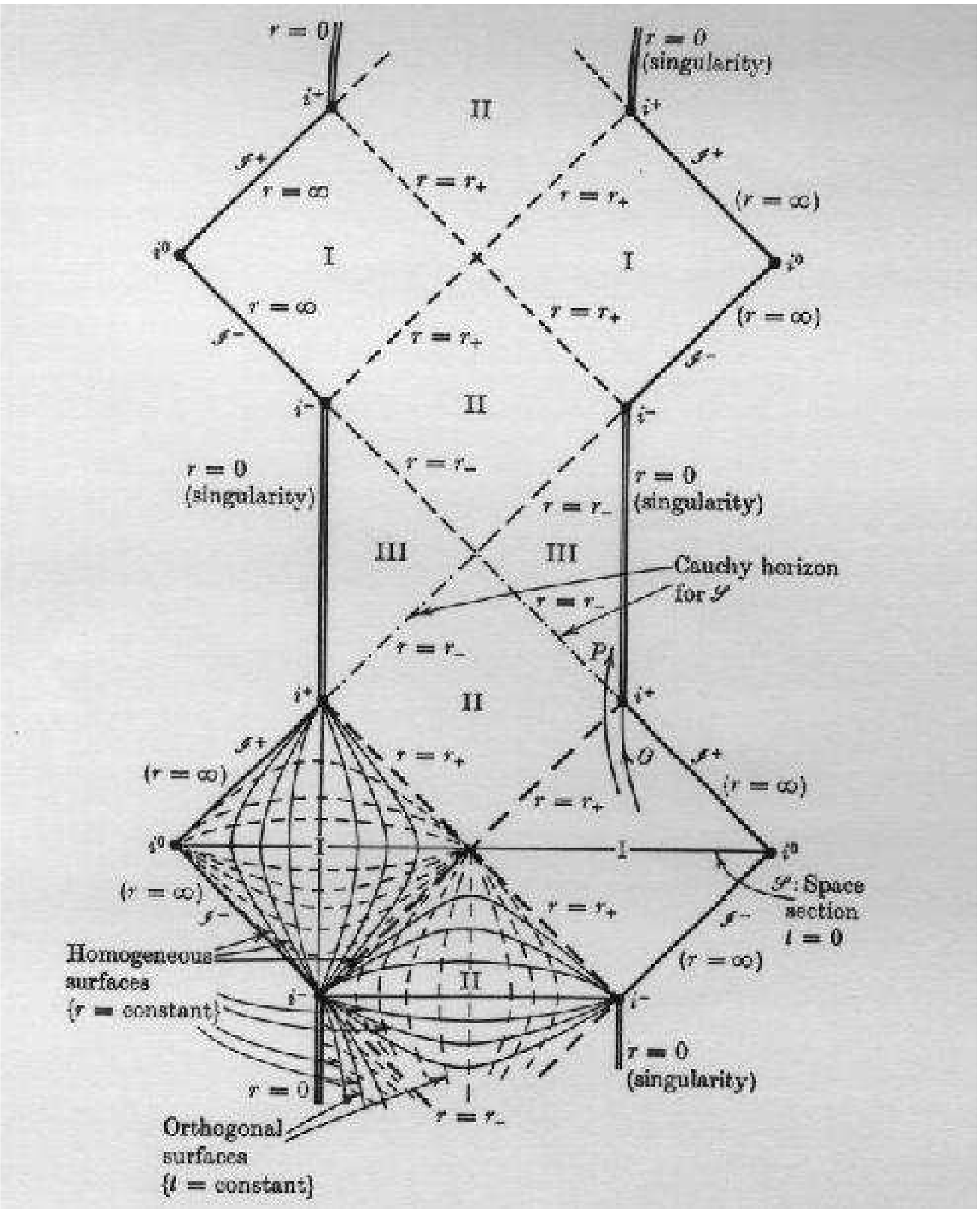


Figure 92: The geometry of a rotating black hole (general relativity) represented by a Penrose diagram.

see Figure 91.

(b) $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$ *is a photon-glued disjoint union of the family* $\langle\, \mathfrak{G}_{\mathfrak{N}}^{\perp_0} \restriction Rng(w_m) \;:\; m \in O\,\rangle$.
Therefore, by item 2b, $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$ *is a photon-glued disjoint union of the family* $\langle\, \mathfrak{G}_{m}^{\perp_0} : m \in O\,\rangle$ *up to isomorphism. See Figure 91.*

4. *(a)–(e) below hold.*

   (a) *Assume* **Ax(diswind)**. *Then two distinct lines meet in at most one point; formally:* $(\forall$ *distinct* $\ell, \ell' \in L)\, |\ell \cap \ell'| \le 1$.

   (b) *Assume we are given two distinct lines such that one of them is time-like or space-like. Then the two lines meet in at most one point. Formally:*

$$(\forall \textit{ distinct } \ell, \ell' \in L)\, (\ell \in L^T \cup L^S \;\Rightarrow\; |\ell \cap \ell'| \le 1).$$

   (c) $L^T \cap L^{Ph} = \emptyset$.

   (d) *Assume* $\mathsf{c}_m(d) < \infty$. *Then* $L^S \cap L^{Ph} = \emptyset$.

   (e) *Assume* $\mathbf{Ax}(\sqrt{\ }) + (\mathsf{c}_m(d) < \infty)$ *and* $(n > 2$ *or* $\mathbf{Ax}(\uparrow\uparrow_0))$. *Then i, ii below hold.*

      i. L^T, L^{Ph}, L^S *are pairwise disjoint.*

      ii. *The irreflexive parts of relations* $\equiv^T, \equiv^{Ph}, \equiv^S$ *are pairwise disjoint.*

5. *Assume* **Ax(diswind)**. *Let* $m \in Obs$ *and* $\ell, \ell' \in L$ *be such that* $w_m^{-1}[\ell] \neq \emptyset$ *and* $w_m^{-1}[\ell'] \neq \emptyset$. *Then* $w_m^{-1}[\ell], w_m^{-1}[\ell'] \in \mathsf{Eucl}$ *(by item 1a), and (a), (b) below hold.*

   (a) $\ell \parallel_{\mathfrak{G}} \ell' \quad\Longleftrightarrow\quad w_m^{-1}[\ell] \parallel w_m^{-1}[\ell']$.

   (b) *Assume* ℓ, ℓ' *are distinct and* $\ell \cap \ell' \neq \emptyset$. *Then*

$$Plane(\ell, \ell') = Plane'(\ell, \ell') = w_m[\mathsf{Plane}(w_m^{-1}[\ell], w_m^{-1}[\ell'])].$$

On the proof: The proof is left to the reader, but we note the following. Items 1b, 1c hold for arbitrary frame model, i.e. the assumption $\mathbf{Bax}^-$ is not needed in these items. The proof of the proposition is based on the following. Assume $\mathbf{Bax}^-$. Let $m, k \in Obs$. Then (i)-(vii) below hold.

(i) w_m is an injection.

(ii) Assume $m \overset{\odot}{\to} k$. Then f_{mk} is a bijective collineation by Thm.3.2.6 (p.110).

(iii) $(Rng(w_m) \cap Rng(w_k) = \emptyset \quad\text{or}\quad Rng(w_m) = Rng(w_k))$ and $(m \overset{\odot}{\to} k \quad\Leftrightarrow\quad Rng(w_m) = Rng(w_k))$. This holds by Thm.3.2.6 and **Ax4**.

(iv) Assume $\mathbf{Ax}(\sqrt{\ })$ and $m \overset{\odot}{\to} k$. Then f_{mk} is betweenness preserving by Prop.4.5.4(i) (p.289).

(v) There are no photons at rest by $\mathbf{AxE_{01}}$.

(vi) Assume **Ax(diswind)** and $Rng(w_m) \cap Rng(w_k) = \emptyset$. Assume ph is a photon such that $(\exists e \in Rng(w_m))\, ph \in e$. Then $(\forall e \in Rng(w_k))\, ph \notin e$.

(vii) Assume $\mathbf{Ax}(\sqrt{\ }) + (\mathsf{c}_m(d) < \infty)$ and $(n > 2$ or $\mathbf{Ax}(\uparrow\uparrow_0))$. Then there are no FTL observers by items 3.2.13 (p.118) and 4.2.31 (p.177) herein. ∎

In connection with the following remark recall that **Pax** is weaker than $\mathbf{Bax}^-$, cf. §3 herein or p.482 in §4.3 of AMN [18].

Remark 4.2.65 The following items remain true if the assumption $\mathbf{Bax}^-$ is replaced by **Pax** in them: Remark 4.2.13 (p.160), Prop.4.2.14 (p.160), Prop.4.2.16 (p.161), Prop.4.2.35 (p.179), Thm.4.2.40(ii) (p.182), and almost the whole of Prop.4.2.64, i.e. Prop.4.2.64 with the exception of items 1h, 4c, 4d, 4e.

$\triangleleft$

Remark 4.2.66 (On Figure 92) Figure 91 shows that our geometries $\mathfrak{G}_{\mathfrak{M}}$ can be viewed as being glued together from "windows" which in turn can be regarded as world-views of individual observers. There is a (deliberate) analogy here with the so-called Penrose diagrams from general relativity. (We will not explain Penrose diagrams here but certain properties are "visible" without explanation.) Figure 92 on p.212 represents a Penrose diagram (of a general relativistic space-time geometry) from Hawking-Ellis [116]. It is visible in Figure 92 that this geometry, too, consists of regions like our windows in Figure 91. (Cf. e.g. regions I, II. III on the diagram.) Roughly, each of these regions can be regarded as the window of some observer (just as in our Fig.91). Of course, besides the similarities there are some dissimilarities which we do not discuss here. We note that the fact that our geometries are glued together from windows is intended to make transitions towards general relativity easier. In passing we note that Figure 92 is the Penrose diagram of a rotating black hole which contains closed time-like geodesics ("time travel"). Therefore it is related to Figure 134 on p.365 which also contains closed time-like geodesics (among other exotic and exciting features). Figure 92 is in "Penrose-diagram form" while Figure 134 is in a more usual space-time diagram form.

$\triangleleft$

The next remark illustrates some properties of our geometries. Intuitively, it says that quite many of our flexible theories of relativity have an interesting geometrical property.

Remark 4.2.67 Let $Th = \mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\|)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6}$. This Th is strong enough to imply that "simultaneities" i.e. space-like hyper-planes[475] are Euclidean geometries. In more detail, assume that $\mathfrak{G} \in \mathsf{Ge}(Th)$ and that H is a space-like hyper-plane of $\mathfrak{G}$. Then the "subgeometry" $\mathfrak{G} \restriction H = \langle H, \mathbf{F_1}, \ldots, eq \restriction H, \ldots \rangle$ [476] of $\mathfrak{G}$ (obtained by restricting the points of $\mathfrak{G}$ to H the natural way) is Euclidean, if $n > 2$. This is proved as Thm.'s 6.6.114–6.6.115 of AMN [18, pp.1130-1131].[477] If we drop **Ax6**, then, roughly, we get disjoint unions of Euclidean

[475] For the definition of space-like hyper-planes we refer to p.1130, Def.6.6.112 in AMN [18].

[476] where $L, Bw, \ldots, \mathcal{T}$ are restricted to H the natural way

[477] In passing we note that general relativistic geometries usually fail to have this property (i.e. "pure" space is already curved). However, the simplified black hole geometry in Andréka et al. [26] enjoys this property at least for some (kinds of) space-like hyper-planes (more precisely space-like "geodesic hyper-surfaces"). Besides being Euclidean, these space-like hyper-surfaces are disjoint from each other, and their union covers the whole of space-time. The same applies to the simplified black hole geometry in Rindler [222] on p.124 given by equation (7.28) describing the metric of the manifold in question. (The two simplified geometries, in [26] and in [222], are obtained via different trains of thought.) As a curiosity we note that one of the main features of the model constructed in Gödel's cosmological papers (and refined in Ozsváth-Schücking [209] for a finite universe) is that the whole of space-time of that model cannot be obtained as a disjoint union of space-like geodesic hyper-surfaces. Such a disjoint union of space-like geodesic hyper-surfaces could be regarded as a kind of "absolute" (even if artificial) temporal structure for the whole universe. We note that universes with rotating black holes have the same "Gödelian" property. (When we write "universe" e.g. in connection with the works of Gödel, Ozsváth etc. we mean a mathematical structure which is in many respects similar to our $\mathfrak{G}$'s but in which

geometries. We did not check what happens if $\mathbf{Bax}^{\oplus}$ is replaced by other weak theories like $\mathbf{Bax}^{-}$ or **Reich**(**Bax**) or **Bax**. But we tend to conjecture that the answer will be negative in the case of **Loc**(**Basax**) substituted to the place of $\mathbf{Bax}^{\oplus}$. Related kinds of results are in items 6.6.111-6.6.118 (pp.1129-1132) in AMN [18].

The machinery for studying $\mathfrak{G}_{\mathfrak{M}}$ developed so far is strong enough for developing a purely algebraic (or model theoretic) characterization of our symmetry axioms (e.g. the ones in §2.8, or $\mathbf{Ax}(\omega)^{0}$ on p.180). We present this characterization in AMN [18, §6.2.8].

4.2.6 Some reducts of our relativistic geometries; connections with the literature

The reader might feel that the geometric object $\mathfrak{G}_{\mathfrak{M}}$ defined in Definition 4.2.3 (pp. 137–146) seems to have too many components. However, we will concentrate on discussing *reducts* of $\mathfrak{G}_{\mathfrak{M}}$ instead of the full structure.

A very nicely streamlined reduct is called the time-like-metric reduct which will be introduced and discussed in §4.6.1 (p.346). About that reduct we note that it is not only mathematically elegant, but also is most useful e.g. can be generalized smoothly such that it becomes a suitable framework for a possible formalization of the basics of general relativity, cf. Busemann [55]. All the same, below we start our discussion with a more “classical”, more “Euclidean” reduct (of the incidence geometry kind).[478]

(1) Perhaps the most well known reduct of $\mathfrak{G}_{\mathfrak{M}}$ is

$$GT_{\mathfrak{M}} := \langle Mn, L; \in, Bw, \perp, eq\rangle$$

which we call the *Goldblatt-Tarski reduct* of $\mathfrak{G}_{\mathfrak{M}}$. This is a geometry of the form

$$\langle Points, Lines; \in, Bw, \perp, eq\rangle.$$

Tarski's axiomatic approach to Euclidean geometries over ordered fields $\mathfrak{F}$, basically, studies structures of this form:

$$\langle Points, Lines; \in, Bw, \perp, eq\rangle.$$

More precisely, there, the first part $\langle Points, Lines; \in\rangle$ is coded up into a one sorted structure[479] $\langle Points, collinear(x, y, z)\rangle$. But as it will be discussed in §4.4 (p.274) below, this causes no

geodesics discussed in §4.7 way below play a dominant role.) Cf. footnote 270 on p.130 for references etc. See Figure 134 on p.365 for an intuitive picture of Gödel's (cosmological) model. What we called above a disjoint union of space-like hyper-surfaces is called a foliation of space-time (or universe) in e.g. Marsden-Tipler [184]. Cf. also p.627 lines 18-17 bottom up in Barrow-Tipler [42] with an introduction by J. A. Wheeler. (We note that, intuitively, this “Gödelian property” prevents the implementation of global Laplacian determinism, cf. e.g. Earman [77, p.44]. Very roughly, the above outlined properties can be summarized by saying that the manifold in question does not contain a so-called Cauchy-surface Σ in the sense of e.g. Hawking-Ellis [116, p.205]. Such a Σ is a space-like hyper-surface with certain conditions.)

[478] Our excuse for starting with this reduct is that, stretching it a little bit, one could say that it was known already by the ancient Greeks.

[479] To be precise, Tarski uses $\langle Points; Bw\rangle$ to “code” $\langle Points; collinear\rangle$. Hence under very mild assumptions the geometries of form $\langle Points; Bw\rangle$ are definitionally equivalent to the geometries of form $\langle Points; collinear, Bw\rangle$. The latter version is used extensively in the literature.

essential difference. Actually, Tarski omitted $\perp$ because it is definable from Bw and eq, and Goldblatt in [102] did not include eq probably because it is definable from the rest of $GT_{\mathfrak{M}}$ in the cases of Minkowskian and Euclidean geometries. From now on we will ignore the fact that Tarski and Goldblatt omitted $\perp$ and eq, respectively.[480]

As we will recall, Hilbert, Tarski and their followers proved that for the Euclidean case the language of $\langle Points, Lines; \in, Bw, \perp, eq\rangle$ is expressive enough in the sense that all familiar concepts of classical geometry, e.g. circles can be defined in the *first-order language* of these structures $\langle Points, \dots, eq\rangle$.[481]

If we assume some conditions on $\mathfrak{M}$, then instead of the $\langle \prec, g, \mathfrak{F}\rangle$-free reduct of $\mathfrak{G}_{\mathfrak{M}}$ it is enough to keep $GT_{\mathfrak{M}}$ because of the following.

THEOREM 4.2.68 *The incidence geometry expanded with $\perp$, $\mathsf{G}_{\mathfrak{M}} = \langle Mn, L; \in, \perp\rangle$ is expressive enough to recover most of the g-free part of $\mathfrak{G}_{\mathfrak{M}}$, under* $\mathbf{Bax}^{\oplus}$ + *auxiliary axioms. More concretely, L^T, L^{Ph}, L^S, Bw are definable in first-order logic over $\mathsf{G}_{\mathfrak{M}}$, assuming $n > 2$ and $\mathfrak{M} \models (\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathbf{diswind}))$.*

Idea of proof: Let $\ell \in L$. Then $\ell \in L^{Ph}$ iff $\ell \perp \ell$. Further $\ell \in L^T$ iff

$$(\forall \text{ 2-dimensional plane}^{482}\ P)$$
$$[\ell \subseteq P \text{ there is a photon line in } P \text{ intersecting } \ell \text{ in a single point}].$$

Of course, one has to prove that these definitions work. The details are available from the author. Definability of Bw follows by Thm.6.7.1 (p.1137) of AMN [18]. ∎

COROLLARY 4.2.69 The $\prec, g$-free reduct of $\mathfrak{G}_{\mathfrak{M}}$ is first-order logic *definable*[483] *over* the Goldblatt-Tarski reduct $GT_{\mathfrak{M}} = \langle Mn, L; \in, \perp, eq\rangle$, under the conditions of Thm.s 4.2.37, 4.2.68 above.

Proof. The topology part follows from the fact that $\mathcal{T}'$ is defined from Bw on p.175. ∎

We note that eq is not definable from $\mathsf{G}_{\mathfrak{M}}$ under the conditions of Thm.4.2.68 (cf. AMN [18, p.1147]). However, if we are willing to assume **Newbasax**, then the situation improves as follows.

THEOREM 4.2.70 *The $\langle \prec, g\rangle$-free reduct of $\mathfrak{G}_{\mathfrak{M}}$ is first-order definable over the "slim" geometry $\langle Mn, \perp\rangle$, hence also over $\mathsf{G}_{\mathfrak{M}}$, assuming $n > 2$ and $\mathfrak{M} \models$* $\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathbf{diswind})$.

[480] For some of our choices of $\mathfrak{M}$, eq is not definable from the rest of $GT_{\mathfrak{M}}$ and $\perp$ is not definable from the rest of $GT_{\mathfrak{M}}$ either. Cf. §4.6.

[481] A difference between Hilbert's and Tarski's approach to axiomatizing geometry is that Tarski insisted on using *purely first-order* logic, and to consider all the models (in the model theoretic sense) of his first-order axioms. (Hilbert used a second-order axiom besides first-order ones.) The approach to studying geometries over arbitrary Euclidean fields was started well before Hilbert's and Tarski's work. Referring to so many people would render our present discussion a little cumbersome. Therefore, instead of writing "Hilbert's, Tarski's, their precursor's and their follower's work" we will simply write Tarski's work or something similar. This is *only* for simplicity and by this we do not want to belittle the importance of Hilbert's, their precursor's and their follower's work. An incomplete list of references includes e.g. [62, 102, 122, 125, 127, 225, 232, 245, 247, 248, 252, 255]. We refer to Appendix ("Why first-order logic?") of AMN [18] for more information, as well as for an explanation of why it is more useful to axiomatize something in first-order logic than in second-order logic.

[482] The notion of a 2-dimensional plane is defined as follows. P is a 2-dimensional plane iff there are distinct $a, b, c \in Mn$ such that they are pairwise connected (i.e. $\sim$-related), $\neg coll(a, b, c)$, and $P = Plane(\{a, b, c\})$.

[483] in the sense elaborated in §4.3 below

Proof. The proof follows from AMN [18, items 6.7.31,6.7.41]. ∎

By items 4.2.68–4.2.70 above, studying $\mathfrak{G}_{\mathfrak{M}}$ can be reduced to studying the sleek geometry $\mathsf{G}_{\mathfrak{M}}$ if we are willing to ignore $\prec, g$ and assume $\mathbf{Bax}^{\oplus}+$ *"some extra"*. Actually, $\mathsf{G}_{\mathfrak{M}}$ is expressive enough to support our $(\mathcal{G}o, \mathcal{M}o)$-duality theory, under **Newbasax** + *auxiliaries*, or if we are willing to assume only $\mathbf{Bax}^{\oplus}$ + *auxiliaries*, then $\langle \mathsf{G}_{\mathfrak{M}}, eq\rangle$ can support our duality theory. So, much what we do in this chapter can be done by using only $GT_{\mathfrak{M}}$ in place of $\mathfrak{G}_{\mathfrak{M}}$, if we are willing to assume $\mathbf{Bax}^{\oplus}$ + *auxiliaries.* This is part of the fulfillment of our promise made above the definition of $\mathfrak{G}_{\mathfrak{M}}$ (first few lines of §4.2.1, p.136). The rest of the promise will come in §4.6.1 (p.346). Cf. also Theorem 6.7.42 of AMN [18] (as a contrast).

We will discuss the remaining interdefinability connections between the basic relations $L^T, L^{Ph}, \ldots, Bw, \perp, eq, g$ of our language for geometries in the section "On the choice of our geometrical vocabulary (or language)" (pp. 342–349) and much more fully in AMN [18, §4.6, pp.342–349].

In passing we note that there is a very natural motivation for Tarski's choice of the primitives of his geometries.[484] Let us pretend for a second that Tarski's geometries are of the form $\langle Points, collinear, eq\rangle$.[485] This choice of primitives matches nicely the traditional "ruler and compass" conception of Euclidean geometry (cf. e.g. Lánczos [151, p.48], namely, *collinear* corresponds to *ruler* (lines) while *eq* corresponds to *compass* (circles).[486] Let *Col* abbreviate *collinear*, cf. p.277.

By the above, it is natural to consider the $\langle Points, Col, eq\rangle$ reducts of our geometries as a distinguished level of abstraction. Since in our relativistic geometries *Bw* and $\perp$ are not as easily definable in terms of *Col* and *eq* as in the Euclidean case (cf. AMN [18, §4.6]), we will often consider the $\langle Points, Col, Bw, \perp, eq\rangle$ reduct as a natural level of abstraction. We will see in our definability section (§4.3) that this level of abstraction is "equivalent" to the level represented in the Goldblatt-Tarski reducts $\langle Mn, L; \in, Bw, \perp, eq\rangle$ of our geometries discussed above.

Further *connections with the literature* are discussed in §§4.6.1, 4.4, 4.5.2, and in AMN [18, §6.5 ("... connections with Tarski's ...") (p.991), §6.2.9 (p.923), §6.7.3 (p.1169]), to mention only a few such places.

---

[484] The same argument motivates our distinguishing the Goldblatt-Tarski reduct of our relativistic geometries.

[485] Tarski uses *Bw* in place of *collinear*, but we will see later (cf. AMN [18, §6.7.1]) that in the presence of *eq*, *Bw* and *collinear* are interdefinable, hence for the sake of the argument, here we may assume that the primitives are *collinear* and *eq*.

[486] Instead of "ruler and compass based conception" we could say "points, lines and circles based conception".

4.3 Definability in many-sorted logic, defining new sorts[487]

On historical background:

The theory of definability as understood in the present work is a branch of mathematical logic (and its model theory) which goes back to Tarski's pioneering work [250]. It goes back even further, to Padoa (1900), Reichenbach 1920-23 [218], and Tarski 1926. Beginning with 1926, 1931, 1934 [250], Tarski did much to help the theory of definability to become a fully developed branch of mathematical logic which is worth studying in its own right. Of the many works illustrating Tarski's concern for the theory of definability we mention only Henkin-Tarski [123], [120, Part I], Tarski-Givant [254], Tarski-Mostowski-Robinson [256] and Tarski [250, 251], cf. also Tarski [249] and [253, Volume 1, pp. 517-548] (which first appeared in 1931 and which already addresses the theory of definability).

In passing we note that the creation of the theory of cylindric algebras can be viewed as a by-product of Tarski's interest in developing and publicizing the theory of definitions (a cylindric algebra over a model can be viewed as the collection of all relations definable in that model).

Below, we try to summarize the theory of definability (allowing definitions of new sorts) in a style tailored to the needs of the present work *and* in a spirit consistent with Tarski's original ideas and views on the subject. Here the emphasis will be on defining new sorts (which is usually not addressed in classical logic books such as e.g. Chang-Keisler [60]).

The subject matter of the present sub-section is relevant to the definability issues discussed in the literature on relativity cf. e.g. Friedman [91, pp. 62–63, 65, 378 (index)]. In Reichenbach's book "Axiomatization of the Theory of Relativity" [218] on the first page of the Introduction (p.3) he already explains the difference between explicit and implicit definitions and emphasizes their importance. (He also traces this distinction (underlying definability theory) to Hilbert's works.) In passing we note that on p.5, Reichenbach [218] also explains in considerable detail why it is desirable to start out with observational concepts when building up our theory (like we do in Chapters 1,2) and later define theoretical concepts over observational ones using definability theory (as we do in the present chapter). For the time being we do not discuss connections between definability theory and definability issues in relativity theory explicitly, but we plan to do so in a later work.[488]

For the physical importance of definability cf. the relevant parts of the introduction of this chapter. Further, we note the following. If in our language we allow using certain concepts and if some other concept is definable from these, this other concept *is* available in our language even if we do not include it (explicitly). So if we allow only such concepts which are definable from observational ones, the effect will be the same as if we allowed only observational concepts. I.e. the physical principle of Occam's razor has been respected.

* * *

[487] Acknowledgement: We would like to express our thanks to Wilfrid Hodges for a careful reading of an earlier version of this section, and for his helpful remarks.

[488] But we note that Reichenbach [218] makes it clear that he considers definability theory very important for relativity, and he also explains rather convincingly why he does so. This is also clear from the relativity works Friedman [91], or Grünbaum [109],[110], to mention only a few. Cf. also Stein [238].

Let $\mathfrak{M} = \langle U_0, \dots, U_j;\ R_1, \dots, R_l \rangle$ be a many-sorted model with universes or sorts $U_0, \dots, U_j$, and relations $R_1, \dots, R_l \quad (j, l \in \omega)$.[489] Since functions are special relations we do not indicate them explicitly in the present discussion.[490] We use the semicolon ";" to separate the sorts (or universes) from the relations of $\mathfrak{M}$.

When discussing many-sorted models, we always assume that they have *finitely many sorts* only.[491] The "big universe" $Uv(\mathfrak{M})$ of the model $\mathfrak{M}$ is the union of its universes (or sorts). Formally

$$Uv \stackrel{\text{def}}{:=} Uv(\mathfrak{M}) \stackrel{\text{def}}{:=} \bigcup \{ U_i \ :\ U_i \text{ is a universe of } \mathfrak{M} \}.^{492}$$

In passing we note that although the sorts $U_0, \dots, U_j$ of $\mathfrak{M}$ need not be disjoint, the following holds. To every many-sorted model $\mathfrak{M}$ there is an isomorphic copy $\mathfrak{M}'$ of $\mathfrak{M}$ such that the sorts $U'_0, \dots, U'_j$ of $\mathfrak{M}'$ *are* mutually disjoint (i.e. $U'_0 \cap U'_1 = \emptyset$ etc.). Therefore we *are permitted to* pretend that the sorts (i.e. universes) of $\mathfrak{M}$ are disjoint from each other whenever we would need this.

By a *reduct* of a many-sorted model $\mathfrak{M}$ we understand a model $\mathfrak{M}^-$ obtained from $\mathfrak{M}$ by omitting some of the sorts and/or some of the relations of $\mathfrak{M}$. I.e. if

$$\mathfrak{M} = \langle U_0, \dots, U_j;\ R_1, \dots, R_l \rangle$$

then the reduct $\mathfrak{M}^-$ may be of the form

$$\langle U_0, \dots, U_{j-1};\ R_1, \dots, R_{l-1} \rangle$$

(assuming $R_1, \dots, R_{l-1}$ do not involve the sort U_j).

A model $\mathfrak{M}^+$ is called an *expansion* of $\mathfrak{M}$ iff $\mathfrak{M}$ is a reduct of $\mathfrak{M}^+$. I.e. an expansion $\mathfrak{M}^+$ is obtained by adding new sorts and/or new relations to $\mathfrak{M}$. We will use the following abbreviation for denoting expansions:

$$\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new};\ \bar{R}^{new} \rangle$$

where U^{new} is the *new sort* and $\bar{R}^{new} = \langle R_1^{new}, \dots, R_r^{new} \rangle$ is the sequence of *new relations*. Of course there may be more new sorts too, then we write

$$\mathfrak{M}^+ = \langle \mathfrak{M}, U_1^{new}, \dots, U_\varrho^{new};\ \bar{R}^{new} \rangle.$$

[489] The assumption that l is finite is irrelevant here in the sense that we will never make use of it (except when we state this explicitly). What we write in this section makes perfect sense if the reader replaces l by an arbitrary ordinal. As a contrast, we do use the assumption that $j \in \omega$.

[490] All the same, we will not hesitate to use functions because it is well known how to eliminate function symbols in first-order logic (without changing the meanings of our formulas). Cf. Bell-Machover [46, §2.10 "elimination of function symbols"].

[491] In some minor items there may be exceptions from this rule but then this will be clearly indicated.

[492] Although, in general, Uv is not a universe of $\mathfrak{M}$, we can *pretend* that it is a universe because there are only finitely many sorts. E.g. if we want to simulate the formula $(\exists x \in Uv)\,\psi(x)$ then we write $[\,(\exists x \in U_0)\,\psi(x)\ \vee\ (\exists x \in U_1)\,\psi(x)\ \vee\ \dots \vee\ (\exists x \in U_j)\,\psi(x)\,]$. Then although the first formula $(\exists x \in Uv)\,\psi(x)$ does not belong to the language of $\mathfrak{M}$, the second formula "$[\,(\exists x \in U_0) \dots]$" does belong to this language (assuming $(\exists x \in U_i)\,\psi(x)$ already belongs to the language) and the meaning of the second formula is the same as the intuitive meaning of the first one. If $(\exists x \in U_i)\,\psi(x)$ did still not belong to our many-sorted language then there is some extra routine work to do in translating this formula into our many-sorted language. This translation is explained in detail in the logic books which reduce many-sorted logic to one-sorted logic (cf. [45, 82, 194]). These books were quoted in §2 where we first encountered many-sorted logic. We also note that the quoted translation is straightforward. For more on why and how we can pretend that $Uv(\mathfrak{M})$ is a universe of $\mathfrak{M}$ we refer to the just quoted logic books.

However, we will concentrate on the case $\varrho = 1$ (for didactical reasons). Informally the general pattern is:

$$\text{“New model”} = \langle \text{“Old model”},\ \text{“New sorts”};\ \text{“New relations/functions”} \rangle.$$

We will ask ourselves when $\mathfrak{M}^+$ will be *(first-order logic) definable* over[493] $\mathfrak{M}$. By *definable* we will *always* (throughout this work) mean first-order logic definable. If $\langle \mathfrak{M}, U^{new}; \bar{R}^{new} \rangle$ is definable over $\mathfrak{M}$ then we will say that the new sort U^{new} together with $\bar{R}^{new}$ are *definable* in $\mathfrak{M}$. When defining a new sort U^{new} (in an “old” model $\mathfrak{M}$) we need the new relations $\bar{R}^{new}$ too because it is $\bar{R}^{new}$ which will specify the connections between the new sort U^{new} and the old sorts of $\mathfrak{M}$.

Although we will start out with discussing definability over a single model $\mathfrak{M}$, the really important part will be when we generalize this to definability (of an expanded class K^+) *over a class* K of models (which is first-order axiomatizable).

We will discuss *two kinds* of definability in many-sorted logic: *implicit* definability in §4.3.1 and *explicit* definability in §4.3.2.[494]

Throughout model theory there is a *distinction* between symbols like Obs and objects like $Obs^{\mathfrak{M}}$ denoted by these symbols in a model $\mathfrak{M}$. This distinction between symbols and objects they denote is even more important in the theory of definitions than in other parts of logic. Therefore, in the next two items we clarify notions and notation connected to this distinction.

CONVENTION 4.3.1 By the *vocabulary* of a model $\mathfrak{M}$ we understand the system of sort-symbols, relation symbols and function symbols interpreted by $\mathfrak{M}$. Since function symbols are special relation symbols, we will restrict our attention to sort symbols and relation symbols. Assume e.g. that $\mathfrak{M}$ is of the form

$$\mathfrak{M} = \langle U_0^{\mathfrak{M}}, \ldots, U_j^{\mathfrak{M}}; R_1^{\mathfrak{M}}, \ldots, R_l^{\mathfrak{M}} \rangle ,$$

and assume that U_i is the sort *symbol* “denoting” $U_i^{\mathfrak{M}}$ and R_i is the relation *symbol* “denoting” $R_i^{\mathfrak{M}}$. Then the vocabulary of $\mathfrak{M}$ is

$$Voc(\mathfrak{M}) \stackrel{\text{def}}{=} \langle \{U_0, \ldots, U_j\}, \{R_1, \ldots, R_l\} \rangle.$$

Throughout we assume that a relation symbol R' contains the extra information which we call the *rank* of R'. This can be implemented by postulating that R' is an ordered pair $R' = \langle R'_0, R'_1 \rangle$ where R'_0 is the symbol we write on paper while R'_1 is the rank of R'. E.g. in the case of the usual model $\mathfrak{N} = \langle \omega, \leq, + \rangle$ the rank of “$\leq$” is 2 while that of “$+$” is 3. If there is more than one sort, then the rank of a relation is a sequence of sort symbols. So, a vocabulary is an ordered pair

$$Voc = \langle \text{“Sort symbols”}, \text{“Relation symbols”} \rangle$$

where “Sort symbols” and “Relation symbols” are two sets as discussed above subject to the condition that the sorts occurring in the ranks of the relation symbols all occur in the set of sort symbols. Now, a model $\mathfrak{M}$ of vocabulary Voc can be regarded as a pair $\mathfrak{M} = \langle \mathfrak{M}_0, \mathfrak{M}_1 \rangle$ of functions such that

$$\mathfrak{M}_0 : \text{“Sort symbols”} \longrightarrow \text{“Universes of } \mathfrak{M}\text{”}$$

[493] “Definable over” is the same as “definable in”.

[494] In passing, we note that in the *special* case of the most traditional one-sorted logic when *only* relations are defined (i.e. defining new sorts is not considered) the distinction between implicit and explicit definability is well investigated and is well understood cf. e.g. Chang-Keisler [60, p.90] or Hodges [130, pp.301-302].

and

$$\mathfrak{M}_1 : \text{“Relation symbols”} \longrightarrow \text{“Relations of } \mathfrak{M}\text{”},$$

with the restriction that $\mathfrak{M}_1$ is “rank-preserving” in a natural sense.
E.g. if $\mathfrak{M} = \langle U_0^{\mathfrak{M}}, \ldots, U_j^{\mathfrak{M}}; R_1^{\mathfrak{M}}, \ldots, R_l^{\mathfrak{M}} \rangle$, then

$$\mathfrak{M}_0 : \{U_i : i \leq j\} \longrightarrow \{U_i^{\mathfrak{M}} : i \leq j\}$$

$$\mathfrak{M}_1 : \{R_i : 0 < i \leq l\} \longrightarrow \{R_i^{\mathfrak{M}} : 0 < i \leq l\}.$$

I.e. with each sort symbol in $Voc(\mathfrak{M})$, $\mathfrak{M}$ associates a universe (i.e. a set) and with each relation symbol R' in $Voc(\mathfrak{M})$, $\mathfrak{M}$ associates a relation (of rank R'_1 as indicated way above).

We call two models $\mathfrak{M}$ and $\mathfrak{N}$ <u>*similar*</u> if they have the same vocabulary, i.e. if $Voc(\mathfrak{M}) = Voc(\mathfrak{N})$.

Let Voc', Voc be two vocabularies. We say that Voc' is a <u>*sub-vocabulary*</u> of Voc, in symbols $Voc' \subseteq Voc$, if the natural conditions $Voc'_0 \subseteq Voc_0$ and $Voc'_1 \subseteq Voc_1$ hold. Assume Voc' is a sub-vocabulary of $Voc(\mathfrak{M})$ for a model $\mathfrak{M}$. Then the <u>*reduct*</u> $\mathfrak{M} \restriction Voc'$ of $\mathfrak{M}$ to the sub-vocabulary Voc' is defined as

$$\mathfrak{M} \restriction Voc' \stackrel{\text{def}}{=} \langle \mathfrak{M}_0 \restriction Voc'_0, \quad \mathfrak{M}_1 \restriction Voc'_1 \rangle.$$

Let $\mathfrak{M} = \langle U_0, \ldots, U_j; R_1, \ldots, R_l \rangle$ and $\mathfrak{M}' = \langle U'_0, \ldots, U'_j; R'_1, \ldots, R'_l \rangle$ be similar models. By a <u>*homomorphism*</u> h between $\mathfrak{M}$ and $\mathfrak{M}'$, in symbols $h : \mathfrak{M} \longrightarrow \mathfrak{M}'$, we mean a system $\langle h_i : 0 \leq i \leq j \rangle$ of mappings $h_i : U_i \longrightarrow U'_i$ such that for all $1 \leq k \leq l$, if R_k is of sort $\langle U_{i_1}, \ldots, U_{i_n} \rangle$, then we have

(*) $R(u_1, \ldots, u_n) \;\Rightarrow\; R'(h_1(u_1), \ldots, h_n(u_n))$

for all $u_1, \ldots, u_n \in Uv(\mathfrak{M})$. We call h <u>*one-to-one*</u> or injective (<u>*onto*</u>, or surjective) if all the h_i's are one-to-one (onto), and the <u>*inverse*</u> of h is $\langle h_i^{-1} : 0 \leq i \leq j \rangle$. As usual, an <u>*isomorphism*</u> between similar many-sorted structures is a one-to-one and onto homomorphism whose inverse is also a homomorphism. Recall that functions f_i and constants c_i are regarded as special relations, hence (*) above also applies to them.[495]

$\lhd$

Remark 4.3.2 (On the intuitive content of Convention 4.3.1 above) On a very intuitive informal level, one can think of a model $\mathfrak{M}$ as a <u>*function*</u> associating objects with symbols. E.g. $\mathfrak{M}$ associates $U_i^{\mathfrak{M}}$ with the symbol U_i and $R_i^{\mathfrak{M}}$ to R_i. It is then a matter of notational convention that we write $U_i^{\mathfrak{M}}$ for the value $\mathfrak{M}(U_i)$ and $R_i^{\mathfrak{M}}$ for $\mathfrak{M}(R_i)$. Then the domain of the function $\mathfrak{M}$ is the collection of those symbols which $\mathfrak{M}$ can interpret. Hence, the <u>*domain*</u> of $\mathfrak{M}$ is the same thing as its vocabulary.

If the best way (from the intuitive point of view) of thinking about a model is regarding it as a function, then why did we formalize the notion of a model as a pair of functions (instead of a single function)? The answer is that *formally* it is easier to handle models as pairs of functions, but *intuitively* we think of models as functions, we think of vocabularies as domains of these functions and we consider two models similar if they have the same domain when they are regarded as functions.[496]

$\lhd$

[495] And it requires e.g. $h_2(f(u)) = f'(h_1(u))$ if f is a unary function of sort $\langle U_1, U_2 \rangle$ and $h_i(c) = c'$ if c is a constant of sort U_i.

[496] We do not claim that it is always the case that the best way of thinking about models is regarding them as functions. What we claim is that in many situations, e.g. in definability theory, this is a rather good way. In other situations it might be better to visualize a model as a set of objects equipped with some relations and functions.

CONVENTION 4.3.3 Throughout, by a class K of models we understand a *class of similar models*, i.e. we always assume $(\forall \mathfrak{M}, \mathfrak{N} \in \mathsf{K})\ Voc(\mathfrak{M}) = Voc(\mathfrak{N})$. For any class K of similar models, $Voc(\mathsf{K}) = Voc\mathsf{K}$ denotes the vocabulary of K, that is, the vocabulary of an arbitrary element of K.

A *reduct* K^- of K is obtained from K by omitting a part of the vocabulary of K, i.e. K^- is a reduct of K iff $Voc(\mathsf{K}^-) \subseteq Voc(\mathsf{K})$ and

$$\mathsf{K}^- = \{\, \mathfrak{M} \restriction Voc(\mathsf{K}^-) \ :\ \mathfrak{M} \in \mathsf{K} \,\}.$$

Expansion is the opposite of reduct. K^+ is an *expansion* of the class K iff K is a reduct of K^+, i.e. K^+ is an expansion of K iff $Voc(\mathsf{K}^+) \supseteq Voc(\mathsf{K})$ and

$$\mathsf{K} = \{\, \mathfrak{M} \restriction Voc(\mathsf{K}) \ :\ \mathfrak{M} \in \mathsf{K}^+ \,\}.$$

Note that forming expansions or reducts of a class K is somehow *uniform* over the members of K. E.g. we forget the *same* symbols (relation symbols or sort symbols) from all models $\mathfrak{M} \in \mathsf{K}$, when taking a reduct of K.

If Voc is a vocabulary with $Voc \subseteq Voc(\mathsf{K})$, then we use the following abbreviation:

$$\mathsf{K} \restriction Voc \stackrel{\text{def}}{=} \{\mathfrak{M} \restriction Voc \ :\ \mathfrak{M} \in \mathsf{K}\}.$$

Examples: $\mathsf{FM}^- = \{\, \mathfrak{F}^{\mathfrak{M}} \ :\ \mathfrak{M} \in \mathsf{FM} \,\}$ is a reduct of our class FM of frame models. Let $\mathsf{L} = \{\, \mathbf{F} \ :\ \mathbf{F} \text{ is a field} \,\}$. Then $\{\, \langle F; + \rangle \ :\ \langle F; +, \cdot, 0, 1 \rangle \in \mathsf{L} \,\}$ is a reduct of L.

Intuitively, we think of $Voc(\mathsf{K})$ as a set of symbols where each symbol contains information about its nature, i.e. about whether it is a sort symbol or a relation symbol of a certain rank. Therefore, we will write $Voc \cap Voc'$ for $\langle Voc_0 \cap Voc'_0, Voc_1 \cap Voc'_1 \rangle$, similarly for $Voc \cup Voc'$, for $Voc \subseteq Voc'$ etc.

$\lhd$

Before getting started, we emphasize that in order to define something over a model $\mathfrak{M}$ or over a class K of models, first of all we need *new symbols* R_i^{new}, U_i^{new} (with i in some index set) not occurring in the language of $\mathfrak{M}$ or of K. (The new symbols may be relation symbols like R_i^{new} or sort symbols U_i^{new} or both.) What we will define then (using definability theory) will be the meanings of the new symbols in $\mathfrak{M}^+$ or K^+. Most of the time we will not talk about the new *symbols* like R_i^{new} because we will identify them with the new relations like $(R_i^{new})^{\mathfrak{M}^+}$ which they denote in the expansion $\mathfrak{M}^+$ of the model $\mathfrak{M}$. Our reason for identifying the "symbol" with the "object" it denotes is to simplify the discussion. However, occasionally it will be useful to remember that an expansion $\mathfrak{M}^+ = \langle \mathfrak{M}, R \rangle$ of a model $\mathfrak{M}$ involves two new things not available in $\mathfrak{M}$, namely: a relation symbol and a relation denoted by this symbol (in $\mathfrak{M}^+$).

4.3.1 Implicit definability in many-sorted (first-order) logic

Let $\mathfrak{M}$ be a many-sorted model. Assume $\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new};\ \bar{R}^{new} \rangle$ is an expansion of $\mathfrak{M}$. We say that $\mathfrak{M}^+$ is *definable implicitly up to isomorphism* over $\mathfrak{M}$ iff

for any model
$$\langle \mathfrak{M}, U';\ \bar{R}' \rangle \models \mathsf{Th}(\mathfrak{M}^+)$$
(expanding $\mathfrak{M}$) there is an isomorphism

($\star$)
$$h : \mathfrak{M}^+ \rightarrowtail\!\!\twoheadrightarrow \langle \mathfrak{M}, U';\ \bar{R}' \rangle$$
such that h is the *identity* function on the sorts of $\mathfrak{M}$ (i.e. for each sort U_i of $\mathfrak{M}$ we have $h \restriction U_i = \mathrm{Id} \restriction U_i$).

$\mathfrak{M}^+$ is said to be *definable implicitly without taking reducts* over $\mathfrak{M}$ iff *in addition* to the above the isomorphism h mentioned above is *unique.*

We say that *U^{new}, $\bar{R}^{new}$ are definable implicitly* over $\mathfrak{M}$ iff $\langle \mathfrak{M}, U^{new};\ \bar{R}^{new} \rangle$ is definable implicitly without taking reducts over $\mathfrak{M}$. Informally we might say in such situations that the new sort U^{new} is definable implicitly in $\mathfrak{M}$ (but then $\bar{R}^{new}$ should be understood from the context, otherwise the definability claim is sort of under-specified).

In the above notion of definability, the *set of formulas defining U^{new}, $\bar{R}^{new}$ implicitly* over $\mathfrak{M}$ is $\mathsf{Th}(\mathfrak{M}^+)$. Hence, $\mathsf{Th}(\mathfrak{M}^+)$ is called an *implicit definition* of U^{new}, $\bar{R}^{new}$ over $\mathfrak{M}$ if ($\star$) above holds and the isomorphism h is unique. Further, for any set Δ of formulas in the language of $\mathfrak{M}^+$, Δ is called an *implicit definition* of $U^{new}, \bar{R}^{new}$ over $\mathfrak{M}$ iff ($\star$) above holds with Δ in place of $\mathsf{Th}(\mathfrak{M}^+)$ in such a way that h is unique.[497]

Remark 4.3.4 The reader might feel that the above notion of (implicit) definability without taking reducts (of $\mathfrak{M}^+$) is not strong enough and he might want to replace h by the identity function (requiring $U^{new} = U'$, $\bar{R}^{new} = \bar{R}'$). However, we claim that the above notion is "best possible" because (i) it is reasonable to assume that the first-order definition of $\mathfrak{M}^+$ (over $\mathfrak{M}$) is included in $\mathsf{Th}(\mathfrak{M}^+)$ and (ii) *any* isomorphic copy $\mathfrak{M}' = \langle \mathfrak{M}, U';\ \bar{R}' \rangle$ of $\mathfrak{M}^+$ will automatically validate $\mathsf{Th}(\mathfrak{M}^+)$ hence, in first-order logic we cannot define the new sort U^{new}, $\bar{R}^{new}$ more closely than up to (a unique) isomorphism.[498]

$\lhd$

$\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new};\ \bar{R}^{new} \rangle$ is said to be *definable implicitly with parameters* over $\mathfrak{M}$ iff there are $s \in \omega$ and $\bar{p} \in {}^sUv(\mathfrak{M})$ such that the expansion $\langle \mathfrak{M}^+, \bar{p} \rangle$ is *definable implicitly without taking reducts* over the expansion $\langle \mathfrak{M}, \bar{p} \rangle$.[499]

* * *

[497] The set Δ of formulas which we call an implicit definition is called a "rigidly relatively categorical" theory in Hodges [130, p.645]. If Δ is an implicit definition up to isomorphism only, then it is called a "relatively categorical" theory on p.638 of [130] (§12.5 therein).

[498] A possible way out of this would be if we required $\bar{R}^{new}$ to contain membership relations "$\in$" and projection functions pj_i (and then add some restrictions postulating e.g. that $\in$ and pj_i are the "real" set theoretic ones etc., cf. p.232 for the definition of the pj_i's). We will not do this because we feel that it would lead to too many complications without yielding enough benefits.

[499] We use "definable implicitly" and "implicitly definable" as synonyms. I.e. we are flexible about word order.

Let us turn to definability over *classes of models*. Let K be a class of models with U^{new}, $\bar{R}^{new}$ in the language of K. For $\mathfrak{M} \in \mathsf{K}$ let $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ obtained by omitting (forgetting) U^{new}, $\bar{R}^{new}$. Let

$$\mathsf{K}^- := \left\{ \mathfrak{M}^- \; : \; \mathfrak{M} \in \mathsf{K} \right\}.$$

We ask ourselves when K is definable over K^- or equivalently (but informally) when U^{new}, $\bar{R}^{new}$ are definable over K^-. We say that the class K of models is definable implicitly without taking reducts over K^- iff there is a set $\Delta \subseteq \mathsf{Th}(\mathsf{K})$ of formulas such that condition $(\star\star)$ below holds.

$(\star\star)$ For every $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\Delta)$ similar to members of K and such that $\mathfrak{M}^- = \mathfrak{N}^- \in \mathsf{K}^-$, there is a unique isomorphism $h : \mathfrak{M} \rightarrowtail\!\!\!\rightarrow \mathfrak{N}$ which is the identity on the universes of $\mathfrak{M}^-$.

If the isomorphism h is not necessarily unique then we say that K is definable implicitly up to isomorphism over K^-. Informally, we say that the new sort U^{new} and $\bar{R}^{new}$ are definable implicitly over K^- iff K as understood above is definable implicitly without taking reducts over K^-. When speaking about definability of U^{new}, $\bar{R}^{new}$ over K^-, it should be clear *from context* how K is obtained from the data K^- and U^{new}, $\bar{R}^{new}$. If $(\star\star)$ holds, then Δ in $(\star\star)$ is called an implicit definition of K over K^-.

We leave it to the reader to generalize the above definitions to the case when we have arbitrary sequences $\bar{U}^{new}$ and $\bar{R}^{new}$ of new sorts and new relations. However, herein we restrict our attention to the case when there are finitely many new symbols (i.e. both $\bar{U}^{new}$ and $\bar{R}^{new}$ are finite sequences of sorts and relations respectively). The classical notion of definability of new relations (without new sorts) is obtained as a special case of our general notion by choosing $\bar{U}^{new} = \emptyset$, i.e. $\bar{U}^{new}$ is the empty sequence.

Let K and L be two classes of models, i.e. L is not necessarily a reduct of K. We say that K *is definable implicitly over* L iff some expansion K^+ of K is definable implicitly without taking reducts over L. (In this case, L will be a reduct of K^+, of course.)[500] This means that statements (i) and (ii) below hold for some expansion K^+ of K:

(i) L is a reduct of K^+,

(ii) K^+ is definable implicitly over L without taking reducts. (Since here L is a reduct of K^+, our *earlier* definition of implicit definability without taking reducts on p.224 can be applied.)

We note that here we have to take seriously that our languages are finite, i.e. K^+ has only finitely many new symbols that do not occur in L.[501] In this case we say that Δ is an *implicit definition of* K *over* L if Δ is an implicit definition of K^+ over L. Thus an implicit definition of K over L may contain symbols not occurring in K.

We will apply the same convention for single models too, i.e. $\mathfrak{N}$ *is definable implicitly over* $\mathfrak{M}$ iff this holds for $\{\mathfrak{N}\}$ and $\{\mathfrak{M}\}$. We will sometimes abbreviate "implicitly definable without taking reducts" by "*nr-implicitly definable*", where "nr" stands for "taking no reducts".

[500] It would be more careful of us if we would call this new implicit definability (which permits taking reducts) weak implicit definability. This is so because when taking reducts then the uniqueness condition, cf. p.223, on isomorphisms may get lost.

[501] Cf. Examples 4.3.9 (2).

Example 4.3.5 The new sort ${}^nF$ together with the projection functions $pj_i : {}^nF \longrightarrow F$ $(i < n)$, cf. p.232, are definable nr-implicitly over the class FM of our frame models.

$\lhd$

Note that $(\star\star)$ above is a straightforward generalization of $(\star)$ on p.223. Therefore $\mathfrak{M}^+$ is definable nr-implicitly over $\mathfrak{M}$ iff the class $\{\mathfrak{M}^+\}$ is definable nr-implicitly over the class $\{\mathfrak{M}\}$.

In situations like the one involving statement $(\star\star)$ above, we also say that U^{new}, $\bar{R}^{new}$ are <u>*uniformly*</u> definable (implicitly) over K^-.[502] The set Δ of formulas is considered as a <u>*uniform (implicit) definition*</u> of U^{new}, $\bar{R}^{new}$ over K^-. Hence in the example above we can also say that ${}^nF$ etc. are uniformly definable over FM. We have not yet discussed *non-uniform* definability which is also called *"local"* or *"one-by-one"* definability: We will discuss this notion below Examples 4.3.9, on p.230.

Although we began this sub-section with discussing definability over a single model $\mathfrak{M}$, the main emphasis in this work will be on definability over a class K of models such that $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ i.e. such that K is axiomatizable in first-order logic.

We note that implicit definability without taking reducts of K over K^- is strictly stronger than implicit definability up to isomorphism. This remains so even if we assume that K and K^- are first-order axiomatizable classes of models. We leave the construction of a simple counterexample to the reader, but cf. Example 4.3.9(8) way below. For the connections between the various notions of definability we refer the reader to Figure 97 on p.270.

Remark 4.3.6 The following are intended to provide a kind of "intuitive" characterization of implicit definability without taking reducts of a class K of models over its reduct K^- (as was defined above).

(1) Assume K^- is a reduct of the class K (i.e. K^- is of the form $\{\mathfrak{M}^- : \mathfrak{M} \in \mathsf{K}\}$).

Then K is definable implicitly over K^- without taking reducts iff (i)–(ii) below hold.

  (i) $(\forall \mathfrak{M} \in \mathsf{K})\mathfrak{M}$ is definable nr-implicitly over its reduct $\mathfrak{M}^-$.

  (ii) There is a single set Δ of formulas such that for every $\mathfrak{M} \in \mathsf{K}$, Δ is an implicit definition of $\mathfrak{M}$ over $\mathfrak{M}^-$. In other words, not only each $\mathfrak{M}$ is nr-implicitly definable over $\mathfrak{M}^-$, but this defining can be done uniformly for the whole of K.

(2) Further, assume K is implicitly definable without taking reducts over its reduct K^-. Then the function

$$\mathsf{rd} \stackrel{\text{def}}{=} \{\langle \mathfrak{M}, \mathfrak{M}^- \rangle : \mathfrak{M} \in \mathsf{K}\}$$

is a bijection up to isomorphism[503]

$$\mathsf{rd} : \mathsf{K} \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\!\succ \mathsf{K}^-$$

such that each $\mathfrak{M} \in \mathsf{K}$ is definable nr-implicitly over $\mathsf{rd}(\mathfrak{M})$ and these definitions coincide for all choices of $\mathfrak{M}$.

$\lhd$

[502] We will explain soon, beginning with item 11 of Examples 4.3.9 (p.227), what aspect of the above situation we are referring to with the adjective "uniform" here.

[503] I.e. $\mathsf{rd}(\mathfrak{M}) = \mathsf{rd}(\mathfrak{N}) \;\Rightarrow\; \mathfrak{M} \cong \mathfrak{N}$. Roughly, something holds "up to isomorphism" iff it holds modulo identifying some of the isomorphic models.

Remark 4.3.7 (properties of "general" definability of classes) Assume K is definable implicitly over L. Then (1)-(2) below hold.

(1) K and L agree on their common vocabulary, i.e.

$$\mathsf{K} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}) = \mathsf{L} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}).$$

(2) There is a surjective function[504] $f : \mathsf{L} \longrightarrow\!\!\!\!\!\rightarrow \mathsf{K}$ such that for all $\mathfrak{M} \in \mathsf{L}$, $f(\mathfrak{M})$ is implicitly definable over $\mathfrak{M}$[505]; moreover the definition of $f(\mathfrak{M})$ over $\mathfrak{M}$ is the same (set of formulas) for all choices of $\mathfrak{M}$.

$\lhd$

Now we turn to giving examples.

Examples 4.3.8 (Traditional, one-sorted examples)

1. Let PA be the class of models of Peano's Arithmetic, cf. any logic book, e.g. Monk [194] or Chang-Keisler [60] for PA. The operation symbols of PA are $+, \cdot, 0, 1$. Consider the extra unary operation symbol "!" intended to denote the factorial. Let $\Delta_!$ be the set of the following two formulas

   $!(0) = 1$

   $\forall x[!(x+1) = (x+1)\cdot !(x)]$.

   I.e. $\Delta_! = \{\ !(0) = 1,\ \forall x[!(x+1) = (x+1)\cdot !(x)]\ \}$. We claim that $\Delta_!$ is a (correct) implicit definition of "!" over PA. (The proof is not easy but is available in almost any logic book.) The point in the above example is that PA is an axiomatizable class and that $\Delta_!$ works over each member of PA. If we want an implicit definition over a single model instead of an axiomatizable class, that is easy:

2. Consider the model $\langle \omega, 0, \mathsf{suc}, + \rangle$.[506] Let Δ_+ be the set of the following formulas:

   $x + y = y + x$

   $0 + x = x$

   $x + \mathsf{suc}(y) = \mathsf{suc}(x + y)$.

   Now, Δ_+ defines $+$ implicitly over the model $\langle \omega, 0, \mathsf{suc} \rangle$. However, it is important to note that over the axiomatizable hull $\mathsf{Mod}(\mathsf{Th}(\langle \omega, 0, \mathsf{suc} \rangle))$ of this model, Δ_+ is not an implicit definition[507], and moreover addition is not nr-implicitly definable in $\mathsf{Mod}(\mathsf{Th}(\langle \omega, 0, \mathsf{suc} \rangle))$.

   This shows that nr-implicit definability over a single model is much weaker than nr-implicit definability over an axiomatizable class of models. (Since primarily we are interested in theories, and theories correspond to axiomatizable classes, we are more interested in definability over axiomatizable classes than over single models.)

3. Let $E = \{2 \cdot n : n \in \omega\}$ be the set of even numbers. Then E as a *unary relation* is definable nr-implicitly over the model $\langle \omega, \mathsf{suc} \rangle$.

[504] f is a function only up to isomorphism, cf. AMN [18, footnote 941 on p.971] for more detail.
[505] I.e. there is an implicit definitional expansion $\mathfrak{M}^+$ of $\mathfrak{M}$ with $f(\mathfrak{M})$ a reduct of $\mathfrak{M}^+$.
[506] Where $\mathsf{suc} : \omega \longrightarrow \omega$ is the usual successor function on ω, i.e. $\mathsf{suc}(n) = n + 1$ for all $n \in \omega$.
[507] i.e. it does not satisfy $(\star\star)$ way above

4. Let BA_0 be the class of Boolean algebras with "$\cap$", "$\cup$", $0, 1$ as basic operations. Now, $\{x \cap -x = 0,\ x \cup -x = 1\}$ is an implicit definition of complementation over BA_0. This implicit definition, however, can easily be rearranged into the form of an explicit definition as follows[508]:

$$-(x) = y \quad \Leftrightarrow \quad [x \cap y = 0 \ \wedge \ x \cup y = 1].$$

5. As an exercise, it is useful to experiment with (i) defining the Boolean partial ordering "$\leq$" over BA_0, (ii) defining "$\cup$" over the basic operations "$\cap, -$" (and the same with the roles of "$\cup$" and "$\cap$" interchanged).

6. The model $\langle \omega, \leq \rangle$ is implicitly definable over $\langle \omega, 0, \mathsf{suc} \rangle$, but it is not nr-implicitly definable because $\langle \omega, \leq \rangle$ is not an expansion of $\langle \omega, 0, \mathsf{suc} \rangle$. If $\mathfrak{M}^+ = \langle \mathfrak{M}; \bar{R}^{new} \rangle$, i.e. if $\mathfrak{M}^+$ does not contain new sorts, then $\mathfrak{M}^+$ is nr-implicitly definable over $\mathfrak{M}$ iff $\mathfrak{M}^+$ is implicitly definable over $\mathfrak{M}$. This is not necessarily true when $\mathfrak{M}^+$ contains new sorts, too.

$\triangleleft$

Examples 4.3.9 (More advanced, many-sorted examples)

1. Let $\mathfrak{F}$ be an ordered field. Then the two-sorted model $\langle \mathfrak{F}, \mathcal{P}(F); \in \rangle$ is *not definable implicitly up to isomorphism* over $\mathfrak{F}$. Hence it is not nr-implicitly definable, either.

   Proof-idea: Assume $|F| = \omega$. Then $|\mathcal{P}(F)| > \omega$. But by the downward Löwenheim-Skolem theorem $\langle \mathfrak{F}, \mathcal{P}(F); \in \rangle$ has an elementary submodel with each sort countable.

2. Let $\bar{R}$ be any countable sequence of relations defined on the sorts $F, \mathcal{P}(F)$ in example 1 above. Then
$$\langle \mathfrak{F}, \mathcal{P}(F); \in, \bar{R} \rangle$$
   is *not definable implicitly up to isomorphism* over $\mathfrak{F}$.

   Hint: The reason remains the same as in example 1.

   This means that $\mathfrak{F}^+ \stackrel{\text{def}}{=} \langle \mathfrak{F}, \mathcal{P}(F); \in \rangle$ is *not implicitly definable* over $\mathfrak{F}$, either.

   However, there is an expansion $\mathfrak{F}^{++}$ of $\mathfrak{F}^+$ with uncountably many new relations such that $\mathfrak{F}^{++}$ is nr-implicitly definable over $\mathfrak{F}$. Indeed, let us take a new constant c_x for each element x of $F \cup \mathcal{P}(F)$. Then $\mathfrak{F}^{++} \stackrel{\text{def}}{=} \langle \mathfrak{F}, \mathcal{P}(F), \in, \langle c_x : x \in F \cup \mathcal{P}(F) \rangle \rangle$ is an nr-implicitly definable expansion of $\mathfrak{F}$. This shows the importance of allowing only finitely many relation symbols in our languages when defining implicit definability, cf. p.224.

3. Let $\mathbf{F}$ be a finite field. Then $\langle \mathbf{F}, \mathcal{P}(F); \in \rangle$ *is definable nr-implicitly* over $\mathbf{F}$. The same applies for any finite structure in place of $\mathbf{F}$.

   Notation: For any set H and cardinal κ we let $\mathcal{P}_\kappa(H)$ be the collection of those subsets of H whose cardinality is smaller than κ. In particular, $\mathcal{P}_\omega(H)$ denotes the set of finite subsets of H.

4. Let $\mathfrak{A}$ be a(n infinite) structure with universe A. Then $\langle \mathfrak{A}, \mathcal{P}_i(A); \in \rangle$ *is nr-implicitly definable* over $\mathfrak{A}$ for any $i \in \omega$.

[508] We have not discussed explicit definitions yet, but they will be discussed soon (beginning with §4.3.2 on p.230).

5. Let $\mathfrak{A} = \langle \omega, \leq \rangle$ be the set of natural numbers with the usual ordering. Then the expansion $\langle \mathfrak{A}, \mathcal{P}_\omega(\omega); \in \rangle$ *is nr-implicitly definable* over $\mathfrak{A}$.

   Hint: An implicit definition is the following set of formulas:

   $$\{\forall x_1 \dots x_n \in \omega \exists y \in \mathcal{P}_\omega(\omega) y = \{x_1, \dots, x_n\} : n \in \omega\} \cup$$
   $$\{\forall y \in \mathcal{P}_\omega(\omega) \exists x \in \omega \forall z \in \omega (z \in y \longrightarrow z \leq x)\} \cup$$
   $$\{\forall y, z \in \mathcal{P}_\omega(\omega)(y = z \leftrightarrow \forall x \in \omega (x \in y \leftrightarrow x \in z))\}.$$

   (In the above, $y = \{x_1, \dots, x_n\}$ abbreviates any formula with the intended meaning.)

   As a contrast, we include the following example.

6. Consider the expansion $\langle \omega, \mathcal{P}_\omega(\omega); \in \rangle$ of the "plain" structure $\langle \omega \rangle$. Then this structure (i.e. $\langle \omega, \mathcal{P}_\omega(\omega); \in \rangle$) *is not implicitly definable up to isomorphism* over $\langle \omega \rangle$.

   Hint: Take any countable elementary submodel $\mathfrak{B}$ of an ultrapower of $\langle \omega, \mathcal{P}_\omega(\omega); \in \rangle$ which contains a "nonstandard" element in $\mathcal{P}_\omega(\omega)$. Then the "$\omega$-part" of $\mathfrak{B}$ is isomorphic to $\langle \omega \rangle$, but $\mathfrak{B}$ is not isomorphic to $\langle \omega, \mathcal{P}_\omega(\omega); \in \rangle$.

7. $\langle \omega, \mathcal{P}_\omega(\omega); \mathsf{suc}, \in \rangle$ is implicitly definable over $\langle \omega, \mathsf{suc} \rangle$. We do not know whether it is nr-implicitly definable over $\langle \omega, \mathsf{suc} \rangle$ or not. (We conjecture that the answer is in the negative.)

8. $\langle \mathfrak{A}, U^{new} \rangle$ *is not implicitly definable up to isomorphism* over $\mathfrak{A}$, for any structure $\mathfrak{A}$ and infinite set U^{new}. Here U^{new} is a new sort, and there are no new relations. If $1 < |U^{new}| < \omega$, then U^{new} is implicitly definable and implicitly definable up to isomorphism, but not implicitly definable without taking reducts. If $|U^{new}| \leq 1$, then U^{new} is implicitly definable without taking reducts.

9. Let $\mathfrak{A}$ be any structure and let $\mathfrak{B}$ be any finite structure. Then $\langle \mathfrak{A}, \mathfrak{B} \rangle$ as a two-sorted structure is impicitly definable over $\mathfrak{A}$.

10. Let $\mathfrak{A}$ be a fixed structure. Consider

    $$\mathsf{K} = \{ \langle \mathfrak{A}; U^{new} \rangle \ : \ |U^{new}| < \omega \}.$$

    Then K is *not nr-implicitly definable* over $\{\mathfrak{A}\}$ (not even up to isomorphism).

Reading the examples below is *not* necessary for understanding the rest of the present work. They are designed to illustrate the distinction between uniform and non-uniform definability.

11. For $k \in \omega$, let $\mathfrak{U}_k$ be the usual $k+1$ element linear ordering $\mathfrak{U}_k = \langle \{0, \dots, k\}, < \rangle$ where "$<$" is the usual ordering of the natural numbers. Recall from set theory that $\aleph_k$ is the k'th infinite cardinal regarded as a special ordinal. E.g. $\aleph_0 = \omega$. Let

    $$\mathsf{K} := \{ \langle \aleph_k, \mathfrak{U}_k \rangle \ : \ k \in \omega \}$$

    where $\langle U^{new}, \bar{R}^{new} \rangle = \mathfrak{U}_k$. I.e. K^- is obtained by forgetting the $\mathfrak{U}_k$-part. Then K is *not* uniformly nr-implicitly definable over K^- although for each $\mathfrak{M} \in \mathsf{K}$, we have that $\mathfrak{M}$ is nr-implicitly definable over $\mathfrak{M}^-$, i.e. $\mathfrak{U}_k$ is nr-implicitly definable over $\langle \aleph_k \rangle$.

12. The following is a generalization of item 11 above. Let $\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots$ $(k \in \omega)$ be *any* ω-sequence of elementarily equivalent one-sorted models.[509] Let $\mathfrak{U}_k$ be as in item 11 above.

$$\mathsf{K} := \{ \langle \mathfrak{A}_k, \mathfrak{U}_k \rangle \, : \, k \in \omega \} .$$

Then K is *not uniformly nr-implicitly definable* over $\mathsf{K}^- = \{ \mathfrak{A}_k \, : \, k \in \omega \}$ while every $\mathfrak{M} \in \mathsf{K}$ *is nr-implicitly definable* over $\mathfrak{M}^-$.

Hint: The key idea can be formulated with using $\mathfrak{A}_1$, $\mathfrak{A}_2$ only. The rest of the $\mathfrak{A}_k$'s serve only as decoration. So, one starts with $\mathfrak{A}_1 \equiv_{ee} \mathfrak{A}_2$ and[510] $|U_1| \neq |U_2|$ are finite. (Where U_i is the universe of $\mathfrak{U}_i$, similarly for A_i.) It is important to note that there are no inter-sort relations permitted here i.e. sort A_i is isolated from sort U_i. Next, one uses the following property of many-sorted logic. Assume $\mathfrak{A}, \mathfrak{B}$ are two structures of *disjoint languages*. Consider the new many-sorted structure $\langle \mathfrak{A}, \mathfrak{B} \rangle$. We claim that $\mathsf{Th}(\langle \mathfrak{A}, \mathfrak{B} \rangle) = \mathsf{Th}(\mathfrak{A}) \cup \mathsf{Th}(\mathfrak{B})$. The reason for this is the fact that an atomic formula xRy belongs to a many-sorted language *only* if x and y are of the *same* sort. Hence e.g. $(\exists x \in U_0)(\exists y \in U_1)\, x \neq y$ is not a (many-sorted) formula.

The present example does not work for "implicitly definable" in place of "implicitly definable without taking reducts".

Someone might think that the reason why the above counterexample works is that all elements of K^- are elementarily equivalent. Below we show that this is *not* the case.

13. Let the language of K^- consist of countably many constant symbols $c_0, \dots, c_i, \dots$ and just one sort. Let $\mathfrak{U}_k$ $(k \in \omega)$ be as in item 11 above.

$$\begin{aligned} \mathsf{K}^- &:= \{ \langle U, c_i \rangle_{i\in\omega} \, : \, \text{the set } \{ i \in \omega \, : \, c_i = c_0 \} \text{ is finite and} \\ &\qquad\qquad U \text{ is a set with } (\forall i \in \omega)\, c_i \in U \} . \\ \mathsf{K} &:= \{ \langle U, c_i; \mathfrak{U}_k \rangle_{i\in\omega} \, : \, k = |\{ i \in \omega \, : \, c_i = c_0 \}| \text{ and } \langle U, c_i \rangle_{i\in\omega} \in \mathsf{K}^- \} . \end{aligned}$$

That is

$$\mathsf{K} = \{ \langle \mathfrak{M}; \mathfrak{U}_k \rangle \, : \, \mathfrak{M} \in \mathsf{K}^- \text{ and } k = |\{ i \in \omega \, : \, \text{in } \mathfrak{M} \text{ we have } c_i = c_0 \}| \} .$$

Now, K is *not uniformly nr-implicitly definable* over K^- *while* each concrete $\mathfrak{M} \in \mathsf{K}$ *is nr-implicitly definable* over $\mathfrak{M}^-$, *further*

$$(\forall \mathfrak{M}, \mathfrak{N} \in \mathsf{K})[\, \mathfrak{M}^- \equiv_{ee} \mathfrak{N}^- \quad \Rightarrow \quad \mathfrak{M} \equiv_{ee} \mathfrak{N} \,].$$

Idea for a proof:

Assume $\Delta = \mathsf{Th}(\mathsf{K})$ defines K implicitly over K^- (up to isomorphisms). Then by using ultraproducts one can show that there is $\mathfrak{N} = \langle U, c_i; \mathfrak{U}_2 \rangle_{i\in\omega} \in \mathsf{Mod}(\Delta)$ such that $(\forall i > 0)(c_i \neq c_0$ holds in $\mathfrak{N})$. But clearly for $\mathfrak{M} := \langle \mathfrak{N}^-; \mathfrak{U}_1 \rangle$ we have $\mathfrak{N}^- = \mathfrak{M}^- \in \mathsf{K}^-$ and $\mathfrak{M} \in \mathsf{K}$ hence by $\mathfrak{M} \not\cong \mathfrak{N}$ we conclude that Δ cannot be a definition of K.

◁

[509] I.e. $(\forall k \in \omega)\, \mathsf{Th}(\mathfrak{A}_0) = \mathsf{Th}(\mathfrak{A}_k)$.

[510] Recall that $\equiv_{ee}$ denotes the binary relation of elementary equivalence defined between models.

The above three examples were designed to illustrate the difference between uniform (nr-implicit) definability and *one-by-one (nr-implicit) definability* where by the latter we understand the case when each $\mathfrak{M} \in \mathsf{K}$ is definable over its reduct $\mathfrak{M}^-$ in K^- (but these definitions might be different for different choices of $\mathfrak{M}$); in more detail: Let K be a class of models with U^{new}, $\bar{R}^{new}$ in the language of K. For $\mathfrak{M} \in \mathsf{K}$ let $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ obtained by omitting (forgetting) U^{new}, $\bar{R}^{new}$. Let $\mathsf{K}^- := \{\mathfrak{M}^- : \mathfrak{M} \in \mathsf{K}\}$. Then we say that K is *one-by-one nr-implicitly definable* over K^- iff each $\mathfrak{M} \in \mathsf{K}$ is nr-implicitly definable over its reduct $\mathfrak{M}^- \in \mathsf{K}^-$. Sometimes, informally we will use instead of one-by-one definability *"non-uniform"* or *"local"* definability as synonyms. We hope that the above three examples illustrate (the generally accepted opinion) that uniform definability is a more useful concept than one-by-one definability (when considering classes K of models) and is closer to what one would intuitively understand under definability.

For completeness, we refer the interested reader to the distinction between the *"local"* and the "usual" versions of explicit definability described in Andréka-Németi-Sain [31] Definitions 55–56 (Beth definability properties) therein. We also note that most standard textbooks concentrate on uniform definability only and they do not mention what we call here one-by-one definability. We too will concentrate on uniform definability and unless otherwise specified, by *definability* we will always understand *uniform definability*.

Remark 4.3.10 A useful refinement of the notion of nr-implicit definability is *finite nr-implicit definability.* Assume K and K^- are as above statement ($\star\star$) on p.224 (definition of nr-implicit definability). Assume K is nr-implicitly definable over K^-. Then K is said to be *finitely nr-implicitly definable* over K^- iff there is a finite set $\Delta_0 \subseteq \mathsf{Th}(\mathsf{K})$ of formulas such that Δ_0 defines K implicitly over K^-, i.e. ($\star\star$) holds for $\Delta = \Delta_0$. In most of our concrete examples and applications we will have *finite* nr-implicit definability, but for simplicity we will write just "definability".

To illustrate the importance of finite nr-implicit definability, consider the simple model $\langle \omega, \mathsf{suc} \rangle$. There are continuum many different implicit definitions (involving one new relation symbol R) over this model while there are only countably many finite implicit definitions (and we will see that there are only countably many explicit definitions over this model). (This example cannot be generalized from a single model like $\mathfrak{M} = \langle \omega, \mathsf{suc} \rangle$ to first-order-axiomatizable classes K of models, assuming there are only finitely many sorts).[511]

$\triangleleft$

4.3.2 Explicit definability in many-sorted (first-order) logic

So far we have discussed implicit definability which is a quite general notion of definability. Below we will turn to a special kind of implicit definability which we call *explicit definability*. Each explicit definition can be considered as an implicit definition. The other direction is not true however, there are implicit definitions which are not explicit definitions. (I.e. there is

[511] The reason for this is the following. In the above reasoning we heavily used the fact that every element of $\langle \omega, \mathsf{suc} \rangle$ is definable "as a constant". (Therefore infinite implicit definitions can be given by listing the elements of R and the non-elements of R.) This does not remain true in $\mathsf{Mod}(\mathsf{Th}(\langle \omega, \mathsf{suc} \rangle))$.

an implicit definition Δ which in its *given form* is not an explicit definition.) In definability theory, the connection between explicit and implicit definitions is an important subject. We will return to this subject in §4.3.5. In particular, we will state a generalization of Beth's theorem, saying that implicit definability is equivalent to explicit definability (even in our general framework where we allow definitions of new sorts, too [besides definitions of new relations], cf. Theorem 4.3.48 and Corollary 4.3.49 on p.268.

Explicit definability will turn out to be (i) a special case of implicit definability and (ii) a strong and useful concept e.g. in the following way. Assume $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ and that K^+ is an expansion of K which is explicitly definable over the class K of models. Then the theories $\mathsf{Th}(\mathsf{K})$ and $\mathsf{Th}(\mathsf{K}^+)$ as well as the languages of K and K^+ will be seen to be equivalent in a rather strong sense to be explained later, see Theorems 4.3.27 and 4.3.29 on p.245.

The key ingredients of explicit definability will be introduced in items (1)–(2.2) below. Then, on p.235, they will be combined into a description of what we mean by explicit definability. The generalization from definability over single models $\mathfrak{M}$ to definability over classes K of models will be given on p.235.

Notation: Assume $\mathfrak{M}$ is a many-sorted model and that ψ is a formula in the language of $\mathfrak{M}$ such that all the free variables of ψ belong to $x_0, \ldots, x_i, \ldots$. Assume $\bar{a} \in {}^{\omega}Uv(\mathfrak{M})$ and that the sort of a_i coincides with the sort of the variable x_i, for every $i \in \omega$. Then

$$\mathfrak{M} \models \psi[\bar{a}]$$

is the standard model theoretic notation for the statement that ψ is true in $\mathfrak{M}$ under the *evaluation* $\bar{a}$ of its free variables cf. e.g. Monk [194], Enderton [82], Chang-Keisler [60]. Sometimes we write $\mathfrak{M} \models \psi[a_1, \ldots, a_n]$ in which case it is understood that the free variables of ψ are among $x_1, \ldots, x_n$. The latter is often indicated by writing $\psi(x_1, \ldots, x_n)$ instead of ψ. I.e. if we write $\psi(x_1, \ldots, x_n)$ in place of ψ then this means that while talking about the formula ψ we want to indicate casually that the free variables of ψ are among $x_1, \ldots, x_n$.

The following is also a standard notation from logic. Assume τ is a term. Then $\psi(x/\tau)$ denotes the formula obtained from ψ by replacing all free occurrences of x by τ. Similarly for $\psi(x_1/\tau_1, \ldots, x_n/\tau_n)$. We could say that "$(x/\tau)$" is the "operator" of substituting τ for x.

If $\psi(x)$ is a formula and y is a variable (of the same sort as x), then $\psi(y)$ denotes $\psi(x/y)$; and similarly for a sequence $\bar{x}$ of variables.

Sometimes below we will write "definable" for "explicitly definable" to save space. Similarly, we write "definitional expansion" for "explicit definitional expansion". In general, we will tend to omit the adjective "explicit", because our primary interest will be explicit definability.

(1) Explicit definability of relations and functions in $\mathfrak{M}$.

Let $\mathfrak{M} = \langle U_0, \ldots, U_j; R_1, \ldots, R_l\rangle$ be a many-sorted model with universes or sorts $U_0, \ldots, U_j$, and relations $R_1, \ldots, R_l$. Let $R^{new} \subseteq U_{i_1} \times \ldots \times U_{i_m}$ be a (new) relation, with $i_1, \ldots, i_m \in (j+1)$. Now, R^{new} is called *(explicitly) definable* (as a relation) over $\mathfrak{M}$ iff there is a formula $\psi(x_{i_1}, \ldots, x_{i_m})$ in the language of $\mathfrak{M}$ such that

$$R^{new} = \{\, \langle a_{i_1}, \ldots, a_{i_m}\rangle \in U_{i_1} \times \ldots \times U_{i_m} \;:\; \mathfrak{M} \models \psi[a_{i_1}, \ldots, a_{i_m}]\,\}\,.$$

Such definable relations can be added to $\mathfrak{M}$ as new basic relations obtaining a(n explicit) *definitional expansion* of $\mathfrak{M}$ in the form

$$\mathfrak{M}^+ := \langle U_0, \ldots, U_j;\; R_1, \ldots, R_l, R^{new}\rangle.$$

To make $\mathfrak{M}^+$ "well defined" we have to add a <u>*new relation symbol*</u> to the language of $\mathfrak{M}$ denoting R^{new}. The formula $R^{new}(\bar{x}) \leftrightarrow \psi(\bar{x})$ is called an <u>*(explicit) definition*</u> of R^{new} (over $\mathfrak{M}$). Notice that $\Delta \stackrel{\text{def}}{=} \{R^{new}(\bar{x}) \leftrightarrow \psi(\bar{x})\}$ is also a(n implicit) definition of $\mathfrak{M}^+$ over $\mathfrak{M}$. We call Δ an <u>*explicit definition of type (1)*</u>. If Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then we say that <u>*$\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by step (1)*</u>. Note that if $\mathfrak{M}'$ is defined over $\mathfrak{M}$ by Δ, then $\mathfrak{M}'$ is $\mathfrak{M}^+$ above.

(2) Explicit definability of new sorts (i.e. universes) in $\mathfrak{M}$.

Defining a new sort explicitly (in $\mathfrak{M}$) takes a bit more care than defining a new relation. This is understandable, since now we want to define (or create) a *new universe* of entities (in terms of the old universes and old relations already available in $\mathfrak{M}$) while when defining a relation we defined only a new property of *already existing* entities (or of tuples of such entities) in $\mathfrak{M}$. If we define a new relation, then this amounts to defining a new property of already existing entities. I.e. we remain on the same <u>*ontological level*</u>. In contrast, if we define <u>*new entities*</u> which "did not exist" before, then we go up to a higher ontological level.[512]

If we want to define a new sort in $\mathfrak{M}$, first of all we need a new sort-symbol, say U^{new}, which does not yet occur in the language of $\mathfrak{M}$. If there is no danger of confusion then we will *identify* a sort-symbol like U^{new} with the universe, say $(U^{new})^{\mathfrak{M}^+}$, which it denotes in a model $\mathfrak{M}^+$.

An explicit definition of a new sort, say U^{new}, describes the elements of U^{new} as being constructed from "old" elements in a systematic, "tangible" and uniform way. More concretely, first we will introduce a few (basic constructions or) basic kinds of explicit definition and then "general" explicit definitions will be obtained by iterating these basic kinds. We will refer to the just mentioned basic kinds (of explicit definition) as basic steps of explicit definitions. Our basic steps for building up explicit definitions of new sorts are described in items (2.1), (2.2) below. Our choice of basic steps might look ad-hoc at first reading, but Theorem 4.3.48 at the end of this section will say that our selected few steps (i.e. examples of explicit definitions) cover (via iteration) all cases of implicit definitions (assuming there is a sort with more than one elements). We will return to a more careful discussion of the present issue of choosing our basic steps in Remark 4.3.53.

(2.1) The first way of defining a new sort U^{new} in $\mathfrak{M}$ explicitly.

The simplest way of defining a new sort U^{new} in a model $\mathfrak{M} = \langle U_0, \ldots, U_j;\, R_1, \ldots, R_l\rangle$ is the following. Let $R \in \{R_1, \ldots, R_l\}$ be fixed. Assume R is an r-ary relation, i.e. $R \subseteq {}^r Uv(\mathfrak{M})$. We want to postulate that U^{new} coincides with R. So the first part of our definition of U^{new} is the postulate:

$$U^{new} \stackrel{\text{def}}{:=} R.$$

But, if we want to expand $\mathfrak{M}$ with U^{new} as a new sort obtaining something like

$$\mathfrak{M}' := \langle U_0, \ldots, U_j, U^{new};\, R_1, \ldots, R_l\rangle$$

then we *need* some new relations or functions *connecting* the new sort U^{new} to the old ones $U_0, \ldots, U_j$. In our present case (of $U^{new} = R$) we use the <u>*projection functions*</u> $pj_i : R \longrightarrow Uv(\mathfrak{M})$ with $i < r$. Formally,

$$pj_i(\langle a_0, \ldots, a_{r-1}\rangle) \stackrel{\text{def}}{:=} a_i. \quad ^{513}$$

[512] In connection with defining new sorts, for completeness, we also refer e.g. to the definition of the "new" many-sorted structure A^{eq} from the "old" structure A in Hodges [130, p.151] (cf. also pp. 148, 212, 213 therein). Cf. also the definition of relative categoricity in Hodges [130] p.638 together with p.638 line 3 bottom up to p.639 line 9.

To identify the domain of pj_i we should write something like pj_i^R, but for brevity we omit the superscript R. Now, the (explicit) *definitional expansion* of $\mathfrak{M}$ obtained by the choice $U^{new} := R$ is

$$\mathfrak{M}^+ := \langle U_0, \ldots, U_j, U^{new};\ R_1, \ldots, R_l, pj_0, \ldots, pj_{r-1}\rangle = \langle \mathfrak{M}, U^{new};\ pj_i\rangle_{i<r}.$$

We note that

$$\mathfrak{M}^+ = \langle U_0, \ldots, U_j, R;\ R_1, \ldots, R_l, pj_i^R\rangle_{i<r}.$$

If x is a variable, then $(\exists\, !x)\psi(x)$ denotes the formula expressing that there is exactly one value for which ψ holds, i.e. it denotes the formula $(\exists x)(\psi(x) \wedge (\forall z)[\psi(z) \to z = x])$. Let

$$\begin{aligned}\Delta \stackrel{\text{def}}{=} \{&(\exists\, !u \in U^{new})(pj_1(u, x_1) \wedge \ldots \wedge pj_r(u, x_r)) \leftrightarrow R(x_1, \ldots, x_r)\ ,\\ &(\exists u \in U^{new})(pj_1(u, x_1) \wedge \ldots \wedge pj_r(u, x_r)) \to R(x_1, \ldots, x_r)\ ,\\ &(\forall u \in U^{new})(\exists\, !x_i)pj_i(u, x_i) : 1 \le i \le r\}\ .\end{aligned}$$

Then Δ is an implicit definition of $\mathfrak{M}^+$ over $\mathfrak{M}$. We call Δ an <u>*explicit definition of type (2.1)*</u>. If Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then we say that <u>$\mathfrak{M}'$ *is obtained from* $\mathfrak{M}$ *by Step (2.1)*</u>. Notice that if Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}^+$ above via an isomorphism which is identity on $\mathfrak{M}$.

Remark 4.3.11 This second form of $\mathfrak{M}^+$ might induce the (misleading) impression that $\mathfrak{M}^+$ contains nothing new: it consists of a rearranged version of the old parts of $\mathfrak{M}$. However, let us notice that as a first step we might define a new relation R^{new} in $\mathfrak{M}$ (in the style of item (1) above) obtaining

$$\mathfrak{M}^+ := \langle U_0, \ldots, U_j;\ R_1, \ldots R_l, R^{new}\rangle$$

and *then* we may define $U^{new} := R^{new}$ obtaining the *definitional expansion*

$$\mathfrak{M}^{++} := \langle U_0, \ldots, U_j, U^{new};\ R_1, \ldots, R^{new}, pj_i\rangle_{i<r}$$

of $\mathfrak{M}^+$. Now, we will *postulate* that a definitional expansion of a definitional expansion of $\mathfrak{M}$ is called a definitional expansion of $\mathfrak{M}$ again. Hence the above obtained $\mathfrak{M}^{++}$ is a definitional expansion of the original $\mathfrak{M}$. Using our abbreviation from p.219 we can write:

$$\langle \mathfrak{M}, U^{new};\ R^{new}, pj_i\rangle_{i<r} := \mathfrak{M}^{++}.$$

Now, *if* we do not want to have R^{new} as a relation, we can take the reduct

$$\mathfrak{M}^{++-} := \langle \mathfrak{M}, U^{new};\ pj_i\rangle_{i<r}$$

by forgetting R^{new} as a relation but not as a sort. We will call $\mathfrak{M}^{++-}$ a *generalized definitional expansion* of $\mathfrak{M}$ (cf. p.235).

$\triangleleft$

[513] By the standard elimination of function symbols, $pj_i(x) = y$ abbreviates $pj_i(x, y)$, hence function symbols like pj_i can occur in atomic formulas only in the form $pj_i(x) = y$.

Example 4.3.12 Let $\mathbf{F} = \langle F, \dots, \cdot \rangle$ be a field. We want to define the plane $F \times F$ over $\mathbf{F}$ as a new *sort* expanding $\mathbf{F}$. First we define the *relation* $R = F \times F$ by the formula $(x_0 = x_0 \ \wedge \ x_1 = x_1)$. Clearly, in $\mathbf{F}$ this formula defines the relation $F \times F$. Then we expand $\mathbf{F}$ with this as a new relation obtaining

$$\mathbf{F}^+ = \langle F; +, \cdot, F \times F \rangle$$

where $F \times F$ is used as a relation interpreting the relation symbol $Rel_{F\times F}$. Now, in $\mathbf{F}^+$ we define the *new sort* $U^{new} := F \times F$ together with the projection functions as indicated above, obtaining the model

$$\mathbf{F}^{++} = \langle F, F \times F; +, \cdot, F \times F, pj_0, pj_1 \rangle$$

where $pj_i : F \times F \longrightarrow F$. Now, we take a reduct of $\mathbf{F}^{++}$ by forgetting the relation symbol $Rel_{F\times F}$, but not the sort $F \times F$. We obtain

$$\mathbf{F}^{++-} = \langle F, F \times F; +, \cdot, pj_0, pj_1 \rangle = \langle \mathbf{F}, F \times F; pj_0, pj_1 \rangle.$$

Clearly this model $\mathbf{F}^{++-}$ is the expansion of the field $\mathbf{F}$ with the plane $F \times F$ as a new sort as we wanted.

The above example shows that the usual expansion of $\mathbf{F}$ with the plane as a *new sort*, is indeed a definitional expansion i.e. the plane as a new sort is *(first-order) definable explicitly* in $\mathbf{F}$.

$\lhd$

Similarly to the above example, ${}^nF$ is first-order definable (explicitly) as a *new sort* in any frame model $\mathfrak{M}$. Later we will introduce uniform explicit definability over a class K of models. Then we will see that ${}^nF$ as a new sort is uniformly (explicitly) definable over the class of all frame models. (In defining ${}^nF$ we use $pj_i : {}^nF \longrightarrow F$, $i \in n$, the same way as we did in the case of $\mathbf{F}^{++-}$.)

(2.2) The second way of defining a new sort U^{new} in $\mathfrak{M}$ explicitly.

To define a new sort U^{new} in a model $\mathfrak{M} = \langle U_0, \dots, U_j; R_1, \dots, R_l \rangle$ explicitly the second way, we begin by selecting an old sort $U := U_i$ and old relation $R := R_k$ $(i \leq j,\ 0 < k \leq l)$ in $\mathfrak{M}$. We proceed *only if* R happens to be an equivalence relation over U (i.e. if $R \subseteq U \times U$ etc.). We define the new sort to be the quotient set of R-equivalence classes[514]

$$U^{new} := U/R.$$

Again, similarly to the case of pj_i's in item (2.1) above, we need a new relation connecting the new sort U^{new} to the old ones. Now we choose the set theoretic membership relation

$$\in \, := \, \in_{U^{new}} \, := \, \in_{U,U^{new}} \, := \{ \langle a, a/R \rangle \ : \ a \in U \}$$

acting between U and U/R. Since $\in_{U^{new}} \subseteq U_i \times U^{new}$, this relation connects the new sort U^{new} with the old one U_i. Let us notice that from the notation $\in_{U,U^{new}}$ we may omit the first index obtaining the simpler notation $\in_{U^{new}}$ or we may omit both indices obtaining $\in$. The (explicit) *definitional expansion* of $\mathfrak{M}$ obtained by the choice $U^{new} = U_i/R_k$ is defined to be the model

$$\begin{aligned}
\mathfrak{M}^+ &= \langle U_0, \dots, U_j, U^{new}; R_1, \dots R_l, \in_{U^{new}} \rangle \\
&= \langle U_0, \dots, U_i/R_k; R_1, \dots, R_l, \in \rangle \\
&= \langle \mathfrak{M}, U^{new}; \in_{U^{new}} \rangle \\
&= \langle \mathfrak{M}, U_i/R_k; \in_{U^{new}} \rangle.
\end{aligned}$$

Let

[514] $U/R \stackrel{\text{def}}{=} \{a/R : a \in U\}$ where $a/R \stackrel{\text{def}}{=} \{b \in U : \langle a, b \rangle \in R\}$. I.e. U/R is the set of all "blocks" of R, and a/R is the "block" of R a is in.

$$\Delta \stackrel{\text{def}}{=} \{(\exists u \in U^{new})(\in(x,u) \land \in(y,u)) \leftrightarrow R(x,y),$$
$$[\in(x,u) \land \in(x,v)] \to u = v \quad \}.$$

Then Δ is an implicit definition of $\mathfrak{M}^+$ over $\mathfrak{M}$. We call Δ an *explicit definition of type (2.2)*. If Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then we say that *$\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by Step (2.2)*. Notice that if Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}^+$ above via an isomorphism which is identity on $\mathfrak{M}$.

* * *

We are ready for defining our notion of explicit definability. We call a new *sort or relation (explicitly) definable in* $\mathfrak{M}$ iff it is definable by repeated applications of the steps described in items (1), (2.1), (2.2) above.

A model $\mathfrak{N}$ is called a *definitional expansion of* $\mathfrak{M}$ iff $\mathfrak{N}$ is obtained from $\mathfrak{M}$ by repeated applications of steps (1), (2.1), (2.2) above (involving finitely many steps only). An *explicit definition of $\mathfrak{N}$ over $\mathfrak{M}$* is the union of the explicit definitions of type (1), (2.1), (2.2) involved in a sequence leading from $\mathfrak{M}$ to $\mathfrak{N}$. We call Δ an *explicit definition* if Δ is an explicit definition of some definitional expansion.

A model $\mathfrak{N}$ is called a *generalized definitional expansion* of $\mathfrak{M}$ if (i), (ii) below hold.

(i) $\mathfrak{N}$ is a reduct of a definitional expansion, say $\mathfrak{M}^+$, of $\mathfrak{M}$.

(ii) $\mathfrak{N}$ is an expansion of $\mathfrak{M}$, i.e. $\mathfrak{M}$ is a reduct of $\mathfrak{N}$.

We call $\mathfrak{N}$ *(explicitly) definable in* $\mathfrak{M}$ iff item (i) above holds. If we want to indicate that we do not take a reduct while defining say $\mathfrak{M}^+$ from $\mathfrak{M}$ explicitly (i.e. that $\mathfrak{M}^+$ is obtainable by repeatedly applying steps (1), (2.1), (2.2) to $\mathfrak{M}$) then we say that $\mathfrak{M}^+$ is *explicitly definable* in $\mathfrak{M}$ *without taking reducts*. Sometimes we write *"definitional expansion without taking reducts"* to emphasize that we mean definitional expansion and not generalized definitional expansion.

We emphasize that a precise statement claiming that U^{new} is definable as a new sort should also mention the relations and/or functions (of $\mathfrak{N}$) connecting U^{new} to the original sorts of $\mathfrak{M}$. Examples of such "connecting relations" are pj_i and $\in_{U^{new}}$ discussed above.

We note that *explicit definability with parameters* is completely analogous with implicit definability with parameters cf. p.223.

Let us turn to (explicit) definability over a *class* K of models (instead of over a single model $\mathfrak{M}$). We say that K is a(n explicit) *definitional expansion* of its reduct K^- iff K can be obtained from K^- by (a finite sequence of) repeated (uniform) appications of the steps described in items (1), (2.1), (2.2) on pp.231–235. This is equivalent to saying that there is an explicit definition which defines K over K^- (as an implicit definition). In this case we also say that *K is (explicitly) definable over (or in) K^- without taking reducts*. We say that K is a *generalized definitional expansion* of K^- if K is an expansion of K^- and K is a reduct of a definitinal expansion of K^-. We say that K is *(explicitly) definable in L* if K is a reduct of a definitional expansion of L.

This is completely analogous with the case of implicit definability. *Uniform (explicit) definability* and *one-by-one (explicit) definability* are obtained from the notion of (explicit) definability for single models the same way as their counterparts were obtained in the case of implicit definability, cf. pp. 225, 230.

Finally, we introduce one more notion of definability which we will call *rigid definability*. We will use this in our examples to come. About the importance of this notion see Theorem 4.3.31 on p.251.

Assume $\mathfrak{M}^+ = \langle \mathfrak{M}, \bar{U}^{new}; \bar{R}^{new} \rangle$ is an expansion of $\mathfrak{M}$ (with new sorts and relations). We say that $\mathfrak{M}^+$ is *(explicitly)* *rigidly definable over* $\mathfrak{M}$ if $\mathfrak{M}^+$ is definable in $\mathfrak{M}$ and the identity is the only automorphism of $\mathfrak{M}^+$ which is the identity on $\mathfrak{M}$. Informally, we will say that the new sorts and relations $\bar{U}^{new}, \bar{R}^{new}$ are rigidly definable over $\mathfrak{M}$ if $\langle \mathfrak{M}; \bar{U}^{new}, \bar{R}^{new} \rangle$ is rigidly definable over $\mathfrak{M}$.

Further, K^+ is *rigidly definable over* K iff K^+ is a generalized definitional expansion of K and each $\mathfrak{M}^+ \in \mathsf{K}^+$ is rigid(ly definable) over its K-reduct.

In our opinion, rigid definability is "just as good" as definability without taking reducts. In other words, we feel that if $\bar{U}^{new}$ etc. are rigidly definable over K then $\bar{U}^{new}$ etc. are almost as well determined by K (or describable in K) as if they were definable without taking reducts. We note that rigid definability seems to be perhaps, our most important (or most central) version of definability[515] (cf. e.g. Theorem 4.3.29, Theorem 4.3.31 and Theorem 4.3.48).

CONVENTION 4.3.13 Assume K^+ is a definitional expansion of K. For $\mathfrak{M}^+ \in \mathsf{K}^+$ the reduct $\mathfrak{M}^+ \restriction Voc\mathsf{K}$ may have more than one definitional expansions in K^+. (However these expansions are isomorphic.) Therefore K may have several different definitional expansions $\mathsf{K}^\oplus$ with the same set of defining formulas say Δ which defines K^+ from K. In such cases, of course we have $\mathbf{I}\mathsf{K}^\oplus = \mathbf{I}\mathsf{K}^+$. The largest such class is called a *maximal* definitional expansion of K. Since most of the time we will be interested in classes of models *closed under isomorphisms*, sometimes, but not always, we will concentrate on maximal definitional expansions. There are important exceptions to this[516], e.g. the class of two-sorted geometries[517] is not closed under isomorphisms and despite of this we will say that it is a definitional expansion of the class of one-sorted geometries (in Tarski's sense), under some conditions of course.

$\lhd$

Remark 4.3.14 (On isomorphism closure) In Convention 4.3.13 above, and in the definition of definitional equivalence "$\equiv_\Delta$" (p.255) way below, we are "navigating around" two different trends both present in the present work (i.e. we are trying to make the consequences of these two trends "consistent" with each other). These are the following.

Trend 1. When discussing definability over $\mathfrak{M}$ or over K, what we are really interested in is definability over $\mathbf{I}\{\mathfrak{M}\}$ or $\mathbf{I}\mathsf{K}$. More generally in the present work, most of the time, we tend to concentrate our attention to isomorphism-closed classes $\mathsf{K} = \mathbf{I}\mathsf{K}$ of models, moreover we are inclined to identify isomorphic models.

A motivation behind Trend 1 (i.e. isomorphism invariance) is that when discussing definability over a structure like $\mathfrak{M}$, we want to regard $\mathfrak{M}$ as an abstract structure (and not a concrete structure).[518] Cf. also the note on p.137 on this and cf. Remark 4.2.5 on p.149.

[515] Our definition of K^+ being explicitly definable over K is strongly related to the notion of K^+ being "*coordinatisable over*" K as defined in Hodges [130, p.644], while K^+ is rigidly definable over K is strongly related to "*coordinatised over*" as defined in [130] (same page). We will return to discussing this connection in the sub-section beginning on p.268.

[516] i.e. to concentrating on maximal definitional expansions

[517] in the sense of $\langle Points, Lines; \in \rangle$, cf. p.274

[518] Recall that a structure is called abstract if it is defined only up to isomorphism. I.e. when discussing an abstract structure we want to abstract from knowing what its elements are. (Since our foundation is set theory the elements of a structure $\mathfrak{A}$ are sets whose elements are again sets etc. When regarding a structure as abstract, we want to disregard these details about the elements of the elements of our structure.)

Trend 2. For purely aesthetical reasons, some of our distinguished classes of models are not quite closed under isomorphisms. E.g. in the definition of our class FM of frame models we insisted that the relation $\in$ connecting ${}^nF$ and G should be the real set theoretical membership relation.[519] This aesthetics motivated decision is the only reason why $\mathsf{FM} \neq \mathbf{I}\mathsf{FM}$. Similarly in our two-sorted geometries of the kind $\langle Points, Lines; \in\rangle$ we insisted that $Lines \subseteq \mathcal{P}(Points)$ and "$\in$" is the real set theoretic one. This is the only reason why our two-sorted geometries are not closed under isomorphisms.

If only Trend 1 were present then we could simplify much of the presentation in this sub-section by discussing only isomorphism closed classes $\mathsf{K} = \mathbf{I}\mathsf{K}$, $\mathsf{K}^+ = \mathbf{I}\mathsf{K}^+$ etc. However, we cannot carry through this simplification because Trend 2 presents a "purely administrative" obstacle to it. We call this obstacle purely administrative because the decision behind Trend 2 is purely aesthetical (everything would go through smoothly if we worked with $\mathbf{I}\mathsf{FM}$ in place of FM). As a consequence we do the following: On the intuitive level we tend to follow the simplifications suggested by Trend 1. At the same time, on the formal level we take Trend 2 into account in order to make our results (and definitions) applicable to classes like FM or to two-sorted geometries even when we take the formal details fully into account. Therefore on the formal level, we try to make sure that our definitions make sense (and mean what they should) even when $\mathsf{K} \neq \mathbf{I}\mathsf{K}$. We suggest that the reader keep in mind the "intuitive level" (when we use only Trend 1 and replace FM by $\mathbf{I}\mathsf{FM}$ etc.) and to treat the "formal level" as secondary, because this simplifies the picture *without* loosing any of the essential ideas.

$\lhd$

We close sub-section 4.3.2 with some examples. As an application, we also will apply the just defined notions to the geometries we defined earlier in this chapter.

Example 4.3.15 (Explicit definability of the rational numbers in the ring Z of integers.)
Let $\mathbf{Z} = \langle \mathrm{Z}; 0, 1, +, \cdot \rangle$ be the (usual) ring of integers. We will discuss how the set Q of rationals is definable explicitly as a *new sort* in $\mathbf{Z}$.[520] (Moreover with a little stretching of our terminology, we can say that the field $\mathbb{Q}$ of rationals is definable in $\mathbf{Z}$.) Here, the new *functions connecting* the new sort Q to the old one Z are (i) the ring-operations $+_{\mathrm{Q}}$ and $\cdot_{\mathrm{Q}}$ on the sort Q, and (ii) an injection $repr : \mathrm{Z} \rightarrowtail \mathrm{Q}$ representing the integers as rationals. The role of *repr* is to tell us which member of sort Z is considered to be equal with which member of the new sort Q. (Although the present "connecting-functions" do not coincide with our standard "explicit definability theoretical" ones pj_i and $\in$, we will see that they are first-order definable from the latter.)

Let us get started! We start out with $\mathbf{Z}$. First we define

$$R = \{ \langle a, b\rangle \ : \ a, b \in \mathrm{Z},\ b \neq 0 \}$$

as a new relation, obtaining the expansion $\langle \mathbf{Z};\ R\rangle$. Then we define the new sort U to be R with projections pj_0, pj_1 *and* for simplicity we forget R as a relation (but we keep it as a sort named U). This yields the definitional expansion

$$\mathbf{Z}^+ = \langle \mathrm{Z}, U;\ 0, 1, +, \cdot\ , pj_0, pj_1\rangle = \langle \mathbf{Z}, U;\ pj_0, pj_1\rangle$$

[519] This is so if we understand the definition of FM in accordance with Convention 2.1.3 on p.10. (Otherwise FM can be understood in such a way that it becomes closed under isomorphisms.)

[520] Although we promised, in §2, not to use the letter Q for other purposes than denoting the "quantity"-sort of our frame language, in the examples of the present sub-section we make an exception (since here there is no danger of creating a confusion).

where $pj_i : U \longrightarrow \mathrm{Z}$ are the usual. Next, we define the equivalence relation $\equiv$ on U as follows

$$\langle a, b\rangle \equiv \langle c, d\rangle \quad \overset{\text{def}}{\Longleftrightarrow} \quad a \cdot d = b \cdot c.$$

Note, that it is this point where we need the operations pj_i, namely, "$\langle a, b\rangle$" is not an expression of our first-order language, but we can simulate it by using the projections as follows. We define $\equiv$ by

$$x \equiv y \quad \overset{\text{def}}{\Longleftrightarrow} \quad pj_0(x) \cdot pj_1(y) = pj_1(x) \cdot pj_0(y),$$

where x, y are of sort Q. By using item (2.2) of our outline for definability, we define the *new sort* Q by $\mathrm{Q} := U/{\equiv}$ together with the usual membership relation $\in$ connecting sort U with sort Q.

Now, using the symbols $\in, pj_0, pj_1$ one can define the operations $+_{\mathrm{Q}}$, $\cdot_{\mathrm{Q}}$, *repr* as follows.

Assume $x \in \mathrm{Z}$ and $y \in \mathrm{Q}$. Then

$$repr(x) = y \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\exists z \in y)\,[\,pj_0(z) = x \ \wedge \ pj_1(z) = 1].$$

Assume $x, y, z \in \mathrm{Q}$. Then

$$\begin{aligned} x \cdot_{\mathrm{Q}} y = z \quad \overset{\text{def}}{\Longleftrightarrow} \quad & (\exists x' \in x)(\exists y' \in y)(\exists z' \in z) \\ & [\,pj_0(x') \cdot pj_0(y') = pj_0(z') \ \wedge \ pj_1(x') \cdot pj_1(y') = pj_1(z')]. \end{aligned}$$

The rest is easy, hence we omit it.

The above shows that the structure

$$\mathbf{Z}^{++} = \langle \mathbf{Z}, \mathrm{Q};\, +_{\mathrm{Q}}, \cdot_{\mathrm{Q}}, repr\rangle$$

is definable over $\mathbf{Z}^+$ hence it is also definable over $\mathbf{Z}$.

In passing, we note that the above definitional expansion makes sense and remains first-order if instead of $\mathbf{Z}$ we start out with an arbitrary ring, say $\mathfrak{A}$.

$\lhd$

Examples 4.3.16 (Geometries over fields)

1. Let $\mathbf{F}$ be a field. Consider the geometric expansion

$$\mathbf{G_F} := \langle \mathbf{F}, Points, Lines;\, pj_0, pj_1, \mathrm{E}\rangle$$

of $\mathbf{F}$ where $Points = F \times F$ and $pj_i : F \times F \longrightarrow F$ and $\mathrm{E} \subseteq Points \times Lines$ is the incidence relation (the usual way) and $Lines \subseteq \mathcal{P}(Points)$ is the set of lines in the Euclidean sense.

Then $\mathbf{G_F}$ *is rigidly definable* over $\mathbf{F}$. See the Hint in Example 2 below.

2. With each field $\mathbf{F}$ let $\mathbf{G_F}$ be associated as in item 1 above. Then

$$\mathsf{K}^+ := \{\, \mathbf{G_F} \ : \ \mathbf{F} \text{ is a field}\,\}$$

is rigidly definable (explicitly) over the class K of fields.[521]

Hint: First we define $Points = F \times F$ (with pj_i) as a new sort. Then we define

$$R = \{\, \langle p, q\rangle \in Points \times Points \ : \ p \neq q\,\},$$

as a new relation. Then we define the new auxiliary sort U to be R with the *new* projections $\overline{pj}_i : R \longrightarrow Points$ *and* we forget R as a relation (but we keep it as a sort named U). Then we define the equivalence relation $\equiv$ on U by saying

[521] From now on we will tend to omit "explicitly" since we agreed that definability automatically means explicit definability.

$$\langle p, q\rangle \equiv \langle r, s\rangle \stackrel{\text{def}}{\iff} (p, q, r, s \text{ are collinear in the Euclidean sense}).$$

Then we define the new sort *Lines* $:= U/\equiv$ together with $\in\ \subseteq U \times$ *Lines*. From these data we define our final incidence relation $\mathrm{E} := \mathrm{E}_{\mathrm{Points,Lines}}$ the usual way.[522]

$\lhd$

In the case of implicit definability we saw that uniform and one-by-one definability are wildly different. The example below is intended to demonstrate, for the case of explicit definability, the same kind of difference between uniform and one-by-one (explicit) definability. In this example we restricted ourselves to the most classical case: one sort only and the defined thing is a relation over the old sort. Besides providing explanation, this example was also designed to provide motivation for consistently sticking with the *uniform* versions of the kinds of definability we consider.

Example 4.3.17 Let $\underline{\omega} = \langle \omega; 0, 1, +, \cdot\ \rangle$ be the usual standard model of Arithmetic. Let us choose $R \subseteq \omega$ such that R is not explicitly definable even in higher-order logic over $\underline{\omega}$ (and even with parameters). Such an R exists.[523] Let

$$\mathsf{K} := \{\, \langle \underline{\omega};\ c, P\rangle \ :\ c \in \omega,\ P \subseteq \omega \text{ and } (c \in R\ \Rightarrow\ P = \{c\}) \text{ and } (c \notin R\ \Rightarrow\ P = \emptyset)\,\}.$$

Let K^- be the P-free reduct of K i.e.

$$\mathsf{K}^- := \{\, \langle \underline{\omega}, c\rangle \ :\ c \in \omega\,\}.$$

<u>*Claim:*</u> Each member $\mathfrak{M} = \langle \underline{\omega};\ c, P\rangle$ of K is *explicitly* definable over its P-free reduct $\mathfrak{M}^- = \langle \underline{\omega}, c\rangle$. I.e. K is one-by-one explicitly definable over its reduct K^-.

We will see that K is very far from *being uniformly explicitly* definable over K^-. (Moreover K is far from being uniformly finitely implicitly definable.)

For $n \in \omega$, we denote the constant-term $\underbrace{1 + \ldots + 1}_{n\text{-times}}$ by $\bar{n}$. Assume P is uniformly explicitly definable over K^-. Then

$$\mathsf{K} \models [\, P(x) \leftrightarrow \psi(c, x)\,],$$

[522] I.e. $p \mathrel{\mathrm{E}} \ell \stackrel{\text{def}}{\iff} (\exists x \in \ell)[\overline{pj_0}(x), \overline{pj_1}(x), p$ are collinear as computed in $\mathbf{F}\,]$.

[523] One can choose R to be so far from being computable that R is not even in the so-called Analytical Hierarchy cf. [44].

for some formula $\psi(x, y)$ in the language of $\underline{\omega}$.[524] Now, for any $n \in \omega$ we have the following:

$$\begin{array}{llllll} n \in R & \Rightarrow & [\,\mathsf{K} & \models & \bar{n} = c \to P(\bar{n}) & \text{hence} \\ & & \mathsf{K} & \models & \bar{n} = c \to \psi(c, \bar{n}) & \text{hence} \\ & & \mathsf{K}^- & \models & \bar{n} = c \to \psi(c, \bar{n}) & \text{hence} \\ & & \mathsf{K}^- & \models & \bar{n} = c \to \psi(\bar{n}, \bar{n}) & \text{hence}^{525} \\ & & \mathsf{K}^- & \models & \psi(\bar{n}, \bar{n}) & \text{hence} \\ & & \underline{\omega} & \models & \psi(\bar{n}, \bar{n})\,]. & \end{array}$$

$$\begin{array}{llllll} n \notin R & \Rightarrow & [\,\mathsf{K} & \models & \bar{n} = c \to \neg P(\bar{n}) & \text{moreover} \\ & & \mathsf{K} & \models & \bar{n} = c \to P = \emptyset & \text{hence} \\ & & \mathsf{K}^- & \models & \bar{n} = c \to \neg\psi(c, \bar{n}) & \text{hence} \\ & & \mathsf{K}^- & \models & \bar{n} = c \to \neg\psi(\bar{n}, \bar{n}) & \text{hence}^{525} \\ & & \underline{\omega} & \models & \neg\psi(\bar{n}, \bar{n})\,]. & \end{array}$$

But then $\psi(x, x)$ explicitly defines $R(x)$ in $\underline{\omega}$, which is a contradiction.

We have seen that while in K^- the new relation P is one-by-one explicitly definable (in other words locally explicitly definable), P is very far from being *uniformly* explicitly definable over the same K^-.

$\lhd$

We hope that the above construction and proof explain why and how one-by-one definability is so much weaker than[526] uniform definability. We also hope that the above example illustrates why most authors simply identify uniform definability with definability.

Application: definability of the observer-independent geometries

Now we turn to the issue of definability of the observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$ over the ("observational") frame models $\mathfrak{M}$, which has already been discussed in §4.2.2 and Remark 4.2.9 (p.153). The propositions and the theorems below serve to illuminate parts of Remark 4.2.9.

The following proposition says, roughly, that the set of points Mn, and our various kinds of lines $L, \dots, L^S$ are definable over the "observational" models $\mathfrak{M}$.

PROPOSITION 4.3.18 *For every frame model $\mathfrak{M}$ let $\mathfrak{M}^+ := \langle \mathfrak{M}, Mn; \in_{Mn} \rangle$ be the expansion of $\mathfrak{M}$ with the set of events $Mn := \bigcup \{ Rng(w_m) : m \in Obs \}$ and the set theoretic membership relation $\in_{Mn} \subseteq B \times Mn$. Let*

$$\mathsf{FM}^+ := \{ \mathfrak{M}^+ : \mathfrak{M} \in \mathsf{FM} \}.$$

Then (i) and (ii) below hold.

(i) FM^+ *is <u>rigidly definable</u> over the class* FM *of frame models.*

[524] This is so because $\psi(c, x)$ is in the language of K^-, which is the same as the language of $\underline{\omega}$ expanded with a constant symbol c.

[525] by $\mathsf{K} \not\models n \neq c$ (i.e. by $(\exists \mathfrak{M} \in \mathsf{K})\mathfrak{M} \models n = c$) and since under any evaluation of the variables (in a member of K) the value of the constant term $\bar{n}$ coincides with the element n of ω.

[526] One-by-one definability is not only weaker than uniform definability, but also it is much *less satisfactory* from the point of view of re-capturing the intuitive idea of definability. In our opinion one-by-one definability does not capture the intuitive notion of definability while uniform definability does. (All the same, one-by-one definability is useful as a mathematical *auxiliary* concept.)

(ii) *For every* $\mathfrak{M}^+ \in \mathsf{FM}^+$ *let* $\mathfrak{M}^{++} := \langle \mathfrak{M}^+, L; L^T, L^{Ph}, L^S, \in_L\rangle$ *be the expansion of* $\mathfrak{M}^+$, *where* L^T, L^{Ph}, L^S, L *are, respectively, the sets of time-like, photon-like, space-like, and all lines as defined in item 4 of Def.4.2.3(I); and* $\in_L \subseteq Mn \times L$ *is the membership (or equivalently the incidence) relation between points (elements of Mn) and lines. Then the class*

$$\mathsf{FM}^{++} := \{\, \mathfrak{M}^{++} \;:\; \mathfrak{M}^+ \in \mathsf{FM}^+ \,\}$$

is rigidly definable over the class FM *of frame models.*

Proof:
Proof of (i): The new sort ${}^nF$ together with the projection functions are rigidly definable over FM, therefore we will pretend that ${}^nF$ is an old sort of FM. In defining FM^+ over FM up to unique isomorphism, first we define

$$R := \{\, \langle m,p\rangle \in B \times {}^nF \;:\; m \in Obs\,\}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1 *and* we forget R as a relation (but we keep it as a sort named U). Then we define the equivalence relation $\equiv$ on U by saying

$$\langle m,p\rangle \equiv \langle k,q\rangle \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad w_m(p) = w_k(q); \quad \text{formally:}$$
$$\langle m,p\rangle \equiv \langle k,q\rangle \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall b \in B)\,[\, W(m,p,b) \leftrightarrow W(k,q,b)\,];$$

while if we want to get rid of the notation "$\langle m,p\rangle$" we can write the following. Let $a, d \in U$. Then

$$a \equiv d \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall b \in B)\,[\, W(pj_0(a), pj_1(a), b) \leftrightarrow W(pj_0(d), pj_1(d), b)\,].$$

Then we define the new sort $Mn := U/\!\equiv$ together with $\in\, \subseteq U \times Mn$. From these data finally we define the "membership" relation $\mathrm{E}_{Mn} \subseteq B \times Mn$ as follows. Let $b \in B$ and $e \in Mn$. Then

$$b \;\mathrm{E}_{Mn}\; e \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists a \in e)\, W(pj_0(a), pj_1(a), b).$$

So far we have defined Mn and E_{Mn}, hence all parts of (an isomorphic copy of) FM^+ have been defined (over FM). The "rigid-ness" (i.e. "uniqueness") part of definability stated in (i) comes from the fact that the axiom of extensionality holds for E_{Mn}, i.e.

$$(\forall e, e_1 \in Mn)\,[\, e = e_1 \;\leftrightarrow\; (\forall b \in B)(b \;\mathrm{E}_{Mn}\; e \leftrightarrow b \;\mathrm{E}_{Mn}\; e_1)\,].$$

Proof of (ii): By item (i) it is sufficient to prove that FM^{++} is rigidly definable over FM^+. In defining FM^{++} over FM^+ up to unique isomorphism, first we define

$$R := \{\langle h,i\rangle \in B \times F \;:\; h \in Obs \cup Ph,\; i \in n,\; (h \notin Obs \;\Rightarrow\; i = 0)\,\}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1 *and* we forget R as a relation (but we keep it as a sort named U). Intuitively, the elements of U will code the lines.[527] We define a kind of incidence relation $E \subseteq Mn \times U$ as follows. Let $e \in Mn$ and $\ell \in U$. Then

[527] We code lines by elements of U according to the following intuition. Photon-like lines and time-like lines are coded by $\langle h,0\rangle$ where h is a photon or an observer (then $\langle h,0\rangle$ codes the life-line of h). Space-like lines are coded by an observer h and an axis $\bar{x}_i$ $(i \neq 0)$ and the coded line is what h sees on the $\bar{x}_i$ axis i.e. it is $w_h[\bar{x}_i]$.

$$e \; E \; \ell \quad \stackrel{\text{def}}{\Longleftrightarrow}$$
$$[\, pj_1(\ell) = 0 \;\wedge\; pj_0(\ell) \in_{Mn} e \,] \;\vee\; \bigvee_{0<i\in n} [\, pj_1(\ell) = i \;\wedge\; (\exists q \in \bar{x}_i)\,(e = w_{pj_0(\ell)}(q))^{528} \,].$$

Then we define the equivalence relation $\equiv$ on U as follows. Let $\ell, \ell' \in U$. Then

$$\ell \equiv \ell' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall e \in Mn)\,(e \; E \; \ell \;\leftrightarrow\; e \; E \; \ell').$$

We define the sort $L := U/\!\equiv$ together with the membership relation $\in \subseteq U \times L$. Now, the "membership" (or incidence) relation $\mathrm{E}_L \subseteq Mn \times L$ is defined as follows. Let $e \in Mn$ and $\ell \in L$. Then

$$e \; \mathrm{E}_L \; \ell \;\stackrel{\text{def}}{\Longleftrightarrow}\; (\exists \ell' \in \ell) \; e \; E \; \ell'.$$

Finally, the unary relations L^T, L^{Ph}, L^S on L are defined as

$$\begin{aligned}
L^T &:= \{\, \ell \in L \,:\, (\exists \ell' \in \ell)\,(\, pj_0(\ell') \in Obs \;\wedge\; pj_1(\ell') = 0 \,)\,\},\\
L^{Ph} &:= \{\, \ell \in L \,:\, (\exists \ell' \in \ell)\,(\, pj_0(\ell') \in Ph \;\wedge\; pj_1(\ell') = 0 \,)\,\},\\
L^S &:= \{\, \ell \in L \,:\, (\exists \ell' \in \ell)\,(\, pj_0(\ell') \in Obs \;\wedge\; pj_1(\ell') > 0)\,\}.
\end{aligned}$$

So far we have defined L, L^T, L^{Ph}, L^S and E_L, hence all parts of (an isomorphic copy of) FM^{++} have been defined (over FM^{+}). The "rigid-ness" (i.e. "uniqueness") part goes exactly as in the case of (i). ∎

Our next proposition says, roughly, that the topology part $\mathcal{T}$ (of our geometries) is definable over the "observational" models $\mathfrak{M}$.

PROPOSITION 4.3.19 *Let* FM^{+} *be as in Proposition 4.3.18 above. For every* $\mathfrak{M}^+ \in \mathsf{FM}^+$ *let* $\langle \mathfrak{M}^+, T_0; \in \rangle$ *be the expansion of* $\mathfrak{M}^+$ *with the subbase*

$$T_0 = \big\{ S(e, \varepsilon) \;:\; e \in Mn,\; \varepsilon \in {}^{+}F \,\big\}$$

for the topology $\mathcal{T}$ *(as defined in item 13 of Def.4.2.3(I)) and with the (standard) membership relation* $\in \subseteq Mn \times T_0$. *Then the class*

$$\mathsf{FM}^{++} := \big\{\, \langle \mathfrak{M}^+, T_0; \in \rangle \;:\; \mathfrak{M}^+ \in \mathsf{FM}^+ \,\big\}$$

is rigidly definable over the class FM *of frame models. Roughly, this means that the ("heart" of the) topology part of our geometries associated with* FM *is also definable over* FM*, but cf. the discussion in* $(\star\star\star)$ *of Remark 4.2.9 on p.155.*

Proof: Let $\mathsf{FM}^+, \mathsf{FM}^{++}$ be as above. By Prop.4.3.18(i) it is sufficient to prove that FM^{++} is rigidly definable over FM^+. In defining FM^{++} over FM^+ up to unique isomorphism, first we define the pseudo-metric $g : Mn \times Mn \stackrel{\circ}{\longrightarrow} F$ as a new relation as it was defined in item 12 of Def.4.2.3(I) (p.145). It can be checked that the just quoted definition of g can be translated into a first-order formula in the language of FM^+. Then we define

$$R := \big\{\, \langle e, \varepsilon \rangle \;:\; e \in Mn,\; \varepsilon \in {}^{+}F \,\big\}$$

[528] We note that "$e = w_{pj_0(\ell)}(q)$" is a formula since the following is a formula.
$(\forall b \in B)\,[\, b \in e \;\leftrightarrow\; W(pj_0(\ell), q, b)\,]$.

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1 *and* we forget R as a relation (but we keep it as a sort named U). Then we define the equivalence relation $\equiv$ on U by saying

$$\langle e, \varepsilon\rangle \equiv \langle e_1, \varepsilon_1\rangle \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall e_2 \in Mn)\,(\,g(e, e_2) < \varepsilon \;\leftrightarrow\; g(e_1, e_2) < \varepsilon_1\,).$$

(Of course one uses the projection functions pj_0, pj_1 to formalize the above definition of $\equiv$.)

Then, we define the new sort $T_0 := U/\equiv$ together with the membership relation $\in_{U,T_0} \subseteq U \times T_0$. Finally we define the "membership" relation $\mathrm{E} \subseteq Mn \times T_0$ as follows. Let $e \in Mn$, $A \in T_0$. Then

$$e\ \mathrm{E}\ A \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists a \in U)\,[\,a \in_{U,T_0} A \;\wedge\; g(pj_0(a), e) < pj_1(a)\,].$$

So far we have defined T_0 and E, hence all parts of (an isomorphic copy of) FM^{++} have been defined (over FM^{+}). The "rigid-ness" (i.e. "uniqueness") part goes exactly as in the case of Prop.4.3.18(i). ∎

In connection with the following two propositions recall that alternative versions $\mathcal{T}'$ and $\mathcal{T}''$ of the topology part $\mathcal{T}$ of our geometries were defined in Def.4.2.30 (p.175). Further, T_0' and T_0'' are subbases for $\mathcal{T}'$ and $\mathcal{T}''$, respectively, as defined in Def.4.2.30.

PROPOSITION 4.3.20 *Proposition 4.3.19 remains true if we replace T_0 by T_0'' in it, where T_0'' was defined in Def.4.2.30(ii).*

We omit the easy **proof**.

Our next proposition says, roughly, that the topology part $\mathcal{T}'$ (of our geometries) is definable over the "observational" models $\mathfrak{M}$, assuming $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$.

PROPOSITION 4.3.21

(i) *For every frame model $\mathfrak{M}$ let $\mathfrak{M}^+$ be defined as in Prop.4.3.18, i.e. $\mathfrak{M}^+ := \langle \mathfrak{M}, Mn; \in_{Mn}\rangle$. Further, let $\langle \mathfrak{M}^+, T_0'; \in\rangle$ be the expansion of $\mathfrak{M}^+$ with the subbase T_0' for $\mathcal{T}'$, where T_0' is defined in Def.4.2.30(i); and with the membership relation $\in \subseteq Mn \times T_0'$. Then the class*

$$\mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }))^+ := \{\, \langle \mathfrak{M}^+, T_0'; \in\rangle \;:\; \mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }))\,\}$$

is <u>rigidly definable</u> over the class $\mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }))$.

(ii) *Since in $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$ T_0' is a base for our topology $\mathcal{T}'$, for all practical purposes (i) "means" that the <u>topology</u> $\mathcal{T}'$ <u>is definable over</u> these <u>models</u> (cf. $(\star\star\star)$ on p.155).*

We omit the **proof**, but we note that a proof can be obtained using Propositions 4.2.16 (p.161), 4.2.64 (p.208), cf. also Thm.4.2.35 (p.179) and the proof of Thm.4.2.33 (p.177). ∎

Our next three theorems say, roughly, that our class $\mathsf{Ge}(Th)$ of relativistic geometries is definable over the corresponding class of observational models.[529]

[529] These three theorems were stated as Theorem 4.2.40 in the previous sub-section on p.182.

THEOREM 4.3.22 *The class* $\mathsf{Ge}(Th)$ *is* <u>*definable*</u> *over the class* $\mathsf{Mod}(Th)$*, assuming that* $n > 2$ *and* Th *is a set of formulas in our frame language such that* $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^{-} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax(diswind)} + \mathbf{Ax}(\sqrt{\ })$.
More precisely, instead of definability of the topology part $\mathcal{T}$ *we claim definability of only a subbase* T_0 *for* $\mathcal{T}$*, together with* $\in \subseteq Mn \times T_0$ *of course.*[530]

Proof: The theorem follows by Propositions 4.3.18 (p.240), 4.3.19 (p.242) and by Theorems 4.2.11 (p.158), 4.2.19 (p.163) and 4.2.22 (p.168). Cf. Remark 4.2.9 (p.153). ∎

Conjecture 4.3.23 *We conjecture that in the above theorem* $\mathbf{Ax(diswind)}$ *is needed (because we conjecture that* $\perp_r$ *is not first-order definable in* $\mathsf{Mod}(Th \setminus \{\mathbf{Ax(diswind)}\} + \mathbf{Ax})$*, where* Th *is as in Thm.4.3.22 above and* $\mathbf{Ax}$ *is*[531] $\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)} + \mathbf{Ax(eqtime)}$*), cf. Figure 93.* ◁

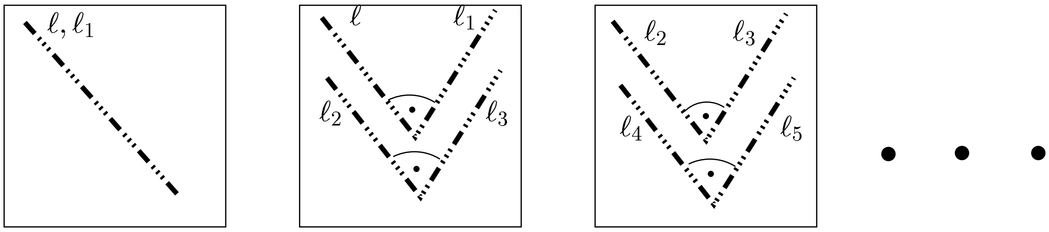


Figure 93: We conjecture that $\mathbf{Ax(diswind)}$ is needed in Thm.4.3.38, i.e. that without assuming $\mathbf{Ax(diswind)}$ $\perp_r$ is not definable. (Hint: $\ell, \ell_1, \ldots \in L^{Ph}$, $\ell \perp_r \ell_1$ by closing $\perp_r$ up under limits; and $\ell \perp_r \ell_1 \Rightarrow \ell_2 \perp_r \ell_3 \Rightarrow \ell_4 \perp_r \ell_5 \Rightarrow \ldots$, by closing $\perp_r$ up under parallelism.)

The theorem below says that if in Thm.4.3.22 above **Basax** is assumed in place of $\mathbf{Bax}^{\oplus}$ then the assumptions $n > 2$, $\mathbf{Ax}(\|)^{-}$ and $\mathbf{Ax(diswind)}$ are not needed.

THEOREM 4.3.24 *The class* $\mathsf{Ge}(Th)$ *is* <u>*definable*</u> *over the class* $\mathsf{Mod}(Th)$*, assuming that* Th *is a set of formulas in our frame language such that* $Th \models \mathbf{Basax} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ })$. *(More precisely instead of definability of* $\mathcal{T}$ *we claim definability of* T_0 *only.)*

Proof: The theorem follows by Propositions 4.3.18 (p.240), 4.3.19 (p.242) and by Theorems 4.2.11 (p.158), 4.2.21 (p.166). Cf. the proof of Thm.4.3.22 and Remark 4.2.9 (p.153). ∎

In connection with the next two theorems recall that $\mathsf{Ge}'(Th)$ and $\mathsf{Ge}''(Th)$ are alternative versions of $\mathsf{Ge}(Th)$ and are introduced in Definition 4.2.39 (p.181).

[530] Cf. the discussion of definability of $\mathcal{T}$ in $(\star\star\star)$ of Remark 4.2.9 on p.155.

[531] For the relevance of **Ax** cf. Thm.4.3.38 (p.261). In brief, we will have to add **Ax** to Th when proving the other direction, i.e. that $\mathsf{Mod}(Th)$ is also definable over $\mathsf{Ge}(Th)$.

THEOREM 4.3.25

(i) $\mathsf{Ge}'(Th)$ *is* *__definable__* *over* $\mathsf{Mod}(Th)$*, for any set* Th *of formulas in our frame language. (More precisely instead of definability of* $\mathcal{T}$ *we claim definability of* T_0 *only.)*

(ii) $\mathsf{Ge}''(Th)$ *is* *__definable__* *over* $\mathsf{Mod}(Th)$*,*[532] *assuming that* Th *is a set of formulas in our frame language such that* $Th \models \mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$.

Proof: The theorem follows by Propositions 4.3.18, 4.3.19, 4.3.20, 4.3.21. Cf. the proof of Thm.4.3.22 and Remark 4.2.9 (p.153). ∎

4.3.3 Eliminability of defined concepts.

Notation 4.3.26 For a class K of (many-sorted, similar) models, $Fm(\mathsf{K})$ denotes the set of formulas of the language of K. Hence $\mathsf{Th}(\mathsf{K}) \subseteq Fm(\mathsf{K})$. Sometimes we refer to $Fm(\mathsf{K})$ as the __language__ of K.[533]

$\lhd$

THEOREM 4.3.27 (First translation theorem) *Let* K *and* K^+ *be two classes of (many-sorted) models. Assume that* K^+ *is a generalized expansion of* K*. Then there is a "natural" translation mapping*

$$Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$$

having the following property (called preservation of meaning):[534]

(⋆) *Assume* $\psi(\bar{x}) \in Fm(\mathsf{K}^+)$ *is such that all its free variables (indicated as* $\bar{x}$*) belong to "old"*[535] *sorts, i.e. to sorts of* K*. Then*

$$\mathsf{K}^+ \models [\,\psi(\bar{x}) \leftrightarrow Tr(\psi)(\bar{x})\,].$$

Further, for all $\psi \in Fm(\mathsf{K}^+)$

$$\mathsf{K}^+ \models \psi \Leftrightarrow \mathsf{K} \models Tr(\psi).$$

Moreover, Tr *is very simple (transparent) from the computational point of view, e.g. it is Turing-computable in linear time.*

[532] Cf. Prop.4.3.21(ii) in connection with definability of $\mathcal{T}'$ in $\mathsf{Ge}''(Th)$.

[533] According to our philosophy, $Fm(\mathsf{K})$ is the language, while the system of basic symbols (like relation symbols, sort symbols etc.) is the *vocabulary* of this language, cf. Convention 4.3.1 on p.220. We note this because some logic books use the word "language" for what we call the vocabulary (of a language or a model).

[534] The existence of such a translation mapping Tr is often called in the literature "*__uniform reduction property__*", cf. Hodges [130, p.640]. A result of Pillay and Shelah is that for first order axiomatizable classes implicit definability without taking reducts implies the reduction property, cf. [213]. Cf. also Lemma 12.5.1 in Hodges [130, p.641].

[535] A symbol (e.g. a sort) is called old if it is available already in K (and not only in K^+).

Theorem 4.3.27 follows from the stronger Theorem 4.3.29 (and its proof) to be stated soon, so we do not prove it here.

COROLLARY 4.3.28 Let K and K^+ be classes of one-sorted models such that the name of their sorts agree. Then K is definable in K^+ in the classical sense, i.e. such that we allow only step (1) in the definitions iff K is definable in K^+ in the new many-sorted sense, i.e. such that we allow the use of steps (1) - (2.2). In other words, the possibility of defining new universes (and then forgetting them) does not create new definitional expansions among one-sorted models. $\lhd$

Before stating the stronger version of Theorem 4.3.27, let us ask ourselves in what sense *Tr* in Thm.4.3.27 preserves the meanings of formulas. To answer this question, let us notice that the conclusion of Theorem 4.3.27 implies (i) and (ii) below.

(i) ψ and $Tr(\psi)$ have the same free variables $\bar{x}$, and in some intuitive sense they say the same thing about these variables $\bar{x}$.

(ii) Let $\mathfrak{M} \in \mathsf{K}^+$, $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ in K and let $\bar{a}$ be a sequence of members of $Uv(\mathfrak{M}^-)$ matching the sorts of $\bar{x}$. In other words $\bar{a}$ is an evaluation of the variables $\bar{x}$. Then

$$\mathfrak{M} \models \psi[\bar{a}] \quad \Longleftrightarrow \quad \mathfrak{M}^- \models (Tr(\psi))[\bar{a}]\,;$$

cf. the notation on p.231. Intuitively, *whatever can be said about* some "old" elements $\bar{a}$ in a model $\mathfrak{M}$ in K^+, *it can be said* (about the same elements $\bar{a}$) <u>*already*</u> in the "old" model $\mathfrak{M}^-$ (in K). This will be generalized to "new" elements also (i.e. to arbitrary elements), in our next theorem.

Recall that K is a reduct of K^+. In some sense (i) and (ii) above mean that the poorer class K and the richer class K^+ of models are equivalent from the point of view of expressive power of language. So, the "language + theory" of K^+ is equivalent to the "language + theory" of K in means of expression. Therefore, on some level of abstraction, we may consider the languages of K and K^+ to be the *same* except that they[536] choose different "basic vocabularies" for representing this language. (In passing we note that a stronger form of this kind of *sameness* will appear in the form of definitional equivalence $\equiv_\Delta$, cf. beginning with p.255 (and the figure on p.260).)

Generalization of Theorem 4.3.27 to permitting free variables of new sorts to occur in ψ and $Tr(\psi)$

Let us turn to discussing the restriction in Theorem 4.3.27 which says (in statement $(\star)$) that the free variables of ψ belong to the sorts of K. The theorem does admit a generalization which is without this restriction on the free variables. This will be stated in Theorem 4.3.29 below. But then two things happen discussed in items (I), (II) below.

(I) Consider the process of defining K^+ over K as a sequence of steps (as described on p.235). Assume that a relation like pj_i or $\in_U$ connecting a new sort to an old one is introduced in one step and then is *forgotten* at the (last) reduct step. Then we call the relation (e.g. pj_i) in question an <u>*auxiliary relation*</u> of the definition of K^+ over K. Now, for the generalization of Theorem 4.3.27 we have in mind, we have to assume that all

[536] i.e. K and K^+

auxiliary relations (of the definition of K^+) remain definable in K^+. We will formulate this condition as "K^+ and K have a common (explicit) definitional expansion (without taking reducts)".

That K^+ and K have a common definitional expansion expresses that K^+ *is definable over* K *with recoverable auxiliaries* because of the following. Assume that K^{++} is a common definitional expansion of K^+ and of K. Then K^+ is a reduct of K^{++} which is a definitional expansion of K, hence K^+ is definable in K. Also, all the relations and sorts that get forgotten in the reduct-forming from K^{++} to K^+ are definable in K^+ since K^{++} is a definitional expansion of K^+.

(II) The formulation of the theorem gets somewhat complicated. Intuitively, the generalized theorem says that all new objects[537] can be represented as equivalence classes of tuples of old objects, and then (using this representation) whatever can be said about elements of $Uv(\mathfrak{M})$ in an expanded model $\mathfrak{M} \in \mathsf{K}^+$ can be already said in the reduct $\mathfrak{M}^- \in \mathsf{K}$ of $\mathfrak{M}$. This intuitive statement is intended to serve as a generalization the text below item (ii) in the discussion of the intuitive meaning of Theorem 4.3.27 (presented immediately below Theorem 4.3.27). Cf. Figure 94.

<u>*Notation:*</u> $Var(U_i)$ denotes the (infinite) set of variables of sort U_i (where U_i is treated as a sort symbol or *equivalently* U_i is the name of one of the universes of the models in K^+).

THEOREM 4.3.29 (Second translation theorem) *Assume* K *is a reduct of* K^+ *and* K *and* K^+ *have a common definitional expansion (without taking reducts). This holds e.g. whenever* K^+ *is a definitional expansion of* K. *Assume* $U_1^{new}, \dots, U_k^{new}$ *are the new sorts.*[538] *Then there is a translation mapping*

$$Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$$

for which the following hold. For each U_i^{new} *there is a formula* $code_i(x, \vec{x}) \in Fm(\mathsf{K}^+)$ *such that the following 1-2 hold.*

1. $x \in Var(U_i^{new})$ *and* $\vec{x}$ *is a sequence of variables of old sorts.*

2. *(a)–(c) below hold.*

   (a) $\mathsf{K}^+ \models \forall x\, \exists \vec{x}\; code_i(x, \vec{x})$, [539]

   (b) $\mathsf{K}^+ \models [\, code_i(x, \vec{x}) \;\wedge\; code_i(y, \vec{x})\,] \;\rightarrow\; x = y$, *where* $y \in Var(U_i^{new})$. [540]

   (c) *Our translation mapping*[541]

$$Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$$

[537] By objects we mean elements of some sort.

[538] I.e. they are available in K^+ but not in K.

[539] Note that here "$\forall x$" means "$\forall x \in U_i^{new}$" automatically since we know that x is of sort U_i^{new} (as a variable symbol of the language of K^+).

[540] Note that items (a), (b) mean that $code_i$ represents an unambiguous coding of elements of U_i^{new} with equivalence classes of tuples of elements of old sorts, cf. (II) preceding the statement of the theorem and the text immediately below the theorem.

[541] fixed at the beginning of the formulation of the present theorem

satisfies the following stronger[542] *property of meaning preservation. Assume* $\psi(y,\bar{z}) \in Fm(\mathsf{K}^+)$ *is such that* $y \in \mathrm{Var}(U_i^{new})$ *and* $\bar{z}$ *is (a sequence of variables) of old sorts such that the variables in* $\bar{z}$ *are* distinct *from those occurring in* $\vec{y}$*. Then*

$$\mathsf{K}^+ \models code_i(y,\vec{y}) \to [\,\psi(y,\bar{z}) \leftrightarrow (\mathrm{Tr}(\psi))(\vec{y},\bar{z})\,].$$

Intuitively, whatever is said by ψ *about* y *and* $\bar{z}$*, the same is said by the translated formula* $\mathrm{Tr}(\psi)$ *about the code* $\vec{y}$ *of* y *and* $\bar{z}$*. The case when* ψ *contains an arbitrary sequence, say* $\bar{y}$*, of variables of various new sorts is a straightforward generalization and is left to the reader.*

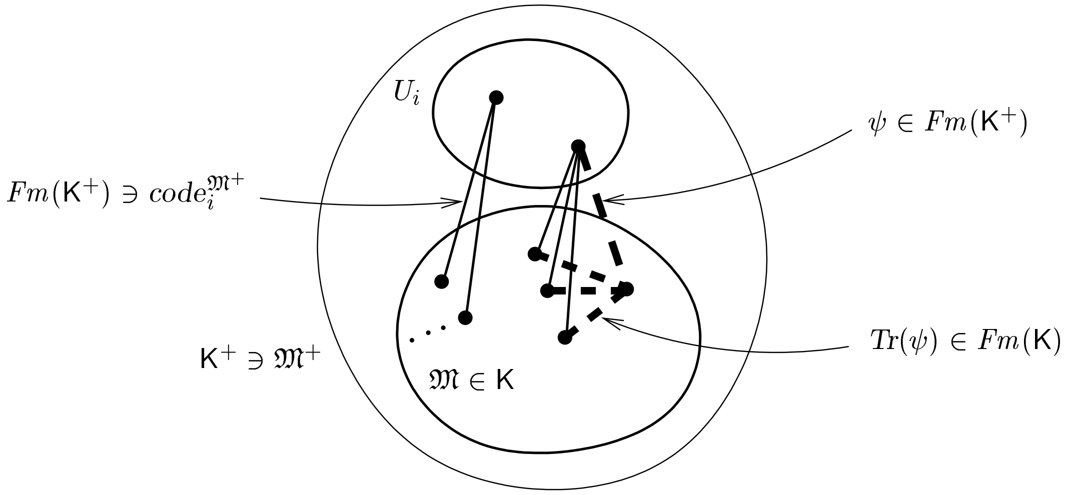


Figure 94: Illustration for the second translation theorem (Thm.4.3.29). Whatever can be said of a new element in $\mathfrak{M}^+$ can be said of its "code" in the old model $\mathfrak{M}$. (In the Figure, the codes of the new elements have length 1.)

We note that the intuitive meaning of "$code_i(x,\bar{y})$" is "$\bar{y}$ codes x". Property (b) then says that "$\bar{y}$ codes only one element", property (a) says that "every new element has a code", and property (c) then tells us that "whatever can be said of a new element x in the new language, can be said of any of its codes $\bar{y}$ in the old language", cf. (II) before the statement of Theorem 4.3.29 and Figure 94.

Proof:

(I) The case of step (2.1): Assume that K^+ is obtained from K by applying step (2.1) so that we defined $U^{new} \stackrel{\text{def}}{=} R$ where R is an old r-ary relation. For simplicity we assume $r=2$ and $R \subseteq U_0 \times U_1$ where U_0, U_1 are old sorts. Then the new symbols (in K^+) are U^{new} and pj_0, pj_1. We want to represent objects (variables) of sort U^{new} with pairs of objects of ("old") sorts. To this end, we fix an injective function

$$rep : Var(U^{new}) \rightarrowtail Var(U_0) \times Var(U_1)$$

such that the values $rep(x)_i$ of rep are all distinct.[543] For simplicity, we will denote $rep(x)_i$ by x_i. We also assume that x_0, x_1 do not occur in the formulas to be translated.

[542]stronger than in Theorem 4.3.27

[543]$rep(x) = \langle rep(x)_0, rep(x)_1\rangle$; and $rep(x)_i = rep(y)_j$ iff $\langle x,i\rangle = \langle y,j\rangle$.

Now, we define Tr by recursion as follows.

- $Tr((\exists x \in U^{new})\psi) := (\exists x_0 \in U_0, x_1 \in U_1)[R(x_0, x_1) \wedge Tr(\psi)]$; if $x \in Var(U^{new})$;
- $Tr((\exists y)\psi) := (\exists y)Tr(\psi)$; if y is a variable of old sort;
- $Tr(\neg\psi) := \neg Tr(\psi)$, $\qquad Tr(\psi \wedge \varphi) := Tr(\psi) \wedge Tr(\varphi)$;
- $Tr(x = y) := (x_0 = y_0 \ \wedge \ x_1 = y_1)$, for any $x, y \in Var(U^{new})$;
- for any other *atomic* formula ψ, $Tr(\psi)$ is obtained from ψ by replacing each occurrence of $pj_i(x)$ with x_i (i.e. with $rep(x)_i$) in ψ for *every* variable $x \in Var(U^{new})$ and $i \in 2$; i.e. $Tr(\psi) := \psi(pj_i(x)/x_i)_{x\in \mathrm{Var}(U^{new}), i<2}$.

We introduce the formula $code(x, x_0, x_1)$ (saying explicitly that the values of x_0, x_1 form really the code of the value of x) as follows:

$$code(x, x_0, x_1) \quad \overset{\text{def}}{\Longleftrightarrow} \quad [\, x_0 = pj_0(x) \ \wedge \ x_1 = pj_1(x) \ \wedge \ R(x_0, x_1)\,].$$

Now, it is not difficult to check that $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ is well defined, and (a)-(c) in the statement of Theorem 4.3.29 hold.

(II) The case of step (2.2): Assume that K^+ is obtained from K by applying step (2.2) so that the only new symbols (in K^+) are $U^{new} = U/R$ and $\in$, where U is an (old) sort of K, and $R(x, y) \in Fm(\mathsf{K})$ where x, y are variables of sort U.

We fix an injective function

$$rep : Var(U^{new}) \rightarrowtail Var(U)$$

and we denote $rep(x)$ by $\underline{x}$. So $\underline{x} \in Var(U)$ if $x \in Var(U^{new})$. As before, we assume that the variables $\underline{x}$ do not occur in the formulas to be translated.

Now, we define Tr by recursion as follows.

- $Tr((\exists x \in U^{new})\psi) := (\exists \underline{x} \in U)Tr(\psi)$; if $x \in Var(U^{new})$;
- $Tr((\exists y)\psi) := (\exists y)Tr(\psi)$; if y is a variable of old sort;
- $Tr(\neg\psi) := \neg Tr(\psi)$, $\qquad Tr(\psi \wedge \varphi) := Tr(\psi) \wedge Tr(\varphi)$;
- $Tr(x = y) := R(\underline{x}, \underline{y})$, where $x, y \in Var(U^{new})$;
- $Tr(\in(z, x)) := R(z, \underline{x})$ and $Tr(\psi) := \psi$; for any other *atomic* formula ψ with no variables of new sort.

We introduce the formula $code(x, \underline{x})$ as follows:

$$code(x, \underline{x}) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \in(\underline{x}, x)\,.$$

Now, it is not difficult to check that $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ is well defined, and (a)-(c) in the statement of Theorem 4.3.29 hold.

(III) The case of explicit definability without taking reducts: If K^+ is obtained from K by step (1) then we have an obvious translation with all the good properties known from classical definability theory.[544]

[544] In the case of step (1), "*code*" is not needed because there are no new sorts involved. Hence (*if* we want to preserve uniformity of the steps) we can choose $code(x, y)$ to be $x = y$.

By this we have covered all the steps (i.e. (1)–(2.2)) which might occur in an explicit definition. I.e. we defined *code*, *Tr* to all three kinds of "one-step" explicit definitions represented by items (1)–(2.2).

Assume now that K^+ is explicitly defined over K without taking reducts. Now, the definition of K^+ is a finite sequence of steps with each step using one of items (1), (2.1), (2.2). Hence by the above, we have a meaning preserving translation mapping Tr_k for the k'th step for each number

$$k < n := \text{“number of steps in the definition of } \mathsf{K}^+\text{”}.$$

Besides Tr_k we also have a formula $code_k$ for each number k. Also for each Tr_k we have that (a)-(c) in the statement of Theorem 4.3.29 hold. But then we can take the composition $Tr := Tr_1 \circ Tr_2 \circ \ldots \circ Tr_n$ of these meaning preserving functions, and then the composition too will be meaning preserving if we also combine the formulas $code_1, \ldots, code_n$ into a single "big" formula *code*.

One can check that for the just defined *Tr* and *code*, (a)-(c) in the statement of Theorem 4.3.29 hold.

(IV) The general case: Assume now that K is a reduct of K^+ and that K^{++} is a common definitional expansion of K and K^+. By the previous case we have translation mappings $Tr_1 : Fm(\mathsf{K}^{++}) \longrightarrow Fm(\mathsf{K})$ and $Tr_2 : Fm(\mathsf{K}^{++}) \longrightarrow Fm(\mathsf{K}^+)$ together with appropriate $code^1, code^2$ which satisfy (a)-(c) in the statement of Theorem 4.3.29. Note that $Fm(\mathsf{K}^+) \subseteq Fm(\mathsf{K}^{++})$. Now we define

$$Tr \stackrel{\text{def}}{=} Tr_1 \restriction Fm(\mathsf{K}^+),\ \ code(x, \vec{x}) \stackrel{\text{def}}{=} Tr_2(code^1(x, \vec{x}))$$

whenever x is a variable of new sort in the language of K^+. One can check that *Tr* and *code* as defined above satisfy (a)-(c). In more detail: Assume that U_i is a new sort of K^+, i.e. U_i is not a sort of K. Then U_i is a new sort of K^{++}, therefore there is $code_i^1(x, \bar{x}) \in Fm(\mathsf{K}^{++})$ which "matches" Tr_1. We cannot use $code_i^1$ in the interpretation from K^+ to K because $code_i^1$ may not be in the language of K^+. We will use Tr_2 to translate $code_i^1$ into the language of K^+ as follows. Since K^+ is an expansion of K, all the variables in $x, \bar{x}$ have sorts which occur in K^+. Thus by the properties of Tr_2 we have

$$\mathsf{K}^{++} \models code_i^1(x, \bar{x}) \leftrightarrow Tr_2(code_i^1(x, \bar{x})).$$

Let $code_i(x, \bar{x}) \stackrel{\text{def}}{=} Tr_2(code_i^1(x, \bar{x}))$. Then $code_i(x, \bar{x}) \in Fm(\mathsf{K}^+)$ and

$$\mathsf{K}^+ \models code_i(x, \bar{x}) \to [\psi(x, \bar{x}) \leftrightarrow Tr(\psi)(\bar{x}, \bar{z})]$$

because

$$\mathsf{K}^{++} \models code_i^1(x, \bar{x}) \to [\psi(x, \bar{z}) \leftrightarrow Tr_1(\psi)(\bar{x}, \bar{z})].$$

This finishes the proof. ∎

More is true than stated in Theorem 4.3.29, namely, the existence of a translation mapping as in the theorem is actually sufficient for definability, as Theorem 4.3.31 below states.

Remark 4.3.30 (In connection with Theorems 4.3.27, 4.3.29.) These theorems state that the expressive powers of two languages $Fm(\mathsf{K}^+)$ and $Fm(\mathsf{K})$ coincide. However, the proofs of these theorems prove more. Namely, there exists a *computable* translation mapping *Tr* acting between the two languages. Even more than this, *Tr* preserves the logical structure of the formulas i.e. in the sense of algebraic logic, *Tr* is a "linguistic homomorphism". (Whether one is interested in this extra property of being a "linguistic homomorphism" is related to a difference between the algebraic logic approach and the abstract model theoretic approach to defining the equivalence of logics [hence, in particular, to how one approaches characterizations of logics like the celebrated Lindström theorems].)

$\lhd$

The following theorem says that *eliminability of new symbols is an essential feature of explicit definability*: If the new relations and sorts are arbitrary but are eliminable in the sense that there exist a mapping $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ together with "coding" formulas $code_i(x, \vec{x})$ for all new sorts U_i of K^+ which satisfy 1,2 in Theorem 4.3.29, then we can explicitly construct these new relations and sorts by using our concrete steps (1) - (2.2) (in such a way that some additional auxiliary new sorts and relations get defined in the way, but then we can forget these).

We note that both (ii) and (iii) in Theorem 4.3.31 say that K^+ is a special reduct of some definitional expansion of K. In (ii) we allow to forget relations and sorts which then can be "defined back" (i.e. K^+ is a reduct of its definitional expansion, so we forget the relations and sorts of a definitional expansion). In (iii) we allow to forget only as many relations and sorts that the remaining ones still "fix" the new sorts and relations.

If Tr and $code_i$ satisfy the conclusion of Theorem 4.3.29, then we say that they ***interpret*** K^+ in K.[545]

THEOREM 4.3.31 *Assume* K *is a reduct of* K^+. *Then (i) and (ii) below are equivalent and they imply (iii). If, in addition,* K^+ *is closed under taking ultraproducts, then (i)-(iii) below are equivalent.*

(i) K^+ *is interpreted in* K *by some* Tr *and* $code_i$*, i.e. the conclusion of Theorem 4.3.29 is true: there are* Tr *and* $code_i$ *satisfying 1-2 of Theorem 4.3.29.*

(ii) K^+ *and* K *have a common definitional expansion.*

(iii) K^+ *is rigidly definable over* K.

Proof: Proof of (i) $\Rightarrow$ (ii): Assume that $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ and $code_i \in Fm(\mathsf{K}^+)$ are such that 1,2 in Theorem 4.3.29 hold. We want to show that K^+ is explicitly definable over K with recoverable auxiliaries, i.e. that K^+ and K have a common definitional expansion K^{++}. Now we set to definining K^{++}.

Let U_i be a new sort of K^+. First from the formula $code_i$ we will extract an explicit definition for U_i, cf. Figure 95.

Consider $code_i(x, \bar{x})$. Define[546]

$$\delta(\bar{x}) \stackrel{\text{def}}{=} Tr(\exists x code_i(x, \bar{x})) \quad \text{and}$$

$$\rho(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} Tr(\exists x(code_i(x, \bar{x}) \wedge code_i(x, \bar{y}))).$$

Now $\delta(\bar{x}), \rho(\bar{x}, \bar{y}) \in Fm(\mathsf{K})$. Let $\bar{x} = \langle x_1, \dots, x_k \rangle$ and let the sorts of $x_1, \dots, x_k$ be $U_{j_1}, \dots, U_{j_k}$. These latter are sorts of K. Fix a model $\mathfrak{M} \in \mathsf{K}$.

First we define the relation S_i^{new} by $S_i^{new} \leftrightarrow \delta$, i.e.

[545] Cf. the definition of interpretations in Hodges [130, p.212, 221]. The existence of a tuple $Tr, code_i$ interpreting K^+ in K (as in Theorem 4.3.29) is strictly stronger than the ***uniform reduction property*** in [130, p.640]. Actually, the existence of $Tr, code_i$ is equivalent to K^+ being coordinatised over K in the sense of [130, p.644]. This equivalence is proved in [21].

[546] δ stands for "domain of $code_i{}^{-1}$" while ρ stands for "equivalence relation defined by $code_i{}^{-1}$".

$S_i^{new} \stackrel{\text{def}}{=} \{\bar{u} \in U_{j_1} \times \ldots \times U_{j_k} : \mathfrak{M} \models \delta(\bar{u})\}.$

Then S_i^{new} is a k-ary relation defined in $\mathfrak{M}$ by step (1). Second, from S_i^{new} we define the new sort D_i^{new} and $pj_1^i, \ldots, pj_k^i$ by step (2.1):

$D_i^{new} \stackrel{\text{def}}{=} S_i^{new}$ and

$pj_r^i \stackrel{\text{def}}{=} \{\langle \bar{u}, u_r \rangle : \bar{u} \in D_i^{new}\}$, for $1 \le r \le k$.

Now we define the new binary relation R_i^{new} by step (1) as follows:

$R_i^{new} \stackrel{\text{def}}{=} \{\langle v, w \rangle \in {}^2D_i^{new} : \mathfrak{M} \models \rho(pj_1^i(v), \ldots, pj_k^i(v), pj_1^i(w), \ldots, pj_k^i(w)).$

Intuitively, R_i^{new} is the relation defined by ρ "projected up" to D_i^{new}. Then R_i^{new} is an equivalence relation on D_i^{new}, by the properties of $code_i$, Tr and by the definitions of ρ, δ. We then can define, as in step (2.2), the factor-sort:

$U_i \stackrel{\text{def}}{=} D_i^{new}/R_i^{new},$

$\in_i \stackrel{\text{def}}{=} \{\langle v, v/R_i^{new} \rangle : v \in D_i^{new}\}.$

Let

$\mathfrak{N} \stackrel{\text{def}}{=} \langle \mathfrak{M}, D_i^{new}, U_i; pj_1^i, \ldots, pj_k^i, \in_i, S_i^{new}, R_i^{new} \rangle.$

Let $\mathfrak{M}^+ \in \mathsf{K}^+$ be any expansion of $\mathfrak{M}$. The name of the sort U_i in $\mathfrak{N}$ is the same as in $\mathfrak{M}^+$, but its "value" may be different, i.e. $U_i^{\mathfrak{N}}$ may be different from $U_i^{\mathfrak{M}^+}$. However, there is a natural bijection between these sets, as follows. Let

$Code_i(u, \bar{x}) \stackrel{\text{def}}{=} (\exists v \in D_i^{new})(\in_i(v, u) \wedge \bigwedge_r pj_r^i(v, x_r) \wedge S_i^{new}(\bar{x})).$

Then $Code_i(u, \bar{x})$ is in the language of $\mathfrak{N}$. By the above construction and by the properties of our translation, there is a bijection $f : U_i^{\mathfrak{N}} \longrightarrow U_i^{\mathfrak{M}^+}$ such that for all $u \in U_i^{\mathfrak{N}}$ and $\bar{a} \in {}^kUv\mathfrak{M}$

$\mathfrak{N} \models Code_i[u, \bar{a}]$ iff $\mathfrak{M}^+ \models code_i[f(u), \bar{a}].$

See Figure 95.

Let $\mathfrak{M}_i$ be the isomorphic copy of $\mathfrak{N}$ where we replace each element u of $U_i^{\mathfrak{N}}$ by $f(u)$. Then $\mathfrak{M}_i$ is a definitional expansion of $\mathfrak{M}$, obtained by steps (1),(2.1),(1),(2.2). Let $U_1, \ldots, U_t$ be all the new sorts of K^+ and let us do the above for all new sorts. Let $\mathfrak{M}'$ be the definitional expansion of $\mathfrak{M}$ we get by expanding $\mathfrak{M}$ with all the new sorts and relations of $\mathfrak{M}_i$, for $1 \le i \le t$. Then $\mathfrak{M}'$ contains all the new sorts of K^+, $Uv\mathfrak{M}^+ \subseteq Uv\mathfrak{M}'$, and moreover, for all $1 \le i \le t$

$\mathfrak{M}' \models Code_i[u, \bar{a}]$ iff $\mathfrak{M}^+ \models code_i[u, \bar{a}].$

Now we set to defining the new relations of K^+ in $\mathfrak{M}'$.

Let T_j be a new relation in K^+ with arity $\langle U_1, \ldots, U_m \rangle$. Assume that, of these, $U_{i_1}, \ldots, U_{i_\ell}$ are sorts of K, while the rest, $U_{j_1}, \ldots, U_{j_s}$ are new sorts of K^+. Let $\tau \stackrel{\text{def}}{=} Tr(T_j(\bar{y}))$. Then by the properties of the translation function Tr we have that

$\mathsf{K}^+ \models code_{j_1}(y_{j_1}, \bar{x}_1) \wedge \ldots \wedge code_{j_s}(y_{j_s}, \bar{x}_s) \rightarrow [T_j(\bar{y}) \leftrightarrow \tau(\bar{x}_1, \ldots, \bar{x}_s, y_{i_1}, \ldots, y_{i_\ell})].$

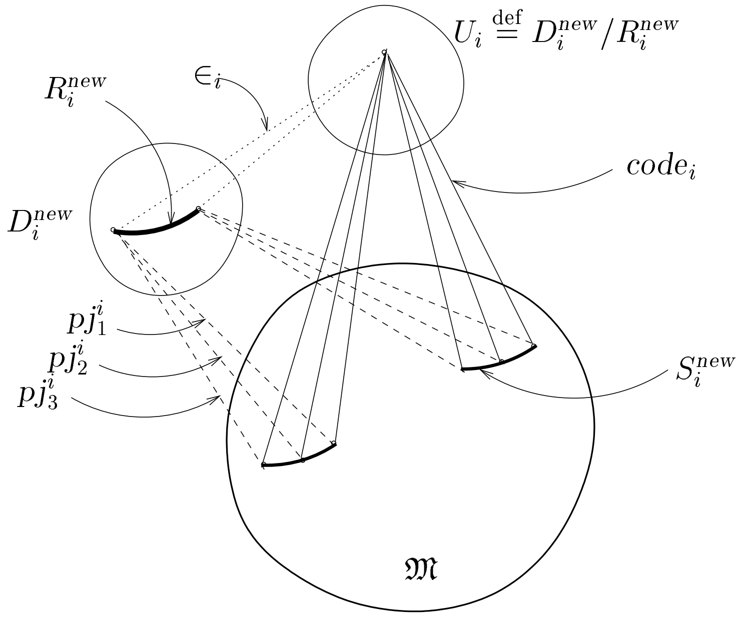


Figure 95: Illustration for the proof of Theorem 4.3.31 (i) ⇒ (ii). From the formula $code_i$ we construct an explicit definition for U_i.

Now we define the new relation T_j in $\mathfrak{M}'$ by the formula

$$T_j(\bar{y}) \leftrightarrow \exists \bar{x}_1 \dots \bar{x}_s (\bigwedge_r Code_{j_r}(y_{j_1}, \bar{x}_r) \wedge \tau(\bar{x}_1, \dots, \bar{x}_s, y_{i_1}, \dots, y_{i_\ell})).$$

By the construction, T_j denotes the same relation in $\mathfrak{M}'$ and in $\mathfrak{M}^+$. Let now $\mathfrak{M}^{++}$ be the definitional expansion of $\mathfrak{M}'$ with all the new relations T_j, let $\mathsf{L} \stackrel{\text{def}}{=} \{\mathfrak{M}^{++} : \mathfrak{M} \in \mathsf{K}\}$ and let $\mathsf{K}^{++} \stackrel{\text{def}}{=} \{\mathfrak{N} \in \mathsf{L} : \mathfrak{N} \restriction Voc\mathsf{K}^+ \in \mathsf{K}^+\}$. Then K^{++} is a definitional expansion of K and K^+ is a reduct[547] of K^{++}. We want to show that K^{++} is a definitional expansion of K^+ also. The new (relative to K^+) sorts and relations of K^{++} are

$$S_i^{new}, D_i^{new}, pj_r^i, R_i^{new} \text{ and } \in_i$$

when U_i is a sort of K^+ which is not present in K. We define $S_i^{new}, D_i^{new}, pj_r^i$, and R_i^{new} by using steps (1),(2.1),(1) as we did in K. Since K^+ is an expansion of K, and these were all definable in K, we immediately have that the same definition will work for them in K^+, too. We then define $\in_i$ by step (1) (and not by step (2.2) since U_i is an "old" sort in K^+) as follows:

$$\in_i(v, u) \leftrightarrow \exists \bar{x}(code_i(u, \bar{x}) \wedge pj_1^i(v, x_1) \wedge \dots \wedge pj_k^i(v, x_k))).$$

By the above, (i) ⇒ (ii) has been proved. (ii) ⇒ (i) was proved as Theorem 4.3.29.

Proof of (i) ⇒ (iii): Assume that K^+ is interpreted in K by some translation mapping Tr and formulas $code_i$. Then K^+ is definable in K, as we have seen above. By the properties of a translation mapping then K^+ is *rigidly* definable over K.

[547] We introduced L into the picture only because we did not assume that K^+ is closed under isomorphism and we want K^+ be a reduct of K^{++}.

Proof of (iii) $\Rightarrow$ (i): Here we will use Beth's definability theorem for one-sorted models (i.e. for defining relations only). Assume that K^+ is rigidly explicitly definable over K. Let K^{++} be a definitional expansion of K and assume that K^+ is a reduct of K^{++}. Let $code_i$, Tr be a translation of K^{++} to K. Since K^+ is a reduct of K^{++}, then $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$. Let U_i be a new sort of K^+. Then U_i is a new (relative to K) sort of K^{++}, therefore there is $code_i \in Fm(\mathsf{K}^{++})$ which has good properties w.r.t. Tr. The problem is that $code_i$ may not be in $Fm(\mathsf{K}^+)$. We will show that $code_i$ is expressible in $Fm(\mathsf{K}^+)$, i.e. it is equivalent in K^+ with a formula in the language of K^+.

For any new sort U_i of K^+ let R_i be a new relation symbol and let $\Delta(R_i)$ be the set of the following three formulas, where $\rho_i(\bar{x},\bar{y})$ is $Tr(\exists x(code_i(x,\bar{x}) \wedge code_i(x,\bar{y}))$, as in the proof of (i) $\Rightarrow$ (ii):

$$\forall x \exists \bar{x} R_i(x,\bar{x})$$

$$\exists x(R_i(x,\bar{x}) \wedge R_i(x,\bar{y})) \leftrightarrow \rho_i(\bar{x},\bar{y})$$

$$(R_i(x,\bar{x}) \wedge R_i(y,\bar{x})) \longrightarrow x = y$$

Then $\Delta(R_i)$ is a set of formulas in the language of K expanded with one new relation symbol R_i. For any new relation T_j of K^+ let $U_{i_1},\dots,U_{i_\ell}$ be the sorts of K, and $U_{j_1},\dots U_{j_t}$ be the new sorts of K^+ occurring in the arity of T_j and let $\Delta(T_j)$ denote the formula

$$T_j(\bar{y}) \leftrightarrow \exists \bar{x}_1 \dots \bar{x}_t (\textstyle\bigwedge_r R_{j_1}(u_{j_r},\bar{x}_r) \wedge Tr(T_j(\bar{y}))(\bar{x}_1,\dots,\bar{x}_t,y_{i_1},\dots,y_{i_\ell})).$$

Let Δ be the set of all the above formulas, i.e.

$$\Delta \stackrel{\text{def}}{=} \bigcup \{\Delta(R_i) : U_i \text{ is a new sort of } \mathsf{K}^+\} \cup \{\Delta(T_j) : T_j \text{ is a new relation of } \mathsf{K}^+\}.$$

Now, Δ is a set of formulas in the language of K^+ expanded with new relation symbols R_i for all new sorts U_i. We will show that Δ is an implicit definition of $\langle R_i : U_i \text{ is a new sort in } \mathsf{K}^+\rangle$ in K^+, in the usual sense. Indeed, let $\mathfrak{M}^+ \in \mathsf{K}^+$, $\bar{R}' \stackrel{\text{def}}{=} \langle R_i'\rangle$ and $\bar{R}'' \stackrel{\text{def}}{=} \langle R_i''\rangle$ be systems of concrete relations in $\mathfrak{M}^+$ such that

$$\langle \mathfrak{M}^+, \bar{R}'\rangle \models \Delta \quad \text{and} \quad \langle \mathfrak{M}^+, \bar{R}''\rangle \models \Delta.$$

Then, by using the construction of Δ, one can show that there is an isomorphism f between $\langle \mathfrak{M}^+, \bar{R}'\rangle$ and $\langle \mathfrak{M}^+, \bar{R}''\rangle$ such that f is identity on the sorts of K, i.e.. f is identity on $\mathfrak{M} \in \mathsf{K}$, where $\mathfrak{M}$ is the reduct of $\mathfrak{M}^+$ in K. Rigidity of K^+ over K implies that then f is identity on $\mathfrak{M}^+$ also, because both Id and f are isomorphisms on $\mathfrak{M}^+$ that are identity on $\mathfrak{M}$. Since f is the identity, we get that $\bar{R}' = \bar{R}''$. Thus in each model $\mathfrak{M}^+ \in \mathsf{K}^+$ there is at most one system $\bar{R}$ of concrete relations satisfying Δ. To be able to use the Beth theorem, we need that this property holds for all $\mathfrak{M}^+$ in the axiomatizable hull $\mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$ of K^+ as well. By the Keisler-Shelah theorem, and by our assumption that K^+ is closed under taking ultraproducts we have that $\mathfrak{N} \in \mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$ iff an ultrapower ${}^I\mathfrak{N}/F$ of $\mathfrak{N}$ is in K^+. Assume that there are two different systems of relations satisying Δ in $\mathfrak{N}$. Then the same is true in ${}^I\mathfrak{N}/F$. This contradicts our earlier argument showing that on each model $\mathfrak{M}^+ \in \mathsf{K}^+$ there is at most one system of relations satisfying Δ. Thus Δ is an implicit definition in the axiomatizable hull $\mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$ of K^+. By Beth's theorem then each of R_i is definable in the language of K^+. Let $\gamma_i(x,\bar{x}) \in Fm(\mathsf{K}^+)$ be such that $\mathsf{Th}(\mathsf{K}^+) \cup \Delta \models R_i(x,\bar{x}) \leftrightarrow \gamma_i(x,\bar{x})$. By the construction of Δ we also have that

$\mathsf{K}^{++} \models code_i(x,\bar{x}) \leftrightarrow R_i(x,\bar{x})$.

Thus, $Tr \restriction Fm(\mathsf{K}^+)$ toghether with the γ_i's is a good translation from K^+ to K. This finishes the proof of Theorem 4.3.31. ∎

Remark 4.3.32 (Discussion of Theorem 4.3.31.) **(i)** Theorem 4.3.31 is true for arbitrary languages, we do not need that there are only finitely many sorts or that we have only countably many symbols in the language. **(ii)** The condition that K^+ is closed under ultraproducts is needed for the direction (ii) ⇒ (i). An example showing this is the following. Let K be the class of finite linear orderings on sort U_0. Let K^+ be the class of two-sorted models where the sorts are U_0, U_1, there is a finite linear ordering both on U_0 and on U_1 and $|U_0| = |U_1|$. Now K^+ is rigidly explicitly definable over K. But the class UpK^+ of all ultraproducts of members of K^+ is not rigid over the class UpK of all ultraproducts of members of K. (To see this, take any infinite ultraproduct of elements of K^+. Then there is a nontrivial automorphism of the linear ordering on U_1.) However, it is not difficult to see that if K and K^+ have a common definitional expansion, then UpK and UpK^+ also have a common definitional expansion, which would imply that UpK^+ is rigid over UpK. Thus K and K^+ do not have a common definitional expansion. **(iii)** Thm4.3.48 together with Thm.4.3.31 will imply that if K^+ is rigidly definable over K and K is axiomatizable, then K^+ is nr-implicitly definable over K. ◁

4.3.4 Definitional equivalence of theories.

In section 4.3.3 we dealt with classes K and L where L was an expansion of K. In this sub-section we turn to the case when L is not necessarily an expansion of K.

Definition 4.3.33 Let K and L be two classes of models. We say that they are *definitionally equivalent*, in symbols $\mathsf{K} \equiv_\Delta \mathsf{L}$, iff they admit a common (explicit) definitional expansion M (without taking reducts).[548]

Further, $\mathfrak{M} \equiv_\Delta \mathfrak{N}$ abbreviates $\{\mathfrak{M}\} \equiv_\Delta \{\mathfrak{N}\}$. If $\mathfrak{M} \equiv_\Delta \mathfrak{N}$, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ are *definitionally equivalent models*. Two *theories* Th_1, Th_2 are called *definitionally equivalent* iff $\mathsf{Mod}(Th_1) \equiv_\Delta \mathsf{Mod}(Th_2)$. ◁

Cf. also in Hodges [130] under the name "*definitional equivalence*" pp. 60–61; cf. also Henkin-Monk-Tarski [120, Part I, e.g. p.56].

We will see that one can say that two definitionally equivalent theories can be regarded as being *essentially the same* theory and the difference between them is only that their *"syntactic decorations"* are different (i.e. they "choose" to represent their [essentially] common language with *different basic vocabularies*).

The same applies to classes of models K, L when $\mathsf{K} \equiv_\Delta \mathsf{L}$. As an example, choose K to be Boolean algebras with $\{\cap, -\}$ as their basic operations while choose L to be Boolean algebras

[548] I.e. M is a definitional expansion (without taking reducts) of K and the same holds for L in place of K. Note that $\mathsf{Th}(\mathsf{M})$ can be regarded as an implicit definition of M over K, and the same for L in place of K.

with $\{\cup, -, 0, 1\}$ as basic operations. (Then $\mathsf{K} \equiv_\Delta \mathsf{L}$.) At a certain level of abstraction, K and L can be regarded as a collection of the *same* mathematical structures (namely, Boolean algebras) and the difference (between K and L) is only in the choice of their basic vocabularies (which is "$\cap, -$" in the one case while "$\cup, -, 0, 1$" in the other). Summing it up: In some sense, definitionally equivalent theories $Th_1 \equiv_\Delta Th_2$ can be considered as just one theory with two *different linguistic representations*. The same applies to definitionally equivalent classes of models.

The relation $\equiv_\Delta$ defined above is symmetric and reflexive. For certain "administrative" reasons it is not transitive, but the counterexamples (to transitivity) are so artificial that we will not meet them (in this work). We could define $\equiv^*_\Delta$ to be the transitive closure of $\equiv_\Delta$ and then use $\equiv^*_\Delta$ as definitional equivalence. If this were a logic book we would do that. However, in the present work we will not need $\equiv^*_\Delta$, hence we do not discuss it, and we call $\equiv_\Delta$ definitional equivalence (though it is $\equiv^*_\Delta$ which is the really satisfactory notion of definitional equivalence.)

Discussion of the definition of $\equiv_\Delta$

(1) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then K and L agree on the common part of their vocabularies.[549] I.e.

$$\mathsf{K} \equiv_\Delta \mathsf{L} \qquad \Rightarrow \qquad \mathsf{K} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}) = \mathsf{L} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}).$$

(2) For any definitional expansion K^+ of K we have $\mathsf{K} \equiv_\Delta \mathsf{K}^+$. In general, if K^+ is an expansion of K and K is closed under taking ultraproducts, then $\mathsf{K}^+ \equiv_\Delta \mathsf{K}$ iff K^+ is rigidly definable over K, see Thm.4.3.31.

(3) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then L and K are definable over each other. Moreover one can choose their definitions over each other to be the same. Indeed, if M is the common definitional expansion of K and L mentioned in the definition of $\equiv_\Delta$, then $\mathsf{Th}(\mathsf{M})$ is a definition of K over L as well as a definition of L over K.

(4) Definitional equivalence is stronger than mutual (explicit) definability: there exist classes K and L such that they are definable over each other, yet $\mathsf{K} \not\equiv_\Delta \mathsf{L}$ (see Examples 4.3.46, p.266). Moreover, this is so even in the one-sorted case: We can choose K and L such that both K and L have only one, common, sort. Such an example can be found in Andréka-Madarász-Németi [22].

(5) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then there is a bijection-up-to-isomorphism

$$f : \mathsf{K} \rightarrowtail\!\!\twoheadrightarrow \mathsf{L}$$

between K and L, and there are definitions $\Delta_\mathsf{K}, \Delta_\mathsf{L}$ such that for all $\langle \mathfrak{M}, \mathfrak{N} \rangle \in f$ the following hold:

(i) $\mathfrak{M} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}) = \mathfrak{N} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L})$

(ii) $\mathfrak{M}$ and $\mathfrak{N}$ have a common definitional expansion $\mathfrak{M}^+$

(iii) Δ_K defines $\mathfrak{M}^+$ over $\mathfrak{M}$ and Δ_L defines $\mathfrak{M}^+$ over $\mathfrak{N}$.

[549] As a contrast, $\mathsf{K} \equiv^*_\Delta \mathsf{L}$ does not imply this, however as we said we will not need the generality of $\equiv^*_\Delta$ in this work.

Indeed, if M is a common definitional expansion of K and L with definitions $\Delta_{\mathsf{K}}, \Delta_{\mathsf{L}}$ over K and L respectively, then we can choose f to be

$$f = \{\langle \mathfrak{M} \restriction Voc\mathsf{K}, \mathfrak{M} \restriction Voc\mathsf{L}\rangle : \mathfrak{M} \in \mathsf{L}\}.$$

(6) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then the bijection-up-to-isomorphism $f : \mathsf{K} \rightarrowtail\!\!\twoheadrightarrow \mathsf{L}$ in (5) above has the following property. For all $\mathfrak{M} \in \mathsf{K}$, the automorphism group of $\mathfrak{M}$ is isomorphic to the automorphism group of $f(\mathfrak{M})$, in symbols[550]

$$\langle Aut(\mathfrak{M}), \circ\rangle \cong \langle Aut(f(\mathfrak{M})), \circ\rangle.$$

This is so because of the following. Let $\mathfrak{M}^+ \in \mathsf{M}$ be such that $\mathfrak{M}^+$ is implicitly definable without taking reducts both over $\mathfrak{M}$ and over $f(\mathfrak{M})$. Since $\mathfrak{M}^+$ is implicitly definable without taking reducts over $\mathfrak{M}$, each automorphism of $\mathfrak{M}$ extends in a unique way to an automorphism of $\mathfrak{M}^+$, and this implies that the automorphism groups of $\mathfrak{M}$ and $\mathfrak{M}^+$ are isomorphic. We get the same for $f(\mathfrak{M})$ and $\mathfrak{M}^+$ completely analogously, and this proves that the automorphism groups of $\mathfrak{M}$ and $f(\mathfrak{M})$ are isomorphic.

(7) Each of the properties in items (3) and (5) are equivalent to $\mathsf{K} \equiv_\Delta \mathsf{L}$. This is proved in Thm.4.3.34 below.

(8) For more on definitional equivalence, its importance, and for motivation for the way we defined and use $\equiv_\Delta$ we refer to [120, pp. 56-57, Remark 0.1.6], [130, pp. 58-61].

$\lhd$

THEOREM 4.3.34 *Let K and L be two classes. Then (i)-(iii) below are equivalent.*

(i) $\mathsf{K} \equiv_\Delta \mathsf{L}$

(ii) *There is a Δ such that Δ defines K over L and Δ defines L over K.*

(iii) *For every $\mathfrak{M} \in \mathsf{K}$ there is $\mathfrak{N} \in \mathsf{L}$ and for every $\mathfrak{N} \in \mathsf{L}$ there is $\mathfrak{M} \in \mathsf{K}$ such that $\mathfrak{M}$ and $\mathfrak{N}$ have a common definitional expansion, and moreover the definitions of the expansion over $\mathfrak{M}$ and over $\mathfrak{N}$ can be chosen uniformly.*

Proof: Proof of (ii) $\Rightarrow$ (i): Assume Δ defines K over L and Δ defines L over K. Then there are K^+ and L^+ such that K^+ is a definitional expansion of K, defined by Δ, and L is a reduct of K^+ and the analogous statement for L^+. It is easy to see that then $Voc\mathsf{K}^+ = Voc\mathsf{L}^+ = Voc\mathsf{K} \cup Voc\mathsf{L} \cup Voc\Delta$, where $Voc\Delta$ denotes the set of sort and relation symbols occurring in Δ. Let $\mathsf{M} \overset{\text{def}}{=} \mathsf{K}^+ \cup \mathsf{L}^+$. Then M is an expansion of both K and L. Also, $\mathsf{M} \models \Delta$ because $\mathsf{K}^+ \models \Delta$ and $\mathsf{L}^+ \models \Delta$. Thus, Δ is an nr-implicit definition of M both over K and over L.

Proof of (iii) $\Rightarrow$ (i): Let $\Delta_{\mathsf{K}}, \Delta_{\mathsf{L}}$ be the uniform definitions in (iii). Let $\mathsf{M} \overset{\text{def}}{=} \{\mathfrak{M} : \mathfrak{M} \restriction Voc\mathsf{K} \in \mathsf{K},\ \mathfrak{M} \restriction Voc\mathsf{L} \in \mathsf{L}$ and Δ_{K} defines $\mathfrak{M}$ over $\mathfrak{M} \restriction Voc\mathsf{K}$, Δ_{L} defines $\mathfrak{M}$ over $\mathfrak{M} \restriction Voc\mathsf{L}\}$. Then M is a class of similar models, namely, $Voc\mathsf{M} = Voc\mathsf{K} \cup Voc\mathsf{L} \cup Voc\Delta_{\mathsf{K}} \cup Voc\Delta_{\mathsf{L}}$. Then both K and L are reducts of M, i.e. $\mathsf{K} = \mathsf{M} \restriction Voc\mathsf{K}$ etc., by (iii). Also, $\mathsf{M} \models \Delta_{\mathsf{K}} \cup \Delta_{\mathsf{L}}$, thus M is a definitional expansion of both K and L.

(i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii) was shown already in the discussion of $\equiv_\Delta$. ∎

[550] $f(\mathfrak{M})$ exists only up to isomorphism, but we talk only about the automorphism group of $f(\mathfrak{M})$ which is defined by the isomorphism type of $\mathfrak{M}$ up to isomorphism. So this makes sense.

THEOREM 4.3.35 *Let* K, L *be two classes of models and assume that* $\mathbf{I}\mathsf{K}$ *is closed under taking ultraproducts. Then (i) and (ii) below are equivalent.*

(i) $\mathsf{K} \equiv_\Delta \mathsf{L}$

(ii) K *and* L *have a common extension which is rigidly definable both over* K *and over* L.

Proof: Let M be a common rigidly definable expansion of K and L. Since $\mathbf{I}\mathsf{K}$ is closed under taking ultraproducts, then $\mathbf{I}\mathsf{M}$ is closed under taking ultraproducts, too. Hence we can apply Thm.4.3.31 to obtain common definitional expansions K^+ and L^+ of K and M and of L and M respectively. We also may assume that the new sorts and relations in K^+ and L^+ have different names, i.e. $Voc\mathsf{K}^+ \cap Voc\mathsf{L}^+ = Voc\mathsf{M}$. Then it is not difficult to see that there is a common definitional expansion M^+ of K^+ and L^+. Now, M^+ is a common definitional expansion of K and L. ∎

In connection with Theorem 4.3.35 we note that it is not difficult to see that if K^+ is nr-implicitly definable over K, then $\mathbf{I}\mathsf{K}$ is closed under ultraproducts iff $\mathbf{I}\mathsf{K}^+$ is closed under ultraproducts.

We will also need the following lemma.

LEMMA 4.3.36 *Let* K, L *and* K^+ *be classes of models. Assume that* K^+ *is rigidly definable over* K, $\mathbf{I}\mathsf{L}$ *is closed under taking ultraproducts,* $Voc\mathsf{K}^+ \cap Voc\mathsf{L} = Voc\mathsf{K} \cap Voc\mathsf{L}$, *and* $\mathsf{K} \equiv_\Delta \mathsf{L}$. *Then* $\mathsf{K}^+ \equiv_\Delta \mathsf{L}$.

Proof: Assume that $\mathsf{K}, \mathsf{L}, \mathsf{K}^+$ satisfy the conditions of the lemma. Let M be a common definitional expansion of L and K and let Δ, Σ be the respective definitions of M over L and K. Since $\mathbf{I}\mathsf{L}$ is closed under taking ultraproducts and $\mathsf{K} \equiv_\Delta \mathsf{L}$, we have that $\mathbf{I}\mathsf{K}$ also is closed under taking ultraproducts, and since K^+ is definable over K, then $\mathbf{I}\mathsf{K}^+$ is closed under taking ultraproducts. Thus by Thm.4.3.31, K and K^+ have a common definitional expansion M^+. Let Δ_1, Σ_1 be the respective definitions of M^+ over K and K^+.

We may assume that $Voc\mathsf{M}$ is disjoint from $Voc\mathsf{K}^+ \setminus Voc\mathsf{K}$ (by our assumption $Voc\mathsf{K}^+ \cap Voc\mathsf{L} = Voc\mathsf{K} \cap Voc\mathsf{L}$), and that $Voc\mathsf{M}^+ \setminus Voc\mathsf{K}^+$ is disjoint from $Voc\mathsf{M}$. Hence $Voc\mathsf{M}^+ \cap Voc\mathsf{M} = Voc\mathsf{K}$.

Let $\mathfrak{N} \in \mathsf{L}$ be arbitrary. There are $\mathfrak{M} \in \mathsf{K}$ and $\mathfrak{M}^+ \in \mathsf{M}$ such that $\mathfrak{M}^+$ is a common definitional expansion of $\mathfrak{N}$ and $\mathfrak{M}$. There are $\mathfrak{M}_1 \in \mathsf{K}^+$ and $\mathfrak{M}_1^+$ in M^+ such that $\mathfrak{M}_1^+$ is a common definitional expansion of both $\mathfrak{M}$ and $\mathfrak{M}_1$. By $Voc\mathsf{M} \cap Voc\mathsf{M}^+ = Voc\mathsf{K}$, the union of $\mathfrak{M}^+$ and $\mathfrak{M}_1^+$ is a model, $\mathfrak{M}^{++}$. Then $\mathfrak{M}^{++}$ is a common expansion of $\mathfrak{N}$ and $\mathfrak{M}_1 \in \mathsf{K}^+$. Since Δ defines $\mathfrak{M}^+$ over $\mathfrak{N}$ and Δ_1 defines $\mathfrak{M}_1^+$ over $\mathfrak{M}$, we have that $\Delta \cup \Delta_1$ defines $\mathfrak{M}^{++}$ over $\mathfrak{N}$. Similarly, $\Sigma \cup \Sigma_1$ defines $\mathfrak{M}^{++}$ over $\mathfrak{M}_1$. The proof of the other direction, $(\forall \mathfrak{M} \in \mathsf{K}^+ \exists \mathfrak{N} \in \mathsf{L}) \ldots$ is completely analogous. $\mathsf{K}^+ \equiv_\Delta \mathsf{L}$ then follows by Thm.4.3.34(iii)⇒(i). ∎

Remark 4.3.37 (How and why can definitionally equivalent theories [and classes of models] be regarded as identical [as a corollary of the translation theorems]?)

In addition to the text below, we also refer the reader to [120, p.56] and [130, pp.58–61] for explanations of why definitionally equivalent classes of models can be regarded as (in some sense) identical.

Let K and L be two definitionally equivalent classes of models (formally, $\mathsf{K} \equiv_\Delta \mathsf{L}$). Then, by the definition of $\equiv_\Delta$, there is a class M which is a definitional expansion (without taking reducts) of both K and L. We will argue below that this M establishes a very strong connection between K and L. (Cf. also item (5) in the discussion of the definition of $\equiv_\Delta$.) Our argument begins with the following: We can apply Theorem 4.3.27 to the pair M and K with M in place of K^+ in that theorem. The same applies to the pair M and L. By Theorem 4.3.27, then we have two translation mappings

$$\begin{tikzcd} & Fm(\mathsf{M}) \arrow[dl, "Tr_1"'] \arrow[dr, "Tr_2"] & \\ Fm(\mathsf{K}) & & Fm(\mathsf{L}) \end{tikzcd}$$

both of which preserve meaning (in the sense of Theorem 4.3.27). Both of Tr_1 and Tr_2 are surjective. Intuitively, Tr_1 identifies K with M while Tr_2 identifies M with L. Hence K *gets identified with* L. (Perhaps the best way of thinking about this is that we identify both K and L with their common expansion M. As a by-product of this we identify K and L with each other, too.)

By surjectiveness of Tr_1 and Tr_2, whatever can be said in the language $Fm(\mathsf{K})$, the same can be said in $Fm(\mathsf{M})$ and hence (using Tr_2) the same can be said in the language $Fm(\mathsf{L})$ of L. Similarly, whatever can be said in $Fm(\mathsf{L})$ the same can be said in $Fm(\mathsf{K})$, too.

Now, if we want some more detail, let $\varphi(\bar{z}) \in Fm(\mathsf{K})$ with a sequence $\bar{z}$ of variables belonging to common sorts K and L. Then there are $\varphi'(\bar{z}) \in Fm(\mathsf{M})$, $\varphi''(\bar{z}) \in Fm(\mathsf{L})$ such that $Tr_1(\varphi') = \varphi$ and $Tr_2(\varphi') = \varphi''$. I.e.

$$\varphi(\bar{z}) \xleftarrow{\ Tr_1\ } \varphi'(\bar{z}) \xmapsto{\ Tr_2\ } \varphi''(\bar{z}).$$

Actually, we can choose $\varphi' = \varphi$ if we want to. Using Theorem 4.3.27 we can conclude

(23) $$\mathsf{M} \models \varphi(\bar{z}) \leftrightarrow \varphi''(\bar{z}).$$

I.e. the same things can be said about the common variables $\bar{z}$ in $Fm(\mathsf{K})$ and in $Fm(\mathsf{L})$. Hence the languages of K and L have the *same* expressive power.

On the basis of (23) above and what was said before (23), we can introduce two, more direct, translation mappings

$$\begin{tikzcd} Fm(\mathsf{K}) \arrow[r, shift left, "T_2"] & Fm(\mathsf{L}) \arrow[l, shift left, "T_1"] \end{tikzcd}$$

defined as follows. In defining T_1 and T_2 we can rely on the fact that

$$Fm(\mathsf{K}) \subseteq Fm(\mathsf{M}) = Dom(Tr_1)$$

and that $Tr_1 \restriction Fm(\mathsf{K}) = \mathrm{Id} \restriction Fm(\mathsf{K})$ which is the identity function. Hence we can choose

$$\begin{aligned} T_1 &:= Tr_1 \restriction Fm(\mathsf{L}) \quad \text{and} \\ T_2 &:= Tr_2 \restriction Fm(\mathsf{K}).\end{aligned}$$ [551]

$$
\begin{tikzcd}
 & Fm(\mathsf{M}) \arrow[dl, "Tr_1"'] \arrow[dr, "Tr_2"] & \\
Fm(\mathsf{K}) \arrow[rr, shift left, "T_2"] & & Fm(\mathsf{L}) \arrow[ll, shift left, "T_1"]
\end{tikzcd}
$$

Assume $\varphi \in Fm(\mathsf{M})$ involves only common free variables of K and L. Then

$$
\begin{aligned}
\mathsf{M} &\models (T_2 Tr_1 \varphi) \leftrightarrow Tr_2 \varphi. \\
\mathsf{M} &\models (T_1 Tr_2 \varphi) \leftrightarrow Tr_1 \varphi.
\end{aligned}
$$

So in this "logical sense" the above diagram commutes.

For completeness, about the above diagram we also note the following commutativity property:

$$
\begin{aligned}
T_2 &\subseteq (Tr_1)^{-1} \circ Tr_2\,, \\
T_1 &\subseteq (Tr_2)^{-1} \circ Tr_1\,.
\end{aligned}
$$

Here we note that $(Tr_1)^{-1} \circ Tr_2$ is a binary relation but not necessarily a function.

Using Theorem 4.3.27, and (23) way above, one can check that for all $\varphi \in Fm(\mathsf{K})$ and for all $\psi \in Fm(\mathsf{L})$, if φ and ψ use only variables of common sorts (of K and L) then:

(24)
$$
\begin{aligned}
\mathsf{M} &\models \varphi(\bar{z}) \leftrightarrow (T_2\varphi)(\bar{z}), \\
\mathsf{M} &\models (T_1\psi)(\bar{z}) \leftrightarrow \psi(\bar{z}), \qquad \text{further}
\end{aligned}
$$

(25)
$$
\begin{aligned}
\mathsf{K} &\models \varphi(\bar{z}) \leftrightarrow (T_1 T_2 \varphi)(\bar{z}), \\
\mathsf{L} &\models \psi(\bar{z}) \leftrightarrow (T_2 T_1 \psi)(\bar{z}).
\end{aligned}
$$

These statements can be interpreted as saying that T_1 and T_2 are kind of *inverses* of each other and that they establish a kind of logical isomorphism between equivalence classes of formulas in $Fm(\mathsf{K})$ and $Fm(\mathsf{L})$ involving free variables of common sorts only. For completeness, we note that (24–25) can be generalized to formulas involving free variables of arbitrary sorts by using Theorem 4.3.29. For formulating this generalized version of (24–25) one needs to use the formulas "*code*" as they were used in Theorem 4.3.29. E.g. the first line of (25) becomes

$$
\mathsf{K} \models code(x, \vec{x}) \to [\varphi(x, \bar{z}) \leftrightarrow (T_1 T_2 \varphi)(\vec{x}, \bar{z})],
$$

where x belongs to a sort of K not in L, and $\bar{z}$ is a sequence of variables of common sorts of K and L. Here $code(x, \vec{x})$ is the formula we get from combining the corresponding formulas

[551] In passing, we also note that Tr_1 can be regarded as *injective in the sense* that if $\psi(\bar{z}), \gamma(\bar{z}) \in Fm(\mathsf{M})$ involve free variables of K only then $[\, Tr_1(\psi) = Tr_1(\gamma) \;\Rightarrow\; \mathsf{M} \models \psi(\bar{z}) \leftrightarrow \gamma(\bar{z})]$. Similarly for Tr_2 and L.

belonging to Tr_1 and Tr_2. We leave the details of generalizing (23–25) to treating free variables not in the common language to the interested reader.

(We note that the generalization of (25) above reminds us of the notion of equivalence between two categories, in the sense of category theory.)

We hope, the above shows how and to what extent we consider two definitionally equivalent classes (and theories) as being essentially identical.

$\lhd$

Application: definitional equivalence of of the observer-independent geometries with frame models.

We proved in this section that $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$, under some conditions on Th (cf. Thm.4.3.22). We will see that if we add some more, reasonable, conditions[552] on Th, then not only $\underline{\mathsf{Mod}(Th)\ \textit{is also definable over}\ \mathsf{Ge}(Th)}$, but the much stronger statement holds that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are *definitionally equivalent*. Definitional equivalence is a very strong connection between our frame-models $\mathsf{Mod}(Th)$ and our observer-independent geometries $\mathsf{Ge}(Th)$, as we have just seen this in Remark 4.3.37 above. The methodological importance of these kinds of theorems (from the point of view of physics) was discussed in the introduction of §4.2.2 (p.152) and in the introduction to the present chapter (§4.1). The theorem below says that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent under some assumptions. But if two theories (or axiomatizable classes of models) are definitionally equivalent then this means that, basically, *they are the same* theory "represented" in two different ways; cf. Remark 4.3.37 (p.258) and the discussion on p.972 (in §4.3). The same applies to classes of models (like $\mathsf{Ge}(Th)$ and $\mathsf{Mod}(Th)$) in place of theories. Therefore our next theorem can be interpreted as saying that our observational world $\mathsf{Mod}(Th)$ is basically the same as our theoretical world $\mathsf{Ge}(Th)$. The theorem implies that our theoretical concepts are already available in $\mathsf{Mod}(Th)$ as "abbreviations" or "shorthands"[553]; and that in the other direction, our observational concepts (like observer, coordinate system etc.) are present in our theoretical world $\mathsf{Ge}(Th)$ as "abbreviations".

THEOREM 4.3.38 $\mathsf{Mod}(Th)$ *and* $\mathsf{Ge}(Th)$ *are definitionally equivalent, in symbols*

$$\boxed{\mathsf{Mod}(Th) \equiv_{\Delta} \mathsf{Ge}(Th),}$$

assuming $n > 2$ *and* $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\|)^{-} + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$.

Proof of Thm.4.3.38: Assume $n > 2$ and that Th satisfies the assumptions of the theorem. Let $\mathsf{Ge}^{-}(Th)$ be the topology free reduct of $\mathsf{Ge}(Th)$. Let $\mathsf{Ge}^{-}(Th) + T_0$ denote the expansion of $\mathsf{Ge}^{-}(Th)$ with the subbase T_0 (of the topology) and the membership relation $\in_{Mn\times T_0}$ as indicated in Prop.4.3.19 on p.242. Hence, the models $\mathsf{Ge}^{-}(Th) + T_0$ are of the form $\langle \mathfrak{G}, T_0; \in_{Mn\times T_0}\rangle$ with $\mathfrak{G} \in \mathsf{Ge}^{-}(Th)$ and $T_0, \in_{Mn\times T_0}$ as indicated on p.242. By

[552] $\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)} + \mathbf{Ax(eqtime)}$. It can be seen that these conditions are sort of necessary for definability in the other direction.

[553] This direction can be interpreted as concluding that our theoretical concepts are acceptable (or well chosen) from the point of view of Machian-Einsteinian-Reichenbachian philosophy of theory making.

the proof of Prop.4.3.19, $\mathsf{Ge}^-(Th) + T_0$ is rigidly definable over $\mathsf{Ge}^-(Th)$. By this and by Lemma 4.3.36, we conclude that it is sufficient to prove $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}^-(Th)$ for proving $\mathsf{Mod}(Th) \equiv_\Delta (\mathsf{Ge}^-(Th) + T_0)$. According to our convention below $(\star\star)$ on p.155 we consider the latter sufficient for proving $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$. Therefore to prove the present theorem, it enough to prove $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}^-(Th)$. We will do just this.

To prove $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}^-(Th)$, by Thm.4.3.35, it is enough to find a class M such that M is rigidly definable both over $\mathsf{Mod}(Th)$ and $\mathsf{Ge}^-(Th)$. Now, we turn to constructing such an M. First, we define the vocabulary of M. (The common vocabulary of $\mathsf{Mod}(Th)$ and $\mathsf{Ge}^-(Th)$ consists of the sort symbol F and relation/function symbols $+, \cdot, \leq$.) Let $Voc\,\mathsf{M} :=$"$Voc\,\mathsf{Mod}(Th) + Voc\,\mathsf{Ge}^-(Th) +$ (relation symbols O and P, where the rank of O is $\langle B, \underbrace{Mn, \ldots, Mn}_{(n+1)\text{-times}}\rangle$, and the rank of P is $\langle B, L\rangle$)". Now,

$$\mathsf{M} :\overset{\text{def}}{=} \mathbf{I}\Big\{ \langle \mathfrak{M}, \mathfrak{G}_{\mathfrak{M}}; O, P\rangle \ : \ \mathfrak{M} \in \mathsf{Mod}(Th), \\ O = \{ \langle m, w_m(\bar{0}), w_m(1_0), \ldots, w_m(1_{n-1})\rangle \ : \ m \in Obs^{\mathfrak{M}} \} \\ P = \big\{ \langle ph, \{ e \in Mn \ : \ ph \in e \}\rangle \ : \ ph \in Ph^{\mathfrak{M}} \big\} \Big\}.$$

By the proof of Prop.4.3.18 (p.240) and Thm.4.3.22 (p.244) it is not hard to see that M is rigidly definable over $\mathsf{Mod}(Th)$. By Def.4.5.38 (p.310), and Claim 4.5.44 (p.315) in the proof of Prop.4.5.43 it is not hard to see that M is rigidly definable over $\mathsf{Ge}^-(Th)$. ∎

Weak definitional equivalence

Definition 4.3.39 Let K and L be two classes of models and let $f : \mathsf{K} \longrightarrow \mathsf{L}$ be a function. We say that f is a *first-order definable meta-function* iff for each $\mathfrak{M} \in \mathsf{K}$ $f(\mathfrak{M})$ is first-order definable over $\mathfrak{M}$ (in the sense of §4.3.2) and the definition of $f(\mathfrak{M})$ over $\mathfrak{M}$ is uniform, i.e. is the same for all choices of $\mathfrak{M} \in \mathsf{K}$.[554]

◁

A typical example for first-order definable meta-functions is e.g.
$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$, where $\mathcal{G} : \mathfrak{M} \mapsto \mathfrak{G}_{\mathfrak{M}}$, if Th is strong enough, cf. Thm.4.3.22 (p.244).
A similar example will be a kind of inverse to this function
$\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$, cf. Prop.4.5.41 (p.313) and Def.4.5.38 (p.310).

We note that if $f : \mathsf{K} \twoheadrightarrow \mathsf{L}$ is a surjective first-order definable meta-function then L is definable over K; and, more generally, if $f : \mathsf{K} \longrightarrow \mathsf{L}$ is a first-order definable meta-function then $Rng(f)$ is definable over K. In the other direction, if $\mathsf{L} = \mathbf{I}\mathsf{L}$ is definable over K then there is a first-order definable meta-function $f : \mathsf{K} \longrightarrow \mathsf{L}$ such that $Rng(f)$ is L up to isomorphism. To be able to claim this for the case when $\mathsf{L} \neq \mathbf{I}\mathsf{L}$ we make the following convention.

[554] A first-order definable meta-function (acting between classes of models) is a rather different kind of thing from an ordinary function like *factorial* : $N \longrightarrow N$ definable in a model, say in $\mathfrak{N} \in \mathsf{Mod}$(Peano's arithmetic), cf. Example 4.3.8(1) on p.226. (This is the reason why we call f a meta-function and not simply a function.)

CONVENTION 4.3.40 (Class form of the axiom of choice)
In connection with the above definition, for simplicity, throughout the present chapter we assume the *class form of the axiom of choice*. More concretely we assume that our set theoretic universe V is well-orderable by the class $\mathsf{Ordinals}$ of ordinal numbers. I.e. there is a bijection

$$f : \mathsf{Ordinals} \rightarrowtail\!\!\twoheadrightarrow V.$$

This implies that any proper class is well-orderable and therefore there exists a bijection between any two proper classes.

$\lhd$

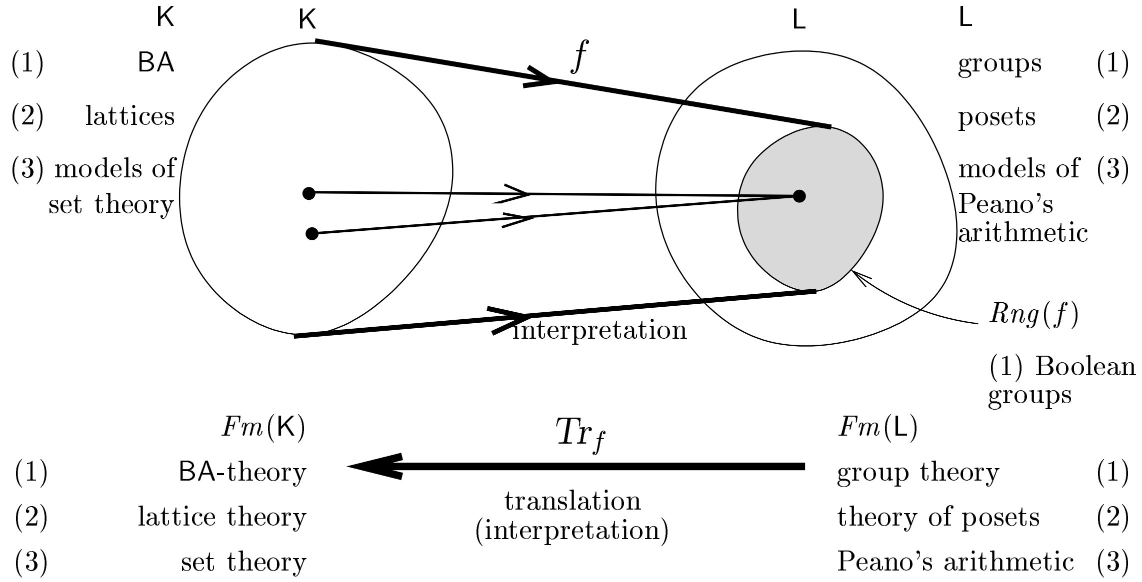


Figure 96: Examples for first-order definable meta-functions f and the induced translations between theories. For more explanation in connection with this picture cf. item (III) of Remark 4.5.14, pp. 294–295. The corresponding theories are labelled by the same numbers. E.g. BA is interpreted in "groups", "lattices" in "posets" etc. Here f, or the pair $\langle f, Tr_f\rangle$, or Tr_f are (often) called *interpretations*, cf. footnote 1022 on p.1023 in AMN [18]. E.g. Tr_f interprets group theory in BA-theory. Equivalently f interprets BA's in groups. (This figure also serves as an illustration for Prop.4.3.41, p.264.)

The following proposition makes connections between the following three things: (i) "interpretations" of one theory in another, (ii) first-order definable meta-functions $f : \mathsf{K} \longrightarrow \mathsf{L}$ between classes of models, and (iii) definability of a class $Rng(f)$ over another class K, see Fig.96. In this context the function Tr_f (in the proposition) below is what we call an interpretation (or translation). Cf. item (III) of Remark 4.5.14 on p.294; and footnote 1022 on p.1023 in AMN [18] for the intuitive idea behind interpretations.[555] In particular the proposition says that any first-order definable meta-function $f : \mathsf{K} \longrightarrow \mathsf{L}$ induces a natural syntactical

[555] In the one-sorted case an interpretation $Tr : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$ is the same thing as a cylindric algebraic homomorphism between the cylindric algebras of formulas $Fm(\mathsf{L})$ and $Fm(\mathsf{K})$. I.e. if we endow $Fm(\mathsf{L})$ with the cylindric algebraic structure (of first-order formulas) and do the same with $Fm(\mathsf{K})$ then the homomorphisms between the two algebras of formulas are typical examples of interpretations.

translation mapping from the language $Fm(\mathsf{L})$ of L to that of K. Moreover, this translation is meaning preserving w.r.t. the semantical function f.[556]

PROPOSITION 4.3.41 *Assume $f : \mathsf{K} \longrightarrow \mathsf{L}$ is a first-order definable meta-function. Then there is a "natural" translation mapping*

$$Tr_f : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{L})$ with all free variables belonging to common sorts of K and L [557], $\mathfrak{A} \in \mathsf{K}$ and evaluation $\bar{a}$ of $\bar{x}$ in the common sorts (i.e. universes) of $\mathfrak{A}$ and $f(\mathfrak{A})$ the following holds.[558]

$$f(\mathfrak{A}) \models \varphi[\bar{a}] \quad \Leftrightarrow \quad \mathfrak{A} \models Tr_f(\varphi)[\bar{a}].$$

Cf. Fig.96.

Proof: The proposition follows easily by Thm.4.3.27 (first translation theorem) on p.245. In more detail: Assume $f : \mathsf{K} \longrightarrow \mathsf{L}$ is a first-order definable meta-function. Then there is an expansion K^+ of $Rng(f)$ such that K^+ is definable over K without taking reducts. Then, by Thm.4.3.27, there is a translation mapping $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ such that $(\star)$ in Thm.4.3.27 holds. Let $Tr_f := Tr \restriction Fm(\mathsf{L})$. One can check that Tr_f has the desired properties. ∎

We will have results analogous to the conclusion of Prop.4.3.41 above at various points in the remaining part of this chapter, cf. e.g. Thm.4.5.42 on p.315.

The following is a weaker form of definitional equivalence. We will use it e.g. in Thm.4.5.26 (p.303).

Definition 4.3.42 (Weak definitional equivalence)
Let K, L be two classes of models. K and L are called <u>*weakly definitionally equivalent*</u>, in symbols

$$\mathsf{K} \equiv^w_\Delta \mathsf{L},$$

iff there are first-order definable meta-functions

$$f : \mathsf{K} \longrightarrow \mathsf{L} \quad \text{and} \quad g : \mathsf{L} \longrightarrow \mathsf{K}$$

such that for any $\mathfrak{M} \in \mathsf{K}$ and $\mathfrak{G} \in \mathsf{L}$, (i) and (ii) below hold.

(i) $(f \circ g)(\mathfrak{M}) \cong \mathfrak{M}$ and $(g \circ f)(\mathfrak{G}) \cong \mathfrak{G}$, and

(ii) moreover there is an isomorphism between the two structures $\mathfrak{M}$ and $(f \circ g)(\mathfrak{M})$ which is the identity map on the reduct $\mathfrak{M} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L})$ [559] of $\mathfrak{M}$. Similarly for structures $\mathfrak{G}$ and $(g \circ f)(\mathfrak{G})$.

[556] Translation functions of the type $Tr : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$ play an important role in the present work. They have two important features: (i) they are meaning preserving, and (ii) they *respect the logical structure* of the languages involved, e.g. $Tr(\neg\varphi) = \neg Tr(\varphi)$ and analogously for the remaining parts of our logic. (We do not discuss property (ii) explicitly, but since it is important we mention that it is discussed in the algebraic logic works e.g. in Andréka et al. [31].) In other words (ii) could be interpreted as saying that our translation mappings are grammatical, i.e. they respect the grammar of the languages involved. Cf. Remark 4.3.30 on p.250.

[557] i.e. to $Voc_0\mathsf{K} \cap Voc_0\mathsf{L}$

[558] We note that the formulas φ and $Tr_f(\varphi)$ have the same free variables (therefore the statement below makes sense).

[559] $Voc\mathsf{K} \cap Voc\mathsf{L}$ is the common part of the vocabularies of K and L.

$\lhd$

Intuitively, K and L are weakly definitionally equivalent iff they are definable over each other and the first-order definable meta-functions induced by these definitions are inverses of each other up to isomorphism.

PROPOSITION 4.3.43 *Assume* K, L *are two classes of models. Then*

$$\mathsf{K} \equiv_\Delta \mathsf{L} \quad \Rightarrow \quad \mathsf{K} \equiv_\Delta^w \mathsf{L},$$

i.e. if K *and* L *are definitionally equivalent then they are also weakly definitionally equivalent.*

We omit the **proof**. ∎

In connection with the above proposition we note that the other direction does not hold in general, i.e.

$$\mathsf{K} \equiv_\Delta^w \mathsf{L} \quad \not\Rightarrow \quad \mathsf{K} \equiv_\Delta \mathsf{L}.$$

This (i.e. $\not\Rightarrow$) is so even if we assume that K and L are both axiomatizable, cf. Examples 4.3.46 (p.266) and Thm.4.5.26 (p.303).

Examples come at the end of this section.

Remark 4.3.44 Assume that $f : \mathsf{K} \longrightarrow \mathsf{L}$ and $g : \mathsf{L} \longrightarrow \mathsf{K}$ are first-order definable meta-functions as in Def.4.3.42. Then $Rng(f)$ is L up to isomorphism and $Rng(g)$ is K up to isomorphism. Moreover, for every $\mathfrak{A} \in \mathsf{L}$ there is $\mathfrak{A}' \in Rng(f)$ such that there is an isomorphism between the structures $\mathfrak{A}$ and $\mathfrak{A}'$ which is the identity map on the reduct $\mathfrak{A} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L})$ of $\mathfrak{A}$; and the analogous statement holds for every $\mathfrak{B} \in \mathsf{K}$.

$\lhd$

The following proposition says that if $\mathsf{K} \equiv_\Delta^w \mathsf{L}$ then the language $Fm(\mathsf{K})$ of K can be translated into the language $Fm(\mathsf{L})$ of L in a meaning preserving way and vice-versa; more precisely these translations work well for the sentences[560] only or more generally for those formulas which contain only such free variables that range over the common sorts of K and L. Moreover these translation mappings are inverses of each other (up to logical equivalence "$\leftrightarrow$"). We note that if in addition we have $\equiv_\Delta$ in place of $\equiv_\Delta^w$ [561] then this nice, meaning preserving translation mapping extends to all formulas, cf. the end of Remark 4.3.37 on p.260.

PROPOSITION 4.3.45 *Assume* $\mathsf{K} \equiv_\Delta^w \mathsf{L}$. *Then there are "natural" translation mappings*

$$T_f : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K}) \quad and \quad T_g : Fm(\mathsf{K}) \longrightarrow Fm(\mathsf{L})$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{L})$, $\psi(\bar{y}) \in Fm(\mathsf{K})$ *with all their free variables belonging to common sorts of* K *and* L, $\mathfrak{A} \in \mathsf{L}$ *and* $\mathfrak{B} \in \mathsf{K}$, *and evaluations* $\bar{a}, \bar{b}$ *of the variables* $\bar{x}, \bar{y}$, *respectively, (i)–(iv) below hold, where* f *and* g *are as in Def.4.3.42.*

(i) $f(\mathfrak{B}) \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathfrak{B} \models T_f(\varphi)[\bar{a}]$ *and* $g(\mathfrak{A}) \models \psi[\bar{b}] \;\Leftrightarrow\; \mathfrak{A} \models T_g(\psi)[\bar{b}]$.

(ii) $\mathfrak{A} \models \varphi[\bar{a}] \;\Leftrightarrow\; g(\mathfrak{A}) \models T_f(\varphi)[\bar{a}]$ *and* $\mathfrak{B} \models \psi[\bar{b}] \;\Leftrightarrow\; f(\mathfrak{B}) \models T_g(\psi)[\bar{b}]$.

(iii) $\mathfrak{A} \models \varphi(\bar{x}) \leftrightarrow (T_f \circ T_g)(\varphi)(\bar{x})$ *and* $\mathfrak{B} \models \psi(\bar{y}) \leftrightarrow (T_g \circ T_f)(\psi)(\bar{y})$.

[560] *Sentence* means closed formula, i.e. formula without free variables.

[561] i.e. $\mathsf{K} \equiv_\Delta \mathsf{L}$

(iv) $\mathsf{L} \models \varphi \;\Leftrightarrow\; \mathsf{K} \models T_f(\varphi)$ *and* $\mathsf{K} \models \psi \;\Leftrightarrow\; \mathsf{L} \models T_g(\psi)$.

Proof: Item (i) of the proposition follows by Prop.4.3.41 above. Items (ii)–(iv) follow by item (i) and Remark 4.3.44. ∎

In connection with Prop.4.3.45 above cf. Remark 4.3.37 on p.258. We will have results analogous to the conclusion of Prop.4.3.45 above at various points in the remaining part of this chapter, cf. e.g. Thm.4.5.12 on p.291.

Examples 4.3.46 In all three examples below we state $\mathsf{K} \not\equiv_\Delta \mathsf{L}$ for some classes K, L. In all three examples we can use item (6) on p.257 to prove $\mathsf{K} \not\equiv_\Delta \mathsf{L}$.

1. Let K be the class of two-element algebras without operations. I.e.

$$\mathsf{K} = \{\, A \;:\; |A| = 2 \,\}.$$

Let L be the class of two-element ordered sets. Important: The sort symbol of K and the sort symbol of L are different. Then

$$\mathsf{K} \equiv^w_\Delta \mathsf{L}, \quad \text{but} \quad \mathsf{K} \not\equiv_\Delta \mathsf{L}.$$

2. Let K_2 be the same as K was in item 1. above. Let K_3 be the class of three element algebras without operations. Let the sort symbols of K_2 and K_3 be different. Then

$$\mathsf{K}_2 \equiv^w_\Delta \mathsf{K}_3, \quad \text{but} \quad \mathsf{K}_2 \not\equiv_\Delta \mathsf{K}_3.$$

3. More sophisticated example, *affine structures*: Let AB be the class of Abelian (i.e. commutative) groups.

Assume $\mathfrak{G} = \langle G;\, +, -, 0\rangle \in \mathsf{AB}$.

We define the affine relation R_+ on G as follows.

$$R_+(a,b,c,d,e,f) \quad \overset{\text{def}}{\Longleftrightarrow} \quad (a-b)+(c-d) = (e-f).$$

The *affine structure* associated with the group $\mathfrak{G}$ is

$$\mathfrak{A}_{\mathfrak{G}} := \langle G;\, R_+\rangle.$$

The class of affine structures is

$$\mathsf{Af} := \{\, \mathfrak{A}_{\mathfrak{G}} \;:\; \mathfrak{G} \in \mathsf{AB} \,\}.$$

Let the sort symbols of AB and Af be different. Claim:

$$\mathsf{AB} \equiv^w_\Delta \mathsf{Af}, \quad \text{but} \quad \mathsf{AB} \not\equiv_\Delta \mathsf{Af}.$$

Hint: Definability of Af over AB is trivial. Definability of AB over Af: Let $\langle G;\, R_+\rangle \in \mathsf{Af}$. We define a new relation eq as follows.

$$\langle a,b\rangle \; eq \; \langle c,d\rangle \quad \overset{\text{def}}{\Longleftrightarrow} \quad R_+(a,b,a,a,c,d).$$

Let us notice that eq is an equivalence relation on $G \times G$. Now, let

$$A := G \times G/eq$$

be a new sort. Further

$$\langle a,b\rangle/eq + \langle c,d\rangle/eq = \langle e,f\rangle/eq \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad R_+(a,b,c,d,e,f).$$

Now, defining the rest of the Abelian group $\langle A,+,\dots\rangle$ over the affine structure $\langle G;\, R_+\rangle$ is left to the reader.

The proof of $\not\equiv_\Delta$ is based on looking at the large number of automorphisms of the affine structure $\langle G;\, R_+\rangle$. We omit the details. (The idea is similar to that of example 1.)

$\triangleleft$

Remark 4.3.47 (Making $\equiv_\Delta^w$ strong by using parameters)
Consider the applications of $\equiv_\Delta^w$ in items (i), (ii) below.

(i) In Thm.4.5.23 (p.301) it is stated that

$$\text{(Fields)} \ \equiv_\Delta^w \ (\textbf{pag}\text{-geometries}).$$

Theorems 4.5.19, 4.5.26 are analogous.

(ii) $\mathsf{Mod}(Th) \equiv_\Delta^w \mathsf{Mog}(TH)$ for certain choices of Th, TH, where the class $\mathsf{Mog}(TH)$ of geometries is defined on p.326. We note that this is not proved or even stated in the present work, but it can be proved.

Now, if in the context (or background) of items (i), (ii) above we replace the notion of definability by parametric definability using finitely many parameters only (in the usual sense cf. p.235 and p.223, immediately below Remark 4.3.4, or e.g. Hodges [130, pp. 27–28])[562] then we will obtain that the classes in question e.g. $\mathsf{Mod}(Th)$ and $\mathsf{Mog}(TH)$ become definitionally equivalent in this weaker parametric sense. (I.e. they have a single common parametrically definable definitional expansion etc.) More concretely we could add $(n+1)$-many new constants to **pag** geometries such that

$$\text{(Fields)} \ \equiv_\Delta \ (\textbf{pag}\text{-geometries} \ + \ \text{these constants}).$$

Completely analogous improved versions of Theorems 4.5.19, 4.5.26 (pp. 300, 303) are also true.

Also we could add $n+1$ new constants to $\mathsf{Mog}(TH)$ and a constant (a distinguished observer) to $\mathsf{Mod}(Th)$ yielding

$$(\mathsf{Mod}(Th) \ + \ \text{new constant}) \ \equiv_\Delta \ (\mathsf{Mog}(TH) \ + \ \text{new constants}),$$

for certain choices of Th and TH. This works even if we assume **Ax(eqtime)** $\in Th$ (cf. Conjecture 4.5.58 on p.329).
It is these new auxiliary constants which are called parameters in the theory of parametric definability.

We leave elaborating the details of this parametric direction to the interested reader.

$\triangleleft$

[562] Parametric definability is a slightly weaker notion than definability.

4.3.5 A generalization of Beth's theorem. Connections with the literature.

Many-sorted definability theory with new sorts (i.e. the notion of implicit and explicit definition) is a generalization of one-sorted definability theory (without new elements) discussed in traditional logic books. This observation leads to several natural questions which we discuss here only tangentially. One of these is the question whether Beth's theorem (about the equivalence of the two notions of definability) generalizes to our present case. Note that Thm.4.3.31 already establishes a connection between "syntactical aspects" (like existence of translation functions) and "semantical aspects" (like constructibility with means of steps (1)-(2.1)) of definability.

THEOREM 4.3.48 *Assume* $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ *is a reduct of* K^+ *such that* K^+ *has only finitely many sorts. Assume that the language of* K^+ *is countable, and that* K *has a sort with more than one element. Then (i) and (ii) below are equivalent.*

(i) K^+ *is implicitly definable over* K *without taking reducts.*

(ii) K^+ *is rigidly explicitly definable over* K.

Outline of proof: The proof uses Thm.4.3.31 and Gaifman's theorem (cf. Hodges [130, Thm.12.5.8, p.645]), which is about one-sorted structures, together with ideas from Pillay & Shelah [213], and can be found in Andréka-Madarász-Németi [21]. Here we outline the main steps in that proof.

First we show that we may assume that K^+ is axiomatizable, too. Here we can use the corollary of Thm.4.3.31 that if K is axiomatizable and $\mathsf{K} \equiv_\Delta \mathsf{K}^+$, then $\mathsf{K} \equiv_\Delta \mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$, too.

Let $T \overset{\text{def}}{=} \overline{\mathsf{Th}(\mathsf{K}^+)}$ denote the one-sorted version of $\mathsf{Th}(\mathsf{K}^+)$ such that there is an additional unary relation P which denotes the union of universes of sort those of K. (I.e. P denotes the "old universes" inside the new universes.)

In the literature there are three properties for one-sorted theories which are useful for us, these are the following: T has the reduction property, T is coordinatized over P and T is rigidly relatively categorical over P, cf. Hodges [130]. We have mentioned these properties in footnotes so far, but see also the following part about connections with the literature and Figure 97. Gaifman's theorem (which is based on the Chang-Makkai definability theorem) states that the latter two properties coincide for countable languages when T is complete and has infinite models. In AMN [21] we showed that the conditions on T can be eliminated. Further, using our Thm.4.3.31 and that we have only finitely many sorts, we proved that $\mathsf{K} \equiv_\Delta \mathsf{K}^+$ iff $\overline{\mathsf{Th}(\mathsf{K}^+)}$ is coordinatized over P and K^+ is implicitly definable over K without taking reducts iff $\overline{\mathsf{Th}(\mathsf{K}^+)}$ is rigidly relatively categorical over P. This finishes the proof of Thm.4.3.48. ∎

COROLLARY 4.3.49 (Beth's theorem generalized to defining new sorts) Assume $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ is a reduct of K^+ such that K^+ has only finitely many sorts. Assume that the language of K^+ is countable, and that K has a sort with more than one element. Then (i) and (ii) below are equivalent.

(i) K^+ is implicitly definable over K.

(ii) K^+ is explicitly definable over K.

QUESTION 4.3.50 *Can Theorem 4.3.48 and Corollary 4.3.49 above be generalized to the case when infinitely many sorts are allowed? (First one has to generalize the definition of explicit definability. This can be done easily, e.g. we may allow iteration of steps (1), (2.1), (2.2) along an infinite ordinal, taking "unions" of ascending chains of expansions in the limit steps.)* $\lhd$

The above question seems to be more about logic than about relativity, so we do not discuss it here.

Connections with the literature

For investigations related to definability of new sorts as discussed in the present section (§4.3 herein) we refer to Hodges [130] Chapter 12, and within that chapter to §12.3 (pp.624-632), §12.5 (pp.638-652). E.g. p.638 last 3 lines – p.639 line 9 discusses generalizability of Beth's theorem, and similarly for p.645 line 6, p.649 lines 5-6. (We would also like to point out Exercises 13, 14 on p.649 of [130].) We also refer to Myers [197], Hodges-Hodkinson-Macpherson [131], Pillay-Shelah [213], Shelah [234]. In passing we note that our subject matter (i.e. definability of new sorts) is related to the directions in recent (one-sorted) model theory called "relative categoricity" or "categoricity over a predicate", and "theory of stability over a predicate".

Below we outline some connections between our notions and the ones used in a substantial part of the above quoted (one-sorted) literature. We will systematically refer to Hodges [130].

Assume $\mathsf{K}^+ = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$ and that K has finitely many sorts $U_0, \ldots, U_k$. Let $P = U_0 \cup \ldots \cup U_k$ be the union of these sorts regarded as a unary predicate. Then:

(1) "K^+ is implicitly definable up to isomorphism over K" is equivalent to "$\mathsf{Th}(\mathsf{K}^+)$ is relatively categorical over P".

(2) "K^+ is implicitly definable without taking reducts over K" is equivalent to "$\mathsf{Th}(\mathsf{K}^+)$ is rigidly relatively categorical over P".

(3) "K^+ is explicitly definable over K" is *not* equivalent to "$\mathsf{Th}(\mathsf{K}^+)$ is coordinatizable over P".

(4) "K^+ is a definitionally equivalent expansion of K" is equivalent to "$\mathsf{Th}(\mathsf{K}^+)$ is coordinatized over P".

In items (1)-(4) above, on the left hand side we have many-sorted notions, while on the right-hand side we have one-sorted notions (like relative categoricity). So it needs some explanation what we mean by claiming their equivalence. The answer is the following: First we translate our many-sorted notions into one-sorted ones (by treating the sorts as unary predicates of one-sorted logic) the usual, natural way, and then we claim that the so translated version of our many-sorted notion is equivalent into the other one-sorted notion quoted from Hodges [130]. E.g., the so elaborated version of item (1) looks like the following. "The one-sorted translation of (K^+ is implicitly definable up to isomorphism over K)" is equivalent to "(the one-sorted version of $\mathsf{Th}(\mathsf{K}^+)$) is relatively categorical over P". The point here is that relative categoricity is defined only for one-sorted logic in Hodges [130]. Therefore, to use it as a possible equivalent of (our many-sorted) "implicit definability up to isomorphism", first we have to translate everything into one-sorted logic, and then make the comparison. Indeed, items (1)-(4) are understood this way. They are proved in AMN [21].

Connections between the various notions of definability

Figure 97 below shows the connections between the various notions introduced in this sub-section. It also indicates the above outlined connections with some notions used in the literature (relative categoricity, coordinatizability). The connections indicated are fairly easy to show, except for the following proposition (and, of course where Theorem 4.3.48 and Corollary 4.3.49 are indicated).

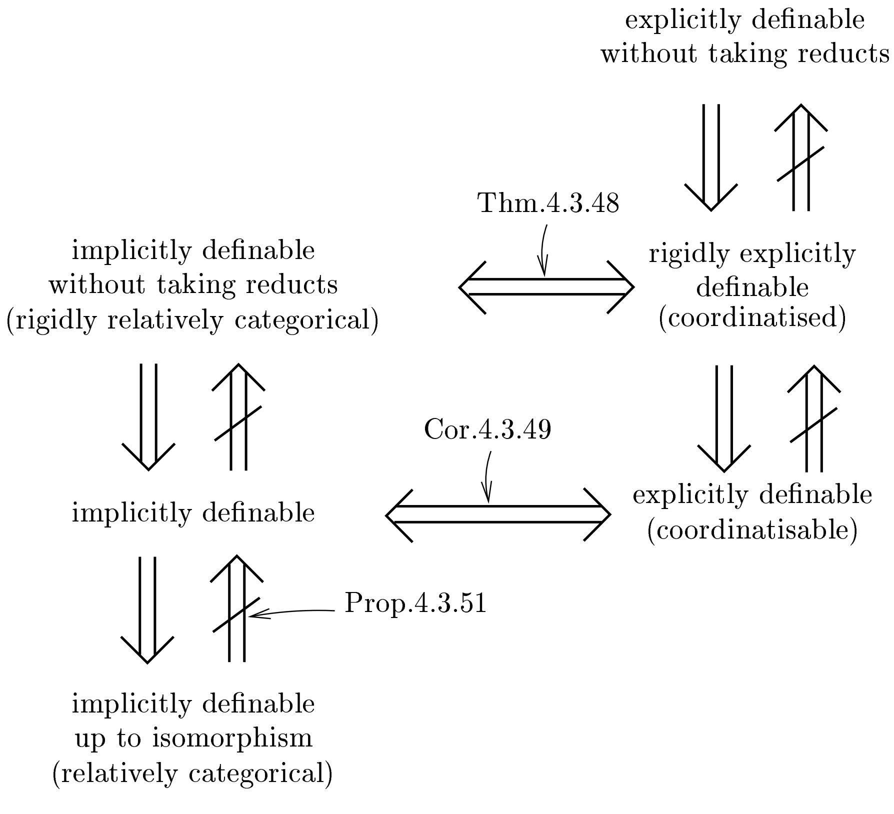


Figure 97: Connections between the various notions of definability. We assume that $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ is a reduct of K^+ such that K^+ has only finitely many new sorts. We also assume that the language of K^+ is countable, and that K has a sort with more than one element. On the figure we write "implicitly definable without taking reducts" for "K^+ is implicitly definable over K without taking reducts", and similarly for the other notions. For the implication "implicitly definable" to "implicitly definable up to isomorphisms" we need the extra assumption $\mathsf{K}^+ = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$.

PROPOSITION 4.3.51 (Hodges [130]) *Assume the hypotheses of Theorem 4.3.48 (which are the same as the hypotheses used in Figure 97). Then "K^+ is implicitly definable over K up to isomorphism" does not imply "K^+ is implicitly definable over K".*

Proof: A 6-element counterexample proving this is given in Hodges [130, Example 2 on p.625]. There two structures are defined, **A** and **B**, with **A** a reduct of **B**. **B** is implicitly definable up to isomorphism over **A** (this follows from the fact that **B** is finite). At the same time, **B** is not definable implicitly over **A**, because **A** has an automorphism α of order 2 (i.e. $\alpha \circ \alpha = Id_A$) which cannot be extended to an automorphism β of **B** of order 2. Indeed, if **B** was implicitly definable over **A**, then an expansion $\mathbf{B}^+$ of **B** would be implicitly definable over **A** without taking reducts. Hence the automorphism α would extend to an automorphism β of $\mathbf{B}^+$. Since the identity of **A** extends to a unique automorphism of $\mathbf{B}^+$, then $\beta \circ \beta = Id_{B+}$ should hold. But then $\beta \restriction B$ would be an automorphism of **B** of order 2 and extending α. (Cf. Thm.12.5.7 in [130, p.644].) Since **A** and **B** are finite structures, we can take $\mathsf{K} = \mathbf{I}\{\mathbf{A}\}$ and $\mathsf{K}^+ = \mathbf{I}\{\mathbf{B}\}$, and then the hypotheses of Proposition 4.3.51 hold for K and K^+. This finishes the proof. ∎

On the choice of basic steps in explicit definability

Remark 4.3.52 (Forming disjoint union of two sorts) For didactical reasons we will refer to items (1)–(2.2) as steps (1)–(2.2) to emphasize their roles in constructing an explicit definition (for some new class K^+) in a step-by-step manner.

We could have included in this list of steps as step (2.3) the definition of a new sort as a disjoint union of two old sorts. This goes as follows:

Assume U_k, U_m are old sorts, i.e. sorts of $\mathfrak{M}$, while U^{new} is not a sort of $\mathfrak{M}$. Then, we can define the new sort as

$$U^{new} := U_k \mathbin{\dot\cup} U_m$$

with two injections

$$i_1 : U_k \rightarrowtail U^{new} \qquad \text{and} \qquad i_2 : U_m \rightarrowtail U^{new}$$

such that U^{new} is the union of $Rng(i_1)$, $Rng(i_2)$ and $Rng(i_1) \cap Rng(i_2) = \emptyset$. Here $k = m$ is permitted. But even if $k = m$, i_1 and i_2 are different. Now the expanded model is

$$\mathfrak{M}^+ := \langle \mathfrak{M}, U^{new}; i_1, i_2 \rangle.$$

We note that such an $\mathfrak{M}^+$ is always implicitly definable over $\mathfrak{M}$, further *all the nice properties* of *explicit definitions*[563] in items (1)–(2.2) hold for this new kind of explicit definition which from now on we will consider as step (2.3) of explicit definability.

All the same, we do not include step (2.3) into the list of permitted steps of building up an explicit definition. We have two reasons for this.

(i) Step (2.3) can be reduced to (or simulated by) steps (1)–(2.2). Namely, assume $\mathfrak{M}^+$ is defined from $\mathfrak{M}$ by using step (2.3). Assume further that $\mathfrak{M}$ has a sort U_i with more than one elements (i.e. $|U_i| > 1$). Then by using steps (1)–(2.2) one can define an expansion $\mathfrak{M}^{++}$ from $\mathfrak{M}$ such that $\mathfrak{M}^+$ is a reduct of $\mathfrak{M}^{++}$.[564]

Further:

(ii) We will not need step (2.3) in the present work. I.e. in the logical analysis of relativity, explicit definitions of form (2.3) did not come up so far.

[563] As an example we mention that explicitly defined symbols can be eliminated from the language, cf. subsection 4.3.3 on p.245.

[564] More precisely there is a *unique* isomorphism h between $\mathfrak{M}^+$ and this reduct of $\mathfrak{M}^{++}$ such that $h \restriction \mathfrak{M}$ is the identity function.

Item (i) above shows that adding step (2.3) to the permitted steps of explicit definitions would increase the collection of sorts and relations definable over $\mathfrak{M}$ *only* in the pathological case when all universes of $\mathfrak{M}$ have cardinalities ≤ 1.

Therefore while noting that step (2.3) could be included without changing the theory of explicit definability significantly, we do *not* include it. However, sometimes (in some intuitive text) when we want to get "dreamy" we might refer to explicit definability as involving four steps (1)–(2.3). $\triangleleft$

Remark 4.3.53 One might want to develop a more systematic understanding of what explicit definitions are. For such a more systematic understanding of explicit definitions let us rearrange the basic steps into steps (1^*)–(5^*) below.

(1^*) Definition of new relations $\bar{R}^{new}$ explicitly the classical way (as in item (1) on p.231).

(2^*) Definition of new sorts as *direct products* of old sorts together with projection functions ($U^{new} := U_i \times U_j$ etc) (as in item (2.1) on p.232).

(3^*) Definition of new sorts as *disjoint unions* of old sorts together with inclusion functions ($U^{new} := U_i \dot{\cup} U_j$ etc) (as in item (2.3) on p.271).

(4^*) Definition of a new sort as a definable *subset* of an old sort together with an inclusion function. I.e.

$$U^{new} := \{\, x \in U_i \ :\ \mathfrak{M} \models \psi(x) \,\}$$

and $i_{new} : U^{new} \longrightarrow U_i$ is the usual inclusion function. The expanded model is $\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new}; i_{new} \rangle$.

(5^*) Definition of a new sort as a definable *quotient* of an old sort exactly as in item (2.2) on p.234 (i.e. $U^{new} = U_i/R$ etc).

Now, an explicit definition in the new sense is given by an arbitrary sequence (i.e. iteration) of steps (1^*)–(5^*) above.

If we disregard the trivial case when all sorts are singletons or empty, then explicit definitions in the new sense are equivalent to explicit definitions as introduced in §4.3.2. We leave checking this claim to the reader.

We would like to point out that explicit definitions as built up from steps (1^*)–(5^*) are not ad-hoc at all. In the category theoretic sense the formation of disjoint unions is the *dual* of the formation of direct products and the formation of sub-universes (or sub-structures) is the dual of the formation of quotients. So, we are left with two basic steps and their duals.

It is interesting to note that our steps (2^*)–(5^*) correspond to basic operations producing new models from old ones. (Indeed if U_i is a universe of $\mathfrak{M}$ then we can restrict $\mathfrak{M}$ to U_i and then we obtain a one-sorted reduct of $\mathfrak{M}$ with universe U_i. Hence creating new sorts from old ones is not unrelated to creating new models from old ones. All the same, we do not want to stretch this analogy too far.)

What we would like to point out here, is that steps (2^*)–(5^*) seem to form a natural, well balanced set of basic operations, while step (1^*) has been inherited from the classical theory of definability.

Further, we note that while selecting our basic steps (e.g. steps (1^*)–(5^*) above) we had to be careful to keep them implicitly definable i.e. they should not lead to "explicitly definable things" which are not implicitly definable. Therefore operations like formation of powersets cf. Example 4.3.9(1) (or all finite subsets of a set cf. Example 4.3.9(6))[565] are ruled out from the beginning.

◁

Further recent results on definability theory (sometimes in algebraic form[566]) are in Madarász [167], [164], [163], Madarász-Sayed [178], Hoogland [134].

Some further results on definability: As we mentioned, the importance, for *relativity*, of the logical theory of definability was pointed out already in 1924 by Reichenbach in his relativity book [218]. Reichenbach credits the origins of definability theory to David Hilbert by referring to Hilbert's 1921 book of the foundations of geometry (cf. [218, p.3]). The recognition of the importance of definability for relativity theory has been more and more recognized ever since, cf. e.g. Friedman [91]. As it turns out in the Appendix ("Why exactly FOL") of AMN [18], it is relevant to the present investigations to study properties of different logical systems. Indeed, this is the subject matter of the general theory of logics (cf. e.g. Barwise-Feferman [45], or Andréka-Németi [205] and also of algebraic logic, cf. e.g. Adréka-Németi-Sain [31] or Madarász [170]. The subject matter of studying definability properties of various logical systems and to characterize them algebraically was initiated by Alfred Tarski and his followers (cf. Pigozzi [212]) and Daigneault [65]. There are more than one definability properties, one of them is the already discussed "Beth definability property" while another rather important one is the so-called Craig interpolation property. Pigozzi [212] formulated the question of finding the precise algebraic counterpart of the Craig property.[567] Several authors worked on this problem, and the final answer was found in Madarász [164, 166] which says that this counterpart is exactly the so-called super-amalgamation property (SUPAP for short). Related results are in Madarász [161],[165]. Having found the algebraic counterpart of this important logical property, the following papers contain results on the question which classes of algebras have the algebraic counterpart of one or the other of the distinguished definability properties: Madarász [163], [167], Madarász-Sayed [178]. Using the above mentioned kinds of equivalence results, these papers answer the question of which logics have the definability property in question. In particular, Madarász [167] answers questions implicit in the classical paper Henkin-Tarski [123] concerning logics of schemas[568] of formulas. In passing we note that all the questions that remained open in the historical paper Pigozzi [212][569] are answered in Madarász [167]. (A survey of all the answers is given in Madarász-Sayed [178].)

[565] Seeing that $\mathcal{P}_\omega(U_i)$ leads to problems (i.e. checking Example 4.3.9(6)) is not obvious, it is not necessary to check this for understanding this work.

[566] Cf. our section on duality theory (in particular §A.3).

[567] The question seems to go back to Tarski.

[568] To illustrate the importance of the question, we note that most mathematicians use schemas of formulas instead of single formulas (think of the induction axiom schema of Peano's Arithmetic as an example).

[569] which is often regarded as a companion of Henkin-Monk-Tarski [120], cf. the references in [120] to this paper

4.4 On the connection between Tarski's language for geometry and ours (both in first-order logic), and some notational convention

In this section we heavily use the results and methods of §4.3 ("Definability of new sorts"), and the content of this section will be used in §4.5.2 ("Coordinatization") way below.

In connection with the present section it might be useful for the non-logician reader to have a look at the Appendix (entitled: "Why first order logic?") on higher-order logic versus first-order logic in AMN [18, p.1245].

In the discussion below we say "the language" and then instead of specifying the language we have in mind we write down a typical structure of the language. We hope, this causes no confusion.

Tarski uses the language

$$\mathbf{G}_{Ta} = \langle \mathit{Points};\ \mathit{Col}, \text{“extra relations”}\rangle,$$ [570]

while we use the language

$$\mathbf{G}_{We} = \langle \mathit{Points}, \mathit{Lines};\ \in, \text{“extra relations”}\rangle$$ [571]

for studying geometry, where $\mathit{Col} \subseteq {}^{3}\mathit{Points}$ is a ternary relation called collinearity, while $\in\ \subseteq \mathit{Points} \times \mathit{Lines}$ is the usual incidence relation of geometry. Since the "extra relations" part is essentially the same for both approaches, let us compare $\langle \mathit{Points};\ \mathit{Col}\rangle$ and $\langle \mathit{Points}, \mathit{Lines};\ \in\rangle$. Intuitively $\mathit{Col}(a, b, c)$ holds iff a, b, c are on the same line. Now, we claim that the two languages (that of $\mathbf{G}_{Ta}$ and $\mathbf{G}_{We}$) are of the *same expressive power* i.e. they are definitionally equivalent[572], under some very mild conditions; this claim will be formulated as Prop. 4.4.2 below (where information on the proof will also be given). Cf. also e.g. Example 4.3.16 (p.238). It remains to simulate the incidence relation "$\in$". We do this by postulating $c \in \langle a, b\rangle$ iff $\mathit{Col}(a, b, c)$ holds.[573]

A detailed explanation of the connections between our two-sorted language and structures $\langle \mathit{Points}, \mathit{Lines};\ \in\rangle$ and Tarski's one-sorted version $\langle \mathit{Points};\ \mathit{Col}\rangle$ is given both in our definability section §4.3 together with items 4.4.2 herein and 6.5.6 in AMN [18], in Givant [98, pp.582-584], and in Appendix A of Goldblatt [102]. Cf. also the first 6 lines on p.viii of [102]. We would like to emphasize that the difference between the two languages is only "notational", cf. Remark 4.3.37 (p.258), the intuitive text above that remark and Prop. 4.4.2.

To formulate the conditions which we need to prove definitional equivalence between Tarski's language and ours we introduce axiom **Det** in the language of $\langle \mathit{Points}, \mathit{Lines};\ \in\rangle$ and axiom **det** in the language of $\langle \mathit{Points};\ \mathit{Col}\rangle$. The acronym "**Det**" abbreviates "points determine lines". Similarly for "**det**".

[570] Actually instead of Col Tarski uses Bw, but Col is definable from Bw (in Tarski's geometries).

[571] Sometimes we write "*Points*, *Lines*" instead of "Mn, L" only to sound more intuitive or more suggestive. Summing it up: *Points* denotes Mn and *Lines* denotes L.

[572] More precisely the theory of the language $\langle \mathit{Points}, \mathit{Lines};\ \in\rangle$ and another one of the language $\langle \mathit{Points};\ \mathit{Col}\rangle$ are definitionally equivalent assuming very mild axioms on both sides. Cf. Def.4.3.33 (p.255) for definitional equivalence.

[573] We note that Col (definable from *Lines*) is slightly different from coll (which was defined from Bw, on p.159 way above). Cf. Item 4.5.36, p.308.

Det $(\forall p,q \in \mathit{Points})(\forall \ell,\ell' \in \mathit{Lines})\,[\,(p \neq q \ \wedge\ p,q \in \ell \cap \ell') \ \rightarrow\ \ell = \ell'\,] \quad \wedge$
$(\forall \ell \in \mathit{Lines})(\exists p,q \in \mathit{Points})\,[\,p \neq q \ \wedge\ p,q \in \ell\,]$.

Intuitively, two different lines intersect each other in at most one point; and on each line there are at least two points.

Note that axiom **Det** is an extra possible assumption about our frame models $\mathfrak{M}$ considered in this section. At the end of this section, we will return to discussing the role of axiom **Det**.

Below we introduce axiom **det** in Tarski's language.

det $\big(\ \mathit{Col}(a,b,c) \quad\rightarrow\quad (\,\mathit{Col}(a,c,b)\ \wedge\ \mathit{Col}(b,a,c)\ \wedge\ \mathit{Col}(a,a,b)\,)\,\big)$ [574] $\quad\wedge$
$\big(\,[\,(\,\mathit{Col}(a,b,c)\ \wedge\ \mathit{Col}(a,b,d)\ \wedge\ a \neq b\,) \quad\rightarrow\quad \mathit{Col}(a,c,d)\,] \quad\wedge$
$[\,\mathit{Col}(a,a,a)\ \rightarrow\ (\exists b)(b \neq a \quad\wedge\quad \mathit{Col}(a,b,b))\,]\,\big)$.

Intuitively, **det** says two things, the second part is basically a translation of our axiom **Det** above, while the first part says that *Col* is invariant under permutations and even "transformations" of its arguments. We note that $\mathsf{Ge}(\textbf{Newbasax}) \not\models \textbf{Det}$, while $\mathsf{Ge}(\textbf{Newbasax}) + \textbf{Ax(diswind)} \models \textbf{Det}$, cf. Fig. 98 (or Prop. 6.5.8 in AMN [18]).

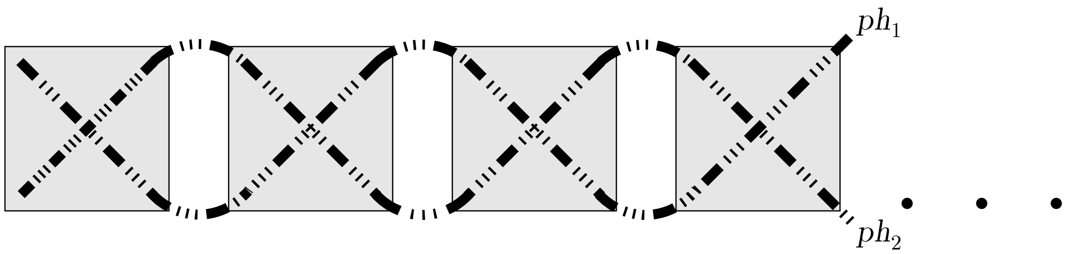


Figure 98: Illustration for a model of **Newbasax** in which axiom **Det** fails.

Now, we can formally define Tarski's class Ge_{Ta} and ours Ge_{We} as of geometries we promised way above.

Definition 4.4.1

$$\mathsf{Ge}_{Ta} \ \stackrel{\text{def}}{:=}\ \{\,\langle \mathit{Points};\ \mathit{Col}\rangle\ :\ \langle \mathit{Points};\ \mathit{Col}\rangle \models \textbf{det}\,\},$$
$$\mathsf{Ge}_{We} \ \stackrel{\text{def}}{:=}\ \{\,\langle \mathit{Points}, \mathit{Lines};\ \in\rangle\ :\ \langle \mathit{Points}, \mathit{Lines};\ \in\rangle \models \textbf{Det}\,\}.$$

$\lhd$

The following proposition says that Tarski's language and our language are definitionally equivalent, under some mild assumptions.

PROPOSITION 4.4.2 Ge_{Ta} *and* Ge_{We} *are definitionally equivalent, i.e.*

$$\mathsf{Ge}_{Ta} \equiv_{\Delta} \mathsf{Ge}_{We}.$$

[574] We note that this part of **det** implies that for any function $\pi : \{a,b,c\} \longrightarrow \{a,b,c\}$, $\mathit{Col}(a,b,c) \ \rightarrow\ \mathit{Col}(\pi(a),\pi(b),\pi(c))$.

Proof of Prop.4.4.2: The proof can be found in AMN [18] on pp. 994-996, but cf. also Example 4.3.16 herein. ∎

Remark 4.4.3 By Prop. 4.4.2 and Remark 4.3.37 in §4.3, we conclude that there are very strong meaning-preserving translation mappings between Tarski's language $Fm(\mathsf{Ge}_{Ta})$ and ours $Fm(\mathsf{Ge}_{We})$ going in both directions. Therefore, as we explained on p. 255, we regard the two languages as two syntactic representations of the same *single language* differing only in what we regard as basic vocabulary and what we regard as convenient abbreviations. ◁

Remark 4.4.4 By Thm.6.5.5 and Remark 6.5.6 of AMN [18], formulas in two-sorted language $\langle Points, Lines;\, \in\rangle$ of our incidence geometries are *abbreviations* for formulas in Tarski's one-sorted language $\langle Points;\, Col\rangle$. Actually, we can introduce some further useful abbreviations making our language even more intuitive and more "compact". For our next definition, we need the expanded version $\langle Points;\, Col, Bw\rangle$ of Tarski's language also due to Tarski. Besides the new sort *Lines* we can extend Tarski's language with new sorts *Planes*, *Half-lines* and the incidence relations $\in_{Pl} \subseteq Points \times Planes$ and $\in_{Hl} \subseteq Points \times Half\text{-}lines$ as follows. We use the rules of explicit definability introduced on pp. 230-235 in our section §4.3.2 on definability. Our definitions of *Planes* etc. are explicit definitions (in the sense of §4.3.2). The details of these definitions are in AMN [18, Remark 6.5.7, p.997]. By the above, the many-sorted geometric structures (and language)

$$\langle Points, Lines, Planes, Half\text{-}lines;\, \in, \in_{Pl}, \in_{Hl}, Bw, Col\rangle$$

are *definitional expansions* of the one-sorted structures (and language) $\langle Points;\, Col, Bw\rangle$.

Our definition of *Planes*, *Half-lines* etc. in AMN [18, Remark 6.5.7] followed the steps prescribed in §4.3.2(2) on pp. 230-235 for constructing explicit definitional expansions of models. Hence our $\langle Points, Lines, Planes, \dots, Col\rangle$ is a definitional expansion of the original Tarskian geometry $\langle Points, Col, Bw\rangle$. In §4.3.2(2) we also indicated how to write up the set of formulas Δ which constitute the explicit definitions themselves. We leave it to the reader to use our above construction (for *Planes* etc.) for writing up the explicit definition of Δ; in view of §4.3.2(2) this is a routine task.

We note that we do not have to stop with introducing *Planes* as a convenient abbreviation. In the same spirit we can introduce the remaining geometric objects, e.g. hyper-planes or 3-dimensional subspaces, or circles, spheres etc. All these remain abbreviations only and we remain in the language $\langle Points;\, Col, Bw, eq\rangle$. In other words the expanded language $\langle Points, Lines, \dots, \text{3-dimensional subspaces}, \dots\rangle$ remains a definitional expansion of Tarski's original language $\langle Points;\, Col, Bw, eq\rangle$.

◁

Convention: Throughout this convention we assume the axiom **Det**. Motivated by Thm.4.4.2 above, we can *identify* our many-sorted geometry

$$\mathfrak{G}^0_{\mathfrak{M}} = \langle Mn, L;\, L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq\rangle$$

with a one-sorted structure

$$\mathfrak{G}^{0\,-}_{\mathfrak{M}} = \langle Mn;\, Col, Col^T, Col^{Ph}, Col^S, \prec, Bw, \perp, eq\rangle^{575}$$

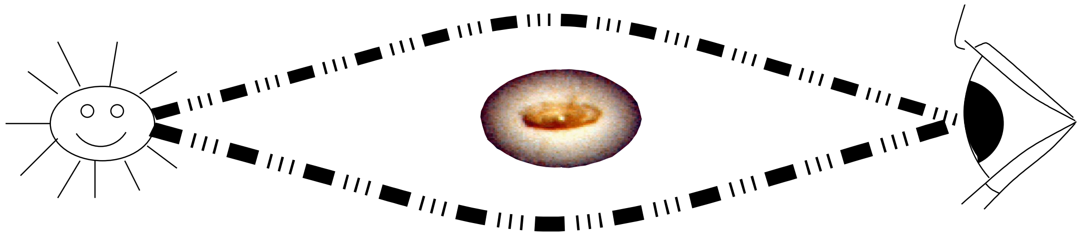


Figure 99: A massive object such as a galaxy, or even a black hole, can act as a giant lens. Light from a distant source (e.g. a quasar) is bent by the gravitational space warp surrounding the object. This effect can produce multiple images of a distant source.

where $Col(a,b,c) \stackrel{\text{def}}{\Longleftrightarrow} (\exists \ell \in L)\ a,b,c \in \ell$ for $a,b,c, \in Mn$. Further, $Col^T \subseteq Mn \times Mn \times Mn$ is T-collinearity, defined from L^T the natural way, i.e. $Col^T(a,b,c) \stackrel{\text{def}}{\Longleftrightarrow} (\exists \ell \in L^T)\, a,b,c \in \ell$, for $a,b,c \in Mn$.[576] Similarly for Col^{Ph} and Col^S. In the other direction, in $\mathfrak{G}^0_{\mathfrak{M}}{}^-$, L^T is a *defined* relation and not a basic symbol, similarly L is a *defined* sort. Further in $\mathfrak{G}^0_{\mathfrak{M}}{}^-$ $\perp$ is a relation between pairs of points, i.e. it is a 4-ary relation on Mn. Intuitively, $\langle a,b,c,d\rangle \in \perp$ iff the lines determined by $\langle a,b\rangle$ and $\langle c,d\rangle$ are $\perp_r$-orthogonal according to $\mathfrak{G}^0_{\mathfrak{M}}$. We emphasize that, as it was explained in §4.3, from the point of view of first-order logic there is no real difference between $\mathfrak{G}^0_{\mathfrak{M}}$ and $\mathfrak{G}^0_{\mathfrak{M}}{}^-$. More precisely, the difference between $\mathfrak{G}^0_{\mathfrak{M}}$ and $\mathfrak{G}^0_{\mathfrak{M}}{}^-$ is *the same* as that between a Boolean algebra

$$\mathfrak{B}_1 = \langle B; \vee, \wedge, -, 0, 1\rangle \quad \text{and} \quad \mathfrak{B}_2 = \langle B; \vee, -, 0\rangle.$$

$\triangleleft$

Let us include g into $\mathfrak{G}^0_{\mathfrak{M}}$ obtaining

$$\mathfrak{G}^0_{\mathfrak{M}}{}^g = \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp, eq, g\rangle.$$

Let us try to make $\mathfrak{G}^0_{\mathfrak{M}}{}^g$ one-sorted in the style of the above discussion. Then we obtain the following structure

$$(\mathfrak{G}^0_{\mathfrak{M}}{}^g)^- = \langle Mn, \mathbf{F_1};\ Col, Col^T, Col^{Ph}, Col^S, \prec, Bw, \perp, eq, g\rangle.$$

This leaves us with two problems listed in (i) and (ii) below.

(i) $(\mathfrak{G}^0_{\mathfrak{M}}{}^g)^-$ remains *many-sorted* because it has two sorts Mn and F.

(ii) We can replace L by Col only when we assume axiom **Det** on our model $\mathfrak{M}$ from which the geometry is obtained. In special relativity (i.e. in the present section) we are allowed to do this and this causes no loss of generality. However in general relativity this is not allowed (because axiom **Det** would kill essential features of the theory). Cf. Fig.99 on p.277.

[575] For completeness we note that $\mathfrak{G}^0_{\mathfrak{M}}{}^-$ is a legitimate structure even in the most classical and most purist version of first-order logic.

[576] Here, $Col, Col^T, Col^{Ph}, Col^S$ are defined from L, L^T, L^{Ph}, L^S, respectively, cf. footnote 571 on p.274.

We will return to the difficulty outlined in item (ii) at the end of this sub-section (p.278).

We will extend the above identification of ${\mathfrak{G}^0_{\mathfrak{M}}}^g$ with $({\mathfrak{G}^0_{\mathfrak{M}}}^g)^-$ to identifying $\mathfrak{G}_{\mathfrak{M}}$ with its variant

$$\mathfrak{G}^-_{\mathfrak{M}} := \langle Mn, \mathbf{F_1};\ Col, Col^T, Col^{Ph}, Col^S, \prec, Bw, \perp, eq, g, \mathcal{T}\rangle.$$

However, we will remain cautious with this identification in connection with generalizations towards general relativity because of item (ii) above.

We will return to the subject of identifying $\mathfrak{G}^0_{\mathfrak{M}}$, ${\mathfrak{G}^0_{\mathfrak{M}}}^g$, $\mathfrak{G}_{\mathfrak{M}}$ with ${\mathfrak{G}^0_{\mathfrak{M}}}^-$, $({\mathfrak{G}^0_{\mathfrak{M}}}^g)^-$, $\mathfrak{G}^-_{\mathfrak{M}}$, respectively, etc. in §4.6, but cf. also §4.3. These issues are studied in more detail in AMN [18, §4.6 (Interdefinability)].

By the above, we will consider our geometries e.g. $\mathfrak{G}_{\mathfrak{M}}$ as natural, definitional expansions of Tarski's geometries $\langle Mn;\ Col,$ "extra relations"$\rangle$. Our reason for doing so is that we would like to use the insights of Tarski's school in our framework.

In our theorems in the present section we used axiom **Det**. As we said, this restricts the class of all frame models to the smaller class

$$\mathsf{M}_{\mathrm{Det}} := \{\mathfrak{M} \in \mathsf{FM} \ :\ \mathfrak{G}_{\mathfrak{M}} \models \mathbf{Det}\}.$$

We note that $\mathsf{M}_{\mathrm{Det}}$ is axiomatizable in its original language too, this follows from Prop.4.3.18 (p.240) and Prop.4.3.41 on p.264. The investigations in Chapters 1–4 in this work do not change essentially if we restrict our attention to $\mathsf{M}_{\mathrm{Det}}$. E.g, the properties of the theories $Th \in \{$**Newbasax**, **Bax**, **Reich**(**Bax**), $\dots\}$ remain basically the same if instead of $\mathsf{Mod}(Th)$ we investigate $\mathsf{Mod}_{\mathrm{Det}}(Th) = \mathsf{M}_{\mathrm{Det}} \cap \mathsf{Mod}(Th)$.

Therefore, the geometrical counterpart of the theory developed in Chapters 1–4 of this work can be built up in the Tarskian one-sorted framework

$${\mathfrak{G}^0_{\mathfrak{M}}}^- = \langle Mn;\ Col, Col^T, Col^{Ph}, Col^S, \prec, Bw, \perp, eq\rangle;$$

or if g plays an important role then in the metric version $\mathfrak{G}^-_{\mathfrak{M}}$ of the Tarskian geometry ${\mathfrak{G}^0_{\mathfrak{M}}}^-$.[577]

However, *when* we *generalize* our approach to *general relativity* theory then it will be essential to use many-sorted geometries of the kind $\langle Mn, L;\ \in\rangle$ for the following reason. As we already said, we can add axiom **Det** to our presently discussed relativity theories like **Newbasax**, **Bax** etc. without changing the essential, characteristic properties of these theories. This will *not* be the case with general relativity cf. Fig.99 (p.277). (See also Figures 134, 83 on pages 365, 187.) Namely, in general relativity it is an essential feature for life-lines ℓ, ℓ' of inertial bodies that the number of intersections of ℓ and ℓ' can be arbitrarily large. That is, in general relativity, for every $n \in \omega$ it is possible to have $\ell \neq \ell'$ such that $|\ell \cap \ell'| > n$. Hence it is impossible to code lines[578] with n-tuples of points.[579] Therefore, the way Tarski represented (or coded) lines with pairs (or n-tuples) of points does not seem to work in general relativity. Therefore, it seems to be the case, that if, for general relativity, we want to carry through the programme represented by "the geometry of Tarski's school[580]", Suppes [242], and Goldblatt [102], then we will have to develop first-order logic of geometry in the many-sorted style $\langle Points, Lines;\ \in\rangle$ and not in the one-sorted style $\langle Points;\ Col\rangle$.

[577] where $\mathfrak{G}^-_{\mathfrak{M}}$ was introduced on p. 278 below item (ii) discussing $({\mathfrak{G}^0_{\mathfrak{M}}}^g)^-$.

[578] more precisely, life-lines of photons (cf. §4.7 entitled "geodesics" herein)

[579] Roughly speaking, adding axiom **Det** to general relativity would basically reduce general relativity to the level of special relativity, cf. Fig.99 (p.277). Hence we do not want to add axiom **Det** to general relativity.

[580] cf. e.g. [255, 252, 245, 232]

As we said before, we will discuss the interconnections between our basic relation (and function) symbols Col^T, Col^{Ph},..., eq, g (i.e. between the ingredients of $\mathfrak{G}_{\mathfrak{M}}$) in §4.6.

For more information, results and proofs on the subject matter of the present section we refer to §6.5 of AMN [18]. We omitted these items for lack of space.

4.5 Duality theory: connections between relativistic geometries ($\mathfrak{G}_{\mathfrak{M}}$) and models ($\mathfrak{M}$) of relativity

Assume we have two essentially different ways of thinking about the world. Assume we can establish some very strong interconnections between these two ways of thinking.[581] (Call this "duality theory" between the two ways.) Then such a system of interconnections (i.e. "duality theory") can be rather useful because then we can use these two ways of thinking combined, and ideas or reasonings formulated in one of these ways of thinking can be translated into the other. One could say that such a duality theory enables us to reason about the world by using the two ways of thinking simultaneously, achieving a kind of "stereo" effect.

Figure 100: A duality theory can be viewed as a bridge connecting two worlds of mathematics, permitting two-way traffic. The bridge idea is explained in great detail in Andréka et al. [31], cf. §II "Bridge ..." therein. Cf. also [30] and Mikulás [191, §1.3 ("Bridge between logics and algebras")].

A second, equally important, motivation for duality theories is the following. Duality theories often establish two-way "translations"

$$\mathsf{World}_1 \quad \underset{T_2}{\overset{T_1}{\rightleftarrows}} \quad \mathsf{World}_2$$

between two "worlds"[582] of mathematics such that problems formulated in World_1 are often easier to solve the following way: (i) translate "problem" into World_2, then solve T_1(problem) in World_2 and translate the result back along T_2 into World_1. With certain other problems

[581] Later we will refer to this interconnection, in a figurative way of speaking, as a bridge, cf. Fig.100.

[582] One world can be a branch, like Boolean algebras, while the other world can be another branch of mathematics, like topological spaces. On p.1103 of AMN [18] we see that these worlds can be arbitrarily far apart, e.g. one can be a part of analysis while the other a part of algebra (Laplace transformation).

(originating from World_2) the other direction might sometimes work better. See Fig.100. With this "pragmatic view" we do *not* mean to diminish the importance of the intellectual pleasure and scientific value of integrating World_1 and World_2 into a unified perspective, we only want to emphasize that this pragmatic, problem-solving-oriented motivation is there, too. An example is the

$$\text{"proof theory"} \;\overrightarrow{\longleftarrow}\; \text{"model theory"}$$

duality built on Gödel's completeness theorem: some proof-theoretic problems are easier to solve in the world of model theory, like proving $Th \not\vdash \varphi$ by constructing a model $\mathfrak{M} \in \mathsf{Mod}(Th)$ with $\mathfrak{M} \not\models \varphi$.

Before starting our particular application of this idea (i.e. that of duality theories) we note, that we list widely used examples of duality theories and motivation for duality theories scattered through section §6.6 (duality theory) of AMN [18]. The main part of these items remain in AMN [18, pp. 1078–1105], and here we recall only a few. For the rest the reader is kindly referred to AMN [18, pp. 1078–1105].

So much for duality theories in general. In the present section we will investigate certain concrete duality theories. More concretely, the subject matter of the present section concerns the connections between the "observation-oriented" models $\mathsf{Mod}(Th)$ and the "theoretically oriented" models $\mathsf{Ge}(Th)$.[583] The investigation of such connections has already been proposed by Reichenbach [218] and has been pursued to some extent in a model-theoretic spirit (similar to ours, in many respects) in Friedman [91, § VI.3 (p.236)] under the title "Theoretical Structure and Theoretical Unification".[584] ("Theoretical structure" in the title can be interpreted as referring to the structures in $\mathsf{Ge}(Th)$,[585] while "theoretical unification" can refer to a unified study of $\mathsf{Ge}(Th)$ and $\mathsf{Mod}(Th)$ and their interconnections [e.g. what we do in the present section].) Cf. the introduction to the present chapter, i.e. §4.1 (p.130).

Among other things, in this section we will complete the proof that our "observation-oriented" models $\mathsf{Mod}(Th)$ are definitionally equivalent to our relativistic geometries $\mathsf{Ge}(Th)$, assuming Th is strong enough. Formally,

$$\mathsf{Mod}(Th) \;\equiv_\Delta\; \mathsf{Ge}(Th),$$

under some assumptions, cf. Thm.4.3.38 (p.261). Besides this, we will also elaborate duality theories between the worlds $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$, cf. Fig.100 (p.280) and e.g. §§ 4.5.1, 4.5.3, A.2. We note that a duality theory between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ means a weaker connection than definitional equivalence. Hence, duality theories (between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$) are more general in the sense that they hold under milder assumptions on Th. (Actually $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$ implies isomorphism between the categories[586] $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ if we choose elementary embeddings as morphisms; which seems to be the strongest possible form of duality, cf. item (5) on p.256.)

[583] As the reader might expect at this point, this connection will appear in the form of a duality theory.

[584] In passing we note that the emphasis on model theory (in connection with studying relativity, of course), characteristic of the present work, is not without precursors, e.g. the relativity theory book Friedman [91] puts quite a bit of emphasis on using model theory in a spirit similar to ours. Cf. e.g. our reference to Friedman's $\mathcal{A}$ and $\mathcal{B}$ on p.130 herein.

[585] Or more "literally" as referring to a common expansion $\langle \mathfrak{M}, \mathfrak{G}_\mathfrak{M} \rangle$ of $\mathfrak{M}$ and $\mathfrak{G}_\mathfrak{M}$, but we are closer to the spirit of the connections between [91] and the present work if we interpret "theoretical structure" as $\mathfrak{G}_\mathfrak{M}$ or equivalently $\mathsf{Ge}(Th)$.

[586] Categories will be introduced later, cf. §A.2 (p.A-6).

The following convention is made only to have a nicer duality theory between the frame models and the observer independent geometries.

CONVENTION 4.5.1

(i) Throughout the present chapter ("Observer independent geometry") we postulate that the *empty model*[587] similar to our frame models is a frame model too (i.e. is a member of FM). Further we postulate that for any $\mathfrak{M} \in \mathsf{FM}$

$$Obs^{\mathfrak{M}} = \emptyset \quad \Rightarrow \quad (\mathfrak{M} \text{ is the empty model}).$$

In the present convention the definition of the class of frame models FM was modified. The definition of $\mathsf{Mod}(Th)$ is modified accordingly, for any set Th of formulas in our frame language.[588]

(ii) Deviating from the convention usually made in Algebra, in the present chapter, in accordance with item (i), algebraic structures with empty universes are allowed, e.g. $\langle \emptyset; +, \cdot \rangle$ with $+, \cdot$ binary operations on $\emptyset$ is a field.[589]

$\lhd$

Let us recall that by a *relativistic geometry* we understand an *isomorphic copy* of $\mathfrak{G}_{\mathfrak{M}}$, for some frame model $\mathfrak{M}$. Let us also recall that for any set Th of formulas in our frame language for relativity theory we defined

$$\mathsf{Ge}(Th) :\stackrel{\text{def}}{=} \{\mathfrak{G} \ : \ (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\ \mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}\}.$$

Let $\mathfrak{G} = \langle Mn, \ldots, L, \in, \ldots \rangle \in \mathsf{Ge}(\emptyset)$. Then we recall that we assumed that the relation $\in$ between Mn and L is the, real, set-theoretic membership relation, and that this does not cause loss of generality.

4.5.1 A duality theory between models and geometries (first part of the first version[590])

Given a relativistic geometry $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$ for a frame model $\mathfrak{M}$, it is a natural question to ask whether we can reconstruct $\mathfrak{M}$ (up to isomorphism) from $\mathfrak{G}$.[591] A possible answer to such a

[587] We call a model empty if all its sorts (i.e. universes) are empty.

[588] The case of the empty model can be handled by making appropriate conventions. (E.g. one defines which formulas are valid in the empty model.) Cf. e.g. Márkusz [183] or Burmeister [52]. To save space, here we do not go into this issue.

[589] The convention of allowing empty algebras and empty models comes from category theory cf. e.g. Adámek et al. [2, p.15, item 3.3(2)(e)]. Also the *motivation* for allowing such structures comes from category theoretic results; but cf. also the model theory book Hodges [130, §1.1, p.2] which does permit empty models, cf. Exercise 10 on p.11 (§1.2) in [130].

[590] By the first version we mean the $(\mathcal{M}, \mathcal{G})$-duality to be introduced soon while by the second version we mean the $(\mathcal{M}o, \mathcal{G}o)$-duality to be introduced in §4.5.4 much later.

[591] Here, the emphasis is on the case when $\mathfrak{G} \neq \mathfrak{G}_{\mathfrak{M}}$; cf. Remark 4.2.5 (p.149).

question consists of elaborating a duality theory[592] acting between the geometrical world $\mathsf{Ge}(\emptyset)$ and the world $\mathsf{Mod}(\emptyset)$ of our frame models. This consists of two functions

$$\mathcal{G} : \mathsf{Mod}(\emptyset) \longrightarrow \mathsf{Ge}(\emptyset) \quad \text{and} \quad \mathcal{M} : \mathsf{Ge}(\emptyset) \longrightarrow \mathsf{Mod}(\emptyset),^{593}$$

see Figure 101. We define $\mathcal{G}$ to be the function $\mathfrak{M} \mapsto \mathfrak{G}_{\mathfrak{M}}$ (specified in Def.4.2.3 way above).

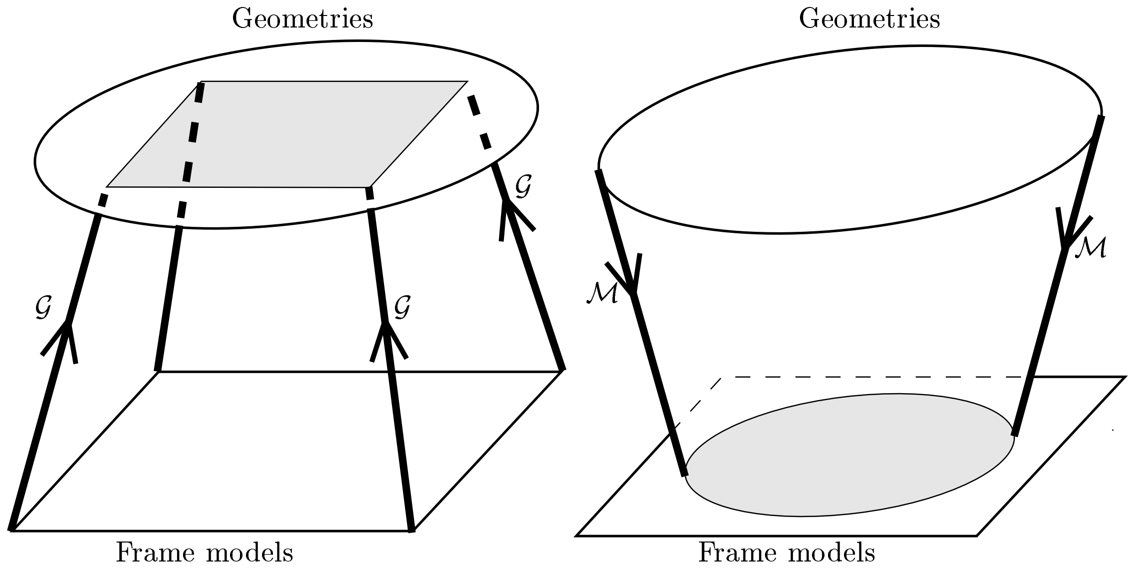


Figure 101: Connecting two worlds, namely, the world of frame models and the world of geometries.

The function $\mathcal{M}$ will be defined later, in §4.5.3. Sometimes we call $\mathcal{G}$ and $\mathcal{M}$ *functors* because (i) they connect classes of structures, (ii) they preserve certain connections between structures, e.g. isomorphisms and embeddability, and (iii) the corresponding "things" in Stone duality theory are called functors for category theoretic reasons. (Cf. item (II) in Remark 6.6.4 on p.1015 of AMN [18] for Stone duality.) Actually, $\mathcal{M}$ and $\mathcal{G}$ will become "real" functors in §A.2 way below.

CONVENTION 4.5.2 If f is a function and $H \subseteq Dom(f)$ then the notation "$f : H \longrightarrow K$" means that $f \restriction H : H \longrightarrow K$.

$\triangleleft$

In the spirit of the above convention $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$, for any set Th of formulas in our frame language.

Besides defining $\mathcal{M}$, a duality theory is supposed to prove some theorems stating that the functors $\mathcal{G}$ and $\mathcal{M}$ behave nicely in some sense. In order to prove such theorems we assume some axioms on our models $\mathfrak{M}$. Therefore the duality theory will be of the form:

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th) \quad \text{and} \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th),$$

[592] For duality theories cf. pp. 280–281, Remark 4.5.14 (pp. 293–296), pp. A-1–A-4, pp. A-11–A-12, pp. A-14–A-18.

[593] Despite of the fact that $\mathcal{G}$ and $\mathcal{M}$ are only proper classes of ordered pairs (as opposed to being a set of ordered pairs) we call them functions.

i.e.

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th)$$

and our theorems will be of the form (A)–(I) below, and they will be stated for certain choices of Th, see Figure 102. Motivation for discussing theorem schemas (A)–(I) can be found in §A.2 (p.A-6) and Remark 4.5.14 (p.293). For formulating items (A)–(I) we will need the following definition.

Definition 4.5.3 (Embeddability, weak submodel) Assume $\mathfrak{A}$ and $\mathfrak{B}$ are similar models.

(i) We say that $\mathfrak{A}$ is *embeddable* into $\mathfrak{B}$, in symbols $\mathfrak{A} \rightarrowtail \mathfrak{B}$ (or $\mathfrak{B} \leftarrowtail \mathfrak{A}$) iff there is an injective homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$. Cf. Convention 2.1.1 (p.2).

(ii) $\mathfrak{A}$ is a *weak submodel* of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B}$ iff $A \subseteq B$ and the identity function Id_A is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.[594] Hence weak submodels are always embeddable. Further, the definition for the many-sorted case is completely analogous. I.e. the existence of an *identical embedding* of say $\mathfrak{M}$ into $\mathfrak{N}$ is equivalent to $\mathfrak{M}$ being a weak submodel of $\mathfrak{N}$.[595]

$\lhd$

(A) Assume $\mathfrak{M} \in \mathsf{Mod}(Th)$. Then

$\mathfrak{M}$ is embeddable into $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$, i.e.

$$\mathfrak{M} \rightarrowtail (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}),$$

cf. Fig.107 (p.292) and Fig.102.

In duality theories similar to our $(\mathcal{G}, \mathcal{M})$-duality, in addition to item (A) it is sometimes required that the embedding (or morphism) "$\rightarrowtail$" occurring in (A) is the "shortest one" in some intuitive sense, cf. Fig.103 (p.287). This will be made precise in Definitions A.2.6 (p.A-11) and A.2.7 (p.A-13) in our category theoretic sub-section §A.2. An analogous remark applies to item (B) below.

(B) Assume $\mathfrak{G} \in \mathsf{Ge}(Th)$. Then

$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$ is embeddable into $\mathfrak{G}$, i.e.

$$\mathfrak{G} \leftarrowtail (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}),$$

cf. Fig.102.

---

[594] $\mathfrak{A}$ is a *strong submodel* of $\mathfrak{B}$ if every weak submodel $\mathfrak{C}$ of $\mathfrak{B}$ with the same universe as that of $\mathfrak{A}$ (i.e. with $C = A$) is a weak submodel of $\mathfrak{A}$, too. In other chapters of the present work we write simply *"submodel"* for "strong submodel". Further, the definition (of weak and strong submodels) for the many-sorted case is completely analogous with the above one. For more on the distinction between strong and weak submodels cf. e.g. [52] or [29]. We note that if $\mathfrak{A} \subseteq \mathfrak{B}$, i.e. if $\mathfrak{A}$ is a strong submodel of $\mathfrak{B}$ then $\mathfrak{A}$ is also a weak submodel of $\mathfrak{B}$, i.e. $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B}$. The other direction does not hold in general, i.e. $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B} \Leftarrow \mathfrak{A} \subseteq \mathfrak{B}$ but $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B} \not\Rightarrow \mathfrak{A} \subseteq \mathfrak{B}$.

[595] Using the notation $Uv(\mathfrak{M})$ on p.219 (§4.3), we could say that $\mathfrak{M}$ is a *weak submodel* of $\mathfrak{N}$ if the inclusion function of $Uv(\mathfrak{M})$ in $Uv(\mathfrak{N})$ is an embedding of $\mathfrak{M}$ in $\mathfrak{N}$.

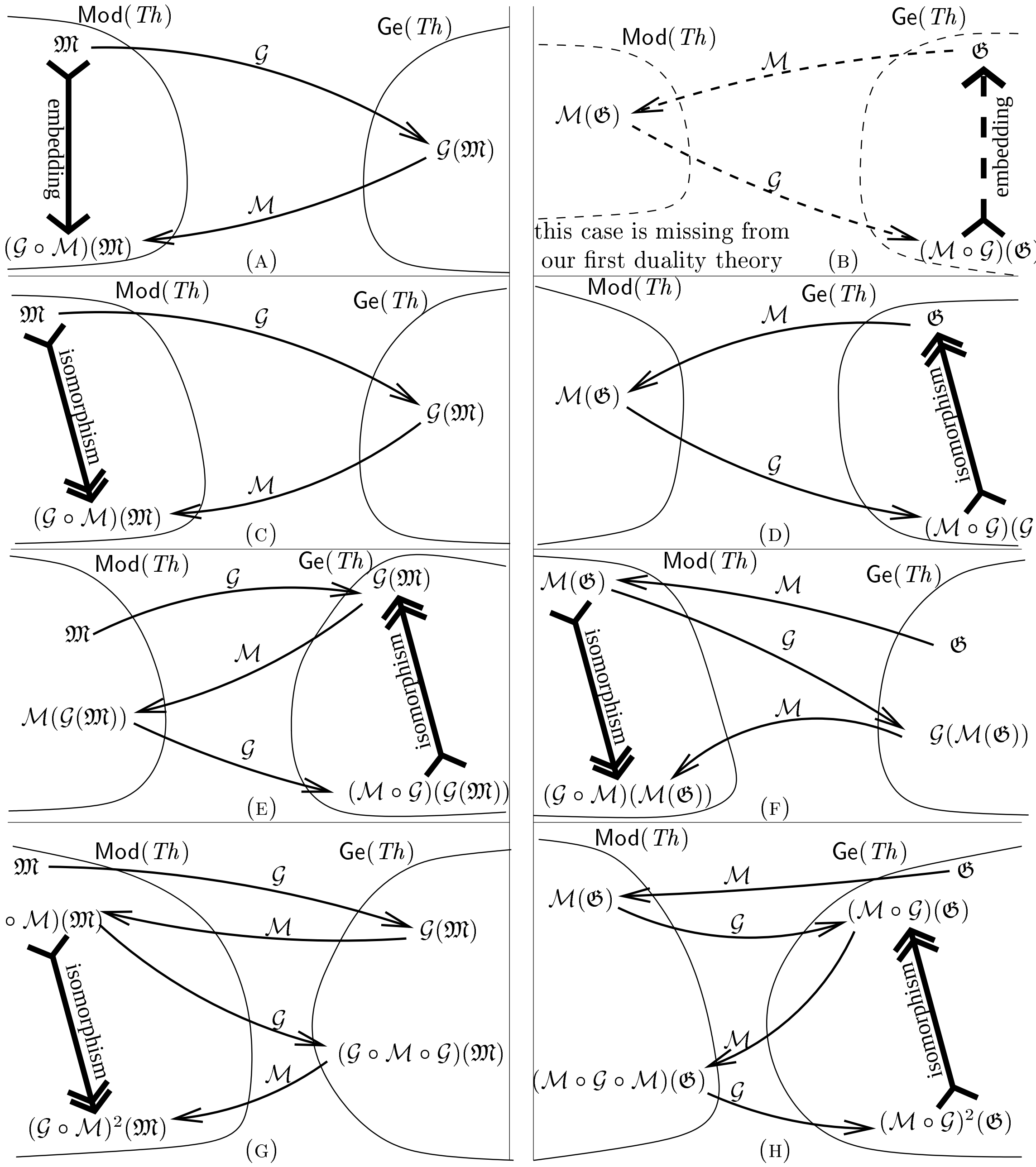


Figure 102: Illustration for theorem schemas (A)–(H) for duality theory.

We will have two kinds of dualities, one represented by $(\mathcal{M}, \mathcal{G})$ and the other represented by $(\mathcal{M}o, \mathcal{G}o)$. In the first case (i.e. in the case of $\mathcal{M}, \mathcal{G})$ the (B)-type theorems will become degenerate in that they will be of the form (D) below.[596]

(C) $\mathcal{G} \circ \mathcal{M}$ has a strong fixed-point property in the sense that for any $\mathfrak{M} \in \mathsf{Mod}(Th)$

$$(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M},^{597}$$

cf. the right-hand side of Fig.105 (p.291) and Fig.102.

(D) $\mathcal{M} \circ \mathcal{G}$ has a strong fixed-point property in the sense that for any $\mathfrak{G} \in \mathsf{Ge}(Th)$

$$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G},$$

cf. the left-hand side of Fig.105 (p.291) and Fig.102.

(E) The members of the range of $\mathcal{G}$ are fixed-points[598] of $\mathcal{M} \circ \mathcal{G}$, formally: For any $\mathfrak{M} \in \mathsf{Mod}(Th)$

$$(\mathcal{M} \circ \mathcal{G})(\mathcal{G}(\mathfrak{M})) \cong \mathcal{G}(\mathfrak{M}),^{599}$$

cf. Fig.102.

(F) The members of the range of $\mathcal{M}$ are fixed-points of $\mathcal{G} \circ \mathcal{M}$, formally: For any $\mathfrak{G} \in \mathsf{Ge}(Th)$

$$(\mathcal{G} \circ \mathcal{M})(\mathcal{M}(\mathfrak{G})) \cong \mathcal{M}(\mathfrak{G}),$$

cf. Fig.102.

For any function f, $f^2 :\stackrel{\text{def}}{=} f \circ f$.

(G) $\mathcal{G} \circ \mathcal{M}$ has a fixed-point property in the sense that for any $\mathfrak{M} \in \mathsf{Mod}(Th)$

$$(\mathcal{G} \circ \mathcal{M})^2(\mathfrak{M}) \cong (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}),$$

cf. the left-hand side of Fig.106 (p.292) and Fig.102.

---

[596] I.e. as a side-effect of our choice $\mathcal{M}$ and $\mathcal{G}$ we will have (B)⇒(D). This effect will disappear when we turn to $\mathcal{M}o$ and $\mathcal{G}o$ (i.e. to our second duality theory).

[597] i.e. $\mathfrak{M}$ is a fixed-point of $\mathcal{G} \circ \mathcal{M}$, up to isomorphism.

[598] i.e. $\mathcal{G}(\mathfrak{M})$ is a fixed-point up to isomorphism of $\mathcal{M} \circ \mathcal{G}$.

[599] This is the typical form of basic statements of Galois connections[600], e.g. $\mathsf{Th}(\mathsf{Mod}(\mathsf{Th}(\mathsf{K}))) = \mathsf{Th}(\mathsf{K})$, or in the case of Galois theory of field extensions $\Delta(H(\Delta(\mathsf{G}))) = \Delta(\mathsf{G})$, cf. items (I), (IV) of Remark 6.6.4 of AMN [18].

[600] The definition of Galois connection is in Def.A.1.2 (p.A-3) and motivation for Galois connection is in Remark A.1.1 (p.A-1).

$\mathcal{M} \circ \mathcal{G}$ has a fixed-point property in the sense that for any $\mathfrak{G} \in \mathsf{Ge}(Th)$

(H)
$$(\mathcal{M} \circ \mathcal{G})^2(\mathfrak{G}) \cong (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}),$$

cf. the right-hand side of Fig.106 (p.292) and Fig.102.

For any $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(Th)$ and $\mathfrak{G}, \mathfrak{H} \in \mathsf{Ge}(Th)$

(I)
$$\begin{array}{lcl} \mathfrak{M} \rightarrowtail \mathfrak{N} & \Rightarrow & \mathcal{G}(\mathfrak{M}) \rightarrowtail \mathcal{G}(\mathfrak{N}), \quad \text{and}^{601} \\ \mathfrak{G} \rightarrowtail \mathfrak{H} & \Rightarrow & \mathcal{M}(\mathfrak{G}) \rightarrowtail \mathcal{M}(\mathfrak{H}).^{602} \end{array}$$

We will refer to items <u>(A)–(I) above as theorem-schemas for</u> our <u>duality</u> theories.

Figure 103 intends to illustrate our $(\mathcal{M}, \mathcal{G})$-duality[603], theorem schemas (A), (B), and the idea of a shortest "$\rightarrowtail$" in the explanation below (A). The figure itself uses the terminology of category theory which will be explained in §A.2 (p.A-6). It also intends to serve as a complement for Fig.102.

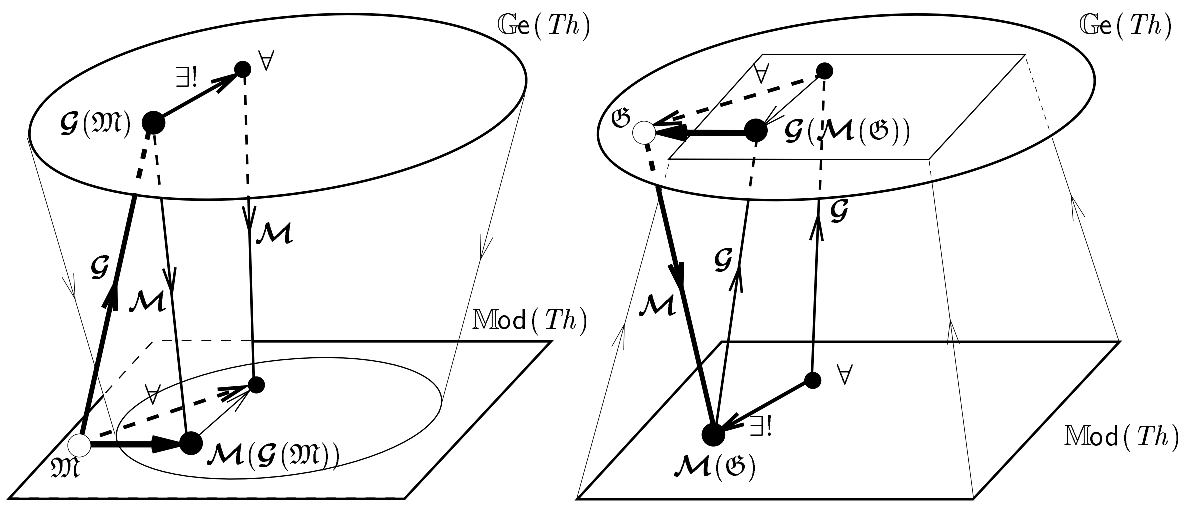


Figure 103: $(\mathcal{M}, \mathcal{G})$ is an adjoint pair of functors, under certain conditions. For the missing definitions (e.g. $\mathbb{M}\mathsf{od}(Th)$, $\mathbb{G}\mathsf{e}(Th)$) cf. §A.2 (p.A-6).

---

[601] On p.1015, p.1016 of AMN [18] we see that the functors like $\mathcal{M}$, $\mathcal{G}$ can be arrow reversing. This means that the $\mathcal{M}$ image of a pattern $\mathfrak{A} \rightarrowtail \mathfrak{B}$ is of the form $\mathcal{M}(\mathfrak{A}) \twoheadleftarrow \mathcal{M}(\mathfrak{B})$. For such arrow reversing dualities schema (I) obtains the form

$$\begin{array}{lcl} \mathfrak{M} \rightarrowtail \mathfrak{N} & \Rightarrow & \mathcal{G}(\mathfrak{M}) \twoheadleftarrow \mathcal{G}(\mathfrak{N}) \\ \mathfrak{M} \twoheadrightarrow \mathfrak{N} & \Rightarrow & \mathcal{G}(\mathfrak{M}) \leftarrowtail \mathcal{G}(\mathfrak{N}) \end{array}$$

etc.

[602] (I) implies that

$$\mathfrak{M} \rightarrowtail \mathfrak{N} \quad \Rightarrow \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \rightarrowtail (\mathcal{G} \circ \mathcal{M})(\mathfrak{N})$$

which corresponds to closure operators (induced by Galois connections) being order preserving cf. footnote 607 on p.288 and p.A-3 (§A.1).

[603] Sometimes we write $(\mathcal{G}, \mathcal{M})$-duality for $(\mathcal{M}, \mathcal{G})$-duality. They are the same thing.

We note that in the case of $(\mathcal{M},\mathcal{G})$, i.e. in our first duality theory,

$$\text{(C)} \;\Rightarrow\; \text{(D)} \;\Leftrightarrow\; \text{(E)} \;\Rightarrow\; \text{(F)} \;\Leftrightarrow\; \text{(G)} \;\Rightarrow\; \text{(H)}.^{604}$$

In the above schema, e.g. "(C) $\Rightarrow$ (D)" means that for each Th for which $(\mathcal{M},\mathcal{G})$ satisfies (C), $(\mathcal{M},\mathcal{G})$ also satisfies (D). (Similarly for the rest of the implications.)

Items (C) and (D) above imply

$$\mathsf{Mod}(Th) \equiv^{w}_{\Delta} \mathsf{Ge}(Th),$$

i.e. that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are weakly definitionally equivalent[605], for any Th in our frame language, assuming $\mathcal{M},\mathcal{G}$ are first-order definable meta-functions[606] with $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$ such that the isomorphisms mentioned in (C) and (D) can be chosen such that they are identity functions on the sort F.

Further, if Th is strong enough, then $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ turns out to be definitionally equivalent, in symbols

$$\mathsf{Mod}(Th) \equiv_{\Delta} \mathsf{Ge}(Th),$$

cf. Thm.4.3.38 (p.261). For the intuitive meaning and methodological importance of this cf. the text above Thm.4.3.38 on p.261.

If for Th items (A) and (G) above hold, then we will say that $\mathcal{G}\circ\mathcal{M}$ is a <u>*closure operator*</u>[607] on $\langle \mathsf{Mod}(Th), \subseteq_{\mathsf{w}}\rangle$ <u>*up to isomorphism*</u>[608] (and the values of $\mathcal{G}\circ\mathcal{M}$ are fixed-points up to isomorphism), assuming it preserves the partial order $\subseteq_{\mathsf{w}}$ up to isomorphism, cf. Fig.104. In this case, we call $(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$ the <u>*closure*</u> (or the $(\mathcal{G},\mathcal{M})$-closure) of $\mathfrak{M}$. Further, if for Th items (B) and (H) hold, then we will say that $\mathcal{M}\circ\mathcal{G}$ is a closure operator on $\langle \mathsf{Ge}(Th), {}_{\mathsf{w}}\!\supseteq\rangle$ up to isomorphism, assuming it preserves ${}_{\mathsf{w}}\!\supseteq$ up to isomorphism, where $\mathfrak{G}\,{}_{\mathsf{w}}\!\supseteq \mathfrak{H}$ iff $\mathfrak{H} \subseteq_{\mathsf{w}} \mathfrak{G}$. In such situations, sometimes, $(\mathcal{M}\circ\mathcal{G})(\mathfrak{G})$ is called the <u>*interior*</u>, which means <u>*dual-closure*</u>, of $\mathfrak{G}$.

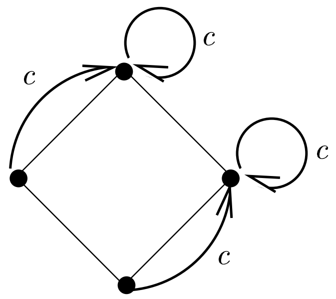


Figure 104: A possible closure operator.

Below we will start developing such a duality theory. For stating our first theorems (of schema (A)–(I)) we introduce two new axioms **Ax(Bw)**, **Ax(∞*ph*)** and the new axiom system **Pax$^+$**.

[604] This is so because, if $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ then, $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism.

[605] cf. Def.4.3.42 (p.264) for the notion of weak definitional equivalence

[606] in the sense of Def.4.3.39 (p.262)

[607] Let $\langle P,\leq\rangle$ be a partially ordered set (or class) and $f : P \longrightarrow P$. Then f is a <u>*closure operator*</u> on $\langle P,\leq\rangle$ iff for all $x,y\in P$, $x \leq f(x) = f^2(x)$ and $(\,x\leq y \;\Rightarrow\; f(x)\leq f(y)\,)$. (In passing we note that this notion admits a natural generalization to pre-ordered sets in place of partially ordered ones.)

[608] The up to isomorphism part is important, because what we know of $\mathfrak{M}\in Rng(\mathcal{G}\circ\mathcal{M})$ is that it is a fixed point of $\mathcal{G}\circ\mathcal{M}$ <u>*only*</u> up to isomorphism and for $\mathfrak{M}\in\mathsf{Mod}(Th)$ "$\mathfrak{M}\subseteq_{\mathsf{w}}(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$" holds only up to isomorphism.

Ax(Bw) $(\forall m,k\in Obs)[m \overset{\odot}{\rightarrow} k \;\Rightarrow\; (\mathsf{f}_{mk}$ is betweenness preserving) [609]].

Ax(∞ph) $(\forall m\in Obs)(\forall ph,ph'\in Ph)\Big([\,\bar{0}\in tr_m(ph)\cap tr_m(ph') \;\wedge\; ($ ph and ph' move in the same direction as seen by m $) \;\wedge\; v_m(ph)=\infty\,] \;\rightarrow\; v_m(ph')=\infty\Big)$.

Intuitively, no observer can emit simultaneously in the same direction two photons one with infinite speed and the other one with finite speed.

In connection with **Ax(Bw)** and **Ax(∞ph)** we state Propositions 4.5.4, 4.5.8 which will be needed later. Recall that **Pax** is weaker than **Bax**$^-$, cf. §3 and AMN [18, p.482 in §4.3]. The proposition below says that **Pax** + **Ax**($\sqrt{\ }$) implies **Ax(Bw)** and that if $n>2$, then **Bax**$^\oplus$ implies **Ax(Bw)**.

PROPOSITION 4.5.4

(i) **Pax** + **Ax**($\sqrt{\ }$) $\models$ **Ax(Bw)**.

(ii) *Assume $n>2$. Then* **Bax**$^\oplus$ $\models$ **Ax(Bw)**.

Proof: Item (i) follows from Thm.3.2.6 on p.110 saying that the word-view transformations are bijective collineations in all models of **Pax**, and from Lemma 3.1.6 on p.163 of AMN [18] saying that a line preserving bijection is an affine transformation composed by a field automorphism (cf. also footnote250 on p.119). Item (ii) follows from Thm.3.4.40 on p.241 of AMN [18] saying that **Bax** implies that $\mathsf{f}_{mk}=\widetilde{\varphi}\circ f$, for some $f\in Aftr$ and $\varphi\in Aut(\mathbf{F})$, from Thm.3.2.13 on p.118 which says that **Bax** does not allow FTL observers, and from Lemma 4.5.5 below. ∎

LEMMA 4.5.5 *Let $\mathfrak{F}=\langle \mathbf{F},\le\rangle$ be an ordered field. Let $\varphi\in Aut(\mathbf{F})$ be such that $(\forall x\in F)\ (|x|<1 \;\Rightarrow\; |\varphi(x)|<1)$.*

Then we have $\varphi\in Aut(\mathfrak{F})$, i.e. φ is order preserving.

We omit the **proof**. ∎

QUESTION 4.5.6 *Assume $n>2$. Does* **Bax**$^{-\oplus}$ $\models$ **Ax(Bw)** *hold?*

$\lhd$

Remark 4.5.7 Many of the theorems of the present work as well as of AMN [18] remain true if we replace the assumption **Ax**($\sqrt{\ }$) by the "weaker" **Ax(Bw)**. An example of such a theorem is Thm.3.2.13 saying that if $n>2$ then **Pax**$^\oplus$ + **Ax**($\sqrt{\ }$) excludes FTL observers. There are similar examples almost in every chapter. By replacing **Ax**($\sqrt{\ }$) with **Ax(Bw)**, usually we obtain theorems stronger than the original one, since usually **Pax** is assumed and then Prop.4.5.4(i) implies that the new theorem is stronger (or equivalent).

$\lhd$

PROPOSITION 4.5.8 **Bax**$^-$ $\models$ **Ax(∞ph)**.

We omit the easy **proof**. ∎

Definition 4.5.9 $\mathbf{Pax}^+ \overset{\text{def}}{:=} \mathbf{Pax} + \mathbf{AxE_{01}} + \mathbf{Ax(Bw)} + \mathbf{Ax}(\infty ph)\ +$
$\Big([\,\mathbf{Ax(eqtime)} \;\wedge\; (\forall m,k\in Obs)(\forall 0<i\in\omega)\ tr_m(k)\neq \bar{x}_i\,] \quad\vee\quad \mathbf{Ax(eqm)}\Big)$.[610]

$\lhd$

[609] This can be formalized as $(\forall p,q,r\in {}^nF)(\mathsf{Betw}(p,q,r) \;\Rightarrow\; \mathsf{Betw}(\mathsf{f}_{mk}(p),\mathsf{f}_{mk}(q),\mathsf{f}_{mk}(r)))$.

[610] Instead of **Ax(eqtime)** we could use the weaker axiom **Ax(eqtime)** $\vee$ **Ax(eqspace)**$^\oplus$. Then we would obtain a weaker axiom system **Pax**$^{+-}$. The theorems of the present sub-section (i.e. §4.5.1) remain true if we replace **Pax**$^+$ by **Pax**$^{+-}$ in them. For an even more general duality theory cf. Remark 4.5.51 (p.322).

If we replace $\mathbf{Ax(Bw)}$ by $\mathbf{Ax}(\sqrt{\ })$ in $\mathbf{Pax}^+$ then we get a stronger axiom system than $\mathbf{Pax}^+$(by Prop.4.5.4(i)). Note that $\mathbf{Pax}^+ \models \mathbf{Ax(eqtime)}$.

The theory $\mathbf{Pax}^+$ above is designed to be weak, just strong enough for defining the function $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$.[611] This is why $\mathbf{Pax}^+$ is so artificial. Our next proposition shows that in our definitions, and statements the assumption $\mathbf{Pax}^+$ can be replaced by more natural (but stronger) theories. In passing we note that $\mathbf{Pax}^+(2)$ allows $\mathbf{Basax} + \mathbf{Ax(symm)}$ models with FTL observers.

PROPOSITION 4.5.10 *Assume $n > 2$. Then (i)–(iii) below hold.*

(i) $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(Bw)} + \mathbf{Ax(eqtime)} \models \mathbf{Pax}^+$.

(ii) $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)} \models \mathbf{Pax}^+$.

(iii) $\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqtime)} \models \mathbf{Pax}^+$.

Proof: Assume $n > 2$. Then, by the proof of Thm.3.2.13 (p.118), $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(Bw)}$ excludes FTL observers. Further, $\mathbf{Bax}^- \models \mathbf{Ax}(\infty ph)$ by Prop.4.5.8. Therefore item (i) of the proposition holds. Item (ii) follows by (i) and by Prop.4.5.4(i). Item (iii) follows by Thm.3.4.19 (p.221) of AMN [18] and by Prop.4.5.4(ii) herein. ∎

Below we state a theorem corresponding to the theorem schemas (C) and (D) on p.286 way above. The theorem below implies that $\mathsf{Mod}(Th) \equiv^w_\Delta \mathsf{Ge}(Th)$, if we assume that Th satisfies $\mathbf{Ax(diswind)}$ and condition $(\star)$ in the theorem. We note that more than this is true, namely, Thm.4.3.38 says that $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$ under the same conditions.

THEOREM 4.5.11
There is a first-order definable meta-function $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ such that (i)–(iii) below hold, for any Th satisfying condition $(\star)$ way below. Moreover, this $\mathcal{M}$ is the natural meta-function defined in Def.4.5.38 (p.310).

(i) $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$ *(and of course $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$).*

(ii) *Both $\mathcal{M} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{M}$ have the strong fixed-point property in the sense that for any $\mathfrak{G} \in \mathsf{Ge}(Th)$ and $\mathfrak{M} \in \mathsf{Mod}(Th)$*

$$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G} \quad \textit{and} \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M},$$

moreover there is an isomorphism between $\mathfrak{G}$ and $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$ which is the identity map on F, and the analogous statement holds for $\mathfrak{M}$ and $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$, see Figure 105 and pictures (C), (D) *in Figure 102 (p.285).*

(iii) *Moreover, $\mathcal{G}$ and $\mathcal{M}$ are first-order definable meta-functions, assuming $Th \models$* $\mathbf{Ax(diswind)}$.

$(\star)$ *$n > 2$ and $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\|)^- + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ })$.*

Proof: The theorem is proved in the proof of Thm.4.5.43 (p.315) way below. ∎

The following theorem implies that the sentences in our frame language can be translated (in a meaning preserving way) into sentences in the language of our observer independent geometries and vice-versa, under some assumptions. Cf. the text above Thm.4.3.38, Remark 4.3.37, introduction of §4.2.2 and the text above Prop.4.3.45 (p.265). In connection with the following theorem we note that F is a common sort of $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$.

[611] That functor $\mathcal{M}$ will be defined later (beginning with p.308).

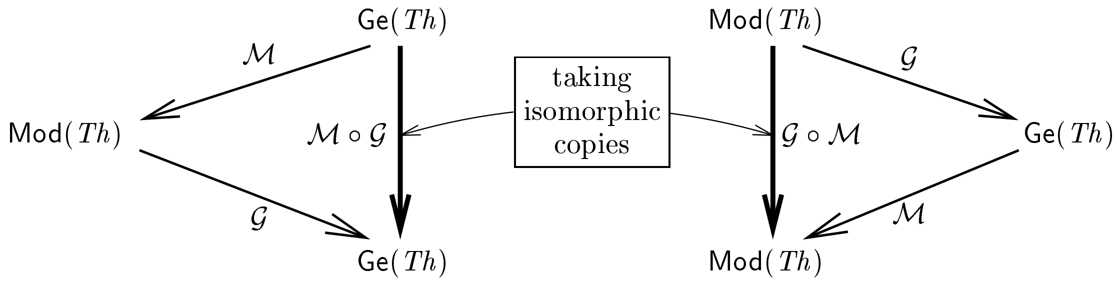


Figure 105: $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G}$ and $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M}$.

THEOREM 4.5.12 *Let $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ be a first-order definable meta-function such that for this choice of $\mathcal{M}$ the conclusions of Thm.4.5.11 above hold. Assume $n > 2$ and that Th is as in Thm.4.3.38 above. Then there are "natural" translation mappings*

$$T_{\mathcal{M}} : Fm(\mathsf{Mod}(Th)) \longrightarrow Fm(\mathsf{Ge}(Th)) \quad \textit{and} \quad T_{\mathcal{G}} : Fm(\mathsf{Ge}(Th)) \longrightarrow Fm(\mathsf{Mod}(Th))$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{Mod}(Th))$, $\psi(\bar{y}) \in Fm(\mathsf{Ge}(Th))$ with all their free variables belonging to sort F, $\mathfrak{M} \in \mathsf{Mod}(Th)$ and $\mathfrak{G} \in \mathsf{Ge}(Th)$, and evaluations $\bar{a}, \bar{b}$ of $\bar{x}, \bar{y}$, respectively (in F of course), (i)–(iv) below hold.[612]

(i) $\mathcal{M}(\mathfrak{G}) \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathfrak{G} \models T_{\mathcal{M}}(\varphi)[\bar{a}]$ *and* $\mathcal{G}(\mathfrak{M}) \models \psi[\bar{b}] \;\Leftrightarrow\; \mathfrak{M} \models T_{\mathcal{G}}(\psi)[\bar{b}]$.

(ii) $\mathfrak{M} \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathcal{G}(\mathfrak{M}) \models T_{\mathcal{M}}(\varphi)[\bar{a}]$ *and* $\mathfrak{G} \models \psi[\bar{b}] \;\Leftrightarrow\; \mathcal{M}(\mathfrak{G}) \models T_{\mathcal{G}}(\psi)[\bar{b}]$.

(iii) $\mathfrak{M} \models \varphi(\bar{x}) \leftrightarrow T_{\mathcal{G}}(T_{\mathcal{M}}(\varphi))(\bar{x})$ *and* $\mathfrak{G} \models \psi(\bar{y}) \leftrightarrow T_{\mathcal{M}}(T_{\mathcal{G}}(\psi))(\bar{y})$.

(iv) $\mathsf{Mod}(Th) \models \varphi \;\Leftrightarrow\; \mathsf{Ge}(Th) \models T_{\mathcal{M}}(\varphi)$ *and* $\mathsf{Ge}(Th) \models \psi \;\Leftrightarrow\; \mathsf{Mod}(Th) \models T_{\mathcal{G}}(\psi)$.

Proof: The theorem follows from Theorems 4.5.11 and by Prop.4.3.45 on p.265 (and by noticing that Thm.4.5.11 implies that $\mathsf{Mod}(Th) \equiv^w_\Delta \mathsf{Ge}(Th)$).

∎

Below we state a theorem corresponding to the theorem schemas (A), (C)–(H) on p.286 way above. In connection with the formulation of the next theorem we note that for any Th, $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ by the definition of $\mathcal{G}$. (Hence, in particular $\mathcal{G} : \mathsf{Mod}(\mathbf{Pax}^+) \longrightarrow \mathsf{Ge}(\mathbf{Pax}^+)$.)

THEOREM 4.5.13
There is a first-order definable meta-function $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ such that (i)–(iv) below hold. Moreover, this $\mathcal{M}$ is the natural meta-function defined in Def.4.5.38 (p.310).

(i) *The members of the range of $\mathcal{M}$ are fixed-points of $\mathcal{G} \circ \mathcal{M}$, formally: For any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$*

$$(\mathcal{G} \circ \mathcal{M})(\mathcal{M}(\mathfrak{G})) \cong \mathcal{M}(\mathfrak{G}),$$

see picture (F) in Figure 102 (p.285).

[612] We note that the formulas φ and $T_{\mathcal{M}}(\varphi)$ have the same free variables (therefore (i) below makes sense). Similarly for $T_{\mathcal{G}}$ etc.

(ii) *Both* $\mathcal{G}\circ\mathcal{M}$ *and* $\mathcal{M}\circ\mathcal{G}$ *have fixed-point property in the sense that for any* $\mathfrak{M}\in$ $\mathsf{Mod}(\mathbf{Pax}^+)$ *and* $\mathfrak{G}\in\mathsf{Ge}(\mathbf{Pax}^+)$

$$(\mathcal{G}\circ\mathcal{M})^2(\mathfrak{M})\cong(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})\quad \textit{and}\quad (\mathcal{M}\circ\mathcal{G})^2(\mathfrak{G})\cong(\mathcal{M}\circ\mathcal{G})(\mathfrak{G}),$$

see Figure 106 and pictures (G) *and* (H) *in Figure 102 (p.285).*

$Th = \mathbf{Pax}^+$

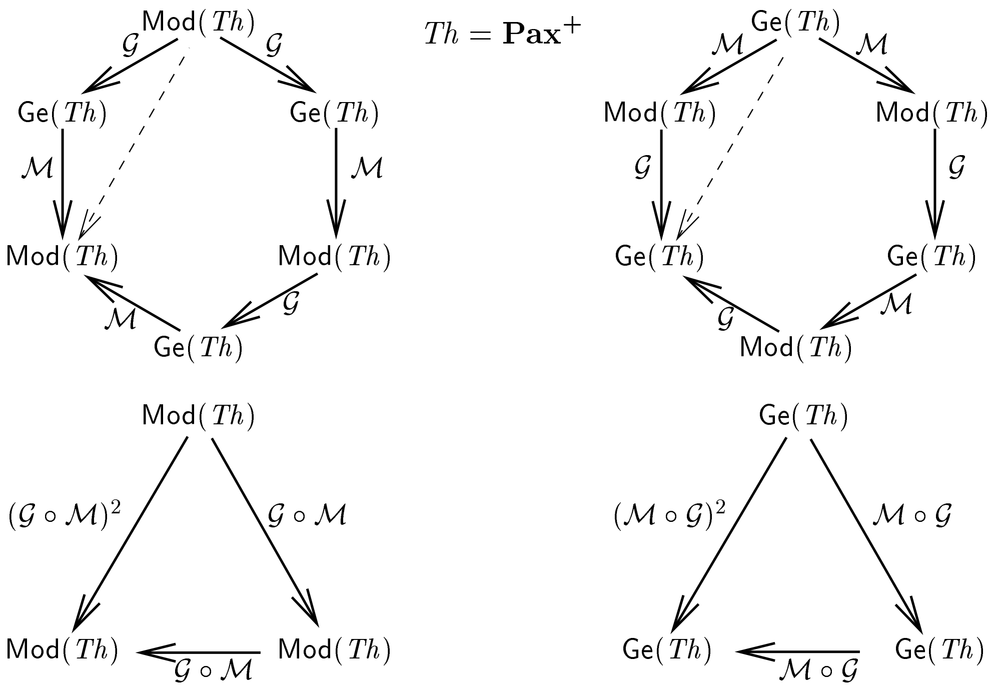


Figure 106: These diagrams commute up to isomorphism.

$Th = \mathbf{Pax}^+ + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit$

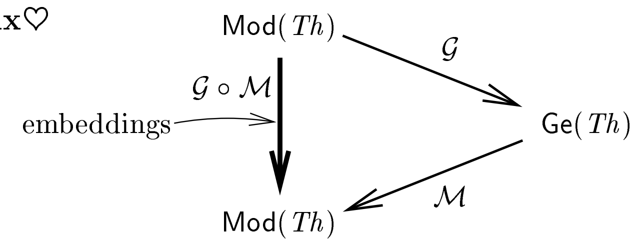


Figure 107: $\mathfrak{M}\rightarrowtail(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$

(iii) $\mathcal{M}:\mathsf{Ge}(\mathbf{Pax}^+ + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit)\longrightarrow\mathsf{Mod}(\mathbf{Pax}^+ + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit)$ *and* *for any* $\mathfrak{M}\in\mathsf{Mod}(\mathbf{Pax}^+ + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit)$ $\mathfrak{M}$ *is embeddable into* $(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$*, i.e.*

$$\mathfrak{M}\rightarrowtail(\mathcal{G}\circ\mathcal{M})(\mathfrak{M}),$$

see Figure 107 and picture (A) *in Figure 102 (p.285).*

(iv) $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)}) \longrightarrow \mathsf{Mod}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$ *(and of course* $\mathcal{G} : \mathsf{Mod}(\mathbf{Pax^+} + \mathbf{Ax(eqm)}) \longrightarrow \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$*), and*

$\mathcal{M} \circ \mathcal{G}$ has a strong fixed-point property in the sense that for any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$

$$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G},$$

(cf. the left-hand side of Fig.105 and picture (D) *in Fig.102).*

Further, the members of the range of $\mathcal{G}$ are fixed-points of $\mathcal{M} \circ \mathcal{G}$, formally: For any $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$

$$(\mathcal{M} \circ \mathcal{G})(\mathcal{G}(\mathfrak{M})) \cong \mathcal{G}(\mathfrak{M}),$$

cf. picture (E) *in Figure 102 (p.285).*

Proof: The theorem follows by Thm.4.5.43 (p.315) way below. ∎

Assume for $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax^+}) \longrightarrow \mathsf{Mod}(\mathbf{Pax^+})$ that the conclusions of Thm.4.5.13 hold and $\mathcal{M}$ is a first-order definable meta-function. Let

$$Th := \mathbf{Pax^+} + \mathbf{Ax(eqm)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit.$$

Then, by Thm.4.5.13, $\mathcal{G} \circ \mathcal{M}$ and $\mathcal{M} \circ \mathcal{G}$ are closure operators on $\langle \mathsf{Mod}(Th), \subseteq_{\mathsf{w}} \rangle$ and $\langle \mathsf{Ge}(Th), {}_{\mathsf{w}}\supseteq \rangle$ up to isomorphism, respectively (cf. p.288), assuming $\mathcal{G} \circ \mathcal{M}$ and $\mathcal{M} \circ \mathcal{G}$ preserve $\subseteq_{\mathsf{w}}$. Further, $\mathcal{M} \circ \mathcal{G}$ is the "identity operator" on $\mathsf{Ge}(Th)$ up to isomorphism, i.e. for any $\mathfrak{G} \in \mathsf{Ge}(Th)$, $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G}$. The analogous statement for $\mathcal{G} \circ \mathcal{M}$ does not hold in general, i.e. there is $\mathfrak{M} \in \mathsf{Mod}(Th)$ such that $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \ncong \mathfrak{M}$. This asymmetry is caused by our choice of $\mathcal{G}$, i.e. by the fact that $\mathcal{G}$ is surjective in the sense that $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism. We will have a duality theory for the $(g, \mathcal{T})$-free reduct of our geometries in §4.5.4 which will be more symmetric.

Further theorems in this line (duality theories, Galois connections etc.) will follow after we elaborate the definitions of e.g. the function $\mathcal{M}$. For that definition we will need some preparation e.g. coordinatization of our geometries summarized in §4.5.2 below.

Remark 4.5.14 (Duality theories, Galois theories, Galois connections all over mathematics, in analogy with the ones in the present work)
In connection with "theorem patterns" (A)–(I) above there is an analogy between our present functors $\mathcal{G}$ and $\mathcal{M}$ and the various Galois theories, duality theories, Galois connections in mathematics in the sense outlined in Remark 6.6.4 of AMN [18], cf. Def.A.1.2 (p.A-3) for Galois connections and Remark A.1.1 (p.A-1) for motivation for studying Galois connections.[613]

(I) As motivating examples[614], we recall from the literature e.g. the Galois theory of fields, that of cylindric and relation algebras, Stone duality and related duality theories in AMN [18, §§ 6.6.5–6.6.7, pp. 1078–1107]. The notion of a duality theory is presented in AMN [18] in such a way that the well known examples of *Laplace transform* and Fourier transform are shown there to be special cases of this notion.

[613]The reader not familiar with abstract algebra may safely skip this discussion of connections with Galois theory.

[614]for duality theories in general, and Galois connections in particular. (We regard category theoretic adjoint situations too as duality theories [cf. AMN [18] for explanation of this].)

In AMN [18], Stone duality is abbreviated by the notation

$$\mathsf{BA} \quad \underset{\mathcal{L}_{\mathsf{BA}}}{\overset{\mathcal{S}}{\rightleftarrows}} \quad \text{Boolean topological spaces,}$$

where $\mathcal{S}$ and $\mathcal{L}_{\mathsf{BA}}$ are two functors. We will use this notation below.

(II) *Connections of Stone duality with the (syntax, semantics)-duality in logic and in particular with parts of definability theory discussed in §4.3 (p.218):*

In connection with Fig.96 (p.263) and Fig.108 (p.295), for the interested reader, we note that the (syntax, semantics)-duality as discussed in this work is an organic part of algebraic logic. Therefore if the reader wants to learn more about this duality he can find more information in works usually classified as algebraic logic (or sometimes as its category-theoretic-oriented parts).

Notation: For any first-order theory Th,

$$Fm(Th) :\stackrel{\text{def}}{=} Fm(\mathsf{Mod}(Th)),$$

i.e. $Fm(Th)$ is the set of formulas of the language of the theory Th. In this definition we assume that the vocabulary of Th is somehow determined by Th. I.e. when specifying a theory one has to specify its vocabulary, too. (We often leave this to context).

Convention: In the present remark (explaining duality theories etc.), we treat interpretations in a somewhat simpler way/form than in the definability section §4.3. The difference is that in the duality item we consider only one-sorted theories. I.e. the objects of the category Theories in Fig.108 are one-sorted theories. In the definability section we concentrated on many-sorted theories. Hence there interpretations were understood between many-sorted theories which made them slightly more complicated objects than interpretations in the present part.[615] Cf. p.263, p.251, footnote 545 (p.251) herein and p.1023 footnote 1022 in AMN [18].

Here we sketch an analogy between two duality theories. One of them is Stone duality, while the other duality acts between the category of first-order theories (and translation mappings between them as morphisms) and the category of axiomatizable model classes (and first-order definable meta-functions i.e. interpretations between them). (For the latter duality see Fig.96 on p.263, while for the analogy with Stone duality cf. Fig.108.) In more detail, the category of Boolean algebras is put into analogy with the category of (first-order) theories (in the syntactical sense) whose objects are the $\langle Fm(Th), Th\rangle$ pairs. The morphisms of this category are the translation mappings or interpretations like Tr_f in Fig.96 on p.263 cf. also footnote 1022 on p.1023 of AMN [18], Prop.4.3.41 (p.264) and Theorems 4.3.27, 4.3.29 (pp. 245, 247). (These translation mappings are often called interpretations.[616]) E.g. we can consider $\langle Fm(Th_1), Th_1\rangle$, $\langle Fm(Th_2), Th_2\rangle$ as two BA's and any translation mapping Tr from Th_1 to Th_2 will be a homomorphism between these BA's. Let, now, K_1 and K_2 be two axiomatizable classes of models. Let $f : \mathsf{K}_2 \longrightarrow \mathsf{K}_1$ be a first-order definable meta-function[617]. Then f induces a

[615] E.g. in §4.3 an interpretation consisted of a function Tr together with something called "*code*" (cf. p.251). In the present item we do not need "*code*" but only Tr.

[616] The choice between using "translation mapping" or "interpretation" depends only on which aspects or which perspective/background we want to emphasize (and also on with which part of the literature we want to emphasize the connections cf. footnote 1022 on p.1023 of AMN [18].

[617] cf. p.262, Def.4.3.39

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\mathsf{BA}\\ \text{homomorphisms}\end{array} \arrow[r, shift left, "\mathcal{S}"] \arrow[d, leftrightarrow] & \begin{array}{c}\text{Boolean topological spaces}\\ \text{continuous functions}\end{array} \arrow[l, shift left, "\mathcal{L}_{\mathsf{BA}}"] \arrow[d, leftrightarrow] \\
\begin{array}{c}\text{Theories}\\ \text{translations}\\ \text{(interpretations)}\end{array} \arrow[r, shift left, "\text{semantics}", "\mathsf{Mod}"'] & \begin{array}{c}\text{Axiomatizable classes of models}\\ \text{first-order definable meta-functions}\\ \text{(interpretations)}\end{array} \arrow[l, shift left, "\mathsf{Th}"', "\text{syntax}"]
\end{tikzcd}
$$

Figure 108: The analogy between Stone duality and (syntax, semantics)-duality.

translation mapping $Tr_f : Fm(\mathsf{K}_1) \longrightarrow Fm(\mathsf{K}_2)$ satisfying the conclusion of Prop.4.3.41, p.264. Let us notice that $Fm(\mathsf{K}_i)$ are theories hence they correspond to BA's (of equivalence classes of formulas) and Tr_f turns out to be a BA-homomorphism. Further K_2, K_1 are Boolean topological spaces[618] and f is a continuous function. Now if we apply Stone duality to the BA-homomorphism Tr_f then we will obtain the continuous function f as its dual. See Fig.108. For more detailed exposition of the above see AMN [18, Rem.6.6.4, p.1014].

Summing it up, what we tried to say in the above discussion is that Stone duality is, basically, <u>the same thing</u> as the (key idea of) (syntax, semantics)-duality of <u>logic</u> which was used implicitly in §4.3 (and which, in particular, makes translation mappings between formulas go in the <u>opposite direction</u> as they go between the models).[619]

* * *

(III) <u>*Analogy with Galois theory of Cylindric algebras:*</u> Let us take the Galois theory of Cylindric algebras as an example, cf. Andréka-Comer-Németi [8, 9] and Comer [61]. Here, $\mathfrak{M}$

[618] if we collapse the elementarily equivalent models

[619] Stone duality concentrates on the category of BA's. Syntax-semantics duality concentrates on the category of theories. But a theory Th induces a Boolean algebra which we denote as $Fm(Th)/Th$. This gives us a connection ...etc.
Else: In passing we note that $Fm(Th)/Th$ is a slightly more complex object than a plain BA. Therefore (in logical (syntax, semantics)-duality) when forming the dual $\mathcal{S}(Fm(Th)/Th)$ of $Fm(Th)/\dots$ we do not take all prime ideals of $Fm(Th)/Th$, but only those ones, say P, whose complements $-P$ form <u>consistent</u> theories of our logic. (To this end we have to view $-P$ as a subset of $Fm(Th)$.) Equivalently, we could use the prime ideals of the subalgebra $Fmclosed(Th)/Th$, but we think that requiring $-P$ to be a consistent theory is more helpful in building good logical intuition.
For completeness: To make the connection with Stone duality even closer, we have the following extra option: We can stick with $Fm(Th)/Th$ on the BA side (using all prime ideals) <u>and</u> on the topology side use model-evaluation pairs $\langle\mathfrak{M}, \bar{a}\rangle$ as points of our topology "$\mathsf{Mod}(Th)$". In this setting the analogy with Stone duality is perfect. This train of thought when pushed to the extreme leads eventually to cylindric algebras (CA's) in place of BA's, and to represented CA's in place of represented BA's (which are nothing but Boolean spaces).

corresponds to a cylindric set algebra, say $\mathfrak{A}$, and $\mathcal{G}(\mathfrak{M})$ corresponds to the Galois group of $\mathfrak{A}$. Recall that a cylindric set algebra $\mathfrak{A}$ is an algebra whose elements are concrete relations on a base set, and the Galois group $\mathcal{G}(\mathfrak{A})$ of $\mathfrak{A}$ is the group of all permutations of this base set which leave all elements of $\mathfrak{A}$ fixed. If G is a group of permutations on a base set U, then $\mathcal{M}(\mathsf{G})$ consists of all those relations on U which remain fixed by all elements of G. Then $\mathcal{M}(\mathcal{G}(\mathfrak{M}))$ corresponds to the Galois closure $\mathfrak{A}^+$ of $\mathfrak{A}$, for which it is true that $\mathfrak{A}^{++} = \mathfrak{A}^+ \supseteq \mathfrak{A}$. The Galois closure $\mathfrak{A}^+$ of $\mathfrak{A}$ contains extra relations whose existence is kind of suggested by the relations in $\mathfrak{A}$. So in a sense, in analogy with this, $\mathcal{M}(\mathcal{G}(\mathfrak{M}))$ is a kind of "Galois closure" of the original model $\mathfrak{M}$ (which will contain extra observers whose existence is kind of suggested by the observers already existing in $\mathfrak{M}$). These ideas on cylindric and related algebras are implicitly used in Madarász [167], [170], [169], [177], [23], [20]. We note that the Galois theory of cylindric algebras is strongly analogous with the Galois theory of fields, cf. item (I) in AMN [18, p.1014].

(IV) Analogy with <u>*algebraic logic*</u> will be discussed in §A.3, p.A-17. Algebraic logic can be regarded as a very important duality theory (actually it is a system or collection of duality theories) as this is shown e.g. in Madarász [170], [165], [164], [166]. Connections with *Galois connections* and *adjoint functors* are discussed in §§ A.1, A.2 pp. A-1–A-17 herein, but cf. also §§ 6.6.5, 6.6.6 in AMN [18, pp. 1078–1105]. For further uses of Galois theories and duality theories (e.g. in connection with differential equations) cf. Janelidze [139, p.369]. For further duality theories in physics we refer to Varadarajan [268], but cf. also Lawvere-Schanuel [156, pp. 5–6, pp. 76–77]. Important additional information is in Remark A.1.1 ("Motivation for Galois connections") item (II) and footnote 1077 (p.1079) in AMN [18]. Duality theories involving C^*-algebras, and Laplace transform are on pp. 1098–1105 in AMN [18]. Further examples of duality theories (in and outside of physics) are given in AMN [18, pp. 1078–1080, pp. 1096–1105].

This concludes Remark 4.5.14 (Galois theories, Galois connections, duality theories all over mathematics, in analogy with the ones in the present work).

$\lhd$

Our next sub-section is on coordinatization. For applications of this kind of coordinatization in physics cf. e.g. Varadarajan [268].

4.5.2 Coordinatization of geometries by ordered fields

In the present sub-section our geometries, in most of the cases, are of the form $\langle Mn; Bw\rangle$, where Mn is the set of points and Bw is a ternary relation (of betweenness) on Mn. We do not assume that our geometries $\langle Mn; Bw\rangle$ are reducts of relativistic geometries. It is known from elementary geometry that if a geometry $\langle Mn; Bw\rangle$ satisfies certain axioms, then it can be coordinatized by an ordered field and this ordered field is unique up to isomorphism (cf. e.g. Hilbert [127] or Goldblatt [102] or Schwabhäuser-Szmielew-Tarski [232]). We will recall this coordinatization procedure from the literature (cf. [102, 127, 232]) in a slightly modified form. Before recalling the coordinatization we collect some axioms obtaining the axiom system

opag which[620] will be sufficient for the coordinatization[621] of $\langle Mn;\, Bw\rangle$ by an ordered field. The "geometrical theory" **opag** and the theory of ordered fields will turn out to be weakly definitionally equivalent, cf. Prop.4.5.26 (p.303).

Roughly speaking, **opag** is an axiomatization of affine geometry. Affine geometry has been thoroughly studied in the literature, and several axiomatizations for affine geometry are available in the literature, cf. Remark 6.7.17 on p.1148 of AMN [18]. (So **opag** is not particularly new, it has been put together to suit our purposes in the present work.)

Beside the geometry $\langle Mn;\, Bw\rangle$ we will also discuss the geometry $\langle Mn;\, coll\,\rangle$. In the case of "$\langle Mn;\, Bw\rangle$" *coll* is a defined relation, i.e. we use the abbreviation *coll* over $\langle Mn;\, Bw\rangle$ exactly as it was introduced in item 4.2.12 on p.159.

The new sort *lines* of $\langle Mn; coll\rangle$ as well as of $\langle Mn;\, Bw\rangle$ together with the incidence relation $\in\, \subseteq Mn\times lines$ are explicitly defined (in the sense of §4.3.2) as follows. (Recall that in the case of "$\langle Mn;\, Bw\rangle$" *coll* is a defined relation.) First we define

$$R := \{\, \langle a,b\rangle \;:\; (\exists c\in Mn)\, coll(a,b,c),\;\; a\neq b\,\}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1. Intuitively, the elements of U will code the elements of *lines*. We define a kind of incidence relation E' between Mn and U as follows. Let $e\in Mn$ and $\ell\in U$. Then

$$e\; E'\; \ell \quad \overset{\text{def}}{\Longleftrightarrow} \quad coll(pj_0(\ell), pj_1(\ell), e).$$

Then we define the equivalence relation $\equiv$ on U as follows. Let $\ell,\ell'\in U$. Then

$$\ell\equiv\ell' \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall e\in Mn)(e\; E'\; \ell\; \leftrightarrow\; e\; E'\; \ell').$$

We define the new sort $lines := U/{\equiv}$ together with $\in_{U,U/\equiv}\ \subseteq U\times U/{\equiv}$. Finally, the incidence relation $\mathrm{E}\subseteq Mn\times lines$ is defined as follows. Let $e\in Mn$ and $\ell\in lines$. Then

$$e\,\mathrm{E}\,\ell \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\exists\ell'\in\ell)\; e\; E'\; \ell'.$$

Since the axiom of extensionality holds for the incidence relation E we identify E with the real set theoretic membership relation $\in$. More precisely, without loss of generality we may assume that $lines\subseteq\mathcal{P}(Mn)$ and that E coincides with the set theoretic $\in$, so we will do this from now on.[622] This completes the explicit definition of the two sorted geometry $\langle Mn, lines;\, \in, coll\,\rangle$ over the one-sorted geometry $\langle Mn;\, coll\rangle$, and the explicit definition of the two sorted geometry $\langle Mn, lines;\, \in, Bw, coll\,\rangle$ over the one-sorted geometry $\langle Mn;\, Bw\rangle$. For the connection of *lines* with L of $\mathfrak{G}_{\mathfrak{M}}$ cf. Item 4.5.36 on p.308.

Next, we introduce axioms $\mathbf{A_0}$–$\mathbf{A_4}$, $\mathbf{P_1}$, $\mathbf{P_2}$, **Pa**. Though these axioms will be in the two-sorted language of $\langle Mn, lines;\, \in, coll\,\rangle$, by Thm.4.3.27 (p.245), they can be translated into the one-sorted languages of both $\langle Mn;\, coll\rangle$ and $\langle Mn;\, Bw\rangle$.

---

620 "**opag**" stands for ordered Pappian affine gometry

621 The coordinatizations (by Hilbert and others) of (synthetic) geometries mentioned above are related to the subject matter of the present section because observer m coordinatizes Mn by the world-view function w_m, i.e. $w_m : {}^nF \longrightarrow Mn$ is a coordinatization of Mn. In passing we note that the coordinatization methods of Hilbert, von Neumann, von Staudt (cf. in [10]), and others are applied in pure logic e.g. in Andréka-Givant-Németi [10, pp. 16–19]. (The reference to von Neumann can be found in [10].) Tarski's school call such coordinatization results representation theorems. The idea is that we represent an abstract axiomatic geometry as a concrete (analytic) geometry in the Cartesian spirit. Cf. Remark 6.6.87 (p.1106) of AMN [18].

622 For more detail on why and how we can do this (with "$\in$", E and *lines*) we refer to Appendix ("Why fist-order logic?") of AMN [18].

$\mathbf{A_0}$ $(\forall a, b, c \in Mn)[\, coll(a, b, c) \leftrightarrow (\exists \ell \in lines)\, a, b, c \in \ell\,]$.

Intuitively, a, b, c are collinear iff there is a line that contains a, b, c.

$\mathbf{A_1}$ $(\forall a, b \in Mn)(\, a \neq b \ \rightarrow\ (\exists! \ell \in lines)\, a, b \in \ell\,)$.

Informally, any two distinct points lie on exactly one line.[623]

Though axioms $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$ below are not first-order formulas in their present form, they can be easily reformulated in the first-order languages of both $\langle Mn; Bw\rangle$ and $\langle Mn; coll\rangle$. Throughout $n \geq 2$ is the dimension of our geometry. If $H \subseteq Mn$ then we will use the definition of $Plane'(H)$ exactly as it was introduced in Def.4.2.15(ii) (p.161). Intuitively, $Plane'(H)$ is the n-long closure of H under $coll$. Recall that the definition of $Plane'(H)$ is a first-order one over both structures $\langle Mn;\ coll, H\rangle$ and $\langle Mn;\ Bw, H\rangle$.

$\mathbf{A_2}$ Intuitively, if H is a less than $n+2$ element subset of Mn then the "n-long closure" $Plane'(H)$ of H under $coll$ will be closed under $coll$, hence the plane $Plane(H)$ generated by H coincides with $Plane'(H)$ (cf. Def.4.2.15, p.160), formally:

$$(\forall H \subseteq Mn)$$
$$\Big((\, |H| \leq n+1 \ \wedge\ a, b \in Plane'(H) \ \wedge\ coll(a, b, c)\,) \ \rightarrow\ c \in Plane'(H) \Big).$$

For introducing axioms $\mathbf{A_3}$ and $\mathbf{A_4}$ we need the following definition.

Definition 4.5.15 Consider a geometry $\langle Mn;\ Bw\rangle$.

(i) Let $H \subseteq Mn$. Then H is called *independent* iff $(\forall e \in H)\, e \notin Plane'(H \setminus \{e\})$.

(ii) Let $P \subseteq Mn$. Then P is called an *i-dimensional plane* iff there is an $i+1$ element independent subset H of Mn such that $Plane'(H) = P$.

$\lhd$

$\mathbf{A_3}$ Intuitively, if $i \leq n$ and H is an $i+1$ element independent subset of Mn then there is exactly one i-dimensional plane that contains H, formally:

$$(\forall H, H' \subseteq Mn)\Big((\, |H| = |H'| \leq n+1 \ \wedge\ \text{(both } H \text{ and } H' \text{ are independent)} \ \wedge\ H \subseteq Plane'(H')\,) \ \rightarrow\ Plane'(H) = Plane'(H') \Big).$$

$\mathbf{A_4}$ Mn is an n-dimensional plane.

Our next two axioms $\mathbf{P_1}$ and $\mathbf{P_2}$ concern "parallel lines". For these axioms we need the notion of parallelism.

Definition 4.5.16 Informally, two lines are parallel if they are in the same 2-dimensional plane, they do not meet or they coincide, formally: Let $\ell, \ell' \in lines$. Then ℓ and ℓ' are *parallel*, in symbols $\ell \parallel \ell'$, iff $(\exists a, b, c \in Mn)\ell, \ell' \subseteq Plane'(\{a, b, c\})$ and $(\ell \cap \ell' \neq \emptyset$ or $\ell = \ell')$.[624]

$\lhd$

[623] Cf. axiom AS1 in Golblatt [102, p.112] and axioms I_1 and I_2 in Hilbert [127, §2].

[624] If we apply these definitions (i.e. the def. of $lines$ and $\parallel$) to $\mathfrak{G}_{\mathfrak{M}}$ then (assuming $\mathbf{Pax} + \mathbf{Ax(diswind)}$):
(i) $lines$ and L are potentially different but $L \subseteq lines$, further
(ii) $\parallel$ and $\parallel_{\mathfrak{G}}$ are potentially different but $\parallel_{\mathfrak{G}}$ is the restriction of $\parallel$ to L. Cf. Item 4.5.36 on p.308.

$\mathbf{P_1}$ $(\forall \ell \in lines)(\forall a \in Mn)(\exists! \ell' \in lines)(a \in \ell' \ \wedge \ \ell \parallel \ell')$.

Informally, if we are given a line ℓ and a point a, then there is exactly one line ℓ' that passes through point a and is parallel to line ℓ.[625] This axiom is called Euclid's axiom in the literature.

$\mathbf{P_2}$ $(\ell \parallel \ell' \ \wedge \ \ell' \parallel \ell'') \ \rightarrow \ \ell \parallel \ell''$.

I.e. the relation of parallelism is transitive.[626]

Definition 4.5.17 (affine geometry)

(i) $\mathbf{ag} :\overset{\text{def}}{=} \{\mathbf{A_0}, \mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_4}, \mathbf{P_1}, \mathbf{P_2}\}$.

(ii) If $\langle Mn;\ coll\rangle \models \mathbf{ag}$ then we say that $\langle Mn;\ coll\rangle$ is an *affine geometry*.

$\lhd$

An algebraic structure $\mathbf{D} = \langle D; +, \cdot\rangle$ with binary operations $+$ (addition) and $\cdot$ (multiplication), is called a *division ring* iff 1–3 below hold.

1. $\langle D; +\rangle$ is an Abelian (i.e. commutative) group. We let 0 denote its neutral (i.e. identity) element.

2. $\langle D \setminus \{0\}; \cdot\rangle$ is a group.

3. The distributive laws

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad (y + z) \cdot x = y \cdot x + z \cdot x$$

hold for all $x, y, z \in D$.

We note that a division ring in which the multiplication is commutative ($x \cdot y = y \cdot x$) is a field.

Assume $\mathbf{D} = \langle D; +, \cdot\rangle$ is a division ring. Then the set of lines $\mathsf{Eucl}(n, \mathbf{D}) \subseteq \mathcal{P}({}^nD)$ of the "coordinate system ${}^nD$" is defined completely analogously to the case of fields on p.18. Further, $coll_{\mathbf{D}}$ is a ternary relation on ${}^nD$ defined as

$$coll_{\mathbf{D}} :\overset{\text{def}}{=} \{\, \langle p, q, r\rangle \in {}^nD \times {}^nD \times {}^nD \ : \ (\exists \ell \in \mathsf{Eucl}(n, \mathbf{D}))p, q, r \in \ell \,\}\,.$$

The following fact (known from geometry) says that a geometry is an affine one iff it can be coordinatized by a division ring.

FACT 4.5.18 *Assume $n > 2$. Then*

$$\langle Mn;\ coll\rangle \models \mathbf{ag}$$
$$\Updownarrow$$
$$(\textit{there is a division ring } \mathbf{D} = \langle D; +, \cdot\rangle \textit{ such that } \langle Mn;\ coll\rangle \cong \langle {}^nD;\ coll_{\mathbf{D}}\rangle).$$

[625] Cf. axiom AS3 in Goldblatt [102, p.113], and axiom IV in Hilbert [127, §7].
[626] Cf. axiom AS4 in Goldblatt [102, p.113].

On the proof: A proof can be recovered from Goldblatt [102, pp. 23-27, 71, 114] and Hilbert [127, §24]. Cf. also the proof of Fact 4.5.25 (p.302). ∎

Fact 4.5.18 above gives hints on how one can try to find relativistic models behind geometries. It also gives an idea for a possible generalization of our approach, namely, in our frame theory for relativity instead of requiring that $\mathfrak{F}$ is an ordered field we could require only that $\mathfrak{F}$ is an ordered division ring.

Our theorem below implies that the theory of division rings and the theory of affine geometries are weakly definitionally equivalent. Therefore, by Prop.4.3.45 (p.265), there are meaning preserving translation mappings between the two theories such that these translation mappings are inverses of each other in some sense. Cf. the discussion of weak definitional equivalence on pp. 263–265 for more intuition for the next theorem.

THEOREM 4.5.19 *Assume $n > 2$. Then*

$$\begin{aligned}
&\text{(the class of division rings)} \;\equiv^{w}_{\Delta}\; \{\langle Mn;\, coll\rangle : \langle Mn;\, coll\rangle \models \mathbf{ag}\}, \quad \textit{but}\\
&\text{(the class of division rings)} \;\not\equiv_{\Delta}\; \{\langle Mn;\, coll\rangle : \langle Mn;\, coll\rangle \models \mathbf{ag}\},
\end{aligned}$$

i.e. the theory of division rings and the theory of affine geometries (if $n > 2$) are weakly definitionally equivalent, but they are not definitionally equivalent.

On the proof: We omit the proof but cf. the proof of Thm.4.5.26. ∎

It is interesting that by the above theorem the "one-sorted" class of division rings is weakly definitionally equivalent with the geometries $\langle Mn, lines;\, \in, coll\rangle$ satisfying **ag**.

To make our division ring **D** in Fact 4.5.18 commutative (i.e. to make it a field) we introduce a new axiom **Pa** called Pappus-Pascal Property in the literature, cf. e.g. Hilbert [127] or Goldblatt [102, p.21]. In the axiom **Pa** we will use the following abbreviation.

Notation 4.5.20 Let $a, b, c, d \in Mn$. Then

$$\langle a, b\rangle \parallel \langle c, d\rangle$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$\Big(a \neq b \;\wedge\; c \neq d \;\wedge\; (\exists \ell, \ell' \in lines)(\ell \parallel \ell' \;\wedge\; a, b \in \ell \;\wedge\; c, d \in \ell')\Big).$$

◁

Pa $(\forall \ell, \ell' \in lines)(\forall a, b, c \in \ell \setminus \ell')(\forall a', b', c' \in \ell' \setminus \ell)$

$$[\,(\,\langle a, b'\rangle \parallel \langle a', b\rangle \;\wedge\; \langle a, c'\rangle \parallel \langle a', c\rangle\,) \;\rightarrow\; \langle b, c'\rangle \parallel \langle b', c\rangle\,],$$

see Figure 109.

Definition 4.5.21 (Pappian affine geometry)

(i) $\mathbf{pag} \overset{\text{def}}{=} \mathbf{ag} + \mathbf{Pa}$.

(ii) If $\langle Mn;\, coll\rangle \models \mathbf{pag}$ then we say that $\langle Mn;\, coll\rangle$ is a *Pappian affine geometry*.

◁

The following fact (known from geometry) says that a geometry is a Pappian affine one iff it can be coordinatized by a field.

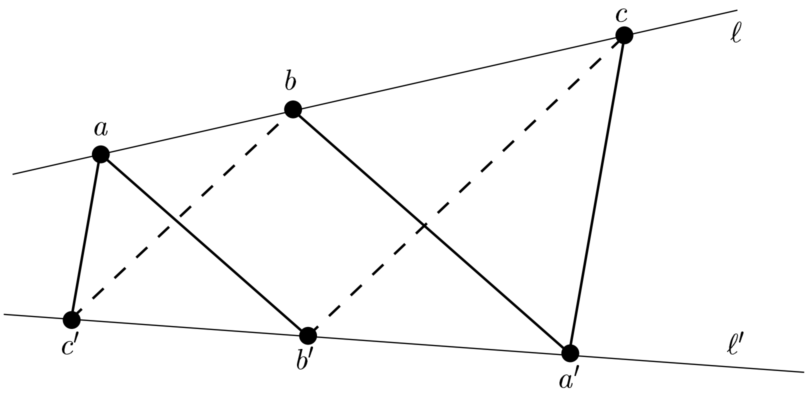


Figure 109: Pappus-Pascal Property.

FACT 4.5.22

$$\langle Mn;\ coll\rangle \models \mathbf{pag}$$
$$\Updownarrow$$
$$(\textit{there is a field } \mathbf{F} \textit{ such that } \ \langle Mn;\ coll\rangle \cong \langle {}^nF, coll_{\mathbf{F}}\rangle).$$

On the proof: A proof can be recovered from Goldblatt [102, pp. 23-27, 71, 114] and Hilbert [127, §24]. Cf. also the proof of Fact 4.5.25 (p.302). ∎

THEOREM 4.5.23

$$\text{(the class of fields)} \ \equiv^w_\Delta \ \{\langle Mn;\ coll\rangle : \langle Mn;\ coll\rangle \models \mathbf{pag}\}, \quad \textit{but}$$
$$\text{(the class of fields)} \ \not\equiv_\Delta \ \{\langle Mn;\ coll\rangle : \langle Mn;\ coll\rangle \models \mathbf{pag}\},$$

i.e. the theory of fields and the theory of Pappian affine geometries are weakly definitionally equivalent, but they are not definitionally equivalent.

On the proof: We omit the proof but cf. the proof of Thm.4.5.26. ∎

To make our field an ordered field in Fact 4.5.22 we need a few further axioms. These further axioms concern betweenness Bw, and they are in the language of $\langle Mn;\ Bw\rangle$. ($coll$ is a defined relation.)

$\mathbf{B_1}$ $Bw(a,b,c) \ \rightarrow \ (a \neq b \neq c \neq a \ \wedge \ Bw(c,b,a) \ \wedge \ \neg Bw(b,a,c))$.

Intuitively, if b lies between a and c then a,b,c are distinct points and b lies between c and a. Further, for any three points a,b,c at most one of them lies between the other two.[627]

$\mathbf{B_2}$ $a \neq b \ \rightarrow \ (\exists c) Bw(a,b,c)$.

Informally, for any two distinct points a,b there is at least one point c such that b lies between a and c.[628]

Axiom $\mathbf{B_3}$ below is called Pasch's Law in the literature.

[627] Cf. axioms B1, B3 in Goldblatt [102, pp. 70-71] and axioms II_1 and II_3 in Hilbert [127, §3].
[628] Cf. axiom B2 in Goldblatt [102, p.70] and axiom II_2 in Hilbert [127, §3].

$\mathbf{B_3}$ Intuitively, if a line ℓ lies in the plane determined by a triangle abc, and passes between a and b but not through c, then ℓ passes between a and c, or between b and c,[629] formally:

$(\neg coll(a,b,c) \ \wedge\ \ell \subseteq Plane'(\{a,b,c\}) \ \wedge\ (\exists d \in \ell) Bw(a,d,b)) \ \rightarrow$
$(\exists e \in \ell)(Bw(a,e,c) \ \vee\ Bw(b,e,c))$, see Figure 110.

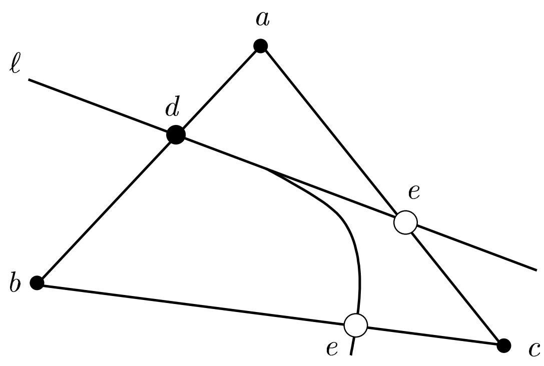


Figure 110: Pasch's Law.

So far it was clear what we meant when we wrote $\langle Mn; coll\rangle \models$ **pag**. Now, beside *coll* we want to use *Bw* too, and we want to write $\langle Mn; coll, Bw\rangle \models$ **pag** + (some new axioms [concerning *Bw*]). Since *coll* is *definable* from *Bw*, we will write $\langle Mn; Bw\rangle \models$"..." instead of $\langle Mn; coll, Bw\rangle \models$"...". We hope that the similarity between the expressions $\langle Mn; coll\rangle$ and $\langle Mn; Bw\rangle$ will create no confusion[630] (because context will help).

Definition 4.5.24 (ordered Pappian affine geometry)

(i) **opag** = **pag** + $\{\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}\}$.

(ii) If $\langle Mn; Bw\rangle \models$ **opag** then we say that $\langle Mn; Bw\rangle$ is an <u>*ordered Pappian affine geometry*</u>.

$\lhd$

The following fact (known from geometry) says that a geometry is an ordered Pappian affine one iff it can be coordinatized by an ordered field.

FACT 4.5.25

$$\langle Mn, Bw\rangle \models \mathbf{opag}$$
$$\Updownarrow$$
$$(\textit{there is an ordered field } \mathfrak{F} \textit{ such that } \langle Mn, Bw\rangle \cong \langle {}^nF, \mathsf{Betw}\rangle).$$

Proof: Proof of direction "$\Uparrow$" goes by checking the axioms, while direction "$\Downarrow$" follows from Prop.4.5.35 (p.308) way below. (Cf. also Def.4.5.34 on p.307). ∎

[629] Cf. axiom B4′ in Goldblatt [102, p.136] and axiom II_4 in Hilbert [127, §3].
[630] Cf. Convention 4.3.1 on p.220.

THEOREM 4.5.26

$$\begin{array}{lll}\text{(the class of ordered fields)} & \equiv^w_\Delta & \{\langle Mn;\, Bw\rangle : \langle Mn;\, Bw\rangle \models \mathbf{opag}\}, \quad \textit{but} \\ \text{(the class of ordered fields)} & \not\equiv_\Delta & \{\langle Mn;\, Bw\rangle : \langle Mn;\, Bw\rangle \models \mathbf{opag}\},\end{array}$$

i.e. the theory of ordered fields and the theory of ordered Pappian affine geometries are weakly definitionally equivalent, but they are not definitionally equivalent.

On the proof: A proof for the "$\equiv^w_\Delta$" part can be obtained by Def.4.5.28, Prop.4.5.29, Def.4.5.31, Prop.4.5.32 and Examples 4.3.16 (p.238).

A proof for the "$\not\equiv_\Delta$" part can be obtained by using item (6) on p.257 and Fact 4.5.25 as follows. It can be seen that $\langle {}^nF, \mathsf{Betw}\rangle$ has many non-trivial automorphisms for any ordered field $\mathfrak{F}$. (E.g. $x \mapsto 2x$ induces such an automorphism of $\langle {}^nF, \mathsf{Betw}\rangle$.) Thus any ordered Pappian affine geometry has many non-trivial automorphisms, in particular, the automorpism group has more than one element, by Fact 4.5.25. On the other hand, there are ordered fields with one-element automorphism groups (e.g. the ordered field $\mathfrak{R}$ of real numbers is such). Then (6) on p.257 implies that the class of ordered fields cannot be definitionally equivalent ($\equiv_\Delta$) with the class of ordered Pappian affine geometries. By this, the "$\not\equiv_\Delta$" part of our theorem is proved, too. ∎

Assume $\langle Mn;\, Bw\rangle \models \mathbf{opag}$. In Def 4.5.28 below, for every $o, e \in Mn$ with $o \neq e$ we will define an "ordered field" $\mathfrak{F}_{oe}$ corresponding to o, e. Prop.4.5.29 says that $\mathfrak{F}_{oe}$ is indeed an ordered field. In Prop.4.5.30 we will see that the ordered field $\mathfrak{F}_{oe}$ does not depend on the particular choice of o, e. Thus, there is a unique ordered field $\mathfrak{F}$ behind the geometry $\langle Mn;\, Bw\rangle$. In Def.4.5.31 we will define this ordered field $\mathfrak{F}$ explicitly over $\langle Mn;\, Bw\rangle$. Finally, in Def.4.5.34 we will define a coordinatization of the geometry $\langle Mn;\, Bw\rangle$ by $\mathfrak{F} = \langle F, \ldots\rangle$ which will be proved to be an isomorphism between $\langle Mn;\, Bw\rangle$ and $\langle {}^nF; \mathsf{Betw}\rangle$ as Prop.4.5.35.

Notation 4.5.27 Let $\langle Mn; Bw\rangle$ be a geometry, and $o, e \in Mn$. Then the <u>half-line</u> $[oe$ with origin o and containing e is defined as follows.

$$[oe \;\stackrel{\text{def}}{:=}\; \{\, a \in Mn \,:\, coll(o,e,a) \,\wedge\, \neg Bw(a,o,e)\,\}\,.^{631}$$

◁

Definition 4.5.28 (The ordered field $\mathfrak{F}_{oe}$)
Assume $\langle Mn;\, Bw\rangle \models \mathbf{opag}$. Let $o, e \in Mn$ with $o \neq e$. We define an "ordered field" $\mathfrak{F}_{oe}$ corresponding to o and e as follows. Our o and e represent 0 and 1, respectively. Let

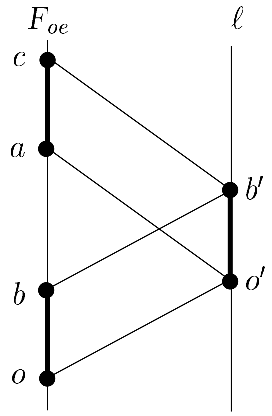
$$F_{oe} \stackrel{\text{def}}{:=} \{\, a \in Mn \,:\, coll(o,e,a)\,\}\,,$$

i.e. F_{oe} is the line determined by o and e. We will first define addition $+_{oe}$ as a ternary relation $+_{oe} \subseteq F_{oe} \times F_{oe} \times F_{oe}$ and later (in Prop.4.5.29) we will see that it is a function $+_{oe} : F_{oe} \times F_{oe} \longrightarrow F_{oe}$. We will define multiplication $\cdot_{oe} \subseteq F_{oe} \times F_{oe} \times F_{oe}$ in an analogous style. Further we will define "ordering" $\leq_{oe} \subseteq F_{oe} \times F_{oe}$.

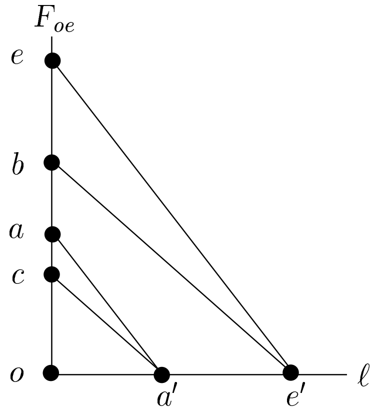
Let $a, b, c \in \ell$.

[631] We note that we have a slightly different notion of a half-line denoted as $\vec{\ell}_{oe}$ in §6.2.6, p.891 of AMN [18]. Our present notion "$[oe$" of a half-line is slightly different (it is tailored for the structure $\langle Mn;\, Bw\rangle$, while the one in AMN [18] is tailored for $\mathfrak{G}_{\mathfrak{M}}$), but the basic idea is the same.

$$+_{oe}(a,b,c)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists \ell \in lines)\Big(o \notin \ell \ \wedge\ \ell \parallel F_{oe} \ \wedge$$
$$(\exists o', b' \in \ell)(\, \langle o, o'\rangle \parallel \langle b, b'\rangle \ \wedge\ \langle o', a\rangle \parallel \langle b', c\rangle \,)\Big).$$



$$\cdot_{oe}(a,b,c)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists \ell \in L)\Big(\ell \cap F_{oe} = \{o\} \ \wedge$$
$$(\exists a', e' \in \ell)(\langle e, e'\rangle \parallel \langle a, a'\rangle \ \wedge\ \langle b, e'\rangle \parallel \langle c, a'\rangle)\Big).$$



$$(\forall d \in F_{oe})(o \leq_{oe} d \ \overset{\text{def}}{\Longleftrightarrow}\ d \in [oe\ ), \quad \text{and}$$
$$a \leq_{oe} b \ \overset{\text{def}}{\Longleftrightarrow}\ (\exists d \in F_{oe})\,(\, a + d = b \ \wedge\ o \leq_{oe} d\,).$$

We define the algebraic structure $\mathfrak{F}_{oe}$ as

$$\mathfrak{F}_{oe} :\overset{\text{def}}{=} \langle F_{oe};\ +_{oe}, \cdot_{oe}, \leq_{oe}\rangle.$$

$\mathfrak{F}_{oe}$ is an ordered field by Prop.4.5.29 below.

$\lhd$

PROPOSITION 4.5.29 *Assume* $\langle Mn;\ Bw\rangle \models$ **opag**. *Assume* $o, e \in Mn$ *with* $o \neq e$. *Then* $\mathfrak{F}_{oe}$ *is an ordered field.*

On the proof: A proof can be recovered from Goldblatt [102, pp. 23-27, 71, 114] and Hilbert [127, §24]. ∎

Item (i) of the following proposition says that the ordered field $\mathfrak{F}_{oe}$ does not depend on the particular choice of o and e. I.e. if we choose o, e differently we obtain an ordered field isomorphic to $\mathfrak{F}_{oe}$. In item (ii) we state that there is an isomorphism between the ordered fields $\mathfrak{F}_{oe}$ and $\mathfrak{F}_{o'e'}$ such that it is (uniformly) first-order definable over the structure $\langle Mn;\ Bw, o, e, o', e'\rangle$.

PROPOSITION 4.5.30 *Assume* $\langle Mn;\ Bw\rangle \models$ **opag**. *Assume* $o, e, o', e' \in Mn$ *are such that* $o \neq e$ *and* $o' \neq e'$. *Then (i)–(iii) below hold.*

(i) $\mathfrak{F}_{oe} \cong \mathfrak{F}_{o'e'}$.

(ii) *There is an isomorphism* $f^{o'e'}_{oe} : \mathfrak{F}_{oe} \longrightarrow \mathfrak{F}_{o'e'}$ *which is first-order definable over the structure* $\langle Mn;\, Bw, o, e, o', e'\rangle$ *and the first-order definition of this isomorphism* $f^{o'e'}_{oe}$ *does not depend on the particular choice of* o, e, o', e'; *i.e.*

(iii) *the definition of the relation* $f^{o'e'}_{oe}$ *is uniform over the class*

$$\{\, \langle Mn;\, Bw, o, e, o', e'\rangle \;:\; \langle Mn;\, Bw\rangle \models \mathbf{opag},\;\; o, e, o', e' \in Mn,\;\; o \neq e,\;\; o' \neq e' \,\}$$

of models; where we note that $f^{o'e'}_{oe} \subseteq Mn \times Mn$.

Outline of proof: Assume the assumptions. Let $f^{o'e'}_{oe} \subseteq F_{oe} \times F_{o'e'}$ be defined as follows. Let $\langle a, a'\rangle \in F_{oe} \times F_{o'e'}$. Before reading the formula below the reader is advised to consult Figure 111. Then

$$\langle a, a'\rangle \in f^{o'e'}_{oe}$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$\Big(\big[\,(\, o = o' \;\wedge\; \neg coll(o, e, e')\,) \;\rightarrow\; (\,(\text{I})\text{ below hold}\,)\big] \;\wedge$$
$$\big[\,(\, o = o' \;\wedge\; coll(o, e, e')\,) \;\rightarrow\; (\,(\text{II})\text{ below hold}\,)\big] \;\wedge$$
$$\big[\, o \neq o' \;\rightarrow\; (\,(\text{III})\text{ below hold}\,)\big]\Big),$$

see Figure 111.

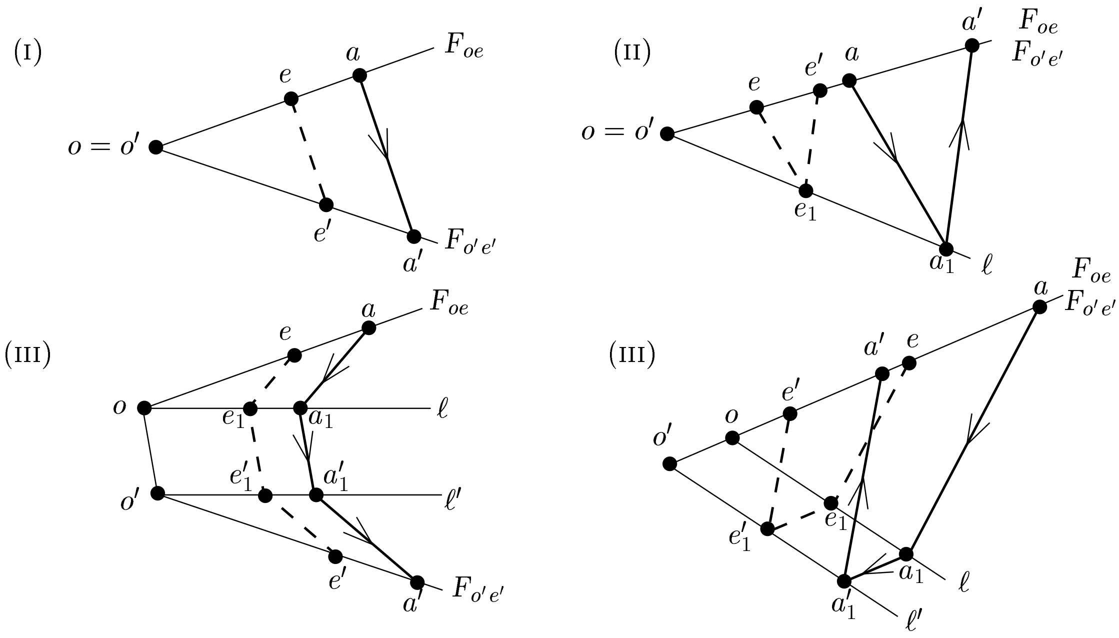


Figure 111: (I) is the easy case when $o = o'$ and o, e, e' are not collinear, (II) is somewhat more complicated because there o, e, e' are collinear, etc.

(I) $\langle e, e'\rangle \parallel \langle a, a'\rangle$.

(II) $(\exists \ell \in lines)(\exists e_1, a_1 \in \ell)(\ell \cap F_{oe} = \{o\} \;\wedge\; \langle e, e_1\rangle \parallel \langle a, a_1\rangle \;\wedge\; \langle e_1, e'\rangle \parallel \langle a_1, a'\rangle\,)$.

(III) $(\exists$ distinct $\ell, \ell' \in lines)(\exists e_1, a_1 \in \ell)(\exists e_1', a_1' \in \ell')$
$(\ell \cap F_{oe} = \{o\} \;\wedge\; \ell' \cap F_{o'e'} = \{o'\} \;\wedge\; \ell \parallel \ell' \;\wedge$
$\langle e, e_1\rangle \parallel \langle a, a_1\rangle \;\wedge\; \langle o, o'\rangle \parallel \langle e_1, e_1'\rangle \parallel \langle a_1, a_1'\rangle \;\wedge\; \langle e_1', e'\rangle \parallel \langle a_1', a'\rangle)$.

Then $f_{oe}^{o'e'}$ is an isomorphism between $\mathfrak{F}_{oe}$ and $\mathfrak{F}_{o'e'}$. A proof of this can be recovered from Goldblatt [102, pp. 23-27, 71, 114] and Hilbert [127, §24].

The present definition of the isomorphism $f_{oe}^{o'e'}$ is somewhat complicated. Probably we would obtain a less complicated definition for this isomorphism if we first defined it for the special cases (I)[632] and $\langle o, o'\rangle \parallel \langle e, e'\rangle \ \wedge \ \langle o, e\rangle \parallel \langle o', e'\rangle$, and then we would obtain an isomorphism for the general case as a composition of three isomorphisms defined for the special cases. ∎

Definition 4.5.31 (The ordered field $\mathfrak{F}$ corresponding to $\langle Mn;\, Bw\rangle$)
Assume $\langle Mn;\, Bw\rangle \models$ **opag**. We define the ordered field $\mathfrak{F}$ explicitly (in the sense of §4.3.2) over $\langle Mn;\, Bw\rangle$ as follows. First, we define the new relation

$$R := \left\{ \langle a, o, e\rangle \in {}^3F \ : \ o \neq e, \ a \in F_{oe} \right\}.$$

Then we define the new auxiliary sort U to be R together with the projection functions pj_0, pj_1, pj_2. Then we define the equivalence relation $\equiv$ on U as follows. Let $\langle a, o, e\rangle, \langle a', o', e'\rangle \in U$. Then

$$\langle a, o, e\rangle \equiv \langle a', o', e'\rangle \ \stackrel{\text{def}}{\Longleftrightarrow} \ \langle a, a'\rangle \in f_{oe}^{o'e'},$$

where $f_{oe}^{o'e'} : \mathfrak{F}_{oe} \longrightarrow \mathfrak{F}_{o'e'}$ is the isomorphism which was defined in (the proof of) Prop 4.5.30. Of course one uses pj_0, pj_1, pj_2 in the formal definition of $\equiv$. We define the sort F to be $U/\equiv$ together with $\in \subseteq U \times F$.[633] Now, we define $+, \cdot \ \subseteq {}^3F$ and $\leq \subseteq {}^2F$ as follows. Let $a, b, c \in F$. Then

$$\begin{gathered} +(a,b,c) \\ \stackrel{\text{def}}{\Longleftrightarrow} \\ (\exists a' \in a)(\exists b' \in b)(\exists c' \in c) \\ \Big(pj_1(a') = pj_1(b') = pj_1(c') \ \wedge \ pj_2(a') = pj_2(b') = pj_2(c') \ \wedge \\ pj_0(a') \ +_{pj_1(a')pj_2(a')} \ pj_0(b') \ = \ pj_0(c')\Big), \end{gathered}$$

$$\begin{gathered} \cdot(a,b,c) \\ \stackrel{\text{def}}{\Longleftrightarrow} \\ (\exists a' \in a)(\exists b' \in b)(\exists c' \in c) \\ \Big(pj_1(a') = pj_1(b') = pj_1(c') \ \wedge \ pj_2(a') = pj_2(b') = pj_2(c') \ \wedge \\ pj_0(a') \ \cdot_{pj_1(a')pj_2(a')} \ pj_0(b') \ = \ pj_0(c')\Big), \end{gathered}$$

$$\begin{gathered} a \leq b \\ \stackrel{\text{def}}{\Longleftrightarrow} \\ (\exists a' \in a)(\exists b' \in b)\Big(pj_1(a') = pj_1(b') \ \wedge \ pj_2(a') = pj_2(b') \ \wedge \\ pj_0(a') \ \leq_{pj_1(a')pj_2(a')} \ pj_0(b')\Big). \end{gathered}$$

Let

$$\mathfrak{F} :\stackrel{\text{def}}{=} \langle F;\, +, \cdot, \leq\rangle.$$

$\mathfrak{F}$ is first-order defined over $\langle Mn;\, Bw\rangle$. $\mathfrak{F}$ is an ordered field by Prop.4.5.32 below.

We will often use the elements of F in the form $\langle a, o, e\rangle/\equiv$ where $o \neq e$ and $a \in F_{oe}$.

◁

[632] i.e. for the case $o = o'$ and $\neg coll(o, e, e')$

[633] We use the notation pj and $\in$ in the style of §4.3.2. If someone wants to avoid this then he can use a notation like $+(\langle a_0, a_1, a_2\rangle/\equiv, \ldots, \langle c_0, c_1, c_2\rangle/\equiv) \ \stackrel{\text{def}}{\Longleftrightarrow} \ \exists a'[a' \equiv a$ etc.

PROPOSITION 4.5.32 *Assume* $\langle Mn; Bw\rangle \models$ **opag**. *Let* $\mathfrak{F} = \langle F, \ldots\rangle$ *be the "ordered field" corresponding to* $\langle Mn; Bw\rangle$ *defined in Def.4.5.31. Assume* $o, e \in Mn$. *Let* $\mathfrak{F}_{oe} = \langle F_{oe}; \ldots\rangle$ *be the ordered field corresponding to* o, e *defined in Def.4.5.28. Let* $f_{oe} : F_{oe} \longrightarrow F$ *be defined by* $a \mapsto \langle a, o, e\rangle/\equiv$.

Then f_{oe} *is an isomorphism between* $\mathfrak{F}_{oe}$ *and* $\mathfrak{F}$.

On the proof: The proposition can be proved by Prop.4.5.30. ∎

Assume $\langle Mn; Bw\rangle \models$ **opag**. We will use $n+1$ tuples $\langle o, e_0, e_1, \ldots, e_{n-1}\rangle$ where $\{o, e_0, \ldots, e_{n-1}\}$ is an $n+1$ element independent subset of Mn to identify potential coordinate systems. We will think of o as the origin and $e_0, \ldots, e_{n-1}$ as the unit vectors. We will define a coordinatization for such $n+1$ tuples in Def.4.5.34 below. In Def.4.5.34 we will use the following notation.

Notation 4.5.33 Assume $\langle Mn; Bw\rangle$ is a geometry. Let $a, b \in Mn$ and $H \subseteq Mn$. Then

$$\langle a, b\rangle \parallel Plane'(H) \ \stackrel{\text{def}}{\Longleftrightarrow}\ (\exists c, d \in Plane'(H))\, \langle a, b\rangle \parallel \langle c, d\rangle.$$

◁

Definition 4.5.34 (coordinatization)
Assume $\langle Mn; Bw\rangle \models$ **opag**. Recall that for every $o, e \in Mn$ with $o \neq e$ the ordered field $\mathfrak{F}_{oe} = \langle F_{oe}; \ldots\rangle$ was defined in Def.4.5.28. Let $\mathfrak{F} = \langle F; \ldots\rangle$ be the ordered field corresponding to $\langle Mn; Bw\rangle$ defined in Def.4.5.31. Let $\langle o, e_0, \ldots, e_{n-1}\rangle \in {}^{n+1}Mn$ be such that $\{o, e_0, e_1, \ldots, e_{n-1}\}$ is an $n+1$ element independent subset of Mn. We define the coordinatization

$$Co_{\langle o, e_0, \ldots, e_{n-1}\rangle} : Mn \longrightarrow {}^nF$$

as follows. Let $a \in Mn$. For every $i \in n$, let $a_i \in F_{oe_i}$ be such that if $a \notin F_{oe_i}$ then $\langle a, a_i\rangle \parallel Plane'(\{o, e_0, \ldots, e_{n-1}\} \setminus \{e_i\})$, otherwise $a_i = a$, see Figure 112. Such a_i's exist and

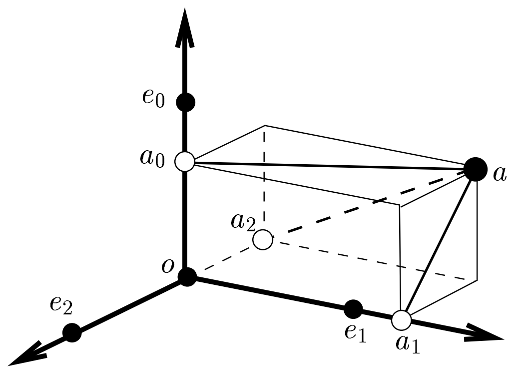


Figure 112:

are unique.

We define

$$Co_{\langle o, e_0, \ldots, e_{n-1}\rangle}(a) :\stackrel{\text{def}}{=} \langle f_{oe_0}(a_0), \ldots, f_{oe_{n-1}}(a_{n-1})\rangle,$$

where $f_{oe_0}, \ldots, f_{oe_{n-1}}$ are as defined in Prop.4.5.32 (p.307).

◁

PROPOSITION 4.5.35 *Assume* $\langle Mn;\ Bw\rangle \models$ **opag**. *Assume* $\langle o, e_0, \ldots, e_{n-1}\rangle \in {}^{n+1}Mn$ *is such that* $\{o, e_0, \ldots, e_{n-1}\}$ *is an* $n+1$ *element independent subset of* Mn. *Let* $\mathfrak{F} = \langle F; \ldots\rangle$ *be the ordered field corresponding to* $\langle Mn;\ Bw\rangle$ *defined in Def.4.5.31.*

Then $Co_{\langle o,e_0,\ldots,e_{n-1}\rangle}$ *is an isomorphism between* $\langle Mn;\ Bw\rangle$ *and* $\langle {}^nF, \mathsf{Betw}\rangle$.

On the proof: A proof can be recovered from Goldblatt [102, pp. 23-27, 71, 114] and Hilbert [127, §24]. ∎

Item 4.5.36 (Summary of some notation)
Let us return to $\mathsf{Ge}(\mathbf{Pax})$. Our definitions of *lines*, $\|$ make sense for the geometries in $\mathsf{Ge}(\mathbf{Pax})$, too. Now, we have strongly related triples of notions L, Col, $\|_{\mathfrak{G}}$ and *lines*, *coll*, $\|$. The differences between these two are rather <u>small</u>. The reason for the differences is that by the construction of $\mathfrak{G}_{\mathfrak{M}}$ some lines (in the sense of *coll*) may be missing from L (in some sense).[634] Assume $\mathbf{Pax} + \mathbf{Ax(diswind)}$. (Recall that L, Col, and $\|_{\mathfrak{G}}$ belong together, while *lines*, *coll* and $\|$ belong together.) Now, $L \subseteq lines$, $Col \subseteq coll$ and $\|_{\mathfrak{G}} \subseteq \|$. Further Col and $\|_{\mathfrak{G}}$ are the natural restrictions (of *coll* and $\|$) to the "world of L". If we assume $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ })$ in addition then L, Col, $\|_{\mathfrak{G}}$ coincide, respectively, with *lines*, *coll*, $\|$.

◁

4.5.3 Recovering frame models from geometries: defining the functor $\mathcal{M}$. (Continuation of duality theory.)

Let us recall from p.293 that our purpose with §4.5.2 was to prepare ourselves to the definition of our functor $\mathcal{M}$.

In Def.4.5.38 below we define the functor $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^{+}) \longrightarrow \mathsf{Mod}(\emptyset)$. In this definition we will use facts and propositions stated in §4.5.2 for ordered Pappian affine geometries (i.e. for **opag**) and notation introduced in §4.5.2. Therefore we include Prop.4.5.37 below. Intuitively, the proposition says that the windows of ($\mathbf{Pax}$+$\mathbf{Ax(Bw)}$)-geometries are ordered Pappian affine geometries.

PROPOSITION 4.5.37 *Assume* $\mathfrak{G} = \langle Mn, \ldots\rangle \in \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. *Assume* $o \in Mn$. *Let* Mn_o *be the "window of o", i.e.* $Mn_o \stackrel{\text{def}}{:=} \{\, e \in Mn \ :\ e \sim o \,\}$.[635] *Then*

$$\langle Mn_o;\ Bw \restriction Mn_o\rangle \ \models\ \mathbf{opag}.$$

Outline of proof: Let $\mathfrak{G} = \langle Mn, \ldots\rangle \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. Then $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}} = \langle Mn_{\mathfrak{M}}, \ldots\rangle$ for some $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. Let this $\mathfrak{M}$ be fixed. Let $o \in Mn_{\mathfrak{M}}$ and $(Mn_{\mathfrak{M}})_o = \{\, e \in Mn_{\mathfrak{M}} \ :\ o \sim e \,\}$. To prove the proposition it is enough to prove

$$(*) \qquad \langle (Mn_{\mathfrak{M}})_o;\ Bw_{\mathfrak{M}} \restriction (Mn_{\mathfrak{M}})_o\rangle \ \models\ \mathbf{opag}.$$

[634] The reason for this is that L was obtained from coordinate axes (and traces of photons) only. If we had defined L such that a set of events is in L if some inertial observer thinks that it is a Euclidean line then we would have obtained all of *lines* as elements of L. In other words L corresponds to inertial coordinate axes (and traces of photons), while *lines* corresponds to Euclidean lines. I.e. $\ell \in L$ if some inertial m thinks it is a coordinate axis (or is a trace of a photon), while $\ell \in lines$ if some inertial m thinks it is a Euclidean line.

[635] Recall that $\sim$ is a binary relation of connectedness on Mn defined in Def.4.2.12 (p.159).

Let $m \in Obs$ be such that $o \in Rng(w_m)$. Then by Thm.3.2.6 (p.110) of AMN [18], w_m is an isomorphism between $\langle {}^nF; \mathsf{Betw}\rangle$ and $\langle (Mn_{\mathfrak{M}})_o; Bw_{\mathfrak{M}} \restriction (Mn_{\mathfrak{M}})_o\rangle$. Cf. Prop.4.2.64 (p.208). But then $(*)$ above holds. ∎

Intuitive idea for the definition of the functor $\mathcal{M}$: geometries $\longrightarrow$ frame models.

Assume we are given a geometry $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$. We want to define (by using first-order logic only) an observational model $\mathcal{M}(\mathfrak{G})$ over this geometry $\mathfrak{G}$. Moreover, we would like to choose $\mathcal{M}(\mathfrak{G})$ such that its geometry $\mathcal{G}(\mathcal{M}(\mathfrak{G}))$ should be as close to the original $\mathfrak{G}$ as possible (cf. potential theorem schemas (A)–(I) for duality on pp. 284–287). (In a sense, one could say, that using the functor $\mathcal{M}$ we would like to recover from $\mathbf{I}\mathfrak{G}$ [but using only the "legitimate geometrical" structure of $\mathfrak{G}$] that "long forgotten" observational model whose geometric counterpart $\mathfrak{G}$ is.) Cf. here the relevant motivational parts of the introduction (pp. 129-778) to the present chapter. What do we need in order to find an observational $\mathfrak{M}$ inside our geometry $\mathfrak{G}$? Surely we need to find a field $\mathfrak{F}^{\mathfrak{M}}$ in $\mathfrak{G}$, but that is no problem as we saw in §4.5.2 ("Coordinatization …"). This is a good start, but what else do we need to find in $\mathfrak{G}$? Certainly we will need to find observers in $\mathfrak{G}$. But what is an observer? We can identify an observer m with his coordinatization[636] $w_m : {}^nF \longrightarrow Mn$ of (a part of Mn). What is w_m? It is a coordinatization of (a part of) Mn [637] by ${}^nF$. For simplicity, in this intuitive remark we fix $n = 3$. We can represent such a coordinatization $w_m : {}^nF \longrightarrow Mn$, by a choice of w_m's origin $o \in Mn$ and by w_m's three unit-vectors 1_t, 1_x, 1_y. More precisely, we are thinking of the w_m-images of the origin, and of the unit-vectors as they appear in Mn. Let us notice that in geometry, i.e. in Mn, vectors are easily represented by pairs of points. Actually, $w_m(\bar{0}), w_m(1_t), w_m(1_x), w_m(1_y)$ are nothing but 4 elements $o, e_t, e_x, e_y \in Mn$ of our geometry satisfying some conditions.[638] So the idea naturally comes to one's mind to try to represent (or code or define) observers as four-tuples $\langle o, \dots, e_y\rangle$ of points (in Mn) satisfying certain conditions.

To make this idea work, we still have to figure out how to reconstruct the whole of the coordinatization w_m from the origin o and the unit vectors $e_t, \dots, e_y$, but having access to the whole geometry $\mathfrak{G}_{\mathfrak{M}}$, one can believe that, one way or another, at least some w_m can be reconstructed from $\langle o, \dots, e_y\rangle$. So, our plan is to code (or represent[639]) observers (found in $\mathfrak{G}$) by tuples $\langle o, \dots, e_y\rangle \in {}^4Mn$ satisfying some conditions. It is natural to identify photons with photon-like lines i.e. elements of L^{Ph}. It is also natural to choose $B = Ib = Obs \cup Ph$. At this point we already have a grasp on what the $\mathfrak{F}^{\mathfrak{M}}$, $B^{\mathfrak{M}}$, $Obs^{\mathfrak{M}}$, $Ph^{\mathfrak{M}}$ parts of our model $\mathcal{M}(\mathfrak{G}) = \mathfrak{M} = \langle B, \dots, \mathfrak{F}, G, \in, W\rangle$ will be. It is, again, natural to choose $G = \mathsf{Eucl}(\mathfrak{F})$. Hence the only remaining part of $\mathfrak{M}$ which we still have to define over $\mathfrak{G}$ is $W^{\mathfrak{M}}$ which in turn is equivalent to defining w_m for each $m \in Obs$. However, by knowing m's unit vectors[640] and having the geometric tools of $\mathfrak{G}$ (e.g. g, $lines$, $\|$)[641] at our hand it is only a matter of patience to work out a definition for w_m. E.g. for $\lambda \in F$, $w_m(\langle \lambda, 0, 0\rangle) \in Mn$ is on the line determined by o, e_t and its g-distance from o is $|\lambda \cdot g(o, e_t)|$. There are only two such points in Mn, and it is easy to figure out (by using e.g. Bw) which one to choose. We leave the details of defining W to the formal definition below. Now, we are ready for the formal (first-order) definition of $\mathcal{M}(\mathfrak{G})$ over $\mathfrak{G}$, which comes below.

636 or world-view function

637 Eventually, we will need a coordinatization of a part of $\mathcal{P}(B)$ instead of Mn but that change will be easy to make, hence we postpone worrying about it.

638 e.g. $o \prec e_t$, $o \neq e_x$ and $\langle o, e_t\rangle \perp_r \langle o, e_x\rangle$ etc.

639 or identify

640 i.e. knowing $w_m(\bar{0}), w_m(1_t), \dots, w_m(1_y)$

641 $L \subseteq lines$, cf. Item 4.5.36 on p.308.

The definition given below becomes simpler and more intuitive if condition (e) is omitted. The so obtained simpler definition still works but less "spectacularly". What we mean by this is explained in footnote 642.

Definition 4.5.38 (the functor $\mathcal{M}$)
We define $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\emptyset)$ as follows. Let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$. Then we define

$$\mathcal{M}(\mathfrak{G}) = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, \mathsf{Eucl}(\mathfrak{F});\ \in, W \rangle$$

as follows:

1. $Obs :\overset{\text{def}}{=} \{ \langle o, e_0, \ldots, e_{n-1} \rangle \in {}^{n+1}Mn\ :\ $(a)–(f) below hold$\}$, see Figure 113.

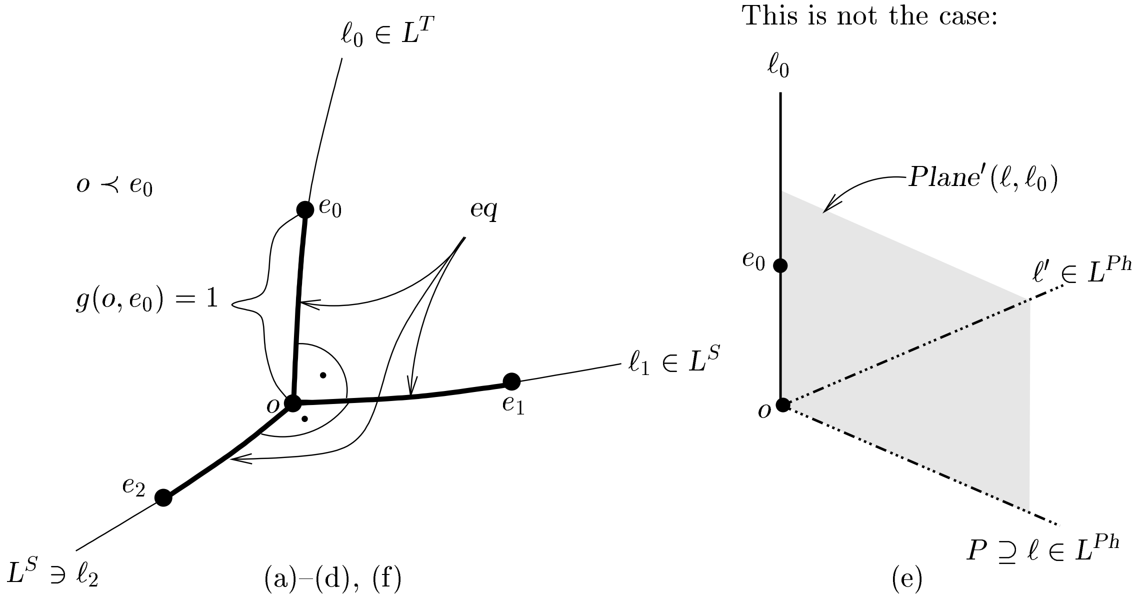


Figure 113: Illustration for the definition of Obs.

(a) $\{o, e_0, \ldots, e_{n-1}\}$ is an $n+1$ element independent subset of Mn.

(b) $o \prec e_0$.

(c) $\langle o, e_0 \rangle\ eq\ \langle o, e_i \rangle$, for all $i \in n$.

(d) $(\exists \ell_0 \in L^T)(\exists \ell_1, \ldots, \ell_{n-1} \in L^S)$
$$\Big((\forall i \in n)\, o, e_i \in \ell_i\ \wedge\ (\forall \text{ distinct } i, j \in n)\, \ell_i \perp_r \ell_j \Big).$$
<u>*Convention:*</u> To each choice of $\langle o, \ldots, e_{n-1} \rangle$ we will use $\ell_0, \ell_1, \ldots, \ell_{n-1}$ as fixed by (d) above.

(e) $P := Plane'(\{o, e_1, \ldots, e_{n-1}\})$ is space-like in the following sense:
$(\forall \ell, \ell' \in L^{Ph}) \Big([o \in \ell \subseteq P\ \wedge\ o \in \ell' \subseteq Plane'(\ell, \ell_0)] \quad \rightarrow \quad \ell = \ell' \Big)$,
see the right-hand side of Figure 113. In $\mathbf{Bax}^-$ geometries, intuitively, this means that if P contains the trace of a photon then the speed of this photon is infinite.[642]

[642] Item (e) is required only in order to make the following statement true: If $\mathfrak{G}$ is a $\mathbf{Bax}^{-\oplus}$ geometry then $\mathcal{M}(\mathfrak{G})$ is a $\mathbf{Bax}^{-\oplus}$ model, assuming $\mathbf{Pax}^+$ of course.

Without assuming $\mathbf{Bax}^-$, condition (e) corresponds to axiom $\mathbf{Ax}(\infty ph)$ on p.289 as part of the theory $\mathbf{Pax}^+$ (Def.4.5.9).

(f) $g(o, e_0) = 1$.

2. $Ph :\stackrel{\text{def}}{=} L^{Ph}$.

3. $B :\stackrel{\text{def}}{=} Ib :\stackrel{\text{def}}{=} Obs \cup Ph$.

4. Definition of the world-view relation W: First for every $m \in Obs$ we define the coordinatization function $w^0_m : {}^nF \rightarrowtail Mn$ as follows.[643] Let $m = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs$. (Notice that, by (c), $o, e_0, \ldots, e_{n-1}$ are pairwise connected, i.e. $\sim$-related.) We use the notation F_{oe} introduced in Def.4.5.28, i.e. F_{oe} is the line determined by the points o and e. First, by using parallel lines[644], we obtain a coordinatization mapping

$$F_{oe_0} \times F_{oe_1} \times \ldots \times F_{oe_{n-1}} \rightarrowtail Mn,$$

as depicted in the left-hand side of Fig.114. Next, for every $i \in n$, we identify F_{oe_i} with F_{oe_0} as depicted in the right-hand side of Fig.114, using lines parallel with $F_{e_i e_0}$. By

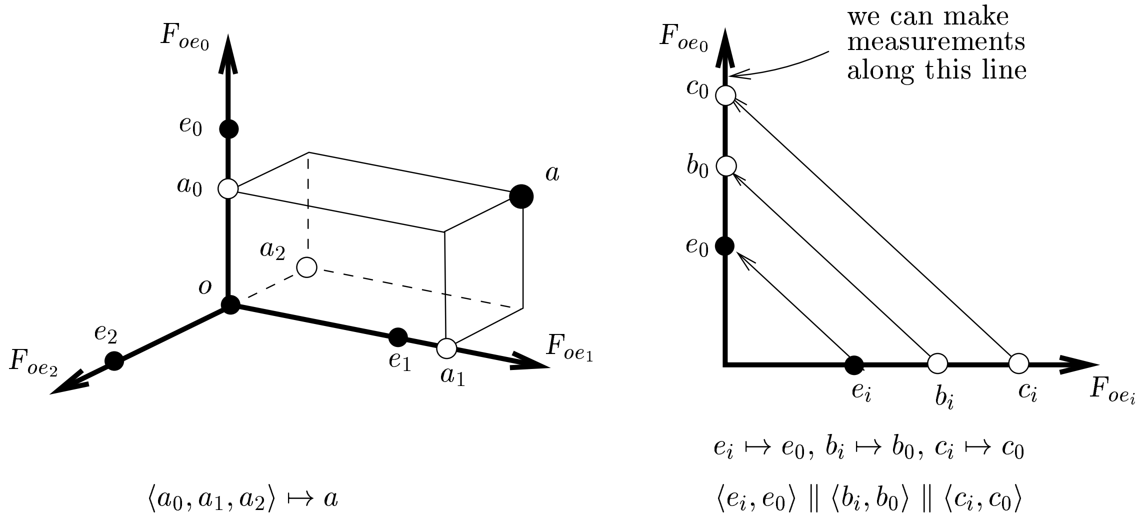


Figure 114: In the left-hand side of the picture we assume that $n = 3$.

these identifications and the above coordinatization, we obtain a coordinatization

$${}^nF_{oe_0} \rightarrowtail Mn.$$

We identify F by F_{oe_0} using g, the natural way, i.e. 0 and 1 get identified with o and e_0, repectively, and $x \in F$ gets identified with $a \in F_{oe_0}$ such that $g(o, a) = |x|$ and $(Bw(a, o, e_0) \Leftrightarrow x < 0)$. (This identification can be done because by the assumption

[643] The problem which we will have to circumnavigate is that by g we can make reliable measurements only on the line determined by o, e_0 (since we assumed $\mathbf{Ax(eqtime)}$ but not $\mathbf{Ax(eqm)}$). I.e. by g we can suitably measure the o, e_0 distance, while by the same g we cannot suitably measure the o, e_1 distance. This is why we will use parallel lines, cf. the right-hand side of Fig.114.

[644] Here we use *lines* and $\parallel$ both definable in $\mathfrak{G}$, cf. Item 4.5.36 (p.308). Note that $L \subseteq$ *lines* but *lines* $\not\subseteq L$ may happen. Cf. footnotedifferent on p.298.

Pax$^+$ we can make reliable measurements along F_{oe_0} by g.) In this way, from the above coordinatization ${}^nF_{oe_0} \rightarrowtail Mn$ we obtain the coordinatization

$$w_m^0 : {}^nF \rightarrowtail Mn.$$

In the next step, from w_m^0 we define the real world-view function w_m (whose range is a subset of $\mathcal{P}(B)$). To this end we "represent" Mn as part of $\mathcal{P}(B)$ i.e. we define a mapping $f : Mn \longrightarrow \mathcal{P}(B)$ the natural way. Let $e \in Mn$. Then we say that a photon $\ell \in Ph$ is present in event e iff $e \in \ell$, and an observer $\langle o', e'_0, \ldots, e'_{n-1}\rangle \in Obs$ is present in event e iff $e \in F_{o'e'_0}$. Let

$$f : Mn \longrightarrow \mathcal{P}(B)$$

be defined by

$$f(e) :\stackrel{\text{def}}{=} \{\, b \in B \;:\; b \text{ is present in } e \,\}, \quad \text{for all } e \in Mn.$$

Let $w_m :\stackrel{\text{def}}{=} w_m^0 \circ f$. The world-view relation W is defined from the w_m's the obvious way, i.e.

$$W :\stackrel{\text{def}}{=} \{\, \langle m, p, b\rangle \in Obs \times {}^nF \times B \;:\; b \in w_m(p) \,\}.$$

Thus, all ingredients of $\mathcal{M}(\mathfrak{G})$ are defined except for the ordered field $\mathfrak{F}$. Now we turn to defining $\mathfrak{F}$.

5. *Definition of $\mathfrak{F}$:* To define the ordered field $\mathfrak{F}$ from the geometry $\mathfrak{G}$ it is enough to define multiplication on F (from $\mathfrak{G}$), since $\mathbf{F_1} = \langle F; 0, 1, +, \le\rangle$ is contained in $\mathfrak{G}$. Now we turn to doing this.

   First let us notice that there is an original ordered field $\mathfrak{F}^{\mathfrak{M}}$ behind $\mathfrak{G}$, since $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$, for some $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^+)$. Let such an $\mathfrak{M}$ be fixed. Let

$$\mathbf{F}_1^{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle \mathbf{F}^{\mathfrak{M}}; 0^{\mathfrak{M}}, 1^{\mathfrak{M}}, +^{\mathfrak{M}}, \le^{\mathfrak{M}}\rangle.$$

   Now, $\mathbf{F_1} \cong \mathbf{F}_1^{\mathfrak{M}}$ by $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Of course we are not allowed to use $\mathfrak{F}^{\mathfrak{M}}$ when we are defining something from $\mathfrak{G}$, since $\mathfrak{F}^{\mathfrak{M}}$ is not explicitly included in $\mathfrak{G}$. (We use $\mathfrak{F}^{\mathfrak{M}}$ only for didactical [i.e. explanatory] purposes.) Now, we start defining multiplication over $\mathfrak{G}$. Assume $o, e \in Mn$, $o \equiv^T e$ and $g(o, e) = 1$. Such o, e exist by **Ax(eqtime)** (and by $\mathbf{AxE_{01}}$ + $(\forall m, k)(\forall 0 < i \in n)\, tr_m(k) \ne \bar{x}_i)$ or by **Ax(eqm)** (and $\mathbf{AxE_{01}}$). Let $Mn_o :\stackrel{\text{def}}{=} \{\, a \;:\; o \sim a \,\}$. Then $\langle Mn_o; Bw \restriction Mn_o\rangle \models$ **opag** by Prop.4.5.37. Let $\mathfrak{F}_{oe} = \langle F_{oe}; +_{oe}, \cdot_{oe}, \le_{oe}\rangle$ be the ordered field corresponding to o, e defined in Def.4.5.28. By Prop.4.5.29, $\mathfrak{F}_{oe}$ is indeed an ordered field (and is isomorphic to $\mathfrak{F}^{\mathfrak{M}}$). Let $g_{oe} : F_{oe} \longrightarrow F$ be defined as follows: Let $a \in F_{oe}$. Then

$$g_{oe}(a) :\stackrel{\text{def}}{=} \begin{cases} g(o, a) & \text{if } a \in [oe \\ -g(o, a) & \text{otherwise.} \end{cases}$$

   Clearly, $g_{oe}(o) = 0$ and $g_{oe}(e) = 1$ by our choice of o, e. We note that $g_{oe} : F_{oe} \longrightarrow F$ is an isomorphism between $\langle F_{oe}; o, e, +_{oe}, \le_{oe}\rangle$ and $\mathbf{F_1}$. Now we use these g_{oe}'s to copy the multiplications $\cdot_{oe}$ on F_{oe}'s to obtain multiplication $\cdot$ on F. We define multiplication $\cdot \subseteq F \times F \times F$ as follows. Let $x, y, z \in F$

$$\cdot(x, y, z) \;\stackrel{\text{def}}{\Longleftrightarrow}\; (\exists o, e \in Mn)\Big[\; o \equiv^T e \;\wedge\; g(o, e) = 1 \;\wedge\; g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y) \,\Big].$$

By this, multiplication $\cdot$ is defined on F. By the above the structure $\mathfrak{F} := \langle F; +, \cdot, \leq \rangle$ is defined. We will prove as Claim 4.5.40 that $\mathfrak{F}$ is an ordered field isomorphic to $\mathfrak{F}^{\mathfrak{M}}$.

By items 1–5 above, the frame model $\mathcal{M}(\mathfrak{G})$ is defined.

END OF DEFINITION OF THE FUNCTOR $\mathcal{M}$.

$\triangleleft$

Remark 4.5.39 We note that, if $n > 2$, then $\mathcal{M}$ is defined on $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqtime)})$ and $\mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)})$, by Proposition 4.5.10 (p.290).

$\triangleleft$

Claim 4.5.40 below serves to prove correctness of Def.4.5.38 above.

Claim 4.5.40 Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$. Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^+)$ be such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Let the structure $\mathfrak{F}$ be defined as in item 5 of Def.4.5.38 above. Then $\mathfrak{F}$ is an ordered field isomorphic to $\mathfrak{F}^{\mathfrak{M}}$.

Outline of proof: Let $\mathfrak{G}, \mathfrak{M}, \mathfrak{F}$ be as in the claim. Without loss of generality we can assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$, because the functor $\mathcal{M}$ was defined in such a style that it associates isomorphic models with isomorphic structures. Assume $o, e \in Mn$ are such that $o \equiv^T e$ and $g(o,e) = 1$. Let $Mn_o := \{ a \in Mn \ : \ a \sim o \}$. Let $m \in Obs$ be such that $o, e \in w_m[\bar{t}]$. It exists. Then

$$w_m : \langle {}^nF;\ \mathsf{Betw} \rangle \rightarrowtail\!\!\!\rightarrow \langle Mn_o;\ Mn_o \restriction Bw \rangle$$

is an isomorphism by Thm.3.2.6 (p.110). Let $o' := w_m^{-1}(o)$ and $e' := w_m^{-1}(e)$. Clearly $o', e' \in \bar{t}$. Let $\mathfrak{F}_{o'e'} = \langle F_{o'e'}; \ldots \rangle$ and $\mathfrak{F}_{oe} = \langle F_{oe}; \ldots \rangle$ be the ordered fields corresponding to o', e' and o, e, respectively defined in Def.4.5.28 (p.303). Then $F_{o'e'} = \bar{t}$ and $|e'_t - o'_t| = 1$. The latter holds by

$$(*) \qquad \begin{array}{l} g(o,e) = 1 \quad \text{and} \quad \mathbf{AxE_{01}} + \\ \Big((\,\mathbf{Ax(eqtime)} \ \wedge\ (\forall m,k)(\forall 0 < i \in n)\, tr_m(k) \neq \bar{x}_i\,) \ \vee\ \mathbf{Ax(eqm)} \Big). \end{array}$$

Without loss of generality we may assume that $e'_t - o'_t = 1$. Let $g_{oe} : F_{oe} \longrightarrow F^{\mathfrak{M}}$ be defined as on p.312. Now, $(w_m \restriction \bar{t}) \circ g_{oe} : F_{o'e'} \longrightarrow F^{\mathfrak{M}}$ and $(w_m \restriction \bar{t}) \circ g_{oe} : p \mapsto p_t - o'_t$ by $(*)$. Thus, $(w_m \restriction \bar{t}) \circ g_{oe} : \mathfrak{F}_{o'e'} \rightarrowtail\!\!\!\rightarrow \mathfrak{F}^{\mathfrak{M}}$ is an isomorphism. By this and by noticing that $w_m \restriction \bar{t} : \mathfrak{F}_{o'e'} \rightarrowtail\!\!\!\rightarrow \mathfrak{F}_{oe}$ is an isomorphism, we conclude that

$$g_{oe} : \mathfrak{F}_{oe} \rightarrowtail\!\!\!\rightarrow \mathfrak{F}^{\mathfrak{M}}$$

is an isomorphism. By this it can be checked that the multiplication defined on $F^{\mathfrak{M}}$ on p.312 coincides with the multiplication of $\mathfrak{F}^{\mathfrak{M}}$. Hence $\mathfrak{F}$ and $\mathfrak{F}^{\mathfrak{M}}$ are isomorphic. (Actually, by our assumption that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$ $\mathfrak{F}$ and $\mathfrak{F}^{\mathfrak{M}}$ coincide.) ∎

Next we state that the functor $\mathcal{M}$ constructed so far is of the kind we need for our duality theory outlined on pp.282-284, cf. Fig.101 (p.283).

PROPOSITION 4.5.41 *$\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ and $\mathcal{M}$ is a first-order definable meta-function. Hence $\mathcal{M}[\mathsf{Ge}(\mathbf{Pax}^+)] \subseteq \mathsf{Mod}(\mathbf{Pax}^+)$ is first-order definable over $\mathsf{Ge}(\mathbf{Pax}^+)$.*

Outline of proof: First-order definability of $\mathcal{M}$ comes immediately from the definition of $\mathcal{M}$ (by using Remark 4.3.52 on p.271). To prove $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax^+}) \longrightarrow \mathsf{Mod}(\mathbf{Pax^+})$ let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax^+})$. Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax^+})$ be such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Without loss of generality we can assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$. The visibility relation $\overset{\odot}{\to}$ is an equivalence relation when restricted to $Obs^{\mathfrak{M}}$ by Thm.3.2.6. Let $O \subseteq Obs^{\mathfrak{M}}$ be a set of representatives for the equivalence relation $\overset{\odot}{\to}$. Recall that for every $k \in Obs^{\mathfrak{M}}$ $\mathfrak{G}_k = \langle {}^nF, \ldots\rangle$ is the observer-dependent geometry defined in Def.4.2.61 (p.206). Then similarly to item 3b of Prop.4.2.64 (p.213) the $\perp_r$-free reduct of $\mathfrak{G}$ is a photon-glued disjoint union of the family

$$\langle \perp_r\text{-free reduct of } \mathfrak{G}_k \; : \; k \in O\rangle.$$

Further $Bw_k = \mathsf{Betw}$ and $L_k \subseteq \mathsf{Eucl}$ by Thm.3.2.6 for every $k \in Obs^{\mathfrak{M}}$. Thus $\mathfrak{G}$ is a photon-glued disjoint union of the familiar ${}^nF$-geometries. By this, it can be checked that $\mathcal{M}(\mathfrak{G}) \models \mathbf{Pax}+\mathbf{Ax(Bw)}+\mathbf{AxE_{01}}+\mathbf{Ax(\infty \mathit{ph})}$. Thus it remains to prove that $\mathcal{M}(\mathfrak{G}) \models (\mathbf{Ax(eqtime)}+ (\forall m,k)(\forall 0 < i \in n) tr_m(k) \neq \bar{x}_i)$ or $\mathcal{M}(\mathfrak{G}) \models \mathbf{Ax(eqm)}$. By $\mathfrak{M} \models \mathbf{Pax^+}$, we have $\mathfrak{M} \models (\mathbf{Ax(eqtime)} + (\forall m,k)(\forall 0 < i \in n) tr_m(k) \neq \bar{x}_i)$ or $\mathfrak{M} \models \mathbf{Ax(eqm)}$. For the case $\mathfrak{M} \models (\mathbf{Ax(eqtime)} + \ldots)$ checking $\mathcal{M}(\mathfrak{G}) \models (\mathbf{Ax(eqtime)} + \ldots)$ is easy and is left to the reader. (Hint: $L^T \cap L^S = \emptyset$ and $L^T \cap L^{Ph} = \emptyset$ hold in this case.)

Assume $\mathfrak{M} \models \mathbf{Ax(eqm)}$. We will prove that $\mathcal{M}(\mathfrak{G}) \models \mathbf{Ax(eqm)}$. Let $g^* : Mn \times Mn \overset{\circ}{\longrightarrow} F$ be the partial function defined as follows. Let $e, e_1 \in Mn$ and $\lambda \in F$. Then

$$g^*(e,e_1) = \lambda$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs^{\mathfrak{M}})(\exists i \in n)(\exists p,q \in \bar{x}_i)(w_m(p) = e \;\wedge\; w_m(q) = e_1 \;\wedge\; |p-q| = \lambda).$$

By $\mathbf{Ax(eqm)}$, g^* is well defined. By $\mathbf{Ax(eqm)}+\mathbf{AxE_{01}}$, it is easy to check that g and g^* agree on time-like separated pairs of points. For every $m \in Obs^{\mathcal{M}(\mathfrak{G})}$ let $w_m^0 : {}^nF \longrightarrow Mn$ be defined as on p.311 in Def.4.5.38. If we prove

$$(*) \qquad (\forall m \in Obs^{\mathcal{M}(\mathfrak{G})})(\forall i \in n)(\forall p,q \in \bar{x}_i)\, |p-q| = g^*(w_m^0(p), w_m^0(q))$$

then $\mathcal{M}(\mathfrak{G}) \models \mathbf{Ax(eqm)}$ will hold (by the definition of W on p.312). Thus it is enough to prove $(*)$ above. For every $o, e \in Mn$ with $o \neq e$ and $o \sim e$ let $F_{oe} = \{a \in Mn \; : \; coll(a,o,e)\}$; and for every $o, e, o', e' \in Mn$ with $o \neq e$, $o' \neq e'$, $o \sim e$ and $o' \sim e'$ let $f_{oe}^{o'e'} : F_{oe} \longrightarrow F_{o'e'}$ be defined as in the proof of Prop.4.5.30 on p.305. Now items 1 and 2 below hold because of the following. It is easy to check that items 1,2 hold when eq is replaced by eq_0 in them. By this, by $f_{oe}^{o'e'} \circ f_{o'e'}^{o''e''} = f_{oe}^{o''e''}$ and since eq is defined to be the transitive closure of eq_0 we have that 1 and 2 below hold. (In proving this, $L^T \cap L^{Ph} = \emptyset$ is used too).

1. $\langle a,b\rangle \; eq \; \langle c,d\rangle \;\Rightarrow\; g^*(a,b) = g^*(c,d)$.

2. $(\forall o,e,o',e' \in Mn)\Big((o \neq e \;\wedge\; o' \neq e' \;\wedge\; \langle o,e\rangle \; eq \; \langle o',e'\rangle) \;\rightarrow$
$$(\forall a,b \in F_{oe})\, \langle a,b\rangle \; eq \; \langle f_{oe}^{o'e'}(a), f_{oe}^{o'e'}(b)\rangle\Big).$$

Now we turn to proving $(*)$ above. Let $m \in Obs^{\mathcal{M}(\mathfrak{G})}$, $i \in n$, $p,q \in \bar{x}_i$. Then $m = \langle o, e_0, \ldots, e_{n-1}\rangle$ for some $o, e_0, \ldots e_{n-1} \in Mn$ satisfying (a)–(f) on p.310. By 2 above and by $\langle o,e_0\rangle \; eq \; \langle o,e_i\rangle$, we have that $\langle w_m^0(p), w_m^0(q)\rangle \; eq \; \langle f_{oe_i}^{oe_0}(w_m^0(p)), f_{oe_i}^{oe_0}(w_m^0(q))\rangle$. Hence, by 1 above, $g^*(w_m^0(p), w_m^0(q)) = g^*(f_{oe_i}^{oe_0}(w_m^0(p), f_{oe_i}^{oe_0}(w_m^0(q)))$. By the definition of $\mathcal{M}(\mathfrak{G})$, [645] we

[645] and by noticing that $f_{oe_i}^{oe_0}(w_m^0(p))$, $f_{oe_i}^{oe_0}(w_m^0(q)) \in F_{oe_0}$, $\langle w_m^0(p), f_{oe_i}^{oe_0}(w_m^0(p))\rangle \parallel \langle e_i, e_0\rangle$, $\langle w_m^0(q), f_{oe_i}^{oe_0}(w_m^0(q))\rangle \parallel \langle e_i, e_0\rangle$

have $|p-q| = g^*(f^{oe_0}_{oe_i}(w^0_m(p)), f^{oe_0}_{oe_i}(w^0_m(q)))$. Thus $|p-q| = g^*(w^0_m(p), w^0_m(q))$ and this proves the proposition. ∎

The following theorem implies that the sentences in our frame language can be translated into sentences in the language of our relativistic geometries (in a meaning preserving way), assuming $\mathbf{Pax}^+$. More intuitively, whatever can be said in the language of the ("observational") frame models can be said in the "theoretical terminology" of relativistic geometries, too. (Cf. Thm.4.5.12 on p.291.)

THEOREM 4.5.42 *There is a "natural" translation mapping*

$$T_{\mathcal{M}} : Fm(\mathsf{Mod}(\mathbf{Pax}^+)) \longrightarrow Fm(\mathsf{Ge}(\mathbf{Pax}^+))$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{Mod}(\mathbf{Pax}^+))$ *with all its free variables belonging to sort* F, $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$ *and evaluation* $\bar{a}$ *of* $\bar{x}$ *(in* F *of course)*

$$\mathcal{M}(\mathfrak{G}) \models \varphi[\bar{a}] \quad \Leftrightarrow \quad \mathfrak{G} \models T_{\mathcal{M}}(\varphi)[\bar{a}].$$

Proof: The theorem follows by Prop.4.5.41 and by Prop.4.3.41 (p.264). ∎

Actually, $T_{\mathcal{M}}$ admits a very natural description which suggests itself if we look at the definition of $\mathcal{M}$ and contemplate Figure 96 (p.263).

The following theorem says that for our $(\mathcal{G}, \mathcal{M})$-duality, theorem schemas (A)–(H), hold under some conditions.

THEOREM 4.5.43 *For the choice of* $\mathcal{M}$ *given in Def.4.5.38 above, the conclusions of Theorems 4.5.11 (p.290) and 4.5.13 (p.291) hold. E.g.* $\mathcal{G}$ *and* $\mathcal{M}$ *are first-order definable meta-functions and*

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th),$$

assuming Th *satisfies condition* $(\star)$ *in Thm.4.5.11 and* $\mathbf{Ax(diswind)}$. *Further, theorem schemas (A)–(H) hold, etc.*

Proof:

<u>*Case of Thm.4.5.11:*</u>

Let Th be as in Thm.4.5.11. Assume $n > 2$. Clearly, $Th \models \mathbf{Pax}^+$ (by Thm.4.5.10(i)). Let $\mathfrak{M} \in \mathsf{Mod}(Th)$. Let $h_{Obs} : Obs^{\mathfrak{M}} \longrightarrow Obs^{(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})}$ be defined by $h_{Obs} : m \mapsto \langle w_m(\bar{0}), w_m(1_0), \ldots, w_m(1_{n-1})\rangle$ and $h_{Ph} : Ph^{\mathfrak{M}} \longrightarrow Ph^{(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})}$ be defined by $h_{Ph} : ph \mapsto \{e \in Mn_{\mathfrak{M}} : ph \in e\}$.

Claim 4.5.44

$$\langle h_{Obs} \cup h_{Ph}, \mathrm{Id} \restriction F, \mathrm{Id} \restriction G\rangle : \mathfrak{M} \rightarrowtail\!\!\!\twoheadrightarrow (\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$$

is an isomorphism, assuming $n > 2$ and $\mathfrak{M} \models \mathbf{Bax}^{\oplus} + \mathbf{Ax(}Triv_t\mathbf{)}^- + \mathbf{Ax(||)}^- + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ })$.

Proof: We will prove Claim 4.5.44 by several steps, which steps we will use in later parts of this proof, too. In the following we will simply write h for h_{Obs}, h_{Ph}. Assume now that $\mathfrak{M} \models \mathbf{Pax}^+$ is arbitrary. Let $\mathfrak{M}^+ \stackrel{\text{def}}{=} \mathcal{M}\mathfrak{G}_{\mathfrak{M}}$.

(1)

$$h : Obs^{\mathfrak{M}} \longrightarrow Obs^{\mathfrak{M}^+}, \quad \text{by } \mathfrak{M} \models \mathbf{Pax^+}.$$

To see (1), it is enough to check that $h(m) = \langle w_m(\bar{0}), \ldots, w_m(1_{n-1})\rangle$ satisfies (a)-(f) in Def.4.5.38(1). In this, one uses $\mathbf{Pax^+}$. E.g., the last axiom of $\mathbf{Pax^+}$ is used when checking $g(w_m(\bar{0}), w_m(1_t)) = 1$.

(2) $h(m) = h(k)$ implies $w_m = w_k$, by $\mathbf{Pax} + \mathbf{Ax(eqtime)}$, for any $m, k \in Obs^{\mathfrak{M}}$.

Indeed, $h(m) = h(k)$ implies $w_m(\bar{0}) = w_k(\bar{0})$ and $w_m(1_t) = w_k(1_t)$. Thus $tr_m(k) = \bar{t}$ and $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. By $\mathbf{Ax(eqtime)}$ then f_{mk} is identity on $\bar{t}$. By $\mathbf{Pax}$ and Thm.3.2.6, f_{mk} is a bijective collineation, hence it is an affine transformation composed with $\widetilde{\varphi}$ for some field-automorphism φ. By $\mathsf{f}_{mk} \restriction \bar{t} \subseteq \mathrm{Id}$ then φ must be the identity, hence $\mathsf{f}_{mk} \in Aftr$. By $h(m) = h(k)$ we have $\mathsf{f}_{mk}(1_i) = 1_i$ for all $i < n$. Since $\mathsf{f}_{mk} \in Aftr$, this implies that $\mathsf{f}_{mk} = \mathrm{Id}$. This means that $w_m = w_k$.

Clearly, $h : Ph^{\mathfrak{M}} \longrightarrow Ph^{\mathfrak{M}^+}$, and h_{Ph} is injective if $\mathfrak{M} \models \mathbf{Ax(ext)}$. Thus by (1),(2) above we have

(3)

$$h : B^{\mathfrak{M}} \rightarrowtail B^{\mathfrak{M}^+} \quad \text{if } \mathfrak{M} \models \mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}.$$

(4)

$$h : Obs^{\mathfrak{M}} \twoheadrightarrow Obs\mathfrak{M}^+ \text{ is surjective, whenever}$$

$\mathfrak{M} \models \mathbf{Bax}^{\oplus} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax(}\sqrt{\ }\mathbf{)} + \mathbf{Ax(eqtime)} + \mathbf{Ax(}\|\mathbf{)}^{-} + \mathbf{Ax(diswind)}$ and $n > 2$.

To prove (4), let $K \stackrel{\text{def}}{=} \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs^{\mathfrak{M}^+}$. Let $\ell_0, \ldots, \ell_{n-1}$ be as in Def.4.5.38(1) (p.310), i.e. $\ell_0 \in L^T$, $o, e_0 \in \ell_0$ etc. By $\mathbf{Bax}^{\oplus} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax(}\sqrt{\ }\mathbf{)}$, there is an observer $m \in Obs^{\mathfrak{M}}$ whose coordinate axes are exactly $\ell_0, \ldots, \ell_{n-1}$, see Remark 4.2.52 (ii) (p.196). Then $o = w_m(\bar{0})$. By $\mathbf{Ax(eqtime)}$ and $g(o, e_0) = 1$ we have that $e_0 = w_m(1_t)$. Let $0 < i < n$. We may assume that $w_m(e_i)$ lies on the half-line determined by o, e_i, by $\mathbf{Ax(}Triv_t\mathbf{)}^{-}$. By the definition of eq, we then have that $\langle o, e_0\rangle\ eq\ \langle o, w_m(1_i)\rangle$. Since $\langle o, e_0\rangle\ eq\ \langle o, e_i\rangle$ (by $K \in Obs^{\mathfrak{M}^+}$), and eq is transitive, we have $\langle o, e_0\rangle\ eq\ \langle o, w_m(1_i)\rangle$. Since e_0 and $w_m(1_i)$ are on the same "side" of o by our choice of m, we have $e_i = w_m(1_i)$ by item 2 on p.167. Thus, $h(m) = K$ and this proves surjectivity of h_{Obs}.

(5) $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{F}^{\mathfrak{M}^+}$, by the proof of Claim 4.5.40 (p.313), see the last sentence there.

(6) h preserves the world-view relation.

Indeed, let $m \in Obs^{\mathfrak{M}}$ and $b \in Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}}$. We will show that for every $p \in {}^nF$,

$$W^{\mathfrak{M}}(m, p, b) \quad \text{iff} \quad W^{\mathfrak{M}^+}(h(m), p, h(b)).$$

It is not hard to check that $w^0_{h(m)}$ as defined in Def.4.5.38(4) (p.311) coincides with w_m. Now, $h(ph) \in w_{h(m)}(p)$ iff $ph \in w^0_{h(m)}(p)$ iff $ph \in w_m(p)$, and $h(k) \in w_m(p)$ iff $w_m(p) \in F_{w_k(\bar{0})w_k(1_t)}$ iff $p \in tr_m(k)$ iff $k \in w_m(p)$. Since $W^{\mathfrak{M}}(m, p, b)$ iff $b \in w_m(p)$, and the same for $\mathfrak{M}^+$ in place of $\mathfrak{M}$, we are done. By this, Claim 4.5.44 has been proved, too. ∎

By Claim 4.5.44 above,

$$(*) \qquad (\forall \mathfrak{M} \in \mathsf{Mod}(Th))(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M}.$$

Let $\mathfrak{G} \in \mathsf{Ge}(Th)$. Then $\mathfrak{G} \cong \mathcal{G}(\mathfrak{M})$ for some $\mathfrak{M} \in \mathsf{Mod}(Th)$. Let this $\mathfrak{M}$ be fixed. Then $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M}$ by $(*)$ above. Hence, $(\mathcal{M} \circ \mathcal{G})(\mathcal{G}(\mathfrak{M})) \cong \mathcal{G}(\mathfrak{M})$. Thus, $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G}$. By the above, item (ii) of Thm.4.5.11 is proved. By $(*)$ above, and by the fact that $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism we conclude that $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$. Further, $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ holds by the definition of $\mathcal{G}$. First-order definability of $\mathcal{M}$ comes from Prop.4.5.41 while first-order definability of $\mathcal{G}$ comes from Thm.4.3.22 (p.244). By this, Thm.4.5.11 is proved.

<u>*Case of Thm.4.5.13:*</u> For any $\mathfrak{G} \in \mathsf{Ge}(\emptyset)$ let $\mathfrak{G}^*$ be the geometry obtained from $\mathfrak{G}$ by omitting $\mathcal{T}$ and replacing g with $g \restriction \{\langle a, b\rangle \in Mn \times Mn \ : \ a \equiv^T b\}$.

Claim 4.5.45 $\mathfrak{G}^* \cong [(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}^*)]^*$, for any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax^+})$, and
$\mathfrak{G} \cong (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}^*)$, for any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$.

Proof: To prove this claim let $\mathfrak{M}, \mathfrak{M}^+$ be such that $\mathfrak{G}^* = (\mathcal{G}(\mathfrak{M}))^*$ and $\mathfrak{M}^+ = \mathcal{M}(\mathfrak{G}^*)(= \mathcal{M}\mathcal{G}(\mathfrak{M}))$. We want to prove $\mathfrak{G}^* = \mathcal{G}(\mathfrak{M}^+)^*$. Let $\mathfrak{G}^+ = \langle Mn^+, \dots, \perp^+, eq^+, \prec^+, \dots\rangle = \mathcal{G}(\mathfrak{M}^+)^*$ and $\mathfrak{G}^* = \langle Mn, \dots, \perp, eq, \prec, \dots\rangle = \mathcal{G}(\mathfrak{M})$.

Recall the function $h : Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}} \longrightarrow B^{\mathfrak{M}^+}$ given at the beginning of this proof. The construction of $\mathcal{M}$ gives us a natural function

$$f : Mn \longrightarrow\!\!\!\!\!\rightarrow Mn^+$$

defined as $f(e) = \{h(m) : m \in Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}}\} \cup \{\langle o, e_0, \dots\rangle \in Obs^{\mathfrak{M}^+} : coll^{\mathfrak{G}}(o, e_0, e)\}$. By $\mathfrak{M} \models \mathbf{Pax}$ we have that $f : Mn \rightarrowtail\!\!\!\!\rightarrow Mn^+$ is a bijection. We want to show that $\langle f, \mathrm{Id} \restriction F\rangle$ is actually an <u>*isomorphism*</u> between $\mathfrak{G}^*$ and $\mathfrak{G}^+$.

In the following, in order to make the proof more intuitive, we will identify Mn with Mn^+ (along f) and we will identify $Obs^{\mathfrak{M}}$ with a subset of $Obs^{\mathfrak{M}^+}$ (along h).[646]

First one checks $L^{T+} = L^T$, $L^{Ph+} = L^{Ph}$, $L^{S+} = L^S$. These follow from the construction (the definitions).

To see that $Bw^+ = Bw$, first we observe that $Bw \subseteq Bw^+$ since in the process of $\mathfrak{M} \mapsto \mathfrak{M}^+$ old observers do not disappear. To see the other direction, assume $\langle a, b, c\rangle \in Bw^+$. Then there is $m^+ \in Obs^{\mathfrak{M}}$ who "thinks" b is between a and c. By Prop.4.5.41 we have $\mathfrak{M} \models \mathbf{Pax^+}$. Since the life-line of m^+ is a time-like line in $\mathcal{G}(\mathfrak{M})^* = \mathfrak{G}^*$, there is an "old" observer $m \in Obs$ who sees m^+, hence by basic properties of $\mathbf{Pax}$, i.e. by Thm.3.2.6 on p.110, m sees events a, b, c. By $\mathbf{Ax(Bw)} \in \mathbf{Pax^+}$, the world-view transformations f_{mm^+}, f_{m^+m} preserve betweenness. Hence,

[646] Recall that $w_m = w^0_{h(m)}$ and so $w_m \circ f = w^0_{h(m)} \circ f = w_{h(m)}$. See also Step (2) in the proof of Claim 4.5.44.

in $\mathfrak{M}^+$, m thinks $\langle a,b,c\rangle \in Bw^+$. But by the construction of $\mathfrak{M}^+$ then, in the original $\mathfrak{M}$, we have $\langle a,b,c\rangle \in Bw$. This completes the proof of $Bw^+ = Bw$.

Next we observe $\perp_0 \subseteq \perp_0^+ \subseteq \perp$. Since the operator $\mathcal{G}$ creates $\perp$ from $\perp_0$ by closing under limits and parallelism[647] we conclude $\perp^+=$ closure of $\perp_0^+ \subseteq \perp=$ closure of $\perp_0$. So $\perp^+=\perp$.

Analogously to the case of $\perp$, we prove $eq_0 \subseteq eq_0^+ \subseteq eq$. Of these, $eq_0^+ \subseteq eq$ is harder than the $\perp$-case was. The proof of $eq_0^+ \subseteq eq$ will go by using item 2 from the proof of Prop.4.5.41 (p.314) together with Figure 114, i.e. the definition of the coordinatization $F_{oe_0} \times \ldots \times F_{oe_{n-1}} \rightarrowtail Mn$ in the construction of $\mathcal{M}$ (p.311). Indeed, assume $\langle a,b\rangle\ eq_0^+\ \langle c,d\rangle$. Then there is $m^+ \in Obs^+$, i,j such that m^+ thinks that $a,b \in \bar{x}_i$ and $c,d \in \bar{x}_j$ and they are of the "same length". Assume first that $0 \notin \{i,j\}$. Now, m^+ measures the ab-distance by projecting $\{a,b\}$ to $\bar{t}$ by using $f_{oe_i}^{oe_0} : \bar{x}_i \longrightarrow \bar{t}$, cf. item 2 in the proof of 4.5.41. Then $e_i \mapsto e_0$, $a \mapsto a'$ and $b \mapsto b'$ where $a', b' \in \bar{t}$. By the quoted item 2 then

(*) $\langle a,b\rangle\ eq\ \langle a',b'\rangle$,

since $\langle o,e_i\rangle\ eq \langle o,e_0\rangle$. Analogously, using $f_{oe_j}^{o,e_0} : \bar{x}_j \longrightarrow \bar{t}$ we obtain $o \mapsto o$, $e_j \mapsto e_0$, $c \mapsto c'$ and $d \mapsto d'$ with

(**) $\langle c,d\rangle\ eq\ \langle c',d'\rangle$.

By the definition of the coordinatization of m^+, i.e. by Figure 114 right side, this implies that m^+ thinks $\langle a',b'\rangle$ and $\langle c',d'\rangle$ are of the same length (this is so because m^+ "thinks" that a,b and c,d are of the same length). Clearly, there is an "old" $m \in Obs$ with $tr_m(m^+) = \bar{t}$. By **Ax(eqtime)**, intervals on $\bar{t}$ are of the same lengths for m^+ and m. Hence m, too, thinks that $\langle a',b'\rangle$ and $\langle c',d'\rangle$ are of the same length. Then $\langle a',b'\rangle\ eq\ \langle c',d'\rangle$. By (*),(**) and transitivity of eq we conclude $\langle a,b\rangle\ eq\ \langle c,d\rangle$.

The other case, when $0 \in \{i,j\}$, is completely analogous, we omit it.

Since eq is the transitive closure of eq_0, analogously to the $\perp$-case, $eq =$ closure of $eq_0 =$ closure of $eq_0^+ = eq^+$.

Proving $\prec = \prec^+$ is based on the fact that $\prec$ is tied up with time-like life-lines of observers and their directions of time. These data are easy to trace through our construction.

To see that $g = g^+$, recall that $g = g \restriction \equiv^T$. Now $g(e,e_1)$ is determined by observers who see e, e_1 on one of their axes. If $\mathfrak{M} \models$ **Ax(eqm)**, then these all will agree both in $\mathfrak{M}$ and in $\mathfrak{M}^+$. Further, any "new" observer has an "old" brother. If $\mathfrak{M} \not\models$ **Ax(eqm)**, then $\mathfrak{M} \models$ **Ax(eqtime)** and no observer in $\mathfrak{M}$ will see e, e_1 on a space-axis, by $\mathfrak{M} \models$ **Pax$^+$**. Hence $g(e,e_1)$ is determined by the clocks of time-like observers connecting e and e_1. But two such observers are "brothers" by definition and by **Ax(eqtime)** they will measure the distance between e and e_1 the same way. This remains true in $\mathfrak{M}^+$ since we checked that $L^T = L^{T+}$. Thus, $g = g^+$.

To prove $\mathfrak{G} \cong (\mathcal{M} \circ \mathcal{G})\mathfrak{G}$ for any $\mathfrak{G} \in \mathsf{Ge}($**Pax$^+$** + **Ax(eqm)**$)$, assume that $\mathfrak{M} \models$ **Pax$^+$** + **Ax(eqm)**. We want to show that $\mathfrak{G}$ and $\mathcal{G}(\mathfrak{M}^+)$ agree on g-distances of all events, not only on time-like separated events. The last part of the previous case just proved this. This finishes the proof of Claim 4.5.45. ∎

For any $\mathfrak{G} \in \mathsf{Ge}($**Pax$^+$**$)$ we have $\mathcal{M}(\mathfrak{G}) = \mathcal{M}(\mathfrak{G}^*)$ because we used only the structure of $\mathfrak{G}^*$ when constructing $\mathcal{M}$. Therefore, for any $\mathfrak{G} \in \mathsf{Ge}($**Pax$^+$**$)$, $\mathcal{M}(\mathfrak{G}) \cong (\mathcal{G} \circ \mathcal{M})(\mathcal{M}(\mathfrak{G}))$. This proves item (i) of the theorem. Item (ii) follows from item (i) by the fact that $Rng(\mathcal{G})$

[647] and since these are basically the same in $\mathfrak{G}^*$ and in $(\mathcal{G}\mathcal{M}(\mathfrak{G}^*))^*$

is $\mathsf{Ge}(Th)$ up to isomorphism. Item (iii) follows from steps (3),(6) and (7) in the proof of Claim 4.5.44. Item (iv) follows by the proof of Prop.4.5.41 and from Claim 4.5.45. ∎

Remark 4.5.46 (Discussion of Thm.4.5.43) **Ax(eqm)** is needed in Claim 4.5.45, i.e. there is a model $\mathfrak{M} \models \mathbf{Pax}^+$ such that $\mathfrak{G}_{\mathfrak{M}} \not\cong (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}_{\mathfrak{M}})$. By Claim 4.5.45 this can happen only in the way that $\mathfrak{G} \stackrel{\text{def}}{=} \mathfrak{G}_{\mathfrak{M}}$ and $\mathfrak{G}^+ \stackrel{\text{def}}{=} (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}_{\mathfrak{M}})$ do not agree on some space-like distance. The idea of this model is illustrated in Figure 115.

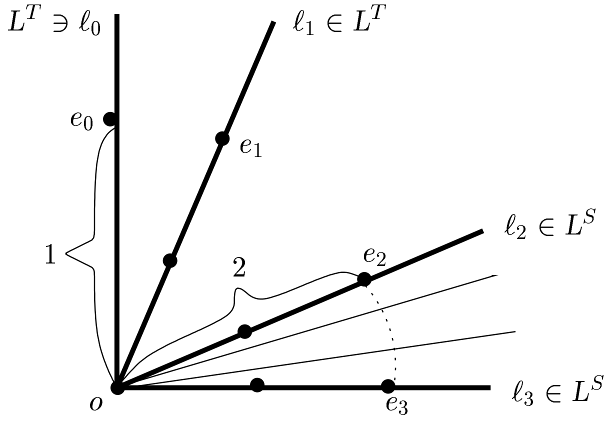


Figure 115: Illustration of $\mathfrak{G}_{\mathfrak{M}}$.

For showing the idea of $\mathfrak{M}$, assume $n = 2$, let $\mathfrak{F}^{\mathfrak{M}} \stackrel{\text{def}}{=} \mathfrak{R}$, and let ℓ_1, ℓ_2 be in $\mathsf{Eucl}(n, \mathfrak{R})$ be as depicted in Figure 115. Let m be an observer in $\mathfrak{M}$. There are two kinds of observers in m's world-view (passing through $\bar{0}$). The first kinds are brothers of m: their time-axis is $\bar{t}$ their space-axes have slope less than that of ℓ_2 and they measure distances on the axes according to the Euclidean distance. In more detail: To any line ℓ with slope less than that of ℓ_2 there is an observer $k \stackrel{\text{def}}{=} m(\bar{t}, \ell)$ such that $1_t^k = 1_t$, 1_x^k is on ℓ and $|1_x^k| = 1$. The life-lines of the second kinds of observers are ℓ_1 (in m's world-view), their space-axes have slopes less than *or equal* to that of ℓ_2 and they measure distances on the axes to be twice of the Euclidean distance. In more detail: To any line ℓ with slope less than equal to that of ℓ_2 there is an observer $k \stackrel{\text{def}}{=} m(\ell_1, \ell)$ such that $1_t^k \in \ell_1$, $1_x^k \in \ell$ and $|1_t^k| = |1_x^k| = 1/2$. Figure 115 shows (part of) the geometry $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$.[648] Let $m \stackrel{\text{def}}{=} m(\bar{t}, \bar{x}), k \stackrel{\text{def}}{=} m(\ell_1, \bar{x})$ and $h \stackrel{\text{def}}{=} m(\ell_1, \ell_2)$. In the figure, $o = w_m(\bar{0})$ and $e_2 = w_h(\langle 0, 2\rangle)$. We will show that $g(o, e_2) = 2$ in $\mathfrak{G}$ while $g(o, e_2) = 1$ in $\mathfrak{G}^+$.

We have $g(o, e_2) = 2$ in $\mathfrak{G}$ because every observer measures the distance oe_2 to be 2 (since there is no observer of first-kind who sees o, e_2 on an axis). In $\mathfrak{G}$ we have $\bar{t} \perp_r \ell_2$ because ℓ_2 is the limit of space-axes of observers with life-line $\bar{t}$. Let $e_0 = w_m(1_t)$. Then in $\mathfrak{G}$ we have $\langle o, e_0\rangle\ eq\ \langle o, e_2\rangle$ because this comes in by transitivity: $e_3 = w_m(\langle 0, 1) = w_k(\langle 0, 2\rangle)$ and $e_1 = w_k(1_t) = w_h 1_t$ (thus $\langle o, e_0\rangle\ eq_0\ \langle o, e_3\rangle\ eq_0\ \langle o, e_1\rangle\ eq_0\ \langle o, e_2\rangle$ according to observers m, k, h respectively). Therefore we will have a new observer m^+ in $\mathcal{M}(\mathfrak{G})$, one whose life-line is $\bar{t}$, space-axis is ℓ_2, time-unit is e_0 and space-unit is e_2. This observer m^+ will measure the distance o, e_2 to be 1, and therefore $g(o, e_2) = 1$ in $\mathfrak{G}^+$.

[648] We identified ℓ_1 and ℓ_2 in the figure with their images according to w_m.

This model $\mathfrak{M}$ is not hard to modify to be a model of $\mathbf{Pax}^+ + \mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}$ with $\mathfrak{G} \not\cong \mathfrak{G}^+$. This shows that theorem schema (D) is not true for $Th = \mathbf{Pax}^+$ or for $Th = \mathbf{Pax}^+ + \mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}$. By using similar models, one can show that theorem schemas (C) and (A) are not true for $\mathbf{Pax}^+ + \mathbf{Ax(eqm)}$ and theorem schema (C) is not true for $\mathbf{Pax}^+ + \mathbf{Ax}\heartsuit + \mathbf{Ax(ext)} + \mathbf{Ax(eqm)}$. ◁

The next proposition says that for certain choices of Th, if $\mathfrak{G}$ is a Th-geometry then $\mathcal{M}(\mathfrak{G})$ is a Th-model. More intuitively, our duality theory works for these choices of Th.

PROPOSITION 4.5.47

$$\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th) \quad \textit{and} \quad \mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th),^{649}$$

assuming

$$Th := Th_1 + \mathbf{Pax}^+,$$

where $Th_1 \in \{\emptyset,\ \mathbf{Bax}^{-\oplus},\ \mathbf{Bax}^{\oplus} + \mathbf{Ax}(||)^- + \mathbf{Ax}(\sqrt{\ }) + Th_2,\ \mathbf{Flxbasax} + Th_2,\ \mathbf{Newbasax} + Th_2,\ \mathbf{Basax} + Th_2,\ \mathbf{Basax} + \mathbf{Ax}(\omega)^0 + Th_2\}$, *where* $Th_2 \subseteq \{\mathbf{Ax}(\mathit{Triv}),\ \mathbf{Ax}(\mathit{Triv}_t)^-,\ \mathbf{Ax}(||)\}$.

Further, for these choices of Th and for $\mathcal{M}$ defined in Def.4.5.38 conclusions (i)–(iii) of Thm.4.5.13 (p.291) hold when $\mathbf{Pax}^+$ *is replaced by Th in them.*

On the proof: We will give a proof for the case $Th_1 = \mathbf{Bax}^{-\oplus}$ and $n > 2$. The proofs for the remaining cases can be obtained by Remark 4.2.52 (ii) (p.196), and Propositions 6.2.88 (p.895) and 6.2.92 (p.901) of AMN [18], and are left to the reader.

Assume $n > 2$. Let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Pax}^+)$. Then $\mathcal{M}(\mathfrak{G}) \in \mathsf{Ge}(\mathbf{Pax}^+)$ by Prop.4.5.41. Thus to prove $\mathcal{M}(\mathfrak{G}) \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Pax}^+)$ it remains to prove $(*)$ below.

$(*)$ In the world-view of any observer $m \in Obs^{\mathcal{M}(\mathfrak{G})}$ for any point p and for any direction d the following holds. There is exactly one photon trace forwards in direction d passing through p and the "speed of this photon trace" is not ∞; and for all speeds less than the speed of this photon trace there is an observer moving in direction d with this speed and passing through point p.

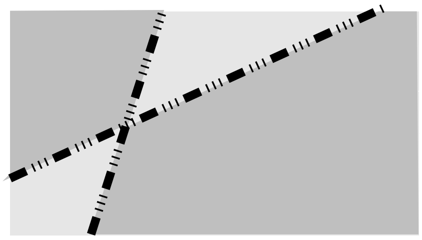
Throughout the proof we tacitly use Prop.4.2.64 (p.208). Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{-\oplus} + \mathbf{Pax}^+)$ be such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Without loss of generality we may assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$.[650] Let $m \in Obs^{\mathcal{M}}(\mathfrak{G})$. Then $m = \langle o, e_0, \dots, e_{n-1}\rangle$ for some $o, e_0, \dots, e_{n-1} \in Mn$ satisfying (a)–(f) on p.310. Let $\ell_0 \in L^T$ be such that $o, e_0 \in \ell_0$. Let P be defined as in item (e) on p.310. Intuitively P is the space part of observer m. We claim that there are no photon-like lines in P. To prove this claim, assume that there is a photon-like line in P. Then, by Thm.4.3.17 (p.488) of AMN [18], there is $\ell \in L^{Ph}$ such that $o \in \ell \subseteq P$. Let this ℓ be fixed. Then by item (e) on p.310 there is exactly one photon-like line in the plane determined by ℓ and ℓ_0 passing through o. ℓ_0 is the life-line of some observer $k \in Obs^{\mathfrak{M}}$, i.e. $\ell_0 = \{e \in Mn : k \in e\}$. Let this k be fixed. Then, since $\mathfrak{M} \models \mathbf{Bax}^-$, and since there is only one photon-like line in the plane determined by ℓ and ℓ_0 passing through o we conclude that for k the photon whose life-line is

[649] The $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ part is easy by the definition of $\mathsf{Ge}(Th)$, so the emphasis is on the $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$ part.

[650] This is so since $\mathcal{M}$ preserves the property of being isomorphic as we already noted.

ℓ moves with infinite speed. This contradicts "$\oplus$", i.e. contradicts $\mathbf{Bax}^{-\oplus}$. Thus there are no photon-like lines in P.

Now, we turn to proving $(*)$ above for m and for $p = \bar{0}$. Let P' be a 2-dimensional plane that contains ℓ_0. Since $\mathfrak{M} \models \mathbf{Bax}^{-\oplus}$ and the life-line of $k \in Obs^{\mathfrak{M}}$ is ℓ_0 there are exactly two photon-like lines in P' passing through o. These two photon-like lines divide the plane P' into two regions as illustrated below.



Let ℓ_P be the intersection of P' and P. Neither one of the two photon-like lines coincides with ℓ_P since in P there are no photon-like lines. We will prove that ℓ_P and ℓ_0 are in different regions. Assume that ℓ_P and ℓ_0 are in the same region. See the left-hand side of Figure 116. Then, since $\mathfrak{M} \models \mathbf{Bax}^-$ and the life-line of k is ℓ_0, we conclude that k sees an observer h on

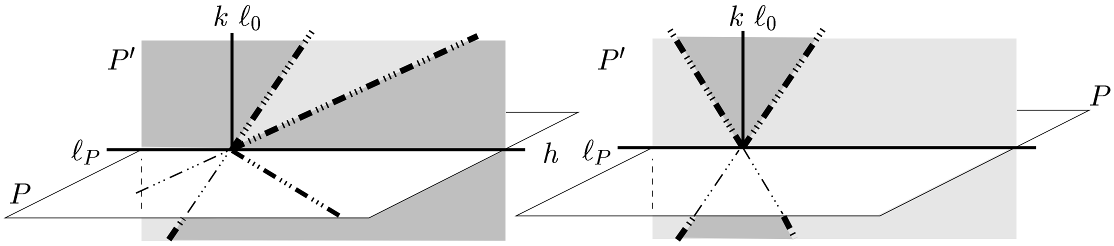


Figure 116:

ℓ_P, i.e. ℓ_P is the life-line of observer $h \in Obs^{\mathfrak{M}}$. Since through any point and in any direction h sees a photon and h's life-line ℓ_P is contained in P we conclude that there is a photon-like line in P. This leads to a contradiction since we proved that there are no photon-like lines in P. Thus, ℓ_0 and ℓ_P are in different regions, cf. the right-hand side of Figure 116. Then any line in the same region as ℓ_0 passing through o is time-like. This can be proved by using the world-view of observer k. But then it can be seen that any line in the same region as ℓ_0 passing through o is a "life-line" of an observer in the model $\mathfrak{M}^{\mathcal{M}(\mathfrak{G})}$, too.[651] Thus we proved that $(*)$ above holds for m and for $p = \bar{0}$. Since, by Thm.4.3.17 (p.488) of AMN [18], straight lines parallel to traces of photons are traces of photons again and since any line parallel to a time-like line is a time-like line by **Ax4**, we conclude that $(*)$ above holds for arbitrary p and not only for $\bar{0}$. ∎

QUESTION 4.5.48 *Does Proposition 4.5.47 above generalize from* $Th_1 = \mathbf{Bax}^{-\oplus}$ *to* $Th_1 = \mathbf{Bax}^-$*?*

$\lhd$

The next proposition says that the operator $\mathcal{G} \circ \mathcal{M}$ makes our models more "puritan" in some sense.

[651] All observers of $\mathfrak{M}$ show up in $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$ in a modified form.

PROPOSITION 4.5.49 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^+)$. *Then*

$$(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \models \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit.$$

We omit the easy **proof.** ∎

It might be interesting to notice that by the above proposition some of the conditions of the categoricity theorem (Thm.3.8.7 on p.299 of AMN [18]) become true in $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$.

Question for future research 4.5.50 It would be interesting to see for which reduct of $\mathfrak{G}_{\mathfrak{M}}$ does the above outlined duality theory still go through. We note that in §4.5.4 we will have an analogous duality theory for the $(g, \mathcal{T})$-free reduct of our geometries.

◁

We close the present sub-section with Remark 4.5.51 below. Further theorems about $(\mathcal{G}, \mathcal{M})$-duality will be stated in §A.1 (p.A-1) and §A.2 (p.A-6).

The following remark shows how to remove the condition **Ax(eqtime)** (or **Ax(eqm)**) from our duality theory $(\mathcal{G}, \mathcal{M})$, i.e. how to reconstruct $\mathfrak{M}$ (at least a version of $\mathfrak{M}$) from the geometry $\mathfrak{G}_{\mathfrak{M}}$ even if **Ax(eqtime)** is not assumed.

Remark 4.5.51 On a possible *more* general function $\mathcal{M}^+$: Geometries → Models (not requiring the whole of $\mathbf{Pax}^+$ to be assumed before the definition):
(A) Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}+\mathbf{Ax(Bw)})$. Let $o, e \in Mn$ with $o \neq e$ and $o \sim e$. Let $\mathfrak{F}_{oe} = \langle F_{oe}, \ldots \rangle$ be the ordered field corresponding to o, e as defined Def.4.5.28. An element a of F_{oe} is called positive iff $o \leq_{oe} a$ and $o \neq a$, as one would expect. Consider the possible properties (i), (ii) below.

(i) $(\forall$ positive $a, b \in F_{oe})$ $[a \neq b \;\Rightarrow\; g(o,a) \neq g(o,b)]$.

Let $g_{oe} : F_{oe} \longrightarrow F$ be defined by

$$g_{oe}(a) :\overset{\text{def}}{=} \begin{cases} g(o,a) & \text{if } a \text{ is positive} \\ -g(o,a) & \text{otherwise.}^{652} \end{cases}$$

(ii) $g_{oe} : \langle F_{oe}; o, e, +_{oe}, \leq_{oe} \rangle \longrightarrow \mathbf{F_1}$ is an isomorphism.

If (i) and (ii) hold for $o, e \in Mn$ with $o \neq e$ and $o \sim e$ then we say that g is nice on F_{oe}.
Question for future research: Do we need (ii) or is (i) enough? That is, is (i) ⇒ (ii) true in *some* sense?
Definition of $\mathcal{M}^+(\mathfrak{G})$: We distinguish two cases.
Case (I): Assume $\mathfrak{G}$ is such that g is nice on *some* F_{oe}. Then we define multiplication "·" on F as follows.

$$\cdot(x,y,z)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists o, e \in Mn)\Big[o \neq e \;\wedge\; o \sim e \;\wedge\; (g \text{ is nice on } F_{oe}) \;\wedge$$
$$g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)\Big].$$

652 This g_{oe} is the same as the g_{oe} on p.312.

Then we construct $\mathcal{M}^+(\mathfrak{G})$ the same way as $\mathcal{M}(\mathfrak{G})$ was constructed, except that we do not require item (f) to hold in the definition of Obs, i.e.

$$Obs := \left\{ \langle o, e_0, \ldots, e_{n-1} \rangle \in {}^{n+1}Mn \;:\; \text{(a)–(e) hold on p.310} \right\}$$

and $\mathfrak{F} := \langle F, +, \cdot, \leq \rangle$, where $\cdot$ is defined above. At the end of the remark we will prove that

$$(\clubsuit) \qquad \mathfrak{F} \text{ is an ordered field.}$$

The rest of the ingredients of $\mathcal{M}^+(\mathfrak{G})$ are defined exactly as those of $\mathcal{M}(\mathfrak{G})$.

<u>*Case (II):*</u> Assume that for any $o, e \in Mn$ with $o \neq e$ and $o \sim e$, g is *not nice* on F_{oe}. Then we throw g away and use an arbitrary $o, e \in Mn$ with $o \neq e$ and $o \sim e$ and an arbitrary isomorphism[653] $i : \langle F_{oe}; o, e, +_{oe}, \leq_{oe} \rangle \longrightarrow \mathbf{F_1}$ to copy the multiplication $\cdot_{oe}$ of $\mathfrak{F}_{oe}$ to F obtaining an ordered field $\mathfrak{F}$. The rest of $\mathcal{M}^+(\mathfrak{G})$ is defined as in Case (I).

We note that in Case (II) $\mathcal{M}^+(\mathfrak{G})$ is not first-order definable over $\mathfrak{G}$ in general while in Case (I) $\mathcal{M}^+(\mathfrak{G})$ is first-order definable over $\mathfrak{G}$.

Now, we conjecture that the theorems stated for $\mathcal{M}$ go through for $\mathcal{M}^+$ with very little change (and the same conditions). Further we guess that some simple theorems like $(\mathcal{G} \circ \mathcal{M}^+)^2(\mathfrak{M}) \cong (\mathcal{G} \circ \mathcal{M}^+)(\mathfrak{M})$ will be true, for $\mathfrak{M} \models \mathbf{Pax} + \mathbf{Ax(Bw)}$.

(B) Item (A) above suggests the <u>following possibility</u> for improving/generalizing our $(\mathcal{G}, \mathcal{M})$-duality theory. First, one formulates an axiom in our frame language which implies about $\mathfrak{M}$ that in $\mathfrak{G}_{\mathfrak{M}}$ g is nice on some F_{oe}, assuming **Pax**. Let us notice that there exist very mild choices for such an axiom, e.g. **Ax(mild)** below is such. We note that **Ax(mild)** is much weaker than $\mathbf{Ax(eqtime)} \vee \mathbf{Ax(eqm)}$, assuming e.g. $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ })$ and $n > 2$.

Ax(mild) $(\exists m \in Obs)(\exists i \in n)\,[(\forall ph \in Ph) tr_m(ph) \neq \bar{x}_i \;\wedge\; (\forall p, q \in \bar{x}_i)(\forall k \in Obs)$
(the distance between events $w_m(p)$ and $w_m(q)$ as measured by k is not smaller than the distance between these two events as measured by m, i.e. if k sees both $w_m(p)$ and $w_m(q)$ on the same coordinate axis then the distance between $w_m(p)$ and $w_m(q)$ as measured by k is not smaller than $|p - q|$)].

Then, one can obtain a duality theory (between frame models and geometries) in which one uses the milder **Ax(mild)** in place of **Ax(eqtime)**. I.e. one defines a first-order definable meta-function $\mathcal{M}^* : \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)} + \mathbf{Ax(mild)}) \longrightarrow \mathsf{FM}$ exactly as $\mathcal{M}^+$ was defined in item (A) for Case (I).

<u>*Proof of ($\clubsuit$):*</u> Now we turn to proving that $\mathfrak{F}$ defined in Case (I) is an ordered field. Let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)})$ be such that g is nice on some F_{oe} and let "$\cdot$" and $\mathfrak{F}$ be defined as in Case (I) above. Then there is $\mathfrak{M} \models \mathbf{Pax} + \mathbf{Ax(Bw)}$ such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Let this $\mathfrak{M}$ be fixed. Without loss of generality we may assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$. Hence $\mathbf{F_1} = \mathbf{F_1^{\mathfrak{M}}}$. To avoid ambiguity we will denote the multiplication of the ordered field $\mathfrak{F}^{\mathfrak{M}}$ by "$*$" (instead of the usual "$\cdot$"). To prove that $\mathfrak{F}$ defined in Case (I) above is an ordered field it is enough to prove that $\cdot$ and $*$ coincide, i.e.

$$(\star) \qquad (\forall x, y, z \in F)\,(\,\cdot(x, y, z) \;\Leftrightarrow\; x * y = z\,).$$

Let $o, e \in Mn$ be fixed such that $o \neq e$ and $o \sim e$. Observer m is called <u>*good for F_{oe}*</u> iff m sees F_{oe} on a coordinate axis (i.e. $w_m[\bar{x}_i] = F_{oe}$ for some $i \in n$) and the distance between o and e

[653] It can be proved that $\langle F_{oe}; o, e, +_{oe}, \leq_{oe} \rangle$ is isomorphic with $\mathbf{F_1}$.

as measured by m is 1 (i.e. $|w_m^{-1}(e) - w_m^{-1}(o)| = 1$). For every observer m which sees F_{oe} on a coordinate axis we define a function $g^m : F_{oe} \longrightarrow F$ as follows. Intuitively, $g^m(a)$ will be the signed distance between o and a as measured by m. Let $m \in Obs$ be such that m sees F_{oe} on a coordinate axis. Let $a \in F_{oe}$. Then

$$g^m(a) :\stackrel{\text{def}}{=} \begin{cases} \ \ |w_m^{-1}(a) - w_m^{-1}(o)| & \text{if } a \text{ is positive} \\ -|w_m^{-1}(a) - w_m^{-1}(o)| & \text{otherwise.} \end{cases}$$

By Thm.3.2.6 (p.110), it is easy to see that

($\star\star$) $g^m : \langle F_{oe};\ o, +_{oe}, \leq_{oe}\rangle \rightarrowtail\!\!\!\!\twoheadrightarrow \langle F;\ 0, +, \leq\rangle$ is an isomorphism and if m is good for F_{oe} then

$$g^m : \mathfrak{F}_{oe} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{F}^{\mathfrak{M}}$$

is an isomorphism.

Claim **4.5.52** Assume that g is nice on F_{oe}. Then for every $x, y, z \in F$ there is an observer m such that m is good for F_{oe}, $(g^m)^{-1}(x) = g_{oe}^{-1}(x)$, $(g^m)^{-1}(y) = g_{oe}^{-1}(y)$, and $(g^m)^{-1}(z) = g_{oe}^{-1}(z)$.

Proof: Assume g is nice on F_{oe}. To prove the claim it is enough to prove that for every $a, b, c \in F_{oe}$ there is an observer m such that m is good for F_{oe} and $g^m(a) = g_{oe}(a)$, $g^m(b) = g_{oe}(b)$, $g^m(c) = g_{oe}(c)$. Let $a, b, c \in F_{oe}$. For every $f \in F_{oe}$ by $-_{oe}f$ we denote the inverse of f taken in the group $\langle F_{oe};\ o, +_{oe}\rangle$. Since for every $f \in F_{oe}$, $g^m(-_{oe}f) = -g^m(f)$ and $g_{oe}(-_{oe}f) = -g_{oe}(f)$ without loss of generality we may assume that a, b, c are non-negative, i.e. that $o \leq_{oe} a$ etc. Let

$$d := e +_{oe} a +_{oe} b +_{oe} c.$$

Let $m \in Obs$ be such that m sees F_{oe} on a coordinate axis and the distance between o and d as measured by m is $g(o, d)$, formally $|w_m^{-1}(d) - w_m^{-1}(o)| = g(o, d)$. Such an m exists by the definition of g. Hence,

$$g^m(d) = g_{oe}(d).$$

By ($\star\star$),

$$g^m(d) = g^m(e) + g^m(a) + g^m(b) + g^m(c),$$

and $g^m(e)$, $g^m(a)$, $g^m(b)$, $g^m(c)$ are non-negative. Further, (since g_{oe} is nice on F_{oe}) we have,

$$g_{oe}(d) = g_{oe}(e) + g_{oe}(a) + g_{oe}(b) + g_{oe}(c),$$

and $g_{oe}(e)$, $g_{oe}(a)$, $g_{oe}(b)$, $g_{oe}(c)$ are non-negative. Further,

$$g_{oe}(e) \leq g^m(e), \quad g_{oe}(a) \leq g^m(a), \quad g_{oe}(b) \leq g^m(b), \quad g_{oe}(c) \leq g^m(c)$$

by the definitions of g, g_{oe}, g^m (i.e. by the fact that for every positive $f \in F_{oe}$ $g^m(f)$ is the distance between o and f as measured by m while $g_{oe}(f)$ is the minimum of the distances between o and f measured by observers who see F_{oe} on a coordinate axis). Therefore, $|w_m^{-1}(e) - w_m^{-1}(o)| =: g^m(e) = g_{oe}(e) = 1$, $g^m(a) = g_{oe}(a)$, etc., i.e. observer m has the desired properties.

(QED Claim 4.5.52)

Now, we turn to proving ($\star$) above. Let $x, y, z \in F$.

Proof of direction "$\Rightarrow$": Assume $\cdot(x, y, z)$. Then there are $o, e \in Mn$ such that $o \neq e$, $o \sim e$, g is nice on F_{oe} and $g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)$. Let such o, e be fixed. Then, by Claim 4.5.52, there is an observer m such that m is good for F_{oe}, $(g^m)^{-1}(x) = g_{oe}^{-1}(x)$, $(g^m)^{-1}(y) = g_{oe}^{-1}(y)$, and $(g^m)^{-1}(z) = g_{oe}^{-1}(z)$. Let this m be fixed. Now, $(g^m)^{-1}(z) = (g^m)^{-1}(x) \cdot_{oe} (g^m)^{-1}(y)$. Thus, by the second part of $(\star\star)$, $z = x * y$.

Proof of direction "$\Leftarrow$": Assume $z = x * y$. Let $o, e \in Mn$ be such that g is nice on F_{oe} (and, of course, $o \neq e$, $o \sim e$). If $m \in Obs$ is good for F_{oe} then by $z = x * y$ and $(\star\star)$, we have $(g^m)^{-1}(z) = (g^m)^{-1}(x) \cdot_{oe} (g^m)^{-1}(y)$. By Claim 4.5.52 there is m such that m is good for o, e, $(g^m)^{-1}(x) = g_{oe}^{-1}(x)$, $(g^m)^{-1}(y) = g_{oe}^{-1}(y)$, and $(g^m)^{-1}(z) = g_{oe}^{-1}(z)$. Therefore $g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)$. Hence $\cdot(x, y, z)$ and this completes the proof of $(\clubsuit)$.

$\triangleleft$

4.5.4 Duality theory for the $(g, \mathcal{T})$-free reducts of our geometries

Motivation for looking at reducts of our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$ is given in §4.5.6 (p.341) and in the introduction of §6.7 ("Interdefinability ...") in AMN [18, pp. 1134–1135]. A further motivation for the physicist might be that depending on which aspect of the physical world we want to concentrate on we will "see" different reducts[654] of our $\mathfrak{G}_{\mathfrak{M}}$.

The main message of our $(\mathcal{G}, \mathcal{M})$-duality is that we can reconstruct the original observational model $\mathfrak{M}$ from the streamlined, more abstract geometry $\mathfrak{G}_{\mathfrak{M}}$ associated with it (under some conditions of course). So, we *do not lose information* if we move from the "detail-rich" world $\mathfrak{M}$ to the geometry abstracted from it. The question naturally comes up: How much of $\mathfrak{G}_{\mathfrak{M}}$ is needed for this reconstruction? In other words, from which reducts of $\mathfrak{G}_{\mathfrak{M}}$ is our "original world" $\mathfrak{M}$ reconstructible? Of course, if we take a too small reduct e.g. $\langle Mn, L; \in \rangle$ then we will not be able to reconstruct $\mathfrak{M}$ from this reduct. Below we will see that if we omit g and $\mathcal{T}$ from $\mathfrak{G}_{\mathfrak{M}}$ then $\mathfrak{M}$ remains reconstructible from this weaker geometry $\mathfrak{G}^0_{\mathfrak{M}} = \langle Mn, \ldots, eq \rangle$, under some conditions.[655] We will do more than just reconstructing $\mathfrak{M}$ from $\mathfrak{G}^0_{\mathfrak{M}}$, namely, we will elaborate a duality theory (analogous to our original one) between $\mathsf{Mod}(Th)$ and our weaker geometries.[656]

In more detail: In the present sub-section we will see that even if we omit g from our geometries we can still develop a duality theory between geometries and models. As a contrast, later (in §4.5.6) we will see that we cannot omit much more from our geometries without losing the possibility for building a (similarly strong) duality theory.

The present duality theory will be more symmetric than the previous one $(\mathcal{M}, \mathcal{G})$, namely, in the new duality the geometries will be axiomatically defined just as the frame models are, cf. the text below Thm.4.5.13 on p.293.

At the same time, we note that at least from a certain point of view, the new duality will involve losing (or forgetting) a bit more "information" than in the case of $(\mathcal{M}, \mathcal{G})$. Namely,

[654] e.g. we may want to concentrate on the so-called conformal structure (i.e. the light-cones) of space-time, or we may want to concentrate on orthogonality, or on the metric g etc.

[655] A price we will have to pay for omitting g is that we will have to add **Ax6** to our assumptions.

[656] We leave it, partially, to the reader to decide exactly which other reducts of $\mathfrak{G}_{\mathfrak{M}}$ are strong enough so that $\mathfrak{M}$ is recoverable from them. In other words: which reducts of $\mathfrak{G}_{\mathfrak{M}}$ are strong enough to support a duality theory analogous to $(\mathcal{G}, \mathcal{M})$-duality and the one below. Cf. also item 4.5.50 (p.322). In §4.5.6 and §4.6 we will obtain some partial information in this direction.

under some assumptions,

$$\mathfrak{M} \models \mathbf{Ax(eqtime)} \quad \Rightarrow \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \models \mathbf{Ax(eqtime)}.$$

I.e. $\mathcal{G} \circ \mathcal{M}$ "preserves" **Ax(eqtime)**. This property will be lost in the case of the new duality. (This can be sometimes be an advantage and some other times a disadvantage).

Definition 4.5.53

(i) For every frame model $\mathfrak{M}$, $\mathfrak{G}^0_{\mathfrak{M}}$ is defined to be the $(g, \mathcal{T})$-free reduct of $\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \ldots \rangle$, i.e.

$$\mathfrak{G}^0_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq \rangle.$$

(ii) For any set Th of formulas in our frame language the corresponding class $\mathsf{Ge}^0(Th)$ of geometries is defined as follows.

$$\mathsf{Ge}^0(Th) :\stackrel{\text{def}}{=} \left\{ \mathfrak{G} \ :\ (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\, \mathfrak{G}^0_{\mathfrak{M}} \cong \mathfrak{G} \right\}.$$

(iii) GEO is defined to be the class of all structures of the <u>similarity type of $\mathsf{Ge}^0(\emptyset)$</u> in which the axiom of extensionality holds for the incidence relation $\in$ ($\in \subseteq Mn \times L$). Because of this, without loss of generality we may assume that our incidence relation is the real set theoretic $\in$. Actually throughout we will assume this.

(iv) For any set TH of formulas in the language of GEO

$$\mathsf{Mog}(TH) :\stackrel{\text{def}}{=} \{\mathfrak{G} \in \mathsf{GEO} \ :\ \mathfrak{G} \models TH\}.$$ [657]

We introduce axioms $\mathbf{L_1}$ and $\mathbf{L_2}$ in the language of GEO. We use the abbreviation *coll* introduced in item 4.2.12 and the new sort *lines* which is first-order defined from *coll* (and Mn) on p.297. Axioms $\mathbf{L_1}$, $\mathbf{L_2}$ below state that L-lines are also *lines*-lines, and that any point is the intersection of two photon-like lines.

$\mathbf{L_1}$ $L \subseteq$ *lines*.

(This is one of the places where we heavily use the assumption in Def.4.5.53(iii), i.e. that the geometric incidence relation is the set theoretic $\in$. Of course the axiom could be formulated without relying on this assumption, but then it would become longer.)

$\mathbf{L_2}$ $(\forall a \in Mn)(\exists \ell, \ell' \in L^{Ph})\, \ell \cap \ell' = \{a\}$.

Recall that **opag** is the axiom system for ordered Pappian affine geometries defined on p.302 in Def.4.5.24.

Definition 4.5.54 $\quad \mathbf{lopag} :\stackrel{\text{def}}{=} \mathbf{opag} + \mathbf{L_1} + \mathbf{L_2}$.

$\lhd$

[657] Since TH is a theory and $\mathsf{Mog}(TH)$ consists of the models of that theory we could have used the notation $\mathsf{Mod}(TH)$ in place of $\mathsf{Mog}(TH)$. However we wanted to emphasize that the language of our present TH is the geometric language of GEO. Therefore the models of TH will be geometries. To emphasize this we use the notation $\mathsf{Mog}(TH)$ to remind the reader that the language is now that of geometries.

In the following definition we define the functors $\mathcal{G}o$ and $\mathcal{M}o$ connecting the two worlds $\mathsf{Mod}(\ldots)$ and $\mathsf{Mog}(\mathbf{lopag})$; according to the pattern

$$\mathsf{Mod}(\ldots) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad \mathsf{Mog}(\mathbf{lopag})$$

and more generally

$$\mathsf{Mod}(\mathit{Th}) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad \mathsf{Mog}(\mathit{TH}),$$

where Th and TH are in two different languages.

Much of the intuitive idea for the definition of $\mathcal{M}$ on p.309 applies to the definition of $\mathcal{M}o$ given below.

Definition 4.5.55 (functors $\mathcal{G}o$ and $\mathcal{M}o$)

(i) We define the functor $\mathcal{G}o : \mathsf{FM} \longrightarrow \mathsf{GEO}$ to be the function $\mathfrak{M} \mapsto \mathfrak{G}^0_{\mathfrak{M}}$.

(ii) We define the functor $\mathcal{M}o : \mathsf{Mog}(\mathbf{lopag}) \longrightarrow \mathsf{FM}$ as follows. Let $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$. Then the model

$\mathcal{M}o(\mathfrak{G}) = \langle (B;\ \mathit{Obs}, \mathit{Ph}, \mathit{Ib}), \mathfrak{F}, \mathsf{Eucl}(\mathfrak{F});\ \in, W \rangle$ is defined as follows.

$\mathit{Obs} :\stackrel{\text{def}}{=} \{\, \langle o, e_0, \ldots, e_{n-1} \rangle \in {}^{n+1}\mathit{Mn} \ :\ \text{(a)–(e) on p.310 hold} \,\}$.

If $\mathit{Obs} = \emptyset$, then $\mathcal{M}o(\mathfrak{G})$ is defined to be the empty model, otherwise the rest of the ingredients of $\mathcal{M}o(\mathfrak{G})$ are defined as follows.

$\mathit{Ph} :\stackrel{\text{def}}{=} L^{\mathit{Ph}}$.

$B :\stackrel{\text{def}}{=} \mathit{Ib} \ :\stackrel{\text{def}}{=} \mathit{Obs} \cup \mathit{Ph}$.

$\mathfrak{F} = \langle F;\ \ldots \rangle$ is the ordered field corresponding to $\langle \mathit{Mn};\ \mathit{Bw} \rangle$ defined in Def.4.5.31 (p.306).

For every $\langle o, e_0, \ldots, e_{n-1} \rangle \in \mathit{Obs}$ the coordinatization

$$\mathit{Co}_{\langle o, e_0, \ldots, e_{n-1} \rangle} : \mathit{Mn} \longrightarrow {}^{n}F$$

is defined in Def.4.5.34 (p.307). By Prop.4.5.35, we have that these coordinatizations are bijections. For every $m = \langle o, e_0, \ldots, e_{n-1} \rangle \in \mathit{Obs}$, we define

$$w^0_m :\stackrel{\text{def}}{=} \mathit{Co}^{-1}_{\langle o, e_0, \ldots, e_{n-1} \rangle}.$$

Now the world-view relation W is defined from the functions w^0_m's exactly as in Def.4.5.38. Let $m \in \mathit{Obs}$ and $p \in {}^{n}F$. Then

$w_m(p) :\stackrel{\text{def}}{=}$
$\{\, \ell \in \mathit{Ph} \ :\ w^0_m(p) \in \ell \,\} \cup \{\, \langle o, e_0, \ldots, e_{n-1} \rangle \in \mathit{Obs} \ :\ \mathit{coll}(o, e_0, w^0_m(p)) \,\}$.[658]

W is defined from the w_m's the obvious way, i.e.

$$W :\stackrel{\text{def}}{=} \{\, \langle m, p, b \rangle \in \mathit{Obs} \times {}^{n}F \times B \ :\ b \in w_m(p) \,\}.$$

[658] For a more intuitive (but longer) formula defining w_m cf. the definition of $\mathcal{M}$, p.312.

$\lhd$

Now we introduce the axiom system **Wax** in our frame language which will nicely "match" with the geometrical axiom system **lopag**. **Ax(**Ph**)** below is one of the axioms of **Wax**.

Ax(Ph**)** $(\forall m \in Obs)(\forall p \in {}^nF)(\exists ph_1, ph_2 \in Ph)\ tr_m(ph_1) \cap tr_m(ph_2) = \{p\}$.

Intuitively, each observer at any point p sees at least two photons, and these two photons do not meet at any point different from p.

Definition 4.5.56 **Wax** $:\stackrel{\text{def}}{=} \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax6}, \mathbf{Ax(Bw)}, \mathbf{Ax}(Ph)\}$.

$\lhd$

We note that the following "weak" axiom systems are stronger than **Wax**. $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6}$, $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(Bw)} + \mathbf{Ax6}$, $\mathbf{Pax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Ph) + \mathbf{Ax6}$, $\mathbf{Pax} + \mathbf{Ax(Bw)} + \mathbf{Ax}(Ph) + \mathbf{Ax6}$; and if $n > 2$ $\mathbf{Bax}^{-}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6}$, $\mathbf{Bax}^{-}(n) + \mathbf{Ax(Bw)} + \mathbf{Ax6}$, $\mathbf{Bax}(n) + \mathbf{Ax6}$.

Item (ii) of the following theorem is of the pattern of theorem-schemas (G), (H) on p.286 way above. (Cf. Thm.4.5.13 for a similar theorem.) The whole theorem is of the pattern

$$\mathsf{Mod}(\mathbf{Wax}) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad \mathsf{Mog}(\mathbf{lopag}).$$

THEOREM 4.5.57

(i)

$$\mathcal{G}o : \mathsf{Mod}(\mathbf{Wax}) \longrightarrow \mathsf{Mog}(\mathbf{lopag}), \qquad \mathcal{M}o : \mathsf{Mog}(\mathbf{lopag}) \longrightarrow \mathsf{Mod}(\mathbf{Wax}),$$

and $\mathcal{M}o$ is a first-order definable meta-function.

(ii) *Both $\mathcal{G}o \circ \mathcal{M}o$ and $\mathcal{M}o \circ \mathcal{G}o$ have fixed-point property in the sense that for any $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax})$ and $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$*

$$(\mathcal{G}o \circ \mathcal{M}o)^2(\mathfrak{M}) \cong (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \quad \text{and} \quad (\mathcal{M}o \circ \mathcal{G}o)^2(\mathfrak{G}) \cong (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}).$$

(iii) *For any $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax})$ and $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$*

$$\mathcal{G}o(\mathfrak{M}) \longleftarrow\!\!\prec (\mathcal{M}o \circ \mathcal{G}o)(\mathcal{G}o(\mathfrak{M})) \quad \text{and} \quad \mathcal{M}o(\mathfrak{G}) \succ\!\!\longrightarrow (\mathcal{G}o \circ \mathcal{M}o)(\mathcal{M}o(\mathfrak{G})).$$

We omit the **proof**, but cf. the proof of Thm.4.5.43. ∎

Galois connections will be introduced on p.A-3, §A.1. Motivated by the above theorem we conjecture that there is a Galois connection between $Rng(\mathcal{G}o)$ and $Rng(\mathcal{M}o)$,[659] cf. Thm.A.1.10 (p.A-5). Actually, this Galois connection can be regarded as an adjoint situation (to be introduced on p.A-13) too according to the following pattern

$$Rng(\mathcal{M}o) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad Rng(\mathcal{G}o),$$

[659] To show that this is a Galois connection one has to define appropriate pre-orderings on the classes $Rng(\mathcal{G}o)$ and $Rng(\mathcal{M}o)$.

cf. Conjecture A.2.9 (p.A-13). Further, we conjecture that between $Rng(\mathcal{G}o \circ \mathcal{M}o)$ and $Rng(\mathcal{M}o \circ \mathcal{G}o)$ the same connection turns out to be an equivalence of categories (cf. p.A-13) of the pattern

$$Rng(\mathcal{G}o \circ \mathcal{M}o) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad Rng(\mathcal{M}o \circ \mathcal{G}o),$$

cf. Conjecture A.2.12 (p.A-14).

Conjecture 4.5.58 *We conjecture that*

$$\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Mog}(TH),$$

for certain natural choices of Th and TH. We note that these choices of Th we have in mind contain the axiom $(\forall m)(\forall h \in Exp)(\exists k)\mathsf{f}_{mk} = h$.[660]

Hint: Use the construction in the proof of Thm.4.3.38 (p.261) omitting of course any references to those parts of the geometry which do not exist in the present case, e.g. g.

$\triangleleft$

Further theorems in this line will be stated in the Appendix.

The following theorem says that the sentences in our frame language can be translated into sentences in the language of our relativistic geometries (not involving the function g and the topology $\mathcal{T}$) in a meaning preserving way, assuming **lopag** on both sides. (Cf. Thm.4.5.42 for a similar theorem.)

THEOREM 4.5.59 *There is a "natural" translation mapping*

$$T_{\mathcal{M}o} : Fm(\mathsf{FM}) \longrightarrow Fm(\mathsf{GEO})$$

such that for every $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$ *and sentence* $\varphi \in Fm(\mathsf{FM})$

$$\mathcal{M}o(\mathfrak{G}) \models \varphi \quad \Leftrightarrow \quad \mathfrak{G} \models T_{\mathcal{M}o}(\varphi).$$

Proof: The theorem follows by item (i) of Thm.4.5.57 and by Prop.4.3.41 (p.264).

∎

The next proposition says that the operators $\mathcal{G}o \circ \mathcal{M}o$ and $\mathcal{M}o \circ \mathcal{G}o$ make our models and geometries "smooth" in some sense. (Cf. Prop.4.5.49 for a similar proposition.) We already know, by Thm.4.5.57, that for any $\mathfrak{M} \models \mathbf{Wax}$ $(\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \models \mathbf{Wax}$. Item (i) of the proposition states that besides **Wax** some further axioms become true when $\mathcal{G}o \circ \mathcal{M}o$ is applied to $\mathfrak{M}$. A similar remark applies to **lopag** and item (ii) below.

PROPOSITION 4.5.60

(i) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax})$. *Then*

$$(\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \models \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + (\forall m, k)(\mathsf{f}_{mk} \in Aftr) +$$

$$+ \mathbf{Ax}(\infty ph) + (\forall m)(\forall h \in Exp)(\exists k)\mathsf{f}_{mk} = h.$$

[660] Intuitively, this means that there are arbitrarily large as well as arbitrarily small animals, cf. Remark 4.2.1 on p.458 of AMN [18].

(ii) *Assume* $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$. *Then*

$$(\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}) \models \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}},$$

where axioms $\mathbf{L_3}, \ldots, \mathbf{L_{10}}$ *are introduced below the present proposition.*

Moreover;

(iii)

$$\begin{array}{rcl} Rng(\mathcal{M}o) & \models & \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + (\forall m,k)(\mathsf{f}_{mk} \in Aftr) + \mathbf{Ax(}\infty ph\mathbf{)} \text{ and} \\ Rng(\mathcal{G}o) & \models & \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}}, \end{array}$$

where axioms $\mathbf{L_3}, \ldots, \mathbf{L_{10}}$ *are introduced below.*

We omit the **proof**. ∎

Now we turn to introducing axioms $\mathbf{L_3}, \ldots, \mathbf{L_{10}}$ in the language of GEO. These axioms are motivated by item (ii) of the above proposition and/or by contemplating the idea that they are very natural (it is hard to imagine a reasonable geometry in which one of them would fail).

$\mathbf{L_3}$ $([\,a \prec b \ \wedge\ (\,Bw(a,b,c) \vee Bw(a,c,b)\,)\,] \ \rightarrow\ a \prec c\,) \ \wedge$
$([\,a \prec b \ \wedge\ (\,Bw(c,a,b) \vee Bw(a,c,b)\,)\,] \ \rightarrow\ c \prec b\,)$.

Intuitively, Bw and $\prec$ are both kinds of orderings. The axiom says that these two are "in harmony". In particular if we know Bw on a line ℓ, and two points of ℓ are $\prec$-related then this fact induces a $\prec$-connection between any two other points of ℓ.

$\mathbf{L_4}$ Intuitively, eq is (very) symmetric, formally:
$\langle a,b\rangle\ eq\ \langle c,d\rangle \quad \rightarrow \quad (\,\langle c,d\rangle\ eq\ \langle a,b\rangle \ \wedge\ \langle b,a\rangle\ eq\ \langle c,d\rangle \ \wedge\ \langle a,a\rangle\ eq\ \langle c,c\rangle\,)$.

$\mathbf{L_5}$ eq is transitive, i.e.
$(\,\langle a,b\rangle\ eq\ \langle c,d\rangle \ \wedge\ \langle c,d\rangle\ eq\ \langle e,f\rangle\,) \quad \rightarrow \quad \langle a,b\rangle\ eq\ \langle e,f\rangle$.

$\mathbf{L_6}$ (For the intuitive meaning of this axiom see Fig.117.)
$(\forall \ell,\ell' \in L)(\forall o,e,e',a,a' \in Mn)\Big([\,\ell \cap \ell' = \{o\} \ \wedge\ e,a \in \ell \ \wedge\ e',a' \in \ell' \ \wedge$
$\langle e,e'\rangle \parallel \langle a,a'\rangle \ \wedge\ \langle o,e\rangle\ eq\ \langle o,e'\rangle\,] \quad \rightarrow \quad \langle o,a\rangle\ eq\ \langle o,a'\rangle\Big)$.

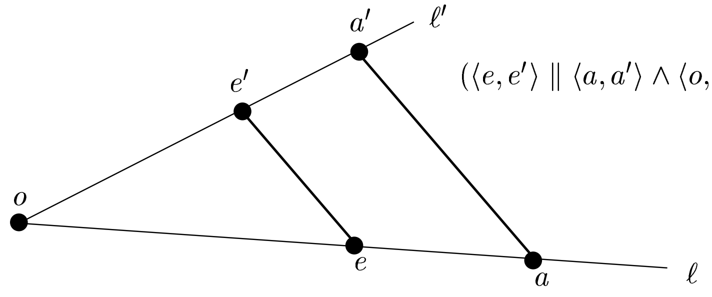


Figure 117: Axiom $\mathbf{L_6}$.

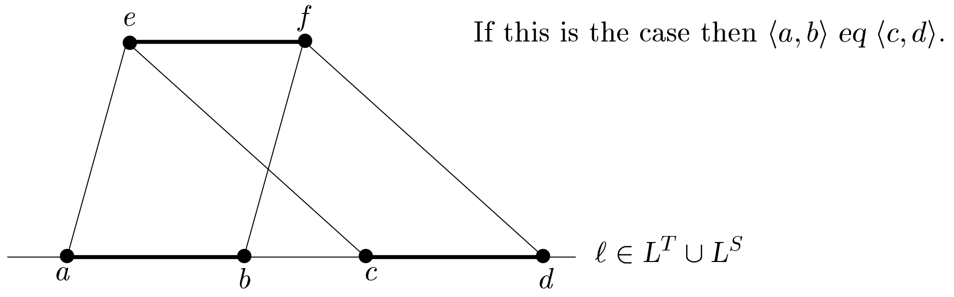


Figure 118: Axiom $\mathbf{L_7}$.

$\mathbf{L_7}$ (For the intuitive meaning of this axiom see Fig.118.)
$(\forall \ell \in L^T \cup L^S)(\forall a,b,c,d,e,f \in Mn)\,[\,(\,a,b,c,d \in \ell \;\wedge$
$\langle a,b\rangle \parallel \langle e,f\rangle \parallel \langle c,d\rangle \;\wedge\; \langle a,e\rangle \parallel \langle b,f\rangle \;\wedge\; \langle c,e\rangle \parallel \langle d,f\rangle\,) \;\rightarrow\; \langle a,b\rangle \; eq \; \langle c,d\rangle\,]$.

$\mathbf{L_8}$ $\perp_r$ is symmetric, i.e.
$(\forall \ell,\ell' \in L)\,(\ell \perp_r \ell' \;\rightarrow\; \ell' \perp_r \ell)$.

$\mathbf{L_9}$ $\perp_r$ is closed under parallelism, i.e.
$(\forall \ell,\ell_1,\ell_2 \in L)\,[\,(\,\ell \perp_r \ell_1 \;\wedge\; \ell_1 \parallel \ell_2\,) \quad\rightarrow\quad \ell \perp_r \ell_2\,]$.

$\mathbf{L_{10}}$ $\perp_r$ is closed under taking limits, i.e. $\perp_r$ satisfies item (ii) on p.141. This property can be formulated in the language of GEO as follows. (See Fig.119.)

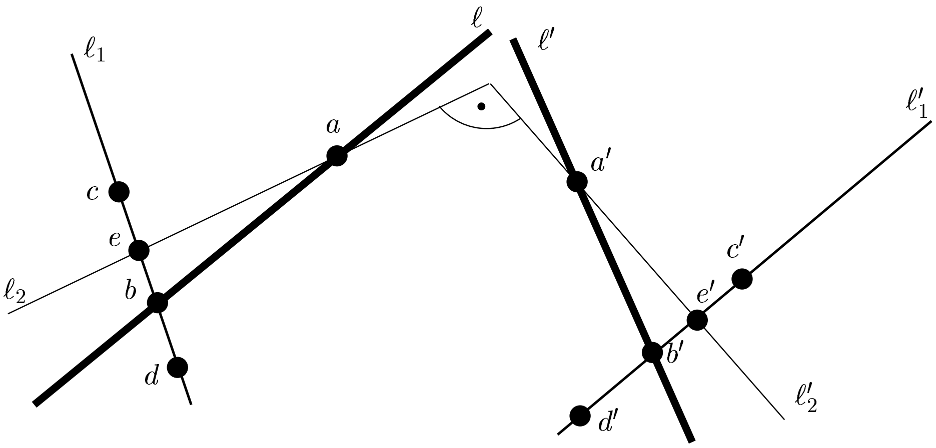


Figure 119: Illustration for axiom $\mathbf{L_{10}}$.

$(\forall \ell,\ell' \in L)\,\Big((\exists \text{ distinct } a,b \in \ell)(\exists \text{ distinct } a',b' \in \ell')(\exists \ell_1,\ell'_1 \in L)$
$\Big[\,\ell \cap \ell_1 = \{b\} \;\wedge\; \ell' \cap \ell'_1 = \{b'\} \;\wedge\; (\forall c,d \in \ell_1)(\forall c',d' \in \ell'_1)$
$[\,(\,Bw(c,b,d) \;\wedge\; Bw(c',b',d')\,) \;\rightarrow\; (\exists e,e' \in Mn)(\exists \ell_2,\ell'_2 \in L)$
$(Bw(c,e,d) \;\wedge\; Bw(c',e',d') \;\wedge\; e,a \in \ell_2 \;\wedge\; e',a' \in \ell'_2 \;\wedge\; \ell_2 \perp_r \ell'_2\,)\,]\Big] \;\rightarrow\; \ell \perp_r \ell'\Big)$, cf. Fig.119.

Improving our relativistic dualities; unification and integration. By "our relativistic dualities" we refer to the $(\mathcal{G}, \mathcal{M})$-duality, $(\mathcal{G}o, \mathcal{M}o)$-duality and their variants discussed so far. In Appendix A we show that our relativistic dualities form what are called "Galois-connections" e.g. in algebra. Moreover, we expand them to the world of category theory where they lead to e.g. "adjoint situations". This way we reach an enriched concept of duality theories which plays an important role all over mathematics and mathematical physics. The discussion (in Appendix A) also reveals how the methods (and results) in the author's earlier papers e.g. Madarász [161, 165, 164, 166, 170, 163], Madarász et al. [177, 176] have lead to the results of the present work (as well as to more obviously related work e.g. AMN [16], AMN et al. [25]). Many of the methods of the present work were published by the author in the above mentioned earlier papers; but because of the terminological differences explained in Appendix A, it is difficult to give precise references at all the relevant points in the dissertation. Therefore we collected these connections into Appendix A, but cf. also the end of §4.3.

4.5.5 Geometric dualities, definability, Gödel incompleteness

The present section is related to the subject matter of §3.8 in AMN [18] ("Making **Basax** complete...", pp.294-346), to the "relativity and Gödel incompleteness papers Andréka-Madarász-Németi [16], [17], and to the "Accelerated observers" materials, e.g. the Accelerated Observers Chapter in Andréka-Madarász-Németi-Sági-Sain [24], and [117], [26].

Notation 4.5.61 For any axiom system Axi, we write $\mathsf{T}(Axi)$ for the <u>theory generated by</u> Axi. I.e.

$$\mathsf{T}(Axi) :\stackrel{\text{def}}{=} \mathsf{Th}(\mathsf{Mod}(Axi)).$$

$\lhd$

Let $\mathfrak{G}^*_{\mathfrak{M}}$ be defined exactly as $\mathfrak{G}_{\mathfrak{M}}$ was defined in Def.4.2.3 (p.137) with the following changes.

$$L :\stackrel{\text{def}}{=} L^T \cup L^{Ph} \cup L^S \cup \text{ "life-lines of inertial bodies"; i.e.}$$
$$L :\stackrel{\text{def}}{=} L^T \cup L^{Ph} \cup L^S \cup \{\{e \in Mn : b \in e\} : b \in Ib\}.$$

Now,

$$\mathfrak{G}^*_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L, L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq, g, \mathcal{T} \rangle.$$

I.e. $\mathfrak{G}^*_{\mathfrak{M}}$ is obtained from $\mathfrak{G}_{\mathfrak{M}}$ by including the <u>life-lines of inertial bodies</u> as extra lines. This is in perfect harmony with our **Ax3** (p.20) (or even $\mathbf{Ax3_0}$) which say that the life-lines of inertial bodies are straight lines.

Instead of $\mathfrak{G}_{\mathfrak{M}}$, we could have investigated $\mathfrak{G}^*_{\mathfrak{M}}$ in the present chapter (Chap.4), the changes would be inessential. The only reason why we chose $\mathfrak{G}_{\mathfrak{M}}$ as a basis of the present chapter (instead of $\mathfrak{G}^*_{\mathfrak{M}}$) was to make it shorter. However, the nature of the present sub-section (§4.5.5) is such that $\mathfrak{G}^*_{\mathfrak{M}}$ is more suitable as a basis for it than $\mathfrak{G}_{\mathfrak{M}}$. So we will concentrate on $\mathfrak{G}^*_{\mathfrak{M}}$ instead of $\mathfrak{G}_{\mathfrak{M}}$ in the present sub-section. Since the differences are small, to avoid complicated, heavy notation, we will simply pretend in the present sub-section that $\mathfrak{G}_{\mathfrak{M}} := \mathfrak{G}^*_{\mathfrak{M}}$ (i.e. that $\mathfrak{G}_{\mathfrak{M}}$ denotes $\mathfrak{G}^*_{\mathfrak{M}}$) and that all the results, definitions etc. of the present chapter are about $\mathfrak{G}^*_{\mathfrak{M}}$.

CONVENTION 4.5.62 In the present sub-section (§4.5.5) we will pretend that $\mathfrak{G}_{\mathfrak{M}} := \mathfrak{G}^*_{\mathfrak{M}}$, hence in particular, that the life-lines of inertial bodies are lines in $\mathfrak{G}_{\mathfrak{M}}$. This convention is valid only inside this sub-section, after the end of this sub-section $\mathfrak{G}_{\mathfrak{M}}$ will retain its original definition. Whenever the present convention would lead to inconsistencies, we leave it to context and the reader's common sense to eliminate these inconsistencies.

⋆ ⋆ ⋆

The purpose of this sub-section is threefold:

(i) We saw, e.g. in Thm.4.3.38 (p.261), that the "world" of observation-oriented models, the $\mathfrak{M}$'s, and the world of observer-independent geometries, the $\mathfrak{G}$'s, are definitionally equivalent (under some assumptions). From [16, 17], and/or from the relevant part of the present work we know that Gödel's incompleteness theorems do apply to many of the $\mathfrak{M}$'s.[661] In brief, the limitative theorems[662] of metamathematics do apply to the "world" of the $\mathfrak{M}$'s. At the same time, one may recall from logic courses, that Gödel's incompleteness theorems have a tendency of not being applicable to geometric structures and in this respect geometries have a tendency of behaving similarly to real-closed fields (or $\mathfrak{R}$ itself) in that they usually do not satisfy the conditions of Gödel's incompleteness theorems (hence, these theorems do not apply to these structures).[663] Cf. e.g. Goldbatt [102, p.169 lines 11-10 bottom up] where it is stated that the theory of Minkowskian geometry over $\mathfrak{R}$ is decidable. In particular, there are natural frame-theories $Th \supseteq$ **Specrel**, such that Gödel's incompleteness theorems apply to Th but do not apply to $\mathsf{Ge}(Th)$ or to $\mathcal{M}[\mathsf{Ge}(Th)] = (\mathcal{G} \circ \mathcal{M})[\mathsf{Mod}(Th)]$. All these lead to the following question: How is it possible that two "worlds" are equivalent and Gödel's theorems apply to one of them but not to the other? Similarly, we could ask, why does the $(\mathcal{G}, \mathcal{M})$-duality not "import" Gödel incompleteness properties from the side (or "world") of the $\mathfrak{M}$'s to the side (or "world") of the geometries, the $\mathfrak{G}_{\mathfrak{M}}$'s.[665] (Below we will see that the answer is in the conditions of our theorems, and that the just outlined "tension"[666] can lead to interesting observations.)

[661] Hence e.g. T(**Basax**) is undecidable, moreover T(**Basax**+ some extra axioms) is hereditarily undecidable, it admits a formulation Con(**Basax** + extra) of its own consistency etc. The techniques of proving this (formalizability of own consistency) ensure that the Liar Paradox expressing "this sentence is not provable from (**Basax** + extra)" can be formulated in "**Basax** + extra", which in turn leads to strong hereditary incompleteness results. If someone wants to make this theory complete, then he will probably try by adding the Liar Paradox to (**Basax** + extra) as a new axiom. But this spectacularly fails, because then there will be a new incarnation of the "Liar" saying "this sentence is not provable from (**Basax** + extra + "Liar formulated for (**Basax** + extra)"). Etc.

[662] See e.g. Bell-Machover [46, Chapter 7, "Logic-limitative results"] or Chaitin [59].

[663] In passing we note that if our field $\mathfrak{F}$ is strange enough (i.e. is far from being a real-closed field) then we can lose decidability of e.g. $\mathsf{Th}(Mink(4, \mathfrak{F}))$. Cf. [17]. But this is not too relevant to our present concerns, so we do not discuss this and we pretend that $\mathsf{Th}(\mathfrak{F})$ is always decidable. Although in the typical well behaved cases Gödel's theorems do not apply to $\mathfrak{G}_{\mathfrak{M}}$ whenever $\mathfrak{F}$ is a real-closed field,[664] we note that there are exotic exceptions. E.g. we conjecture that either for the geometry $\mathfrak{G}_{\mathfrak{M}}$ constructed in the proof of Thm.4.2.23 (p.168) Gödel's incompleteness theorems do apply, or one can construct an analogous $\mathfrak{G}_{\mathfrak{M}}$ for which Gödel's incompleteness theorems apply.

[664] E.g. in Minkowskian geometries this is always so (i.e. [$\mathfrak{F}$ is real-closed] $\Rightarrow$ [Gödel's incompleteness theorems do not apply to $Mink(\mathfrak{F})$]), cf. e.g. Goldblatt [102, p.169] for this.

[665] Of course, there are structures in $\mathsf{Ge}(Th)$ to which the conditions of Gödel's theorems do apply, but they are the exceptional ones, in some sense (from the physical point of view they are somewhat strange); while on the $\mathsf{Mod}(Th)$ side it is much more typical, frequent (and natural) to have these conditions satisfied, cf. Andréka-Madarász-Németi [16],[17] (e.g. having a periodically moving body is sufficient).

[666] By tension we mean something which looks like a contradiction (but is not one).

(ii) Can we extend our $(\mathcal{G}, \mathcal{M})$-duality to handling non-inertial bodies (or at least non-inertial observers) well? (I.e. can we extend our duality such that non-inertial bodies or observers would not necessarily disappear from $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$?)

(iii) We will briefly ask ourselves whether the life-lines of some non-inertial bodies are definable in $\mathsf{Ge}(Th)$, for nice enough choices of Th.

Before going on, we note that the above three issues (i)-(iii) are interconnected as follows: If all non-inertial bodies of $\mathfrak{M}$ would reappear in $\mathcal{G} \circ \mathcal{M}(\mathfrak{M})$ [667] then probably all non-inertial bodies of $\mathfrak{M}$ would be (at least parametrically) definable in $\mathcal{G}(\mathfrak{M})$. (This would answer item (iii).) But, if this would be the case, then applicability of Gödel's incompleteness theorems for $\mathfrak{M}$ would probably be inherited by $\mathcal{G}(\mathfrak{M})$,[668] because non-inertial bodies of $\mathfrak{M}$ played an essential role in applying these theorems to $\mathfrak{M}$ in [16], [17]. So items (i)-(iii) are interconnected.

A perspective on items (i)-(iii): In connection with item (i), in Statement $(\star)$ below, we will see that $(\mathcal{G} \circ \mathcal{M})$ tends to streamline our models, it tends to make our originally complicated, "untidy" $\mathfrak{M}$ into a "streamlined", "tidy", and smooth variant $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$ of the original $\mathfrak{M}$. As a byproduct, it may happen that $\mathfrak{M}$ satisfies the conditions of Gödel's incompleteness theorems but $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$ does not.

Now, in items (ii), (iii) we ask ourselves: Is this good for us or is this bad for us? Roughly, the answer will be the following. At the present level of investigations this is not bad at all. However, in later generalizations towards general relativity, e.g. in the theory of accelerated observers[669] this might create inconveniencies (which we will have to be careful to avoid).

Let us turn to discussing (some of) the questions (i)-(iii) above.

In §4.5.3 we had a proposition saying, roughly, that the operator $\mathcal{G} \circ \mathcal{M}$ makes our possibly complicated and "inhomogeneous"[670] models $\mathfrak{M}$ (which might contain random features) symmetric, "tidy" and "smooth", e.g.

$$(\star) \qquad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \models \mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}.^{671}$$

In the "Gödel incompleteness" papers Andréka-Madarász-Németi [16], [17] related to the present work[672], we saw that, roughly, such "smooth" models usually have a decidable theory to which Gödel's incompleteness theorems do *not* apply (assuming $\mathfrak{F}$ is a real-closed field).[673]

Though $(\star)$ can be viewed as a positive result, in a certain other sense it will turn out to be a *limitative* one, cf. e.g. Thm.4.5.66, Conj.4.5.68.

[667] This would be a positive answer to (ii).

[668] at least in most of the cases (i.e. when non-inertial bodies were responsible for "incompleteness")

[669] Cf. e.g. [16], [17], [26], [24, Chap. "Accelerated Observers"], [117], [192].

[670] We mean here that on some (but not all) life-lines there may be many indistinguishable observers in a random manner, and that there may be many non inertial bodies with complicated life-lines in one part of $\mathfrak{M}$ but not in another etc.

[671] In passing we note that many other duality theories tend to do this "streamlining" of their objects. E.g. in the case of Galois connections (pp. A-1–A-4) if $p \in P$ then $g(f(p))$ is the "closure" of p and usually has more symmetry properties than p. A similar remark applies to the $(\mathsf{Mod}, \mathsf{Th})$-duality on p.1026 of AMN [18] where to the possibly "untidy" or "random" $\Sigma \subseteq Fm$, the streamlined $\mathsf{Th}(\mathsf{Mod}(\Sigma))$ is associated (which is closed under "$\models$").

[672] cf. also the Gödel incompleteness chapter of a future edition [19] of AMN [18]

[673] As we already mentioned in connection with geometries (in footnote 663), there might be exceptional models $\mathfrak{M}$ which are "smooth" in the above sense with $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ and still have an undecidable theory. Cf. Andréka et al. [16, Thm.9(iii)].

Independently of this, we saw in §§ 4.5.3, 4.5.4 that the function

$$\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

is a *first-order definable meta-function* (assuming Th is strong enough), i.e. that $\mathcal{M}(\mathfrak{G})$ is uniformly first-order definable over $\mathfrak{G}$. Moreover $\mathsf{Mod}(Th)$ is definable over $\mathsf{Ge}(Th)$ if Th is strong enough, cf. Theorems 4.5.11, 4.3.38 and Prop.4.5.41.

First-order definability of $\mathcal{M}(\mathfrak{G})$ over $\mathfrak{G}$ includes the claim that (intuitively speaking) *every observer* m of $\mathcal{M}(\mathfrak{G})$ *is* first-order *definable* from $\mathfrak{G}$ by using $n+1$ parameters. Namely, each m is definable by using (as *parameters*) $n+1$ points $o, e_0, \ldots, e_{n-1}$ satisfying (a)–(f) on p.310. (This kind of definability is called *parametrical definability* in standard mathematical logic, cf. §4.3 [pp. 235, 223].)

Summing it up, every observer of $\mathcal{M}(\mathfrak{G})$ is parametrically definable in $\mathfrak{G}$. Moreover

($\star\star$) every body of $\mathcal{M}(\mathfrak{G})$ is parametrically definable in $\mathfrak{G}$.

All the bodies in $\mathcal{M}(\mathfrak{G})$ are *inertial*. But in our relativity theories, e.g. ($\mathbf{Basax}+\mathbf{Ax}(\omega)^\sharp$) *non-inertial* bodies also play some important role, cf. e.g. the formalization of the Twin Paradox in §2 (p.13 and Figure 7 on p.13) and the continuation of this work on accelerated observers [26], the accelerated observers chapter in [24], [117], and the related discussions in the present work.

Therefore, as we already said, the following question naturally comes up: *Can* we *define* (by first-order means) *strongly non-inertial bodies*[674] from $\mathfrak{G}$? Further, can we extend our duality theory

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th), \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

by possibly strengthening Th (and improving the definition of $\mathcal{M}$) such that it would "handle" strongly non-inertial bodies too? We will see that the answer is no, at least if we want to keep our geometries $\mathfrak{G}_{\mathfrak{M}}$ at least remotely similar to the geometries considered in the literature, e.g. if we want to stick with the three sorts *Points*, *Lines* and *Quantities* (i.e. F) only.[675] On the other hand, we will indicate in Remark 4.5.63 that a positive answer is possible in the framework of first-order logic on the expense of making our structures richer than "geometries". Since accelerated observers with constant acceleration will play an important role later in generalizing our theory,[676] we note the following. (Life-lines of) accelerated bodies with constant acceleration are parametrically definable in most of our geometries $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax})$.

Remark 4.5.63 We note that to recover strongly non-inertial bodies from $\mathfrak{G}_{\mathfrak{M}}$ we will need to add, among others, an extra sort representing, roughly, a possibly nonstandard model of Peano's Arithmetic as it was done in the development of nonstandard temporal logics and nonstandard dynamic logic cf. e.g. Sain [227], Andréka-Goranko et al. [12] and the references therein. We plan to do such things in a later work related to the present one. Such developments will also represent connections with nonstandard analysis.[677] We note that in this approach we will add the following extra sorts to $\mathfrak{G}$. (i) A sort usually denoted as I which

[674] Cf. Def.4.5.67 for strongly non-inertial bodies.

[675] or anything in the spirit of ⟨*Points*, *Lines*, *Planes*, *Quantities*⟩-like arrangement to which e.g. our definition of $\mathfrak{G}^*_{\mathfrak{M}}$ does conform

[676] in the direction of general relativity theory

[677] This would mean a connection between the presently discussed kind of "logic-based relativity" and nonstandard analysis.

represents functions from the sort $\mathbf{F_1}$ into itself. I.e. $I \subseteq {}^F F$. (ii) Further, a binary operation $\mathsf{value} : I \times F \longrightarrow F$ such that for $f \in I$, $\mathsf{value}(f, x) \in F$ is considered to be the value "$f(x)$" for $x \in F$. (iii) A unary relation $N \subseteq F$ which plays the role of the positive integer elements of F, e.g. $0, 1 \in N$ and N is closed under $+, \cdot$ of $\mathbf{F_1}$, moreover $\langle N, 0, 1, +, \cdot\rangle$ is a model of Peano's Arithmetic. (iv) We will postulate the comprehension axiom-schema for I saying that all functions $f : F \longrightarrow F$ which are definable in the language of the so expanded model $\mathfrak{G}$ appear as elements of I. I.e. all first-order definable[678] functions $f : F \longrightarrow F$ show up in I, roughly $f \in I$. The purpose of all this machinery is to enable us to express in first-order logic (i.e. in the first-order language of the so expanded $\mathfrak{G}$) the things which we want to express in order to develop our theory of, say, accelerated observers (and/or motion in general). This approach will be described in [19].[679]

$\triangleleft$

Notation 4.5.64 $\mathbf{Mink}(n, rc)$ denotes the class

$$\mathbf{I}\{Mink(n, \mathfrak{F}) \;:\; \mathfrak{F} \text{ is a real-closed field}^{680}\}$$

of all n-dimensional Minkowskian geometries over real-closed fields.

$\triangleleft$

Items 4.5.66, 4.5.68 below can be interpreted as saying that *not all important aspects of* (special) *relativity can be recovered from the geometries* $\mathfrak{G}_{\mathfrak{M}}$ (or from Minkowskian geometry).

A body $b \in B$ is called periodically moving, or periodical for short, if there is $m \in Obs$ such that $tr_m(b)$ can be interpreted as a function $tr_m(b) : \bar{t} \longrightarrow {}^{n-1}F$ and this function is periodical. See Figure 120. For simplicity we will use the following simpler definition.

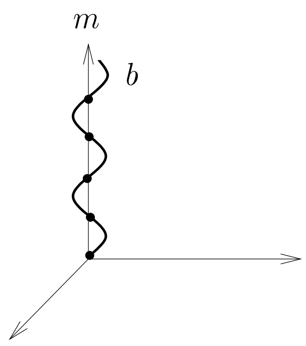


Figure 120: b is a periodically moving body in m's world-view.

Definition 4.5.65 Let $\mathfrak{M}$ be fixed. Body b is called *periodical* iff there is $m \in Obs$ such that letting $H := \bar{t} \cap tr_m(b)$ the set $H \subseteq F$ is discrete[681] and cofinal in $\mathfrak{F}$, and for any two

[678] We mean definable in the many-sorted structure $\langle \mathfrak{G}, I, \mathsf{value}, \text{etc.}\rangle$.

[679] We note that at this point we did not explain why and how adding such extra sorts including an extra arithmetical sort will help. Consulting Sain [227], Andréka-Goranko-et al [12], Montague [195], Gallin [94] may give useful hints.

[680] Cf. p.301 of AMN [18] for the notion of real-closed fields.

[681] We use the language of $\mathfrak{F}$. H is *discrete* if any point in H has a successor and a predecessor in H unless it is an endpoint of H.

neighboring pairs $a, b, a', b' \in H$ we have $|b-a| = |b'-a'|$ (where a and b are neighbors[682] and the same holds for a', b').

$\lhd$

Intuitively, the following theorem says that life-lines of periodical bodies are *not definable* in our geometries like $\mathsf{Ge}(\mathbf{Bax})$. Recall that $\mathbf{Ax(rc)}$ is the usual axiom system for real-closed fields defined in the List of axioms and on p.301 in §3.8 of AMN [18].

THEOREM 4.5.66

(i) *Let $n > 1$ and consider the class $\mathbf{Mink}(n, rc)$ of Minkowskian geometries.*

Then, there exists $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp})$ such that $\mathfrak{G}_{\mathfrak{M}} = \langle Mn, F, \dots \rangle \in \mathbf{Mink}(n, rc)$ and for no periodical body b of $\mathfrak{M}$ is the life-line $\{ e \in Mn : b \in e \}$ of b definable parametrically in the geometry $\mathfrak{G}_{\mathfrak{M}}$.[683]

(ii) *Statement (i) remains true if we replace $\mathbf{Mink}(n, rc)$ by any one of our distinguished classes $\mathsf{Ge}(Th)$ of geometries. (Here Th ranges over our hierarchy $\mathbf{Bax}^{-}, \mathbf{Bax}, \dots, \mathbf{Basax}$).*

Outline of proof: In AMN [16], [17] as well as in the "decidability ... Gödel incompleteness" part of AMN [19] we see that if we add the existence of a periodical body as an extra axiom (this extra axiom is denoted by $\boldsymbol{\nu}$ there) to any one of our distinguished theories Th, then the so obtained $(Th + \boldsymbol{\nu})$ becomes essentially undecidable as a theory, it satisfies the conditions of Gödel's incompleteness theorems, hence the conclusions of Gödel's incompleteness theorems (both of them) apply to the theory $(Th + \boldsymbol{\nu})$.[684]

Therefore, if a periodical body was parametrically definable in $\mathsf{Mod}(Th)$ then this would render $\mathsf{Th}(\mathsf{Mod}(Th))$ essentially undecidable etc. (The parameters [in our notion of definability] cause no problem in this argument because we can use quantifiers in our language to make the parameters "disappear" when translating number theoretic formulas to formulas in the language of $\mathsf{Mod}(Th)$. This technique [for getting rid of the parameters] was used e.g. in Németi [202]).

Having seen that $\mathsf{Mod}(Th)$ would become essentially undecidable if formula $\boldsymbol{\nu}$ was added to it, one can push the same argument through to show that $\mathsf{Ge}(Th)$ would become hereditarily *undecidable if* $\boldsymbol{\nu}$ was *expressible* in the language of $\mathsf{Ge}(Th)$. Since we know that $\mathsf{Ge}(Th +$ ($\mathfrak{F}$ is a real-closed field)) can be extended to a *decidable* consistent theory, cf. the "Making **Basax** complete ..." section of AMN [18], i.e. §3.8 pp.294-346 of AMN [18], we conclude that $\boldsymbol{\nu}$ cannot be expressible in the first-order language of $\mathsf{Ge}(Th)$. But this implies that *no* periodical body can be parametrically defined[685] in $\mathsf{Ge}(Th)$. This finishes the proof. ∎

---

[682] I.e., $[a < b$ and $(\nexists c \in H)\ a < c < b]$.

[683] I.e. for no finite number of parameters $p_1, \dots, p_k$ from $\mathfrak{G}_{\mathfrak{M}}$ (i.e. from $Uv(\mathfrak{G}_{\mathfrak{M}}) = Mn \cup F \cup L$) is the life-line of any <u>periodical body</u> of $\mathfrak{M}$ (first-order) definable in $\mathfrak{G}_{\mathfrak{M}}$ by using $p_1, \dots, p_k$ as parameters. That is, let $\bar{p} = \langle p_1, \dots, p_k \rangle$. Then *no* first-order formula $\varphi(x, \bar{p})$ in the language of $\mathfrak{G}_{\mathfrak{M}}$ defines the trace $\{ e \in Mn : b \in e \}$ of a periodical body b of $\mathfrak{M}$.

[684] This can be seen by interpreting Robinson's arithmetic denoted by R in Monk [194, Def.14.9, p.247] in the theory $(Th + \boldsymbol{\nu})$. Note that this version R of arithmetic is much weaker than Peano's arithmetic, in particular, it involves <u>no</u> induction axiom schema. Hereditary undecidability etc. of R is in Thm.16.1, p.280 of Monk [194]. For more detail on $(Th + \boldsymbol{\nu})$ cf. Andréka-Madarász-Németi [17].

[685] Here, we mean uniform definability for the whole class $\mathsf{Ge}(Th)$. However one can refine the present argument to prove that there is a geometry $\mathfrak{G} \in \mathsf{Ge}(Th)$ in which no such body is parametrically definable.

Definition 4.5.67 Let $\mathfrak{M}$ be fixed. A body b is called <u>*strongly non-inertial*</u> iff there is an observer m such that $tr_m(b) \cap \bar{t}$ is a nonempty set and is gapy in the following sense:

$$(*) \qquad (\forall p \in tr_m(b) \cap \bar{t})(\exists q, r \in \bar{t})(p_t < q_t < r_t \ \wedge \ q \notin tr_m(b) \ \wedge \ r \in tr_m(b)).^{686}$$

◁

If a body b is <u>periodical</u> (in the sense of Def.4.5.65) <u>then</u> it is <u>strongly non-inertial</u>.

Intuitively, the next conjecture says that the life-lines of strongly non-inertial bodies are not definable in our geometries, e.g. $\mathsf{Ge}(\mathbf{Bax})$.

Conjecture 4.5.68 *Theorem 4.5.66 remains true for strongly non inertial bodies in place of periodical ones.*

Possible idea of proof: We choose a model $\mathfrak{M} \in \mathsf{Mod}(Th)$ such that $\mathfrak{F}^{\mathfrak{M}}$ is a real-closed field and such that $\mathfrak{G}_{\mathfrak{M}}$ is the Minkowskian geometry over $\mathfrak{F}^{\mathfrak{M}}$, up to isomorphism. Then $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{F}^{\mathfrak{M}}$. Assume that the life-line of a strongly non-inertial body b of $\mathfrak{M}$ is parametrically definable over $\mathfrak{G}_{\mathfrak{M}}$. Then $(*)$ above holds for some $m \in Obs$. Let this m be fixed. Then the intersection $\{ e \in Mn : b, m \in e \}$ of the life-lines of b and m is parametrically definable over $\mathfrak{G}_{\mathfrak{M}}$ by a formula $\varphi(x, \bar{p})$ with parameters $\bar{p}$. Since $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{F}^{\mathfrak{M}}$, there is a definitional expansion $\mathfrak{G}^+_{\mathfrak{M}}$ of $\mathfrak{F}^{\mathfrak{M}}$ such that $\mathfrak{G}_{\mathfrak{M}}$ is a reduct of $\mathfrak{G}^+_{\mathfrak{M}}$. Now, by Thm.4.3.29 (p.247) there is a translation mapping $Tr : Fm(\mathfrak{G}^+_{\mathfrak{M}}) \longrightarrow Fm(\mathfrak{F}^{\mathfrak{M}})$ such that the conclusion of Thm.4.3.29 holds for this Tr. Now, if we apply this Tr to our formula $\varphi(x, \bar{p})$ then we obtain a new formula which defines a relation on $\mathfrak{F}^{\mathfrak{M}}$ parametrically. We conjecture that from this, one can obtain a further formula which defines a subset H of $\mathfrak{F}^{\mathfrak{M}}$ parametrically which is gapy in the sense of Def.4.5.67 immediately above. Now, to such a gapy H one can apply the proof of Lemma 4.2.27 (p.172) to derive a contradiction. Namely, by the proof of Lemma 4.2.27 it follows that H is not parametrically definable over $\mathfrak{F}^{\mathfrak{M}}$. (The proof of Lemma 4.2.27 goes through for the present case if one uses arbitrary polynomials in the proof and not only polynomials with rational coefficients and if one uses "gapy" in the sense of Def.4.5.67 above and not in the sense of Def.4.2.26).

◁

Remark 4.5.69 (On Gödel's logic proofs, relativity proof, and Escher:)
Some of Escher's pictures can be associated <u>both</u> with Gödel's <u>incompleteness</u> proof (logic) and to his rotating <u>universe</u> construction for general relativity. So these two seemingly distant creations of Gödel are more closely related than is usually acknowledged in the literature. But cf. Yourgrau [270], Dawson [70, pp. 176–177] for positive exceptions (where the "two Gödel's" <u>are</u> connected). See Figure 121. For Gödel's rotating universe see Figure 134 on p.365.

◁

Items 4.5.66, 4.5.68 above seem to say that our duality theory

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th), \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

cannot be easily extended to a duality theory consisting of some $\mathcal{G}^+$ and $\mathcal{M}^+$ which would satisfactorily handle periodically moving (or strongly non-inertial) bodies present in the models $\mathfrak{M} \in \mathsf{Mod}(Th)$. Or in other words, the duality theory based on $\mathcal{G}$ and $\mathcal{M}$ abstracts from

[686] Cf. Def.4.2.26 (p.172) and note that though the two definitions are similar they are not the same.

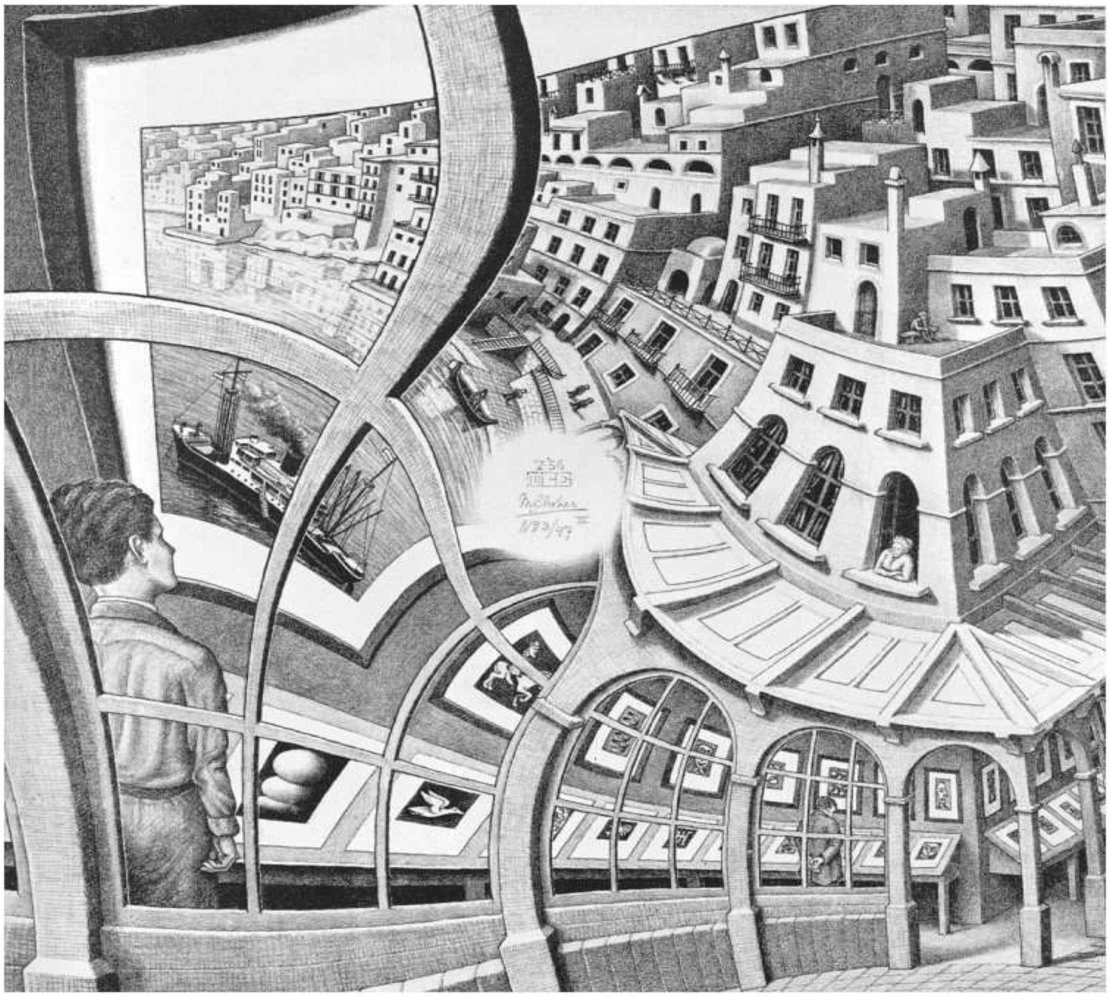


Figure 121: Print Gallery, by M.C. Escher. Cf. Fig.122 for the "logic" of this picture and for its connections with Gödel's proof. A key idea in Gödel's proof is self reference: "this sentence is not provable" (a variant of the well-known Liar paradox).

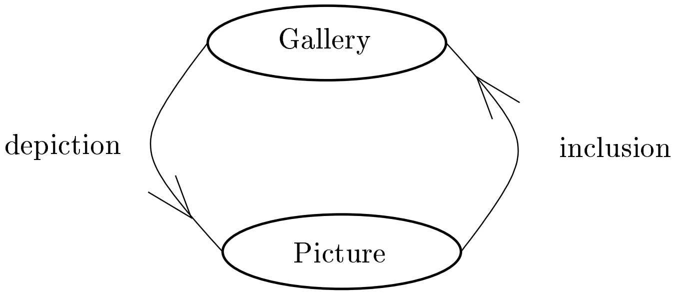


Figure 122: A collapsed version of Fig.121 (i.e. of Escher's Print Gallery).

strongly non-inertial bodies (and therefore also from strongly non-inertial observers (!)), and this feature seems to be unavoidable in view of items 4.5.66, 4.5.68. More precisely, this seems to be so unless we expand our geometries in the "nonstandard dynamic logic" style mentioned/promised in Remark 4.5.63 way above.

Let us return to answering items/questions (i)-(iii) on p.333 close to the beginning of this sub-section. The above discussion, theorem, etc. answer items (ii), (iii)[687].

To answer (i), let us assume some nice, strong frame-theory[688] e.g. $Th^+ :\stackrel{\text{def}}{=} \mathbf{Basax} + \mathbf{Ax}(\omega) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(rc)} + \mathbf{Ax(eqm)} + \mathbf{Ax(eqtime)}$.

Now, we are looking at $\mathsf{Mod}(Th^+)$ and at $\mathcal{G}^*[\mathsf{Mod}(Th^+)] = \{\mathfrak{G}^*_{\mathfrak{M}} : \mathfrak{M} \models Th^+\}$ where $\mathcal{G}^* : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ with $\mathcal{G}^*(\mathfrak{M}) :\stackrel{\text{def}}{=} \mathfrak{G}^*_{\mathfrak{M}}$ for all $\mathfrak{M}$. According to the proofs in [16, 17], there are many models $\mathfrak{M} \models Th^+$ satisfying the conditions of Gödel's incompleteness theorems. At the same time, $\mathfrak{G}^*_{\mathfrak{M}}$ fails to satisfy the conditions of Gödel's theorems for many[689] choices of the above $\mathfrak{M}$. The reason for this is item ($\star$) on p.334 together with the fact that in Thm.10 of [16] we used the presence of <u>periodically moving bodies</u> to prove the conditions of Gödel's theorems (for models satisfying Th^+). But the functor $\mathcal{G}^*$ <u>removes</u> (or forgets) the traces of such bodies. Hence the "periodical body method" in [16],[17] is no longer applicable to the structure $\mathfrak{G}^*_{\mathfrak{M}}$.[690] Recall that here we pretend that the $(\mathcal{G}, \mathcal{M})$-duality is really some $(\mathcal{G}^*, \mathcal{M}^*)$-duality where $\mathcal{G}^*$ corresponds to $\mathfrak{G}^*_{\mathfrak{M}}$ defined on p.332 (beginning of §4.5.5) and $\mathcal{M}^*$ matches $\mathcal{G}^*$ the same way and spirit as $\mathcal{M}$ matched $\mathcal{G}$. In summary, we can say that the apparent paradox in (i) is caused by the following. It is true that $\mathcal{G} \circ \mathcal{M}(\mathfrak{M})$ is almost the same as $\mathfrak{M}$ (hence almost all properties of $\mathfrak{M}$ should probably hold for $\mathcal{G} \circ \mathcal{M}(\mathfrak{M})$), but it is exactly that remaining little difference between $\mathfrak{M}$ and $\mathcal{G} \circ \mathcal{M}(\mathfrak{M})$ which really matters in the Gödel incompleteness issue. Namely, $(\mathcal{G} \circ \mathcal{M})$ preserves all nice properties but it forgets the non-inertial bodies. And it are exactly these bodies which are used in the proof in [16], [17].[691]

So this is why our $(\mathcal{G}, \mathcal{M})$-duality or $(\mathcal{G}^*, \mathcal{M}^*)$-duality does not preserve the Gödel incompleteness properties of the structures involved.[692] One still can ask why the definitional equivalence theorem[693]

$$\mathsf{Mod}(Th) \equiv_{\Delta} \mathsf{Ge}(Th)$$

does not export Gödel incompleteness properties (e.g. hereditary undecidability) from $\mathsf{Mod}(Th)$ to $\mathsf{Ge}(Th)$. The answer is simple: The condition of the just quoted theorem (Thm.4.3.38) on Th excludes the kinds of applicability of Gödel's incompleteness theorems even to $\mathsf{Mod}(Th)$ which we used in e.g. [16]. Indeed, it is indicated in [16],[17] that $\mathbf{Ax}\heartsuit, \mathbf{Ax(ext)}, \mathbf{Ax}(\sqrt{\ }), \mathbf{Ax(diswind)}, \mathbf{Ax(eqtime)}$[694] are all axioms working <u>against</u> satisfiability of the conditions of Gödel's theorems. E.g. $\mathbf{Ax}\heartsuit$ excludes periodic (hence non-inertial) bodies.

[687] at least to some extent

[688] The purpose of assuming such a theory is to avoid being side-tracked by some, more-or-less, inessential detail.

[689] We are inclined to write "for most choices".

[690] There are other "Gödel incompleteness methods" in [16], but they are less important from the physical point of view. (And even most of these are "killed" by the $\mathcal{G} \circ \mathcal{M}$-transition, with the exception of one or two.) Anyway, these alternative methods from [16] are excluded now by our choice of Th^+.

[691] There were other incompleteness methods in [16], [17], but that is, so to speak, beside the point here, for various reasons.

[692] There are also similar minor effects, e.g. $\mathcal{G} \circ \mathcal{M}$ makes $\mathbf{Ax(ext)}$ true which, by [16], eliminates further possibilities of applicability of Gödel's theorems, but to save space we do not discuss these here.

[693] Thm.4.3.38, p.261

[694] The condition of Thm.4.3.38 requires all these axioms to be provable from Th.

Our next sub-section (4.5.6) is related to section 4.6 which in turn, is concerned with streamlining our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$ (among others), as was promised in the introduction.

4.5.6 Recoverability of relativistic models from reducts of geometry

Let us return to the question, formulated at the beginning of this section of whether we can reconstruct $\mathfrak{M}$ from $\mathfrak{G}_{\mathfrak{M}}$ or from a reduct of $\mathfrak{G}_{\mathfrak{M}}$. In the duality theory developed in §§ 4.5.1–4.5.4 above we saw that $\mathfrak{M}$ can be reconstructed from $\mathfrak{G}_{\mathfrak{M}}$ (under some conditions on $\mathfrak{M}$). Below, we will look at the *same question* somewhat differently. We will look at *reduct* geometries $\mathfrak{G}^i_{\mathfrak{M}}$ and we will prove things which might be interpreted as saying that $\mathfrak{M}$ cannot be reconstructed from $\mathfrak{G}^i_{\mathfrak{M}}$. In *this form* these sound like negative results. However, in the form we will state them they will sound like positive results. Roughly speaking, assume we introduced the notation $\mathsf{Ge}^i(Th) = \mathbf{I}\{\mathfrak{G}^i_{\mathfrak{M}} : \mathfrak{M} \models Th\}$. Then for certain choices of Th_1 and Th_2 we will state that

$$(\star) \qquad \mathsf{Ge}^i(Th_1) = \mathsf{Ge}^i(Th_2);$$

(for certain choices of i). This might be interpreted as a representation result stating that every geometry in $\mathsf{Ge}^i(Th_1)$ is *representable* as a geometry of some Th_2-model (and vice-versa). Theorems of style $(\star)$ above can be read of from Fig.84 (p.192).

Intuitively, from a *relativity theoretic* point of view these results (of form $(\star)$) can be used the following way. Consider certain kinds of thought-experiments the *characteristic feature* of which is that they can be formulated in the language of $\mathfrak{G}^i_{\mathfrak{M}}$. Then a result of the type $(\star)$ above can be interpreted by saying that the relativity theories Th_1 and Th_2 cannot be distinguished by thought-experiments of "type $\mathfrak{G}^i$". A result of this kind might be of interest e.g. when Th_1 is Reichenbachian version of relativity like **Reich**(**Basax**) and Th_2 is something more "classical" like **Basax**, cf. e.g. Theorems 6.6.107–6.6.110 in AMN [18].

In AMN [18, §6.6.10] we define progressively weaker reducts $\mathfrak{G}^0_{\mathfrak{M}}$–$\mathfrak{G}^5_{\mathfrak{M}}$ of our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$ (including the Goldblatt-Tarski geometry $GT_{\mathfrak{M}}$ as one of the "levels"). The physical motivation for looking at such reducts is given at the beginning of §4.5.4 on p.325. The main idea is that at different times one may want to concentrate at different aspects of the world, and later one might want to compare the results and/or experiences so obtained. Concrete works on physics are listed in the preface of Schutz [231] which indeed concentrate on different aspects of the world e.g. on $\perp_r$, or on, $\prec$, or g. Some relatively significant physical conclusions (of the investigation of $\mathfrak{G}^0_{\mathfrak{M}}, \dots, \mathfrak{G}^i_{\mathfrak{M}}$) are summarized on p.1147 at the end of item (2) of §6.7.1 in AMN [18]. After this, in AMN [18], we ask ourselves whether $\mathfrak{M}$ is recoverable from $\mathfrak{G}^i_{\mathfrak{M}}$, and for which choices of i and of Th_1, Th_2 is (*) above true.

The issue presented above is elaborated in detail (in the form of theorems, definitions etc) in AMN [18, §6.6.10]. Further, the related issue of characterizing the kinds of geometries we obtain by restricting $\mathfrak{G}_{\mathfrak{M}}$ to hyperplanes is investigated in AMN [18, §6.6.11]. For lack of space we omit these results.

4.6 Interdefinability questions; on the choice of our geometrical vocabulary (or language $L, L^T, \ldots, g, \mathcal{T}$)

Our $\mathfrak{G}_{\mathfrak{M}}$ has a large number of components. As we have indicated in the introduction (§4.1), in AMN [18, §6.7] the present author explored how $\mathfrak{G}_{\mathfrak{M}}$ can be streamlined so that it will consist only of a few components and each remaining component will either be definable in terms of these or turn out to be superfluous. Our criteria here are that (i) the theory of the streamlined geometry be simple and perspicuous and (ii) the streamlined geometry be a familiar mathematical structure.[695] In other words: In [18, §6.7] we investigate how the various ingredients (i.e. non-logical symbols) of our geometries in $\mathsf{Ge}(Th)$ are *definable from each other*. Among other things, this amounts to asking ourselves whether one or another ingredient is superfluous (in presence of the others). Below we present a sample of these results of the present author.

Convention. For brevity, we will refer to AMN [18, §6.7] simply as §6.7. This will cause no misunderstanding since the present work has no §6.7.

In §6.7 we concentrate on two basic versions of geometry associated with $\mathfrak{M}$, these are $\mathfrak{G}_{\mathfrak{M}}$ and $\mathfrak{G}^0_{\mathfrak{M}}$ = *"the $g, \mathcal{T}$-free reduct of $\mathfrak{G}_{\mathfrak{M}}$"* defined in Def.4.5.53 (p.326). The reason why $\mathfrak{G}^0_{\mathfrak{M}}$ can compete with $\mathfrak{G}_{\mathfrak{M}}$ as *the right geometry* associated with $\mathfrak{M}$ is that (i) in $\mathfrak{G}^0_{\mathfrak{M}}$ we can define a topology $\mathcal{T}'$ or $\mathcal{T}''$ as shown in Def.4.2.30 and on pp.175-179. By Thm.4.2.37, $\mathcal{T}$ and $\mathcal{T}', \mathcal{T}''$ coincide under some reasonable assumptions, hence we do not lose much by omitting $\mathcal{T}$ from $\mathfrak{G}^0_{\mathfrak{M}}$. Further, by Prop.4.2.35, $\mathcal{T}'$ is as nice as we can wish, under very mild[696] assumptions, hence we lose practically nothing by omitting $\mathcal{T}$ from $\mathfrak{G}^0_{\mathfrak{M}}$. The justification for omitting g from $\mathfrak{G}^0_{\mathfrak{M}}$ is slightly weaker, namely, it goes by saying that we can use eq in place of g for measuring relativistic distances between events. We prove in §6.7 that what we lose by forgetting g is information about what the *units of measurement* were in $\mathfrak{M}$. We prove there that this information is really lost, e.g. $\mathfrak{G}_{\mathfrak{M}}$ is *not definable* from $\mathfrak{G}^0_{\mathfrak{M}}$ even if we assume our strongest conditions (like **BaCo** etc) on $\mathfrak{M}$.

Among other things, we prove in §6.7 that, for all n, there is $\mathfrak{M} \models$ **Flxbasax** + **Ax(diswind)** such that Bw is not definable from the $\langle Mn; Col, \perp, eq\rangle$ reduct of $\mathfrak{G}_{\mathfrak{M}}$. Moreover, Bw is not definable from the Bw-free reduct of $\mathfrak{G}_{\mathfrak{M}}$. As a *contrast*, Bw is first-order definable from $\langle Mn; Col\rangle$ under assuming $\mathfrak{M} \models$ **Newbasax** + **Ax(diswind)** and $n > 2$. Actually, Bw is definable from $\langle Mn; Col\rangle$ iff $\leq$ is definable in $\mathbf{F}^{\mathfrak{M}}$, under some extremely mild conditions on $\mathfrak{M}$. These results are among Thm.s 6.7.4, 6.7.8, 6.7.10, 6.7.13.

Then, beginning with §6.7.2 we prove that $\mathfrak{G}^0_{\mathfrak{M}}$ is definitionally equivalent to some of its very "slim" and streamlined reducts, under some conditions. One of these is the famous "causality" reduct $\langle Mn; \prec\rangle$, cf. e.g. Thm.6.7.20 therein. Further such reducts are $\langle Mn; Col^{Ph}\rangle$ and $\langle Mn; Col^T\rangle$, under extremely mild assumptions[697] (here we consider the $\prec$-free reduct of $\mathfrak{G}^0_{\mathfrak{M}}$). Cf. Thm.s 6.7.30-6.7.32. The above is only a small sample from the "*streamlineability*" results on $\mathfrak{G}^0_{\mathfrak{M}}$ in §6.7. There we prove analogous results for $\mathfrak{G}_{\mathfrak{M}}$ too, where $\langle Mn, \mathbf{F_1}, g\rangle$ is one of the various reducts of $\mathfrak{G}_{\mathfrak{M}}$ from which $\mathfrak{G}_{\mathfrak{M}}$ is recoverable (definable), cf. e.g. AMN [18, Thm.6.7.39 (p.1167)].

[695] These two criteria were kept in mind by Tarski and his followers while building up algebraic logic. Cf. §A.3.

[696] $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$

[697] $\mathbf{Bax}^\oplus$ + *some of our auxiliary axioms like* $\mathbf{Ax}(\sqrt{\ })$.

Motivated by the last result, next we briefly discuss uses of eq, g and their visualizations.[698]

On circles or spheres (and drawing them)

We note that having eq around is nice because it enables us to speak about circles or spheres.[699] We note that for $n > 2$ in **Basax** + $\mathbf{Ax}(\mathit{Triv}_t)^-$ in terms of eq a sphere looks like as in Figure 123 when intersected with $\mathsf{Plane}(\bar{t}, \bar{x})$. So far we talked about circles based on eq. Let us call them eq-circles. Similarly we can consider circles based on g. Let us call these second kind of circles g-circles.[700] We use the expression "circles" in 2-dimensional models and "spheres" in $n > 2$ dimensional ones. We note that the set of neighborhoods $T_0 :\stackrel{\text{def}}{=} \{\, S(e, \varepsilon) \,:\, e \in Mn,\ \varepsilon \in {}^+F \,\}$ defined on p.146 coincides with the set of g-circles (in any $\mathfrak{G} \in \mathsf{Ge}(\emptyset)$).

(i) A g-circle in $\mathbf{Basax}(2) + \mathbf{Ax}(\omega)^\sharp$ looks like as in Figure 123 (where the lines of our sheet of paper represent the lines in $\mathfrak{G}_\mathfrak{M}$).

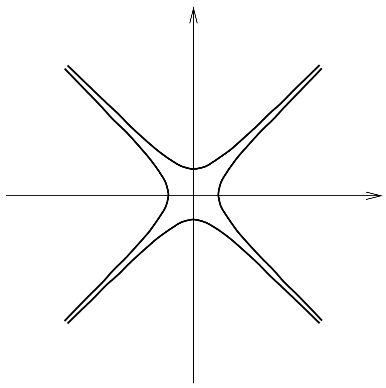


Figure 123: A g-circle in $\mathbf{Basax}{+}\mathbf{Ax}(\omega)^\sharp$. An eq-circle in **Basax** may look like this. Cf. also Fig.29 on p.51.

(ii) However, a g-circle in $\mathbf{Basax}(2)$ may look like as any one of those in Figure 124.

(iii) A g-circle in $\mathbf{Bax}(2)$ may even look like as in Figure 125.

(iv) If $n > 2$, a g-sphere as well as an eq-sphere in $\mathbf{Bax}^\oplus + \mathbf{Ax(eqspace)} + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\mathit{Triv}_t)^-$ may look like as in Figure 126. We note that the hyperboloid part is necessary, and the horizontal part is an (almost) arbitrary surface. Under these axioms the sides of the sphere always form a hyperboloid, while the top may be an arbitrarily complicated surface. The bottom surface is the reflection of the top one w.r.t. the origin. This g-sphere is typical of $\mathbf{Bax}^\oplus$ + "auxiliaries". If we throw **Ax(eqtime)** away then

[698] We just stated that $\mathfrak{G}_\mathfrak{M}$ is recoverable from g, under some assumptions on $\mathfrak{M}$. Hence if we learn intuitive ways of drawing g, then we also learn how to draw $\mathfrak{G}_\mathfrak{M}$ hence by the earlier duality theory, to draw $\mathfrak{M}$ itself.

[699] For completeness we note that circles were already touched upon in Chapter 2 (cf. p.52).

[700] By a g-sphere we understand a maximal set of such points of Mn whose g-distance is the same (constant) from a given point. Similarly for g-circles and for eq in place of g.

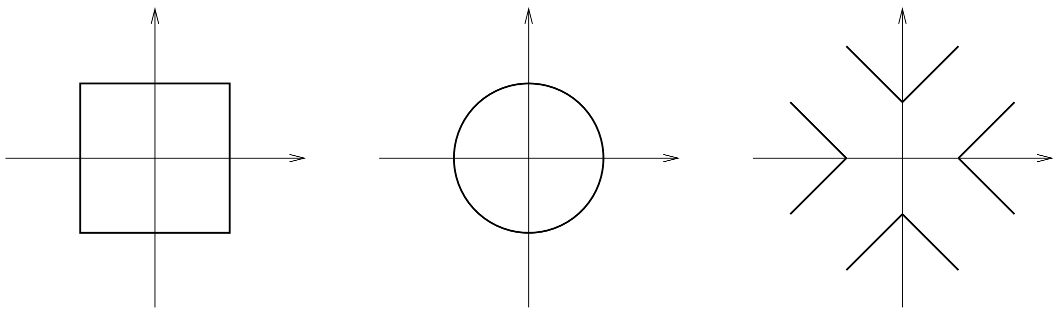


Figure 124: A g-circle in **Basax** may look like any of these. No one of these can be an eq-circle of **Basax**, cf. also Fig.29 on p.51.

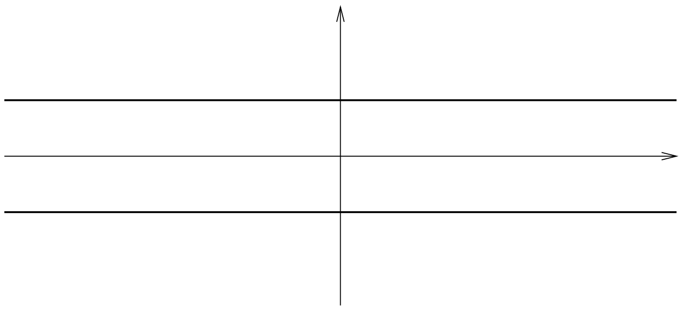


Figure 125: A g-circle in **Bax** may even look like this.

the top and bottom surfaces of the g-sphere may be replaced by clouds of points. If we throw $\mathbf{Ax}(\mathit{Triv}_t)^-$ away then the sides of the g-sphere may become "gapy".

A possible way of visualizing a relativistic geometry say $\mathfrak{G}_{\mathfrak{M}}$ (or equivalently the model $\mathfrak{M}$) is to draw a g-sphere or g-circle as in Figures 123–126. More precisely if we do not assume any "symmetry" property on $\mathfrak{M}$ then this picture will represent the model or geometry from the point of view of a certain observer. However assuming the axioms listed in item (iv) together with $\mathbf{Ax}(\uparrow\uparrow)$ ensure that such a drawing contains information about the world-views of all other observers too, hence about the whole model $\mathfrak{M}$ (or geometry), assuming $\mathbf{Ax}\heartsuit$ and $\mathbf{Ax(ext)}$ of course. Cf. Figure 29 on p.51 for more information in this direction.

Finally, we turn to connections with works of Busemann, e.g. Busemann [55], where Busemann shows how to modify (actually, localize) structures like our $\mathfrak{G}_{\mathfrak{M}}$ to obtain models for general relativity. So, in a sense, what comes below shows explicit connections with the structures $\mathfrak{G}_{\mathfrak{M}}$ studied in the present section and general relativity. At the same time, what comes below is an example of the streamlinings[701] of $\mathfrak{G}_{\mathfrak{M}}$ we elaborated in §6.7.

[701] "streamlining" means finding a streamlined reduct of $\mathfrak{G}_{\mathfrak{M}}$ definitionally equivalent to the original $\mathfrak{G}_{\mathfrak{M}}$.

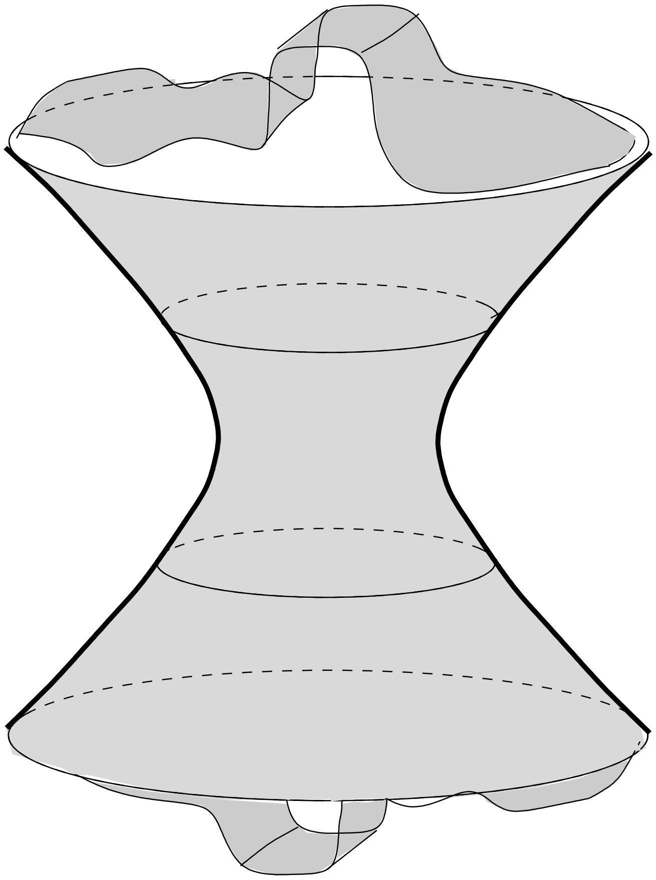


Figure 126: A g-sphere or an eq-sphere in $\mathbf{Bax}^{\oplus}(3)$ + **Ax(eqspace)** + **Ax(eqtime)** + $\mathbf{Ax(}\mathit{Triv}_t\mathbf{)^-}$.

4.6.1 The streamlined, partial metric $g^{\prec}$

Recall that the Reichenbachian relativistic geometry[702] $\mathfrak{G}^R_{\mathfrak{M}} = \langle Mn, \dots, g^R, \mathcal{T}^R \rangle$ associated with $\mathfrak{M}$ is defined in item (VI) of Def.4.2.3 on p.147 and is motivated by §4.5 of AMN [18]. $\mathsf{Ge}^R(Th)$ is the class of Reichenbachian relativistic geometries associated with Th, i.e.

$$\mathsf{Ge}^R(Th) :\stackrel{\text{def}}{=} \mathbf{I}\{\, \mathfrak{G}^R_{\mathfrak{M}} \ :\ \mathfrak{M} \in \mathsf{Mod}(Th)\,\}.^{703}$$

Definition 4.6.1 Assume $\mathfrak{G}$ is a relativistic (or a Reichenbachian relativistic) geometry.

(i) The reflexive hull $\preceq \, := \, \prec \cup \, \mathrm{Id}$ of $\prec$ is defined as follows:

$$a \preceq b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad [\, a \prec b \ \text{ or } \ a = b\,], \qquad a, b \in Mn.$$

(ii) The <u>*time-like-metric*</u>[704] $g^{\prec}$ is defined to be $g \restriction (\preceq)$, i.e.

$$g^{\prec} \ \stackrel{\text{def}}{=} \ \{\, \langle a, b, \lambda\rangle \in Mn \times Mn \times F \ :\ a \preceq b \ \text{ and } \ g(a,b) = \lambda\,\}.^{705}$$

(iii) $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ is called the <u>*time-like-metric reduct*</u> of $\mathfrak{G}$. For "time-like-metric reduct" we will also use the expressions "time-like-metric geometry", "time-like-metric structure", and "time-like-metric relativistic geometry".

$\lhd$

We will see that under some assumptions on $\mathfrak{M}$, $g^{\prec}$ satisfies certain very nice and familiar looking axioms, e.g. is more "streamlined" than g is, from the mathematical point of view, cf. p.347. Therefore we will often refer to $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ as the <u>streamlined partial metric</u> reduct of $\mathfrak{G}_{\mathfrak{M}}$. Beginning with p.347 we will see that in many regards $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ is the most streamlined reduct of $\mathfrak{G}_{\mathfrak{M}}$ and at the same time it seems to be rather suitable (to serve as a stepping-stone) for generalizations in the direction of general relativity.

The next theorem says that the Reichenbachian geometry $\mathfrak{G}^R_{\mathfrak{M}}$ is definable from its streamlined, time-like-metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$, under mild assumptions on $\mathfrak{M}$. The second theorem (Thm.4.6.3) says the same for the <u>full</u> geometry $\mathfrak{G}_{\mathfrak{M}}$, under some stronger conditions on $\mathfrak{M}$.

THEOREM 4.6.2

(i) $\mathsf{Ge}^R(Th)$ *is* <u>*definable*</u> *from its streamlined, simple reduct* $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ *more precisely from its reduct of language* $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$, *assuming* $n > 2$ *and* $Th \models \mathbf{Bax}^{-\oplus} + \mathbf{Ax(TwP)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$.

[702] Reichenbachian relativistic geometry is a short name for Reichenbachian version of the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$.

[703] We note that $\mathsf{Ge}^R(Th)$ coincides with $\mathsf{Ge}^5(Th)$, where $\mathsf{Ge}^5(Th)$ was defined in AMN [18, p.1125].

[704] "Time-like-metric" is the same as "streamlined partial metric".

[705] I.e. $g^{\prec}(a,b) = g(a,b)$ if $a \preceq b$ else is undefined.

(ii) *Statement (i) above remains true if the assumption* $\mathbf{Ax(TwP)}$ *is replaced by any one of* $\mathbf{R}(\mathbf{Ax\ syt_0}) + \mathbf{Ax}(\mathit{Triv})$, $\mathbf{Bax} + \mathbf{Ax(syt_0)}$.

(iii) *Statements (i) and (ii) above remain true if we omit the assumption* $n > 2$ *and assume instead* $\mathbf{Ax}(\uparrow\uparrow_0)$ *as a substitute.*

Idea of proof:

Case of (i): Assume the assumptions. Then $Th \models \mathbf{Ax(eqtime)}$ by Prop.6.8.25 on p.1201 in AMN [18] and there are no FTL observers by Thm.3.2.13 on p.118. By these (and by the assumptions, of course), one can check that the following definitions work.

$$\begin{aligned} Col^T(a,b,c) \;&\overset{\text{def}}{\Longleftrightarrow}\; \Big(g^+(a,b) = g^+(a,c) + g^+(c,b) \;\vee \\ &\qquad g^+(a,c) = g^+(a,b) + g^+(b,c) \;\vee \\ &\qquad g^+(b,c) = g^+(b,a) + g^+(a,c)\Big), \quad \text{where} \\ g^+(a,b) = \lambda \;&\overset{\text{def}}{\Longleftrightarrow}\; g^{\prec}(a,b) = \lambda \;\vee\; g^{\prec}(b,a) = \lambda. \end{aligned}$$

Bw is definable from Col^T by the proof of AMN [18, Thm.6.7.1 (p.1137)] and Fig.344 on p.1161.[706]

$$\begin{aligned} a \equiv^T b \;&\overset{\text{def}}{\Longleftrightarrow}\; (\exists c \in Mn)\, Col^T(a,b,c). \\ a \equiv^{Ph} b \;&\overset{\text{def}}{\Longleftrightarrow}\; a = b \;\vee\; \Big(a \not\equiv^T b \;\wedge\; (\exists c \in Mn)[c \neq b \;\wedge\; c \sim b \;\wedge \\ &\qquad\qquad (\forall d \in Mn)(Bw(b,d,c) \;\to\; a \equiv^T d)]\Big). \\ Col^{Ph}(a,b,c) \;&\overset{\text{def}}{\Longleftrightarrow}\; a \equiv^{Ph} b \equiv^{Ph} c \equiv^{Ph} a. \\ a \prec b \;&\overset{\text{def}}{\Longleftrightarrow}\; a \neq b \;\;\wedge\;(\exists \lambda \in F)\, g^{\prec}(a,b,\lambda). \\ g^R(a,b,\lambda) \;&\overset{\text{def}}{\Longleftrightarrow}\; g^{\prec}(a,b,\lambda) \;\vee\; g^{\prec}(b,a,\lambda) \;\vee (a \equiv^{Ph} b \;\wedge\; \lambda = 0). \end{aligned}$$

$\mathcal{T}^R$ is defined by g^R.

Case of (ii): Item (ii) follows by item (i), and by AMN [18, Thm.4.7.15 (p.622), Thm.4.2.9 (p.461)].

Case of (iii): Item (iii) follows by the proof of item (i) and Prop.4.2.31 on p.177. ∎

THEOREM 4.6.3 $\mathsf{Ge}(Th)$ *is definable from* $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ *i.e. from its reduct of language* $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$, *assuming* $n > 2$ *and* $Th \models \mathbf{Newbasax} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$.

Idea of proof: Assume the assumptions. By Thm.4.2.46 (p.191) and by Examples 4.2.54 (p.201), the $\prec$-free reducts of members of $\mathsf{Ge}(Th)$ are disjoint unions of $\prec$-free reducts of Minkowskian geometries. Using this fact together with Thm.4.6.2 and the theorems in §6.7.2 one can complete the proof. ∎

Axiomatics of $g^{\prec}$

Under some mild assumptions on $\mathfrak{M}$,[707] the following simple axioms $\mathbf{G_1}$–$\mathbf{G_4}$ hold in the time-like-metric reduct $\langle Mn, \mathbf{F_1};\, g\rangle$ of $\mathfrak{G}_{\mathfrak{M}}$.

[706] To avoid misunderstandings we note that this is Bw for all lines and not only for e.g. L^T or $L^T \cup L^{Ph}$.

[707] e.g. $\mathbf{Bax}^{-\oplus}, \mathbf{Ax(TwP)}, \mathbf{Ax}(\sqrt{\ }), \mathbf{Ax}(\uparrow\uparrow_0)$ are sufficient

$\mathbf{G_1}$ The domain $\preccurlyeq \; := \; Dom(g^{\prec})$ is a reflexive partial ordering.

$\mathbf{G_2}$ $g^{\prec}(x,y) \geq 0$ if it is defined.

$\mathbf{G_3}$ $g^{\prec}(x,y) = 0 \quad \Leftrightarrow \quad x = y.$

$\mathbf{G_4}$ $g^{\prec}(x,y) + g^{\prec}(y,z) \leq g^{\prec}(x,z) \qquad$ if $x \preccurlyeq y \preccurlyeq z$.

We define the axiom system **busg** as follows.

$$\mathbf{busg} :\stackrel{\text{def}}{=} \mathbf{G_1} + \mathbf{G_2} + \mathbf{G_3} + \mathbf{G_4}.$$

It is interesting to compare **busg** with the usual[708] axiomatizations of metric spaces (we feel that **busg** is closer to the usual axiomatizations of metrics[709] than e.g. the axioms which could describe g).

The above axiomatization **busg** is not unrelated to the one given in Busemann [55, p.7]. Unlike Busemann, however, we regard the topology on $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ to be defined from the partial metric $g^{\prec}$ (or from $\prec$) in the style of either Def.4.2.30(ii) (p.175) or of Def.4.2.3(VI) (p.148), i.e. in the style of our defining the Reichenbachian topology $\mathcal{T}^R$ from the Reichenbachian partial metric g^R.[710] I.e. for $e \in Mn$ and $\varepsilon \in {}^+F$ we let

$$S^{\prec}(e,\varepsilon) :\stackrel{\text{def}}{=} \{\, e_1 \in Mn \;:\; 0 < g^{\prec}(e,e_1) < \varepsilon \,\}.$$

Now, our topology $\mathcal{T}^{\prec}$ is the one generated by the subbase

$$\{\, S^{\prec}(e,\varepsilon) \;:\; e \in Mn,\; \varepsilon \in {}^+F \,\}.$$

When the topology $\mathcal{T}^{\prec}$ is present, we add to **busg** the extra axiom

$\mathbf{G_5}$ $\langle Mn, \mathcal{T}^{\prec}\rangle$ is a Hausdorff (i.e. T_2) space[711] and $g^{\prec} : Mn \times Mn \xrightarrow{\circ} \mathbf{F_0}$ is continuous.

It is shown in Busemann [55] that the topological structure

$$\langle Mn, \mathbf{F_1};\, g^{\prec}, \mathcal{T}^{\prec}\rangle$$

has many desirable properties from the point of view of mathematical elegance, and at the same time admits a relatively natural generalization in the direction of general relativity theory (cf. e.g. Busemann [55, p.7, axioms T_1–T_4]).

The generalization in the "local" direction of **busg** tailored for general relativity theory states only that first we are given a Hausdorff topology $\mathcal{T}^{\prec}$ and then for any point $e \in Mn$ there is a neighborhood U_e of e such that a partial ordering $\preccurlyeq_e$ and a partial function $g_e^{\prec}$ are defined on U_e. Then the axioms of **busg** are stated only for the little structures $\langle U_e, \mathbf{F_1};\, \preccurlyeq_e, g_e^{\prec}\rangle$, $e \in Mn$.[712] In addition to these axioms one has to add some consistency axioms for the case

[708] non-relativistic

[709] both in complexity and in spirit

[710] The difference between g^R and $g^{\prec}$ seems to be minor but is not negligible. Else: We note that instead of $g^{\prec}$ we could use $\prec$ for defining the topology in the style of Fig.81, p.176. Cf. Def.4.2.30 (ii), p.175.

[711] For Hausdorff spaces cf. footnote 1009 on p.1018 in AMN [18].

[712] It would be sufficient to write $\langle U_e, \mathbf{F_1};\, g_e^{\prec}\rangle$, $e \in Mn$ for these structures, since $\preccurlyeq_e$ is obviously definable from $g_e^{\prec}$.

when U_e and $U_{e'}$ overlap. These consistency axioms are rather simple and natural, we do not recall them, they can be found in Busemann [55, p.7] axiom T_4. The so obtained local version of **busg** is completely consistent with (and is applicable to) general relativity theory, cf. Busemann [55] for more information on this. Summing it up, the general relativistic versions of the time-like-metric structures $\langle Mn, \mathbf{F_1};\, g^{\prec}, \mathcal{T}^{\prec}\rangle$ look like $\langle Mn, \mathbf{F_1};\, \mathcal{T}^{\prec}, \preccurlyeq_e, g_e^{\prec}\rangle_{e\in Mn}$ (cf. the definition of ${}^n\mathbf{F}_1$ on p.16 for the $\langle \dots, g_e^{\prec}\rangle_{e\in Mn}$ notation). Further, the class of these structures is axiomatized by the list of axioms just quoted from Busemann [55, p.7] (ending with T_4).

In connection with the general relativistic (i.e. localised) structures $\langle Mn, \mathbf{F_1};\, \mathcal{T}^{\prec}, \preccurlyeq_e, g_e^{\prec}\rangle_{e\in Mn}$ we note that although we included the topology $\mathcal{T}^{\prec}$ into the structure, it is definable from the rest $GG := \langle Mn, \mathbf{F_1};\, \preccurlyeq_e, g_e^{\prec}\rangle_{e\in Mn}$. Therefore one can define GG without $\mathcal{T}^{\prec}$ and then later one can define $\mathcal{T}^{\prec}$ from GG. Namely, assume $e \in Mn$ and $\varepsilon \in {}^{+}F$. Then

$$S^{\prec}(e,\varepsilon) :\stackrel{\text{def}}{=} \{\, e_1 \in Mn \,:\, 0 < g_e^{\prec}(e, e_1) < \varepsilon \,\}$$

is an open set, and it is an element of the subbase of $\mathcal{T}^{\prec}$ we want to define. Now, we postulate that

$$\{\, S^{\prec}(e,\varepsilon) \,:\, e \in Mn,\ \varepsilon \in {}^{+}F \,\}$$

is a subbase of our topology $\mathcal{T}^{\prec}$. We note this only as a possibility; we do not explore the general relativistic time-like-metric structures GG, in this section any further.

Remark 4.6.4 In the language of time-like-metric structures $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ we could define a kind of collinearity relation $coll^{\prec}$ the following way and could enrich the axiom-system **busg** by adding natural conditions on this collinearity: First we define

$$Bw^{\prec}(a,b,c) \quad \stackrel{\text{def}}{\iff} \quad g^{\prec}(a,c) = g^{\prec}(a,b) + g^{\prec}(b,c).$$

Then we define $coll^{\prec}$ from $Bw^{\prec}$ basically the same way as $coll$ was defined from Bw on p.159. $\lhd$

It would be interesting to know how many further axioms we need to add to **busg** in order to ensure that the partial metric structure $\langle Mn, \mathbf{F_0};\, g^{\prec}\rangle$ comes from a model of one of our relativity theories $\mathsf{Mod}(Th)$. Looking into this might be a nice future research task.

4.7 Geodesics

In the present section (taken together with its counterpart, §6.8, in AMN [18]) we discuss geodesics which, among other things, will help us to understand the connections between g and L. In later work, in moving in the direction of general relativity, geodesics will play an important role (they do so already in the case of accelerated observers even in "flat" space-time).[713] In moving towards general relativity geodesics will *replace* L as possible life-lines of inertial bodies. (They will play other important roles, e.g. they can be used for recognizing curvature of space-time). At the same time, studying geodesics may be considered as a continuation of §4.6 discussing recoverability of various parts (or reducts) of our relativistic geometries from each other. Geodesics can be regarded as an attempt to recover the lines of our geometry, basically, from g, in a style different from the Alexandrov-Zeeman style proofs in AMN [18, §6.7.2].[714] For completeness we note that by Corollary 6.7.15 in AMN [18, p.1145], the present author proved that L and $\perp$ are first-order logic definable from eq as well as from g under some reasonable assumptions on $\mathfrak{M}$ (e.g. $(\mathbf{Basax}+\mathbf{Ax}(\mathit{Triv})+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax}(\mathbf{eqtime}))$ is sufficient for this).

Though we will not prove this, by using geodesics one can recover from $g, \equiv^T, \equiv^{Ph}, \equiv^S$ [715] the potential life-lines of inertial bodies even when the axiom **Det** [716] is not assumed (but certain conditions are still needed, of course). Roughly speaking, in generalizations of our geometries in the direction of general relativity (cf. e.g. the geometries GG on p.349 in §4.6.1), geodesics will remain suitable for representing life-lines of inertial bodies. Further, time-like geodesics will be the possible life-lines of inertial observers, photon-like geodesics will be the life-lines of photons, while space-like geodesics can be regarded as potential life-lines of hypothetical FTL particles called tachyons in the literature (assuming such things exist); all this is understood under sufficient conditions. Already in the world-view of an accelerated observer[717], say m, it will be convenient to say that for m the life-lines of inertial bodies are geodesics [determined by $g, \mathbf{F_0}, \equiv^T, \equiv^{Ph}$] because in the world-view $w_m : {}^nF \longrightarrow Mn$ of m the Euclidean lines of ${}^n\mathbf{F}$ do not necessarily correspond to inertial bodies (if m is really accelerated).[718]

To make a long story short, the present section on geodesics intends to prepare the road

[713] Cf. e.g. [24], [19], [26]. For completeness we note that sometimes geodesics are used in special relativity, too, cf. e.g. Friedman [91, pp.125-126, 128ff].

[714] To be able to use g, we will need its codomain $\mathbf{F_0}$, too. To make our life easier, we will also use $\equiv^T$, $\equiv^{Ph}$, $\equiv^S$, but with sufficient (coding) effort these data could be recovered from g, where g is understood together with its domain Mn and codomain $\mathbf{F_0}$. We will not discuss here how, under sufficient conditions $\equiv^T$, $\equiv^{Ph}$ are recoverable from $\langle Mn, \mathbf{F_0}; g\rangle$. Cf. Remark 4.2.41 on p.183. Cf. also the first 15 lines of (III) on p.1166 of AMN [18]. On p.1150 of AMN [18] we used $\mathbf{F_1}$ as the codomain of g. The reason for the difference is that here we think of g slightly differently than we did there. So this is not an inconsistency, but simply a change in perspective. The choice of perspective depends on for what purposes we want to use g. (Once we identify it with $\langle Mn, \mathbf{F_0}; g\rangle$ and once with $\langle Mn, \mathbf{F_1}; g\rangle$.) For completeness we note that $\equiv^T, \equiv^{Ph}, \equiv^S$ are definable from g (more precisely, from $\langle Mn, \mathbf{F_0}; g\rangle$) if $n > 2$ and some conditions hold, cf. items 6.7.38-6.7.39 (p.1167) in AMN [18].

[715] and $\mathbf{F_0}$, Mn of course

[716] Cf. §4.4, p.275 for **Det** (**Det** says that "points determine lines").

[717] Cf. e.g. [26] and the relevant parts of this work.

[718] A more important point will be that in general relativity the life-lines of inertial bodies do not satisfy the axiom **Det**, i.e. different geodesics can meet in several points. This is true in the approximation of general relativity built on "special relativity"+"accelerated observers"+"Newtonian approximations" in Rindler [222, §7.7, e.g. item (7.28) on p.124].

for generalizations (in the direction of general relativity). For further motivation we refer to Figure 134 (p.365), Figure 83 (p.187) and Figure 99 (p.277).

Remark 4.7.1 We note that we could have based our theory of geodesics entirely on the streamlined, time-like metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ of $\mathfrak{G}_{\mathfrak{M}}$. This would have advantages (i) from the point of view of aesthetics and (ii) from the point of view of generalizability towards general relativity (as the latter is illustrated in Busemann [55]). To save space we use below a "bigger" reduct. We leave it as a future research task to elaborate a version of the present section (§4.7 "Geodesics") based entirely on the streamlined, time-like metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$.

$\lhd$

We base our definition of geodesics in $\mathfrak{G}_{\mathfrak{M}}$ (Def.4.7.2) below on the definition of geodesics in e.g. Busemann [54], [55], cf. also Busemann-Beem [56]. Part of the relevant mathematical literature uses the same kind of definition while another part uses a definition (of geodesics) which goes e.g. via using derivatives[719]. (Within this, they distinguish "affine geodesics" and "metric geodesics" which distinction is nicely illuminated e.g. in Friedman [91, pp.349,357].)[720] Busemann's version is simpler (as far as we have a metric around). One might think that a large part of the literature uses the derivatives oriented version because that is needed for general relativity. However, this is not the case since Busemann [55] shows that general relativity can be based on his simple definitions.[721] So, here we stick with Busemann's simple definition (especially because in the introduction to AMN [18] we adopted a policy to keep things as simple as possible, postponing the introduction of more complicated ideas to the point where they become useful/needed). A further motivation for adopting Busemann's definition of geodesics is that Busemann [54] is an ambitious mathematics (modern geometry) book whose main subject matter is the study of geodesics.

The definition of geodesics (Def.4.7.2) below is <u>not</u> intended to be a first-order logic definition over (a reduct of) the structure $\mathfrak{G}$. This causes no harm to our first-order logic oriented philosophy (for building up physical theories). We will return to discussing this briefly in Remark 4.7.3 below the definition.

Definition 4.7.2 (Geodesics) Assume $\mathfrak{G}$ is a relativistic geometry.

1. Throughout $\mathbf{F_0} = \langle F;\, 0, +, \leq\rangle$ is the ordered group reduct of the sort $\mathbf{F_1}$ of $\mathfrak{G}$.

2. The <u>*pseudo-metric reduct*</u> M of $\mathfrak{G}$ is defined as follows.

$$M \stackrel{\text{def}}{:=} \langle Mn, \mathbf{F_0};\, g, \equiv^T, \equiv^{Ph}, \equiv^S\rangle.$$

In the definition of geodesics of the geometry $\mathfrak{G}$ we will use <u>only</u> its pseudo-metric reduct. If we wanted to concentrate on the time-like geodesics, then it would be sufficient to use the streamlined, time-like-metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ discussed in §4.6.1 (p.346).

[719] Cf. e.g. d'Inverno [73, pp.75, 83, 99] or Misner-Thorne-Wheeler [192], or Hawking-Ellis [116], or Hicks [124, pp.19,27].

[720] In Friedman [91, p,357] it is explained that the above "metric-affine" distinction behaves differently in non-relativistic geometries and in relativistic ones (this might perhaps be related to our Corollary 4.7.13).

[721] It seems a more likely explanation that the derivatives-oriented version is suitable for discussing the metric geodesic affine geodesic distinction and that it can be used on a level of abstraction where we throw g and eq away (i.e. we don't have a metric) e.g. in differential topological approaches to relativity.

3. Let $\ell \subseteq Mn$. Then ℓ is called a *photon-like geodesic* iff
$$(\forall a, b \in \ell)\, a \equiv^{Ph} b.$$
Any photon-like geodesic is also called a *photon-like quasi geodesic*, and a *photon-like Archimedean geodesic*.

4. By an *interval of* $\mathbf{F_0}$ we mean an open interval
$$(x, y) := \{\, z \in F \,:\, x < z < y \,\},$$
where $x, y \in F \cup \{-\infty, \infty\}$, and $x < y$.[722]

5. Let $\ell \subseteq Mn$. By a *parameterization* of ℓ we understand a function h mapping an interval of $\mathbf{F_0}$ onto ℓ, such that h is locally distance preserving, i.e. for any $z \in Dom(h)$ there is $\varepsilon \in {}^{+}F$ such that, letting $D := (z - \varepsilon, z + \varepsilon)$, $(*)$ below holds.[723]

   $(*)$ $h \restriction D$ is distance preserving, i.e.
$$(\forall x, y \in D)\, g(h(x), h(y)) = |x - y|.$$
   If ℓ admits such a parametrization, then we call it a *parametrizable curve*.

6. Let $\ell \subseteq Mn$. ℓ is called a *time-like quasi geodesic* iff there is a parameterization h of ℓ such that for every $z \in Dom(h)$ there is $\varepsilon \in {}^{+}F$ such that, for $D := (z - \varepsilon, z + \varepsilon)$, $(**)$ below holds.

   $(**)$
$$(\forall x, y \in D)\, h(x) \equiv^{T} h(y).$$

7. A time-like quasi geodesic ℓ is called a *short time-like geodesic* iff there is a parameterization h of ℓ such that, for $D := Dom(h)$, $(*)$ and $(**)$ above hold.

8. Let ℓ, h be as in item 5 above. Intuitively, ℓ is a *space-like quasi geodesic* if it is a union of "intervals" $h[D]$ each one of which consists of events 1/2-simultaneous for some observer, cf. Figure 127. Formally:

   ℓ is called a *space-like quasi geodesic* iff there is a parameterization h of ℓ such that for any $z \in Dom(h)$ there is $\varepsilon \in {}^{+}F$ such that, for $D := (z - \varepsilon, z + \varepsilon)$, $(***)$ below holds. Intuitively, the second part of $(***)$ says that there is an observer who thinks that all the events in $h[D]$ are 1/2-simultaneous, cf. Figure 127.

   $(***)$ $(\forall x, y \in D)\, h(x) \equiv^{S} h(y)$ and there are a short time-like geodesic ℓ' and $a \in \ell'$ such that $(\forall x \in D)(\exists c, d \in \ell')$ $[c \neq d \;\wedge\; g(a, c) = g(a, d) \;\wedge\; c \equiv^{Ph} h(x) \equiv^{Ph} d\,]$,[724]

   see Figure 127.

---

[722] In this section $-\infty \neq \infty$ deviating from our convention on p.534 of AMN [18]. As usual, $-\infty < x < \infty$, for any $x \in F$.

[723] Note that such a parameterization h : "interval of $\mathbf{F_0}$" $\longrightarrow \ell$ is always continuous w.r.t. the natural topology on $\mathbf{F_0}$ and the topology induced by g on ℓ. I.e. condition $(*)$ (postulated for every D as above) implies this kind of continuity. This continuity is slightly weaker than continuity w.r.t. the topology $\mathcal{T}$ of $\mathfrak{G}$; the latter amounts to viewing h as h : "interval of $\mathbf{F_0}$" $\longrightarrow Mn$.

[724] We note for "general relativitists" that if we make the above condition local by requiring $\ell' \cap D \neq \emptyset$ then the condition will get only stronger which means that our theorems will get weaker, i.e. omitting this locality condition makes our theorems stronger.

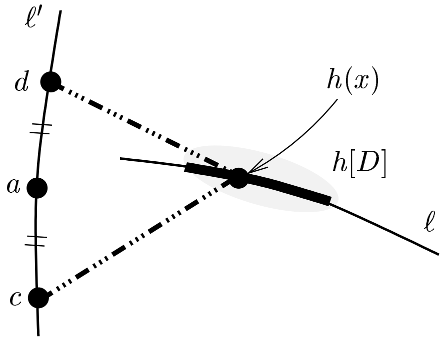


Figure 127: Illustration for $(***)$.

9. Let $\ell \subseteq Mn$. ℓ is called a *quasi geodesic* iff it is a time-like or a photon-like or a space-like quasi geodesic.

10. A quasi geodesic ℓ is called a *time-like geodesic* iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $\varepsilon \in {}^{+}F$ such that for any $z \in (x, y)$, letting $D := (z - \varepsilon, z + \varepsilon)$, $(*)$ and $(**)$ above hold.

11. A quasi geodesic ℓ is called a *space-like geodesic* iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $\varepsilon \in {}^{+}F$ such that for any $z \in (x, y)$, letting $D := (z - \varepsilon, z + \varepsilon)$, $(*)$ and $(***)$ above hold.

12. A quasi geodesic ℓ is called a *geodesic* iff it is a time-like or a photon-like or a space-like geodesic.

13. A geodesic ℓ is called a *time-like Archimedean geodesic* iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $k \in \omega$ and intervals $I_0, \dots, I_k$ of $\mathbf{F_0}$ such that

    $(x, y) \subseteq I_0 \cup \ldots \cup I_k \quad \wedge \quad (\forall i \in k)\, I_i \cap I_{i+1} \neq \emptyset \quad \wedge$
    $(\forall i \in (k+1))\, [\,(*)$ and $(**)$ above hold for $D := I_i\,]$.

14. A geodesic ℓ is called a *space-like Archimedean geodesic* iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $k \in \omega$ and intervals $I_0, \dots, I_k$ of $\mathbf{F_0}$ such that

    $(x, y) \subseteq I_0 \cup \ldots \cup I_k \quad \wedge \quad (\forall i \in k)\, I_i \cap I_{i+1} \neq \emptyset \quad \wedge$
    $(\forall i \in (k+1))\, [\,(*)$ and $(***)$ above hold for $D := I_i\,]$.

15. A geodesic ℓ is called an *Archimedean geodesic* iff it is a time-like or a photon-like or a space-like Archimedean geodesic.

16. A space-like geodesic ℓ is called a *short space-like geodesic* iff there is a parameterization h of ℓ such that, for $D := Dom(h)$, $(*)$ and $(***)$ above hold.

17. A geodesic ℓ is called a *short geodesic* iff it is a photon-like geodesic or it is a time-like short geodesic or it is a space-like short geodesic.

18. A geodesic ℓ is called a *strong geodesic* iff it is either photon-like or there is a parameterization h of ℓ which is continuous w.r.t. the natural topology on $\mathbf{F_0}$ and the relativistic

topology $\mathcal{T}$ of $\mathfrak{G}$,[725] and h satisfies the conditions in the definition of geodesics (in items 10–12 above).[726] We define the *strong* versions of our special kinds of geodesics defined in items 6–17 completely analogously, i.e. by requiring that the parameterization h occuring in their definitions is continuous w.r.t. the relativistic topology $\mathcal{T}$ of $\mathfrak{G}$.

19. A geodesic ℓ is called a *maximal geodesic* iff

$$(\forall \text{ geodesic } \ell')[\ell' \supseteq \ell \ \rightarrow \ \ell' = \ell].$$

The definition of "maximality" remains completely analogous for special kinds of geodesics in place of just geodesics. (E.g. a maximal strong space-like quasi geodesic is a strong space-like quasi geodesic which is maximal among the strong space-like quasi geodesics.)

20. Let $e \in Mn$ and $\varepsilon \in {}^+F$. Let us recall that the ε-neighborhood of e is defined as

$$S(e,\varepsilon) := \{\, e_1 \in Mn \ : \ g(e,e_1) < \varepsilon \,\}.$$ [727]

21. Let $\ell \subseteq Mn$. Then ℓ is called a *weak geodesic* iff

$$(\forall e \in \ell)(\exists \varepsilon \in {}^+F)\,[\, g \restriction (\ell \cap S(e,\varepsilon)) \text{ is } \underline{additive}\,],$$

where *additivity* means that, letting $D := \ell \cap S(e,\varepsilon)$,

$$(\forall a,b \in D)\ (g(a,b) \text{ is defined}) \ \wedge$$
$$(\forall a,b,c \in D)\ [\, g(a,b), g(b,c) \le g(a,c) \ \rightarrow \ g(a,c) = g(a,b) + g(b,c)\,].$$

A quasi geodesic which is also a weak geodesic will be called *locally additive*.[728]

22. Let $\ell \subseteq Mn$. ℓ is called *additive* iff $g \restriction \ell$ is additive.

23. A weak geodesic ℓ is called a *continuous weak geodesic* iff there is a continuous function h mapping an interval of $\mathbf{F_0}$ onto ℓ.

$\triangleleft$

We will see in Thm.4.7.12 (p.361) that the second part of condition $(***)$ on space-like quasi geodesics and geodesics in the above definition is needed, cf. Figure 133 (p.362).

---

[725] i.e. h is continuous from an interval of $\mathbf{F_0}$ into the topology $\langle Mn; \mathcal{T}\rangle$

[726] Assume ℓ is a strong geodesic with parameterization h. Then h is a "local" homeomorphism in the sense that $(\forall x \in Dom(h))(\exists \varepsilon \in {}^+F)\,[\, h \restriction (x-\varepsilon, x+\varepsilon)$ is a homeomorphism w.r.t. the relativistic topology $\mathcal{T}$ of $\mathfrak{G}\,]$. Cf. the notion of a parameterized curve in Hicks [124, p.10] and curves in Kurusa [150]. In passing we note that in general, continuity w.r.t. $\langle Mn; \mathcal{T}\rangle$ is stronger than continuity w.r.t. the topology on ℓ induced by g as discussed in footnote 723 (p.352). Hence, in general, there are fewer strong geodesics than geodesics.

[727] In the case of Minkowskian geometry our notation $S(e,\varepsilon)$ might be ambiguous since it both denotes the relativistic "ε-sphere" and the Euclidean "ε-sphere". Throughout the present section by $S(e,\varepsilon)$ we always mean the relativistic sphere.

[728] Therefore being locally additive is a property of geodesics and in some situations there may be fewer locally additive geodesics than geodesics.

Remark 4.7.3 (Connections with first-order logic definability) We also note that our definition of geodesics over $\langle Mn, \mathbf{F_0}; g, \ldots, \equiv^S\rangle$ is not a first-order logic definition in the sense of §4.3. To save space, here we do not address the question of how and under what price[729] we could turn the definition of geodesics into a first-order logic one. We only note that if we include the geodesics together with their parameterization into $\mathfrak{G}$ obtaining a structure $\langle \mathfrak{G}, geodesics, pameterizations \ldots\rangle$ as extra sorts[730], then things can be arranged so that the class of so expanded structures does admit a first-order logic axiomatization. We note that by the above we do not mean that the $\mathfrak{G}$-reduct of $\langle \mathfrak{G}, geodesics, \ldots\rangle$ would determine the rest of the structure (e.g. the sort geodesics) uniquely. We only mean to say that in the expanded structure $\langle \mathfrak{G}, geodesics, \ldots\rangle$ the geodesics would behave well enough for our working with them in accordance with our intuition and for basing our relativity theoretic ideas on them. (This is very much like the difference between standard models of higher-order logic and Henkin-style nonstandard models of that logic. Our expanded structures $\langle \mathfrak{G}, geodesics, \ldots\rangle$ are very much like Henkin-style nonstandard models.)

In passing we note that if we assume enough axioms of special relativity on $\mathfrak{M}$, then geodesics become first-order logic definable over $\langle Mn; g, \equiv^T, \equiv^{Ph}, \equiv^S\rangle$, but the greatest value of geodesics is in their use in general relativity where these axioms are not assumed. Hence we do not discuss this direction here. $\lhd$

In passing we note that for the purposes of accelerated observers and general relativity (to come in a later work) "quasi geodesic", "short geodesic" and "geodesic" are "local" concepts while "maximal geodesic" seems to be more on the "global" side. Further, in general relativity the emphasis is on time-like and photon-like geodesics, cf. e.g. Busemann [54, 55] or Ehlers-Pirani-Schild [78].

In the present section we will concentrate on quasi geodesics, geodesics, Archimedean geodesics, and the maximal versions of these geodesics. By our definition,

$$\ell \text{ is an Archimedean geodesic} \quad \Rightarrow \quad \ell \text{ is a geodesic} \quad \Rightarrow \quad \ell \text{ is a quasi geodesic.}$$

The analogous statements hold, respectively, for time-like, space-like, and photon-like versions of Archimedean geodesics, geodesics and quasi geodesics, e.g. (ℓ is an Archimedean time-like geodesic) $\Rightarrow$ (ℓ is a time-like geodesic) $\Rightarrow$ (ℓ is a time-like quasi geodesic). In some of the cases these implications hold in the other direction, too. In connection with this we include Propositions 4.7.4 and 4.7.6 below.

PROPOSITION 4.7.4 *Assume $M = \langle Mn, \mathbf{F_0}; \ldots\rangle$ is the pseudo-metric reduct of a relativistic geometry. Assume that $\mathbf{F_0}$ is isomorphic with the ordered additive group reduct of the ordered field $\mathfrak{R}$ of reals. Let $\ell \subseteq Mn$. Then*

$$\ell \textit{ is an Archimedean geodesic} \quad \Leftrightarrow \quad \ell \textit{ is a geodesic} \quad \Leftrightarrow \quad \ell \textit{ is a quasi geodesic.}$$

The analogous statements hold, respectively, for space-like, time-like and photon-like versions of Archimedean geodesics, geodesics and quasi geodesics.

[729] we mean under what extra conditions and what modification of the definition of $\mathfrak{G}_{\mathfrak{M}}$

[730] Actually, it is enough to include *parameterizations* of geodesics as an extra sort Geod together with an extra inter-sort operation $\mathsf{valueof} : \mathsf{Geod} \times F \to Mn$. Intuitively, for $h \in \mathsf{Geod}$, $e = \mathsf{valueof}(h, x)$ means that $e = h(x)$, i.e. e is the value of parameterization h at value $x \in F$. Actually, $\mathsf{valueof}$ is a partial function only since we do not want to require $Dom(h) = F$. The details are analogous with the style of Nonstandard Dynamic Logic, cf. e.g. Sain [227], Andréka, Goranko et al. [12].

We omit the easy **proof.** ∎

For stating our next proposition we need the following definition.

Definition 4.7.5 An ordered group $\langle G;\, 0, +, \leq\rangle$ is said to be <u>Archimedean</u> iff for any $a, b \in G$

$$(\forall i \in \omega) ia < b \quad \Rightarrow \quad a = 0.$$ [731]

$\lhd$

We note that an ordered field $\mathfrak{F}$ is Archimedean iff its ordered additive group reduct $\mathbf{F_0}$ is Archimedean in the sense of the above definition.

PROPOSITION 4.7.6 *Assume $M = \langle Mn, \mathbf{F_0};\, \ldots\rangle$ is the pseudo-metric reduct of a relativistic geometry. Assume that $\mathbf{F_0}$ is Archimedean. Let $\ell \subseteq Mn$. Then*

$$\ell \textit{ is an Archimedean geodesic} \quad \Leftrightarrow \quad \ell \textit{ is a geodesic.}$$

The analogous statements hold, respectively, for space-like, time-like and photon-like versions of Archimedean geodesics and geodesics.

We omit the easy **proof.** ∎

Let us consider how the notion of geodesics helps us to recover the "truly geometric parts" L^T, L^{Ph} etc. from g and $\equiv^T, \equiv^{Ph}$.

Let us recall that the geometry $\mathfrak{G}_{\mathfrak{M}}$ associated with a model $\mathfrak{M}$ looks like

$$\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \mathbf{F_1}, L;\, L^T, L^{Ph}, L^S, \in, \prec, \equiv^T, \equiv^{Ph}, \equiv^S, Bw, \perp_r, eq, g, \mathcal{T}\rangle.$$ [732]

Among other things, below we compare <u>lines</u> (i.e. elements of L) with <u>geodesics</u>.[733] We have time-like, photon-like and space-like lines and the same applies to geodesics. This gives us, roughly, 4 kinds of comparisons, lines with geodesics in general, and then special lines with special geodesics.

Recall that we call the elements of L <u>lines</u> of $\mathfrak{G}_{\mathfrak{M}}$. Above we defined the geodesics of $\mathfrak{G}_{\mathfrak{M}}$, but they are <u>not</u> necessarily the same as lines of $\mathfrak{G}_{\mathfrak{M}}$. We will elaborate this subject in the following discussion of the theorems which will come soon. We will see that, under some assumptions on Th, all elements of L turn out to be geodesics, i.e.

$$L \subseteq \text{(geodesics)},$$

in $\mathsf{Ge}(Th)$ of course (cf. Prop.4.7.7).[734] Under stronger assumptions, L coincides with the set of maximal geodesics, i.e.

$$L = \text{(maximal geodesics)}$$

(AMN [18, Cor.6.8.33, p.1204]). Under somewhat milder assumptions, we will already have

$$L^T = \text{(maximal time-like geodesics)}$$

[731] Here ia is understood in the sense $3a = a + a + a$.

[733] We mean to compare lines of $\mathfrak{G}$ with geodesics of the same $\mathfrak{G}$.

[734] As a contrast, we will have <u>no</u> theorem saying the reverse of this, i.e. that under some assumptions on Th, say, $L \supseteq$ (maximal geodesics) without claiming equality, i.e. without claiming $L =$ (maximal geodesics).

(AMN [18, Thm.6.8.24, p.1200 and Corollary 6.8.27, p.1202]).

$$L^{Ph} \;=\; \text{(maximal photon-like geodesics)},$$

under some (reasonably mild) assumptions on Th (item (v) of Prop.4.7.7). The conditions in the above quoted theorems are quite strong, hence we will address the issue whether they are really needed. We will do this in the form of conjectures, open problems, etc.

(We will also see that the various kinds of geodesics (e.g. "maximal geodesics") introduced in Def.4.7.2 are needed for forming a clear picture of the subject of this section.)

The following proposition says that lines (i.e. elements of L) are geodesics under certain assumptions.

PROPOSITION 4.7.7 *Assume* $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqm)}$. *Then the elements of* L *are geodesics. Further,*

(i)

$$\begin{aligned} L^T &\subseteq \textit{(time-like Archimedean geodesics)},^{735} \quad \textit{and} \\ L^T &\subseteq \textit{(short time-like geodesics)} \\ L^T &\subseteq \textit{(maximal locally additive time-like geodesics)}. \end{aligned}$$

(ii) $L^{Ph} \subseteq \textit{(photon-like geodesics)} = \textit{(photon-like Archimedean geodesics)}$.

(iii)

$$L^S \subseteq \textit{(space-like Archimedean geodesics)}.$$

(iv) *Assume* $\mathbf{Reich(Bax)}^{\oplus} + \mathbf{Ax(diswind)}$. *Then*

$$L^{Ph} \;=\; \textit{(maximal photon-like geodesics)}.$$

For a **proof** and an essentially sharper formulation of this proposition we refer to AMN [18, 4.7.7, 6.8.8]. Here we recall only Figure 128 illustrating ideas for the proof.

Remark 4.7.8 (Discussing some of the conditions of Prop.4.7.7)
Item (iv) of Prop.4.7.7 does not generalize form $\mathbf{Reich(Bax)}^{\oplus}$ to $\mathbf{Bax}^{-\oplus}$. The idea of a proof is illustrated in the right-hand side of Figure 129. In the figure ℓ is a maximal photon-like geodesic. We note that in **Reich**(**Bax**) the right-hand side of Figure 129 is excluded by the characterization theorem of **Reich**(Th)-models in AMN [18, §4.5]. This is the theorem which states that every model of **Reich**(Th) can be obtained from some model of Th by "relativizing" with an artificial simultaneity (under some conditions on Th). $\lhd$

In connection with Proposition 4.7.7 above we ask the following.

[735] Note that (Archimedean geodesics) $\subseteq$ (geodesics) and the same holds for time-like ones etc. Therefore the conclusions of the present proposition remain true if the adjective Archimedean is omitted. Later we will not return to indicating the consequences of this observations explicitly.

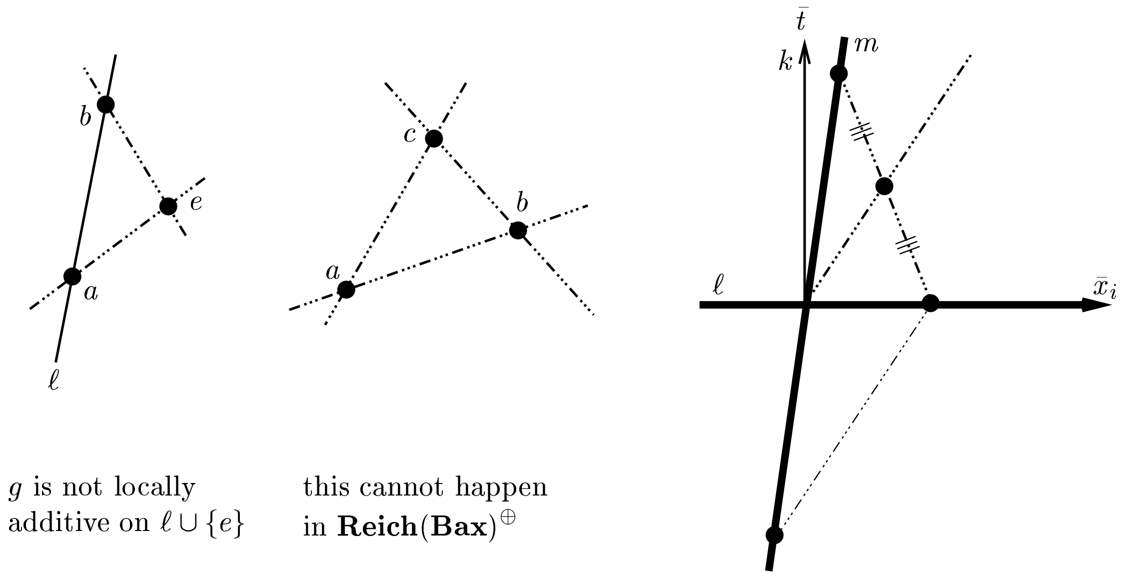


Figure 128: Illustration for the proof of Prop.4.7.7.

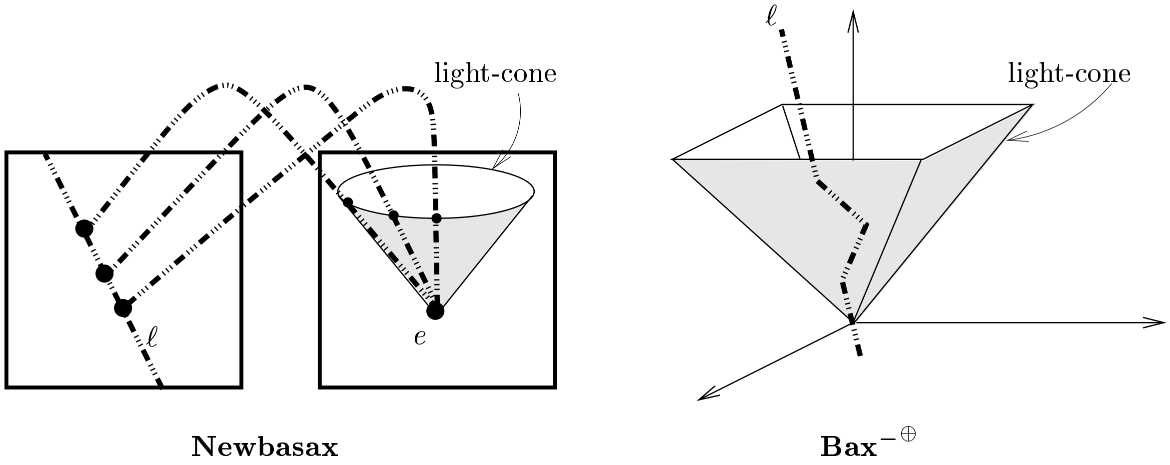


Figure 129: Illustration for Remark 4.7.8. On the right-hand side ℓ is a maximal photon-like geodesic. (Here ℓ is on the surface of the light-cone.) On the left-hand side, $\ell \cup \{e\}$ is a photon-like geodesic.

QUESTION 4.7.9 *Assume $n > 2$, $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)})$ and that $\mathbf{F_0}$ is isomorphic with the ordered additive group reduct of $\mathfrak{R}$. Is there a time-like geodesic ℓ such that ℓ has a non-injective parameterization?* $\triangleleft$

In connection with the above question we note that if we assume that $\mathbf{F_0}$ is non-Archimedean (instead of $\mathbf{F_0} \cong \langle \mathrm{R}; 0, +, \leq \rangle$), then the answer is "yes", cf. the proof of Theorem 4.7.11 (p.359), and Fig.130 (p.360).

Intuitively, Question 4.7.9 is equivalent with the following question. Does there exist a geodesic *time-travel*, i.e. can the life-line of a time-traveler who meets his younger himself be a geodesic?

Remark 4.7.10 In AMN [18, Thm.6.8.11] the present author proved that (i) in Minkowskian geometries the maximal Archimedean geodesics are exactly the lines, (ii) in Minkowskian geometries over Archimedean fields the maximal geodesics are exactly the lines, and (iii) in the Minkowskian geometry over the field $\mathfrak{R}$ of reals the maximal quasi geodesics are exactly the lines. This remains true for the L^T, L^{Ph}, L^S cases, too (when these are treated separately). For the details we refer to AMN [18, Thm.6.8.11].

Roughly, the just quoted theorem (Thm.6.8.11 of AMN [18]) says that in $Mink(n, \mathfrak{F})$ the various geodesics behave as one would expect them to behave, assuming $\mathfrak{F}$ is Euclidean. Among other things, the following theorem says that the condition that $\mathfrak{F}$ is Euclidean cannot be omitted from this theorem.

THEOREM 4.7.11 *Assume $\mathfrak{F}$ is a non-Archimedean and Euclidean ordered field. Then (i)–(iv) below hold.*

(i) *For any $n \geq 2$ in $Mink(n, \mathfrak{F})$*

$$\begin{array}{lll} L^T & \cap & (\textit{maximal time-like geodesics}) \ = \ \emptyset, \qquad \underline{\textit{but}} \\ L^T & \subseteq & (\textit{maximal locally additive time-like geodesics}), \qquad \textit{while} \\ L^T & \not\supseteq & (\textit{maximal locally additive time-like geodesics}). \end{array}$$

(ii) *For any $n \geq 2$ in $Mink(n, \mathfrak{F})$*

$$L^S \cap (\textit{maximal space-like geodesics}) \ = \ \emptyset,$$

i.e. if $\ell \in L^S$ then ℓ is not a maximal space-like geodesic.

(iii) *In $Mink(2, \mathfrak{F})$*

$$\begin{array}{lll} L^S & \subseteq & (\textit{maximal locally additive space-like geodesics}), \qquad \textit{while} \\ L^S & \not\supseteq & (\textit{maximal locally additive space-like geodesics}). \end{array}$$

(iv) *As a contrast with item (iii), for any $n > 2$ in $Mink(n, \mathfrak{F})$*

$$\begin{array}{lll} L^S & \cap & (\textit{maximal locally additive space-like geodesics}) \ = \emptyset \quad \textit{and} \\ L^S & \cap & (\textit{maximal additive space-like geodesics}) \ = \ \emptyset. \end{array}$$

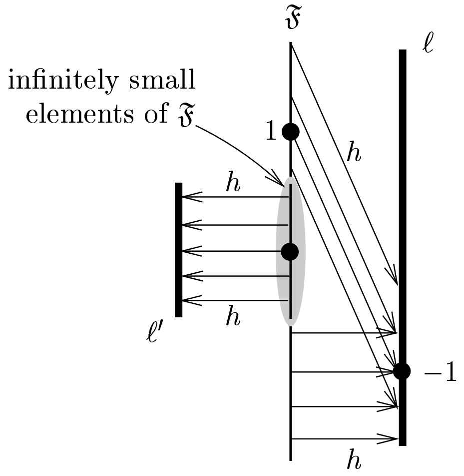


Figure 130: $\ell \in L^T$ and $\ell \cup \ell'$ is a time-like geodesic.

Outline of proof: Assume $\mathfrak{F}$ is non-Archimedean and Euclidean. Idea of proof for

$$L^T \ \cap \ \text{(maximal time-like geodesics)} \ = \ \emptyset \quad \text{in } \mathit{Mink}(\mathfrak{F})$$

is depicted in Figure 130. In the figure $\ell \in L^T$ and $\ell \cup \ell'$ is a time-like geodesic. Hence, ℓ is not a maximal time-like geodesic. By Prop.4.7.7(ii) on p.357 (and by Thm.4.2.45 on p.190), in $\mathit{Mink}(\mathfrak{F})$

$$L^T \ \subseteq \ \text{(maximal locally additive time-like geodesics).}$$

Idea for the proof of

$$L^T \ \not\supseteq \ \text{(maximal locally additive time-like geodesics)} \quad \text{in } \mathit{Mink}(\mathfrak{F})$$

is depicted in Figure 131. In the figure ℓ is a maximal locally additive time-like geodesic. This holds by the proof of item (ii) of Prop.4.7.7. Clearly, ℓ (in Fig.131) is not a time-like line. By

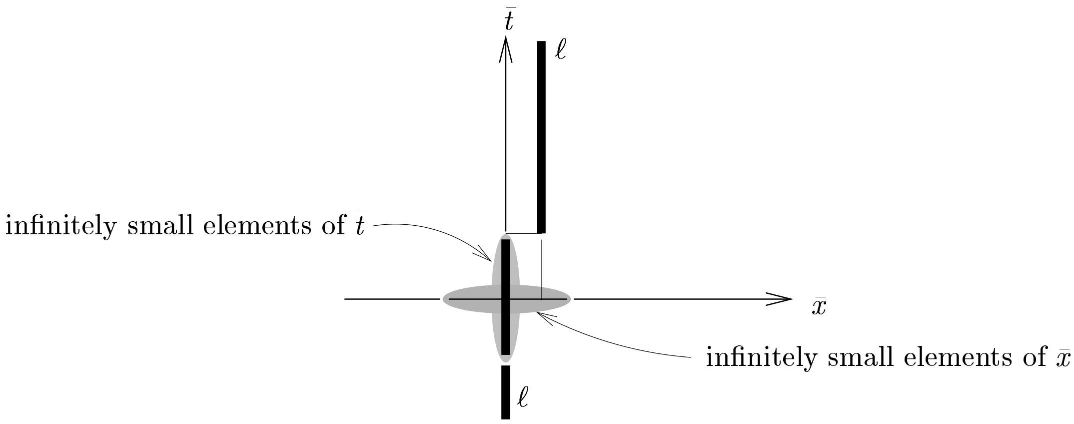


Figure 131: ℓ is a maximal locally additive time-like geodesic in the Minkowskian geometry over a non-Archimedean $\mathfrak{F}$.

these item (i) of our theorem is proved. Proofs for items (ii) and (iii) can be obtained by the proof of item (i). (The proofs of (ii) and (iii) are left to the reader.)

To prove item (iv) let $\ell \in L^S$. Consider a Robb plane[736] that contains ℓ. Let ℓ' be constructed as in Figure 130 but such that ℓ' is contained in the Robb plane, see Figure 132. Then, in Figure 132, $\ell \cup \ell'$ is an additive space-like geodesic, cf. hint for the proof of Thm.4.7.12 on p.362. Hence, ℓ is not a maximal locally additive space-like geodesic and is not a maximal additive space-like geodesic. ∎

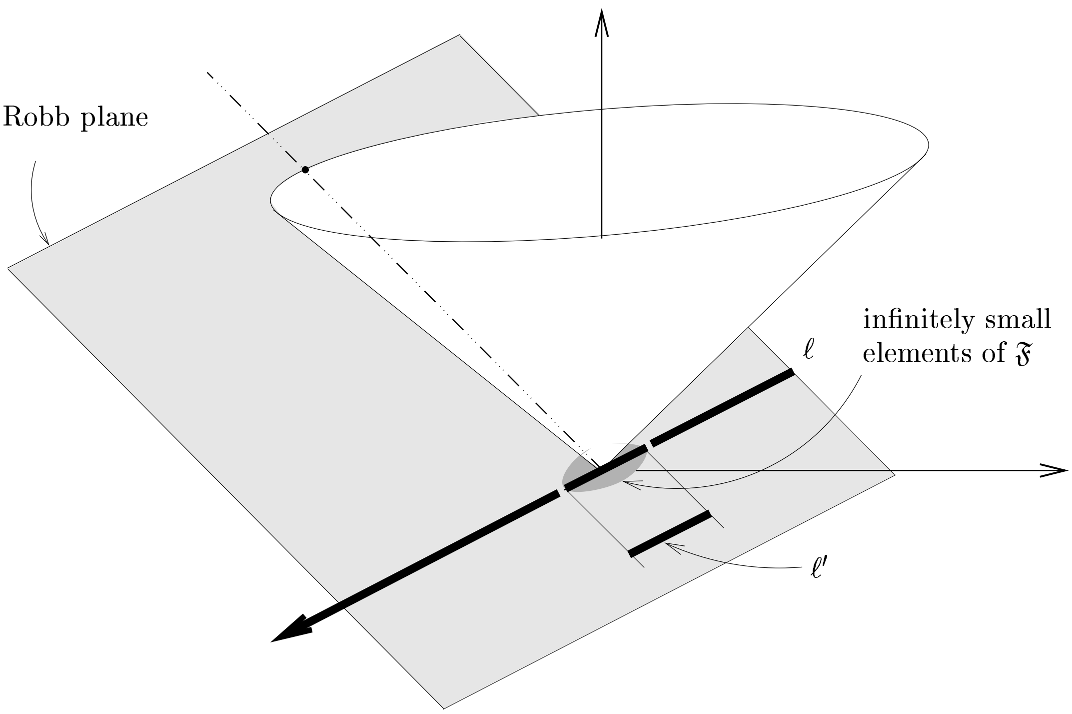


Figure 132: $\ell \cup \ell'$ is an additive space-like geodesic in the Minkowskian geometry over a non-Archimedean $\mathfrak{F}$.

The following theorem says that the second part of condition $(***)$ (on p.352) is needed in the definition of space-like quasi geodesics, geodesics and Archimedean geodesics. In other words, if we omit condition $(***)$ from the definition of geodesics, then they *do not "work"* in relativistic geometries, e.g. in Minkowskian space-times. Although they do work in Euclidean geometry and more generally in Riemannian geometries. This further implies that if we use the definition of geodesics as given e.g. in the book "Geometry of Geodesics" (Busemann [54]), then they do not work in relativistic geometries ($n > 2$), e.g. in Minkowskian geometry.[737]

[736] cf. e.g. Goldblatt [102] or p.1163 in AMN [18] for the notion of a Robb plane. If $n > 3$ then we can talk about ***Robb hyper-planes*** (cf. p.804 in AMN [18]) which in Goldblatt [102] are called Robb threefolds (if $n = 4$). However, there still exist *Robb planes*, too, which are (two-dimensional) and planes with the Robb property. In the above proof of Thm.4.7.12 it is important that we talk about Robb planes and not about Robb hyper-planes.

[737] This entails nothing negative about Busemann [54], since it does not deal with relativistic geometries. Caution is needed with the word "Minkowskian geometry", since here (cf. also Goldblatt [102], Schutz [231]) we use it for certain relativistic geometries while e.g. in Busemann [54, §17] it is used for other kinds of (non-relativistic) spaces.

THEOREM 4.7.12 *Assume $n \geq 3$. Then in the Minkowskian geometry $Mink(n, \mathfrak{R})$ there is a "curve" $\ell \subseteq {}^n\mathrm{R}$ such that $(\forall p, q \in \ell)\, p \equiv^S q$,*

$$(\forall e \in \ell)(\exists \varepsilon \in {}^+F)\,[\,\underline{no}\text{ three distinct points of } \ell \cap S(e,\varepsilon)\ \underline{\text{are collinear}}],$$

and there is a homeomorphism $h : \mathfrak{R} \rightarrowtail\!\!\twoheadrightarrow \ell$ which is differentiable infinitely many times and is distance preserving in the sense that

$$(\forall x, y \in \mathrm{R})\, |x - y| = g_\mu(h(x), h(y)),$$

see Figure 133. Moreover this function h is a homeomorphism w.r.t. (the usual topology on $\mathfrak{R}$ and) any one of the following topologies on ℓ: the topology induced by g_μ, the relativistic topology $\mathcal{T}_\mu$ of $Mink(\mathfrak{R})$ and the Euclidean topology on ${}^n\mathrm{R}$. Actually these topologies coincide on ℓ.

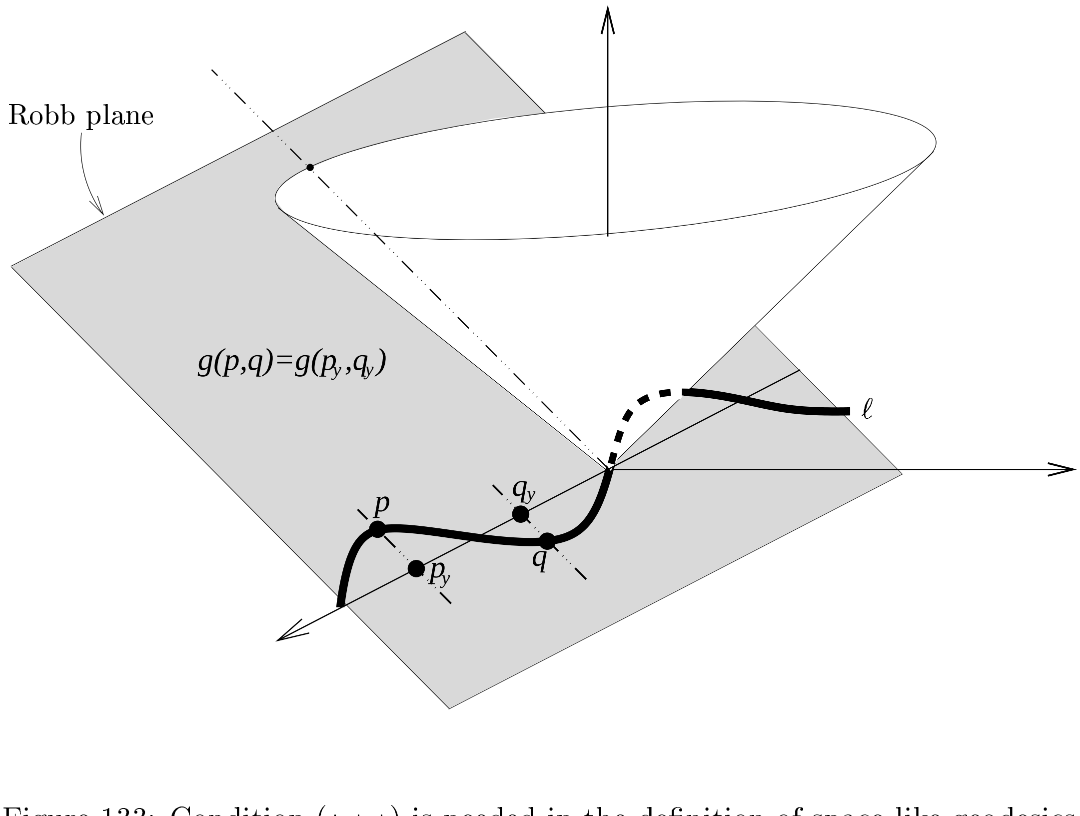


Figure 133: Condition $(***)$ is needed in the definition of space-like geodesics.

On the proof: Hint: Assume $n \geq 3$. The Robb planes[738] have the following "exotic" property in $Mink(\mathfrak{R})$ (in connection with the metric g_μ and geodesics). Let P be a Robb plane containing the $\bar{y}$ axis. Then the relativistic distance $g_\mu(p,q)$ between points $p, q \in P$ coincides with the absolute value of the difference between the y-coordinates p_y and q_y of p and q, respectively. Cf. Figure 133. Therefore the metric g_μ is additive on the whole Robb plane. Actually this idea works in many of our relativistic geometries, e.g. in the case of $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqspace)} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ }))$ they do. ∎

[738] cf. e.g. Goldblatt [102] or p.1163 in AMN [18] for the notion of a Robb plane.

COROLLARY 4.7.13 Assume $n \geq 3$ and consider $\mathfrak{G} \stackrel{\text{def}}{=} Mink(n, \mathfrak{R})$ as in Thm.4.7.12 above. Then

(i) If we omit condition $(***)$ from the definition of geodesics, then there are geodesics in $\mathfrak{G}$ which are not straight lines. Further,

(ii) there exist many Robb planes[739] in $\mathfrak{G}$, and

(iii) almost[740] every curve in every Robb plane counts as a geodesic if we omit condition $(***)$ from the definition of geodesics.

Discussion of Thm.4.7.12 and Corollary 4.7.13. The condition $(***)$ is not present in the usual definition of geodesics. Items 4.7.12, 4.7.13 say that this condition is needed in relativistic geometries if we want to discuss space-like geodesics, too.

The definition of *usual geodesics* is obtained from Def.4.7.2 by replacing all occurrences of condition $(***)$ by $(\forall x, y \in D) x \equiv^S y$.

What we obtain this way is more or less the usual definition of geodesics (cf. Busemann [54]) adapted to the relativistic situation where we have $\equiv^T, \equiv^{Ph}, \equiv^S$.[741] Now, what items 4.7.12, 4.7.13 say is that even in the most classical, most standard form of special relativity, i.e. in Minkowskian space-time with $n > 2$, usual geodesics (as defined above) do not "work". (They do not behave as we wanted them to behave when defining them.)

COROLLARY 4.7.14 Let $n > 2$ and consider $\mathfrak{G} \stackrel{\text{def}}{=} Mink(n, \mathfrak{R})$. Then there are *usual geodesics* ℓ in $\mathfrak{G}$ which are not straight lines, moreover ℓ can be chosen to be continuous and differentiable such that $(\forall p \in \ell)(\forall \varepsilon \in {}^+F)$*the* ε*-neighborhood of* p *in* ℓ *is not straight.* Moreover, this ℓ is an *Archimedean*, short, usual geodesic, cf. Def.4.7.2 items 14, 17. Further, it is a *maximal* geodesic, and a *strong* geodesic. Through any two distinct space-like separated points of $\mathfrak{G}$ there are continuum many such usual geodesics.

Proof. The proof goes by inspecting Figure 133 (and the proof of Thm.4.7.12) and by checking all the items quoted from Def.4.7.2. ∎

COROLLARY 4.7.15 Let $n > 2$. A statement analogous to items 4.7.12-4.7.14 applies to our geometries in $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + ax(eqspace) + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ }))$.

Proof. The proof goes by checking that already under the axioms $\mathbf{Bax}^{\oplus} + \ldots + \mathbf{Ax}(\sqrt{\ })$ listed above, the Robb plane exhibits the strange properties illustrated in Figure 133. ∎

Items 4.7.12-4.7.15 show that condition $(***)$ is really needed and is not easily replaced by something "more traditional". Further, they indicate that the (simplest) usual notion of geodesics[742] does not work in relativistic situations for *space-like geodesics*. This might be

[739] each photon line is contained in a Robb plane which is unique iff $n = 3$. So, if $n > 3$, then the Robb plane in question is not unique.

[740] Instead of defining precisely which curves in the Robb plane we mean, we give only an intuitive description: Let ℓ be a "continuous, differentiable" connected curve in the Robb plane as illustrated in Figure 133. Assume $(\forall p, q \in \ell) p \equiv^S q$. Assume further that ℓ is a homeomorphic image of some connected interval of $\mathbf{F_0}$. Then ℓ counts as a geodesic (without $(***)$).

[741] i.e., so to speak, adapted from Riemannian geometries to pseudo-Riemannian ones; or in other words, adapted to so-called "indefinite metrics".

[742] Cf. the definition of usual geodesics above.

connected to the historical fact that *in general relativity* much less attention is paid to space-like geodesics than to time-like or photon-like ones. E.g. the basic book Hawking-Ellis [116] does not even mention space-like geodesics.[743] A further indication of this[744] is that in the world-famous basic book of relativity Misner-Thorne-Wheeler [192] the statement of Exercise 13.6 on p.324 (discussing space-like geodesics) seems to be either false or not very carefully formulated. (We mean this of course wrt. the definitions given in that book.[745]) Further, as far as we know, this (about the book) has not yet been pointed out in the literature. With this we stop discussing items 4.7.12-4.7.15 (and return to discussing our notion of geodesics in our relativity theories).

In the second part of AMN [18, §6.8(geodesics)] the present author generalizes our earlier positive results from the concrete case of Minkowskian geometries to a broader class of our observer independent geometries of the "axiomatic form" $\mathsf{Ge}(Th)$. For lack of space we omit these results and refer the reader to items 6.8.24–6.8.36 in AMN [18].

Summary of some further results by the author on geodesics and our "more flexible theories of relativity": Assume $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqm)}$. Then all the elements of L turn out to be geodesics (cf. Prop.4.7.7). If in addition we assume $n > 2$, $\mathbf{Basax} + \mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^- + \mathbf{Ax}(\sqrt{\ })$ and that $\mathfrak{F}$ is Archimedean, then the set L of lines coincides with the set of maximal geodesics (cf. Corollary 6.8.33, p.1204 of AMN [18]). We conjecture that the condition $n > 2$ is needed in the previous sentence, namely, we conjecture that there is a model $\mathfrak{M}$ of $\mathbf{Basax}(2) + \mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^- + \mathbf{Ax(eqm)}$ with $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ the ordered field of reals in which some maximal time-like geodesics of $\mathfrak{G}_{\mathfrak{M}}$ are <u>not</u> in $L_{\mathfrak{M}}$ (cf. Conjecture 6.8.35, p.1205 of AMN [18]). Further, we conjecture that for any $n \geq 2$, maximal time-like geodesics are not necessarily lines even if we assume **Basax** and $\mathfrak{F} = \mathfrak{R}$ (cf. Conjecture 6.8.35 of AMN [18]). As a contrast, if $n > 2$ and if we assume $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)}$ <u>or</u> $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{R}(\mathbf{Ax\ syt_0})$ together with some auxiliary axioms and that $\mathfrak{F}$ is Archimedean, then the set of maximal time-like geodesics coincides with the set L^T of time-like lines, cf. AMN [18, Thm.6.8.24 (p.1200) and Corollary 6.8.27 (p.1202)]. The latter condition (i.e. that $\mathfrak{F}$ is Archimedean) is needed even if we assume $\mathbf{BaCo} + \mathbf{Ax(rc)}$, cf. AMN [18, Thm.4.7.11 (p.359)] and Figure 130 (p.360) herein. Assuming $\mathbf{Reich(Bax)}^{\oplus} + \mathbf{Ax(diswind)}$, the maximal photon-like geodesics are exactly the members of L^{Ph} (cf. item (iv) of Prop.4.7.7 herein).

[743] This in turn might be motivated by the famous quotation for Eddington [55, p.22] "Assuming that a material particle cannot travel faster than light ... we ourselves are limited by material bodies and have direct experience of time-like intervals."

[744] i.e. that relativity theorists seem to pay little attention to space-like geodesics

[745] But it seems to remain false for any usual definition of geodesics known to the present author.

[747] This figure is from Hawking-Ellis [116].

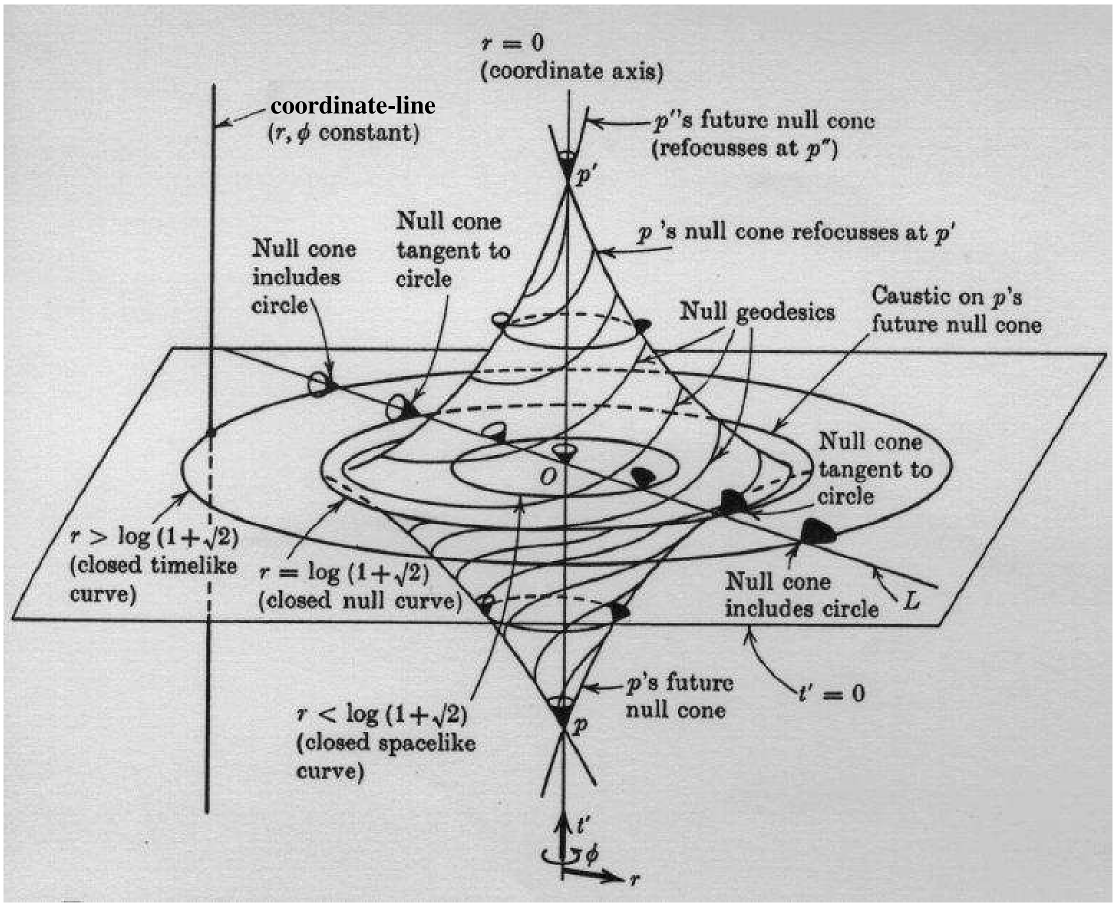


Figure 134: Gödel's rotating universe with the irrelevant coordinate z suppressed. The space is rotationally symmetric about any point; the diagram represents correctly the rotational symmetry about the axis $r = 0$, and the time invariance. The light cone opens out and tips over as r increases (see line L) resulting in closed time-like curves. The diagram does not correctly represent the fact that all points are in fact equivalent.[747]

A APPENDIX: Integrating our duality theories into the landscape of dualities all over mathematics

A.1 Galois connections

In this section we will see that $(\mathcal{G}, \mathcal{M})$ and $(\mathcal{G}o, \mathcal{M}o)$ form "Galois connections". We will mention e.g. in item A.1.1(II) below that this fact indicates one of the various connections between the present work and the more algebraic papers of the present author, e.g. Madarász [170] or [165]. In Def.A.1.2 below, we will recall from the literature the notion of a Galois connection cf. e.g. Adámek-Herrlich-Strecker [2, item 6.26(4), p.81]. We compare Galois connections with adjoint functors and with further related concepts in the mathematical literature in item entitled "Connections between adjoint situations, Galois connections, …" on p.1096 at the end of §6.6.6 of AMN [18]. Cf. also Remark 4.5.14 (pp. 293–296) herein and Remark 6.6.4 (Galois theories, Galois connections, duality theories all over mathematics, in analogy with the ones in the present work) in AMN [18] as motivations for studying Galois connections. That remark also serves as a kind of mathematical perspective/background for the present section.

Remark A.1.1
(Motivations for Galois connections [for the physicist reader])

(I) Galois connection is a simplified form of adjoint situation (from category theory)[748] which in turn is regarded as one of the most important[749] conceptual tools of category theory. (To understand adjoint situations well, the first step is to understand Galois connections [as special adjoint situations].) Galois connections are obtained from adjointness by considering the simple kinds of categories called pre-orderings (where between any two objects there is at most one morphism); for these kinds of categories etc. cf. the subtitle "Connections between adjoint situations, Galois connections, …" on p.A-15.

Galois connection is a generalization of isomorphism. The idea is that isomorphism is very useful but it is a too rigid concept (and therefore it occurs rarely). So let us make isomorphism a little bit more flexible so that it would retain most of its useful properties[750] but would become more flexible (more often applicable). The result i.e. the flexible version of isomorphism is called Galois connection (in the case when it connects pre-orderings). The definition is given in Def.A.1.2 below. In the general case (of categories) the name of "flexible isomorphisms" is adjoint situations or adjoint pairs of functors. To see a glimpse of the idea let us recall that an isomorphism from $\langle P, \leq\rangle$ onto $\langle Q, \leq\rangle$ is a homomorphism f such that there is a backward

[748] Cf. §A.2 (p.A-6) for category theory.
[749] Cf. e.g. Adámek-Herrlich-Strecker [2], p.283 first sentence (Chap.18, Adjoint functors). There they write: "Perhaps the most successful concept of category theory is that of adjoint functor. Adjoint functors occur frequently in many branches of mathematics … surprising range of applications." Cf. also (†) on p.1096 of AMN [18] for importance of adjointness in physics.
[750] e.g. we can transfer "constructions" from one side to the other

homomorphism g

$$\langle P, \leq \rangle \quad \underset{g}{\overset{f}{\rightleftarrows}} \quad \langle Q, \leq \rangle$$

with $(f \circ g)(x) = x$ and $(g \circ f)(y) = y$. For easier formulation (of what comes) we replace homomorphism by dual-homomorphism (i.e. order reversing map). Now, to make the concept less rigid, we replace the condition $(f \circ g)(x) = x$ by the weaker one $(f \circ g)(x) \geq x$ and similarly for $g \circ f$. The result is summarized in Fact A.1.3 below, but cf. also ($\star$) on p.A-15 which might be a more suggestive (equivalent) definition of "flexible isomorphism". Then Fact A.1.5 indicates that the resulting notion of "flexible isomorphisms" (i.e. Galois connections) retains many of the useful properties of isomorphisms.[751]

(II) Galois connections can serve as a unified theory of the research-branches mentioned on pp. 1096–1105 of AMN [18] ranging from Boolean algebras with operators, residuated-residual pairs, conjugates of operators, linear logic, Lambek calculus, relation algebras, closure operators, geometry, vector spaces, C^*-algebras, but cf. also Janelidze [139] for more daring applications via Galois theories (which are of course strongly tied up[752] with Galois connections). Some of these examples are explained in more detail in the present work, too (on pp.A-15–A-17). Some of the above are investigated and used in Madarász [161], [165], [164], [167], [169], [170], [176], [20], [23], [178].

In particular, studying Galois connections can serve as an abstract, *unified study of duality theories* or adjoint situations, which in turn, according to Adámek et al. [2], Lawvere [153] and others[753] pervade much of mathematics and modern mathematical physics. We hope, recalling the patterns:

$$\langle P, \leq \rangle \quad \underset{g}{\overset{f}{\rightleftarrows}} \quad \langle Q, \leq \rangle \qquad \text{Galois connection}$$

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th) \qquad \text{duality theory}^{754}$$

gives a hint for the above idea (of Galois connections serving as a unified, abstract study of dualities).

(III) Whenever we are given two sets or classes say K, L and a binary relation $R \subseteq K \times L$ between them then R induces a natural Galois connection between $\mathcal{P}(K)$ and $\mathcal{P}(L)$ as follows. For $X \subseteq K$, $f_R(X) = \{ y \in L : (\forall x \in X)\, x\, R\, y \}$. So $f_R : \mathcal{P}(K) \longrightarrow \mathcal{P}(L)$ is order

[751] The same idea in different words: A homomorphism f is called an isomorphism iff it admits a two-sided inverse g ($g \circ f = \text{Id}$ and $f \circ g = \text{Id}$). Now, in order to be a flexible isomorphism it is enough to admit a quasi-inverse as sketched in footnote 767 on p.A-15.

[752] A Galois theory is always a (special) Galois connection, cf. items (I), (V) of Remark 6.6.4 of AMN [18] (pp. 293, 295)

[753] A sample of the references claiming and illustrating with examples that duality theories, i.e. adjoint situations ***are very broadly applicable*** (and applied) throughout mathematics and also in mathematical physics is Lawvere [153, 155, 154], Arbib-Manes [34, 33], Manes [180], Guitart [111], Mac Lane [159], Goldblatt [101], Handbook of Categorical Algebra [50], Barr-Wells [41, §1.9, p. 50–63], Freyd-Scedrov [88], Adámek et al. [2], [3], Varadarajan [268], Lawvere-Schanuel [156], Nel [201], Pelletier-Rosický [210], Dimov-Tholen [72], Janelidze [139], Davey-Priestley [67]. These references give examples ranging from algebraic geometry, compact Galois groups, geometry and analysis, sheaves of continuous maps, metric spaces, tensor algebra, Banach spaces and spaces of generalized Lipschitz functions, computability & automata & linear systems. (Cf. the works of Arbib, Manes, Guitart for the latter four topics.) Cf. also (†) on p.1096 of AMN [18].

[754] We have not yet defined a structure like "$\leq$" on $\mathsf{Mod}(Th)$, $\mathsf{Ge}(Th)$ but that will come later (and is kind of implicit already in schemas (A)–(I) on pp.284–287).

reversing. $g_R : \mathcal{P}(L) \longrightarrow \mathcal{P}(K)$ is defined analogously. Cf. item (IV) of Remark 6.6.4 of AMN [18] (p.1026) which is about the $(\mathsf{Mod}, \mathsf{Th})$-Galois connection induced by the relation $\models$. Cf. also p.453 of AMN [18]. This kind of Galois-connection is generalized to the case when R is a so-called partial relation in Madarász [165] establishing a representation theorem for non-normal Boolean algebras with operators which in turn solves a problem left open in 1952 in Jónsson-Tarski [142]. This problem appears implicitly already in the 1948 version of the Jónsson-Tarski paper (in Abstracts of Amer. Math. Soc.).[755]

(IV) Cf. Remark 4.5.14, pp. 293–296.

END OF MOTIVATION FOR GALOIS CONNECTIONS.

◁

Definition A.1.2 (Galois connection)
Let $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ be pre-ordered classes and

$$f : P \longrightarrow Q \quad \text{and} \quad g : Q \longrightarrow P.$$

The pair (f, g) is called a *Galois connection* between $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ iff for all $p \in P$ and $q \in Q$

$$p \leq g(q) \quad \Leftrightarrow \quad q \leq f(p).$$

◁

The following fact states a (known) equivalent reformulation of the definition of Galois connections.

FACT A.1.3 *Assume $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ are pre-ordered clases and that $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$. Then the pair (f, g) is a Galois connection between $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ iff (a) and (b) below hold.*

(a) *f and g are both order-reversing, i.e. if $p \leq p' \in P$ then $f(p) \geq f(p')$, and if $q \leq q' \in Q$ then $g(q) \geq g(q')$.*

(b) *$f \circ g$ and $g \circ f$ are both monotonous, i.e.*
$p \leq (f \circ g)(p)$ *for all* $p \in P$ *and* $q \leq (g \circ f)(q)$ *for all* $q \in Q$. ∎

Notation A.1.4 Assume that $\langle P, \leq \rangle$ is a pre-ordered class. Then the binary relation $\simeq$ on P is defined as

$$p \simeq p' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (p \leq p' \ \wedge \ p' \leq p).$$

We note that $\simeq$ is an equivalence relation.

◁

[755] The non-normal character of the algebras in question is important for the foundation of several logics that became important recently. Some of these are logics of action and dynamic logic. Interestingly, nonstandard dynamic logic (cf. e.g. Sain [227]) is closely related to the subject of the present work, e.g. because both are concerned with processes happening in time, both handle temporality by a certain way of using many-sorted logic, both can be related to nonstandard analysis in a certain way, etc. In particular, the techniques we plan to use in generalizing the present approach to handling accelerated observers in [26] are borrowed from the just quoted version of dynamic logic. (Logic of actions can also, perhaps, be developed to a platform adequate for handling the pseudo-paradoxes associated with closed timelike loops in general relativity.)

Fact A.1.5 below is known from algebra. Items (i)–(iii) of this fact say that if (f,g) is a Galois connection then both $f\circ g$ and $g\circ f$ are closure operators up to the equivalence relation $\simeq$ (cf. the notion of a closure operator up to isomorphism on p.288.) Further, item (iv) says that the closed "up to $\simeq$" elements of $f\circ g$ are the elements of the range of g ("up to $\simeq$"). Similarly for $g\circ f$.

FACT A.1.5 *Assume* $\langle P,\leq\rangle$ *and* $\langle Q,\leq\rangle$ *are pre-ordered classes and* (f,g) *is a Galois connection between them. Then for all* $p\in P$ *and* $q\in Q$, *(i)–(iv) below hold.*

(i) $p\leq (f\circ g)(p)$ *and* $q\leq (g\circ f)(q)$.

(ii) *Both* $f\circ g$ *and* $g\circ f$ *have fixed-point property in the sense* $(f\circ g)^2(p)\simeq (f\circ g)(p)$ *and* $(g\circ f)^2(q)\simeq (g\circ f)(q)$.

(iii) *If* $p\leq p'\in P$ *and* $q\leq q'\in Q$ *then* $(f\circ g)(p)\leq (f\circ g)(p')$ *and* $(g\circ f)(q)\leq (g\circ f)(q')$.

(iv) $(g\circ f)(f(p))\simeq f(p)$ *and* $(f\circ g)(g(q))\simeq g(q)$. ∎

For the motivation of the following definition cf. Propositions 4.5.49 (p.322) and 4.5.60 (p.329).

Definition A.1.6

$$\begin{aligned}
\mathbf{Pax}^{++} &\overset{\text{def}}{:=} \mathbf{Pax}^{+} + \mathbf{Ax(eqm)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit,\\
\mathbf{Wax}^{+} &\overset{\text{def}}{:=} \mathbf{Wax} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\infty ph) + (\forall m,k)(\mathsf{f}_{mk}\in \mathit{Aftr})\\
\mathbf{lopag}^{+} &\overset{\text{def}}{:=} \mathbf{lopag} + \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}}.
\end{aligned}$$

◁

Remark A.1.7 We note that item (iii) of Prop.4.5.60 (p.329) states, by Thm.4.5.57 (p.328), that

$$\mathit{Rng}(\mathcal{M}o)\models \mathbf{Wax}^{+} \qquad \text{and} \qquad \mathit{Rng}(\mathcal{G}o)\models \mathbf{lopag}^{+}.$$

◁

We will prove that $(\mathcal{G}o,\mathcal{M}o)$ forms a Galois connection between the classes $\mathsf{Mod}(\mathbf{Wax}^{+})$ and $\mathsf{Mog}(\mathbf{lopag}^{+})$ for a certain choice of pre-orderings $\leq_{\mathcal{M}o}$ and $\leq_{\mathcal{G}o}$ of these two classes. (I.e. $\leq_{\mathcal{M}o}$ is a pre-ordering of $\mathsf{Mod}(\mathbf{Wax}^{+})$, and similarly for $\leq_{\mathcal{G}o}$ and $\mathsf{Mog}(\mathbf{lopag}^{+})$). We will prove an analogous statement about $(\mathcal{G},\mathcal{M})$ and $\mathsf{Mod}(\mathbf{Pax}^{++})$, $\mathsf{Ge}(\mathbf{Pax}^{++})$.

Definition A.1.8 $(\leq_{\mathcal{M}o},\ \leq_{\mathcal{G}o},\ \leq_{\mathcal{M}},\leq_{\mathcal{G}})$

(i) We define $\leq_{\mathcal{M}o}$ to be the smallest transitive binary relation on $\mathsf{Mod}(\mathbf{Wax}^{+})$ for which 1 and 2 below hold.

1. $\mathfrak{M}\leq_{\mathcal{M}o}(\mathcal{G}o\circ\mathcal{M}o)(\mathfrak{M})$, and
2. $\mathfrak{M}\cong\mathfrak{N}\ \Rightarrow\ \mathfrak{M}\leq_{\mathcal{M}o}\mathfrak{N}$, for all $\mathfrak{M},\mathfrak{N}\in\mathsf{Mod}(\mathbf{Wax}^{+})$.

(ii) We define $\leq_{\mathcal{G}o}$ to be the smallest transitive binary relation on $\mathsf{Mog}(\mathbf{lopag}^{+})$ for which 1 and 2 below hold.

1. $\mathfrak{G}\leq_{\mathcal{G}o}(\mathcal{M}o\circ\mathcal{G}o)(\mathfrak{G})$, and
2. $\mathfrak{G}\cong\mathfrak{H}\ \Rightarrow\ \mathfrak{G}\leq_{\mathcal{G}o}\mathfrak{H}$, for all $\mathfrak{G},\mathfrak{H}\in\mathsf{Mog}(\mathbf{lopag}^{+})$.

(iii) We define $\leq_{\mathcal{M}}$ to be the smallest transitive binary relation on $\mathsf{Mod}(\mathbf{Pax}^{++})$ for which 1 and 2 below hold.

1. $\mathfrak{M} \leq_{\mathcal{M}} (\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$, and
2. $\mathfrak{M} \cong \mathfrak{N} \;\Rightarrow\; \mathfrak{M} \leq_{\mathcal{M}} \mathfrak{N}$, for all $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Pax}^{++})$.

(iv) We define $\leq_{\mathcal{G}}$ to be the smallest transitive binary relation on $\mathsf{Ge}(\mathbf{Pax}^{++})$ for which 1 and 2 below hold.

1. $\mathfrak{G} \leq_{\mathcal{G}} (\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$, and
2. $\mathfrak{G} \cong \mathfrak{H} \;\Rightarrow\; \mathfrak{G} \leq_{\mathcal{G}} \mathfrak{H}$, for all $\mathfrak{G}, \mathfrak{H} \in \mathsf{Ge}(\mathbf{Pax}^{++})$.

$\lhd$

Next we state some simple properties of the pre-orderings $\leq_{\mathcal{M}o}$ etc.

PROPOSITION A.1.9

(i) *Let* $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Wax}^{+})$. *Then*

$$\begin{aligned}(\mathfrak{M} \leq_{\mathcal{M}o} \mathfrak{N} \;\wedge\; \mathfrak{N} \leq_{\mathcal{M}o} \mathfrak{M}) \quad &\Rightarrow \quad \mathfrak{M} \cong \mathfrak{N}, \quad \textit{and}\\ \mathfrak{M} \leq_{\mathcal{M}o} \mathfrak{N} \quad &\Rightarrow \quad \mathfrak{M} \rightarrowtail \mathfrak{N}.\end{aligned}$$

(ii) *Let* $\mathfrak{G}, \mathfrak{H} \in \mathsf{Mog}(\mathbf{lopag}^{+})$. *Then*

$$\begin{aligned}(\mathfrak{G} \leq_{\mathcal{G}o} \mathfrak{H} \;\wedge\; \mathfrak{H} \leq_{\mathcal{G}o} \mathfrak{G}) \quad &\Rightarrow \quad \mathfrak{G} \cong \mathfrak{H}, \quad \textit{and}\\ \mathfrak{G} \leq_{\mathcal{G}o} \mathfrak{H} \quad &\Rightarrow \quad \mathfrak{G} \longleftarrow\!\!\prec \mathfrak{H}.\end{aligned}$$

(iii) *Let* $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Pax}^{++})$. *Then*

$$\begin{aligned}(\mathfrak{M} \leq_{\mathcal{M}} \mathfrak{N} \;\wedge\; \mathfrak{N} \leq_{\mathcal{M}} \mathfrak{M}) \quad &\Rightarrow \quad \mathfrak{M} \cong \mathfrak{N}, \quad \textit{and}\\ \mathfrak{M} \leq_{\mathcal{M}} \mathfrak{N} \quad &\Rightarrow \quad \mathfrak{M} \rightarrowtail \mathfrak{N}.\end{aligned}$$

(iv) *Let* $\mathfrak{G}, \mathfrak{H} \in \mathsf{Ge}(\mathbf{Pax}^{++})$. *Then*

$$\begin{aligned}(\mathfrak{G} \leq_{\mathcal{G}} \mathfrak{H} \;\wedge\; \mathfrak{H} \leq_{\mathcal{G}} \mathfrak{G}) \quad &\Rightarrow \quad \mathfrak{G} \cong \mathfrak{H}, \quad \textit{and}\\ \mathfrak{G} \leq_{\mathcal{G}} \mathfrak{H} \quad &\Rightarrow \quad \mathfrak{G} \longleftarrow\!\!\prec \mathfrak{H}.\end{aligned}$$

We omit the **proof**. ∎

THEOREM A.1.10

(i)

$$\begin{aligned}\mathcal{G}o \;&:\; \mathsf{Mod}(\mathbf{Wax}^{+}) \longrightarrow \mathsf{Mog}(\mathbf{lopag}^{+}) \quad \textit{and}\\ \mathcal{M}o \;&:\; \mathsf{Mog}(\mathbf{lopag}^{+}) \longrightarrow \mathsf{Mod}(\mathbf{Wax}^{+}).\end{aligned}$$

(ii) *$(\mathcal{G}o, \mathcal{M}o)$ is a Galois connection between* $\langle \mathsf{Mod}(\mathbf{Wax}^{+}), \leq_{\mathcal{M}o}\rangle$ *and* $\langle \mathsf{Mog}(\mathbf{lopag}^{+}), \leq_{\mathcal{G}o} \rangle$.

We omit the **proof**. ∎

We suggest that the reader compare Theorem A.1.10 with the intuitive text on p.328 below Thm.4.5.57 together with Remark A.1.7 (p.A-4).

The following corollary is of the pattern of theorem schemas (A), (B), (E)–(H) and it is a corollary of Theorem A.1.10, Fact A.1.5, and Prop.A.1.9.

COROLLARY A.1.11
For any $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax}^+)$ and $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag}^+)$, (i)–(iii) below hold.

(i)

$$\mathfrak{M} \rightarrowtail (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \quad \text{and} \quad \mathfrak{G} \leftarrowtail (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}).$$

(ii) The members of the range of $\mathcal{G}o$ are fixed-points of $\mathcal{M}o \circ \mathcal{G}o$ and the members of the range of $\mathcal{M}o$ are fixed-points of $\mathcal{G}o \circ \mathcal{M}o$, i.e.

$$(\mathcal{M}o \circ \mathcal{G}o)(\mathcal{G}o(\mathfrak{M})) \cong \mathcal{G}o(\mathfrak{M}) \quad \text{and} \quad (\mathcal{G}o \circ \mathcal{M}o)(\mathcal{M}o(\mathfrak{G})) \cong \mathcal{M}o(\mathfrak{G}).$$

(iii) Both $\mathcal{G}o \circ \mathcal{M}o$ and $\mathcal{M}o \circ \mathcal{G}o$ have fixed-point property in the sense

$$(\mathcal{G}o \circ \mathcal{M}o)^2(\mathfrak{M}) \cong (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \quad \text{and} \quad (\mathcal{M}o \circ \mathcal{G}o)^2(\mathfrak{G}) \cong (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}).$$

∎

THEOREM A.1.12

(i) $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^{++}) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^{++})$ *(and* $\mathcal{G} : \mathsf{Mod}(\mathbf{Pax}^{++}) \longrightarrow \mathsf{Ge}(\mathbf{Pax}^{++})$*).*

(ii) $(\mathcal{G}, \mathcal{M})$ *is a Galois connection between* $\langle \mathsf{Mod}(\mathbf{Pax}^{++}), \leq_{\mathcal{G}} \rangle$ *and* $\langle \mathsf{Ge}(\mathbf{Pax}^{++}), \leq_{\mathcal{M}} \rangle$.

Proof: The theorem follows by Thm.4.5.13 (p.291) and Fact A.1.3. ∎

At this point we could formulate a corollary of Thm.A.1.12 which would be analogous with Corollary A.1.11 of Thm.A.1.10. This corollary of Thm.A.1.12 basically coincides with our Thm.4.5.13 formulated on p.291.

A.2 Adjoint functors, categories

Motivation for adjoint functors for the physicist reader is found in Remark A.1.1 (p.A-1). Cf. also p.1096 of AMN [18]. For adjoint situations in physics cf. e.g. Lawvere-Schanuel [156, pp. 5–6, pp. 76–77]; but see also the references in footnote 753, p.A-2.[756]

[756] Category theory has been becoming increasingly popular and often used in physics recently, cf. e.g. Baez-Dolan [37], Crane [64], Freed [87], Andai [4], Kassel [147], Baez [38]. Cf. also Lawvere-Schanuel [156].

The subject matter of this section is strongly connected to Remark 4.5.14 (p.293) entitled "Galois theories, Galois connections, duality theories all over mathematics ...."

In this section we will see that $(\mathcal{M}, \mathcal{G})$ and $(\mathcal{M}o, \mathcal{G}o)$ are "adjoint pairs of functors" in the category theoretic sense, under certain conditions.

We use the notion of a *category* in the usual category theoretic sense, cf. e.g. Mac Lane [159]. Assume $\mathbb{C}$ is a category. Then $\mathsf{Ob}\,\mathbb{C}$ and $\mathsf{Mor}\,\mathbb{C}$ denote the classes of *objects* and *morphisms* of $\mathbb{C}$, respectively. $f : A \longrightarrow B$ means that f is a morphism with *domain* $A \in \mathsf{Ob}\,\mathbb{C}$ and *codomain* $B \in \mathsf{Ob}\,\mathbb{C}$. For any $A, B \in \mathsf{Ob}\,\mathbb{C}$,

$$\hom(A,B) :\stackrel{\text{def}}{=} \{\, f \in \mathsf{Mor}\,\mathbb{C} \;:\; (f : A \longrightarrow B) \,\}.$$

Further, *composition* $\circ$ is a partial binary operation on $\mathsf{Mor}\,\mathbb{C}$, and if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ then $f \circ g : A \longrightarrow C$. We use the notion of a *functor* in the usual sense, i.e. a functor is a map from a category to a category which takes objects to objects, morphisms to morphisms, preserves domains and codomains, identities[757] and composition $\circ$. If $\mathbb{C}$ and $\mathbb{D}$ are categories and $\mathcal{D}$ is a functor from $\mathbb{C}$ to $\mathbb{D}$, then we will write $\mathcal{D} : \mathbb{C} \longrightarrow \mathbb{D}$.

Definition A.2.1 (strong embedding)
Terminology from model theory: Let $f : \mathfrak{A} \rightarrowtail \mathfrak{B}$ be an embedding of model $\mathfrak{A}$ into model $\mathfrak{B}$. By the *f-image* $f[\mathfrak{A}]$ of $\mathfrak{A}$ we understand the unique (weak) submodel of $\mathfrak{B}$ such that f is an isomorphism between $\mathfrak{A}$ and $f[\mathfrak{A}]$.

Now, $f : \mathfrak{A} \rightarrowtail \mathfrak{B}$ is called a *strong embedding* iff it is an embedding and the f-image $f[\mathfrak{A}]$ of $\mathfrak{A}$ is a strong submodel of $\mathfrak{B}$.

$\lhd$

Definition A.2.2 (categories $\mathbb{M}\mathsf{od}(Th)$, $\mathbb{G}\mathsf{e}(Th)$, $\mathbb{M}\mathsf{og}(TH)$)
Let Th be a set of formulas in our frame language.

(i) $\mathsf{Mod}(Th)$ forms a category $\mathbb{M}\mathsf{od}(Th)$ the following way. The class of objects of $\mathbb{M}\mathsf{od}(Th)$ is $\mathsf{Mod}(Th)$ and the morphisms are those embeddings

$$f : \mathfrak{M}_0 \rightarrowtail \mathfrak{M}_1$$

which are surjective on the sets of photons (i.e. $f[Ph_0] = Ph_1$), unless $\mathfrak{M}_0$ is the empty model.[758] More precisely, the morphisms of the category $\mathbb{M}\mathsf{od}(Th)$ are triples of the form $\langle \mathfrak{M}, f, \mathfrak{N} \rangle$, where $f : \mathfrak{M} \rightarrowtail \mathfrak{N}$ is such that $f[Ph^{\mathfrak{M}}] = Ph^{\mathfrak{N}}$ or $\mathfrak{M}$ is the empty model. The reason why we need triples instead of f in itself is that when looking at a morphism we have to be able to tell what its domain and codomain are. For simplicity, if there is no danger of confusion we will use f as a morphism instead of the triple $\langle \mathfrak{M}, f, \mathfrak{N} \rangle$. We hope context will help. The composition $\circ$ is the usual one.[759]

[757] A morphism $f : A \longrightarrow A$ is called an *identity* if for every morphism g with domain A, $f \circ g = g$ and for every morphism g' with codomain A, $g' \circ f = g'$.

[758] Surjectiveness on the sets of photons is required only because eventually we want $\mathcal{M}$ to be a functor between $\mathbb{G}\mathsf{e}(Th)$ and $\mathbb{M}\mathsf{od}(Th)$. It is not quite obvious to see why this purpose (functoriality of $\mathcal{M}$) makes us to need the surjectiveness condition. Hint: this is connected to condition (e) on p.310. If we omitted item (e) on p.310 from the definition of $\mathcal{M}$, then we could define morphisms of $\mathbb{M}\mathsf{od}(Th)$ to be the embeddings. The reader is invited to elaborate an alternative version to our $(\mathcal{M}, \mathcal{G})$-duality by omitting condition (e) from the definition of $\mathcal{M}$ and then dropping the present surjectiveness condition w.r.t. Ph.

[759] I.e. $\langle \mathfrak{M}, f, \mathfrak{N} \rangle \circ \langle \mathfrak{M}_1, g, \mathfrak{N}_1 \rangle = \langle \mathfrak{M}, f \circ g, \mathfrak{N}_1 \rangle$ if $\mathfrak{N} = \mathfrak{M}_1$ and is *undefined* otherwise.

(ii) $\mathsf{Ge}(Th)$ forms a category $\mathbb{G}\mathsf{e}(Th)$ in the following way. The class of objects of $\mathbb{G}\mathsf{e}(Th)$ is $\mathsf{Ge}(Th)$ and the morphisms are those embeddings

$$h : \mathfrak{G}_0 \rightarrowtail \mathfrak{G}_1$$

which are (i) strong embeddings on the $\langle Mn;\, Bw\rangle$ reducts and are (ii) surjective on the sets of photon-like lines (i.e. $h[L_0^{Ph}] = L_1^{Ph}$), unless $\mathfrak{G}_0$ is the empty model. (The composition $\circ$ is the usual one.)

(iii) For any set TH of formulas in the language of GEO, $\mathsf{Mog}(TH)$ forms a category $\mathbb{M}\mathsf{og}(TH)$ in a completely analogous way with item (ii), i.e. the class of objects of $\mathbb{M}\mathsf{og}(TH)$ is $\mathsf{Mog}(TH)$ and the morphisms are those embeddings

$$h : \mathfrak{G}_0 \rightarrowtail \mathfrak{G}_1$$

which are (i) strong embeddings on the $\langle Mn;\, Bw\rangle$ reducts and are (ii) surjective on the sets of photon-like lines (i.e. $h[L_0^{Ph}] = L_1^{Ph}$), unless $\mathfrak{G}_0$ is the empty model.

$\lhd$

Definition A.2.3 $\quad \mathbf{Pax}^+_+ \stackrel{\text{def}}{=} \mathbf{Pax}^+ + \mathbf{Ax(diswind)}$.

$\lhd$

The functions $\mathcal{M}$, $\mathcal{G}$, $\mathcal{M}o$, $\mathcal{G}o$ are defined on the objects of the categories $\mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$, $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$, $\mathbb{M}\mathsf{og}(\mathbf{lopag})$, $\mathbb{M}\mathsf{od}(\mathbf{Wax})$, respectively. In the following definition we extend these functions to the morphisms. In this way we obtain functors

$$\begin{array}{ll} \boldsymbol{\mathcal{M}} : \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+), & \boldsymbol{\mathcal{G}} : \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+), \\ \boldsymbol{\mathcal{M}o} : \mathbb{M}\mathsf{og}(\mathbf{lopag}) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Wax}), & \boldsymbol{\mathcal{G}o} : \mathbb{M}\mathsf{od}(\mathbf{Wax}) \longrightarrow \mathbb{M}\mathsf{og}(\mathbf{lopag}). \end{array}$$

Definition A.2.4 (the functors $\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{M}o}, \boldsymbol{\mathcal{G}o}$)
To define a functor, one has to define what it does with the objects and what it does with the morphisms (of the category in question). On the objects $\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{M}o}, \boldsymbol{\mathcal{G}o}$ agree with $\mathcal{M}, \mathcal{G}, \mathcal{M}o, \mathcal{G}o$, respectively. It remains to define our functors on the morphisms.

$\boldsymbol{\mathcal{M}}$. For every morphism $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ of $\mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$ we will define the morphism $\boldsymbol{\mathcal{M}}(h) : \boldsymbol{\mathcal{M}}(\mathfrak{G}_0) \longrightarrow \boldsymbol{\mathcal{M}}(\mathfrak{G}_1)$ of $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$, see the left-hand side of Figure 135. Since the definition looks somewhat "longish" we note that in it we will do the "natural thing" (following the structure of the definition of $\mathcal{M}$). Let $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ be a morphisms of $\mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$, i.e. $\mathfrak{G}_0 = \langle Mn_0, \ldots\rangle$, $\mathfrak{G}_1 = \langle Mn_1, \ldots\rangle \in \mathsf{Ge}(\mathbf{Pax}^+_+)$ and h is an embedding satisfying the conditions in the definition of the category $\mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$, i.e. in Def.A.2.2(ii). Then h is a tuple $\langle h_M, h_F, h_L\rangle$ with $h_M : Mn_0 \rightarrowtail Mn_1$, $h_F : F_0 \rightarrowtail F_1$ and $h_L : L_0 \rightarrowtail L_1$. Further, $\boldsymbol{\mathcal{M}}(\mathfrak{G}_0) = \langle B_0, \ldots\rangle$, $\boldsymbol{\mathcal{M}}(\mathfrak{G}_1) = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Pax}^+_+)$ by Prop.4.5.41 (p.313). Then $\boldsymbol{\mathcal{M}}(h) := \langle \boldsymbol{\mathcal{M}}(h)_B, \boldsymbol{\mathcal{M}}(h)_F, \boldsymbol{\mathcal{M}}(h)_G\rangle$, where $\boldsymbol{\mathcal{M}}(h)_B : B_0 \longrightarrow B_1$, $\boldsymbol{\mathcal{M}}(h)_F : F_0 \longrightarrow F_1$ and $\boldsymbol{\mathcal{M}}(h)_G : G_0 \longrightarrow G_1$ are defined as follows. To define $\boldsymbol{\mathcal{M}}(h)_B$ let $b \in B_0$. Then either $b = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs_0 \subseteq {}^{n+1}Mn_0$, for some $o, \ldots, e_{n-1}$ or $b \in Ph_0 = L_0^{Ph}$. Now,

$$\boldsymbol{\mathcal{M}}(h)_B(b) :\stackrel{\text{def}}{=} \begin{cases} \langle h_M(o), h_M(e_0), \ldots, h_M(e_{n-1})\rangle & \text{if } b = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs_0 \\ h_L(b) & \text{if } b \in Ph_0. \end{cases}$$

$$\begin{tikzcd}
\mathfrak{G}_0 \arrow[d, "h"'] \arrow[r, phantom, "\xrightarrow{\boldsymbol{\mathcal{M}}}"] & \boldsymbol{\mathcal{M}}(\mathfrak{G}_0) \arrow[d, "\boldsymbol{\mathcal{M}}(h)"] \\
\mathfrak{G}_1 & \boldsymbol{\mathcal{M}}(\mathfrak{G}_1)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathfrak{M}_0 \arrow[d, "f"'] \arrow[r, phantom, "\xrightarrow{\boldsymbol{\mathcal{G}}}"] & \boldsymbol{\mathcal{G}}(\mathfrak{M}_0) \arrow[d, "\boldsymbol{\mathcal{G}}(f)"] \\
\mathfrak{M}_1 & \boldsymbol{\mathcal{G}}(\mathfrak{M}_1)
\end{tikzcd}$$

Figure 135:

$\boldsymbol{\mathcal{M}}(h)_B$ takes observers to observers and photons to photons. $\boldsymbol{\mathcal{M}}(h)_F$ is defined to be h_F and $\boldsymbol{\mathcal{M}}(h)_G$ is naturally induced by $\boldsymbol{\mathcal{M}}(h)_F$, i.e. $\boldsymbol{\mathcal{M}}(h)_G : \mathsf{Eucl}(\mathfrak{F}_0) \longrightarrow \mathsf{Eucl}(\mathfrak{F}_1)$ is defined by $\ell \mapsto \widetilde{\boldsymbol{\mathcal{M}}(h)_F}[\ell]$.

We will prove as Claim A.2.5(i) that $\boldsymbol{\mathcal{M}}(h) : \boldsymbol{\mathcal{M}}(\mathfrak{G}_0) \longrightarrow \boldsymbol{\mathcal{M}}(\mathfrak{G}_1)$ is indeed a morphism of $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$, moreover that $\boldsymbol{\mathcal{M}} : \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$ is a functor.

$\boldsymbol{\mathcal{G}}$. For every morphism $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ of $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$ we will define the morphism $\boldsymbol{\mathcal{G}}(f) : \boldsymbol{\mathcal{G}}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}}(\mathfrak{M}_1)$, see the right-hand side of Figure 135. Let $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ be a morphism of $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$, i.e. $\mathfrak{M}_0 = \langle B_0, \ldots\rangle$, $\mathfrak{M}_1 = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Pax}^+_+)$ and f is an embedding satisfying the conditions in the definition of the category $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$, i.e. in Def.A.2.2(i). Then f is a tuple $\langle f_B, f_F, f_G\rangle$ with $f_B : B_0 \rightarrowtail B_1$, $f_F : F_0 \rightarrowtail F_1$ and $f_G : G_0 \rightarrowtail G_1$. Further, $\boldsymbol{\mathcal{G}}(\mathfrak{M}_0) = \langle Mn_0, \ldots\rangle$, $\boldsymbol{\mathcal{G}}(\mathfrak{M}_1) = \langle Mn_1, \ldots\rangle \in \mathsf{Ge}(\mathbf{Pax}^+_+)$. Then $\boldsymbol{\mathcal{G}}(f) := \langle \boldsymbol{\mathcal{G}}(f)_M, \boldsymbol{\mathcal{G}}(f)_F, \boldsymbol{\mathcal{G}}(f)_L\rangle$, where $\boldsymbol{\mathcal{G}}(f)_M \subseteq Mn_0 \times Mn_1$, $\boldsymbol{\mathcal{G}}(f)_F : F_0 \longrightarrow F_1$ and $\boldsymbol{\mathcal{G}}(f)_L \subseteq L_0 \times L_1$ are defined as follows. Let $\langle e_0, e_1\rangle \in Mn_0 \times Mn_1$ and $\langle \ell_0, \ell_1\rangle \in L_0 \times L_1$. Then

$$\langle e_0, e_1\rangle \in \boldsymbol{\mathcal{G}}(f)_M$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs_0)(\exists p \in {}^nF_0)\Big(w_m(p) = e_0 \ \land \ w_{f_B(m)}(\widetilde{f_F}(p)) = e_1\Big).$$

Further

$$\langle \ell_0, \ell_1\rangle \in \boldsymbol{\mathcal{G}}(f)_L$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs_0)\Big((\exists i \in n)\,(\,\ell_0 = w_m[\bar{x}_i] \ \land \ \ell_1 = w_{f_B(m)}[\bar{x}_i]\,)^{760} \lor$$
$$(\exists ph \in Ph)\,(\,\ell_0 = w_m[tr_m(ph)] \ \land \ \ell_1 = w_{f_B(m)}[tr_{f_B(m)}(f_B(ph))]\,)\Big).$$

$\boldsymbol{\mathcal{G}}(f)_F$ is defined to be f_F.

We will prove as Claim A.2.5(ii) that $\boldsymbol{\mathcal{G}}(f) : \boldsymbol{\mathcal{G}}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}}(\mathfrak{M}_1)$ is indeed a morphism of $\mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$, moreover that $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$ is a functor.

[760] The first $\bar{x}_i$ is the i-th coordinate axis in ${}^nF_0$ while the second $\bar{x}_i$ is the i-th coordinate axis in ${}^nF_1$.

$\boldsymbol{\mathcal{M}o}$. For every morphism $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$ we will define the morphism $\boldsymbol{\mathcal{M}o}(h) : \boldsymbol{\mathcal{M}o}(\mathfrak{G}_0) \longrightarrow \boldsymbol{\mathcal{M}o}(\mathfrak{G}_1)$ of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$. Let $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ be a morphism of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$, i.e. $\mathfrak{G}_0 = \langle Mn_0, \ldots\rangle$, $\mathfrak{G}_1 = \langle Mn_1, \ldots\rangle \in \mathsf{Mog}(\mathbf{lopag})$ and h is an embedding satisfying the conditions in the definition of the category $\mathbb{M}\mathrm{og}(\mathbf{lopag})$, i.e. in Def.A.2.2(iii). Then h is a pair $\langle h_M, h_L\rangle$ with $h_M : Mn_0 \rightarrowtail Mn_1$ and $h_L : L_0 \rightarrowtail L_1$. Further, $\boldsymbol{\mathcal{M}o}(\mathfrak{G}_0) = \langle B_0, \ldots\rangle$, $\boldsymbol{\mathcal{M}o}(\mathfrak{G}_1) = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Wax})$ by Thm.4.5.57. Then $\boldsymbol{\mathcal{M}o}(h) := \langle \boldsymbol{\mathcal{M}o}(h)_B, \boldsymbol{\mathcal{M}o}(h)_F, \boldsymbol{\mathcal{M}o}(h)_G\rangle$ where $\boldsymbol{\mathcal{M}o}(h)_B : B_0 \longrightarrow B_1$, $\boldsymbol{\mathcal{M}o}(h)_F \subseteq F_0 \times F_1$ and $\boldsymbol{\mathcal{M}o}(h)_G \subseteq G_0 \times G_1$ are defined as follows. $\boldsymbol{\mathcal{M}o}(h)_B$ is defined analogously to the case of $\boldsymbol{\mathcal{M}}$, i.e. as follows. Let $b \in B_0$. Then either $b = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs_0 \subseteq {}^{n+1}Mn_0$, for some $o, \ldots, e_{n-1}$ or $b \in Ph_0 = L_0^{Ph}$. Now,

$$\boldsymbol{\mathcal{M}}(h)_B(b) :\overset{\text{def}}{=} \begin{cases} \langle h_M(o), h_M(e_0), \ldots, h_M(e_{n-1})\rangle & \text{if } b = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs_0 \\ h_L(b) & \text{if } b \in Ph_0. \end{cases}$$

To define $\boldsymbol{\mathcal{M}o}(h)_F$ let $\langle p, q\rangle \in F_0 \times F_1$. In the definition below, we will use $F_0, F_1, \mathfrak{F}_0, \mathfrak{F}_1$ which were introduced in Definitions 4.5.31 (p.306) and 4.5.55 (p.327). Then,

$$\langle p, q\rangle \in \boldsymbol{\mathcal{M}o}(h)_F$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists p' \in p)(\exists q' \in q)\Big(pj_i(q') = h_M(pj_i(p')), \quad \text{for all } i \in 3\Big).$$

$\boldsymbol{\mathcal{M}o}(h)_G$ is induced by $\boldsymbol{\mathcal{M}o}(h)_F$ the natural way, cf. the definition of $\boldsymbol{\mathcal{M}}(h)_G$ in item $\boldsymbol{\mathcal{M}}$. above.

We will prove as Claim A.2.5(iii) that $\boldsymbol{\mathcal{M}o}(h) : \boldsymbol{\mathcal{M}o}(\mathfrak{G}_0) \longrightarrow \boldsymbol{\mathcal{M}o}(\mathfrak{G}_1)$ is indeed a morphism of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$, moreover that $\boldsymbol{\mathcal{M}o} : \mathbb{M}\mathrm{og}(\mathbf{lopag}) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax})$ is a functor.

$\boldsymbol{\mathcal{G}o}$. For every morphism $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$ we will define the morphism $\boldsymbol{\mathcal{G}o}(f) : \boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}o}(\mathfrak{M}_1)$ of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$. Let $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ be a morphism of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$, i.e. $\mathfrak{M}_0 = \langle B_0, \ldots\rangle$, $\mathfrak{M}_1 = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Wax})$ and f is an embedding satisfying the conditions in the definition of the category $\mathbb{M}\mathrm{od}(\mathbf{Wax})$, i.e. in Def.A.2.2(i). Further, $\boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) = \langle Mn_0, \ldots\rangle$, $\boldsymbol{\mathcal{G}o}(\mathfrak{M}_1) = \langle Mn_1, \ldots\rangle \in \mathsf{Mog}(\mathbf{lopag})$ by Thm.4.5.57. We define the morphism

$$\boldsymbol{\mathcal{G}o}(f) : \boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}o}(\mathfrak{M}_1)$$

of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$ to be $\langle \boldsymbol{\mathcal{G}}(f)_M, \boldsymbol{\mathcal{G}}(f)_L\rangle$, where $\boldsymbol{\mathcal{G}}(f)_M$ and $\boldsymbol{\mathcal{G}}(f)_L$ are defined as in item $\boldsymbol{\mathcal{G}}$. above.

We will prove as Claim A.2.5(iv) that $\boldsymbol{\mathcal{G}o}(f) : \boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}o}(\mathfrak{M}_1)$ is indeed a morphism, moreover that $\boldsymbol{\mathcal{G}o} : \mathbb{M}\mathrm{od}(\mathbf{Wax}) \longrightarrow \mathbb{M}\mathrm{og}(\mathbf{lopag})$ is a functor.

$\triangleleft$

Claim A.2.5 below serve to prove correctness of Def.A.2.4 above.

Claim A.2.5 **($\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{G}}$, $\boldsymbol{\mathcal{M}o}$, $\boldsymbol{\mathcal{G}o}$ are functors)**

(i) $\boldsymbol{\mathcal{M}} : \mathbb{G}\mathrm{e}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Pax}^+_+)$ is a functor.

(ii) $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathrm{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathrm{e}(\mathbf{Pax}^+_+)$ is a functor.

(iii) $\boldsymbol{\mathcal{M}o} : \mathbb{M}\mathrm{og}(\mathbf{lopag}) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax})$ is a functor.

(iv) $\mathcal{G}o : \mathbb{M}\mathsf{od}(\mathbf{Wax}) \longrightarrow \mathbb{M}\mathsf{og}(\mathbf{lopag})$ is a functor.

The **proof** is available from the author. ∎

Next, we recall the notion of adjoint pair of functors from category theory e.g. from Mac Lane [159]. For this, first we introduce the notion of a reflection and coreflection in Def.A.2.6 below. We will use the notion of a subcategory in the usual way, cf. e.g. Mac Lane [159].

Definition A.2.6 (reflection, coreflection) Let $\mathbb{C}$ and $\mathbb{D}$ be two categories.

(i) Assume $\mathbb{D}$ is a subcategory of $\mathbb{C}$. Let $A \in \mathsf{Ob}\,\mathbb{C}$.

(a) $B \in \mathsf{Ob}\,\mathbb{D}$ is called the *reflection* of A in $\mathbb{D}$ iff B is the "nearest" object to A in $\mathbb{D}$, i.e. iff there is a morphism $f : A \longrightarrow B$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(A, B'))(\exists! g \in \hom(B, B'))\, f \circ g = f',$$

see the left top picture in Figure 136.

(b) $B \in \mathsf{Ob}\,\mathbb{D}$ is called a *coreflection* of A in $\mathbb{D}$ iff there is a morphism $f : B \longrightarrow A$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(B', A))(\exists! g \in \hom(B', B))\, g \circ f = f',$$

see the right top figure in Figure 136.

(ii) Assume $\mathcal{C} : \mathbb{D} \longrightarrow \mathbb{C}$ is a functor. Let $A \in \mathsf{Ob}\,\mathbb{C}$.

(a) $B \in \mathsf{Ob}\,\mathbb{D}$ is called a *reflection* of A in $\mathbb{D}$ iff B is the nearest object to A in $\mathbb{D}$, i.e. there is a morphism $f : A \longrightarrow \mathcal{C}(B)$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(A, \mathcal{C}(B')))(\exists! g \in \hom(B, B'))\, f \circ \mathcal{C}(g) = f',$$

see the left bottom picture in Figure 136.

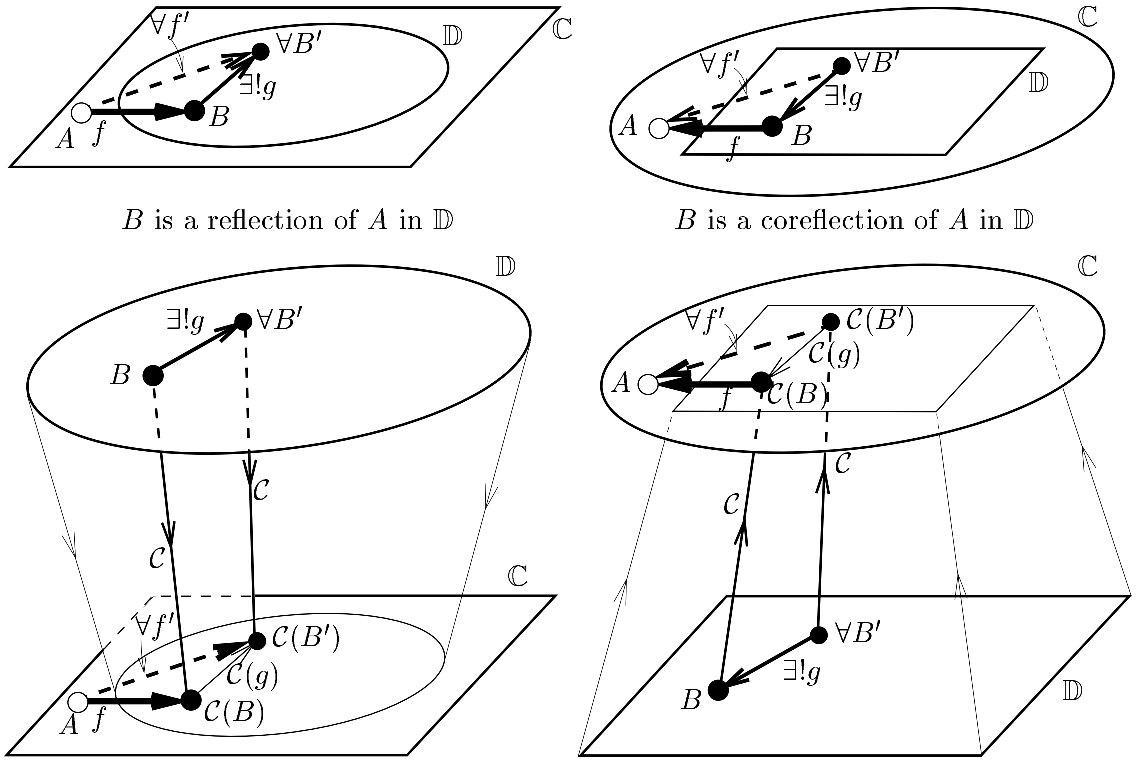
The morphism $f : A \longrightarrow \mathcal{C}(B)$ above is called the *$\mathcal{C}$-reflection arrow*[761] *of the object A*.

(b) $B \in \mathsf{Ob}\,\mathbb{D}$ is called a *coreflection* of A in $\mathbb{D}$ iff there is a morphism $f : \mathcal{C}(B) \longrightarrow A$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(\mathcal{C}(B'), A))(\exists! g \in \hom(B', B))\, \mathcal{C}(g) \circ f = f',$$

see the right bottom picture in Figure 136.

The morphism $f : \mathcal{C}(B) \longrightarrow A$ above is called the *$\mathcal{C}$-coreflection arrow of the object A*. ◁



B is a reflection of A in $\mathbb{D}$

B is a coreflection of A in $\mathbb{D}$

Figure 136: Reflection and coreflection.

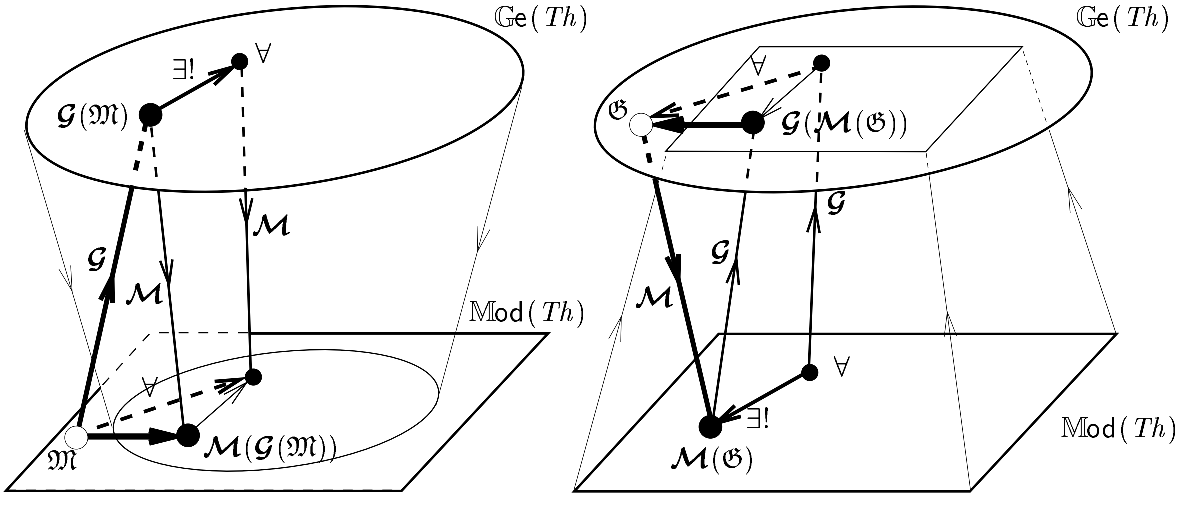


Figure 137: $(\mathcal{M}, \mathcal{G})$ is an adjoint pair of functors, under certain conditions.

Definition A.2.7 (adjoint situation)[762]
Let $\mathbb{C}$ and $\mathbb{D}$ be two categories and let

$$(\star) \qquad \mathcal{C} : \mathbb{D} \longrightarrow \mathbb{C} \quad \text{and} \quad \mathcal{D} : \mathbb{C} \longrightarrow \mathbb{D}$$

be two functors. Then $(\mathcal{C}, \mathcal{D})$ is called an *adjoint pair* iff for every $A \in \mathsf{Ob}\,\mathbb{C}$, $\mathcal{D}(A)$ is the reflection of A in $\mathbb{D}$ and for every $B \in \mathsf{Ob}\,\mathbb{D}$, $\mathcal{C}(B)$ is the coreflection of B in $\mathbb{C}$, cf. Figure 137.

Further, we say that $(\star)$ above is an *adjoint situation* iff $(\mathcal{C}, \mathcal{D})$ is an adjoint pair of functors.

$\lhd$

Definition A.2.8 $\quad \mathbf{Pax}^{++}_{+} \stackrel{\text{def}}{=} \mathbf{Pax}^{++} + \mathbf{Ax(diswind)}$.

$\lhd$

For the following conjectures recall that $\mathcal{M}$, $\mathcal{G}$, $\mathcal{Mo}$, $\mathcal{Go}$ are functors by Claim A.2.5 (p.A-10).

Conjecture A.2.9 *We strongly conjecture that (i) and (ii) below hold.*

(i) $\mathcal{M} : \mathbb{G}\mathsf{e}(\mathbf{Pax}^{++}_{+}) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Pax}^{++}_{+})$ *and* $\mathcal{G} : \mathbb{M}\mathsf{od}(\mathbf{Pax}^{++}_{+}) \longrightarrow \mathbb{G}\mathsf{e}(\mathbf{Pax}^{++}_{+})$ *is an adjoint situation,*[763] *cf. Figure 137.*

(ii) $\mathcal{Mo} : \mathbb{M}\mathsf{og}(\mathbf{lopag}^{+}) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Wax}^{+})$ *and*
$\mathcal{Go} : \mathbb{M}\mathsf{od}(\mathbf{Wax}^{+}) \longrightarrow \mathbb{M}\mathsf{og}(\mathbf{lopag}^{+})$ *is an adjoint situation.*

$\lhd$

Let $f : A \longrightarrow B$ be a morphism of the category $\mathbb{C}$. We call f an *isomorphism* (of $\mathbb{C}$) if

$$(\exists g \in \hom(B, A))(f \circ g \text{ and } g \circ f \text{ are identity morphisms}),$$

cf. footnote 757 on p.A-7 for identity morphisms.

Definition A.2.10 (equivalence of categories)[764]
The categories $\mathbb{C}$ and $\mathbb{D}$ are called *equivalent* iff there is an adjoint pair of functors

$$\mathcal{C} : \mathbb{D} \longrightarrow \mathbb{C} \quad \text{and} \quad \mathcal{D} : \mathbb{C} \longrightarrow \mathbb{D}$$

such that the following holds. For every object A of $\mathbb{C}$ the $\mathcal{C}$-reflection arrow is an isomorphism and the same holds for the $\mathcal{D}$-coreflection arrows of objects $B \in \mathsf{Ob}\,\mathbb{D}$. In such situations the pair $(\mathcal{C}, \mathcal{D})$ of functors is called an *equivalence of categories* ($\mathbb{C}$ and $\mathbb{D}$).[765]

$\lhd$

---

[761] We could call this f intuitively $\mathbb{D}$-reflection arrow.

[762] We refer to e.g. Mac Lane [159] for the "official" definition of adjointness. Cf. also Adámek [1, p. 138–148, (sub-section 3F)], and Adámek-Herrlich-Strecker [2, pp. 283-300] where a large number of mathematical applications/examples of adjointness and what we call here duality theories is given.

[763] In accordance with our Convention 4.5.2 (p.283) here we are talking about the restrictions of $\mathcal{M}$ and $\mathcal{G}$ to $\mathbb{G}\mathsf{e}(\mathbf{Pax}^{++}_{+})$ and $\mathbb{M}\mathsf{od}(\mathbf{Pax}^{++}_{+})$. We will use this convention throughout the present appendix.

[764] We refer to e.g. Mac Lane [159] or Adámek et al [2, p.26, Def.3.33] for the "official" definition of equivalence of categories. Officially a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is an equivalence iff it is a bijection on every $\hom(A, B)$, i.e. $F : \hom_{\mathbb{C}}(A, B) \rightarrowtail\!\!\!\twoheadrightarrow \hom_{\mathbb{D}}(F(A), F(B))$, and it is surjective with respect to isomorphisms.

[765] An adjoint situation $(\mathcal{C}, \mathcal{D})$ could be called a *Galois-adjoint situation* iff $Rng(\mathcal{C})$ and $Rng(\mathcal{D})$ are categories and $(\mathcal{C}, \mathcal{D})$ is an equivalence between categories $Rng(\mathcal{C})$ and $Rng(\mathcal{D})$. The so obtained notion could be considered as a special kind of adjoint situations and at the same time as a generalization of Galois connections.

Conjecture A.2.11 *We strongly conjecture that* $\mathbb{M}\mathrm{od}(Th)$ *and* $\mathbb{G}\mathrm{e}(Th)$ *are equivalent categories, and* $(\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}})$ *is an equivalence between these two categories, assuming* $n > 2$ *and* $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(||)^- + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. ◁

In connection with the above conjecture cf. Thm.4.3.38 (p.261) saying that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent, assuming the assumptions of the above conjecture. Thm.4.3.38 already implies isomorphism, hence equivalence, between categories $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ if we choose elementary embeddings as morphisms, cf. p.281.

Before stating our next conjecture we note the following. Consider the functor $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathrm{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathrm{e}(\mathbf{Pax}^+_+)$. Then $\boldsymbol{\mathcal{G}}$ is surjective in the sense that $Rng(\boldsymbol{\mathcal{G}})$ is $\mathbb{G}\mathrm{e}(\mathbf{Pax}^+_+)$ up to isomorphism. This holds for any Th with $Th \models \mathbf{Pax}^+_+$ in place of $\mathbf{Pax}^+_+$.

Conjecture A.2.12 *We strongly conjecture that (i) and (ii) below hold.*

(i) *Consider the functors* $\boldsymbol{\mathcal{M}} : \mathbb{G}\mathrm{e}(\mathbf{Pax}^{++}_+) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Pax}^{++}_+)$ *and* $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathrm{od}(\mathbf{Pax}^{++}_+) \longrightarrow \mathbb{G}\mathrm{e}(\mathbf{Pax}^{++}_+)$. *Then* $Rng(\boldsymbol{\mathcal{M}})$ *is a category and* $Rng(\boldsymbol{\mathcal{M}})$ *and* $\mathbb{G}\mathrm{e}(\mathbf{Pax}^{++}_+)$ *are equivalent categories, and* $(\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}} \restriction Rng(\boldsymbol{\mathcal{M}}))$ *is an equivalence between these two categories.*

(ii) *Consider the functors* $\boldsymbol{\mathcal{M}o} : \mathbb{M}\mathrm{og}(\mathbf{lopag}^+) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax}^+)$ *and* $\boldsymbol{\mathcal{G}o} : \mathbb{M}\mathrm{od}(\mathbf{Wax}^+) \longrightarrow \mathbb{M}\mathrm{og}(\mathbf{lopag}^+)$. *Then* $Rng(\boldsymbol{\mathcal{M}o})$ *and* $Rng(\boldsymbol{\mathcal{G}o})$ *are equivalent categories and* $(\boldsymbol{\mathcal{M}o} \restriction Rng(\boldsymbol{\mathcal{G}o}), \boldsymbol{\mathcal{G}o} \restriction Rng(\boldsymbol{\mathcal{M}o}))$ *is an equivalence between these two categories.*

◁

Items (i) of Conjectures A.2.9 and A.2.12 together say that $(\mathcal{M}, \mathcal{G})$ is a Galois-adjoint situation in the sense of footnote 765, assuming $\mathbf{Pax}^{++}_+$; while items (ii) of the same conjectures say that $(\mathcal{M}o, \mathcal{G}o)$ is a Galois-adjoint situation, assuming $\mathbf{Wax}^+$ and $\mathbf{lopag}^+$. Cf. also the intuitive text on p.328 above Conjecture 4.5.58 together with Remark A.1.7 and compare them with Conjectures A.2.9, A.2.12.

For the idea of systematically extending the duality theories studied herein to "category theoretic adjoint situations" we refer to AMN [18] p.1095 (Exercise 6.6.85) and to related parts therein.

Notation A.2.13 Let A be a set and let $\tau(x)$ be a term with input variable x, defined for $x \in A$. Recall that then $f := \langle \tau(x) : x \in A \rangle$ denotes a function $f : A \longrightarrow Rng(f)$, cf. p.2 where we used *expr* in place of τ.

We will use the intuitive notation $\tau(-)$ for denoting this function f. I.e.

$$\tau(-) \stackrel{\text{def}}{:=} \langle \tau(x) : x \in A \rangle.$$

This notation is somewhat under-specified since A, i.e. the domain of $\tau(-)$, is not explicitly indicated. This intuitive notation $\tau(-)$ comes from category theory. Cf. also the notational convention $g(-, y, z)$ above Def.4.3.35 (partial derivative) on p.518 (in §4.3) of AMN [18]. That convention is the <u>same</u> as the present one (with some extra parameters added).

◁

Connections between adjoint situations, Galois connections, and other duality theories: Before getting started, we note that Remark A.1.1 (p.A-1) is also about our present subject.

Assume that in our category $\mathbb{C}$ there is at most one morphism between any two objects, i.e. assume $|\hom(A, B)| \leq 1$ is valid in $\mathbb{C}$. Then $\mathbb{C}$ becomes a pre-ordering. (Hint: We use $A \leq B$ to denote $\hom(A, B) \neq \emptyset$.) Assume the same for category $\mathbb{D}$. Then functors $\mathcal{C} : \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathcal{D} : \mathbb{D} \longrightarrow \mathbb{C}$ become order preserving mappings between pre-orderings $\mathbb{C}$ and $\mathbb{D}$. Then it is a natural question to ask which pairs (f, g) of order preserving mappings between pre-orderings P, Q are actually adjoint situations. Translating the definition of adjoint situations way above (from the language of categories to that of pre-orderings) gives us a natural answer to this question. Assume for simplicity that our pre-orderings are actually partial orderings (posets for short). Then (f, g) forms an adjoint pair iff $(\star)$ below holds.

$$(\star) \qquad \begin{aligned} f(p) &= \inf\{\, q \in Q \;:\; p \leq g(q) \,\} \\ g(q) &= \sup\{\, p \in P \;:\; q \geq f(p) \,\}. \end{aligned}$$

Actually, we note that $(\star)$ works for characterizing adjointness even in the more general case of pre-orderings, too. More precisely, if we want $(\star)$ to work for pre-orderings too, then it is enough to replace "$f(p) = \inf\{\ldots\}$" by "$f(p)$ is a smallest element[766] of the set $\{\, q \in Q \;:\; p \leq g(q) \,\}$" and similarly for "$g(q) = \sup\{\ldots\}$".

Summing it up, $(\star)$ is the order-theoretic counterpart of adjointness. The paper Andréka-Greechie-Strecker [13] discusses and investigates equivalent versions and applications of (order-theoretic) adjointness of the form $(\star)$ above. In that paper $(\star)$ shows up in the fourth line beginning with "If (f, g) is such a pair, then $f(p) = \ldots$". (This is the second, equivalent definition they give for order-theoretic adjointness.) They call an (order-theoretic) adjoint pair (f, g) satisfying $(\star)$ a *residuated-residual pair*. Among other things, they show that residuated-residual pairs are equivalent with Galois connections. They discuss the connections with Galois theory, too. Residuatedness plays an extremely important role in many branches of algebra, in sophisticated duality theories, and in Algebraic Logic. One of the slogans in a large part of Algebraic Logic says that all extra Boolean operators in Algebraic Logic are residuated.[767] Cf. e.g. Jónsson-Tarski [142], Jónsson [141], Jónsson-Tsinakis [143], Thompson [258, p.340] and Jipsen-Jónsson-Rafter [140] and the references in the latter. Actually, Birkhoff in his famous Lattice Theory book [47] introduces relation algebras as "residuated Boolean lattices" (where we note that relation algebras are one of the main themes in the literature of Tarskian Algebraic Logic). In passing we note that the *residual* g of f is very strongly related to what is called the *conjugate* of f in a large part of abstract algebra, cf. e.g. Jónsson [141, pp. 129-130], Thompson [258, p.340] and Henkin-Monk-Tarski [120, Part I, p.175]. If our posets are Boolean algebras then for any mapping g its dual[768] g^∂ is also defined. Now, if (f, g) are residuated then g^∂ is exactly the conjugate of f. I.e. the conjugate of f is the dual g^∂ of the residual g of f. Therefore, conjugates of mappings are extremely close to residuals of mappings, e.g. in

[766] In pre-orderings, x is a smallest element of H iff $x \in H$ and $(\forall y \in H)\, x \leq y$.

[767] An operator f on a Boolean algebra, or more generally a function f : pre-order $\longrightarrow$ pre-order is called *residuated* iff it is part of a residuated-residual pair (f, g). Then g is called the *residual* of f. (We could call g the "right residual" of f and f the "left-residual" of g, but we do not do this e.g. because it would cause confusion with the slashes to be discussed soon [the slashes are called left and right residuals of $\circ$].) In passing we note that the area we are discussing is known as "Boolean algebras with operators", or more recently as lattices with operators, cf. e.g. Bahl et al. [39]. The originators and promoters of this area are e.g. Tarski, Jónsson, van Benthem and other classics of mathematical logic.
It is sometimes useful to think of the residual g of f as a kind of *quasi-inverse* (w.r.t. the pre-ordering $\leq$) of f. Hence f is residuated iff it is quasi-invertible w.r.t. the pre-order in question.

[768] $g^\partial(x) := -g(-x)$

the case of Boolean algebras the two concepts are term-definable from each other.[769] (More generally, the mathematical idea of a "conjugate" in general is strongly related to the idea of a residual pair, i.e. of an adjoint situation.) In the literature of Algebraic Logic and in that of Sub-structural Logics (e.g. Lambek calculus, linear logic etc.) the residuals of any fixed "central" binary operation, say $\circ$, are denoted by the slashes $/, \backslash$ while the conjugates of the same central operation are denoted by the triangles $\triangleleft, \triangleright$, cf. Andréka-Mikulás [28], Jipsen-Jónsson-Rafter [140], Marx-Venema [186], van Benthem [265, pp. 194, 195, 230, 231], [267, p.246] and Bahls-Cole-Galatos-Jipsen-Tsinakis [39][770].

The paper Andréka et al. [13] discusses further important applications and variants of adjointness of the form $(\star)$ above. About this subject cf. also our next section on Algebraic Logic. The present subject is continued in a broader perspective in the part entitled "On the importance ... duality theories" on pp. 1098–1105 of AMN [18].

Madarász-Németi [176] solves a classical, distinguished problem (Problem 2.10) of Henkin-Monk-Tarski [120, Part I (1971)], using the level of abstraction of Boolean algebras with operators (BAO's for short) mentioned several times above implicitly. In the literature the above outlined area involving residuation, conjugates, the slashes $/, \backslash, \triangleleft, \triangleright, \circ$ etc. (related e.g. to Lambek calculus, linear logic) is often referred to as BAO-theory [because it goes back to the historical paper Jónsson-Tarski [142] (entitled BAO's) the abstract of which appeared in 1948]. Madarász [165] refutes a conjecture in the original 1952 version of the historic paper Jónsson-Tarski [142].[771] This was a conjecture which influenced the development of BAO theory and modal logic until the publication of [165] in a kind of misleading way.[772] After this, [165] elaborates (i) a syntax-semantics duality theory for not necessarily normal modal logics, and (ii) a representation theory for all BAO's (including non-normal ones).

On the importance, "omnipresence" and literature of duality theories:

For a discussion of this subject we refer to AMN [18] pp. 1098-1105, but here we also note the following.

Further examples, applications, explanations and motivation for duality theories i.e. adjointness can be found in the following references. Most of the expository works on categories emphasize that adjoint situations (hence duality theories) are extremely important for (almost) the whole of mathematics and that besides this they turn out to be a successful vehicle for unifying and deepening mathematical thought.[773] Cf. Guitart [111], Mac Lane [159], Goldblatt [101], Handbook of Categorical Algebra [50], Barr-Wells [41, §1.9, p.

[769] Though the residual of f is its quasi-inverse, the conjugate of f is not a quasi-inverse (but the dual of a quasi-inverse). E.g. if c is a complemented ($c(-c(x)) = -c(x)$) closure operator on a Boolean algebra (cf. Fig.104) then c is its own conjugate, cf. Henkin-Monk-Tarski [120, Part I, p.175], while the residual c^{∂} of c is the interior operator $c^{\partial}(x) = -c(-x)$ naturally corresponding to c.

[770] In this paper, though the residuals "/", "\" are defined, the lattice we are working in is not required even to be distributive.

[771] The conjecture in question says that all BAO's are term-definitionally equivalent with normal BAO's. Methodological consequences of this conjecture are the conjectures that (i) the representation theory of normal BAO's provides a representation theory of all BAO's, and (ii) the theory of normal BAO's provides a theory for all BAO's hence investigating non-normal BAO's is superfluous. These ((i),(ii)) were also disproved, cf. e.g. Thm.3.6 (pp.79-80) in [165]. But cf. also the remark mentioning Goldblatt on p.78 immediately above § 3 therein. (Cf. also items 5.6, 5.12, 5.17 there.) Naming the refuted conjecture in the 1952 version of [142] remained implicit in [165] (for certain considerations). The refutation of the conjecture in question is discussed in more detail in Andréka-Madarász-Németi-Sain [27], items (*2), 3.13-3.15.

[772] The conjecture itself was removed from the new version of the paper but the formulation of its methodological conclusions remained there, cf. [142, p.379 lines 10-7 bottom up].

[773] Typical examples of this are e.g. Lawvere [153], Mac Lane [159] (but almost all the remaining references say this with differences only in emphasis).

50–63], Freyd-Scedrov [88], Adámek et al. [2], [3], Varadarajan [268], Lawvere-Schanuel [156], Nel [201], Pelletier-Rosický [210], Dimov-Tholen [72], Janelidze [139], Davey-Priestley [67], Marx [185, Fig.1.2 (p.12) and §2.2 (... "duality theory")] and Mikulás [191, §1.3 "Bridge between..."(p.18)]. Several examples of application of duality theories and similar algebraic ideas in physics can be found in Shafarevich [233] cf. e.g. Example 2 in §21 or Example 8 in §5, or the parts on groups of symmetry and laws of nature, or on elementary particles and group representations in §18 item E, or the Galois theory of linear differential equations in §18 item B.

The study of duality theories is an active, fruitful and steadily growing branch of mathematics and mathematical physics nowadays. To illustrate this we mention only (i-v) below. (i) The duality between not-necessarily normal Boolean algebras with operators (non-normal BAO's for short) on the one side and partial Kripke-frames on the other was discovered only recently[774], cf. Madarász [165]. This duality extends to a duality for non-normal modal logics with modalities of higher ranks. (Cf. e.g. Marx-Venema [186] or Blackburn et at. [48] for the latter.) Further duality results relevant to the present work are in Madarász [161, 164, 166, 170]. (ii) The results in the very recent Hirsch-Hodkinson book [129] contains new developments on dualities under the name "representation theorems". (iii) The recent duality paper Goldblatt [103]. (iv) Makkai's duality for ultra-categories and first-order-logic theories [179]. (v) As a further illustration that duality theories are dynamically evolving with applications in physics, we include here a small sample of further references: Stinespring [239], Sankaran [229], Joyal-Street [144], Schauenburg [230], Gootman-Lazar [105].

As we indicated in Remark A.1.1 item (II), footnote 753 on p.A-2 herein, and in footnote 1077 on p.1079 of AMN [18] the application areas range from geometry, analysis, algebra, through to sheaves, computability, logic and other things.

A.3 Algebraic Logic as a duality theory, in analogy with the ones in the present work

There is a methodological connection here with algebraic logic (for the latter cf. e.g. Madarász [170], Andréka-Németi-Sain [31]), Madarász [161], [165], [164], [167], Madarász-Sayed [178] as follows.

In algebraic logic, a logical system $\mathcal{L}$ is a tuple $\mathcal{L} = \langle Fm, \dots, \vdash \rangle$ which, in some sense, is close to a certain intuitive conception of logic. Then a function Alg is defined which with each logic $\mathcal{L}$ associates a class $\mathsf{Alg}(\mathcal{L})$ of algebras. The idea is that $\mathsf{Alg}(\mathcal{L})$ is a mathematically more streamlined object than $\mathcal{L}$, while $\mathcal{L}$ is closer to a certain intuition. Therefore it is worthwhile to develop a so-called duality theory "Logical systems"$\rightleftarrows$"Classes of algebras" which enables us to "translate" problems and results in both directions cf. Andréka-Németi-Sain [31], and Madarász [170].[775]

For discussing the case of our present theory, let $\mathcal{G}$ and $\mathcal{M}$ be the functions as defined above. Then our frame models $\mathfrak{M}$ are in analogy with logical systems $\mathcal{L}$, $\mathfrak{M} \stackrel{\mathcal{G}}{\longmapsto} \mathfrak{G}_{\mathfrak{M}}$ is in analogy with the function $\mathcal{L} \mapsto \mathsf{Alg}(\mathcal{L})$ and $\mathcal{M}$ is in analogy with the construction of

[774] but already receives applications e.g. in connection with partial correctness of programs

[775] This methodology has been further refined in Madarász [161, 165, 164, 166]. Its algebraic "side" has been further explored in Madarász [163, 167], Madarász-Sayed [178].

a logical system from a class of algebras (which we did not recall from Algebraic Logic). Indeed, as in the case of algebraic logic, $\mathfrak{M}$ is also close to a certain intuitive picture of bodies, motion, observation etc, while $\mathfrak{G}_{\mathfrak{M}}$ is a mathematically more streamlined object. (Just as our geometries $\mathfrak{G}_{\mathfrak{M}}$ ($\mathfrak{M} \in \mathsf{Mod}(Th)$) form a category the natural way, the same applies to the $\mathsf{Alg}(\mathcal{L})$'s [for $\mathcal{L} \in$ *Logics*]. I.e. the $\mathsf{Alg}(\mathcal{L})$'s form a category.) In this connection cf. the observational/theoretical distinction in the introduction to Chapter 4, e.g. p.129.

To pursue the analogy, for many logics, $\mathsf{Alg}(\mathcal{L})$ is a class of cylindric algebras (e.g. this is the case for classical first-order logic). It is customary to investigate "reducts" of $\mathsf{Alg}(\mathcal{L})$ e.g. a certain reduct of $\mathsf{Alg}(\mathcal{L})$ is a class of Boolean algebras, while another is a class of distributive lattices. The experience is that investigating these reducts helps us in understanding the behavior of $\mathsf{Alg}(\mathcal{L})$ and even the original object $\mathcal{L}$ itself. In analogous manner, in relativity theory it seems to be interesting to investigate reducts of $\mathfrak{G}_{\mathfrak{M}}$ one $\mathfrak{G}^1_{\mathfrak{M}} = \langle Mn, L; L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, \mathcal{T}\rangle$ of which is obtained by omitting g and eq while another one $\mathfrak{G}^2_{\mathfrak{M}} = \langle Mn, L; L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r\rangle$ is obtained by omitting (or forgetting) g, $\mathcal{T}$ and eq.

A point to make here is the observation that *none* of the two worlds (that of $\mathcal{L}$ and that of $\mathsf{Alg}(\mathcal{L})$) is better than the other. The useful and illuminating thing is that we can move between the two (without making one superior to the other). Similar observation applies here to $\mathfrak{M}$ and $\mathfrak{G}_{\mathfrak{M}}$, the important thing is that we can reconstruct one from the other (i.e. move between them) without thinking that one is superior and the other should be forgotten forever. Recent results in the above kind of algebraic logic, relevant to the present work, are in e.g. Madarász [161, 165, 164, 166, 163, 170, 167], Madarász-Németi [176], Andréka-Madarász-Németi [23].

Applications of duality theories to definability theory (as used in the present work) are e.g. in Madarász [167, 164, 166, 163], Madarász-Sayed [178], Hoogland [134].

The connections between our *duality theories*, *representation theorems*[776], *adjoint functors* and the subject of the logical connections between physical and mathematical theories will be further discussed in a later work (cf. [171]). But we emphasize already here the following: Duality theories, adjoint situations, representation theorems are different words for the same thing. One uses different words for putting the emphasis on different aspects of the (same) subject.[777]

[776] cf. e.g. Madarász [165] for a representation theorem in this spirit

[777] Baez [38], too, treats duality theories, representation theorems and adjointness as belonging together. He also writes about these concepts being important for physics.

List of axioms and axiom systems

Convention: In this list the axiom systems (i.e. theories) to be recalled will be boxed in. The only purpose of this is to make searching in the list easier.

(1) Main axiom systems

$\boxed{\textbf{Basax}} \stackrel{\text{def}}{=} \{\, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{Ax6}, \mathbf{AxE} \,\}$ (cf. p.23), where:

Ax1 $G = \mathsf{Eucl}(n, \mathbf{F})$, p.18.

Ax2 $Obs \cup Ph \subseteq Ib$, p.20.

Ax3 $(\forall h \in Ib)(\forall m \in Obs)\ tr_m(h) \in G$, p.20.

Ax4 $(\forall m \in Obs)\ tr_m(m) = \bar{t}$, p.20.

Ax5 $(\forall m \in Obs)(\forall \ell \in G)\ \Big(ang^2(\ell) < 1\ \Rightarrow\ (\exists k \in Obs)\, \ell = tr_m(k)$ and

$$ang^2(\ell) = 1\ \Rightarrow\ (\exists ph \in Ph)\, \ell = tr_m(ph)\Big),\ \text{p.22.}$$

Ax6 $(\forall m, k \in Obs)\ Rng(w_m) = Rng(w_k)$, p.22.

AxE $(\forall m \in Obs)(\forall ph \in Ph)\ v_m(ph) = 1$, p.23.

$\boxed{\textbf{Newbasax}} \stackrel{\text{def}}{=} (\mathbf{Basax} \setminus \{\, \mathbf{Ax6}, \mathbf{Ax3}, \mathbf{AxE} \,\}) \cup \{\, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{Ax3_0}, \mathbf{AxE_0} \,\} = \{\, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_0} \,\}$ (cf. p.191 of AMN [18] and p.122 herein), where:

Ax6$_{00}$ $(\forall m, k \in Obs)\ w_m[tr_m(k)] \subseteq Rng(w_k)$, p.110.

Intuitively, observer k sees all those events which are seen by another observer m on k's life-line.

Ax6$_{01}$ $(\forall m, k \in Obs)\ Dom(\mathsf{f}_{mk}) \in Open$, p.110 herein and p.190 of AMN [18].

Ax3$_0$ $(\forall h \in Ib)\ (tr_m(h) \in G \cup \{\emptyset\}\ \ \wedge\ \ (\exists k \in Obs) tr_k(h) \neq \emptyset)$, p.109.

AxE$_0$ $(\forall m \in Obs)(\forall ph \in Ph)(tr_m(ph) \neq \emptyset\ \ \Rightarrow\ \ v_m(ph) = 1)$, p.191 of AMN [18].

$m \stackrel{\odot}{\to} b \stackrel{\text{def}}{\Longleftrightarrow} tr_m(b) \neq \emptyset$, p.110.

$\boxed{\textbf{Bax}} \stackrel{\text{def}}{=} (\mathbf{Newbasax} \setminus \{\, \mathbf{Ax5}, \mathbf{AxE_0} \,\}) \cup \{\, \mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}} \,\} = \{\, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}} \,\}$ (cf. p.219 of AMN [18] and also p.121 herein), where:

Ax5$^{\mathbf{Obs}}$ $(\exists ph)(\forall \ell)\Big(m \stackrel{\odot}{\to} ph\ \wedge\ [ang^2(\ell) < v_m(ph)\ \Rightarrow\ (\exists k)\ell = tr_m(k)]\Big)$, p.218 of AMN [18].

Ax5$^{\mathbf{Ph}}$ $ang^2(\ell) = v_m(ph)\ \Rightarrow\ (\exists ph)\ell = tr_m(ph)$, p.219 of AMN [18].

AxE$_{00}$ $(m \overset{\odot}{\to} ph_1, ph_2) \;\Rightarrow\; v_m(ph_1) = v_m(ph_2)$, p.218 of AMN [18].

AxE$_{01}$ $v_m(ph) \neq 0$, p.115.

$\boxed{\textbf{Flxbasax}} \overset{\text{def}}{=} \textbf{Bax} + \textbf{AxE}_{\mathbf{02}} =$
$\{\,\textbf{Ax1}, \textbf{Ax2}, \textbf{Ax3}_{\mathbf{0}}, \textbf{Ax4}, \textbf{Ax5}^{\mathbf{Obs}}, \textbf{Ax5}^{\mathbf{Ph}}, \textbf{Ax6}_{\mathbf{00}}, \textbf{Ax6}_{\mathbf{01}}, \textbf{AxE}_{\mathbf{00}}, \textbf{AxE}_{\mathbf{01}}, \textbf{AxE}_{\mathbf{02}}\}$ (cf. p.428 of AMN [18] and p.121 herein), where:

AxE$_{02}$ $(\forall m, k \in Obs)(\forall ph, ph_1 \in Ph)$
$$[(m \overset{\odot}{\to} ph \;\wedge\; k \overset{\odot}{\to} ph_1) \;\Rightarrow\; v_m(ph) = v_k(ph_1)],$$ p.427 of AMN [18].

$\boxed{\textbf{Bax}^-} \overset{\text{def}}{=} (\textbf{Bax} \setminus \{\,\textbf{Ax5}^{\mathbf{Obs}}, \textbf{Ax5}^{\mathbf{Ph}}, \textbf{AxE}_{\mathbf{00}}\,\}) \cup \{\,\textbf{Ax5}_{\mathbf{Obs}}, \textbf{Ax5}_{\mathbf{Ph}}, \textbf{AxP1}\,\} =$
$\{\,\textbf{Ax1}, \textbf{Ax2}, \textbf{Ax3}_{\mathbf{0}}, \textbf{Ax4}, \textbf{Ax5}_{\mathbf{Obs}}, \textbf{Ax5}_{\mathbf{Ph}}, \textbf{Ax6}_{\mathbf{00}}, \textbf{Ax6}_{\mathbf{01}}, \textbf{AxP1}, \textbf{AxE}_{\mathbf{01}}\,\}$ (cf. p.479 of AMN [18] and p.117 herein), where:

AxP1 Intuitively, starting out from one point p of space-time, in every direction (forwards) there is at most one "speed of light" (i.e. photon-trace), formally:

$(\forall m \in Obs)(\forall ph_1, ph_2 \in Ph)(\forall d \in \mathsf{directions})$[778] $\Big((ph_1$ and ph_2 are moving forwards in direction d as seen by m and $tr_m(ph_1) \cap tr_m(ph_2) \neq \emptyset) \;\Rightarrow$
$$tr_m(ph_1) = tr_m(ph_2)\Big),$$ p.115.

Ax5$_{\mathbf{Ph}}$ Intuitively, from any point p of space-time in any direction there is a photon moving forwards in that direction, cf. Fig.138 (p.477) of AMN [18], formally:

$(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})(\exists ph \in Ph)$
$[p \in tr_m(ph) \;\wedge\;$ (ph is moving forwards in direction d as seen by m)],
p.115.

Ax5$_{\mathbf{Obs}}$ Intuitively: Let us fix an observer m, a direction d, and a point p of space-time. We will speak about things moving forwards in direction d through point p as seen by m (without mentioning all this data). Assume there is a photon moving in direction d. Then there is a photon in the same direction which is limiting in the following sense: For all speeds less than this limiting photon, there is an observer moving with this speed, cf. Fig.139 (p.478) of AMN [18]. Formally:

$(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})$
$\Big(\Big[(\exists ph \in Ph)(p \in tr_m(ph) \;\wedge\;$ (ph is moving forwards in d as seen by m)$\Big] \;\Rightarrow$
$\Big[(\exists ph \in Ph)\Big(p \in tr_m(ph) \;\wedge\;$ (ph is moving forwards in d as seen by m) $\wedge$
$(\forall \lambda \in F)(0 \le \lambda < v_m(ph) \;\Rightarrow\; (\exists k \in Obs)(p \in tr_m(k) \;\wedge\; v_m(k) = \lambda \;\wedge$
(k is moving forwards in direction d as seen by m)$))\Big)\Big]\Big)$, p.117.

$\boxed{\textbf{Pax}} \overset{\text{def}}{=} \{\,\textbf{Ax1}, \textbf{Ax2}, \textbf{Ax3}_{\mathbf{0}}, \textbf{Ax4}, \textbf{Ax5}_{\mathbf{Obs}}{}^{--}, \textbf{Ax6}_{\mathbf{00}}, \textbf{Ax6}_{\mathbf{01}}\,\}$ (cf. p.482 of AMN [18] and p.109 herein) where:

[778] Let us recall that directions are (nonzero) space-vectors, i.e. $\mathsf{directions} = {}^{n-1}F \setminus \{\bar{0}\}$, cf. p.108.

Ax1, Ax2, Ax3$_0$, Ax4, Ax6$_{00}$, Ax6$_{01}$ have already been listed.

Ax5$_{\mathbf{Obs}}$ $^{--}$ Intuitively, for each direction d there is a positive λ such that through any point there are observers moving forwards in direction d with all speeds smaller than λ. More precisely, for any observer m and for any plane P parallel with $\bar{t}$ there is $\lambda \in {}^+F$ such that for any straight line ℓ in P, with $ang^2(\ell) < \lambda$, ℓ is the trace of an observer (as seen by m, of course). In other words:

$(\forall m \in Obs)(\forall d \in \mathsf{directions})(\forall p \in {}^nF)(\exists \lambda \in {}^+F)$
$(\forall q \in {}^nF)\Big[space(p) - space(q) = \delta \cdot d$ for some $\delta \in F \;\Rightarrow\; (\forall 0 \le \varepsilon < \lambda)$
$(\exists k \in Obs)(k$ moves forwards in direction d with speed ε and $q \in tr_m(k))\Big]$, p.109.

* * *

Assume **Ax1**, **Ax2**, **Ax3$_0$**, **AxP1**. Let $m \in Obs$. Then

$$c_m : {}^nF \times \mathsf{directions} \xrightarrow{\circ} F \cup \{\infty\}$$

is a partial function such that $c_m(p, d)$ is defined iff m sees a photon at point p moving forwards in direction d, and $c_m(p, d)$ is the speed of this photon,[779] cf. p.116 herein and pp. 473, 535 of AMN [18].

Let Th be one of our theories such that $Th \models \{\,\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{AxP1}\,\}$. Then

$\boxed{Th^{\oplus}} \overset{\text{def}}{=} Th + c_m(p, d) < \infty$, p.117.

Next, we turn to listing the Reichenbachian versions of our theories. For this we recall some notation.

Assume **Bax**$^-$. By Thm.4.3.17 (p.488) of AMN [18], the speed $c_m(p, d)$ does not depend on p. This motivates the following:

$$\mathsf{c}_m(d) \overset{\text{def}}{=} c_m(\bar{0}, d),$$

cf. p.488 of AMN [18]. Intuitively, $\mathsf{c}_m(d)$ is the (square of the) speed of light in direction d as seen by observer m.

<u>*Notation:*</u> Let $m \in Obs$ and $d \in \mathsf{directions}$. Then

$$T_m(d) \overset{\text{def}}{=} \begin{cases} 1/\sqrt{\mathsf{c}_m(d)} & \text{if } 0 \ne \mathsf{c}_m(d) < \infty, \\ \infty & \text{if } \mathsf{c}_m(d) = 0, \\ 0 & \text{if } \mathsf{c}_m(d) = \infty, \end{cases}$$

cf. p.555 of AMN [18]. $T_m(d)$ is the reciprocal of the "speed of light", i.e. it is the time needed for a photon to cover the unit distance in direction d (as seen by observer m).

$\boxed{\mathbf{Reich_0(Bax)}} \overset{\text{def}}{=}$ **Bax**$^-$ + **R(AxE$_{00}$)** (cf. p.562 of AMN [18]), where

R(AxE$_{00}$) **Ax**$(\sqrt{\ })$ and
$(\forall d, d_1 \in \mathsf{directions})[T_m(d) + T_m(-d) = T_m(d_1) + T_m(-d_1)]$, p.557 of AMN [18].

[779] There is only one such speed because of **AxP1**.

$\boxed{\mathbf{Reich}_0(\mathbf{Flxbasax})} \stackrel{\text{def}}{=} \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE_{02}})$ (cf. p.562 of AMN [18]), where

$\mathbf{R}(\mathbf{AxE_{02}})$ $(\forall m, k \in Obs)(\forall d, d_1 \in \mathsf{directions})$
$T_m(d) + T_m(-d) = T_k(d_1) + T_k(-d_1)$, and $\mathbf{Ax}(\sqrt{\ })$, p.557 of AMN [18].

$\boxed{\mathbf{Reich}_0(\mathbf{Newbasax})} \stackrel{\text{def}}{=} \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE})$ (cf. p.562 of AMN [18]), where

$\mathbf{R}(\mathbf{AxE})$ $\mathbf{Ax}(\sqrt{\ })$ and
$(\forall m \in Obs)(\forall d \in \mathsf{directions})T_m(d) + T_m(-d) = 2$, p.557 of AMN [18].

$\boxed{\mathbf{Reich}_0(\mathbf{Basax})} \stackrel{\text{def}}{=} \mathbf{Reich}_0(\mathbf{Newbasax}) + \mathbf{Ax6}$, p.562 of AMN [18].

Let $Th \in \{\,\mathbf{Bax}, \mathbf{Flxbasax}, \mathbf{Newbasax}, \mathbf{Basax}\,\}$. Then

$\boxed{\mathbf{Reich}(Th)} \stackrel{\text{def}}{=} \mathbf{Reich}_0(Th) + \mathbf{R}_\Delta(E)$ (cf. p.576 of AMN [18]), where

$\mathbf{R}_\Delta(E)$ $(\forall m \in Obs)(\exists r \in F)(\forall d_1, d_2, d_3 \in \mathsf{directions})\Big[d_1 + d_2 + d_3 = \bar{0} \Rightarrow$

$$\frac{|d_1| \cdot T_m(d_1) + |d_2| \cdot T_m(d_2) + |d_3| \cdot T_m(d_3)}{|d_1| + |d_2| + |d_3|} = r\Big],$$ p.574 of AMN [18].

(2) Axioms concerning the direction of flow of time

The binary relation $\uparrow \subseteq Obs \times Obs$ is defined as follows.

$$m \uparrow k \stackrel{\text{def}}{\iff} \mathsf{f}_{km}(1_t)_t > \mathsf{f}_{km}(\bar{0})_t,$$ p.176.

Intuitively, $m \uparrow k$ means that m sees k's clock running forwards. Cf. p.176. Further, if $m, k \in Obs$ then $m\ STL\ k$ means that m sees k moving more slowly than light (cf. p.91).

Ax(↑) $(\forall m, m' \in Obs)\ (\,tr_m(m') = \bar{t} \quad \Rightarrow \quad m \uparrow m'\,)$, p.296 of AMN [18].

Ax(↑↑) $(\forall m, k \in Obs) m \uparrow k$, p.176.

$\mathbf{Ax(\uparrow\uparrow_0)}$ $(\forall m, k \in Obs)\,(m \stackrel{\odot}{\to} k \ \to\ m \uparrow k)$, p.176.

$\mathbf{Ax(\uparrow\uparrow_{00})}$ $(\forall m, k \in Obs)\,(m\ STL\ k \ \to\ m \uparrow k)$, p.176.

(3) Auxiliary axioms

Recall that

$$Triv = \{ f \;:\; f \text{ is an isometry of } {}^n\mathbf{F} \text{ and } f(1_t) - f(\bar{0}) = 1_t \},$$

cf. p.81.

Ax($Triv$) $(\forall m \in Obs)(\forall f \in Triv)(\exists k \in Obs)\ \mathsf{f}_{mk} = f$, p.82.

Ax($Triv_t$) $(\forall m \in Obs)(\forall f \in Triv)\ \Big(f[\bar{t}] = \bar{t} \quad \Rightarrow \quad (\exists k \in Obs)\ \mathsf{f}_{mk} = f\Big)$, p.82.

Ax($Triv_t$)$^-$ Assume we are given an observer m and a $Triv$ transformation f that leaves the time-axis fixed. Then m has a brother, call it k, such that m thinks that (i) the coordinate axes of k are the f-images of the original coordinate axes $\bar{x}_i$, and (ii) the clock of k runs forwards, formally:

$(\forall m \in Obs)(\forall f \in Triv)\, [\, f[\bar{t}] = \bar{t} \quad \Rightarrow$
$\qquad (\exists k \in Obs)(\forall i \in n)(\, \mathsf{f}_{km}[\bar{x}_i] = f[\bar{x}_i] \quad \wedge \quad m \uparrow k\,)\,]$, p.157.

Ax($\|$) $(\forall m, k \in Obs)\Big(tr_m(k) \parallel \bar{t} \quad \Rightarrow \quad (\mathsf{f}_{mk} \text{ is an isometry})\Big)$, p.82.

Ax($\|$)$^-$ $(\forall m, k \in Obs \cap Ib)$
$[\, tr_m(k) = \bar{t} \quad \Rightarrow \quad (\mathsf{f}_{mk} = h \circ I, \quad \text{for some expansion } h \text{ and isometry } I)\,]$, p.167.

Ax($\sqrt{\ }$) $(\forall 0 < x \in F)(\exists y \in F)\ y^2 = x$, p.55.

Ax(rc) (Axiom schema for real-closed fields)

Ax($\sqrt{\ }$) $+ \{\, \phi_{2n+1} \;:\; n \in \omega \,\}$, where

(ϕ_n) $\qquad \forall x_0 \ldots \forall x_n \exists y\, [x_n \neq 0 \;\rightarrow\; (x_0 + x_1 \cdot y + \ldots + x_n \cdot y^n = 0)]$,

p.301 of AMN [18].

Ax(diswind) (Axiom of disjoint windows)

$(\forall m, k \in Obs \cap Ib)\ [(m \overset{\odot}{\to} ph \;\wedge\; k \overset{\odot}{\to} ph) \;\Rightarrow\; m \overset{\odot}{\to} k]$, p.157.

(4) Axioms concerning measuring distances

Ax(eqtime) Observers with common life-line agree on time-like distances, i.e.
$(\forall m, m' \in Obs)$
$$\Big(tr_m(m') = \bar{t} \quad \Rightarrow \quad (\forall p, q \in \bar{t}\,)\ |p - q| = |\mathsf{f}_{mm'}(p) - \mathsf{f}_{mm'}(q)|\Big), \text{ p.77.}$$

Ax(eqspace) Observers agree on spatial distances, i.e.
$(\forall m, k \in Obs)(\forall p, q \in {}^nF)$
$$\Big(\,(p_t = q_t \;\wedge\; \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \quad \Rightarrow \quad |p - q| = |\mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q)|\,\Big),$$
p.83.

Ax(eqm) Inertial observers agree on distances, i.e.
$(\forall m, k \in Obs \cap Ib)(\forall i, j \in n)(\forall p, q \in \bar{x}_i)(\forall p', q' \in \bar{x}_j)$
$$\Big([w_m(p) = w_k(p') \;\wedge\; w_m(q) = w_k(q')] \;\Rightarrow\; |p - q| = |p' - q'|\Big), \text{ p.145.}$$

(5) Axiom systems $\mathbf{Pax^+}$, $\mathbf{Pax^{++}}$, $\mathbf{Pax^+_+}$, $\mathbf{Pax^{++}_+}$, Wax, $\mathbf{Wax^+}$

Ax(Bw) $(\forall m,k\in Obs)[m \overset{\odot}{\to} k \;\Rightarrow\; (\mathsf{f}_{mk}$ is betweenness preserving$)\,]$, p.289.

Ax♡ $B = Obs\cup Ph$, p.125.

Ax(∞*ph*) $(\forall m\in Obs)(\forall ph,ph'\in Ph)\Big(\,[\,\bar 0\in tr_m(ph)\cap tr_m(ph')\;\wedge\;(\,ph$ and ph' move in the same direction as seen by $m\,)\;\wedge\; v_m(ph)=\infty\,]\;\to\; v_m(ph')=\infty\Big)$, p.289.

Intuitively, no observer can emit simultaneously in the same direction two photons one with infinite speed and the other one with finite speed.

Ax(ext) $(\forall m,k\in Obs)\,[w_m=w_k\;\Rightarrow\; m=k]\quad\wedge$
$(\forall b,b_1\in B\setminus Obs)(\forall m\in Obs)\,[tr_m(b)=tr_m(b_1)\;\Rightarrow\; b=b_1]$,
p.125.

Ax(*Ph*) $(\forall m\in Obs)(\forall p\in {}^nF)(\exists ph_1,ph_2\in Ph)\; tr_m(ph_1)\cap tr_m(ph_2)=\{p\}$, p.328.

$\boxed{\mathbf{Pax^+}}\overset{\text{def}}{=}$ **Pax** + $\mathbf{AxE_{01}}$ + **Ax(Bw)** + **Ax(∞*ph*)** +
$\Big([\,\mathbf{Ax(eqtime)}\;\wedge\;(\forall m,k\in Obs)(\forall 0<i\in\omega)\; tr_m(k)\neq\bar x_i\,]\quad\vee\quad \mathbf{Ax(eqm)}\Big)$,
p.289.

$\boxed{\mathbf{Pax^{++}}}\overset{\text{def}}{=}$ $\mathbf{Pax^+}$ + **Ax(eqm)** + **Ax(ext)** + **Ax♡**, p.A-4.

$\boxed{\mathbf{Pax^+_+}}\overset{\text{def}}{=}$ $\mathbf{Pax^+}$ + **Ax(diswind)**, p.A-8.

$\boxed{\mathbf{Pax^{++}_+}}\overset{\text{def}}{=}$ $\mathbf{Pax^{++}}$ + **Ax(diswind)**, p.A-13.

$\boxed{\mathbf{Wax}}\overset{\text{def}}{=}$ { **Ax1**, **Ax2**, **Ax3**, **Ax4**, **Ax6**, **Ax(Bw)**, **Ax(*Ph*)** }, p.328.

$\boxed{\mathbf{Wax^+}}\overset{\text{def}}{=}$ **Wax** + **Ax(ext)** + **Ax♡** + **Ax(∞*ph*)** + $(\forall m,k)(\mathsf{f}_{mk}\in Aftr)$, p.A-4.

(6) Symmetry axioms

$\mathbf{Ax(symm_0)}$ $(\forall m,k\in Obs)(\exists m',k'\in Obs)$
$\Big(tr_m(m')=tr_k(k')=\bar t\quad\wedge\quad \mathsf{f}_{mk}=\mathsf{f}_{k'm'}\Big)$, p.75.

Ax(symm) is defined to be $\mathbf{Ax(symm_0)}$ + **Ax(eqtime)**, p.77.

$\mathbf{Ax(syt_0)}$ $(\forall m,k\in Obs)\Big(tr_m(k)\neq\emptyset\quad\Rightarrow$
$(\forall p\in\bar t)\;|\mathsf{f}_{mk}(p)_t-\mathsf{f}_{mk}(\bar 0)_t|=|\mathsf{f}_{km}(p)_t-\mathsf{f}_{km}(\bar 0)_t|\Big)$, p.81.

$\mathbf{Ax(syt_{00})}$ $(\forall m,k\in Obs)\;[\mathsf{f}_{mk}(\bar 0)=\bar 0\;\Rightarrow\;|\mathsf{f}_{mk}(1_t)_t|=|\mathsf{f}_{km}(1_t)_t|]$, p.90.

Ax(syt)* $\mathsf{f}_{mk}(\bar 0)=\bar 0\;\Rightarrow\;\mathsf{f}_{mk}(1_t)_t=\mathsf{f}_{km}(1_t)_t$, p.721 of AMN [18].

Ax(syx)* (m, k are in pre-standard configuration[780]) $\Rightarrow$ $|\mathsf{f}_{mk}(1_x)_x| = |\mathsf{f}_{km}(1_x)_x|$, p.725 of AMN [18].

Ax(speedtime) $(\forall m, k, m', k' \in Obs)\Big(v_m(k) = v_{m'}(k') \quad \Rightarrow$
$$(\forall p \in \bar{t})\,|\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(\bar{0})_t| = |\mathsf{f}_{m'k'}(p)_t - \mathsf{f}_{m'k'}(\bar{0})_t|\Big),$$ p.83.

Ax□1 $(\forall m, k, m' \in Obs)(\exists k' \in Obs)\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$, p.87.

Ax□2 $(\forall m, k, m', k' \in Obs)(tr_m(k) = tr_{m'}(k') \rightarrow$ there is an isometry N of ${}^n\mathbf{F}$ such that $N[\bar{t}] \parallel \bar{t}$ and $\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ N)$, p.350 of AMN [18].

Ax△1 $(\forall m, k \in Obs)(\exists k' \in Obs)(tr_m(k) = tr_m(k') \;\wedge\; \mathsf{f}_{mk'} = \mathsf{f}_{k'm})$, p.351 of AMN [18].

Ax△2 $(\forall m, k \in Obs)$ (there is an isometry N of ${}^n\mathbf{F}$ such that $N[\bar{t}] \parallel \bar{t}$ and $\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N)$, p.351 of AMN [18].

Ax(ω)^0 is defined to be the <u>disjunction</u> of the following symmetry axioms: **Ax(syt$_0$)**, **Ax(symm)**, **Ax(speedtime)**, **Ax△1+Ax(eqtime)**, **Ax△2**, **Ax□1+Ax(eqtime)**, **Ax□2**, p.180.

Ax(ω)^{00} is defined to be the <u>disjunction</u> of the following symmetry axioms: **Ax(ω)^0**, **Ax(eqspace)**, **Ax(eqm)+Ax($Triv_t$)$^-$**, p.180.

Ax(ω)$^\sharp$ is defined to be **Ax(ω)^0+Ax($Triv_t$)$^-$+Ax($\sqrt{\ }$)**, p.180.

Ax(ω)$^{\sharp\sharp}$ is defined to be **Ax(ω)^{00} + Ax($Triv_t$)$^-$ + Ax($\sqrt{\ }$)**, p.180.

Ax(symm)† is defined to be **Ax(symm) + Ax($Triv$) + Ax($\parallel$)**, p.100.

Ax(ω) **Ax□1** $\wedge$ **Ax□2** $\wedge$ **Ax△1** $\wedge$ **Ax△2**, p.351 of AMN [18].

Ax(ω^-) **Ax□1** $\vee$ **Ax□2** $\vee$ **Ax△1** $\vee$ **Ax△2**, p.351 of AMN [18].

(7) Symmetry axioms adequate for Reichenbachian theories

R$^+$(Ax eqsp) Intuitively, the thickness of spaceships do not change in the direction orthogonal to movement (cf. pp. 608–614 of AMN [18]), formally:

Assume $m, k \in Obs$ such that $m \overset{\odot}{\to} k$. Assume P, Q are parallel planes of ${}^nF$ such that they are parallel with both $\bar{t}$ and $tr_m(k)$. Then

$$\begin{aligned} \mathsf{Eudist}(P,Q) &= \mathsf{Eudist}(\mathsf{f}_{mk}[P], \mathsf{f}_{mk}[Q]), \text{ p.614 of AMN [18], where} \\ \mathsf{Eudist}(H, H_1) &\overset{\text{def}}{=} \inf\{\, \|p - q\| \;:\; p \in H \text{ and } q \in H_1 \,\}. \end{aligned}$$

[780] m and k are said to be in *pre-standard configuration* iff $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ and $\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t}, \bar{x})] = \mathsf{Plane}(\bar{t}, \bar{x})$. Cf. Def.4.6.5 (p.602) of AMN [18] and Fig.201 (p.603) of AMN [18].

R(Ax eqsp) Intuitively, the thickness of spaceships do not change in the direction orthogonal to movement (cf. pp. 608–614 of AMN [18]), formally:

Assume m and k are in pre-standard configuration[781]. Let P be a (2-dimensional) plane parallel with $\mathsf{Plane}(\bar{t},\bar{x})$. Then the distance between P and $\mathsf{Plane}(\bar{t},\bar{x})$ is the same as the distance between $\mathsf{f}_{mk}[P]$ and $\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t},\bar{x})]$. Formally,

$$\begin{aligned}\mathsf{Eudist}(P,\mathsf{Plane}(\bar{t},\bar{x})) &= \mathsf{Eudist}(\mathsf{f}_{mk}[P],\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t},\bar{x})]), \text{ p.611 of AMN [18], where}\\ \mathsf{Eudist}(H,H_1) &\stackrel{\text{def}}{=} \inf\{\, \|p-q\| \;:\; p\in H \text{ and } q\in H_1\,\}, \text{ cf. p.609 of AMN [18].}\end{aligned}$$

See Fig.205 on p.611 of AMN [18].

R(Ax syt$_0$) Intuitively m and k *literally* see, via photons, each other's clocks slowing down with the same rate, see Fig.207 (p.616) of AMN [18], formally:

$$(\forall m,k\in Obs)[\mathsf{f}_{mk}(\bar{0})=\bar{0} \;\Rightarrow$$
$$(\forall p\in\bar{t})|view_m(\mathsf{f}_{km}(p))| = |view_k(\mathsf{f}_{mk}(p))|] \text{ (cf. p.615) of AMN [18],}$$

where $view_m \stackrel{\text{def}}{=} \{\, \langle p,q\rangle \in {}^nF\times\bar{t} \;:\; p_t\le q_t \text{ and } (\exists ph\in Ph)\; p,q\in tr_m(ph)\,\}$, cf. Fig.206 (p.615) of AMN [18].

R(sym) is defined to be **R(Ax eqsp)**+ **R(Ax syt$_0$)**, p.616 of AMN [18].

(8) Twin paradox

Let $m,k\in Obs$. Then $m\ STL\ k$ means that m sees k moving more slowly than light, cf. p.91.

Ax(TwP) $(\forall m,k_1,k_2\in Obs)(\forall p,q,r\in {}^nF)$

$$\Big([\, m\ STL\ k_1 \;\wedge\; m\ STL\ k_2 \;\wedge\; p_t<q_t<r_t \;\wedge$$
$$\{p\}=tr_m(m)\cap tr_m(k_1) \;\wedge\; \{q\}=tr_m(k_1)\cap tr_m(k_2) \;\wedge\; \{r\}=tr_m(m)\cap tr_m(k_2)\,] \;\Rightarrow$$
$$|p_t-r_t| > |\mathsf{f}_{mk_1}(p)_t-\mathsf{f}_{mk_1}(q)_t| + |\mathsf{f}_{mk_2}(q)_t-\mathsf{f}_{mk_2}(r)_t|\Big), \text{ p.92}$$

(cf. Fig.42 on p.93).

(9) Axiom systems Specrel, Flxspecrel, BaCo, Compl, NewtK$^-$, NewtK

$\boxed{\textbf{Specrel}} \stackrel{\text{def}}{=}$ **Basax** + **Ax(symm)**†, p.100.

$\boxed{\textbf{Flxspecrel}} \stackrel{\text{def}}{=}$ **Bax** + **Ax6** + **Ax(symm)**† + **AxE$_{02}$**, p.428 of AMN [18].

$\boxed{\textbf{Compl}} \stackrel{\text{def}}{=}$ $\{$ **Ax(symm)**, **Ax$\heartsuit$**, **Ax($\uparrow$)**, **Ax5$^+$**, **Ax(ext)**, **Ax(*Triv$_t$*)** $\}$ (cf. p.298 of AMN [18]), where

Ax5$^+$ $\quad \ell\in\mathsf{SlowEucl} \;\Rightarrow\; (\exists k\in Obs)\,(\ell=tr_m(k) \;\wedge\; m\uparrow k)$, p.297 of AMN [18].

$\boxed{\textbf{BaCo}} \stackrel{\text{def}}{=}$ **Basax** + **Compl**, p.125 herein and p.298 in AMN [18].

$\boxed{\textbf{NewtK}^-} \stackrel{\text{def}}{=}$ **Bax** + **Ax6** + **Ax(symm)**† + $(\forall m\in Obs)c_m=\infty$ (cf. p.426 of AMN [18]), where c_m is the speed of light for observer m, assuming **Bax**.

$\boxed{\textbf{NewtK}} \stackrel{\text{def}}{=}$ **NewtK$^-$** + **Ax($\uparrow\uparrow$)** + **Ax$\square$1**, p.426 of AMN [18].

[781] Cf. footnote 780 on p.A-25 for the notion of a pre-standard configuration.

(10) Local versions of our axiom systems for relativity

These axiom systems are introduced to prepare generalization toward general relativity.[782] The key idea is that observer m uses only a subset $Dom(w_m^-) \subseteq {}^nF$ of ${}^nF$ for coordinatizing events as this is explained at the end of §3 and in much more detail in AMN [18, §4.9]. The total world-view functions w_m are replaced with their partial versions w_m^-, where

$$w_m^- \stackrel{\text{def}}{=} \{\langle p, e\rangle \in w_m \ : \ e \neq \emptyset\}.$$

This move will cause "partiality" to dominate our local theories. Therefore we use the adjectives "local" and "partial" interchangeably. ("Local" refers more to the philosophical aspects while "partial" to the technical aspects of our approach.)[783] The partial version of f_{mk} is

$$\mathsf{f}_{mk}^- :\stackrel{\text{def}}{=} (w_m^-) \circ (w_k^-)^{-1}.$$

We obtain the local version of an axiom by replacing all w_m's in it with their partial versions (the w_m^-'s) and *relativizing* the whole axiom to $Dom(w_m^-)$ (or to the domains if there were more observers involved). E.g. **Ax4** said $tr_m(m) = \bar{t}$, by relativizing this statement to $Dom(w_m^-)$ we obtain its partial version

$\mathbf{Ax4}^{\mathrm{par}}$ $tr_m(m) \ = \ \bar{t} \cap Dom(w_m^-) \ \neq \ \emptyset.$

$\boxed{\mathbf{Loc(Bax^-)}}$

$$\stackrel{\text{def}}{=} \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0^{\mathrm{par}}}, \mathbf{Ax4^{\mathrm{par}}}, \mathbf{Ax5_{Obs}^{\mathrm{par}}}, \mathbf{Ax5_{Ph}^{\mathrm{par}}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}^{\mathrm{par}}}, \mathbf{AxP1}, \mathbf{AxE_{01}}\}.$$

where

$\mathbf{Ax3_0^{\mathrm{par}}}$ $(\forall h \in Ib)(\exists \ell \in G)[\, tr_m(h) \ = \ \ell \cap Dom(w_m^-) \quad \text{or} \quad tr_m(h) = \emptyset\,].$

$\mathbf{Ax5_{Obs}^{\mathrm{par}}}$ Intuitively, everywhere in $Dom(w_m^-)$ in each direction d m sees a photon ph which is limiting in the sense that for all speeds $v < v_m(ph)$ there is an observer k moving with speed v in direction d. Formally:

$(\forall p \in Dom(w_m^-))(\forall d \in \mathsf{directions})$
$\Big(\Big[(\exists ph \in Ph)(p \in tr_m(ph) \ \wedge \ (ph \text{ is moving forwards in } d \text{ as seen by } m))\Big] \ \Rightarrow$
$\Big[(\exists ph \in Ph)\Big(p \in tr_m(ph) \ \wedge \ (ph \text{ is moving forwards in } d \text{ as seen by } m) \ \wedge$
$(\forall \lambda \in F)(0 \leq \lambda < v_m(ph) \ \Rightarrow \ (\exists k \in Obs)(p \in tr_m(k) \ \wedge \ v_m(k) = \lambda \ \wedge$
$(k \text{ is moving forwards in direction } d \text{ as seen by } m))\Big)\Big]\Big).$

$\mathbf{Ax5_{Ph}^{\mathrm{par}}}$ Intuitively, everywhere in $Dom(w_m^-)$ there are photons moving forwards, in all directions. Formally:

$(\forall p \in Dom(w_m^-))(\forall d \in \mathsf{directions})(\exists ph \in Ph)$
$[\, p \in tr_m(ph) \ \wedge \ (ph \text{ is moving forwards in direction } d \text{ as seen by } m)].$

$\mathbf{Ax6_{01}^{\mathrm{par}}}$ $(\forall m, k \in Obs)\ Dom(\mathsf{f}_{mk}^-) \in Open.$

I.e., we replaced f_{mk} with f_{mk}^- in this axiom.

[782] Similar preparations are e.g. §4.7 on geodesics, the section on accelerated observers in AMN et al. [24]–[26], and the Reichenbachianizations of our theories in AMN [18, SS4.5–4.7].
[783] We use "local" in the same sense as Einstein did.

$\boxed{\mathbf{Loc}(\mathbf{Bax}^{-\oplus})}$ $= \mathbf{Loc}(\mathbf{Bax}^{-})$ + "the speed of photons is not ∞"

We use the following extra axiom in our local no FTL theorem in Chapter 3. The axiom says that $\mathsf{f}^-_{mk} \restriction S(p,\varepsilon)$ preserves betweenness both forward and backward, as follows.

$\mathbf{Ax(syBw)}^{\mathrm{par}}$ $(\forall p \in Dom(\mathsf{f}^-_{mk}))(\exists \varepsilon \in {}^+F)$
$(\forall q, r, s \in S(p,\varepsilon))\ \big[\mathsf{Betw}(q,r,s) \Leftrightarrow \mathsf{Betw}(\mathsf{f}^-_{mk}(q), \mathsf{f}^-_{mk}(r), \mathsf{f}^-_{mk}(s))\big]$.

(11) Geometrical axioms and axiom systems

Axioms $\mathbf{A_0}$–$\mathbf{A_4}$ and $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{Pa}$ below apply to geometries with reducts $\langle Mn; Bw\rangle$ or $\langle Mn; coll\rangle$. In the case of "$\langle Mn; Bw\rangle$" *coll* is a defined relation, cf. p.159. The new universe (or sort) *lines* is (explicitly) defined over $\langle Mn; coll\rangle$ on p.297. For $H \subseteq Mn$, $Plane'(H)$ is intuitively the "n-long closure of H under *coll*" (cf. Def.4.2.15 on p.160), where throughout n is the dimension of our geometry and $n \geq 2$.

$\mathbf{A_0}$ $(\forall a,b,c \in Mn)[\,coll(a,b,c) \leftrightarrow (\exists \ell \in lines)\, a,b,c \in \ell\,]$, p.298.

$\mathbf{A_1}$ $(\forall a,b \in Mn)(\,a \neq b \ \rightarrow\ (\exists! \ell \in lines)\, a,b \in \ell\,)$, p.298.

$\mathbf{A_2}$ Intuitively, if H is a less than $n+2$ element subset of Mn then the "n-long closure" $Plane'(H)$ of H under *coll* will be closed under *coll*, hence the plane $Plane(H)$ generated by H coincides with $Plane'(H)$ (cf. Def.4.2.15, p.160), formally:

$(\forall H \subseteq Mn)$
$\Big((\,|H| \leq n+1 \ \wedge\ a,b \in Plane'(H) \ \wedge\ coll(a,b,c)\,) \ \rightarrow\ c \in Plane'(H)\Big)$, p.298.

$\mathbf{A_3}$ Intuitively, if $i \leq n$ and H is an $i+1$ element independent subset[784] of Mn then there is exactly one i-dimensional plane[785] that contains H, formally:

$(\forall H, H' \subseteq Mn)\Big((\,|H| = |H'| \leq n+1 \ \ \wedge\ \ \text{(both } H \text{ and } H' \text{ are independent)} \ \ \wedge\ \ H \subseteq Plane'(H')\,) \ \rightarrow\ Plane'(H) = Plane'(H')\Big)$, p.298.

$\mathbf{A_4}$ Mn is an n dimensional plane, p.298.

In connection with axioms $\mathbf{P_1}$, $\mathbf{P_2}$ below we note that the relation of parallelism $\parallel$ on *lines* is defined the usual way in Def.4.5.16 on p.298.

$\mathbf{P_1}$ (Euclid's axiom)

$(\forall \ell \in lines)(\forall a \in Mn)(\exists! \ell' \in lines)(a \in \ell' \ \wedge\ \ell \parallel \ell')$, p.299.

$\mathbf{P_2}$ $(\ell \parallel \ell' \ \wedge\ \ell' \parallel \ell'') \ \rightarrow\ \ell \parallel \ell''$, p.299.

$\boxed{\mathbf{ag}} \stackrel{\text{def}}{=} \{\mathbf{A_0}, \mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_4}, \mathbf{P_1}, \mathbf{P_2}\}$, p.299. **ag** is the axiom system for *affine geometries*.

For $a,b,c,d \in Mn$, the abbreviation $\langle a,b\rangle \parallel \langle c,d\rangle$ means that $a \neq b$, $c \neq d$, and there are $\ell, \ell' \in lines$ such that $a,b \in \ell$, $c,d \in \ell'$ and $\ell \parallel \ell'$, cf. p.300.

[784] Let $H \subseteq Mn$. H is called *independent* iff $(\forall e \in H)\, e \notin Plane'(H \setminus \{e\})$, cf. p.298.

[785] Let $P \subseteq Mn$. P is called an i-dimensional iff there is an $i+1$ element independent subset H of Mn such that $Plane'(H) = P$, where for the notion of an independent subset cf. footnote 784. Cf. Def.4.5.15(ii) on p.298.

Pa (Pappus-Pascal property)

$(\forall \ell, \ell' \in lines)(\forall a, b, c \in \ell \setminus \ell')(\forall a', b', c' \in \ell' \setminus \ell)$

$$[\,(\,\langle a, b'\rangle \parallel \langle a', b\rangle \;\wedge\; \langle a, c'\rangle \parallel \langle a', c\rangle\,) \quad\to\quad \langle b, c'\rangle \parallel \langle b', c\rangle\,],$$

see Fig.109, p.300.

$\boxed{\textbf{pag}}$ $\stackrel{\text{def}}{=}$ **ag** + **Pa**, p.300. **pag** is the axiom system for <u>*Pappian affine geometries*</u>.

Axioms $\mathbf{B_1}$–$\mathbf{B_3}$ below apply to geometries with reducts $\langle Mn;\, Bw\rangle$. (*coll* is a defined relation.)

$\mathbf{B_1}$ $Bw(a,b,c) \;\to\; (\,a \neq b \neq c \neq a \;\wedge\; Bw(c,b,a) \;\wedge\; \neg Bw(b,a,c)\,)$, p.301.

$\mathbf{B_2}$ $a \neq b \;\to\; (\exists c) Bw(a,b,c)$, p.301.

$\mathbf{B_3}$ (Pasch's Law)

Intuitively, if a line ℓ lies in the plane determined by a triangle abc, and passes between a and b but not through c, then ℓ passes between a and c, or between b and c, formally:

$(\,\neg coll(a,b,c) \;\wedge\; \ell \subseteq Plane'(\{a,b,c\}) \;\wedge\; (\exists d \in \ell) Bw(a,d,b)\,) \;\to$
$(\exists e \in \ell)(Bw(a,e,c) \;\vee\; Bw(b,e,c))$, p.302 (cf. Fig.110 on p.302).

$\boxed{\textbf{opag}}$ $\stackrel{\text{def}}{=}$ **pag** + $\{\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}\}$, p.302. **opag** is the axiom system for <u>*ordered Pappian affine geometries*</u>.

Axioms $\mathbf{L_1}, \ldots, \mathbf{L_{10}}$ below apply to geometries with reducts

$$\langle Mn, L;\; L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq\rangle.$$

Further, *coll* is a defined relation and it is defined over $\langle Mn;\, Bw\rangle$, cf. p.159, and *lines* is the new sort first-order defined from *coll*.

$\mathbf{L_1}$ $L \subseteq lines$, p.326.

$\mathbf{L_2}$ $(\forall a \in Mn)(\exists \ell, \ell' \in L^{Ph})\, \ell \cap \ell' = \{a\}$, p.326.

$\boxed{\textbf{lopag}}$ $\stackrel{\text{def}}{=}$ **opag** + $\mathbf{L_1}$ + $\mathbf{L_2}$, p.326.

$\mathbf{L_3}$ $(\,[\,a \prec b \;\wedge\; (\,Bw(a,b,c) \vee Bw(a,c,b)\,)\,] \;\to\; a \prec c\,) \;\wedge$
$(\,[\,a \prec b \;\wedge\; (\,Bw(c,a,b) \vee Bw(a,c,b)\,)\,] \;\to\; c \prec b\,)$, p.330.

$\mathbf{L_4}$ Intuitively, eq is (very) symmetric, formally:
$\langle a,b\rangle\; eq\; \langle c,d\rangle \quad\to\quad (\,\langle c,d\rangle\; eq\; \langle a,b\rangle \;\wedge\; \langle b,a\rangle\; eq\; \langle c,d\rangle \;\wedge\; \langle a,a\rangle\; eq\; \langle c,c\rangle\,)$, p.330.

$\mathbf{L_5}$ eq is transitive, i.e.
$(\,\langle a,b\rangle\; eq\; \langle c,d\rangle \;\wedge\; \langle c,d\rangle\; eq\; \langle e,f\rangle\,) \quad\to\quad \langle a,b\rangle\; eq\; \langle e,f\rangle$, p.330.

$\mathbf{L_6}$ (For the intuitive meaning of this axiom see Fig.117 on p.330.)
$(\forall \ell, \ell' \in L)(\forall o, e, e', a, a' \in Mn)\Big(\,[\,\ell \cap \ell' = \{o\} \;\wedge\; e, a \in \ell \;\wedge\; e', a' \in \ell' \;\wedge$
$\langle e, e'\rangle \parallel \langle a, a'\rangle \;\wedge\; \langle o,e\rangle\; eq\; \langle o,e'\rangle\,] \quad\to\quad \langle o,a\rangle\; eq\; \langle o,a'\rangle\Big)$, p.330.

$\mathbf{L_7}$ (For the intuitive meaning of this axiom see Fig.118 on p.331.)
$(\forall \ell \in L^T \cup L^S)(\forall a,b,c,d,e,f \in Mn)\,[\,(\,a,b,c,d \in \ell \ \wedge$
$\langle a,b\rangle \parallel \langle e,f\rangle \parallel \langle c,d\rangle \ \wedge\ \langle a,e\rangle \parallel \langle b,f\rangle \ \wedge\ \langle c,e\rangle \parallel \langle d,f\rangle\,) \ \rightarrow\ \langle a,b\rangle\ eq\ \langle c,d\rangle\,]$, p.331.

$\mathbf{L_8}$ $\perp_r$ is symmetric, i.e.
$(\forall \ell,\ell' \in L)\,(\ell \perp_r \ell' \ \rightarrow\ \ell' \perp_r \ell)$, p.331.

$\mathbf{L_9}$ $\perp_r$ is closed under parallelism, i.e.
$(\forall \ell,\ell_1,\ell_2 \in L)\,[\,(\,\ell \perp_r \ell_1 \ \wedge\ \ell_1 \parallel \ell_2\,) \quad \rightarrow \quad \ell \perp_r \ell_2\,]$, p.331.

$\mathbf{L_{10}}$ $\perp_r$ is closed under taking limits, p.331.

$\boxed{\mathbf{lopag^+}} \stackrel{\text{def}}{=} \mathbf{lopag} + \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}}$, p.A-4

(12) "Speed of light free" axiom systems for relativity (axioms and axiom systems used in Chapter 5 of AMN [18])

$\boxed{\mathbf{Relnoph_0}} \stackrel{\text{def}}{=} (\mathbf{Ax1}\text{–}\mathbf{Ax4})^{786} + \mathbf{Ax6} + \mathbf{Ax\Box 1} + \mathbf{Ax\triangle 1} + \mathbf{Ax(}\sqrt{\ }\mathbf{)} + \mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax(\parallel)}$, p.705 of AMN [18].

Ax(5nop) $\forall m,k\ (\forall \lambda \in {}^+F)[\lambda < v_m(k) \ \Rightarrow\ \exists k'\ (v_m(k') = \lambda)]$, p.706 of AMN [18].

The intuitive idea of **Ax(5nop)** is that if a certain speed is realized by some observer then the *smaller speeds* are also realized by some observers.

$\boxed{\mathbf{Relnoph}} \stackrel{\text{def}}{=} \mathbf{Relnoph_0} + \mathbf{Ax(5nop)}$, p.707 of AMN [18].

$\mathbf{Ax(5nop)^-}$ $\forall m\ (\exists c \in {}^+F)(\forall \lambda \in {}^+F)[\lambda < c \ \Rightarrow\ (\exists k) v_m(k) = \lambda]$, p.761 of AMN [18].

$\mathbf{Ax(5nop)^{-+}}$ $(\forall m)(\exists c > 0)(\forall \ell)[ang^2(\ell) < c \Rightarrow (\exists k \in Obs)\ell = tr_m(k)]$, p.763 of AMN [18].

$\boxed{\mathbf{Relnoph^-}}$ is obtained from **Relnoph** by replacing **Ax(5nop)** with $\mathbf{Ax(5nop)^-}$, p.761 of AMN [18].

$\boxed{\mathbf{Bax^-_{nobs}}} \stackrel{\text{def}}{=} \mathbf{Bax^-} \setminus \{\mathbf{Ax5_{Obs}}\} + \mathbf{Ax(5nop)^-}$, p.762 of AMN [18].

$\boxed{\mathbf{Relnoph^{--}}} \stackrel{\text{def}}{=} \mathbf{Relnoph^-} \setminus \{\mathbf{Ax\triangle 1}\} + \mathbf{Ax(symm)}$, p.764 of AMN [18].

Assume **Ax1**, **Ax2**, $\mathbf{Ax3_0}$, **AxP1**. Let $m \in Obs$. Then

$$c_m : {}^nF \times \mathsf{directions} \xrightarrow{\circ} F \cup \{\infty\}$$

is a partial function such that $c_m(p,d)$ is defined iff m sees a photon at point p moving forwards in direction d, and $c_m(p,d)$ is the speed of this photon,[787] cf. pp. 473, 535 of AMN [18]. Further, for any $m \in Obs$ and $d \in \mathsf{directions}$

$$\mathsf{c}_m(d) \stackrel{\text{def}}{=} c_m(\bar{0},d),$$

cf. p.488 of AMN [18]. Hence, $\mathsf{c}_m : \mathsf{directions} \xrightarrow{\circ} F \cup \{\infty\}$ is a partial function. For axioms and axiom systems not listed here we refer to the Index.

[786] **Ax1**, **Ax2**, **Ax3**, **Ax4**.
[787] There is only one such speed because of **AxP1**.

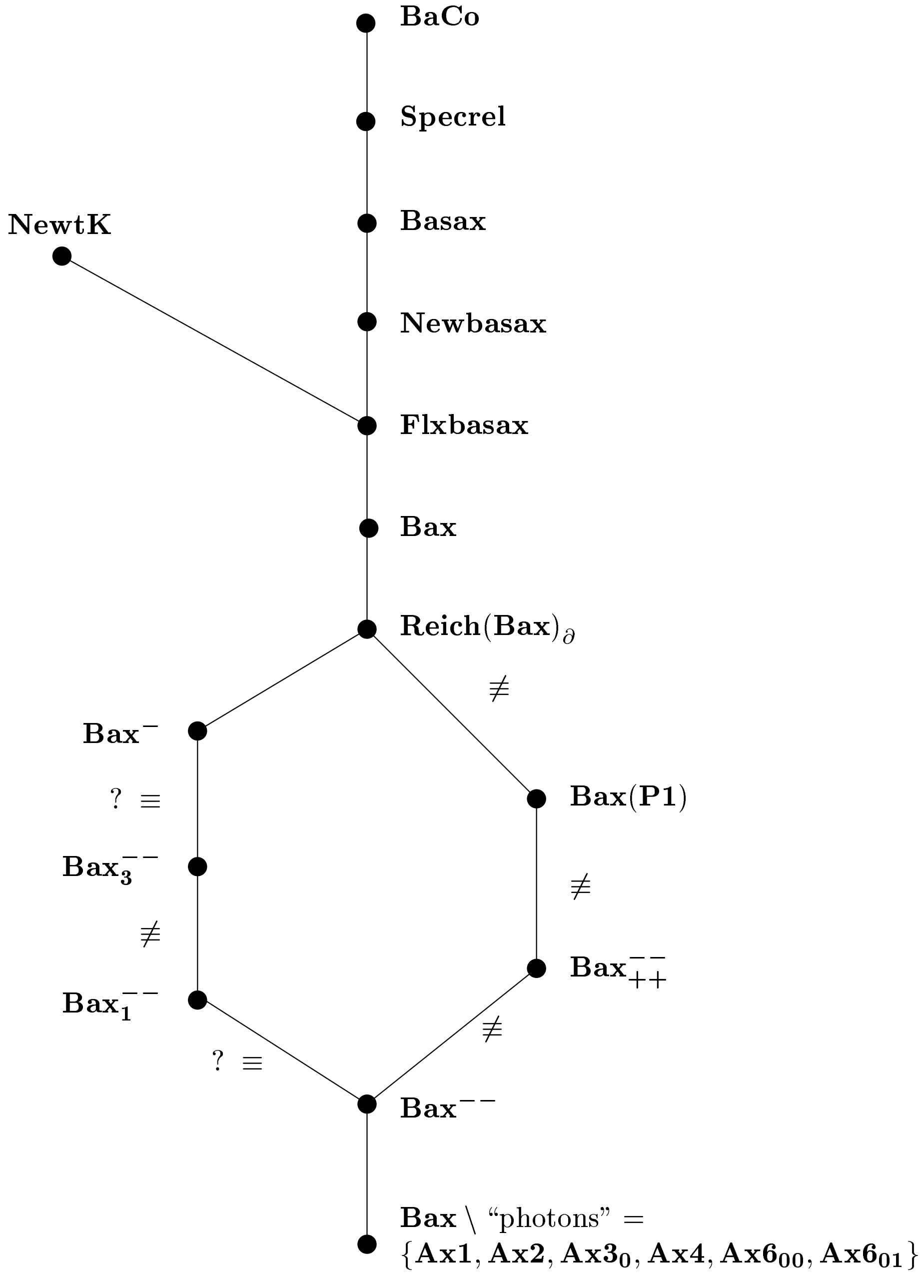


Figure 138: The lattice of our theories introduced so far, where $Th_1 < Th_2$ means $Th_2 + c_m(p,d) < \infty + \mathbf{Ax}(\sqrt{\ }) \models Th_1$. Some parts of this lattice represent conjectures only (while others are theorems).

Index

References

[1] J. Adámek. *Theory of mathematical structures.* Reidel, Dordrecht, Boston, Lancaster, 1983.

[2] J. Adámek, H. Herrlich, and G.E. Strecker. *Abstract and Concrete Categories.* A Wiley-interscience publication. John Wiley & Sons, Inc, New York, 1990.

[3] J. Adámek and S. Mac Lane, editors. *Categorical Topology and its relation to analysis, algebra and combinatorics.* World Scientific, Singapore, New Jersey, London, Hong Kong, 1989.

[4] A. Andai. Category theoretic concepts for physics. Available from Technical University of Budapest, www.math.bme.hu/˜andai/kutatas/kut.html.

[5] A. Andai. Lorentz-transformation for tachyons. Manuscript, Eötvös Loránd University Budapest, 2000.

[6] A. Andai. On the mathematical foundations of quantum mechanics (A kvantum-mechanika matematikai alapjairól). Master's thesis, Eötvös Loránd Univ., Budapest, 1998. (in Hungarian). Available from www.math.bme.hu/˜andai/kutatas/kut.html.

[7] A. Andai, H. Andréka, J. X. Madarász, and I. Németi. Logical approach to relativity theory (a relativitáselmélet logikai megközelítése). (In Hungarian). Abstract for a talk in the Special Meeting on Matheamtical Physics for Celebrating the 60th Birthday of Tamás Matolcsi, Technical University Budapest, Sept. 22, 2001. Available as www.kfki.hu/ elft-term/matfiz01.html.

[8] H. Andréka, S.D. Comer, and I. Németi. Epimorphisms in cylindric algebras. Manuscript, 1983.

[9] H. Andréka, S.D. Comer, and I. Németi. Surjectiveness of epimorphisms of cylindric algebras. *Abstr. Amer. Math. Soc.*, 4, 1983.

[10] H. Andréka, S. Givant, and I. Németi. Decision problems for varieties of relation algebras. *Memoirs of the AMS*, 126(604):126+xiv pages, March 1997.

[11] H. Andréka, R. Goldblatt, and I. Németi. Relativised quantification: some canonical varieties of sequence-set algebras. *The Journal of Symbolic Logic*, 63(1):163–184, 1998.

[12] H. Andréka, V. Goranko, Sz. Mikulás, I. Németi, and I. Sain. Effective first order temporal logics. In *Time and Logic, a computational approach*, pages 51–129. Ed. by L. Bolc and A. Szałas. UCL Press Ltd., London, 1995.

[13] H. Andréka, R.J. Greechie, and Strecker G.E. On residuated approximations. In *Categorical Methods in Computer Science (With aspects from Topology)*, volume 393 of *Lecture Notes in Computer Science*, pages 333–339. Springer–Verlag, Berlin, 1989.

[14] H. Andréka, Á. Kurucz, and I. Németi. Connections between axioms of set theory and basic theorems of universal algebra. *Journal of Symbolic Logic*, 59, 3:912–922, 1994.

[15] H. Andréka, J. X. Madarász, and I. Németi. Craig interpolation does not imply amalgamation after all. Technical report, Math. Inst. Budapest, 2001.

[16] H. Andréka, J.X. Madarász, and I. Németi. Logical analysis of special relativity. In J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, editors, *JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday*, http://www.wins.uva.nl/~j50/cdrom, 1999. Vossiuspers, Amsterdam University Press. CD-ROM, ISBN: 90 5629 104 1.

[17] H. Andréka, J.X. Madarász, and I. Németi. Decidability, undecidability and Gödel incompleteness in relativity theories. Technical report, Alfréd Rényi Institute of Mathematics, Budapest, 2001.

[18] H. Andréka, J.X. Madarász, and I. Németi. On the logical structure of relativity theories. Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, 2001. With contributions from A. Andai, G. Sági, I. Sain and Cs. Tőke. http://www.math-inst.hu/pub/algebraic-logic/Contents.html. 1200 pp.

[19] H. Andréka, J.X. Madarász, and I. Németi. with contributions from A. Andai, G. Sági, I. Sain and Cs. Tőke. On the logical structure of relativity theories, 2001. Future edition of [18].

[20] H. Andréka, J.X. Madarász, and I. Németi. Algebraic Logic. In *Encyclopaedia of Mathematics, Supplement III*, pages 31–34. Kluwer Academic Publishers, 2002.

[21] H. Andréka, J.X. Madarász, and I. Németi. Defining new universes in many-sorted logic, January 2001. Alfréd Rényi Institute of Mathematics, Budapest.

[22] H. Andréka, J.X. Madarász, and I. Németi. A note on definability and definitional equivalence in model theory, June 2000. Alfréd Rényi Institute of Mathematics, Budapest.

[23] H. Andréka, J.X. Madarász, and I. Németi. Relational algebra. In A.V. Mikhalev and G.F. Pilz, editors, *Handbook on the Core of Algebra*. Kluwer, to appear.

[24] H. Andréka, J.X. Madarász, I. Németi, G. Sági, and I. Sain. Analyzing the logical structure of relativity theory via model theoretic logic. Lecture notes for A'dam course., 1998 December 12. http:www.math-inst.hu/pub/algebraic-logic/relativity1.ps, relativity2.ps. Technical Report, University of Amsterdam, CCSOM, 1998. 500 pp.

[25] H. Andréka, J.X. Madarász, I. Németi, G. Sági, and I. Sain. Analyzing the logical structure of relativity theory via model theoretic logic. Lecture notes for A'dam course., 1998 early March. (This was distributed at CCSOM of Univ. of Amsterdam).

[26] H. Andréka, J.X. Madarász, I. Németi, G. Sági, and Cs. Tőke. Accelerated observers, a continuation of On the logical structure of relativity theories, in preparation. Alfréd Rényi Institute of Mathematics, Budapest.

[27] H. Andréka, J.X. Madarász, I. Németi, and I. Sain. On interpolation, amalgamation, universal algebra and Boolean algebras with operators. Technical report, Math. Inst. Budapest, 2001.

[28] H. Andréka and Sz. Mikulás. Lambek calculus and its relational semantics: completeness and incompleteness. *Journal of Logic, Language and Information*, 3(1):1–38, 1994.

[29] H. Andréka and I. Németi. Generalization of variety and quasivariety-concept to partial algebras through category theory. *Dissertationes Mathematicae (Rozprawy Math)*, 204:1–56, 1976.

[30] H. Andréka, I. Németi, and I. Sain. Applying algebraic logic to logic. In M. Nivat, C. Rattray, T. Rus, and G. Scollo, editors, *Algebraic Methodology and Software Technology* (AMAST'93, Proceedings of the Third International Conference on Algebraic Methodology and Software Technology, The Netherlands, 21-25 June 1993), Workshops in Computing series, pages 5–26. University of Twente, Springer-Verlag, London, 1994.

[31] H. Andréka, I. Németi, and I. Sain. Algebraic Logic. In *Handbook of Philosophical Logic Vol. 2, second edition*, pages 133–247. D.M. Gabbay and F. Guenthner (eds), Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

[32] H. Andréka, J.A.F.K. van Benthem, and I. Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 27:217–274, 1998.

[33] M.A. Arbib and E.G. Manes. Adjoint machines, state-behavior machines and duality. *J. Pure Appl. Algebra*, 6:313–345, 1975.

[34] M.A. Arbib and E.G. Manes. A categorist's view of automata and systems. In *Lecture Notes in Computer Science*, volume 25, pages 51–64. Springer–Verlag, Berlin, 1975.

[35] R. Audi, editor. *The Cambridge Dictionary of Philosophy.* Cambridge University Press, 1995. ISBN 0-521-48328-X (pbk.).

[36] J. Ax. The elementary foundations of spacetime. *Found. Phys.*, 8(7–8):507–546, 1978.

[37] J. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory. *Jour. Math. Phys.*, 36:6073–6105, 1995.

[38] J.C. Baez. Higher-dimensional algebra II: 2-Hilbert spaces. Research report, Department of Mathematics, University of California at Riverside, California, USA, 1996.

[39] P. Bahl, J. Cole, N. Galatos, P. Jipsen, and C. Tsinakis. Cancellative residuated lattices. Research report, Department of Mathematics, Vanderbilt University, Nashville, 2001.

[40] J.B. Barbour. *Absolute or relative motion?* Cambridge University Press, Cambridge, 1989.

[41] M. Barr and C. Wells. *Toposes, Triples and Theories.* A Series of Comprehensive Studies in Mathematics. Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.

[42] J. D. Barrow and F. J. Tipler. *The anthropic cosmological principle.* Oxford University Press, Oxford New York, 1986.

[43] A. O. Barut and H. C. Chandola. *Phys. Lett.*, A180:5, 1993.

[44] J. Barwise. *Handbook of Mathematical Logic.* North-Holland, Amsterdam, 1977.

[45] J. Barwise and S. Feferman, editors. *Model-Theoretic Logics.* Springer–Verlag, Berlin, 1985.

[46] J. Bell and M. Machover. *A course in mathematical logic.* North–Holland, Amsterdam, 1977.

[47] G. Birkhoff. *Lattice theory*, volume 25 of *Colloq. Publ.* Amer. Math. Soc., 1967.

[48] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Kluwer, to appear.

[49] J. Blészer, P. Gnädig, and Zs. Varga. Benne van-e az elektrodinamika törvényeiben a relativitáselmélet? Dept. Physics, ELTE University, Budapest, 2001.

[50] F. Borceaux. *Handbook of Categorical Algebra 1. Basic Category Theory.* Cambridge University Press, 1994.

[51] W.L. Burke. *Spacetime, geometry, cosmology.* Books in Astronomy. University Science Books, Mill Valley, California, 1980.

[52] P. Burmeister. *A model theoretic oriented approach to partial algebras.* Akademie-Verlag, Berlin, 1986.

[53] S. Burris and H.P. Sankappanavar. *A course in universal algebra.* Graduate Texts in Mathematics. Springer-Verlag, New York, 1981.

[54] H. Busemann. *The geometry of geodesics.* Academic Press, New York, 1955.

[55] H. Busemann. *Time like spaces.* Dissertationes Mathematicae (Rozprawy Math). Mathematical Institute of Polish Academy of Sci., Warsaw, 1967.

[56] H. Busemann and J. K. Beem. Axioms for indefinite metrics. Rend. Circolo Mat. Palermo.

[57] A. Carlini and I. D. Novikov. Self-consistency in time travel, 1996. Preprint TIT/HEP-338/COSMO-75.

[58] R. Carnap. *The logical structure of the world.* University of California Press, Berkeley, 1967. Translated by R. G. George (Original German edition published in 1928).

[59] G.J. Chaitin. *The Limits of Mathematics – Course Outline & Software.* IBM, Watson Center, Yorktown Heights, December 12, 1993.

[60] C.C. Chang and H.J. Keisler. *Model Theory.* North–Holland, Amsterdam, 1973, 1990.

[61] S.D. Comer. Galois-theory of cylindric algebras and its applications. *Trans. Amer. Math. Soc.*, 286:771–785, 1984.

[62] H.S.M. Coxeter. *Introduction to Geometry.* Wiley, New York, 1969.

[63] L. Crane. Categorical physics, 1993. hep-th/9301061.

[64] L. Crane. Clock and category: is quantum gravity algebraic? *Jour. Math. Phys.*, 36:6180–6193, 1995.

[65] A. Daigneault. Freedom in polyadic algebras and two theorems of Beth and Craig. *Michigan Math. J.*, 11:129–135, 1964.

[66] A. Daigneault and A. Sangalli. Einstein's static universe: an idea whose time has come back? *Notices of the AMS*, 48:9–16, 2001.

[67] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order.* Cambridge mathematical textbooks. Cambridge University Press, 1990.

[68] Gy. Dávid. Relativity without speed of light. Manuscript (Dept. of General Physics, Eötvös Loránd Univ., Budapest), in preparation.

[69] P. C. W. Davies. The physics of time asymmetry, 1976.

[70] J.W. Dawson, Jr. *Logical Dilemmas. The life and work of Kurt Gödel.* A. K. Peters, Ltd., 1997.

[71] D. Deutsch. *The fabric of reality (The Science of Parallel Universes - and Its Implications).* Penguin books, 1997.

[72] G.D. Dimov and W. Tholen. A characterization of representable dualities. In Adámek and Mac Lane [3], pages 336–357.

[73] R. d'Inverno. *Introducing Einstein's Relativity.* Oxford University Press, 1983.

[74] R. Donelly and R. Ziolkowski. *Proc. R. Soc. London*, A460:541, 1993.

[75] F.J. Dyson. *Infinite in all directions.* Penguin Books, London, 1990.

[76] J. Earman, editor. *Inference, Explanation, and Other Frustrations.* California University Press, Berkeley, 1992.

[77] J. Earman. *Bangs, Crunches, Whimpers, and Shrieks (Singularities and Acausalities in Relativistic Spacetimes).* Oxford University Press, New York, Oxford, 1995.

[78] J. Ehlers, F.A.E. Pirani, and A. Shild. The geometry of free fall and light propagation. In *General relativity, Papers in Honor of J.L. Synge*, pages 63–84. Clarendron press, Oxford, 1972.

[79] A. Einstein. On the electrodynamics of moving bodies (in german). *Annalen der Physik*, 17:891, 1905.

[80] A. Einstein. *On the special and the general theories of relativity.* Verlag von F. Vieweg & Son, Braunschweig, 1921. In German.

[81] A. Einstein. The foundation of the general theory of relativity. In W. Perrett and G. B. Jeffery (trans.), editors, *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity.* Dover, New York, 1923. (Original German version published in *Annalen der Physik* 49 [1916]).

[82] H.B. Enderton. *Introduction to Mathematical Logic.* North–Holland, Amsterdam, 1972.

[83] R. Engelking. *General topology.* PWN-Polish Scientific Publishers, Warszawa, 1977.

[84] G. Etesi and I. Németi. On the effective computability of non-recursive functions via Malament-Hogarth space-times. *International Journal of Theoretical Physics*, 2002. http://www.bu.edu/wcp/Papers/Scie/ScieDarv.htm.

[85] G. Feinberg. Possibility of faster-than-light particles. *The Physical Review*, 159(5):1089–1105, 1967.

[86] R. Feynman. *The character of physical law.* The modern library, New York, 1994.

[87] D. Freed. Higher algebraic structures and quantization. *Commun. Math. Phys.*, 159:343–398, 1994.

[88] P.J. Freyd and A. Scedrov. *Categories, Allegories.* North-Holland, Amsterdam, New York, Oxford, Tokyo, 1990.

[89] H.M. Friedman, S.G. Simpson, and R.L. Smith. Countable algebra and set existence axioms. *Annals of Pure and Applied Logic*, 25:141–181, 1983.

[90] J. Friedman, M. S. Morris, I. D. Novikov, F. Echeverria, G. Klinkhammer, and K. S. Thorne. *Physical Review D 42*, 1990.

[91] M. Friedman. *Foundations of Space-Time Theories. Relativistic Physics and Philosophy of Science.* Princeton University Press, 1983.

[92] D. Gabbay and F. Guenthner, editors. *Handbook of Philosophical Logic.* D. Reidel Publishing Company, Dordrecht, 1984. Second edition. Kluwer Academic Publishers, Dordrecht, 2001.

[93] Galileo Galilei. Dialogue concerning the two chief world systems - Ptolemaic and Copernican, 1632.

[94] D. Gallin. *Intensional and Higher-Order Modal Logic.* North Holland/-American Elsevier, Amsterdam-New York, 1975.

[95] M. Gardner. *Relativity Simply Explained.* Dover, 1997.

[96] R. Geroch. *General relativity from A to B.* The University of Chicago Press, 1978.

[97] P. Gibbs. Is faster than light travel or communication possible?, 1998. Usenet Relativity Frequently Asked Questions, http://math.ucr.edu/home/baez/physics/relativity.html.

[98] S. Givant. Universal classes of simple relation algebras. *The Journal of Symbolic Logic*, 64:575–589, 1999.

[99] Kurt Gödel. *Collected Works. Volume II. Publications 1938-1974.* Oxford University Press, 1990. Editors: S. Feferman, J.W. Dawson, Jr., S.C. Kleene, G.H. Moore, R.M. Solovay, and J. van Heijenoort.

[100] Kurt Gödel. *Collected Works. Volume III. Unpublished Essays and Lectures.* Oxford University Press, 1995. Editors: S. Feferman, J.W. Dawson Jr., W. Goldfarb, and C. Parsons.

[101] R. Goldblatt. *Topoi. The Categorial Analysis of Logic.* North-Holland, Amsterdam, New York, Oxford, 1984.

[102] R. Goldblatt. *Orthogonality and Spacetime Geometry.* Springer–Verlag, Berlin, 1987.

[103] R. Goldblatt. Algebraic polymodal logic: a survey. *Logic Journal of IGPL*, 8(4):393–450, 2000.

[104] N. Goodman. *Fact, fiction, and forecast.* Bobbs-Merrill, New York, 1973.

[105] E.C. Gootman and A.J. Lazar. Quantum groups and duality. *Rev. Math. Phys.*, 5:417–451, 1993.

[106] J. R. Gott. On tachyons. *Astrophysical Journal*, 187, 1974.

[107] J. R. Gott. *Time travel in Einstein's universe (The physical possibilities of travel through time).* Houghton Mifflin Company, Boston, New York, 2001.

[108] G. Grätzer. *Universal Algebra, Second edition.* Springer–Verlag, New York, 1979.

[109] A. Grünbaum. Simultaneity by slow clock transport in the special theory of relativity. *Philosophy of Science*, 36:5–43, 1969.

[110] A. Grünbaum. *Philosophical problems of space and time.* D. Reidel Publishing Co., Boston, 1974.

[111] R. Guitart. On the geometry of computations. *II. Cahiers de Topologie et Géométrie Différentielle Catégoriques*, XXIX(4):297–326, 1988.

[112] P. Hájek, editor. *Gödel'96 (Logical Foundations of Mathematics, Computer Science and Physics - Kurt Gödel's Legacy, Brno, Czech Republic August 1996, Proceedings).* Lecture Notes in Logic Vol. 6, Springer–Verlag, Berlin, 1996.

[113] P. Hájek and P. Pudlák. *Mathematics of First Order Arithmetic.* Springer–Verlag, Berlin, 1993.

[114] P.R. Halmos. *Finite-dimensional vector spaces.* Springer–Verlag, Berlin, 1974.

[115] M. Hausner. *A vector space approach to geometry.* Prentice-Hall Inc., Englewood Cliff N. J., 1965.

[116] S.W. Hawking and G.F.R. Ellis. *The large scale structure of space-time.* Cambridge University Press, 1973.

[117] G. Helzer. Special relativity with acceleration. *American Mathematical Monthly*, 107:219–237, 2000.

[118] C. Hempel. *Aspects of scientific explanation.* The Free Press, New York, 1965.

[119] C. G. Hempel and P. Oppenheim. A tudományos magyarázat logikája. In *Tudományfilozófia szöveggyűjtemény*, pages 109–127. Áron Kiadó, Budapest, 1999. Originally appeared as: Studies in logic of explanation, in Philosophy of Science, vol 15, 1948, pp.135-175.

[120] L. Henkin, J.D. Monk, and A. Tarski. *Cylindric Algebras Parts I and II.* North–Holland, Amsterdam, 1971 and 1985.

[121] L. Henkin, J.D. Monk, A. Tarski, H. Andréka, and I. Németi. *Cylindric Set Algebras*, volume 883 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[122] L. Henkin, P. Suppes, and A. Tarski, editors. *The axiomatic method with special reference to geometry and physics*, Studies in Logic and the Foundations of Mathematics. North–Holland, Amsterdam, 1959.

[123] L. Henkin and A. Tarski. Cylindric algebras. In *Lattice Theory, Proceedings of Symposia in Pure Mathematics*, pages 83–113. American Mathematical Society, Providence, 1961. Vol.2, Ed. R. P. Dilworth.

[124] N.J. Hicks. *Notes on differential geometry.* Van Nostrand Co. Inc., 1965.

[125] D. Hilbert. *Grundlagen der Geometrie.* Lepzig / B. G. Teubner Verlag, Stuttgart, 1899/1977.

[126] D. Hilbert. *Foundation of Geometry.* The Open Court Publishing Co., 1921.

[127] D. Hilbert and P. Bernays. *Grundlagen der Geometrie.* Teubner Verlag, Stuttgart, 1956.

[128] J. Hintikka and I. Halonen. Semantics and pragmatics of why-questions. *The Journal of Philosophy*, pages 636–657, 1995.

[129] R. Hirsch and I. Hodkinson. *Relation algebras by games.* North-Holland, to appear.

[130] W. Hodges. *Model Theory.* Cambridge University Press, 1993.

[131] W.A. Hodges, I. Hodkinson, and H.D. Macpherson. Omega-categoricity, relative categoricity and coordinatisation. *Annals of Pure and Applied Logic*, 46(4):169–199, 1990.

[132] M. Hogarth. *Predictability, Computability, and Spacetime.* PhD thesis, University of Cambridge (UK), 2000.

[133] M. Hogarth. Deciding arithmetic in Malament-Hogarth spacetimes, 2001. University of Cambridge (UK), preprint.

[134] E. Hoogland. *Definability and interpolation. Model-theoretic investigations.* PhD thesis, Amsterdam, 2001. University of Amsterdam, ILLC Dissertation Series DS-2001-05.

[135] P. Hraskó. Introduction to general relativity. Technical report, Technical University, Budapest, 1997. (in Hungarian).

[136] F. Huoranszki. *Modern Metafizika (Modern Metaphysics).* Osiris Kiadó, Budapest, 2001.

[137] C. J. Isham and J. Butterfield. Some possible roles for topos theory in quantum theory and quantum gravity, 1999. gr-qc/9910005.

[138] G. James and R.C. James. *Mathematical Dictionary.* Van Nostrand, 1942.

[139] G. Janelidze. Galois theory in categories: the new example of differentiable fields. In Adámek and Mac Lane [3], pages 369–380.

[140] P. Jipsen, B. Jónsson, and J. Rafter. Adjoining units to residuated Boolean algebras. *Algebra Universalis*, 34:118–127, 1995.

[141] B. Jónsson. The preservation theorem for canonical extensions of Boolean algebras with operators. In K.A. Baker and R. Wille, editors, *Lattice theory and its applications*, pages 121–130. Heldermann Verlag, 1995.

[142] B. Jónsson and A. Tarski. Boolean algebras with operators. In *Alfred Tarski Collected Papers Vol. 3*, pages 369–420. Birkhauser Verlag, 1986.

[143] B. Jónsson and C. Tsinakis. Relation algebras as residuated Boolean algebras. *Algebra Universalis*, 30:469–478, 1993.

[144] A. Joyal and R. Street. An introduction to Tannaka duality and quantum groups. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Mathematics*, pages 413–492. Springer–Verlag, Berlin, 1991.

[145] I. Kant. **Kritik der reinen Vernunft**. Johann Friedrich Hardknoch, Riga, 1781. Second revised edition: Druck und Verlag von Philipp Reclam jun, Leipzig, 1787. Later edition: Akademie Verlag (Kant's Schriften), Berlin, 1910.

[146] I. Kant. *Critique of Pure Reason*, volume XLII, 48 of *Great Books of the Western World*. Encyclopaedia Britannica Inc., Chicago, London, Toronto, 1952. Original German edition is [145].

[147] C. Kassel. *Quantum Groups*. Springer-Verlag, New York, 1995.

[148] A.I. Kostrikin and Yu.I. Manin. *Linear Algebra and Geometry*. Gordon and Breach, New York, 1988.

[149] T. S. Kuhn. *The structure of scientific revolutions*. University of Chicago Press, Chicago, 1970.

[150] A. Kurusa. *Introduction to differential geometry (in Hungarian)*. Polygon (Bolyai Institute of University of Szeged), Hungary, 1999.

[151] K. Lánczos. *Space Through The Ages*. Academic Press, London and New York, 1970.

[152] L.D. Landau and E.M. Lifsic. *Theoretical Physics Part II (theory of fields)*. Nauka, Moscow, 1973. in Russian.

[153] F.W. Lawvere. Adjointness in foundations. *Dialectica*, 23(3-4):281–296, 1969.

[154] F.W. Lawvere. Metric spaces, generalized logic, and closed categories. Rendiconti Seminario Matematico e Fisico di Milano 43, 1973.

[155] F.W. Lawvere. Variable quantities and variable structures in topoi. In *Algebra, topology, and category theory. A collection of papers in honor of Samuel Eilenberg*, pages 101–131. Academic Press, New York, 1976.

[156] F.W. Lawvere and S.H. Schanuel, editors. *Categories in continuum physics*, volume 1174 of *Lecture Notes in Mathematics*. Springer–Verlag, Berlin, 1986.

[157] D. Lewis. The paradoxes of time travel. In *David Lewis Philosophical Papers Volume II*, pages 67–80. Oxford University Press, New York, Oxford, 1986.

[158] J. Loose. *A historical introduction to the Philosophy of science*. Oxford Univ. Press, 1980.

[159] S. Mac Lane. *Categories for the working mathematician*. Springer–Verlag, Berlin, 1971.

[160] J.X. Madarász. Craig interpolation in algebraizable logics; meaningful generalization of modal logic. Master's thesis, Eötvös Loránd University, Budapest, 1995.

[161] J.X. Madarász. The Craig interpolation theorem in multi-modal logics (extended abstract). *Bulletin of the Section of Logic*, 24(3):147–154, 1995.

[162] J.X. Madarász. Classification of models of first-order logic version of relativity theory. Rényi Institute of Mathematics, Budapest. Manuscript, 1998.

[163] J.X. Madarász. Epimorphisms in discriminator varieties. In *Proc. of Workshop on Abstract Algebraic Logic*, 1998. Centre de Recerca Matemàtica, Bellaterra (Spain), Quaderns nùm 10/ gener 1998, pp. 106-108.

[164] J.X. Madarász. Interpolation and amalgamation; pushing the limits, Part I. *Studia Logica*, 61(3):311–345, 1998.

[165] J.X. Madarász. Interpolation in algebraizable logics; semantics for non-normal multi-modal logic. *Journal of Applied Non-Classical Logics*, 8(1-2):67–105, 1998.

[166] J.X. Madarász. Interpolation and amalgamation; pushing the limits, Part II. *Studia Logica*, 62(1):1–19, 1999.

[167] J.X. Madarász. Surjectivity of epimorphisms in varieties of algebraic logic. Preprint, Alfréd Rényi Institute of Mathematics, Budapest, 2000.

[168] J.X. Madarász. The symmetry axioms are equivalent in Basax, 2000.

[169] J.X. Madarász. Definability in relativized first-order logic G_α. In preparation, Budapest, 2001.

[170] J.X. Madarász. Equivalence theorems, Section 6 in Chapter Algebraic Logic. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 208–215. Kluwer Academic Publishers, Second edition, 2001.

[171] J.X. Madarász, A. Andai, H. Andréka, I. Németi, I. (with contributions from Sain, and G.) Sági. Duality theories, adjoint situations are everywhere in mathematics. Rényi Institute of Mathematics, Budapest. Manuscript. (Expanded version is in preparation.), 2001.

[172] J.X. Madarász and I. Németi. On faster than light observers, Part I. Manuscript, Budapest, 1998.

[173] J.X. Madarász and I. Németi. FTL observers are at most two-dimensional. Manuscript in preparation, 1999.

[174] J.X. Madarász and I. Németi. On FTL observers in a generalized version of first-order relativity (The generalization is in the direction of partiality coming from general relativity). Manuscript in preparation, 1999.

[175] J.X. Madarász and I. Németi. Bodies with inner clocks and FTL phenomena. Rényi Institute of Mathematics, Budapest. Manuscript in preparation, 2001.

[176] J.X. Madarász and I. Németi. Free Boolean algebras with closure operators and a conjecture of Henkin, Monk, and Tarski. *Studia Sci. Math. Hungar. (Special issue on the 50th anniversary of the Alfréd Rényi Institute of Mathematics)*, 38:273–278, 2001.

[177] J.X. Madarász, I. Németi, and G. Sági. On the finitization problem of relation algebras. *Bulletin of the Section of Logic*, 26(3):pp. 140–145, 1997.

[178] J.X. Madarász and T. Sayed-Ahmed. Amalgamation, interpolation, epimorphisms: solutions of all problems from Pigozzi's paper. Manuscript, Alfréd Rényi Institute of Mathematics, Budapest, 2001.

[179] M. Makkai. Stone duality for first-order logic. *Adv. in Math.*, 65:97–170, 1987.

[180] E.G. Manes. *Algebraic theories*, volume 26 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[181] M. Manzano. Introduction to many-sorted logic. In Meinke and Tucker [190], pages 3–89.

[182] M. Markopoulou. The internal description of a causal set: What the universe looks like from the inside. *Commun. Math. Phys.*, 211:559–583, 2000.

[183] Zs. Márkusz. Different validity concepts in many-sorted logic, 1983. Research Report 151/1983 of Computer Science Research Institute of Hungarian Academy of Science, Budapest.

[184] J. E. Marsden and F. Tipler. *Phys. Rep.*, 66:109, 1980.

[185] M. Marx. *Algebraic Relativization and Arrow Logic*. PhD thesis, University of Amsterdam, appeared as ILLC Dissertation Series 1995–3. ILLC, Plantage Muidergracht 24, 1018 TV Amsterdam, e-mail: illc@fwi.uva.nl, 1995.

[186] M. Marx and Y. Venema. *Multi-dimensional Modal Logic*. Kluwer Academic Press, 1997.

[187] T. Matolcsi. *Spacetime without reference frames*. Akadémia Kiadó, Budapest, 1993.

[188] T. Matolcsi and W. A. Jr. Rodrigues. Spacetime model with superluminal phenomena. Eötvös Loránd University, Budapest, preprint, submitted, 2001.

[189] R.N. McKenzie, G.F. McNulty, and W.F. Taylor. *Algebras, Lattices, Varieties, Vol I*. The Wadsworth and Brooks/Cole Mathematics Series. Monterey, California, 1987.

[190] K. Meinke and J.V. Tucker, editors. *Many-sorted Logic and its Applications*. John Wiley & Sons Ltd, 1993.

[191] Sz. Mikulás. *Taming Logics*. PhD thesis, University of Amsterdam, appeared as ILLC Dissertation Series 1995–12. ILLC, Plantage Muidergracht 24, 1018 TV Amsterdam, e-mail: illc@fwi.uva.nl, 1995.

[192] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. W. H. Freeman, San Francisco, 1973.

[193] C. I. Mocanu. *Found. Phys. Lett.*, 5:444, 1992.

[194] J.D. Monk. *Mathematical Logic.* Springer Verlag, 1976.

[195] R. Montague. Deterministic theories. In *Decisions, Value and Groups, 2*, 1962. Oxford, Pergamon Press; pp.325-370.

[196] J. Mundy. The physical content of Minkowski geometry. *Britisch J. Philos. Sci.*, 37(1):25–54, 1986.

[197] D. Myers. An interpretive isomorphism between binary and ternary relations. In *Structures in logic and computer science*, 1997. ed. Jan Mycielski, Grzegorz Rozenberg, and Arto Salomaa, Lecture Notes in Computer Science 1261, Springer, Berlin 1997, pp.84-105.

[198] G.L. Naber. *Spacetime and singularities, an introduction.* Cambridge University Press, 1988.

[199] E. Nagel. *The structure of science.* Routledge & Kegan Paul, London, 1961.

[200] K. Nagy. *Elektrodynamics (in Hungarian).* Tankönyvkiadó, 1977.

[201] L.D. Nel. Categorical differential calculus and Banach-Steinhaus. In Adámek and Mac Lane [3], pages 149–162.

[202] I. Németi. On varieties of cylindric algebras with applications to logic. *Annals of Pure and Applied Logic*, 36:235–277, 1987.

[203] I. Németi. Fine-structure analysis of first-order logic. In *Arrow Logic and Multi-Modal Logic*, pages 221–247, 1996. CSLI Publications, Stanford, California.

[204] I. Németi. Algebraization of quantifier logics, an introductory overview. Available from www.math-inst.hu/pub/algebraic-logic/survey.ps, 1997. (A shorter, incomplete version of this appeared in Studia Logica, 50(3/4)[a special issue devoted to Algebraic Logic, eds.: W.J. Blok and D.L. Pigozzi]:485–570, 1991).

[205] I. Németi and H. Andréka. General algebraic logic: a perspective on 'what is logic'. In *What is a logical system*, pages 393–444. Ed. by D. M. Gabbay, Clarendron Press, Oxford, 1994.

[206] R. P. A. C. Newman. Black holes without singularities. In *General Relativity and Gravitation*, volume 21, pages 981–995. 1989.

[207] I. D. Novikov. *The river of time.* Cambridge University Press, Cambridge, England, 1998.

[208] B. O'Neil. *The geometry of Kerr black holes.* A. K. Peters, 1995. ISBN: 1-56881-019-9.

[209] I. Ozsváth and E.L. Schücking. The finite rotating universe. *Annals of Physics*, 55:166–204, 1969.

[210] J.W. Pelletier and J. Rosický. Generating the equational theory of C^*-algebras and related categories. In Adámek and Mac Lane [3], pages 163–180.

[211] R. Penrose. *The Emperors's New Mind.* Oxford University Press, 1989.

[212] D.L. Pigozzi. Amalgamation, congruence-extensions, and interpolation properties in algebras. *Algebra Universalis*, 1, fasc.3:269–349, 1972.

[213] A. Pillay and S. Shelah. Classification theory over a predicate I. *Notre Dame Journal of Formal Logic*, 26(4):361–376, 1985.

[214] S. Pulmanová. Quantum measurements and quantum logics. *Journal of Mathematical Physics*, 35:1555–1572, April, 1994.

[215] N. Rakić. *Common sense time and special relativity.* University of Amsterdam, ILLC-Dissertation Series 97-02, Amsterdam, 1997.

[216] T. Regge. *Infinito - Viaggi ai limiti dell'universo.* Mondadori, Milano, 1994.

[217] H. Reichenbach. *The theory of relativity and a priori knowledge.* University of California Press, 1960. Translated by M. Reichenbach. (Original German edition published in 1920).

[218] H. Reichenbach. *Axiomatization of the theory of relativity.* University of California Press, Berkeley, 1969. Translated by M. Reichenbach. Original German edition published in 1924.

[219] J. Rembieliński. Tachyon neutrino? Preprint KFT 5/94, Univ. Lodz, Poland, subm. for publication.

[220] J. Rembieliński. *Phys. Lett.*, A78:33, 1980.

[221] E. (editor) Ricami. *Tachyons, monopoles, and related topics.* North–Holland, Amsterdam, 1978.

[222] W. Rindler. *Essential relativity. Special, general and cosmological.* Springer-Verlag, Berlin, 1969, 1977.

[223] A.A. Robb. *A Theory of Time and Space.* Cambridge University Press, 1914. Revised edition, *Geometry of Time and Space*, published in 1936.

[224] W. A. Rodrigues and J. Jr. Vaz. Subluminal and superluminal solutions in vacuum of the maxwell equations and the massless dirac equation. *Adv. Appl. Clifford Algebras*, 1995. RP 44/95 IMECC-UNICAMP.

[225] H.L. Royden. Remarks on primitive notions for elementary Euclidean and non-Euclidean plane geometry. In Henkin et al. [122], pages 86–96.

[226] P. Rózsa. *Linear algebra and its applications.* Tankönyvkiadó, Budapest, 1991. (in Hungarian).

[227] I. Sain. *Dynamic logic with nonstandard model theory.* PhD thesis, Hungarian Academy of Sciences, Budapest, 1986. In Hungarian.

[228] I. Sain. There are general rules for specifying semantics: observations on abstract model theory. *CL&CL*, 13:195–250, 1990.

[229] S. Sankaran. Hochschild-Tannaka duality theorem for homogeneous spaces. *Rend. Sem. Mat. Univ. Politec. Torino*, 36:59–85, 1979.

[230] P. Schauenburg. *Tannaka duality for arbitrary Hopf algebras*, volume 66 of *Algebra Berichte*. Verlag Reinhard Fischer, Munich, 1992. ii+57 pp.

[231] J.W. Schutz. *Independent axioms for Minkowski space-time*. Longoman, London, 1997.

[232] W. Schwabhäuser, W. Szmielew, and A. Tarski. *Metamathematische Methoden in der Geometrie*. Hochschul text. Springer–Verlag, Berlin, 1983.

[233] I.R. Shafarevich. *Basic Notions of Algebra*, volume 11 of *Encyclopedia of Mathematical Sciences*. Springer–Verlag, Berlin, 1997.

[234] S. Shelah. *Around classification theory of models*. Lecture Notes in Mathematics 1182, Springer-Verlag, Berlin, 1986.

[235] S.G. Simpson. Which set existence axioms are needed to prove the Cauchy/Peano theorem? *The Journal of Symbolic Logic*, 49:783–802, 1984.

[236] L. Smolin. The bekenstein bound, topological field theory and pluralistic quantum cosmology, 1995. gr-qc/950806.

[237] L. Smolin. *Three roads to quantum gravity*. Basic Books, 2001.

[238] H. Stein. On relativity theory and openness of the future. *Philosophy of Science*, 58:147–167, 1991.

[239] W.F. Stinespring. Integration theorems for gauges and duality for unimodular groups. *Trans. Amer. Math. Soc.*, 90:15–56, 1959.

[240] M. Stöltzner. Gödel and the theory of everything. In Hájek [112], pages 291–306.

[241] P. Suppes. Axioms for relativistic kinematics with or without parity. In *Symposium on the Axiomatic Method*, pages 291–307.

[242] P. Suppes. The axiomatic method in the empirical sciences. In L. Henkin, J. Addison, C.C. Chang, W. Craig, D. Scott, and R. Vaught, editors, *Proceedings of the Tarski Symposium*, Proceedings of Symposia in Pure Mathematics Volume XXV, pages 465–479, Providence, Rhode Island, 1974. American Mathematical Society.

[243] L.E. Szabó. *A nyitott jövő problémája - véletlen kauzalitás és determinizmus a fizikában*. Typotex, Budapest. (to appear).

[244] L.E. Szabó. The problem of open future, determinism in the light of relativity and quantum theory. Manuscript, Dept. of Theor. Physics, Eötvös Loránd University, Budapest, 1998.

[245] L.W. Szczerba and A. Tarski. Metamathematical properties of some affine geometries. In Y. Bar-Hiller, editor, *Proceedings of the 1964 International Congress for Logic, Methodology and Philosophy of Science*, Studies in Logic and the Foundations of Mathematics, pages 166–178, 1965.

[246] G. Székely. Around the twin paradox. Manuscript, ELTE University Budapest, 2001.

[247] W. Szmielew. The role of the Pasch axiom in the foundations of Euclidean geometry. In *Proc. of the Tarski Symp. held in Berkeley in 1971*, pages 123–132. Amer. Math. Soc., Providence, RI, 1974.

[248] W. Szmielew. *From Affine to Euclidean Geometry.* Reidel Publ. Co., 1983.

[249] A. Tarski. A problem concerning the notion of definability. In [253], Vol. 3, pp. 163-170.

[250] A. Tarski. Some methodological investigations on the definability of concepts. (in German), in [253], Voll 1, pp.637-639. The Polish version contains more detail: in Revue Philosophique, Vol.37 (1934).

[251] A. Tarski. Fundamentale begriffe der methodologie der deduktiven wissenchaften, i. *Monatsh. Math. Phys.*, 37:361–404, 1930. Also in "Logic, Semantics, Metamathematics. Papers from 1923 to 1938. Clarendon Press, Oxford, 1956. pp.60–109.

[252] A. Tarski. What is elementary geometry? In Henkin et al. [122], pages 16–29.

[253] A. Tarski. *Collected Papers, Volumes 1–4.* Birkhäuser (Basel–Boston–Stuttgart), 1986. Editors: Steven R. Givant and Ralph N. McKenzie.

[254] A. Tarski and S. Givant. *A Formalization of Set Theory Without Variables*, volume 41 of *AMS Colloquium Publications.* Providence, Rhode Island, 1987.

[255] A. Tarski and S. Givant. Tarski's system of geometry. *Bulletin of Symbolic Logic*, 5,2:175–214, 1999.

[256] A. Tarski, A. Mostowski and R.M. Robinson. *Undecidable theories.* North-Holland, Amsterdam, 1953. xii+98pp.

[257] E.F. Taylor and J.A. Wheeler. *Spacetime Physics, second edition, fourth printing.* W. H. Freeman and Company, New York, 1997.

[258] R.J. Thompson. Complete description of substitutions in cylindric algebras and other algebraic logics. In *Algebraic Methods in Logic and in Computer Science* (Proceedings of the '91 Banach Semester, Banach Center), pages 327–342, Warsaw, 1993. Polish Academic Publishers, Banach Center Publications, Volume 28.

[259] K.S. Thorne. *Black holes and time warps.* W. W. Norton and Company, 1994.

[260] F. J. Tipler. *The physics of immortality.* Anchor Books Doubleday, 1994.

[261] Cs. Tőke. Group properties of the world-view transformations (future section for [19]). Manuscript, 1999.

[262] Cs. Tőke. Some problems in the analysis of relativity theory in first-order logic. Master's thesis, Eötvös Loránd University, Budapest, 2000. 58pp.

[263] S. Vályi. Modal version of basic axiom system for special relativity. Manuscript, Institute of Mathematics, Kossuth Lajos University Debrecen, 1998.

[264] J.A.F.K. van Benthem. *The Logic of Time.* D. Reidel, 1983.

[265] J.A.F.K. van Benthem. *Language in Action.* Elsevier Science Publishers, Amsterdam, 1991.

[266] J.A.F.K. van Benthem. Content versus wrapping: an essay in semantic complexity. In M. Marx, L. Pólos, and M. Masuch, editors, *Arrow Logic and Multimodal Logic*, pages 203–219. CSLI Publications, Stanford, 1996.

[267] J.A.F.K. van Benthem. *Exploring logical dynamics.* Studies in Logic, Language and Information. CSLI Publications, Stanford, 1996.

[268] V.S. Varadarajan. *Geometry of Quantum Theory*, volume I. D. Van Nostrand Co. Inc., Princeton, New Jersey, 1968.

[269] R.M. Wald. *General relativity.* The University of Chicago Press, Chicago, 1984.

[270] P. Yourgrau. *Gödel meets Einstein. (Time travel in the Gödel universe.).* Open Court, Chicago and LaSalle, Illinois, 1999.

Alfréd Rényi Institute of Mathematics
Budapest, Pf. 127
H–1364, Hungary
e-mail: madarasz@renyi.hu

SUMMARY

The subject matter of the dissertation is a mathematical logical investigation of the logical structure of (mainly special) relativity theories with an emphasis on the branch of logic called definability theory.

Combining mathematical logic and relativity is not a new discipline, it goes back e.g. to Hilbert, Reichenbach, Carnap, Gödel, Tarski, Suppes, Goldblatt. Definability theory, in particular, was initiated around 1920 because of the needs of relativity theory. Then Tarski took a life-long interest in developing definability theory. In the dissertation we extend definability theory to the case when new elements and new universes of new elements can also be defined. We do this because we need it for studying relativity. We extend (and prove) the main theorems of definability from the classical case to the new situation. E.g. we prove a Beth style theorem about eliminability of defined concepts. We apply these results to proving definitional equivalence of major, seemingly "disjoint" approaches to relativity. In particular, we prove a strong equivalence between purely geometrical theories and observation-oriented versions of relativity. Then we build so-called duality theories (on top of these equivalences) like adjoint situations in category theory. These duality theories connect various parts/versions of relativity with other, purely mathematical theories like Busemann's streamlined time-like-metric spaces.

Besides the above, we strive to build up relativity as a theory purely in first-order logic using as simple and as transparent axioms as we can. One of our goals is to prove strong theorems of relativity from a small number of easily understandable, convincing axioms. We try to eliminate all tacit assumptions from relativity and replace them with explicit axioms in the spirit initiated by Tarski in his first-order axiomatization of geometry.

ÖSSZEFOGLALÓ

A disszertáció matematikai logikai (és matematikai) eszközökkel vizsgálja a relativitáselméletek (elsősorban de nem kizárólag a speciális relativitáselmélet) logikai struktúráját.

A matematikai logika és relativitáselmélet ötvözése nem újkeletű gondolat, Hilbert, Reichenbach, Carnap, Gödel, Tarski, Suppes, Goldblatt és mások munkáit, kezdeményezéseit kell például említenünk. A logika definícióelmélet nevű fejezetét pl. kifejezetten a relativitáselmélet szükségletei hívták életre 1920 táján (Reichenbach, Hilbert). Tarski egész életén végigvonul ezután a definícióelmélet mint egyik fő motiváló erő. A dolgozatban továbbfejlesztjük a definícioelméletet olyan irányban, hogy új elemekből álló új univerzumokat is lehessen definiálni (ne csak régi elemeken értelmezett új relációkat). A definícióelmélet centrális tételeit kiterjesztjük, bizonyítunk az új szituációra pl. egy Beth-tipusú tételt a definiált fogalmak eliminálhatóságáról. (A tétel előtörténete: Padoa 1900, Tarski 1926, Beth 1953, Makkai Mihály és mai modellelméleszek, pl. Shelah és Pillay.) Az így nyert eredményeket relativitáselméletre alkalmazzuk, pl. bizonyítjuk, hogy a relativitáselmélet két fő (látszólag eltérő szellemű) közelítése definíciósan ekvivalens. Tisztán geometriai elméletek és megfigyelés orientált relativitáselmélet változatok között szoros ekvivalenciatételt bizonyítunk. Bizonyítjuk, hogy a relativitáselmélet un. elméleti fogalmai mat. logikai értelemben expliciten definiálhatók megfigyelés orientált fogalmaiból. Ezekre az eredményekre egy átfogóbb dualitáselméletet építünk, mely a relativitáselmélet különböző részeit/változatait kapcsolja össze egymással is és más tisztán matematikai elméletekkel is.

Fentiek mellett egy további célunk a relativitáselmélet felépítése tisztán logikai elméletként az elsőrendű logika keretein belül. Ezt a programmot Tarski és Suppes hirdették meg annak mintájára, ahogy Tarski a geometriát elsőrendű logikában felépítette. Törekszünk arra, hogy kevés, könnyen érthető, meggyőző, jól átlátható axiómából erős relativitáselméleti tételeket bizonyítsunk. Ez többek között azt a célt is szolgálja, hogy megértsük a relativitáselmélet egzotikus, köznapi intuícióval ellenkező predikcióinak "logikai okát".