COMPLEXITY OF EQUATIONS VALID IN ALGEBRAS OF RELATIONS

Part II: Finite axiomatizations.

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Abstract. We study algebras whose elements are relations, and the operations are natural "manipulations" of relations. This area goes back to 140 years ago to works of De Morgan, Peirce, Schröder (who expanded the Boolean tradition with extra operators to handle algebras of binary relations). Well known examples of algebras of relations are the varieties RCA_n of cylindric algebras of n-ary relations, $RPEA_n$ of polyadic equality algebras of n-ary relations, and RRA of binary relations with composition. We prove that any axiomatization, say E, of RCA_n has to be very complex in the following sense: for every natural number k there is an equation in E containing more than k distinct variables and all the operation symbols, if $2 < n < \omega$. Completely analogous statement holds for the case $n \geq \omega$. This improves Monk's famous non-finitizability theorem for which we give here a simple proof. We prove analogous nonfinitizability properties of the larger varieties SNr_nCA_{n+k}. We prove that the complementation-free (i.e. positive) subreducts of RCA_n do not form a variety. We also investigate the reason for the above "non-finite axiomatizability" behaviour of RCA_n. We look at all the possible reducts of RCA_n and investigate which are finitely axiomatizable. We obtain several positive results in this direction. Finally, we summarize the results and remaining questions in a figure. We carry through the same programme for $RPEA_n$ and for RRA. By looking into the reducts we also investigate what other kinds of natural algebras of relations are possible with more positive behaviour than that of the well known ones. Our investigations have direct consequences for the logical properties of the n-variable fragment L_n of first order logic. The reason for this is that RCA_n and $RPEA_n$ are the natural algebraic counterparts of L_n while the varieties SNr_nCA_{n+k} are in connection with the proof theory of L_n.

This paper appears in two parts. The first part contains the non-finite axiomatizability results. The present second part contains finite axiomatizations of some fragments (reducts) together with a figure summarizing the finite and non-finite axiomatizability results in this area and the problems left open.

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CONTENTS

This is second part of the paper Complexity of equations valid in algebras of relations. This part is self-contained. We use the notation of the first part, but we introduce less usual notation.

In this part we prove some "positive" axiomatizability theorems. These theorems are complementing the results in Part I. We prove that the Boolean operations together with the permutations p_{ij}, or together with the diagonals d_{ij}, or together with both, are finitely axiomatizable. So in a sense, yet the cylindrifications cause the complexity of the equations valid in algebras of relations. At the end of the paper we summerize both the positive and the negative axiomatizability theorems on two figures, where we also indicate the questions that remained open.

Let U be a set. Then $\mathcal{P}(U)$ denotes the powerset of U and $\mathfrak{P}(U)$ denotes the Boolean algebra of all subsets of U. Let n be an ordinal. Then ${}^nU$ is the set of all U–termed sequences of length n, and thus $\mathcal{P}({}^nU)$ is the set of all n–ary relations on U. We consider n as the set of all smaller ordinals, and a sequence $s \in {}^nU$ to be a map from n to U. Thus, if $\sigma : n \to n$, then $s \circ \sigma : n \to U$ is another sequence, the sequence s "rearranged" according to σ. Let $s \in {}^nU, i < n$ and $u \in U$. Then $s(i/u)$ denotes the sequence we obtain from s by replacing its i–th value with u. Let $i, j < n$ and $\sigma : n \to n$. The unary operations $p^U_{ij}, p^U_\sigma, \mathsf{s}^U_{ij}, c^U_i$ on n–ary relations over U and the constant $d^U_{ij} \in \mathcal{P}({}^nU)$ are defined as

$$\begin{aligned}
&p^U_{ij}(X) = \{s(i/s_j)(j/s_i) : s \in X\}\\
&p^U_\sigma(X) = \{s \circ \sigma : s \in X\}\\
&\mathsf{s}^U_{ij}(X) = \{s(i/s_j) : s \in X\}\\
&c^U_i(X) = \{s \in {}^nU : s(i/u) \in X \text{ for some } u \in U\},\\
&d^U_{ij} = \{s \in {}^nU : s_i = s_j\}.
\end{aligned}$$

We often omit the upper indices U. We will need c_i and s_{ij} only at the end of the paper, in Figure 1.

If K is any class of algebras, then $\mathbf{S}K$, $\mathbf{I}K$, $\mathbf{P}K$, $\mathbf{H}K$, $\mathbf{Up}K$ denote the classes of all subalgebras, isomorphic copies, direct products, homomorphic imeges, and ultraproducts of elements of K, respectively. In this paper we shall characterize the (equations valid in the) classes

$$\begin{aligned}
&RA^d_n = \mathbf{SI}\{\langle \mathfrak{P}({}^nU), d^U_{ij}\rangle_{i,j<n} : U \text{ is a set}\},\\
&RA^p_n = \mathbf{SI}\{\langle \mathfrak{P}({}^nU), p^U_{ij}\rangle_{i,j<n} : U \text{ is a set}\}, \quad \text{and}\\
&RA^{dp}_n = \mathbf{SI}\{\langle \mathfrak{P}({}^nU), d^U_{ij}, p^U_{ij}\rangle_{i,j<n} : U \text{ is a set}\}.
\end{aligned}$$

We note that these classes are all axiomatizable by sets of quantifier–free formulas, because it is quite easy to show that they are closed under ultraproducts, and by definition, they are closed under subalgebras. (For the techniques of these proofs see Németi[90].)

Throughout this paper we assume that n is finite. If $i, j < n$ then $[i, j]$ denotes the permutation of n which interchanges i and j and leaves all other elements of n fixed. $S(n)$ denotes the set of all permutations of n. The permutations $[i, j]$ are usually called transpositions, and it is known that each permutation $\sigma \in S(n)$ is a product of transpositions. For any $\sigma \in S(n)$ let us fix a sequence $i_0, j_0, i_1, j_1, ..., i_k, j_k < n$ such that

$$\sigma = [i_0, j_0] \circ [i_1, j_1] \circ ... \circ [i_k, j_k].$$

(Such a sequence exists for all $\sigma \in S(n)$.) We then define the term

$$p_\sigma(x) = p_{i_k j_k} ... p_{i_1 j_1} p_{i_0 j_0}(x).$$

We will need the terms p_σ in writing up the axioms in (BP2) below.

We begin with characterizing p_{ij} together with the Boolean operations. Consider the following formulas. Let $i, j, k < n$.

(B) a finite equational axiom system for Boolean algebras.

(P) $p_{ij}x = p_{ji}x, \quad p_{ij}p_{ij}x = x, \quad p_{ij}p_{ik}x = p_{jk}p_{ij}x, \quad p_{ii}x = x \quad$ for $k \notin \{i, j\}$.

(BP1) $p_{ij}(x + y) = p_{ij}x + p_{ij}y, \quad p_{ij} - x = -p_{ij}x.$

(BP2) $\sum\{p_\sigma x : \sigma \in S(n)\} = 1 \wedge \prod\{p_\sigma x : \sigma \in S(n)\} = 0 \ \rightarrow \ 1 = 0.$

In the following, we will consider (B),(P) etc. to be sets of formulas, e.g.

$$(P) = \bigcup\{\{p_{ij}x = p_{ji}x, \ p_{ij}p_{ij}x = x, \ p_{ij}p_{ik}x = p_{jk}p_{ij}x, \ p_{ii}x = x\} : \\ i, j, k < n, \ k \notin \{i, j\}\}.$$

It follows from the results in Jónsson[62] that the equations in (P) axiomatize the operations p_{ij} (see also [HMT85]3.2.17). Thus (P) implies (P') below.

(P') $p_\delta p_\sigma x = p_{\sigma \circ \delta} x \qquad$ for all $\delta, \sigma \in S(n)$.

We shall make use of this fact several times. We also note that (BP1) just states that the p_{ij}'s are Boolean homomorphisms, and thus (BP1) is equivalent to (BP1') below:

(BP1') $p_{ij}(x+y) = p_{ij}x + p_{ij}y,\ p_{ij}(x \cdot y) = p_{ij}x \cdot p_{ij}y,\ p_{ij}1 = 1,\ p_{ij}0 = 0.$

We will also use this fact several times.

Let $\quad Ax_1 = (B) \cup (P) \cup (BP1)$, $\quad$ and $\quad Ax_2 = Ax_1 \cup (BP2)$.

PROPOSITION 1.

(i) $RA_n^p = Mod(Ax_2)$.

(ii) *Ax_2 is an axiomatization of both the quantifier–free theory and of the quasi–equational theory of RA_n^p, while Ax_1 is an axiomatization of the equational theory of RA_n^p. In Ax_2, (BP2) cannot be replaced with any set of equations if $n \geq 2$, i.e. $Ax_1 \not\models Ax_2$. Hence $RA_n^p = \mathbf{SPUp}RA_n^p \neq \mathbf{H}RA_n^p$, i.e. RA_n^p is closed under subalgebras, products and ultraproducts but RA_n^p is not closed under taking homomorphic images, if $n \geq 2$.*

Proof: (i): $RA_n^p \models Ax_1$ is easy to check, therefore we leave it to the reader. To check $RA_n^p \models (BP2)$, let $\mathfrak{A} \subseteq \langle \mathfrak{P}({}^nU), p_{ij}^U \rangle_{i,j<n}$, $a \in A$ and assume that $\sum\{p_\sigma a : \sigma \in S(n)\} = 1 \neq 0$ holds in $\mathfrak{A}$. Then $U \neq \emptyset$. Let $u \in U$ be arbitrary and let $s = \langle u, ..., u\rangle = n \times \{u\}$ be the constant sequence of length n and with values u. Then $\sigma \circ s = s$ for all $\sigma \in S(n)$. Thus $s \in {}^nU = \bigcup\{p_\sigma a : \sigma \in S(n)\}$ implies $s \in \bigcap\{p_\sigma a : \sigma \in S(n)\}$, i.e. $\prod\{p_\sigma a : \sigma \in S(n)\} \neq 0$ in $\mathfrak{A}$.

To prove $Mod(Ax_2) \subseteq RA_n^p$, let $\mathfrak{A} = \langle \mathfrak{B}, p_{ij}\rangle_{i,j<n}$ be any algebra satisfying (B),(P),(BP1),(BP2). Then $\mathfrak{B}$ is a Boolean algebra by $\mathfrak{A} \models (B)$. We may assume that $\mathfrak{B}$ is countable because RA_n^p is an axiomatizable class. We may also assume $|B| > 1$. Let Uf denote the set of all ultrafilters of $\mathfrak{B}$. First we prove that $\mathfrak{B}$ has an ultrafilter which is closed under all the p_{ij}'s, i.e. we will prove

$$(*) \qquad (\exists F \in Uf)(\forall i,j < n) {p_{ij}}^* F \subseteq F.$$

To prove (*), let $x \in B$. We say that x is a <u>fixpoint</u> if $p_{ij}x = x$ for all $i,j < n$. We say that x is <u>decomposable</u> if there exists a $y \in B$ for which $x = \sum\{p_\sigma y : \sigma \in S(n)\}$ and $\prod\{p_\sigma y : \sigma \in S(n)\} = 0$. We say that x is indecomposable (<u>indec</u> for short) if x is not decomposable. Thus x is indecomposable if $x = \sum\{p_\sigma y : \sigma \in S(n)\}$ implies $\prod\{p_\sigma y : \sigma \in S(n)\} \neq 0$. If $x, x_0, \ldots, x_k$ are elements of a Boolean algebra, then we say that $x_0, \ldots, x_k$ is a <u>partition</u> of x provided that $x = \sum\{x_i : i \leq k\}$

and $x_i \cap x_j = 0 \neq x_i$ for all distinct $i, j \leq k$. The following is an easy observation, as we shall see:

(1) Assume that x is an indec fixpoint and the fixpoints $x_0, ..., x_k$ constitute a partition of x. Then one of $x_0, ..., x_k$ is indecomposable.

Indeed, assume that no one of $x_i, i \leq k$ is indecomposable, i.e. that all of them are decomposable. For all $i \leq k$ let y_i be such that $x_i = \sum\{p_\sigma y_i : \sigma \in S(n)\}$ and $\prod\{p_\sigma y_i : \sigma \in S(n)\} = 0$. Let $y = \sum\{y_i : i \leq k\}$. Then $p_\sigma y = \sum\{p_\sigma y_i : i \leq k\}$ by (BP1), hence $x = \sum\{x_i : i \leq k\} = \sum\{p_\sigma y : \sigma \in S(n)\}$. But $\prod\{p_\sigma y : \sigma \in S(n)\} = 0$ because $p_\sigma y_i \cdot p_\delta y_j = 0$ if $\sigma, \delta \in S(n)$ and $i \neq j$, since x_i, x_j are disjoint fixpoints and $y_i \leq x_i$ (by $p_{Id} y_i = y_i \leq x_i$). This contradicts the fact that x is indecomposable. (1) has been proved.

We will construct a descending chain $x_0 \geq x_1 \geq ...$ of indec fixpoints such that

(2) $\qquad$ for all $y \in B$ there is $k < \omega$ such that $x_k \leq y \quad$ or $\quad x_k \leq -y$

will hold. Let $B = \{b_k : k \in \omega\}$. We let

$$x_0 = 1.$$

Then x_0 is an indec fixpoint by $\mathfrak{A} \models (BP1), (BP2)$ and by $|B| > 1$. Assume that $x = x_k$ has already been constructed such that x_k is an indec fixpoint. Let $y = x_k \cdot b_k$, $\overline{y} = x_k - b_k$. Let $I = \mathcal{P}(S(n))$. For every $i \in I$ define

$$a_i = \prod\{p_\sigma y : \sigma \in i\} \cdot \prod\{p_\sigma \overline{y} : \sigma \in S(n) \smallsetminus i\}.$$

Let $\mathcal{A} = \{a_i : i \in I\}$. We will prove

(3) $\mathcal{A} \smallsetminus \{0\}$ is a partition of x and $\mathcal{A}$ is closed under p_σ, i.e. $p_\sigma{}^*\mathcal{A} \subseteq \mathcal{A}$ for all $\sigma \in S(n)$.

Indeed, if $i \neq j$ then $a_i \cdot a_j = 0$ (by $p_\sigma y \cdot p_\sigma \overline{y} = p_\sigma(y \cdot \overline{y}) = p_\sigma 0 = 0$). By $x = y + \overline{y} = p_\sigma x$ we have $x = p_\sigma y + p_\sigma \overline{y}$ for all $\sigma \in S(n)$. Thus $x = \prod\{p_\sigma y + p_\sigma \overline{y} : \sigma \in S(n)\}$. The latter equals $\sum \mathcal{A}$ by the distributivity law, and the definitions of $a_i, \mathcal{A}$. Thus $x = \sum \mathcal{A}$. Therefore $\mathcal{A} \smallsetminus \{0\}$ is a partition of x. Let $\delta \in S(n)$ and $i \in I$. Then $p_\delta a_i = \prod\{p_\delta p_\sigma y : \sigma \in i\} \cdot \prod\{p_\delta p_\sigma \overline{y} : \sigma \notin i\} = a_j$ where $j = \{\sigma \circ \delta : \sigma \in i\}$ (by $p_\delta p_\sigma y = p_{\sigma \circ \delta} y$). We have proved (3).

Let $\quad e_i = \sum\{p_\sigma a_i : \sigma \in S(n)\}, \quad$ for all $i \in I$.

(4) e_i is a fixpoint and $\quad e_i \cdot e_j \neq 0 \Longrightarrow e_i = e_j \quad$ for all $i, j \in I$.

Indeed, let $\delta \in S(n)$, $i \in I$. Then $p_\delta e_i = \sum\{p_\delta p_\sigma a_i : \sigma \in S(n)\} = \sum\{p_\sigma a_i : \sigma \in S(n)\} = e_i$ because $\{\sigma \circ \delta : \sigma \in S(n)\} = S(n)$. Therefore e_i is a fixpoint. Assume that $e_i \cdot e_j \neq 0$ for $i, j \in I$. Then $p_\varepsilon a_i \cdot p_\delta a_j \neq 0$ for some $\varepsilon, \delta \in S(n)$, therefore $p_\varepsilon a_i = p_\delta a_j$ by (3). Now $e_i = \sum\{p_\delta a_i : \sigma \in S(n)\} = \sum\{p_\sigma p_\varepsilon a_i : \sigma \in S(n)\} = \sum\{p_\sigma p_\delta a_j : \sigma \in S(n)\} = \sum\{p_\sigma a_j : \sigma \in S(n)\} = e_j$. (4) has been proved.

(5) If e_i is indec then $e_i = a_i$, for all $i \in I$.

Indeed, assume $a_i \neq e_i = \sum\{p_\sigma a_i : \sigma \in S(n)\}$. Then $a_i \neq p_\sigma a_i$ for some $\sigma \in S(n)$, but then $a_i \cdot p_\sigma a_i = 0$, therefore e_i is decomposable by $\prod\{p_\sigma a_i : \sigma \in S(n)\} = 0$. (5) has been proved.

By (3),(4), $\{e_i : i \in I\} \smallsetminus \{0\}$ is a partition of x consisting of fixpoints. By (1) then e_i is indec for some $i \in I$. Let e_i be indec. Then $e_i = a_i$ by (5). We now define

$$x_{k+1} = a_i, \quad \text{for this } i.$$

Then x_{k+1} is an indec fixpoint (by $a_i = e_i$), $x_{k+1} \leq x_k$ and $x_{k+1} \leq b_k$ or $x_{k+1} \leq -b_k$ (by the definition of a_i).

By induction we have constructed the descending chain $x_0 \geq x_1 \geq \ldots$ of fixpoints with property (2). Let $F = \{b \in B : (\exists k \in \omega) x_k \leq b\}$. Then $F \in Uf$ by (2) and $p_\sigma{}^* F \subseteq F$ because x_k is a fixpoint for all $k \in \omega$. (*) has been proved.

Let $\mathfrak{C}$ be the complete atomic extension of $\mathfrak{A}$ constructed from ultrafilters, i.e. let

$$\mathfrak{C} = \langle \mathfrak{P}(Uf), p_{ij} \rangle_{i,j<n}$$

where $p_{ij} X = \{p_{ij}{}^* F : F \in X\}$ for any $X \subseteq Uf$. (It is easy to check that $p_{ij}{}^* F \in Ff$ for any $F \in Uf$.) Then $\mathfrak{A}$ is embeddable into $\mathfrak{C}$, because the usual embedding $em(b) = \{F \in Uf : b \in F\}$ works (to see this, one has to check $p_{ij} em(b) = em(p_{ij} b)$ for all $i, j < n$ and $b \in B$). Therefore it is enough to represent $\mathfrak{C}$ (i.e. to show $\mathfrak{C} \in RA_n^p$). $\mathfrak{C}$ is atomic and, by (*), $\mathfrak{C}$ has an atom, say c, which is a fixpoint. Also, $\mathfrak{C} \models (P), (BP1)$ is easy to check by using the definition of $\mathfrak{C}$.[2] Let At denote the set of atoms of $\mathfrak{C}$. Then $p_\sigma : At \rightarrowtail\!\!\!\!\!\rightarrow At$ is a bijection for all $\sigma \in S(n)$. Let us define

$$a \equiv b \quad \text{iff} \quad (\exists \sigma \in S(n)) b = p_\sigma a.$$

Then $\equiv$ is an equivalence relation on At. Let $At' \subseteq At$ be a set of representatives for $\equiv$ (i.e. At' contains exactly one element from each block of $\equiv$). Let $\langle U_a :$

[2] This follows also from Jónsson–Tarski[51] Thm.2.18, which says that any equation not containing Boolean complementation is preserved from $\mathfrak{A}$ to $\mathfrak{C}$; and in any Boolean algebra (BP1) is easily seen to be equivalent to (BP1') which contains no complementation.

$a \in At'\rangle$ be a system of disjoint sets such that $|U_a| \geq n$ for all $a \in At'$. Let $U = \bigcup\{U_a : a \in At'\}$ and for all $a \in At'$ let $s_a \in {}^n(U_a)$ be an arbitrary but fixed repetition–free sequence. We define $rep : C \longrightarrow \mathcal{P}({}^nU)$ as follows:

$$\begin{aligned} rep(a) &= \{s_{a'} \circ \sigma : a = p_\sigma a',\ \sigma \in S(n),\ a' \in At'\} \ \text{ if } a \in At,\ a \neq c, \\ rep(c) &= {}^nU \setminus \bigcup\{rep(a) : a \in At,\ a \neq c\}, \\ rep(x) &= \bigcup\{rep(a) : a \in At,\ a \leq x\} \quad \text{for } x \in C. \end{aligned}$$

We will show the following for all $a, b \in At$:

(I) $rep(a) \neq 0$.
(II) $rep(a) \cap rep(b) = 0$ if $a \neq b$.
(III) $p_\sigma^U rep(a) = rep(p_\sigma a)$ for all $\sigma \in S(n)$.

(I) is clear. Assume $rep(a) \cap rep(b) \neq 0$, $a \neq c$. Then $b \neq c$ by the definition of $rep(c)$. Thus $s_{a'} \circ \sigma = s_{b'} \circ \delta$, $a = p_\sigma a'$, $b = p_\delta b'$ hold for some $\sigma, \delta \in S(n)$, $a', b' \in At'$. Then $Rng(s_{a'}) \cap Rng(s_{b'}) \neq 0$, therefore $a' = b'$. Since $s_{a'}$ is repetition–free, then also $\sigma = \delta$ by $s_{a'} \circ \sigma = s_{a'} \circ \delta$. Thus $a = p_\sigma a' = p_\delta a' = p_\delta b' = b$. Therefore (II) holds. To show (III), let $\sigma \in S(n)$, $a \in At$, $a \neq c$, $a' \in At'$ be such that $a \equiv a'$. Then $p_\sigma a \neq c$, $p_\sigma a \equiv a'$. Now $p_\sigma rep(a) = p_\sigma\{s_{a'} \circ \delta : a = p_\delta a',\ \delta \in S(n)\} = \{s_{a'} \circ \delta \circ \sigma : a = p_\delta a',\ \delta \in S(n)\} = \{s_{a'} \circ \varepsilon : p_\sigma a = p_\varepsilon a',\ \varepsilon \in S(n)\} = rep(p_\sigma a)$. To show $p_\sigma rep(c) = rep(p_\sigma c) = rep(c)$, it is enough to notice that $rep(c) = \{s \circ \sigma : s \in rep(c)\}$ for all $\sigma \in S(n)$. Therefore (III) holds, too. By (I)–(III) then rep is a representation for $\mathfrak{C}$, i.e. $rep : \mathfrak{C} \rightarrowtail \langle \mathfrak{P}({}^nU), p_{ij}^U\rangle_{i,j<n}$ is a homomorphic embedding. $Mod(Ax_2) \subseteq RA_n^p$ has been proved.

<u>(ii):</u> Since Ax_2 consists of quasi–equations, by (i) we only have to show that Ax_1 is an axiomatization of the equational theory of RA_n^p and that $RA_n^p \neq \mathbf{H}RA_n^p$.

First we show that RA_n^p is not closed under taking homomorphic images. Let $n \geq 2$ and let $\mathfrak{A} = \langle \mathfrak{P}({}^nU), p_{ij}^U\rangle_{i,j<n} \in RA_n^p$ with $|U| = n$. We will show that a homomorphic image of $\mathfrak{A}$ is not in RA_n^p. Let $V = \{s \in {}^nU : s \text{ is repetition–free}\}$ and let $rl(V) : \mathcal{P}({}^nU) \longrightarrow \mathcal{P}(V)$ be defined by

$$rl(V)(x) = V \cap x \qquad \text{for all } x \subseteq {}^nU.$$

Clearly, $rl(V)$ is a Boolean homomorphism. Let $\sigma \in S(n)$ and $x \in A$. Then $rl(V)(p_\sigma x) = V \cap p_\sigma x = p_\sigma V \cap p_\sigma x = p_\sigma(V \cap x) = p_\sigma rl(V)(x)$, because $V = p_\sigma V$. Thus $rl(V)$ is indeed a homomorphism on $\mathfrak{A}$. Let $\mathfrak{B}$ be the image of $\mathfrak{A}$ under this homomorphism $rl(V)$. We will show $\mathfrak{B} \notin RA_n^p$ by showing $\mathfrak{B} \not\models (BP2)$. By $|U| \geq n \geq 2$ we have $V \neq \emptyset$. Let $s \in V$ be arbitrary and let $x = \{s\} \in B$. Then $\sum\{p_\sigma x : \sigma \in S(n)\} = V = 1^{\mathfrak{B}}$, but $\prod\{p_\sigma x : \sigma \in S(n)\} = 0 \neq 1^{\mathfrak{B}}$ in $\mathfrak{B}$, showing $\mathfrak{B} \not\models (BP2)$.

Next we show that Ax_1 is an axiomatization of the equational theory of RA_n^p. To this end, it is enough to show that the finitely generated free Ax_1–algebras are in RA_n^p. Let $k < \omega$ and let $\mathfrak{A}$ be an Ax_1–free algebra freely generated by $g_0, ..., g_k$. We will show that $\mathfrak{A} \models (BP2)$. By (i), this will imply $\mathfrak{A} \in RA_n^p$. Let $x \in A$. We will show that either $\prod\{p_\sigma x : \sigma \in S(n)\} \neq 0$ or $\sum\{p_\sigma x : \sigma \in S(n)\} \neq 1$. Let U be any set with $|U| \geq 2$, and let $D = \{s \in {}^nU : s_i = s_j \text{ for all } i,j < n\}$. Then $0 \neq D \neq {}^nU$ and $p_\sigma D = D$ for all $\sigma \in S(n)$. Let $\mathfrak{C} = \langle \mathfrak{P}({}^nU), p_{ij}^U \rangle_{i,j<n}$. Then $\mathfrak{C} \models Ax_1$ by $\mathfrak{C} \in RA_n^p$. Since $\mathfrak{A}$ is an Ax_1–free algebra, there is a homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{C}$ such that $h(g_j) = D$ for all $j \leq k$. By $\mathfrak{A} \models Ax_1$ we have that every element of A is a Boolean combination of elements from $\{p_\sigma g_i : \sigma \in S(n),\ i \leq k\}$. By $h(p_\sigma g_j) = p_\sigma D = D$ for all $\sigma \in S(n),\ j \leq k$ then $h(x)$ is a Boolean combination of $D, -D$, i.e. $h(x) \in \{0, D, -D, {}^nU\}$. Then $h(x) = p_\sigma h(x)$ for all $\sigma \in S(n)$, therefore $h(x) = \prod\{p_\sigma h(x) : \sigma \in S(n)\} = \sum\{p_\sigma h(x) : \sigma \in S(n)\}$. This shows that either $\prod\{p_\sigma x : \sigma \in S(n)\} \neq 0$ or $\sum\{p_\sigma x : \sigma \in S(n)\} \neq 1$ in $\mathfrak{A}$. **QED(Proposition 1)**

REMARK 1. Proposition 1 can be generalized as follows. Let $G \subseteq S(n)$ be the universe of any finitely presentable subgroup of $\langle S(n), \circ \rangle$. Let

$$RA_n^G = \mathbf{SI}\{\langle \mathfrak{P}({}^nU), p_\sigma^U \rangle_{\sigma \in G} : U \text{ is a set}\}.$$

Then Proposition 1 remains true if we replace RA_n^p with RA_n^G and (P),(BP1) with the similar equations obtained from a finite presentation of G. ■

Next we characterize the diagonal constants d_{ij} together with the Boolean operations.

Let $Ekv(n)$ denote the set of all equivalence relations on $n = \{0, ..., n-1\}$. If $e \in Ekv(n)$, and $R \subseteq n \times n$ then

$$\|e\| \text{ denotes the number of blocks of } e,$$

and

$$d(R) \quad \text{denotes the term} \quad \prod\{d_{ij} : i,j < n,\ i\ R\ j\} \cdot \prod\{-d_{ij} : i,j < n,\ i \not\!R\ j\}.$$

Consider the following formulas. Let $i, j, k < n,\ e, e' \in Ekv(n)$.

(BD1) $d_{ii} = 1, \quad d_{ij} = d_{ji}, \quad d_{ij} \cdot d_{jk} \leq d_{ik}$.

(BD2) $d(e) = 0 \rightarrow d(e') = 0 \quad \text{if } \|e\| \leq \|e'\|$.

(BD3) $d(e) = 0 \wedge x_0 + ... + x_l \leq d(e') \rightarrow \bigvee\{x_i = x_j : i < j \leq l\}$, where $l = 2^m,\ m = (\|e\| - 1) \cdot ... \cdot (\|e\| - \|e'\|)$, whenever $\|e\| \geq \|e'\|$.

As before, we will consider (BD1),...,(BD3) to be sets of formulas.
Let $Ax_1 = (B) \cup (BD1), \quad Ax_2 = Ax_1 \cup (BD2), \quad Ax_3 = Ax_2 \cup (BD3).$

PROPOSITION 2.

(i) $RA_n^d = Mod(Ax_3)$.

(ii) Ax_1, Ax_2, Ax_3 *are axiomatizations of the equational theory, of the quasi–equational theory and of the quantifier–free theory of* RA_n^d*, respectively. If* $n > 2$ *then these theories are strictly stronger in this order, i.e.*
$Ax_1 \not\models Ax_2$ *and* $Ax_2 \not\models Ax_3$*, or equivalently,*

$$RA_n^d \subsetneqq \mathbf{SP}RA_n^d \subsetneqq \mathbf{HSP}RA_n^d. \textit{ Further, } \mathbf{HSP}RA_n^d = \mathbf{H}RA_n^d.$$

Proof: $RA_n^d \models Ax_2$ is easy to check. To check $RA_n^d \models (BD3)$, let $\mathfrak{A} \subseteq \langle \mathfrak{P}(^nU), d_{ij}^U \rangle_{i,j<n}$. We note that in $\mathfrak{A}$, $d(e) = \{s \in {}^nU : ker(s) = e\}$ for all $e \in Ekv(n)$ where $ker(s) = \{(i,j) : s_i = s_j,\ i,j \in n\}$. Assume that $d(e) = 0$ holds in $\mathfrak{A}$ and let $e' \in Ekv(n)$, $\|e'\| \leq \|e\|$. Then $|U| \leq \|e\| - 1$ by $d(e) = 0$ and hence $|d(e')| \leq (\|e\| - 1) \cdot ... \cdot (\|e\| - \|e'\|) = m$. Then $d(e')$ has at most $l = 2^m$ subsets, and $x_0 + ... + x_l \leq d(e') \ \rightarrow\ \bigvee\{x_i = x_j : i < j \leq l\}$ expresses exactly this.

To prove $Mod(Ax_2) \subseteq \mathbf{SP}RA_n^d$, let $\mathfrak{A} = \langle \mathfrak{B}, d_{ij} \rangle_{i,j<n}$ and assume $\mathfrak{A} \models Ax_2$. We may assume $|B| > 1$. By $\mathfrak{A} \models (B)$ we have that $\mathfrak{B}$ is a Boolean algebra, hence $1 = \sum\{d(R) : R \subseteq n \times n\}$. By $\mathfrak{A} \models (BD1)$ we have that $d(R) = 0$ if $R \notin Ekv(n)$, thus $1 = \sum\{d(e) : e \in Ekv(n)\}$. Then $d(e) \neq 0$ for some $e \in Ekv(n)$, by $|B| > 2$. Let

$$\kappa = max\{\|e\| : d(e) \neq 0\}.$$

Then by $\mathfrak{A} \models (BD2)$ we have

$$d(e) \neq 0 \quad \text{iff} \quad \|e\| \leq \kappa.$$

Let $\langle U_a : a \in B \rangle$ be a system of disjoint sets each of cardinality κ. Let $V = \bigcup\{^n(U_a) : a \in B\}$. For any $e \in Ekv(n)$ let

$$D(e) = \{s \in V : ker(s) = e\}$$

and let $\mathfrak{B}(e)$ be the Boolean algebra $\mathfrak{B}$ relativized to $d(e)$, i.e. $\mathfrak{B}(e)$ is the homomorphic image of $\mathfrak{B}$ along the homomorphism h defined by

$$h(x) = x \cdot d(e) \quad \text{for all } x \in B.$$

Then either $1 < |\mathfrak{B}(e)| \leq |A| \leq |D(e)|$ or ($1 = |\mathfrak{B}(e)|$ and $0 = D(e)$). For each $e \in Ekv(n)$ let $h_e : \mathfrak{B}(e) \rightarrowtail \mathfrak{P}(D(e))$ be any representation of the Boolean algebra $\mathfrak{B}(e)$. Such representations exist by the above facts. Keeping in mind that $D(e) \subseteq V$ for all $e \in Ekv(n)$, we may define $h : B \to \mathcal{P}(V)$ by defining for all $x \in B$

$$h(x) = \bigcup \{h_e(x \cdot d(e)) : e \in Ekv(n)\}.$$

By $1 = \sum\{d(e) : e \in Ekv(n)\}$ we have that $\{d(e) : e \in Ekv(n),\ \|e\| \leq \kappa\}$ is a partition of 1. By $V = \bigcup\{D(e) : e \in Ekv(n),\ \|e\| \leq \kappa\}$ then it is not difficult to check that $h : \mathfrak{B} \rightarrowtail \mathfrak{P}(V)$ is a one–one homomorphism and $h(d(e)) = D(e)$ for all $e \in Ekv(n)$. For any $i, j < n$ let

$$D^V_{ij} = \{s \in V : s_i = s_j\}.$$

It is not difficult to check that $d_{ij} = \prod\{d(e) : e \in Ekv(n),\ (i,j) \in e\}$ and $D^V_{ij} = \bigcap\{D(e) : e \in Ekv(n),\ (i,j) \in e\}$. Thus $h(d_{ij}) = D^V_{ij}$. Finally, it is easy to check that

$$\langle \mathfrak{P}(V), D^V_{ij}\rangle_{i,j<n} \cong \underset{a\in A}{\times} \langle \mathfrak{P}({}^nU_a), d_{ij}\rangle_{i,j<n}.$$

Thus $h : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}(V), D^V_{ij}\rangle_{i,j<n} \in \mathbf{P}RA^d_n$, showing that $\mathfrak{A} \in \mathbf{SP}RA^d_n$.

To show $Mod(Ax_3) \subseteq RA^d_n$, let $\mathfrak{A} \models (BD3)$. First suppose that $\kappa < n$. Then there is $e \in Ekv(n)$ with $d(e) = 0$ and $\|e\| = \kappa + 1$. For any $\lambda \leq \kappa$ let

$$m(\lambda) = \kappa \cdot (\kappa - 1) \cdot \ldots \cdot (\kappa - \lambda + 1).$$

Then $\mathfrak{A} \models (BD3)$ implies that $|\mathfrak{B}(e')| \leq 2^{m(\|e'\|)}$, for all $e' \in Ekv(n)$. Let U be any set with cardinality κ, $V = {}^nU$ and define $D(e)$ as before. Then for any $e \in Ekv(n),\ \|e\| \leq \kappa$

$$|D(e)| = m(\|e\|).$$

Then $|\mathfrak{B}(e)| \leq 2^{|D(e)|}$, thus there is a representation $h_e : \mathfrak{B}(e) \rightarrowtail \mathfrak{P}(D(e))$ of the Boolean algebra $\mathfrak{B}(e)$. Then we can construct, as in the previous case, a homomorphism $h : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}({}^nU), d^U_{ij}\rangle_{i,j<n}$, showing that $\mathfrak{A} \in RA^d_n$.

Assume now $\kappa = n$. Then $d(e) \neq 0$ for all $e \in Ekv(n)$. Let U be any set with cardinality $\geq |A|$. Let $V = {}^nU$, and $\mathfrak{B}(e), D(e)$ as before. Then $1 < |\mathfrak{B}(e)| \leq |D(e)|$ for all $e \in Ekv(n)$, and then as in the previous cases we can construct a one–one homomorphism $h : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}({}^nU), d^U_{ij}\rangle_{i,j<n}$ showing $\mathfrak{A} \in RA^d_n$.

We proved $Mod(Ax_3) \subseteq RA^d_n$ and $Mod(Ax_2) \subseteq \mathbf{SP}RA^d_n$.

To show $Mod(Ax_1) \subseteq \mathbf{H}RA^d_n$, let $\mathfrak{A}$ be any Ax_1–free algebra. We will show $\mathfrak{A} \in RA^d_n$ by showing $\mathfrak{A} \models (BD2), (BD3)$. We will show that $d(e) \neq 0$ in $\mathfrak{A}$, for all $e \in Ekv(n)$. Let U be any set with $|U| \geq n$ and let $\mathfrak{C} = \langle \mathfrak{P}({}^nU), d^U_{ij}\rangle_{i,j<n}$. Then

$\mathfrak{C} \models Ax_1$ by $\mathfrak{A} \in RA_n^d$, therefore there is a homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{C}$ since $\mathfrak{A}$ is free. Then $h(d(e)^{\mathfrak{A}}) = d(e)^{\mathfrak{C}} \neq 0$ shows that $d(e) \neq 0$ in $\mathfrak{A}$. We have proved $Mod(Ax_1) \subseteq \mathbf{H}RA_n^d$.

Finally we show $Ax_1 \not\models Ax_2$ and $Ax_2 \not\models Ax_3$, if $n > 2$. Let $\mathfrak{C} = \langle \mathfrak{P}({}^nU), d_{ij}^U \rangle_{i,j<n} \in RA_n^d$ with $|U| \geq n$. Let $e, e' \in Ekv(n)$, $\|e\| < \|e'\|$. Let $V = {}^nU \setminus D(e)$ and define

$$rl(V)x = V \cap x \quad \text{for all} \quad x \in C.$$

Then $rl(V)$ is a homomorphism of $\mathfrak{C}$. Let $\mathfrak{B}$ be the homomorphic image of $\mathfrak{C}$ along $rl(V)$. Then $\mathfrak{B} \models Ax_1$ by $\mathfrak{C} \models Ax_1$. But by $h(d(e)) = 0 \neq h(d(e'))$ we have $\mathfrak{B} \not\models (BD2)$, showing that $Ax_1 \not\models Ax_2$. Let u, v be different, $V = {}^n\{u\} \cup {}^n\{v\}$, and $\mathfrak{C} = \langle \mathfrak{P}(V), D_{ij}^V \rangle_{i,j<n}$. Then $\mathfrak{C} \models Ax_2$ and $d(e) \neq 0$ in $\mathfrak{C}$ iff $\|e\| = 1$. Let $e \in Ekv(n)$ be such that $\|e\| = 2$, and let $e' = n \times n$. Then $|d(e')| = 2$ and $|\{x : x \leq d(e')\}| = 4$ in $\mathfrak{C}$. Then $1 = (\|e\| - 1) \cdot \ldots \cdot (\|e\| - \|e'\|)$ but there are more than 2^1 elements below $d(e')$ in $\mathfrak{C}$, showing that $\mathfrak{C} \not\models (BD3)$. Thus $Ax_2 \not\models Ax_3$. **QED(Proposition 2)**

Finally we characterize the operations p_{ij}, d_{ij} together with the Boolean operations. Consider the following formulas. Let $i, j, k, l < n$.

(BDP)
$$p_{ij}(x \cdot d_{ij}) = x \cdot d_{ij}, \quad p_{ik}d_{ij} = d_{kj}, \quad p_{lk}d_{ij} = d_{ij} \quad \text{for} \quad \{l, k\} \cap \{i, j\} = \emptyset, i \neq j.$$

We will need also a more complex form of (BD3). To formulate this, we need some preparation. Let $k, \kappa < \omega$. Then $\eta(k)$ denotes the formula

$$\sum_{i<k} x_i = 1 \wedge \bigwedge_{i<j<k} x_i \cdot x_j = 0 \wedge \bigwedge_{\substack{i<k \\ \sigma \in S(n)}} \left(x_i \neq 0 \wedge \bigvee_{e \in Ekv(n)} x_i \leq d(e) \wedge \bigvee_{j<k} p_\sigma x_i = x_j \right).$$

$\mathcal{K}(k, \kappa)$ denotes the set of all bijections $f : X \rightarrowtail\!\!\!\!\rightarrow At\mathfrak{A}$ where $X = \{x_i : i < k\}$ and $\mathfrak{A} \subseteq \langle \mathfrak{P}({}^nU), d_{ij}^U, p_{ij}^U \rangle_{i,j<n}$ for some U with $U \subseteq \kappa$. Note that $\mathcal{K}(k, \kappa)$ is finite. For $f \in \mathcal{K}(k, \kappa)$ define the formula $\delta(f)$ as

$$\bigwedge_{\substack{i,j<k \\ e \in Ekv(n) \\ \sigma \in S(n)}} \left(\bigwedge_{f(x_i) \leq d(e)} x_i \leq d(e) \wedge \bigwedge_{p_\sigma f(x_i) = f(x_j)} p_\sigma x_i = x_j \right).$$

Let now (BDP3) denote the following formulas. Let $e \in Ekv(n)$, $\kappa = \|e\| - 1$ and $k \leq 2^{(\kappa^n)}$. Then

$$\text{(BDP3)} \qquad d(e) = 0 \wedge \eta(k) \rightarrow \bigvee\{\delta(f) : f \in \mathcal{K}(k, \kappa)\}.$$

Let $Ax_1 = (B) \cup (P) \cup (BD1) \cup (BP1) \cup (BDP)$, $Ax_2 = Ax_1 \cup (BD2)$, $Ax_3 = Ax_2 \cup (BD3) \cup (BDP3)$.

PROPOSITION 3.

(i) $RA_n^{dp} = Mod(Ax_3)$.

(ii) *Ax_1, Ax_2, Ax_3 are axiomatizations of the equational theory, of the quasi–equational theory and of the quantifier–free theory of RA_n^{dp}, respectively. If $n > 2$ then these theories are strictly stronger in this order, i.e.* $Ax_1 \not\models Ax_2$ *and* $Ax_2 \not\models Ax_3$, *or equivalently,*

$$RA_n^{dp} \subsetneqq \mathbf{SP}RA_n^{dp} \subsetneqq \mathbf{HSP}RA_n^{dp}. \text{ Further, } \mathbf{HSP}RA_n^{dp} = \mathbf{H}RA_n^{dp}.$$

Proof: $RA_n^{dp} \models (BDP)$ is easy to check. To check $RA_n^{dp} \models (BDP3)$, let $\mathfrak{A} \subseteq \langle \mathfrak{P}({}^nU), d_{ij}^U, p_{ij}^U \rangle_{i,j<n} \in RA_n^{dp}$ and let $e \in Ekv(n)$, $\kappa = \|e\| - 1$, $k \leq 2^{(\kappa^n)}$. Let $a_0, ..., a_{k-1} \in A$, $\vec{a} = \langle a_0, ..., a_{k-1} \rangle$ and assume $\mathfrak{A} \models d(e) = 0 \wedge \eta(k)[\vec{a}]$. By $\mathfrak{A} \models d(e) = 0$ then $|U| \leq \kappa$. We may assume $U \subseteq \kappa$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_0, ..., a_{k-1}\}$. By $\mathfrak{A} \models \eta(k)[\vec{a}]$ we have that $\{a_0, ..., a_{k-1}\} = At\mathfrak{B}$. Let $f : \{x_i : i < k\} \longrightarrow \mathfrak{B}$ be such that $f(x_i) = a_i$ for all $i < k$. Then $f \in \mathcal{K}(k, \kappa)$, and $\mathfrak{A} \models \delta(f)[\vec{a}]$. Thus $\mathfrak{A} \models (BDP3)$. We have seen that $RA_n^{dp} \models Ax_3$.

To prove $Mod(Ax_2) \subseteq \mathbf{SP}RA_n^{dp}$ and $Mod(Ax_3) \subseteq RA_n^{dp}$, let $\mathfrak{A} = \langle \mathfrak{B}, d_{ij}, p_{ij} \rangle_{i,j<n}$, and assume $\mathfrak{A} \models Ax_2$. Let $\kappa = max\{\|e\| : d(e) \neq 0,\ e \in Ekv(n)\}$. Then by $\mathfrak{A} \models (B), (BD1), (BD2)$,

$$d(e) \neq 0 \quad \text{iff} \quad \|e\| \leq \kappa$$

for all $e \in Ekv(n)$.

Assume $\kappa < n$, and $\mathfrak{A} \models Ax_3$. Then $|A| < \omega$ by $\mathfrak{A} \models (BD3)$, thus $\mathfrak{A}$ is atomic. Let $k = |At\mathfrak{A}|$. Let $X = \{x_i : i < k\}$ and let $h : X \rightarrowtail\!\!\!\!\rightarrow At\mathfrak{A}$ be any bijection. Let $e \in Ekv(n)$ be such that $d(e) = 0$ and $\|e\| = \kappa + 1$. Such e exists by the definition of κ, and by $\kappa < n$. Then $\mathfrak{A} \models d(e) = 0 \wedge \eta(k)$ under the evaluation h of variables, therefore by $\mathfrak{A} \models (BDP3)$ we have that $\mathfrak{A} \models \delta(f)$ under the evaluation h, for some $f : X \rightarrowtail\!\!\!\!\rightarrow At\mathfrak{C}$ and $\mathfrak{C} \subseteq \langle \mathfrak{P}({}^nU), d_{ij}, p_{ij} \rangle_{i,j<n}$. Define

$$rep(a) = f(h^{-1}(a)) \qquad \text{if } a \in At\mathfrak{A}, \qquad \text{and}$$
$$rep(b) = \bigcup\{rep(a) : a \leq b,\ a \in At\mathfrak{A}\}.$$

We will show that $rep : \mathfrak{A} \rightarrowtail\!\!\!\!\rightarrow \mathfrak{C}$ is an isomorphism. Clearly,

$$rep : At\mathfrak{A} \rightarrowtail\!\!\!\!\rightarrow At\mathfrak{C}$$

by the conditions on f and h. Let $a, b \in At\mathfrak{A}$, $a = h(x_i)$, $b = h(x_j)$. Then $rep(a) = f(x_i)$, $rep(b) = f(x_j)$. Thus by $\mathfrak{A} \models \delta(f)[h]$ we have that for all $e \in Ekv(n)$ and $\sigma \in S(n)$

$$\begin{array}{lll} rep(a) \leq d(e) & \text{implies} & a \leq d(e), \quad \text{and} \\ p_\sigma rep(a) = rep(b) & \text{implies} & p_\sigma a = b. \end{array}$$

Since for every atom there is a unique $e \in Ekv(n)$ such that the atom is below $d(e)$, both in $\mathfrak{A}$ and in $\mathfrak{C}$, then

$$\begin{array}{lll} rep(a) \leq d(e) & \text{iff} & a \leq d(e), \quad \text{and} \\ p_\sigma rep(a) = rep(p_\sigma a). & & \end{array}$$

The above imply that rep is an isomorphism. Then $\mathfrak{A} \in RA_n^{dp}$ by $\mathfrak{C} \in RA_n^{dp}$.

Assume now only $\mathfrak{A} \models Ax_2$. We show that $\mathfrak{A}$ can be embedded into an atomic $\mathfrak{C}$ such that $\mathfrak{C} \models Ax_2$. As in the proof of Proposition 1, let Uf denote the set of all ultrafilters of $\mathfrak{B}$ and let

$$\mathfrak{C} = \langle \mathfrak{P}(Uf), d_{ij}, p_{ij} \rangle_{i,j<n}$$

where $d_{ij} = \{F \in Uf : d_{ij} \in F\}$ and $p_{ij}(X) = \{p_{ij}{}^*F : F \in X\}$ for all $X \subseteq Uf$. In Proposition 1 we showed $\mathfrak{C} \models (B), (P), (BP1)$. By the same argument, $\mathfrak{C} \models (BD1), (BDP)$ since these are equations not containing Boolean complementation. Finally, we have to show $\mathfrak{C} \models (BD2)$. By $em : \mathfrak{B} \rightarrowtail \mathfrak{C}$ we have $d(e)^{\mathfrak{B}} = 0$ iff $em(d(e)) = d(e)^{\mathfrak{C}} = 0$ for all $e \in Ekv(n)$. This immediately implies $\mathfrak{C} \models (BD2)$. Also, by $d(e)^{\mathfrak{C}} = d(e)^{\mathfrak{B}}$ we have that $max\{\|e\| : d(e)^{\mathfrak{C}} \neq 0\} = \kappa$. Therefore, from now on we assume that $\mathfrak{A}$ is atomic.

Assume $\mathfrak{A} \models Ax_2$ and $\mathfrak{A}$ is atomic. We will show $\mathfrak{A} \in \mathbf{SP}RA_n^{dp}$. Let At denote the set of all atoms of $\mathfrak{A}$. Define $\equiv$ on At as follows. For any $a, b \in At$

$$a \equiv b \qquad \text{iff} \qquad (a = p_\sigma b \ \text{ for some } \sigma \in S(n)).$$

Then $\equiv$ is an equivalence relation on At. Let $At' \subseteq At$ be a set of representatives for $\equiv$ (i.e. At' contains exactly one element from each block of $\equiv$). Let $\langle U_a : a \in At' \rangle$ be a system of disjoint sets each having cardinality κ. Let $V = \bigcup\{{}^n(U_a) : a \in At'\}$. For any $a \in At$ define

$$ker(a) = \{(i,j) : a \leq d_{ij}\}.$$

Then $ker(a) \in Ekv(n)$ by $\mathfrak{A} \models (BD1)$, and in fact, $ker(a)$ is the unique $e \in Ekv(n)$ for which $a \leq d(e)$. For any $a \in At'$ let $s_a \in {}^n(U_a)$ be such that $ker(s_a) = ker(a)$. Let $rep_0 : At \to \mathcal{P}(V)$ be defined by defining for all $a \in At$

$$rep_0(b) = \{s_a \circ \sigma : b = p_\sigma a, \sigma \in S(n), a \in At'\}.$$

Then $rep_0(b) \neq 0$ for all $b \in At$ since At' contains an atom from each block of $\equiv$.

Having achieved $rep(b) \neq 0$ for all $b \in At$, we now distribute the remaining sequences among the atoms. Let $Z = \bigcup\{rep_0(b) : b \in At\}$ and $W = V \smallsetminus Z$. Then $Z = \{s_a \circ \sigma : a \in At', \sigma \in S(n)\}$, hence $p_\sigma Z = Z$ for all $\sigma \in S(n)$. Then $p_\sigma W = W$ for all $\sigma \in S(n)$, since $p_\sigma V = V$. We define an equivalence relation on W. For all $s, z \in W$

$$s \approx z \quad \text{iff} \quad z = s \circ \sigma \quad \text{for some} \quad \sigma \in S(n).$$

Then $\approx$ is an equivalence relation on W. Let $W' \subseteq W$ be a set of representatives for $\approx$, i.e. W' contains exactly one sequence from each block of $\approx$. For any $s \in W'$ let $at(s) \in At$ be such that $ker(at(s)) = ker(s)$. (Such an atom exists, because $\|ker(s)\| \leq \kappa$ and $d(e) \neq 0$ for all $e \in Ekv(n)$ with $\|e\| \leq \kappa$.) We now define $rep_1 : At \longrightarrow \mathcal{P}(W)$ and $rep : At \longrightarrow \mathcal{P}(V)$ as follows. For all $b \in At$

$$rep_1(b) = \{s \circ \sigma : b = p_\sigma(at(s)), \sigma \in S(n), s \in W'\}.$$
$$rep(b) = rep_0(b) \cup rep_1(b).$$

For all $a \in A$ define

$$rep(a) = \bigcup\{rep(b) : b \leq a, b \in At\}.$$

We want to show that $rep : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}(V), D^V_{ij}, p_{ij}\rangle \cong \underset{a \in At'}{\times} \langle \mathfrak{P}({}^nU_a), d_{ij}, p_{ij}\rangle_{i,j<n}$. This will imply $\mathfrak{A} \in \mathbf{SP}RA^{dp}_n$.

To show that rep is a homomorphism, we will prove the following for all $a, b \in At$, $s \in V$ and $\sigma \in S(n)$.

(I) $rep(a) \neq 0$.

(II) $rep(a) \cap rep(b) \neq 0$ implies $a = b$.

(III) $\bigcup\{rep(a) : a \in At\} = V$.

(IV) $s \in rep(a)$ implies $ker(s) = ker(a)$.

(V) $s \in rep(a)$ implies $s \circ \sigma \in rep(p_\sigma a)$.

First we show that (I)–(V) imply that rep is a one–one homomorphism. Now (I)–(III) mean that $\langle rep(a) : a \in At\rangle$ is a partition of V, hence rep is a one–one Boolean homomorphism. Let $s \in V$. Then there is a unique $a \in At$ for which $s \in rep(a)$, and by (IV) we have $s_i = s_j$ iff $a \le d_{ij}$. Hence $rep(d_{ij}) = \bigcup\{rep(a) : a \le d_{ij}, a \in At\} = D^V_{ij}$. Let $\sigma \in S(n)$. To show that rep is a homomorphism w.r.t. p_σ, it is enough to show $p_\sigma rep(a) = rep(p_\sigma a)$ for $a \in At$ because both p_σ and rep are additive. By (V), $p_\sigma rep(a) = \{s \circ \sigma : s \in rep(a)\} \subseteq rep(p_\sigma a)$. Let $\delta = \sigma^{-1}$. Then $p_\delta rep(p_\sigma a) \subseteq rep(p_\delta p_\sigma a) = rep(a)$ by the above, hence $rep(p_\sigma a) = p_\sigma p_\delta rep(p_\sigma a) \subseteq p_\sigma rep(a)$. Thus $p_\sigma rep(a) = rep(p_\sigma a)$.

We start checking (I)–(V). (I) holds because $rep_0(a) \neq 0$ for all $a \in At$. Next we check (III). Let $s \in W$. Then there is $z \in W'$ with $s = z \circ \sigma$. Let $a = at(z)$. Then $s \in rep_1(p_\sigma a) \subseteq rep(p_\sigma a)$. Thus (III) holds. It is easy to check that (V) holds, by inspecting the definitions of rep_0 and rep_1.: Assume $s \in rep(a)$. If $s \in Z$ then $s \in rep_0(a)$, i.e. $s = s_{a'} \circ \delta, a = p_\delta a'$ and $a' \in At'$ for some δ. Then $s \circ \sigma = s_{a'} \circ (\delta \circ \sigma), p_\sigma a = p_\sigma p_\delta a' = p_{(\delta\circ\sigma)} a', a' \in At'$, hence $s \circ \sigma \in rep_0(p_\sigma a)$ by the definition or rep_0. The case $s \in W$ is completely analogous and we omit it. To check (IV), let $\sigma \in S(n)$ and for any $e \in Ekv(n)$ let $\sigma e = \{(\sigma^{-1}i, \sigma^{-1}j) : (i,j) \in e\}$. Then $ker(s \circ \sigma) = \sigma ker(s)$, and by $\mathfrak{A} \models (BDP)$ and by the definition of p_σ, we have $ker(p_\sigma a) = \sigma ker(a)$. By the definition of rep, this implies (IV).

To check (II), we will need to prove the following statement. Let $a \in At$, $s \in V$ and $\sigma \in S(n)$. Then

$$(*) \qquad s \circ \sigma = s \circ \delta \quad \text{and} \quad ker(s) = ker(a) \qquad \text{imply} \qquad p_\sigma a = p_\delta a.$$

Indeed, assume $s \circ \sigma = s \circ \delta$, $ker(s) = ker(a)$. Then $s = s \circ \pi$ for $\pi = \delta \circ \sigma^{-1}$. By $s = s \circ \pi$ we have that π acts inside the blocks of $ker(s)$ only, thus there are $i_0, j_0, ..., i_k, j_k$ such that

$$\pi = [i_0, j_0] \circ ... \circ [i_k, j_k] \quad \text{and} \quad (i_l, j_l) \in ker(s) \ \text{ for all } l \le k.$$

Then $p_\pi a = p_{i_k j_k} ... p_{i_0 j_0} a$ by $\mathfrak{A} \models (P)$. By $ker(a) = ker(s)$ and $\mathfrak{A} \models (BDP)$ then $p_\pi a = a$. Then $p_\sigma a = p_\delta a$ by $\pi = \delta \circ \sigma^{-1}$, as desired.

We are ready for checking (II). Assume $s \in rep(a) \cap rep(b)$. We have to show $a = b$. Assume first $s \in W$. Then $s \in rep_1(a) \cap rep_1(b)$. Then $s = z \circ \sigma$, $a = p_\sigma at(z)$, $z \in W'$, and $s = z' \circ \delta$, $b = p_\delta at(z')$, $z' \in W'$, for some $z, z' \in W', \sigma, \delta \in S(n)$. By $z \circ \sigma = z' \circ \delta$ and $z, z' \in W'$ we have $z = z'$. Then by $z \circ \sigma = z \circ \delta$, $ker(z) = ker(at(z))$, (*) implies $a = p_\sigma at(z) = p_\delta at(z) = p_\delta at(z') = b$. Assume next $s \notin W$. Then $s \in rep_0(a) \cap rep_0(b)$. Then $s = s_{a'} \circ \sigma$, $a = p_\sigma a'$ and $s = s_{b'} \circ \delta$, $b = p_\delta b'$ for some σ, δ and $a', b' \in At'$. By $Rng(s_{a'}) = Rng(s_{b'})$ then $a' = b'$. Let $z = s_{a'} = s_{b'}$. Then $z \circ \sigma = z \circ \delta$, $ker(z) = ker(a')$ imply, by (*), that $a = p_\sigma a' = p_\delta a' = p_\delta b' = b$. We have checked (II). Thus rep is a one–one homomorphism and $\mathfrak{A} \in \mathbf{SP}RA^{dp}_n$.

Assume finally $\mathfrak{A} \models Ax_2$ and $\kappa = n$. We will show that $\mathfrak{A} \in RA_n^{dp}$. Let U be a set with $|U| \geq n \cdot |A|$. Let $V = {}^nU$. Let At' be as in the previous case. For any $a \in At'$ let $s_a \in {}^nU$ be such that $Rng(s_a) \cap Rng(s_b) = \emptyset$ if $a \neq b$. We define rep exactly as in the previous case: For any $a \in At$ let

$$rep_0(a) = \{s_{a'} \circ \sigma : a = p_\sigma a', \sigma \in S(n), a' \in At'\}.$$

Let W, W' be as in the previous case and define for $a \in At$

$$rep_1(a) = \{s \circ \sigma : a = p_\sigma at(s), \sigma \in S(n), s \in W'\},$$

and define

$$\begin{aligned} rep(a) &= rep_0(a) \cup rep_1(a) \quad \text{if } a \in At, \\ rep(b) &= \bigcup\{rep(a) : a \leq b,\ a \in At\} \quad \text{if } b \in A. \end{aligned}$$

Then one can see, exactly as in the previous case, that
$rep : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}({}^nU), d_{ij}, p_{ij}\rangle_{i,j<n} \in RA_n^{dp}$.

So far we have proved $Mod(Ax_2) \subseteq \mathbf{SP}RA_n^{dp}$ and $Mod(Ax_3) \subseteq RA_n^{dp}$. Now we prove $Mod(Ax_1) \subseteq \mathbf{H}RA_n^{dp}$. The proof of this is very simple: Let $\mathfrak{A}$ be any Ax_1–free algebra. Then $d(e) \neq 0$ in $\mathfrak{A}$ for any $e \in Ekv(n)$, hence $\mathfrak{A} \models (BD2), (BD3), (BDP3)$. Thus $\mathfrak{A} \models Ax_3$, hence $\mathfrak{A} \in RA_n^{dp}$. Since any Ax_1–algebra is a homomorphic image of a free one, this implies $Mod(Ax_1) \subseteq \mathbf{H}RA_n^{dp}$.

To finish the proof, we show $Ax_1 \not\models Ax_2$ and $Ax_2 \not\models Ax_3$ if $n > 2$. Let $n \geq 2$ and let U be a set with $|U| \geq 2$ and let $V = \{s \in {}^nU : s_i = s_j \text{ for all } i, j < n\}$, $W = {}^nU \setminus V$. Let $\mathfrak{A} = \langle \mathfrak{P}({}^nU), d_{ij}, p_{ij}\rangle_{i,j<n}$ and $rl(W)(a) = W \cap a$ for all $a \in A$. Then $rl(W)$ is a homomorphism on $\mathfrak{A}$ by $p_\sigma W = W$ for all $\sigma \in S(n)$. Let $\mathfrak{B}$ be the homomorphic image of $\mathfrak{A}$ taken along $rl(W)$. Then $\mathfrak{B} \models Ax_1$ but $\mathfrak{B} \not\models (BD2)$ showing $Ax_1 \not\models Ax_2$. Let $n \geq 3$ and let $\mathfrak{A}$ be as above with $|U| = 2$ and let $\mathfrak{B} = \mathfrak{A} \times \mathfrak{A}$. Then $\mathfrak{B} \models Ax_2$ but $\mathfrak{B} \not\models (BD3)$ showing $Ax_2 \not\models Ax_3$. **QED**(Proposition 3)

We note that the above proof of Proposition 3 is somewhat similar to the proof in Andréka–Thompson[88].

We conclude with summarizing the known facts on a figure.

On the figure there is a tree. The nodes represent classes of algebras of n–ary relations (on some set U) and the operations are those indicated along the path leading to the node. In other words, a node represents the class of all algebras of n–ary relations, up to isomorphisms, where the greatest relation is of form ${}^nU$ for some U, and the operations are those indicated along the unique path leading to the node. So the classes on different nodes have different similarity types. On the figure we concentrate only on the equational theories of the classes involved.

Let K, L be classes of algebras such that the similarity type of K is contained in that of L. Let $Eq(K), Eq(L)$ denote the sets of all equations valid in K and L respectively. We say that L is finitely axiomatizable over K if $Eq(K) \cup \Sigma$ is an (equational) axiomatization of $Eq(L)$ for some finite Σ. If K consists of all appropriate subreducts of elements of L, as in our cases, then this means that $Eq(L)$ has an axiomatization in which the operation symbols not present in K occur only finitely many times.

On the figure, a broken edge like - - - - between two nodes means that the bigger one of the corresponding classes is finitely axiomatizable over the other one. A bold edge like ====== between two nodes means that the bigger one of the corresponding classes is not finitely axiomatizable over the other one. Normal line ——– means that we (the author) do not know the answer.

The most interesting open case presently seems to be whether the operations s_{ij} are finitely axiomatizable over $\cup, \setminus, c_i$ or not.

There are labels on some of the edges. These indicate where the proof (of the represented statement) can be found. Propositions 1–3 refer to statements in the present paper, while Theorems 1–6 refer to theorems of Part I of the present work.

Now we list the results not in this work that we will use on the figure.

Nonfinite axiomatizability of the operations p_{ij} over $\cup, -, c_i, d_{ij}$ $(i, j < n)$ was proved by Andréka and Tuza, see Andréka–Tuza[88]. This is a solution of Problem 2.b in Johnson[69] and of Problems 5.7,5.8 in Henkin–Monk–Tarski[HMT85].

Finite axiomatization of substitutions s_{ij}, and of the substitutions together with the diagonals over the Boolean operations are given in Sági–Németi[96].

It is not difficult to see that the proof in Monk[69] also proves that the Boolean operations together with cylindrifications and substitutions are not finitely axiomatizable. This together with the results in Sági–Németi[96] proves e.g. nonfinite axiomatizability of cylindrifications over the Booleans and the substitutions.

Monk[89] proved that the Boolean operations together with substitutions, permutations, and the same together with diagonals are finitely axiomatizable.

On the figure, the root of the tree contains the Boolean operations $\cup, \setminus$. This means that the Boolean operations are present in every class considered on the figure. There are known axiomatizations of the other operations when considered without or with only some of the Boolean operations. We now list some results of this kind.

Jónsson[62] gives a finite set of equations characterizing the operations p_{ij}, s_{ij} and Thompson[87,93] give a finite axiom system for the operations s_{ij} in themselves. The equations for s_{ij} are much more involved when we cannot use the p_{ij}'s. For a simple proof of Thompson's theorem see Sági[96].

An axiomatization for the operations c_i in themselves is due to Hansen[92,95] and is the following. For all $i, j < n$

$$c_i c_i x = c_i x, \quad c_i c_j x = c_j c_i x,$$
$$c_0 ... c_{n-1} x = x \wedge c_0 ... c_{n-1} y = y \wedge c_0 ... c_{n-1} z = z \;\rightarrow\; (x = y \vee x = z \vee y = z),$$
$$c_0 ... c_{n-1} x' = x \wedge c_0 ... c_{n-1} y' = y \;\rightarrow\; (x' = x \vee y' = y \vee x = y),$$

and an axiomatization of the operations d_{ij} in themselves is the following. For all $i, j < n$

$$d_{ii} = d_{jj}, \quad d_{ij} = d_{ji}, \quad d_{ij} = d_{pq} \rightarrow d_{kl} = d_{pq} \quad \text{for } p \neq q.$$

Interesting are the results when we keep only part of the Boolean operations or weaker structures such as Stone algebras. More such results can be found in Hansen[92,95]. Or, as another example, Comer[89] proved that the positive reducts of cylindric algebras (i.e. the equational theory of the class $\{\langle \mathfrak{P}({}^nU), \cup, \cap, \emptyset, {}^nU, c_i^U, d_{ij}^U \rangle_{i,j<n} : U \text{ is a set}\}$) is not finitely axiomatizable. These and similar reducts (also for binary relations) are widely investigated recently, e.g. in connection with modal logic ("arrow logic"), and in connection with computer science (theory of relational databases, theory of incomplete information, rough sets). For a sample we list Düntsch[90,93,94], Cirulis [88], Comer[90,93], Cosmadakis[87], Marx[95], Mikulás[95,96], Andréka–Mikulás[94], Andréka–Ryan–Schobbens[95], Karger[94][3], Karger–Hoare[95], Andréka–Bredikhin[95], Jónsson[90], Jipsen–Jónsson–Rafter [95]. Open problems concerning these reducts can be found in the Problems part of Andréka–Monk–Németi[88].

In the following figure, in a node all indices range over n, e.g. d_{ij} means $\{d_{ij} : i, j < n\}$.

[3]We note that sequential algebras are RRA's relativized to reflexive and transitive relations and endowed with the two residuals of relation composition.

FIGURE 1

The corresponding picture for relation algebras (algebras of binary relations with composition) is radically different. While the "strength of RCA_n" is distributed among the operations of RCA_n quite evenly, all the strength of RRA is concentrated in relation composition: this operation is so strong that relative to it all the other non–Boolean operations are finitely axiomatizable.

Figure 2 below summarizes the interconnection between the RRA–operations $\cup, -, |, ^{-1}, Id$. If we take the Boolean operations $\cup, -$ for granted, then the only cause of nonfinite axiomatizability of RRA is $|$, namely $^{-1}$ and Id are finitely axiomatizable over any other subsets of the operations, while $|$ is only infinitely axiomatizable over any sets of others. Of these, the finite axiomatizability results are proved in Andréka-Németi[96] (see also Jipsen–Jónsson–Rafter[95]), where the finite axiomatizations are also given. Since RRA is not finitely axiomatizable (by Monk[64], or by Theorem 8 in Part I of this work), it follows from these finite axiomatizability results that relation composition is non–finitely axiomatizable relative any other sets of the non–Boolean operations of RRA.

The "reading" of Figure 2 is exactly the same as that of Figure 1. I.e.. a node represents the class of all algebras of binary relations, up to isomorphisms, where the greatest relation is an equivalence relation and the operations are those indicated along the path leading to the node, etc. We note that all classes represented by the nodes are varieties except the ones inside a box (those are only quasi–varieties).

FIGURE 2

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