COMPLEXITY OF EQUATIONS VALID IN ALGEBRAS OF RELATIONS

Part I: Strong nonfinitization.

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Abstract. We study algebras whose elements are relations, and the operations are natural "manipulations" of relations. This area goes back to 140 years ago to works of De Morgan, Peirce, Schröder (who expanded the Boolean tradition with extra operators to handle algebras of binary relations). Well known examples of algebras of relations are the varieties RCA_n of cylindric algebras of n-ary relations, $RPEA_n$ of polyadic equality algebras of n-ary relations, and RRA of binary relations with composition. We prove that any axiomatization, say E, of RCA_n has to be very complex in the following sense: for every natural number k there is an equation in E containing more than k distinct variables and all the operation symbols, if $2 < n < \omega$. Completely analogous statement holds for the case $n \geq \omega$. This improves Monk's famous non-finitizability theorem for which we give here a simple proof. We prove analogous nonfinitizability properties of the larger varieties SNr_nCA_{n+k}. We prove that the complementation-free (i.e. positive) subreducts of RCA_n do not form a variety. We also investigate the reason for the above "non-finite axiomatizability" behaviour of RCA_n. We look at all the possible reducts of RCA_n and investigate which are finitely axiomatizable. We obtain several positive results in this direction. Finally, we summarize the results and remaining questions in a figure. We carry through the same programme for $RPEA_n$ and for RRA. By looking into the reducts we also investigate what other kinds of natural algebras of relations are possible with more positive behaviour than that of the well known ones. Our investigations have direct consequences for the logical properties of the n-variable fragment L_n of first order logic. The reason for this is that RCA_n and $RPEA_n$ are the natural algebraic counterparts of L_n while the varieties SNr_nCA_{n+k} are in connection with the proof theory of L_n.

This paper appears in two parts. This is the first part, it contains the non-finite axiomatizability results. The second part contains finite axiomatizability results together with a figure summarizing the results in this area and the problems left open.

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CONTENTS

INTRODUCTION

Subject: This work deals with algebras of relations: the elements of our algebras are relations and the operations on these relations are those that arise naturally from the nature of relations. This topic has been investigated for almost 140 years with several, equally important, kinds of motivation (or intuition). Here we briefly sketch some of these motivations. (1) We can look at these algebras with purely algebraic "eyes", we can investigate them from purely algebraic point of view. E.g. these algebras served as the starting point for the universal algebraic concept "discriminator variety", and indeed, the theory of discriminator varieties can fruitfully be applied to algebras of relations (see e.g. Andréka–Jónsson–Németi[88], Németi[90]). Works in this line are e.g. Jónsson[82],[84], Jónsson–Tarski[52], Henkin–Monk–Tarski[HMT71],[HMT85], Givant[94], Andréka–Givant–Németi[95], Andréka–Bredikhin[95], Schein[70], Bredikhin–Schein[78], Bredikhin[77], Wagner[56], Börner[86], but already De Morgan[1860] can be considered as such. (2) These kinds of algebras arise frequently and naturally in computer science. E.g., the meaning, or semantics, of a program is often defined as the relation it defines between its inputs and outputs. Ways of building new programs from existing ones correspond then to natural operations on relations, e.g. the meaning of concatenation of two programs is just the so called relation composition of the meanings of the two programs. Kleene–algebras and dynamic algebras are typically such algebras. Works in this line are e.g. Pratt[79],[90],[91], Kozen[79],[91], Trnková–Reiterman[87], Jónsson[90], Redko[64], Németi[80],[81], [82], Daneczki[84], Fischer–Ladner[79]. Algebras of relations arise in computer science in numerous other ways, too, Andréka–Schobbens-Ryan[95] is an example. Another one is the following. A relational data–base at a given time can be considered as a relation, and the operations we perform on the data–base are natural operations on relations. Works in this line are e.g. Imielinski–Lipski[84], Düntsch[90],[90a],[93],[94], Comer[90],[93], Cosmadakis[87], Cirulis[88], Plotkin [88],[94]. For a survey on algebras of relations in computer science see e.g. Schmidt– Ströhlein[89] or Németi[90]. (3) In the investigation of first–order predicate logic, algebras of relations play an important role. The intuition here is very natural: the meanings of predicates are relations and logical connectives mean operations between these relations. For example, the meaning of the conjunction of two predicates is just the intersection of the meanings of the two predicates. It is very natural therefore to investigate algebras of relations in connection with first–order logic. This is so in such an extent, that the first versions of first–order logic were just algebras of relations (De Morgan, Peirce, Schröder, Löwenheim beginning with 1859), and first–order logic was defined on the basis of these investigations, cf. in this direction Pratt[92], Maddux[90a], and Anellis-Houser[88]. For the connection between first–order logic and our algebras we refer to section 4.3 of [HMT85], and Andréka-Németi-Sain[93], or Németi-Andréka[94]. (4) Algebras of relations play an important role in nonclassical logic, too. See e.g. van Benthem[89],[90], Orlowska[90], Venema[89],[90], Goldblatt[90], Marx–Pólos–Masuch[96], Mikulás[95], Marx[95].

Historical background: Algebras of relations of higher ranks have been investigated since the middle of the last century beginning with the works of De Morgan, Peirce, Macfarlane, Schröder and Löwenheim. For example, Peirce[1870] investigates n–ary relations for *arbitrary* n. There has always been an emphasis on investigating and/or trying to characterize the equations valid in these algebras. In the last century, these valid equations were called laws (e.g. "distributive law", "De Morgan law", "Peircean law" etc). The goal of obtaining a mathematically transparent, elegant characterization of these laws (equations valid in the algebras of relations) appeared very early. Schröder's impressive book contains a very large number of such laws, and it was sometimes conjectured that all laws (equations valid in algebras of relations) could be deduced from the laws in Schröder's book. In modern terminology, this conjecture would amount to conjecturing that the equations valid in the algebras of relations are finitely axiomatizable. This conjecture was open for a very long time[2] and there have been efforts trying to prove the conjecture. Among others, Tarski and his co–workers made efforts in the direction of trying to prove the conjecture. Monk[69] proved that the conjecture in this form is not true. The equations valid in the algebras of relations of higher ranks (defined the usual way) do not admit a finite axiomatization[3]. For the special case of binary relations he proved the result already five years earlier in Monk[64] (confirming a conjecture of Roger Lyndon and improving Jónsson's and Lyndon's method of connecting projective planes to algebraic logic i.e. to algebras of relations, cf. Jónsson[59], Lyndon [61]). Monk's negative results gave rise to two kinds of new questions raised more or less independently by Craig, Henkin, Jónsson, Tarski, and others. These two questions are the following: (i) If in this form the conjecture above is not true, then in what other form is it true? (ii)

[2] We will see that in a certain special form the conjecture is still open. Cf. e.g. in Németi[90] the subsections devoted to the finitization problem.

[3] A possible choice for the algebra of n-ary relations is the class RCA_n of representable cylindric algebras. Other choices are the class $RSCA_n$ of representable substitution-cylindric algebras and the class $RQPEA_n$ of representable quasi-polyadic equality algebras. These choices are strongly tied together, cf. Németi[90], Andréka–Givant–Mikulás–Németi–Simon[96], and Sain–Thompson[88] for discussions and comparisons of these choices.

What is the complexity (from various points of view) of the equations valid in the algebras of relations of higher ranks? (These questions will be made somewhat more concrete later.)

These two (groups of) questions have been studied for a long time. For example, a partial positive result in connection with (i) is in Lyndon[56] which gives a recursive enumeration of the equations valid in algebras of binary relations. (A different enumeration was given by Ralph McKenzie and was generalized by Monk to algebras of relations of arbitrary ranks.) A recent work giving recursive axiomatizations is Hirsch-Hodkinson[95]. The present work is also devoted to these two (groups of) questions.

One of the more concrete versions of (i) is the following. (i.1) If there is no finite set of axioms, perhaps still there is a finite *schema* (in some satisfactory sense) of axioms axiomatizing the equations valid in the algebras of relations (e.g. Jónsson[59], Monk[69], Henkin–Monk[74], [HMT85] Problems 4.1,4.16, Németi[90]). The book [HMT85] §4.1, pp.115–119 summarizes positive results in this direction each of which is found in the book unsatisfactory from some important point of view. That is why the quoted problems are stated in the same book at a later point. Monk[69] proved that a certain kind of schema will not work, and recently Németi–Sági[96] proved that no schema at all works for the infinite dimensional polyadic case. In this work we restrict ourselves to the finite-dimensional case to which the above Németi–Sági result does not apply. Positive results about both the algebraic form and the logical equivalent of the problem were proved in Simon[90] and Venema[90], Mikulás[95], Sain–Gyuris[96], Sain[87,92], Németi–Simon[96], Németi[96]. In the last two references it is shown, among others, that in a certain non-well founded set theory, some of the nonfinitely axiomatizable classes will become finitely axiomatizable. These are major improvements but do not settle the problem completely.

<u>In the present work</u> Thm.s 3,4,6,7,8 point in the direction that it will be quite hard to find a finite schema of axioms with the desired property. Roughly speaking, if Σ is a set of equations axiomatizing the class RCA_ω of algebras of relations of arbitrary ranks, then for every number n, there is an equation $\sigma \in \Sigma$ such that σ contains more than n operation symbols and more than n variables. Moreover, complementation "-", and either "$\cup$" or "$\cap$", and one of the identity relations d_{ij} must occur in σ. Further, to every choice of n and d_{ij} there must exist such a σ in Σ. Similar results are obtained for the other distinguished kinds (RCA_n with $2 < n < \omega$, $RPEA_n, RRA$) of algebras of relations. The above quoted results on the complexity of Σ (longer and longer equations etc.) can be interpreted to show what kind of schema will not work in solving part (i.1) of problem (i) mentioned earlier. This generalizes the negative result on schemata in Monk[69] and solves the

problem on p.342 there. These theorems also provide solutions for problems formulated in Jónsson[59], Johnson[69] and in Henkin–Monk–Tarski[HMT71],[HMT85].

The above quoted Thm.s 3,4,6,7,8 (concerning Σ) are also relevant to question (ii) which concerns the complexity of the (possible axiomatizations of the) equational theory of the algebras of relations. Further results in this work concerning question (ii) are summarized on Figures 1,2. Roughly speaking, Figure 1 addresses the question "Which ones of our operations on relations bring in an infinity of new axioms?". The answer may depend on which operations are added to our algebras first. Since we wanted to represent all the possibilities, Figure 1 is of the form of a tree.

The second part of question (i) is the following. (i.2) Could we choose the basic operations on relations of higher ranks such that

(a) the operations would remain invariant under permutations (see Thm.5 for definition),
(b) the new class of algebras of relations would become a finitely axiomatizable variety or quasi–variety, and
(c) the most important classical operations on relations would be term definable.

This problem was raised independently e.g. by Bjarni Jónsson in 1984, in Henkin –Monk[74], in Tarski–Givant[87], and is discussed in Németi[90] beginning with Remark 2 therein.

In Theorem 5 we prove a negative result in this direction, improving Biró's one Biró[89] and showing that Sain's positive result can not be improved in certain ways. (Sain[87a] was able to give a positive result to the variant of (i.2) in which condition (c) is not extended to the identity relation but is extended to the substitution and permutation operations, like $R \mapsto R^{-1}$, instead. Cf. also Sain–Gyuris[96], Sain[92], Németi[96].)

Let us briefly return to problem (i.1). In this connection, Jónsson[59] investigated the subreducts of the usual algebras RRA of binary relations obtained by omitting the operations "-" and "$\cup$" from RRA. So, the extra–Boolean operations remain the same and of the Booleans we keep only intersection (or meet). For these subreducts of RRA, Jónsson[59] gave an infinite set Σ_∞ of quasi–equations axiomatizing the class in question. Though Σ_∞ does not follow from any of its finite subsets, its mathematical content is more explicit and understandable than that of the axiomatizations discussed way above. Jónsson[59] raised the question whether an axiomatization of RRA can be obtained from Σ_∞ by adding finitely many quasi–equations to Σ_∞. Because of the relative simplicity of Σ_∞, a positive answer to this question would have yielded a kind of a positive solution to Problem (i.1). We show that the answer to Jónsson's question is in the negative. (Theorem

8) Moreover, we prove that if $\Sigma_\infty \cup \Sigma_1$ is an axiomatization of RRA consisting of quasi–equations, then there is a $\sigma \in \Sigma_1$ such that all Boolean operations together with relation composition "$\circ$" occur in σ as operation symbols. (Since $\cup$ is expressible with "-" and "$\cap$", by all Booleans we mean either "-" and "$\cup$", or "-" and "$\cap$", or "$\cap$" and "$\cup$".)

We try to use conventional notation. We introduce less usual notation at their first occurrence in the text. We refer to items in the bibliography by names of authors and by years. There are two exceptions: [HMT71] and [HMT85] refer to Henkin-Monk-Tarski[HMT71] and to Henkin-Monk-Tarski[HMT85] respectively.

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NONFINITE AXIOMATIZABILITY OF ALGEBRAS OF RELATIONS OF HIGHER RANK.
(Simple proof for non–finitizability in algebraic logic.)

Boolean set algebras can be considered as algebras of unary relations, and the theory of Boolean algebras proved to be very useful in different areas of mathematics. Beginning with De Morgan[1860], many mathematicians made efforts to generalize Boolean algebras to algebras of relations of higher arity (e.g. Peirce[1870], Schröder[1895]). Continuing this line of research, Tarski, around 1949, defined the class RCA_n of algebras of n–ary relations as a candidate for playing the same role in first–order logic what Boolean algebras played in propositional logic. The first, most natural task was to generalize the finite axiom system known for Boolean algebras to axiom systems for RCA_n. Tarski proposed several such strengthenings of the axiom system of Boolean algebras, but it always turned out that the proposed axiom system was not strong enough to axiomatize RCA_n. Finally, Monk proved in 1969 that RCA_n is not finitely axiomatizable for $n \geq 3$. (The hypothesis $n \geq 3$ is important here because Henkin gave finite axiom systems for $RCA_n, n \leq 2$.) Monk's result is still one of the most important theorems in algebraic logic and trying to understand the equational theory of RCA_n is one of the central questions of the field.

If RCA_n is not finitely axiomatizable, then the next question is to see how complicated its equational theory is. W. Craig asked around 1968 whether RCA_n can be axiomatized with equations using only three variables. This was a natural question because of the following. Boolean algebras are axiomatized with equations using three variables, and Diamond–McKinsey[47] proved that Boolean algebras cannot be axiomatized with two variables, thus RCA_n cannot be axiomatized with two variables, either. At the same time, all the natural non–trivial equations valid in RCA_n that one could think of were consequences of ones using only three variables[4], so there was a possibility of obtaining an affirmative answer to Craig's question.

Craig's above problem is the first one in the last section of Monk[69], where he raises several open problems concerning the structure of the equational theory of RCA_n. We will give a negative answer to Craig's problem published in Monk[69], see Theorem.1 below.

Theorem 1 is a corollary of the much stronger Theorem 3 in this section. Yet, we give a separate proof for Theorem 1. The reason for this is that the proof we give for Thm.1 is a proof for Monk's original theorem, too, and at the same time this proof is much simpler than the ones existing in the literature. Because of its importance in algebraic logic, there already were efforts to simplify Monk's original proof. Works in this line are e.g. Comer[85], Maddux[89]. The proofs of Monk, Comer, Maddux all use ultraproducts, they construct nonrepresentable algebras an ultraproduct of which is representable. Our proof does not use ultraproduct, we construct big nonrepresentable algebras the small–generated subalgebras of which are representable. This latter method was used in Jónsson[84] Thm.3.5.6 to show that the class of representable relation algebras cannot be axiomatized by using finitely many variables.

Another reason for including a separate proof for Theorem 1 is that the proofs of later stronger theorems are refinements of the one of Theorem 1, and we believe that these refinements can be understood easier after reading the simple proof of Theorem 1.

Let U be a set. Then $\mathcal{P}(U)$ denotes the powerset of U and $\mathfrak{P}(U)$ denotes the Boolean algebra of all subsets of U. Let n be an ordinal. Then ${}^nU$ is the set of all U–termed sequences of length n, and thus $\mathcal{P}({}^nU)$ is the set of all n–ary relations on U. Let $s \in {}^nU, i < n$ and $u \in U$. Then $s(i/u)$ denotes the sequence we obtain from s by replacing its i–th value with u. Let $i, j < n$. The unary operation c_i^U on n–ary relations over U and the constant $d_{ij}^U \in \mathcal{P}({}^nU)$ are defined as

[4] The equations in [HMT71]2.6.11 use many variables, but they all follow from $c_0(x \cdot y) \cdot c_0(x - y) \leq c_0 c_1(c_1 x \cdot s_1^0 c_1 x - d_{01})$.

$c_i^U(X) = \{s \in {}^nU : s(i/u) \in X \text{ for some } u \in U\}$,
$d_{ij}^U = \{s \in {}^nU : s_i = s_j\}$.

We often omit the upper indices U. Because of their geometrical meaning, the operations c_i and d_{ij} are also called cylindrifications and diagonal constants, respectively.

Let

$$FullRA_n = \{\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U \rangle_{i,j<n} : U \text{ is a set}\}.$$

The class of all subalgebras of elements of $FullRA_n$ is called the class Cs_n of n–dimensional cylindric set algebras. The variety generated by $FullRA_n$ is the class of all subdirect products of Cs_n's[5] and is called RCA_n, the class of representable cylindric algebras of dimension n. ω denotes the smallest infinite ordinal.

THEOREM 1. *Let $n \geq 3$ be an arbitrary (possibly infinite) ordinal. Then RCA_n is not axiomatizable with a set Σ of quantifier–free formulas such that Σ contains only finitely many variables.*

Proof: PLAN: For all $k < \omega$ we will construct an algebra $\mathfrak{A}_k$ such that

a) $\mathfrak{A}_k \notin RCA_n$
b) every k–generated subalgebra of $\mathfrak{A}_k$ is in RCA_n.

This will prove the theorem because of the following: Assume that Σ is a set of quantifier–free formulas such that Σ contains at most k variables ($|var(\Sigma)| \leq k < \omega$) and $RCA_n \models \Sigma$. Then $\mathfrak{A}_k \models \Sigma$ is easily seen as follows. Every k–generated subalgebra of $\mathfrak{A}_k$ is in RCA_n, and the validity of Σ in any algebra $\mathfrak{B}$ depends only on the validity of Σ in the k–generated subalgebras of $\mathfrak{B}$ because $|var(\Sigma)| \leq k$ and Σ contains no quantifiers. Thus $\mathfrak{A}_k \models \Sigma$. However, $\mathfrak{A}_k \notin RCA_n$, hence $Mod(\Sigma) \nsubseteq RCA_n$ showing that Σ does not axiomatize RCA_n.

First we will give the proof for $n \geq \omega$, because in this case we can use a simpler construction, and thus the idea of the proof will be easier to see. After this, we will give a proof that works for all $n \geq 3$.

The proof for $n \geq \omega$:

[5] For $n < \omega$, this is a result of Tarski[55], originally proved by using representation theory. Today, using the theory of discriminator varieties, a fairly simple proof can be given for $n < \omega$, see e.g. Németi[90]. For $n \geq \omega$, this is a result in Henkin–Tarski[61] (Thm.2.19), where they ask for a purely algebraic proof which can be found in Henkin–Monk–Tarski–Andréka–Németi[81], p.100, Thm.I.7.15, or in [HMT85]3.1.103. This proof further can be simplified following the outline in Németi[90].

CONSTRUCTION OF $\mathfrak{A}_k$: Let $m \geq 2^k$, $m < \omega$ and let $\langle U_i : i < n \rangle$ be a system of disjoint sets each of cardinality m. Let

$U = \bigcup\{U_i : i < n\}$, let
$q \in \underset{i<n}{\times} U_i = \{s \in {}^nU : s_i \in U_i \text{ for all } i < n\}$ be arbitrary,
$R = \{z \in \underset{i<n}{\times} U_i : |\{i < n : z_i \neq q_i\}| < \omega\}$, and let
$\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{P}({}^nU), c_i, d_{ij} \rangle_{i,j<n}$ generated by R.

Then R is an atom of $\mathfrak{A}'$ because of the following: For any two sequences $s, z \in R$ there is a permutation $\sigma : U \rightarrowtail\!\!\!\!\twoheadrightarrow U$ of U taking s to z and fixing R, i.e. $\sigma \circ s = z$ and $R = \{\sigma \circ p : p \in R\}$. If σ is a permutation of U fixing R, then σ fixes all the elements generated by R because the operations of RCA_n are permutation invariant. Thus if $a \in A'$ and $s \in a \cap R$ then $R \subseteq a$, showing that R is an atom of $\mathfrak{A}'$.

We now "split R into $m+1$ new 'abstract' atoms R_j each imitating R". I.e. let $R_j, j \leq m$ be $m+1$ distinct elements, not in A', and let $\mathfrak{A}$ be an algebra of the same similarity type as $\mathfrak{A}'$ such that

$\mathfrak{A}' \subseteq \mathfrak{A}$, the Boolean part of $\mathfrak{A}$ is a Boolean algebra,
R_j are atoms of $\mathfrak{A}$ and $c_iR_j = c_iR$ for all $j \leq m, i < n$,
each element of $\mathfrak{A}$ is a join of an element of $\mathfrak{A}'$ and of some R_j's,
c_i distributes over joins, for any $i < n$, i.e. $c_i(x + y) = c_ix + c_iy$ for all $x, y \in A$.

By the above, we have constructed our algebra $\mathfrak{A} = \mathfrak{A}_k$.

CLAIM 1. $\mathfrak{A} \notin RCA_n$.

Proof: For $i, j < n, i \neq j$, let $s^i_j(x) = c_i(d_{ij} \cap x)$ and let $s^i_i(x) = x$. Let

$$\tau(x) = \bigcap_{i \leq m} s^0_i c_1 ... c_m x \cap \bigcap_{i<j\leq m} -d_{ij}.$$

Then $\mathfrak{A}' \models \tau(R) = 0$, because of the following. Let ${}^nU^{(q)} = \{z \in {}^nU : |\{i < n : z_i \neq q_i\}| < \omega\}$.

$$c_1...c_m R = U_0 \times \underbrace{U \times \cdots \times U}_{m \text{ times}} \times U_{m+1} \times ... \cap {}^nU^{(q)},$$
$$s^0_i c_1 ... c_m R = U \times ... \times U_0 \times ... \times U \times U_{m+1} \times ... \cap {}^nU^{(q)},$$
$$\bigcap_{i\leq m} s^0_i c_1...c_m R = \underbrace{U_0 \times \cdots \times U_0}_{m+1 \text{ times}} \times U_{m+1} \times ... \cap {}^nU^{(q)} \ .$$

Then by $|U_0| \leq m$ we have that there is no repetition–free sequence in $U_0 \times ... \times U_0$ ($m + 1$ times). Thus $\mathfrak{A}' \models \tau(R) = 0$.

Then $\mathfrak{A} \models \tau(R) = 0$ by $\mathfrak{A}' \subseteq \mathfrak{A}$. Assume that $\mathfrak{A}$ is represented somehow. Then there is a homomorphism $h : \mathfrak{A} \to \langle \mathfrak{P}({}^nW), c_i, d_{ij}\rangle_{i,j<n}$ for some set W such that $h(R) \neq \emptyset$. We will derive a contradiction. By $h(R) \neq \emptyset$ there is $s \in h(R)$. By $R \leq c_0 R_i$ we have $h(R) \subseteq c_0 h(R_i)$, so there is w_i such that $s(0/w_i) \in h(R_i)$, for all $i \leq m$. These w_i's are different from each other since the R_i's are disjoint from each other, and so the $h(R_i)$'s are disjoint from each other. Consider the sequence

$$z = \langle w_0, w_1, ..., w_m, s_{m+1}, ...\rangle.$$

Now $z \in \tau(h(R))$ can be seen as follows. Let $i, j \leq m, i \neq j$. Then $z \in -d_{ij}$ by $w_i \neq w_j$. Next we show $z \in s^0_i c_1...c_m h(R)$. By the definition of w_i, $\langle w_i, s_1, ...\rangle \in h(R_i) \subseteq h(R)$, so $\langle w_i, w_1, ..., w_m, s_{m+1}, ...\rangle \in c_1...c_m h(R)$, and thus $z \in c_0(d_{0i} \cap c_1...c_m h(R)) = s^0_i c_1...c_m h(R)$. Therefore $z \in \tau(h(R))$, a contradiction by $\mathfrak{A} \models \tau(R) = 0$. **QED**(Claim 1)

CLAIM 2. *The k–generated subalgebras of $\mathfrak{A}$ are in RCA_n.*

Proof: Let $G \subseteq A$, $|G| \leq k$. For all $i, j \leq m$ define

$$R_i \equiv R_j \quad \text{iff} \quad (\forall g \in G)[R_i \leq g \leftrightarrow R_j \leq g].$$

Then $\equiv$ is an equivalence relation on $\{R_j : j \leq m\}$ which has $\leq 2^k$ blocks by $|G| \leq k$. Let p denote the number of blocks of $\equiv$, i.e. $p = |\{R_j/\equiv : j \leq m\}| \leq 2^k \leq m$. Define

$$B = \{a \in A : (\forall i, j \leq m)([R_i \equiv R_j \text{ and } R_j \leq a] \Rightarrow R_i \leq a)\}.$$

We now show that B is closed under the operations of $\mathfrak{A}$. Let $i < l < n$.

1) B clearly is closed under the Boolean operations.
2) $d_{il} \in B$ since $R_j \not\leq d_{il}$ for all $j \leq m$.
3) Clearly, $A' \subseteq B$ (since R is an atom of $\mathfrak{A}'$), and $c_i a \in A'$ for all $a \in A$. Thus $c_i b \in B$ for all $b \in B$.

Let $\mathfrak{B} \subseteq \mathfrak{A}$ be the subalgebra of $\mathfrak{A}$ with universe B. By $G \subseteq B$, it is enough to show that $\mathfrak{B} \in RCA_n$.

We will define an embedding $h : \mathfrak{B} \rightarrowtail \langle \mathfrak{P}({}^nU), c_i, d_{ij}\rangle_{i,j<n}$. Notation: If $\mathfrak{B}$ is any Boolean algebra, $X \subseteq B$ and $x \in B$, then we say that X is a <u>partition</u> of x (in $\mathfrak{B}$) provided that $\sum X = x$, and for all different $a, b \in X$ we have $a \cdot b = 0 \neq a$. Let $\{y_j : j < p\} = \{\sum(R_j/\equiv) : j \leq m\}$. Then $\{y_j : j < p\}$ is a partition of R in $\mathfrak{B}$, $c_i y_j = c_i R$ for all $j < p$ and $i < n$ and every element of $\mathfrak{B}$ is a join of some element of A' and of some y_j's. So, $\mathfrak{B}$ looks like

We are going to define the images of the y_j's under the embedding h. Let $Q = \{0, 1, ..., m-1\}$ and let $(Q, +, 0)$ be any commutative group. For each $i < n$ let $f_i : U_i \rightarrowtail\!\!\!\!\rightarrow Q$ be a bijection such that $f_i(q_i) = 0$. For $j < m$ define

$$R''_j = \{z \in R : \sum\langle f_i(z_i) : i < n\rangle = j\}.$$

(Here Σ denotes taking sum in the group $(Q,+,0)$.) Then $R''_j \subseteq {}^nU$ and it is not difficult to check that the R''_j's are disjoint from each other and

$$c_i R''_j = c_i R \quad \text{for all } i < n.$$

Define for all $j < p-1$

$$R'_j = R''_j$$
$$R'_{p-1} = \bigcup \{R''_j : p-1 \leq j < m\}.$$

We are ready to define the embedding h of B.: We define for all $b \in B$

$$h(b) = (b - R) \cup \bigcup \{R'_j : j < p, y_j \leq b\}.$$

It is not difficult to check that h is an embedding $h : \mathfrak{B} \rightarrowtail \langle \mathfrak{P}({}^nU), c_i, d_{ij}\rangle_{i,j<n}$.

In more detail: h preserves $\cup, -$. $h(b) = 0$ implies $b = 0$. $h(d_{il}) = d_{il}$. Now we check $c_i h(b) = h(c_i b)$.

$$\begin{aligned} c_i h(b) &= c_i[(b - R) \cup \bigcup \{R'_j : y_j \leq b\}] &&= \\ &= c_i(b - R) \cup \bigcup \{c_i R'_j : y_j \leq b\} &&= \\ &= c_i(b - R) \cup \bigcup \{c_i y_j : y_j \leq b\} &&= \\ &= c_i[(b - R) \cup \bigcup \{y_j : y_j \leq b\}] &&= c_i b. \end{aligned}$$

$h(c_i b) = (c_i b - R) \cup \bigcup \{R'_j : y_j \leq c_i b\} = c_i b$, since $(\exists j) y_j \leq c_i b$ iff $R \leq c_i b$, and $R \not\leq c_i b$ iff $c_i b = c_i b - R$. **QED**(Claim 2)

By the above we have proved Theorem 1 for $n \geq \omega$.

REMARK 1. In the above proof, we used $n \geq \omega$ only in the proof of Claim 1, where we expressed $|U_0| \leq m$ by translating the formula

$$\neg \exists v_0 \ldots v_m (U_0(v_0) \wedge \ldots \wedge U_0(v_m) \wedge \bigwedge_{i<j\leq m} v_i \neq v_j)$$

into an RCA_n–term. ($\bigcap_{i \leq m} s^0_i c_1 \ldots c_m R \cap \bigcap_{i<j\leq m} - d_{ij} = 0$ is a direct translation of this.) In the above formula we use $m+1$ variables, therefore we need $m+1$ "indices" in the translated term, i.e. we need $n \geq m+1$.

Therefore in the case $n \leq m$ we cannot use the above straightforward method for "counting U_0". In the case $n \leq m$ we will use the following idea: If $U_0 \times U_0$ is the union of m functions, then $|U_0| \leq m$. The formula expressing $U_0 \times U_0$ is the union of m functions $F_0, ..., F_{m-1}$ is the following.

$$(U_0(v_0) \wedge U_0(v_1) \rightarrow \bigvee_{j<m} F_j(v_0 v_1)) \wedge \bigwedge_{j<m} (F_j(v_0 v_1) \wedge F_j(v_0 v_2) \rightarrow v_1 = v_2).$$

The translation of this formula will be the equations

$$c_1 R \cap s_1^0 c_1 R \subseteq \bigcup \{F_j : j < m\}, \quad F_j \cap s_2^1 F_j \subseteq d_{12}, \quad for\ j < m.$$

■

The following proof works for all $n \geq 3$. However, in order to make the proof shorter (i.e. to avoid writing down some details needed only in the case $n \geq \omega$), we will assume $n < \omega$.

The proof for $3 \leq n < \omega$:

CONSTRUCTION OF $\mathfrak{A}_k$: Let $m > 2^k$, $m < \omega$ and let $\langle U_i : i < n \rangle$ be a system of disjoint sets each of cardinality m such that $U_0 = \{0, ..., m-1\}$. Let

$U = \bigcup \{U_i : i < n\}$,
$R = \times_{i<n} U_i = \{s \in {}^nU : s_i \in U_i \text{ for all } i < n\}$,
$F = \{s \in {}^nU : s_0, s_1 \in U_0,\ s_1 = s_0 + 1 (mod\ m)\}$, and let
$\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{P}({}^nU), c_i, d_{ij} \rangle_{i,j<n}$ generated by R, F.

Now R is an atom of $\mathfrak{A}'$ and this can be seen exactly as in the case of $n \geq \omega$. Let $\mathfrak{A}$ be the algebra we obtain from $\mathfrak{A}'$ by splitting R into $m+1$ new atoms R_j, $j \leq m$.

CLAIM 3. $\mathfrak{A} \notin RCA_n$.

Proof: For any j define

$$F_j = \{s \in {}^nU : s_0, s_1 \in U_0,\ s_1 = s_0 + j (mod\ m)\}.$$

We will show that $F_j \in A'$, for all j. $F = F_1 \in A'$ by definition. Assume that $F_j \in A'$, we will show that $F_{j+1} \in A'$, too. It is easy to check that

$$F_{j+1} = c_2(s_2^1 F_j \cap s_2^0 F)$$

as follows. $s_2^1 F_j = \{s \in {}^nU : s_2 = s_0 + j (mod\ m)\}$, $s_2^0 F = \{s \in {}^nU : s_1 = s_2 + 1 (mod\ m)\}$, hence $s_2^1 F_j \cap s_2^0 F = \{s \in {}^nU : s_0, s_1, s_2 \in U_0,\ s_2 = s_0 +$

$j(mod\ m),\ s_1 = s_2 + 1(mod\ m)\}$, therefore $c_2(s_2^1 F_j \cap s_2^0 F) = \{s \in {}^nU : s_0, s_1 \in U_0,\ s_1 = s_0 + j + 1(mod\ m)\} = F_{j+1}$. By $F_0 = F_m$, we have shown

$$(1) \qquad F_0, ..., F_{m-1} \in A'.$$

Also, each F_j satisfies

$$(2) \qquad F_j \cap s_2^1 F_j \subseteq d_{12}$$

by $F_j \cap s_2^1 F_j = \{s \in {}^nU : s_0, s_1, s_2 \in U_0,\ s_1 = s_0 + j(mod\ m),\ s_2 = s_0 + j(mod\ m)\} \subseteq d_{12}$. Finally, we also have

$$U_0 \times U_0 \times {}^{n-2}U = \bigcup\{F_j : j < m\}$$

since for any $u, v \in U_0$ if $j = v - u(mod\ m)$ then $v = u + j$, hence $s(0/u, 1/v) \in F_j$, for any $s \in {}^nU$. Thus

$$(3) \qquad c_1 R \cap s_1^0 c_1 R \subseteq \bigcup\{F_j : j < m\}$$

since $c_1 R \cap s_1^0 c_1 R \subseteq \{s \in {}^nU : s_0, s_1 \in U_0\}$. By $\mathfrak{A}' \subseteq \mathfrak{A}$, (2)–(3) hold in $\mathfrak{A}$, too.

Assume that $\mathfrak{A}$ is represented somehow. Then for some W there is a homomorphism $h : \mathfrak{A} \to \langle \mathfrak{P}({}^nW), c_i, d_{ij}\rangle_{i,j<n}$ such that $h(R) \neq 0$. We will derive a contradiction. By $h(R) \neq 0$ there is $s \in h(R)$. By $R \leq c_0 R_i$ we have $h(R) \subseteq c_0 h(R_i)$, so there is w_i such that $s(0/w_i) \in h(R_i)$, for all $i \leq m$. These w_i's are different from each other since the R_i's are disjoint from each other, and so the $h(R_i)$'s are disjoint from each other. Consider the sets

$H = \{w_i : i \leq m\}$ and
$G = \{w \in H : s(1/w) \in h(F_j)$ for some $j < m\}$.

Then $|H| = m + 1$. $|G| \leq m$ can be seen as follows. Assume $w, w' \in G$ such that $s(1/w), s(1/w') \in h(F_j)$. Then $s(1/w, 2/w') \in c_2 h(F_j) \cap s_2^1 c_2 h(F_j) = h(c_2 F_j \cap s_2^1 c_2 F_j) \subseteq h(d_{12}) = d_{12}$, therefore $w = w'$. We have seen $|G| \leq m$. By $|H| = m+1$ then there is $i \leq m$ with $w_i \in H \setminus G$. Consider the sequence

$$z = \langle s_0, w_i, s_2, s_3, ...\rangle = s(1/w_i).$$

Then $z \in c_1 h(R) \cap s_1^0 c_1 h(R)$ by $s \in h(R), s(0/w_i) \in h(R_i) \subseteq h(R)$. But $z \notin \bigcup\{h(R_j) : j < m\}$ by $w_i \notin G$. Thus $c_1 h(R) \cap s_1^0 c_1 h(R) \not\subseteq \bigcup\{h(F_j) : j < m\}$ contradicting $c_1 R \cap s_1^0 c_1 R \subseteq \bigcup\{F_j : j < m\}$. Therefore $\mathfrak{A}$ is not representable, i.e. $\mathfrak{A} \notin RCA_n$. **QED**(Claim 3)

CLAIM 4. *The k–generated subalgebras of $\mathfrak{A}$ are in RCA_n.*

The proof of Claim 4 is exactly the same as that of Claim 2. We omit the proof of Claim 4.

QED(Theorem 1)

REMARK 2. We can obtain from the proof of Theorem 1 a set of equations "witnessing" nonfinite axiomatizability of RCA_n. Below, we will give a sequence $\langle e_m : m < \omega\rangle$ of stronger and stronger equations valid in RCA_n such that any first–order formula valid in RCA_n can imply only finitely many of the equations $e_m, m < \omega$.

The equations we can get from the proof of Theorem 1 if $n \geq \omega$ are

$$\prod_{i\leq m} c_0(x \cdot x_i \cdot \prod_{i\neq j\leq m} -x_j) \ \leq\ c_0 ... c_m(\prod_{i,j\leq m, i\neq j} s^0_i c_1 ... c_m x \cdot -d_{ij}).$$

Let us denote the above equation by e_m. Then $RCA_n \models \{e_m : m < \omega\}$ follows by the proof of Claim 1. Let $\mathfrak{A}$ be the algebra constructed in the proof of Theorem 1 (case $n \geq \omega$) with $|U_0| = m$. Then $\mathfrak{A} \not\models e_m$ by the proof of Claim 1, while $\mathfrak{A} \models \{e_i : i < m\}$ can be proved by a similar argument as in the proof of Claim 2. Thus the sequence $\langle e_m : m < \omega\rangle$ is strictly "getting stronger" in the sense that $\{e_i : i < m\} \not\models e_m$ for all $m < \omega$.

Let Σ be a set of quantifier–free formulas valid in RCA_n, and using only k variables. Let $m \geq 2^k$. Then $\mathfrak{A} \models \Sigma$ by Claim 2, while $\mathfrak{A} \not\models e_m$, showing that $\Sigma \not\models e_m$ for all $m \geq 2^k$. Thus Σ can imply only finitely many of the equations $e_m, m < \omega$. Let φ be any first–order formula valid in RCA_n. Then φ follows from a finite set Σ of equations valid in RCA_n, because RCA_n is a variety (and by the compactness theorem). Thus φ can imply only finitely many of the equations $e_m, m < \omega$.

The analogous equations for all $n \geq 3$ are the following. Let $x, x_0, ..., x_m, y_0, ... y_{m-1}$ be variables and let ε_m denote the following equation[6]

$$\prod_{i\leq m} c_0(x \cdot x_i \cdot \prod_{i\neq j\leq m} -x_j) \ \leq\ c_0 c_1 c_2 \prod_{j<m} c_1 x \cdot s^0_1 c_1 x - [c_2 y_j - c_2(s^1_2 c_2 y_j - d_{12})].$$

■

[6] The following simplification is due to Ágnes Kurucz: $lhs \leq c_0 c_1 c_2([c_1 x \cdot s^0_1 c_1 x - \sum c_2 y_j] + \sum_{j<m}(c_2 y_j \cdot s^1_2 c_2 y_j - d_{12}))$.

The class CA_n of "n–dimensional cylindric algebras" is defined with the following finite set (of schemas) of equations valid in RCA_n. For any $i, j, k < n$, i, j, k different

the Boolean equations,

$x \leq c_i x = c_i c_i x, \quad c_i(x + y) = c_i x + c_i y, \quad c_i - c_i x = -c_i x,$

$c_i c_j x = c_j c_i x,$

$d_{ii} = 1, \quad d_{ij} = d_{ji} = c_k(d_{ik} \cdot d_{kj}), \quad d_{ij} \cdot c_i(d_{ij} \cdot x) \leq x.$

Clearly, $RCA_n \subset CA_n$. The class CA_n is considered as an "approximation" of RCA_n, and it is of interest to see what kinds of equations are valid in RCA_n that do not hold in CA_n. The equations in Remark 2 are all such[7]. CA_n is the first member in a sequence of varieties SNr_nCA_m approximating RCA_n. We now define the classes SNr_nCA_m.

If $\mathfrak{A} \in CA_m$ and $n \leq m$ then it is easy to check that the subset $Nr_n\mathfrak{A} = \{x \in A : x = c_j x \text{ for all } j \geq n, j < m\}$ is closed under the operations of CA_n. Then one defines[8]

$$\mathfrak{Nr}_n\mathfrak{A} = \langle Nr_n\mathfrak{A}, +^{\mathfrak{A}}, -^{\mathfrak{A}}, c_i^{\mathfrak{A}}, d_{ij}^{\mathfrak{A}} \rangle_{i,j<n} \quad \text{and}$$
$$SNr_nCA_m = \{\mathfrak{A} : \mathfrak{A} \subseteq \mathfrak{Nr}_n\mathfrak{B} \text{ for some } \mathfrak{B} \in CA_m\}.$$

It is easy to check that $CA_n = SNr_nCA_n \supseteq SNr_nCA_{n+1} \supseteq \ldots \supseteq RCA_n$. Henkin[55] proved that $RCA_n = \bigcap\{SNr_nCA_{n+m} : m < \omega\} = SNr_nCA_{n+k}$ for all $k \geq \omega$ and Monk[69] proved that $RCA_n \neq SNr_nCA_{n+m}$ for all $m < \omega$. Monk[61] proved that SNr_nCA_m are varieties[9].

So the varieties SNr_nCA_{n+m} approximate the variety RCA_n better and better as m approaches infinity, but they never reach RCA_n. Monk[69] asked whether SNr_nCA_{n+m} is finitely axiomatizable (if $m > 0$). The following theorem gives a negative answer for $m \geq 2$. For $m = 1$ an affirmative answer is given in Andréka[90c]. We note that Thm.2 is a generalization of Thm.1, by $RCA_n = SNr_nCA_{n+\omega}$.

[7] With the exception of e_0, ε_0.

[8] Cf. e.g. [HMT71]2.6.28. This class was introduced by Henkin.

[9] For proof see also [HMT71]2.6.32(ii). Ferenczi[92] investigates the question: for which superclasses $\mathcal{K}$ of $CA_{n+\omega}$ is it true that $RCA_n = SNr_n\mathcal{K}$. These questions are related to the proof theory of first–order logic.

THEOREM 2. *Let $n \geq 3$ and $m \geq 2$. Then SNr_nCA_{n+m} is not finitely axiomatizable. Moreover SNr_nCA_{n+m} is not axiomatizable with any set of quantifier-free formulas containing only finitely many variables.*

Proof. Let $\langle \varepsilon_m : m < \omega \rangle$ be the second sequence of equations in Remark 2. We will show that

(1) $SNr_nCA_{n+2} \models \varepsilon_m$ for all $m < \omega$.
(2) $\mathfrak{A} \not\models \varepsilon_m$, where $\mathfrak{A}$ is the algebra constructed in the proof of Thm.1 for the case $n \geq 3$, with $|U_0| = m$.

This will prove the theorem because of the following. Let $2 \leq p$ and assume that Σ is a set of quantifier-free formulas valid in SNr_nCA_{n+p} and using only k variables. Let $m \geq 2^k$ and let $\mathfrak{A}$ be the algebra constructed in the proof of Thm.1 for the case $n \geq 3$ with $|U_0| = m$. Then $\mathfrak{A} \models \Sigma$ by Claim 4 because $RCA_n \subseteq SNr_nCA_{n+p}$. But $\mathfrak{A} \not\models \varepsilon_m$ by (2). Thus $\mathfrak{A} \notin SNr_nCA_{n+p}$ by (1) because $SNr_nCA_{n+p} \subseteq SNr_nCA_{n+2} \models \varepsilon_m$. This shows that Σ is not an axiomatization of SNr_nCA_{n+p}.

We now start proving (1). Being in SNr_nCA_{n+2} means that, in deriving equations, we can use the operations c_i, d_{ij} for $i, j \in \{0, 1, \dots, n, n+1\}$, and apart from the cylindric equations for these we also can use $x = c_nx = c_{n+1}x$ if x is a variable. Let us define the following term[10]:

$$x \circ y = c_n(s^1_n x \cdot s^0_n y).$$

Clearly, $x \circ 0 = 0$. We will prove the following:

(3) $x \cdot c_0 y \leq (c_1 x \cdot s^0_1 c_1 y) \circ y$
(4) $x \circ y \leq c_1 x$
(5) $x \circ (y + z) = (x \circ y) + (x \circ z), \quad (x + y) \circ z = (x \circ z) + (y \circ z)$
(6) $x \circ (y \cdot z) = (x \circ y) \cdot (x \circ z)$ if $x \cdot s^1_2 x \leq d_{12},\ x = c_2 x.$

(Of these, (3)–(5) hold in SNr_nCA_{n+1}, but (6) does not hold in SNr_nCA_{n+1}.) Assuming now (3)–(6) above we prove ε_m, and after this we will derive (3)–(6).

To simplify notation, let us introduce

$$X_i = x \cdot x_i \cdot \prod_{j \neq i} - x_j,$$
$$Y_j = c_2 y_j - c_2(s^1_2 c_2 y_j - d_{12}),$$
$$X = \sum\{X_i : i \leq m\},$$
$$u = c_1 X \cdot s^0_1 c_1 X,$$
$$g = u \cdot \prod\{-Y_j : j < m\}.$$

[10] This is the n–ary version of composition defined e.g. in [HMT85]5.3.7.

Then $X \leq x$, and therefore it is enough to prove $\prod_{i \leq m} c_0 X_i \leq c_0 c_1 g$. Clearly the X_i's are disjoint, $X_i \leq X$. Also, $u \leq g + \sum_{j<m} Y_j$ and we note that $c_i z \cdot c_i y = c_i(z \cdot c_i y)$ holds in CA_n. Also, it is easy to show that $Y_j \cdot s^1_2 Y_j \leq d_{12}$, $Y_j = c_2 Y_j$ as follows[11] . Let $z = s^1_2 c_2 y_j - d_{12}$. Then $c_2 Y_j = c_2(c_2 y_j - c_2 z) = c_2 y_j \cdot c_2 - c_2 z = c_2 y_j - c_2 z = Y_j$. Also

$$\begin{aligned}
&Y_j \cdot s^1_2 Y_j \cdot -d_{12} \leq \\
&-c_2(s^1_2 c_2 y_j - d_{12}) \cdot s^1_2 c_2 y_j \cdot -d_{12} \leq \\
&-c_2(s^1_2 c_2 y_j - d_{12}) \cdot c_2(s^1_2 c_2 y_j - d_{12}) = 0.
\end{aligned}$$

Now by the above and (3),(5) we obtain

$$\begin{aligned}
&\prod_{i \leq m} c_0 X_i = c_0 X_0 \cdot c_0 \prod_{i \leq m} c_0 X_i = c_0(X_0 \cdot \prod_{i \leq m} c_0 X_i) = c_0 \prod_{i \leq m} (X_0 \cdot c_0 X_i) \leq \\
&c_0 \prod_{i \leq m} (u \circ X_i) \leq c_0 \prod_{i \leq m} (g \circ X_i + \sum_{j<m} Y_j \circ X_i) = \\
&c_0 \sum \{ \prod_{i \leq m} (f_i \circ X_i) : f \in {}^{m+1}\{g, Y_0, ..., Y_{m-1}\}\}.
\end{aligned}$$

Let $f \in {}^{m+1}\{g, Y_0, ..., Y_{m-1}\}$. If $g \in Rng\ f$ then by (5) and (4), $\prod\{f_i \circ X_i : i \leq m\} \leq g \circ X \leq c_1 g$. Assume $g \notin Rng\ f$. Then by $|\{Y_j : j < m\}| < m+1$, there are $i < j < m+1$ with $f_i = f_j$. Then by (6), by $Y_k \cdot s^1_2 Y_k \leq d_{12}$, $Y_k = c_2 Y_k$ and by $X_i \cdot X_j = 0$ we obtain $(f_i \circ X_i) \cdot (f_j \circ X_j) = 0$, hence $\prod\{f_i \circ X_i : i \leq m\} = 0$. Thus $\prod_{i \leq m} c_0 X_i \leq c_0 c_1 g$ and we are done.

To finish showing $SNr_n CA_{n+2} \models \varepsilon_m$, we are going to prove (3)–(6). First we note that the following hold in every CA_m: Let $i, j, k < m$.

$s^i_j(x \cdot y) = s^i_j x \cdot s^i_j y$	[HMT71]1.5.3.
$s^i_j c_i x = c_i x$	[HMT71]1.5.8(i).
$s^i_j s^k_i c_i y = s^k_j c_i y$	[HMT71]1.5.11(i).
$c_i x \cdot c_i y = c_i(x \cdot c_i y)$	[HMT71]p.162.
$c_i s^k_i c_i x = c_k c_i x$	
$s^i_j s^i_k x = s^i_k x$	[HMT71]1.5.10(i).
$s^i_k d_{ij} = d_{kj}$ if $i \neq j$	[HMT71]1.5.4(i).
$d_{ij} \cdot s^k_i x = d_{ij} \cdot s^k_j x$	[HMT71]1.5.6.
$s^i_j c_k x = c_k s^i_j x$ if $k \notin \{i, j\}$.	[HMT71]1.5.8(ii).

[11] Actually, it is true in CA_n, $n \geq 3$ that $(y = c_2 y \wedge y \cdot s^1_2 y \leq d_{12})$ iff $y = c_2 y - c_2(s^1_2 c_2 y - d_{12})$.

Proof of (3):

$$\begin{aligned}
&(c_1x \cdot s_1^0 c_1 y) \circ y = c_n(s_n^1(c_1x \cdot s_1^0 c_1 y) \cdot s_n^0 y) = \\
&c_n(s_n^1 c_1 x \cdot s_n^1 s_1^0 c_1 y \cdot s_n^0 y) = \\
&c_n(c_1 x \cdot s_n^0 c_1 y \cdot s_n^0 y) = \\
&c_n(c_1 x \cdot s_n^0 y) = \\
&c_n(c_1 c_n x \cdot s_n^0 c_n y) = \\
&c_n(c_n c_1 x \cdot s_n^0 c_n y) = \\
&c_n c_1 x \cdot c_n s_n^0 c_n y = \\
&c_1 c_n x \cdot c_0 c_n y = c_1 x \cdot c_0 y \geq x \cdot c_0 y.
\end{aligned}$$

Proof of (4):

$$\begin{aligned}
&x \circ y = c_n(s_n^1 x \cdot s_n^0 y) \leq c_n s_n^1 x = \\
&c_n s_n^1 c_n x = c_1 c_n x = c_1 x.
\end{aligned}$$

Proof of (5): This is true because c_n, s_n^1, s_n^0 are "additive".

Proof of (6): Assume $x \cdot s_2^1 x \leq d_{12},\ c_2 x = x$ and let $k = n+1$. First we show that $s_n^1 x \cdot s_k^1 x = s_n^1 x \cdot d_{nk}$.

$$\begin{aligned}
&s_n^1 x \cdot s_k^1 x = s_n^1 x \cdot s_n^1 s_k^1 x = \\
&s_n^1(x \cdot s_k^1 x) = s_n^1(c_2 x \cdot s_k^1 c_2 x) = \\
&s_n^1(s_k^2 c_2 x \cdot s_k^2 s_2^1 c_2 x) = \\
&s_n^1 s_k^2(x \cdot s_2^1 x) \leq s_n^1 s_k^2 d_{12} = d_{nk}, \ \text{ hence} \\
&s_n^1 x \cdot s_k^1 x = s_n^1 x \cdot s_k^1 x \cdot d_{nk} = s_n^1 x \cdot d_{nk}.
\end{aligned}$$

It is proved in [HMT85]p.216 that

$$c_n(s_n^1 x \cdot s_n^0 y) = c_k(s_k^1 x \cdot s_k^0 y) \ \text{ if } \ x = c_n c_k x, y = c_n c_k y.$$

Now

$$\begin{aligned}
&(x \circ y) \cdot (x \circ z) = c_n(s_n^1 x \cdot s_n^0 y) \cdot c_n(s_n^1 x \cdot s_n^0 z) = \\
&c_n(s_n^1 x \cdot s_n^0 y \cdot c_n(s_n^1 x \cdot s_n^0 z)) = \\
&c_n(s_n^1 x \cdot s_n^0 y \cdot c_k(s_k^1 x \cdot s_k^0 z)) = \\
&c_n(s_n^1 c_k x \cdot s_n^0 c_k y \cdot c_k(s_k^1 x \cdot s_k^0 z)) = \\
&c_n(c_k s_n^1 x \cdot c_k s_n^0 y \cdot c_k(s_k^1 x \cdot s_k^0 z)) = \\
&c_n c_k(s_n^1 x \cdot s_n^0 y \cdot s_k^1 x \cdot s_k^0 z) = \\
&c_n c_k(d_{nk} \cdot s_n^1 x \cdot s_n^0 y \cdot s_k^0 z) = \\
&c_n c_k(d_{nk} \cdot s_n^1 x \cdot s_n^0 y \cdot s_n^0 z) =
\end{aligned}$$

$$c_n c_k(d_{nk} \cdot s_n^1 x \cdot s_n^0(y \cdot z)) =$$
$$c_n c_k(d_{nk} \cdot s_n^1 c_k x \cdot s_n^0(c_k y \cdot c_k z)) =$$
$$c_n c_k(d_{nk} \cdot c_k(s_n^1 x \cdot s_n^0(y \cdot z)) =$$
$$c_n(s_n^1 x \cdot s_n^0(y \cdot z)) = x \circ (y \cdot z).$$

Finally, we prove (2). Let $\mathfrak{A}$ and m be as in the statement of (2). Let us evaluate the variables $x, x_0, \ldots, x_m, y_0, \ldots, y_{m-1}$ to $R, R_0, \ldots, R_m, F_0, \ldots, F_{m-1}$ respectively. Then for all $i \leq m$ we have $R_i = R \cap R_i \cap \bigcap_{i \neq j \leq m} - R_j$, and so the left-hand-side of ε_m will be $c_0 R \neq \emptyset$, by $c_0 R_i = c_0 R$ for all $i \leq m$. On the other side, $c_2 F_j - c_2(s_2^1 c_2 F_j - d_{12}) = F_j$ for all $j < m$, and thus the right-hand-side of ε_m is $\emptyset$ by $c_1 R \cap s_1^0 c_1 R = U_0 \times U_0 \times U_2 \times \cdots \subseteq F_0 \cup \cdots \cup F_{m-1}$. This finishes the proof of (2).

QED(Theorem 2)

REMARK 3. Let e be an equation valid in RCA_n. We say that the "complexity of e is m" if e can be derived with m extra dimensions but e cannot be derived with $m-1$ extra dimensions, i.e. if $SNr_nCA_{n+m} \models e$ but $SNr_nCA_{n+m-1} \not\models e$ if $m > 0$. It is showed in Andréka[90c] that the members of the second sequence of equations in Remark 2 have all complexity 2 (except ε_0 whose complexity is 0).

It follows from the results of Henkin and Monk quoted just before Thm.2 that every equation valid in RCA_n has a finite complexity and that there are equations of arbitrary big complexities. The question naturally arises whether there are equations of complexity m, for all $m < \omega$. This is asked in Monk[69], and this is Problem 2.12 in [HMT71]. The answer is affirmative for $n \geq \omega$, this is an unpublished result of Don Pigozzi. The problem for $n < \omega$ is still open, only the following partial results are known: Maddux[90] proved that if $n = 3$ then for every $m \geq 3$ there is an equation whose complexity is between m and $(3m-7)$. Andréka[90c] proved that[12] for every $n \geq 3$, $m \leq n$, $m < \omega$ there is an equation valid in RCA_n whose complexity is m. To prove this, the members e_m of the first sequence of equations given in Remark 2 are used, namely it is proved in Andréka[90c] that $SNr_nCA_{n+m+1} \models e_m$ while $SNr_nCA_{n+m} \not\models e_m$. We note that the equation e_m is "meaningful" whenever $m < n$. ■

The next question to ask is which operations cause the nonfinite axiomatizability. One immediately thinks of the cylindrifications as responsible for this, and

[12] As special cases of the result, it was already known that there are equations of complexity 1 (see Monk[69]), if $n = 3$ then there are equations of complexities 2 and 3 (see Maddux[90]), and if $n \geq \omega$, then there are equations of complexity m for all $m < \omega$ (Don Pigozzi).

indeed Johnsson[69] proved that already the diagonal–free reducts of RCA_n are not finitely axiomatizable. (The cylindrifications in themselves are finitely axiomatizable see Hansen[92,95], their interconnection with the Boolean operations is so complex as to cause nonfinite axiomatizability.) Do the diagonal constants contribute to nonfinite axiomatizability of RCA_n? In other words: Can the behaviour of the diagonal constants be described by finitely many formulas assuming that we know (as an oracle) all the formulas holding for the other operations? An equivalent formulation of this question is whether there is an axiom system for RCA_n in which the diagonal constants occur finitely many times only. This question is Problem 1 in Johnson[69] and is restated as Problem 5.4 in the monograph Henkin–Monk–Tarski[HMT85]. The next two theorems give a negative solution to these problems. They state that the diagonal constants are not so "simple" as they seem to be, their interconnections with the Boolean and cylindric operations cannot be described with finitely many variables. (We note that the interconnections of the diagonal constants with the Boolean operations can be described with finitely many formulas (see Proposition 2 in Part II) and the interconnections of the diagonal constants with the cylindrifications clearly can be described with one variable, since the cylindrifications are unary operations.)

The proofs of the theorems to come are variations of the one of Thm.1. Therefore we state some parts of the proof of Thm.1 as lemmas, because we want to use them several times. In the following lemmas, let α be any set.

LEMMA 1. *Let $\mathfrak{A}$ be the subalgebra of $\langle \mathfrak{P}({}^\alpha U), c_i^U, d_{ij}^U \rangle_{i,j\in\alpha}$ generated by $G \subseteq \mathcal{P}({}^\alpha U)$. Let $S(U)$ denote the set of all permutations of U and let*

$$Fix(G) = \{\sigma \in S(U) : g = \{\sigma \circ s : s \in g\} \text{ for all } g \in G\}.$$

Assume that $R \in A$ is such that

$$(*) \qquad (\forall s, z \in R)(\exists \sigma \in Fix(G))\sigma \circ s = z.$$

Then R is an atom of $\mathfrak{A}$.

Proof: For $x \subseteq {}^\alpha U$ let us write σx for $\{\sigma \circ s : s \in x\}$.[13] It is easy to check that if $\sigma x = x, \sigma y = y$ then for all $i, j \in \alpha$

$$\sigma(x + y) = x + y, \quad \sigma(-x) = -x, \quad \sigma(d_{ij}) = d_{ij}, \quad \sigma(c_i x) = c_i x.$$

Thus $\sigma x = x$ for all $x \in A$, $\sigma \in Fix(G)$. Let $x \in A$, and assume that $x \cap R \neq \emptyset$. We will show $R \subseteq x$, and this will prove that R is an atom of $\mathfrak{A}$. Let $s \in x \cap R$ and let $z \in R$ be arbitrary. Then there is $\sigma \in Fix(G)$ such that $z = \sigma \circ s$, by (*). Then $z \in x$ by $x \subseteq \sigma x$, $s \in x$, $z = \sigma \circ s$. Thus $R \subseteq x$ and we are done. **QED**(Lemma 1)

[13] This is denoted by $\tilde{\sigma} x$ in [HMT85],3.1.37.

LEMMA 2. *Let m be any cardinal, $\langle U_i : i \in \alpha \rangle$ be a system of sets each having cardinality $\geq m$, and let $U \supseteq \bigcup\{U_i : i \in \alpha\}$. Then there is a partition $\langle R_j : j < m \rangle$ of $R = \underset{i \in \alpha}{\times} U_i$ such that*

$$c_i^U R_j = c_i^U R \quad \textit{for all } i \in \alpha \textit{ and } j < m.$$

Proof: We define an equivalence relation on R: For any $s, z \in R$

$$s \equiv z \quad \Longleftrightarrow \quad |\{i \in \alpha : s_i \neq z_i\}| < \omega.$$

Let $S \subseteq R$ be a set of representatives for $\equiv$, i.e. S contains exactly one sequence from each block of $\equiv$. Let $(Q, +, 0)$ be any commutative group with $Q = \{j : j < m\}$. For any $s \in S$ and $i \in \alpha$ let $f_i^s : U_i \twoheadrightarrow Q$ be an onto mapping such that $f_i^s(s_i) = 0$. If x is a finite subset of Q, then we let Σx be the sum of the elements of x, computed in the group $(Q, +, 0)$. We note that $s \equiv z$ implies that $f_i^s(z_i) \neq 0$ for only finitely many $i < \alpha$, so that for $s \equiv z$, the sum $\Sigma\{f_i^s(z_i) : i \in \alpha\}$ is always defined. For $j < m$ define

$$R_j^s = \{z \in R : z \equiv s \ \text{ and } \ \Sigma\{f_i^s(z_i) : i \in \alpha\} = j\},$$
$$R_j = \bigcup\{R_j^s : s \in S\}.$$

Now clearly, $\langle R_j : j < m \rangle$ is a partition of R.

Let $i \in \alpha$, $j < m$ and $z \in R$. We want to show $z \in c_i^U R_j$. Let $s \in S$ be such that $z \equiv s$. Let $u \in Q$ be such that $u + \Sigma\{f_l^s(z_l) : l \in \alpha, l \neq i\} = j$. Then $z(i/u) \in R_j$, hence $z \in c_i^U R_j$. **QED**(Lemma 2)

DEFINITION. *If $\mathfrak{B}$ is a Boolean algebra and $f : B \longrightarrow B$ then we say that f is additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in B$. Let $\mathfrak{A}$ be any Boolean algebra with additive operations c_i, $i \in \alpha$ and possibly with some extra constants. Let m be any cardinal and let R be any atom of $\mathfrak{A}$ (i.e. any atom of the Boolean part of $\mathfrak{A}$). We say that the algebra $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ by splitting R into m parts R_j, $j < m$ if the following (i)-(iv) hold*[14].

(i) *$\mathfrak{A}'$ is a Boolean algebra with additive operations c_i', $i \in \alpha$ and with the same constants as $\mathfrak{A}$.*
(ii) *$\mathfrak{A} \subseteq \mathfrak{A}'$.*
(iii) *$c_i' R_j = c_i R$ for all $j < m$ and $i \in \alpha$, and $\langle R_j : j < m \rangle$ is a partition of R.*
(iv) *Each element of $\mathfrak{A}'$ is a (Boolean) join of an element of $\mathfrak{A}$ and of finitely or cofinitely (in m) many R_j's, i.e. for any $x \in A'$ there are $a \in A$ and $J \subseteq m$ such that $x = a + \sum\{R_j : j \in J\}$ and either J or $m \smallsetminus J$ is finite.*

[14]Though we call $\mathfrak{A}'$ obtained from $\mathfrak{A}$ by splitting, this is not a special case of splitting as described in [HMT71]2.6.12. This is a special case of "dilation" as described in [HMT85]3.2.69 (we take $a_\kappa = a_\lambda$ for all $\kappa, \lambda \in \alpha$). The name "splitting" is justified by aiming in both constructions at having disjoint elements $\langle a_j : j < m \rangle$ such that $c_i a_j = c_i a_0$ for all $i \in \alpha, j < m$.

It is easy to check that for any $\mathfrak{A}, R, m$ as in the hypothesis part of the definition, there is, up to isomorphism, exactly one $\mathfrak{A}'$ satisfying (i)–(iv). We shall denote the algebra obtained from $\mathfrak{A}$ by splitting R into m parts with $split(\mathfrak{A}, R, m)$. ■

Since $split(\mathfrak{A}, R, m)$ is defined only up to isomorphism, everything in the following lemma is understood up to isomorphism. Let $\mathfrak{A}, \mathfrak{B}$ be not necessarily similar algebras and let $h : A \longrightarrow B$. Let f be a common operation symbol of $\mathfrak{A}$ and $\mathfrak{B}$. We say that $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, or $h : A \longrightarrow B$, is a homomorphism w.r.t. f if $h : (A, f^{\mathfrak{A}}) \longrightarrow (B, f^{\mathfrak{B}})$ is a homomorphism. We say that $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a Boolean homomorphism if h is a homomorphism w.r.t. the Boolean operations $+, -$.

LEMMA 3. *Let $\mathfrak{A}, \mathfrak{B}$ be Boolean algebras with additive operations c_i, $i \in \alpha$ and, possibly, with additional constants, let R be an atom of $\mathfrak{A}$, and m, m_1, m_2, k be nonzero cardinals, $k < \omega$. Then (1)–(5) below hold.*

(1) $\mathfrak{A} = split(\mathfrak{A}, R, 1)$ *and*
$split(\mathfrak{A}, R, m_1) \subseteq split(\mathfrak{A}, R, m_2)$ *if* $m_1 \leq m_2$.
(2) $c_i a \in A$ *for all* $i \in \alpha$ *and* $a \in split(\mathfrak{A}, R, m)$.
(3) *Any k–generated subalgebra of $split(\mathfrak{A}, R, m)$ is a subalgebra of $split(\mathfrak{A}, R, 2^k)$.*
(4) *Let Z be an atom of $\mathfrak{B}$ and let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a Boolean homomorphism taking R to Z. Let $m_1 \leq m_2$. Then there is an extension $\overline{h}$ of h with the following properties for all $i \in \alpha$:*
$\overline{h} : split(\mathfrak{A}, R, m_1) \longrightarrow split(\mathfrak{B}, Z, m_2)$ *is a Boolean homomorphism.*
$\overline{h}$ *is a homomorphism w.r.t. c_i iff h is such.*
$\overline{h}$ *is one–to–one iff h is such.*
(5) *Assume that $h : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}({}^{\alpha}U), c_i^U, d_{ij}^U \rangle_{i,j \in \alpha}$ is a Boolean embedding and $h(R) = \underset{i \in \alpha}{\times} U_i$ such that $\langle U_i : i \in \alpha \rangle$ is a system of disjoint sets each having cardinality $\geq m$. Then h can be extended to a Boolean embedding $\overline{h} : split(\mathfrak{A}, R, m) \rightarrowtail \langle \mathfrak{P}({}^{\alpha}U), c_i^U, d_{ij}^U \rangle_{i,j \in \alpha}$ such that $\overline{h}$ is a homomorphism w.r.t. the same operations of $split(\mathfrak{A}, R, m)$ w.r.t. which h is such.*

Proof: Checking (2) and $\mathfrak{A} = split(\mathfrak{A}, R, 1)$ is straightforward by using the definition of $split(\mathfrak{A}, R, m)$.

(4): For any set H, let $Cof(H)$ denote the set of all finite and cofinite subsets of H. Assume $m_1 \leq m_2$ and let $\chi : m_1 \longrightarrow Cof(m_2)$ be such that

the sets $\chi(j), j < m_1$ are nonempty and pairwise disjoint, and
$\bigcup\{\chi(j) : j < m_1\} = m_2$ if $m_1 < \omega$.

Let $\rho = m_2 \smallsetminus \bigcup\{\chi(j) : j < m_1\}$. For any $x \in split(\mathfrak{A}, R, m_1)$ let $J(x) = \{j < m_1 : R_j \leq x\}$ and define

$$\overline{h}(x) = \begin{cases} h(x - R) + \sum\{Z_i : i \in \bigcup\{\chi(j) : j \in J(x)\}\} & \text{if } |J(x)| < \omega \\ h(x - R) + \sum\{Z_i : i \in \bigcup\{\chi(j) : j \in J(x)\} \cup \rho\} & \text{if } |J(x)| \geq \omega. \end{cases}$$

It is easy to check that $\overline{h}$ is a Boolean homomorphism into $split(\mathfrak{B}, Z, m_2)$, $\overline{h}$ is an extension of h and $\overline{h}(x) \neq 0$ whenever $0 \neq x \leq R$. Thus $\overline{h}$ is one–to–one iff h is such. Let $i \in \alpha$ and $x \in split(\mathfrak{A}, R, m_1)$. Assume that h is a homomorphism w.r.t. c_i. If $x \cdot R = 0$ then $x \in A$, hence $\overline{h}(c_i x) = h(c_i x) = c_i h(x) = c_i \overline{h}(x)$. So assume $x \cdot R \neq 0$. Then $c_i(x \cdot R) = c_i R$ and $c_i(\overline{h}(x \cdot R)) = c_i Z$ by $0 \neq \overline{h}(x \cdot R) \leq Z$. Now

$$\overline{h}(c_i x) = \overline{h}(c_i(x - R) + c_i(x \cdot R)) = c_i h(x - R) + c_i h(R), \text{ and}$$
$$c_i \overline{h}(x) = c_i(\overline{h}((x - R) + (x \cdot R)) = c_i(h(x - R) + \overline{h}(x \cdot R)) = c_i h(x - R) + c_i Z.$$

Thus $\overline{h}(c_i x) = c_i \overline{h}(x)$ by $h(R) = Z$. (4) has been proved.

The second part of (1), i.e. $split(\mathfrak{A}, R, m_1) \subseteq split(\mathfrak{A}, R, m_2)$ if $m_1 \leq m_2$ follows immediately from (4).

(5): Let everything be as in the hypothesis part of (5). Let $\langle R'_j : j < m\rangle$ be a partition of $R' = \underset{i \in \alpha}{\times} U_i$ which exists by Lemma 2, i.e. $c_i^U R'_j = c_i^U R'$ for all $i \in \alpha$ and $j < m$. Define for all $x \in split(\mathfrak{A}, R, m)$

$$\overline{h}(x) = h(x - R) + \sum\{R'_j : R_j \leq x\}.$$

It is easy to check, exactly as above in the proof of (4), that $\overline{h}$ satisfies the requirements in (5).

(3): Let $\mathfrak{A}'$ be obtained from $\mathfrak{A}$ by splitting R into m parts $R_j, j < m$ and let $\mathfrak{A}''$ be the subalgebra of $\mathfrak{A}'$ generated by some $G \subseteq \mathfrak{A}'$, $|G| \leq k$. We want to show $\mathfrak{A}'' \subseteq split(\mathfrak{A}, R, 2^k)$.

We define an equivalence relation $\equiv$ on m as follows. For all $i, j < m$

$$i \equiv j \quad \text{iff} \quad (\forall g \in G)[R_i \leq g \iff R_j \leq g].$$

Then $\equiv$ has $\leq 2^k$ blocks by $|G| \leq k$. Define

$$B = \{a \in \mathfrak{A}' : (\forall i, j < m)[i \equiv j \text{ and } R_j \leq a \implies R_i \leq a]\}.$$

We now show that B is closed under the operations of $\mathfrak{A}'$. B clearly is closed under the Boolean operations. Clearly, $A \subseteq B$ since R is an atom of $\mathfrak{A}$. Hence $c_i b \in B$ for all $i \in \alpha, b \in B$ by (2), and all the constants of $\mathfrak{A}$ are in B. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}'$ with universe B. Clearly, $G \subseteq B$, hence $\mathfrak{A}'' \subseteq \mathfrak{B}$ and so it is enough to show $\mathfrak{B} \subseteq split(\mathfrak{A}, R, 2^k)$.

First we show that there is at most one infinite block of $\equiv$. Indeed, assume that J_1, J_2 are infinite blocks of $\equiv$, and let $i \in J_1, j \in J_2$. By $i \not\equiv j$ there is $g \in G$ such that$R_i \leq g$ and $R_j \not\leq g$ or $R_j \leq g$ and $R_i \not\leq g$. We may assume $R_i \leq g, R_j \not\leq g$. Then $R_l \leq g$ for all $l \in J_1$ and $R_l \not\leq g$ for all $l \in J_2$. This is a contradiction by $\{l < m : R_l \leq g\} \in Cof(m)$. Thus there is at most one infinite block of $\equiv$. Let p be the number of blocks of $\equiv$. Since $k < \omega$ and $p \leq 2^k$, then $\equiv$ has finitely many blocks. Thus if $\equiv$ has an infinite block, it has to be cofinite. Let $\langle J_l : l < p\rangle$ be the partition belonging to $\equiv$. For any $l < p$ define

$$R'_l = \sum\{R_j : j \in J_l\}.$$

This sum always exists because $J_l \in Cof(m)$ by the above. Then $\langle R'_l : l < p\rangle$ is a partition of R in $\mathfrak{B}$, i.e. R'_l are disjoint, nonzero elements of $\mathfrak{B}$ such that $\sum\{R'_l : l < p\} = R$. Also, $c'_i R'_l = c_i R$ for all $l < p$ and $i \in \alpha$, and $|\{R'_l : l < p\}| = p$. Clearly, $\mathfrak{A} \subseteq \mathfrak{B} \subseteq split(\mathfrak{A}, R, m)$, hence conditions (i)–(iii) in the definition of $split(\mathfrak{A}, R, p)$ are satisfied. Finally, we check condition (iv). Let $x \in B$. Then $x = a + \sum\{R_j : j \in J\}$ for some $a \in A$ and $J \in Cof(m)$ such that J is a union of some blocks of $\equiv$. Let $J' = \{l < p : J_l \subseteq J\}$. Then clearly $\sum\{R_j : j \in J\} = \sum\{R'_j : j \in J'\}$ and we are done. Therefore condition (iv) also holds, so $\mathfrak{B} \cong split(\mathfrak{A}, R, p)$. **QED**(Lemma 3)

We will need one more lemma. For any $\sigma \in {}^\alpha\alpha$ and any set x of α–sequences let $\mathsf{s}_\sigma(x) = \{s : s \circ \sigma \in x\}$. We will use the part of Lemma 4 below concerning s_σ later in Theorem 6.

LEMMA 4. *Let U, W be sets and let $g : {}^\alpha W \longrightarrow {}^\alpha U$. Let $h : \mathcal{P}({}^\alpha U) \longrightarrow \mathcal{P}({}^\alpha W)$ be defined by*

$$h(x) = \{s \in {}^\alpha W : g(s) \in x\}$$

for all $x \subseteq {}^\alpha U$. Then (i)–(iii) below hold for all $i, j \in \alpha$ and $x \subseteq {}^\alpha U$:

(i) *h is a Boolean homomorphism.*
(ii) *$h(d^U_{ij}) = d^W_{ij}$ iff $(\forall s \in {}^\alpha W)[s_i = s_j \iff g(s)_i = g(s)_j]$.*
(iii) *$c^W_i h(x) = h(c^U_i x)$ iff for all $s \in {}^\alpha W$ we have $[(\exists u \in U) g(s)(i/u) \in x \iff (\exists w \in W) g(s(i/w)) \in x]$.*
(iv) *Assume that $t_i : W \longrightarrow U$ for all $i \in \alpha$, $W \neq \emptyset$ and g is such that $g(s)_i = t_i(s_i)$ for all $s \in {}^\alpha W, i \in \alpha$. Let $i \in \alpha$ and $\sigma \in {}^\alpha\alpha$. Then*
h is a homomorphism w.r.t. c_i iff t_i is onto U,
h is one–one iff (t_i is onto U for all $i \in \alpha$),
h is a homomorphism w.r.t. s_σ iff ($t_i = t_{\sigma(i)}$ for all $i \in \alpha$).

Proof: The proofs of (i),(ii) are straightforward, therefore we omit them. The proof of (iii) is straightforward, too, but we include it. Let $s \in {}^{\alpha}W$. Then

$$s \in c_i^W h(x) \text{ iff } (\exists w \in W)s(i/w) \in h(x) \text{ iff } (\exists w \in W)g(s(i/w)) \in x$$

and

$$s \in h(c_i^U x) \text{ iff } g(s) \in c_i^U x \text{ iff } (\exists u \in U)g(s)(i/u) \in x,$$

which immediately imply (iii). <u>(iv):</u> It is easy to check that

$$g(s(i/w)) = g(s)(i/t_i w) \text{ for all } s \in {}^{\alpha}W, i \in \alpha, w \in W$$

therefore h is a homomorphism w.r.t. c_i if t_i is onto U by (iii). Let $x \subseteq {}^{\alpha}W, s \in x$. If t_i is onto U for all $i \in \alpha$, then there is $z \in {}^{\alpha}W$ such that $s_i = t_i z_i$ for all $i \in \alpha$, thus $z \in h(x)$. This shows that h is one–one if t_i are onto U for all $i \in \alpha$. Assume that $t_i = t_{\sigma i}$ for all $i \in \alpha$. Then

$$g(s \circ \sigma) = g(s) \circ \sigma \text{ for all } s \in {}^{\alpha}W$$

because for $i \in \alpha$, $g(s\circ\sigma)_i = t_i(s\circ\sigma)_i = t_i(s_{\sigma i})$ and $(g(s)\circ\sigma)_i = g(s)_{\sigma i} = t_{\sigma i}(s_{\sigma i})$. Then $s \in h(\mathsf{s}_\sigma x)$ iff $g(s) \in \mathsf{s}_\sigma(x)$ iff $g(s) \circ \sigma \in x$ iff $g(s \circ \sigma) \in x$ iff $s \circ \sigma \in h(x)$ iff $s \in \mathsf{s}_\sigma h(x)$. Thus h is a homomorphism w.r.t. s_σ. We are going to check the "only if" parts. Assume that $u \in U \smallsetminus Rng\ t_i$ and let $s \in {}^{\alpha}W$ be arbitrary. Let $z = g(s)(i/u)$. Then $h(\{z\}) = 0$ and $s \in h(c_i\{z\})$, $s \notin c_i h(\{z\})$ showing that h is not one–one and is not a homomorphism w.r.t. c_i. Let $\sigma \in {}^{\alpha}\alpha, i \in \alpha$ and assume that $t_i \neq t_{\sigma i}$, say $t_i(w) \neq t_{\sigma i}(w)$. Let $z \in {}^{\alpha}W$ be such that $z_i = w$ and let $x = \{g(z \circ \sigma)\}$. Then $g(z \circ \sigma) \neq g(z) \circ \sigma$ and so $z \in \mathsf{s}_\sigma h(x)$ but $z \notin h(\mathsf{s}_\sigma x)$. **QED**(Lemma 4)

We are ready to state Thm.3. Let ${}_{\infty}Cs_n$ denote the class of all Cs_n's with greatest elements of form ${}^{n}U$ with U infinite.

THEOREM 3. *Let Σ be a set of quantifier–free formulas axiomatizing $RCA_n, n \geq \omega$. Let k be any natural number and let $\ell < n$. Then Σ contains infinitely many formulas in which at least one diagonal constant with index ℓ, more than k cylindrifications, and more than k variables occur. The same holds with $\mathcal{K}$ in place of RCA_n for any $\mathcal{K}$ such that ${}_{\infty}Cs_n \subseteq \mathcal{K} \subseteq RCA_n$.*

Proof: <u>PLAN:</u> Assume first that $\ell = 0$. At the end of the proof we will show how to eliminate this assumption.

Let Σ and k be as in the statement of the theorem. Let $\mathfrak{A}_k$ be the algebra constructed in the proof of Theorem 1. We will prove the following:

(i) For any $I \subseteq n$, $|I| \leq k$ there is a representation of $\mathfrak{A}_k$ as a ${}_\infty Cs_n$ in which all the operations are the natural ones except for c_i, $i \notin I$.
(ii) There is a representation of $\mathfrak{A}_k$ as a ${}_\infty Cs_n$ in which all the operations are the natural ones except for d_{0i}, d_{i0}, $i < n$.

$\mathbf{I}_\infty Cs_n$ denotes the class of all algebras isomorphic to an element of ${}_\infty Cs_n$. We proved in Theorem 1 that

(iii) Any k–generated subalgebra of $\mathfrak{A}_k$ is representable, in fact in $\mathbf{I}_\infty Cs_n$.

This will prove the theorem because of the following: Let ${}_\infty Cs_n \subseteq \mathcal{K} \subseteq RCA_n$. Let Σ^c denote the set of all quantifier–free formulas valid in $\mathcal{K}$ which contain at most k cylindrifications, Σ^d denote the set of all quantifier–free formulas valid in $\mathcal{K}$ which contain no diagonal with index 0 and let Σ^v denote the set of all quantifier–free formulas valid in $\mathcal{K}$ which contain at most k variables. By (i)–(iii) above and ${}_\infty Cs_n \subseteq \mathcal{K}$ we have that $\mathfrak{A}_k \models \Sigma^c \cup \Sigma^d \cup \Sigma^v$ (an argument for this is given below). Since $\mathfrak{A}_k$ is not in RCA_n by Claim 1, and since $\mathcal{K} \subseteq RCA_n$, we have $\mathfrak{A}_k \not\models \Sigma$. Thus $\Sigma \nsubseteq \Sigma^c \cup \Sigma^d \cup \Sigma^v$, which means that Σ contains a formula with more than k cylindrifications, more than k variables and with a diagonal constant with index 0.

We can show that (i)–(iii) imply $\mathfrak{A}_k \models \Sigma^c \cup \Sigma^d \cup \Sigma^v$ as follows. Let $\mathfrak{A} = \mathfrak{A}_k$. First we show that (i) implies $\mathfrak{A} \models \Sigma^c$. Let $\varphi \in \Sigma^c$, say φ contains only c_i with $i \in I$ where $|I| \leq k$. By (i), there is a representation of $\mathfrak{A}$ as a ${}_\infty Cs_n$ in which all the operations are the natural ones, except for c_i, $i \notin I$. This means that there is a one-one homomorphism $h : \mathfrak{A}^c \rightarrowtail \mathfrak{P}^c$ where $\mathfrak{A}^c = \langle A, +^{\mathfrak{A}}, -^{\mathfrak{A}}, c_i^{\mathfrak{A}}, d_{jk}^{\mathfrak{A}} \rangle_{i \in I, j,k<n}$ and $\mathfrak{P}^c = \langle \mathfrak{P}({}^nW), c_i^W, d_{jk}^W \rangle_{i \in I, j,k<n}$ for some infinite set W. Let $\mathfrak{P} = \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$. Now $\mathfrak{P} \models \varphi$ because φ is valid in $\mathcal{K}$ and $\mathfrak{P} \in {}_\infty Cs_n \subseteq \mathcal{K}$, thus $\mathfrak{P}^c \models \varphi$ because no c_i with $i \notin I$ occurs in φ. Then $\mathfrak{A}^c \models \varphi$ because $\mathfrak{A}^c$ is isomorphic to a subalgebra of $\mathfrak{P}^c$ and φ is quantifier–free. Therefore $\mathfrak{A} \models \varphi$. The proofs of $\mathfrak{A} \models \Sigma^d$, $\mathfrak{A} \models \Sigma^v$ are similar, we omit them.

We could prove (i)–(iii) above for the algebras $\mathfrak{A}_k$ constructed in the proof of Theorem 1. However, to make the proof simpler, we will use modified versions of the algebras used in the proof of Theorem 1.

<u>CONSTRUCTION OF $\mathfrak{A}$:</u> Let $m \geq 2^k$, $m < \omega$ and let $\langle U_i : i \leq n \rangle$ be a system of disjoint sets such that $|U_0| = m$ and $|U_i| > n$ for all $i > 0$. Let

$$U = \bigcup \{U_i : i \leq n\},$$

$R = \underset{i<n}{\times} U_i,$ and let

$\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U \rangle_{i,j<n}$ generated by R.

Then R is an atom of $\mathfrak{A}'$, because it satisfies (*) in Lemma 1. Let $\mathfrak{A}$ be the algebra we obtain from $\mathfrak{A}'$ by splitting R into $m+1$ new atoms $R_j, j \leq m$. The proof of $\mathfrak{A} \notin RCA_n$ is exactly the same as the proof of Claim 1. The proof of (iii) (i.e. that the k–generated subalgebras of $\mathfrak{A}$ are in $\mathbf{I}_\infty Cs_n$'s) is basically the same as that of Claim 2, but now we can use our lemmas: Let $\mathfrak{B}$ be a k–generated subalgebra of $\mathfrak{A}$. Then $\mathfrak{B} \subseteq split(\mathfrak{A}', R, 2^k) \subseteq split(\mathfrak{A}', R, m)$ by Lemma 3(3),(1), and $split(\mathfrak{A}', R, m)$ is in ${}_\infty Cs_n$ by Lemma 3(5), hence $\mathfrak{B}$ is in ${}_\infty Cs_n$, too.

We now want to show that $\mathfrak{A}$ can be represented in such a way that only some cylindrifications are not "real".

Let W be any set such that $U \subsetneqq W$ and $|W \setminus U| < \omega$.

CLAIM 5. *For any $I \subseteq n$, $|I| = m$ there is an embedding $h : A \rightarrowtail \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$ which is a homomorphism w.r.t. all operations of $\mathfrak{A}$ except for c_i, $i \notin I$.*

Proof: Let $W_0 = U_0 \cup (W \setminus U)$, and $W_i = U_i$ for all $0 < i \leq n$. First we define an embedding $h : A' \rightarrowtail \mathcal{P}({}^nW)$ with the above properties such that $h(R) = \underset{i<n}{\times} W_i$.

Let $S \subseteq W_0 \times W_0$ be such that every element of W_0 is in relation with exactly m elements including itself, i.e.

$$\{(w,w) : w \in W_0\} \subseteq S \qquad \text{and} \qquad |\{v \in W_0 : v\ S\ w\}| = m \ \text{ for all } \ w \in W_0.$$

Some notation: For any functions f, f' and set Ω we define $f[\Omega/f'] = f \restriction (Dom\ f \setminus \Omega) \cup f' \restriction \Omega$. For a function f, $ker f = \{(i,j) : i,j \in Dom f \text{ and } f(i) = f(j)\}$. We note that if n is an ordinal, then we consider $n = \{m : m < n\}$ and a sequence $s \in {}^nW$ is considered to be a function mapping n to W.

Using S, we will define a function $g : {}^nW \longrightarrow {}^nU$. Let $s \in {}^nW$ be arbitrary. Let $\Omega_s = \Omega = \{i < n : s_i \in W_0\}$.

Assume $\Omega \subseteq I$. Let $s' \in {}^\Omega U_0$ be such that $ker(s') = ker(s \restriction \Omega)$. Such an s' exists by $|\Omega| \leq |I| = m = |U_0|$. We define

$$g(s) = s[\Omega/s'].$$

Assume $\Omega \nsubseteq I$. Let $\mu = min(\Omega \setminus I)$, the smallest element of the set $\Omega \setminus I$ of ordinals. Let $s' \in {}^\Omega U$ be such that for all $j \in \Omega$,

$s'_j \in U_0$ if $s_j \; S \; s_\mu$
$s'_j \in U_n \setminus Rng(s)$ if $s_j \; \not S \; s_\mu$, and
$ker(s') = ker(s \restriction \Omega)$.

Such an s' exists by $|\{v : v \; S \; s_\mu\}| \leq m = |U_0|$ and by $|U_n| > n$. We again define

$$g(s) = s[\Omega/s'].$$

By the above, we defined $g : {}^nW \longrightarrow {}^nU$. We define $h : A' \longrightarrow \mathcal{P}({}^nW)$ by

$$h(x) = \{s \in {}^nW : g(s) \in x\} \quad \text{for all} \quad x \in A'.$$

We begin with showing $h(R) = \underset{i<n}{\times} W_i$. Let $s \in {}^nW$ and $\Omega = \{i < n : s_i \in W_0\}$. Proof of $s \in \underset{i<n}{\times} W_i \Longrightarrow s \in h(R)$: (Here we will use $Id_{W_0} \subseteq S$.) Assume $s \in \underset{i<n}{\times} W_i$. Then $\Omega = \{0\}$. If $0 \in I$ then $\Omega \subseteq I$ and hence $g(s)_0 \in U_0$. If $0 \notin I$ then $0 = min(\Omega \setminus I)$ and hence $g(s)_0 \in U_0$ by $s_0 \; S \; s_0$. Thus, in both cases $g(s)_0 \in U_0$. Let $0 < i < n$. Then $i \notin \Omega$, hence $g(s)_i = s_i \in W_i = U_i$. We have seen $g(s) \in \underset{i<n}{\times} U_i = R$. Proof of $s \in h(R) \Longrightarrow s \in \underset{i<n}{\times} W_i$: Assume $g(s) \in R$ and let $i < n$. Then $g(s)_i \in U_i$, hence $g(s)_i \notin U_n$ because U_i and U_n are disjoint from each other. By inspecting the definition of g, we can see that this implies $s_i \in W_i$. Thus $h(R) = \underset{i<n}{\times} W_i$ is proved.

We turn to proving that h is a one–one homomorphism w.r.t. all operations except for c_i, $i \notin I$.

By Lemma 4, h is a homomorphism w.r.t. the Boolean operations. Therefore to show that h is one–one, it is enough to show that $h(x) \neq 0$ whenever $x \neq 0$.

We define an equivalence relation $\equiv$ on ${}^HW$, for any set H. Let $s, z \in {}^HW$. Then we define

$$s \equiv z \quad \text{iff} \quad [ker(s) = ker(z) \text{ and } (\forall l \leq n)(\forall i \in H)(s_i \in W_l \Leftrightarrow z_i \in W_l)].$$

In other words, $s \equiv z$ iff there is a permutation π of W such that $\pi^* W_l = W_l$ for all $l \leq n$ and $z = \pi \circ s$. This is true because $|W \setminus U| < \omega$. Therefore each element of A' is closed w.r.t. $\equiv$, i.e. for all $s, z \in {}^nU$

$$s \in x \quad \text{and} \quad z \equiv s \quad \text{implies} \quad z \in x$$

for all $x \in A'$. In the following, we will use this fact several times. If $u, v \in W$ then we define $u \equiv v$ to hold iff $\langle u \rangle \equiv \langle v \rangle$.

In the rest of the proof we will often use the following properties of the function g (these properties are easy to check). Let $s \in {}^nW$, $\Omega = \{i < n : s_i \in W_0\}$, $w \in W$ and $i \in I$.

(G1) $ker(g(s)) = ker(s)$
(G2) $g(s) \equiv s$ if $\Omega \subseteq I$.
(G3) $g(s) \restriction (n \setminus \{i\}) \equiv g(s(i/w)) \restriction (n \setminus \{i\})$.

We now turn to proving that h is one–one. Let $x \in A'$, $x \neq 0$. Let $s \in x$ be arbitrary and $\Omega = \{i < n : s_i \in U_0\}$. If $\Omega \subseteq I$ then $s \equiv g(s)$ by (G2). Hence $g(s) \in x$, therefore $s \in h(x)$ showing that $h(x) \neq 0$. Assume $\Omega \nsubseteq I$ and let $\mu = min(\Omega \setminus I)$. Let $u \in U_0$ be arbitrary and let $s' \in {}^{\Omega}(\{v \in W_0 : v\ S\ u\})$ be such that $s'_\mu = u$ and $ker(s') = ker(s \restriction \Omega)$. Such an s' exists by $s \restriction \Omega \in {}^{\Omega}U_0$ and $|U_0| = |\{v \in W_0 : v\ S\ u\}|$. Let $z = s[\Omega/s']$. Then $g(z) \equiv s$, hence $z \in h(x)$ showing $h(x) \neq 0$.

By Lemma 4 and (G1), h is a homomorphism for all d_{ij}, $i, j < n$.

Let $i \in I$ and $x \in A'$. We want to show $h(c_i^U x) = c_i^W h(x)$. By Lemma 4, we have to show that for all $s \in {}^nW$,

$$(\exists u \in U) g(s)(i/u) \in x \qquad \text{iff} \qquad (\exists w \in W) g(s(i/w)) \in x.$$

Let $s \in {}^nW$ be arbitrary. Let $\Omega = \{i < n : s_i \in W_0\}$, and let $\mu = min(\Omega \setminus I)$ if $\Omega \setminus I$ is nonempty. Let $u \in U$ and $w \in W$ be arbitrary. First we show that

$$g(s)(i/u) \;\equiv\; g(s(i/w))$$

whenever one of (1),(2) below hold.

(1) $u = g(s)_j$ and $w = s_j$ for some $j \in n \setminus \{i\}$.
(2) $u \notin \{g(s)_j : j \in n \setminus \{i\}\}$, $w \notin \{s_j : j \in n \setminus \{i\}\}$, and
either $u \equiv w$, $(w \in W_0, \Omega \nsubseteq I \Longrightarrow w\ S\ s_\mu)$,
or $u \in U_n$, $w \in W_0$, $\Omega \nsubseteq I$, $w \not\!S\ s_\mu$.

Indeed, let $p = g(s)(i/u)$ and $q = g(s(i/w))$. By $i \in I$ and (G3) we have $p \restriction (n \setminus \{i\}) \equiv q \restriction (n \setminus \{i\})$. Thus we have only to show $p_i \equiv q_i$ and ($p_i = p_l$ iff $q_i = q_l$) for all $l \in n \setminus \{i\}$.

Assume that (1) holds. Then $u = p_i = p_j$. By $w = s_j$ we have $(i, j) \in ker(s(i/w)) = ker(q)$, thus $q_i = q_j$. By $p \restriction (n \setminus \{i\}) \equiv q \restriction (n \setminus \{i\})$, $j \neq i$ we have $p_j \equiv q_j$, therefore $p_i = p_j \equiv q_j = q_i$. Let $l \in n \setminus \{i\}$ be arbitrary. Then by (G1), $p_i = p_l$ iff $u = p_j = p_l$ iff(by $ker(g(s)) = ker(s)$) $s_j = s_l$ iff(by $ker(s(i/w)) = ker(q)$) $q_j = q_l$ iff(by $q_i = q_j$) $q_i = q_l$.

Assume now that (2) holds. Let $l \in n \smallsetminus \{i\}$ be arbitrary. Then $p_i = u \neq g(s)_l = p_l$ and, by letting $z = s(i/w)$, $z_i = w \neq s_l = z_l$, hence $q_i \neq q_l$ by (G1). Assume $u \equiv w$ and $(w \in W_0, \Omega \nsubseteq I \Longrightarrow w\ S\ s_\mu)$. Then either $w \notin W_0$ or $\Omega \subseteq I$ or $w \in W_0, w\ S\ s_\mu$. In all these three cases $q_i \equiv w$ (by $q = g(s(i/w)), i \in I$), thus $q_i \equiv w \equiv u = p_i$. Assume $u \in U_n, w \in W_0, \Omega \nsubseteq I, w \not S\ s_\mu$. Then $g(s(i/w))_i \in U_n$ therefore $q_i \equiv u = p_i$.

By the above, to show that h is a homomorphism w.r.t. c_i, it is enough to show that for any $u \in U$ there is $w \in W$ satisfying (1) or (2), and vice versa, for any $w \in W$ there is $u \in U$ satisfying (1) or (2). We are now going to check this.

Let $u \in U$ be arbitrary. If $u = g(s)_j$ for some $j \in n \smallsetminus \{i\}$ then let $w = s_j$. Then (1) is satisfied. Assume that $u \notin \{g(s)_j : j \in n \smallsetminus \{i\}\}$. If $u \notin U_0$ then there is $w \equiv u$ with $w \notin \{s_j : j \in n \smallsetminus \{i\}\}$ by $|U_l| > n$ for all $l > 0$. This w will satisfy (2). Assume now $u \in U_0$. Assume further $\Omega \subseteq I$. Then $|U_0 \cap \{s_j : j \in n \smallsetminus \{i\}\}| = |U_0 \cap \{s_j : j \in I \smallsetminus \{i\}\}| < m$ by $|I| = m, i \in I$. Thus there is $w \in U_0 \smallsetminus \{s_j : j \in n \smallsetminus \{i\}\}$ by $|U_0| = m$, and this w will then satisfy (2). Assume now $\Omega \nsubseteq I$. By $u \in U_0 \smallsetminus \{g(s)_j : j \in n \smallsetminus \{i\}\}$ and $|U| = m$ we have $|U_0 \cap \{g(s)_j : j \in n \smallsetminus \{i\}\}| < m$. By our construction, $g(s)_j \in U_0$ iff $s_j\ S\ s_\mu$, for all $j \in n$. By (G1) then $|\{s_j : s_j\ S\ s_\mu, j \in n \smallsetminus \{i\}\}| < m$. Since $|\{w \in W_0 : w\ S\ s_\mu\}| = m$ by our assumption on S, there is $w \in W_0 \smallsetminus \{s_j : j \in n \smallsetminus \{i\}\}$ such that $w\ S\ s_\mu$. This w satisfies (2).

Conversely, let $w \in W_0$ be arbitrary. If $w = s_j$ for some $j \neq i$ then let $u = g(s)_j$. Then (1) is satisfied. Assume that $w \notin \{s_j : j \in n \smallsetminus \{i\}\}$. If $w \notin W_0$ then there is $u \equiv w$ with $u \notin \{g(s)_j : j \in n \smallsetminus \{i\}\}$ by $|U_l| > n$ for all $l > 0$. Assume $w \in W_0$. Assume further $\Omega \subseteq I$. Then by $g(s)_j \in U_0 \Longrightarrow s_j \in W_0$ for all j, we have $\{j \in n \smallsetminus \{i\} : g(s)_j \in U_0\} \subseteq \{j \in n \smallsetminus \{i\} : s_j \in W_0\} \subseteq I \smallsetminus \{i\}$, then by $|I \smallsetminus \{i\}| = m - 1$, $|U_0| = m$ there is $u \in U_0 \smallsetminus \{g(s)_j : j \in n \smallsetminus \{i\}\}$. This u will satisfy (2). Assume now $\Omega \nsubseteq I$, $w\ S\ s_\mu$. Then $\{j \in n \smallsetminus \{i\} : g(s)_j \in U_0\} \subseteq J = \{j \in n \smallsetminus \{i\} : s_j\ S\ s_\mu\}$, then by $w \notin \{s_j : j \in n \smallsetminus \{i\}\}$, $w\ S\ s_\mu$, we have $|\{s_j : j \in J\}| < m$, thus there is $u \in U_0 \smallsetminus \{g(s)_j : j \in n \smallsetminus \{i\}\}$. This u satisfies (2). Assume finally $\Omega \nsubseteq I$, $w \not S\ s_\mu$. Then let $u \in U_n \smallsetminus \{g(s)_j : j \in n \smallsetminus \{i\}\}$ be arbitrary. Such a u exists by $|U_n| > n$, and this u satisfies (2).

We have checked that h is a homomorphism w.r.t. c_i, $i \in I$.

By the above we have shown that h is a one–to–one homomorphism for all operations of $\mathfrak{A}'$ except perhaps for c_i, $i \notin I$. We note that h is not a homomorphism w.r.t. c_i if $i \notin I, i \neq 0$. To see this, let $i \notin I$, $i \neq 0$, $j \in I$, $j \neq 0$. (We may assume $m > 2$, so we do have such a j.) Let $x = c_0(d_{0j} \cdot c_j R)$. Let $s \in {}^nW$ be such that

$s_i, s_j \in W_0,\ s_j \not S\ s_i$
$s_k \in U_k$ if $k \neq i, j, 0,\ s_0 \notin W_0$.

Then $\Omega_s = \{i,j\} \nsubseteq I$, $\mu = i$. By $g(s)_j \in U_n$ then $g(s)_j \notin U_0$, so $g(s) \notin c_i x$, hence $s \notin h(c_i x)$. On the other hand, let $w \in U_i$, and $z = s(i/w)$. Then $\Omega_z = \{j\} \subseteq I$, so $g(z) \equiv z \in x$. Hence $s \in c_i h(x)$.

Now, Lemma 3(5) finishes the proof of Claim 5. **QED**(Claim 5)

CLAIM 6. *There is an embedding* $h : A \rightarrowtail \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$ *such that* h *is a homomorphism w.r.t. all operations of* $\mathfrak{A}$ *except for* $d_{0i},\ d_{i0},\ i < n$.

Proof: By Lemma 3(5), it is enough to show that there is an embedding $h : A' \rightarrowtail \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$ such that $h(R) = \underset{i\in n}{\times} W_i$ and h is a homomorphism w.r.t. all operations of $\mathfrak{A}'$ except for $d_{0i},\ d_{i0},\ i < n$.

Let $t, r : W \longrightarrow U$ be functions such that

t, r are identity on $U \setminus U_n$,
$t^*(W \setminus U) \subseteq U_0,\ \ t^*(U_n) = U_n$ and $t \restriction U_n$ is bijective,
$r^*((W \setminus U) \cup U_n) = U_n$ and $r \restriction ((W \setminus U) \cup U_n)$ is bijective.

Such an r exists because $|U_n| \geq \omega$, $|W \setminus U| \leq |U_n|$. Then $t, r : W \twoheadrightarrow U$ are onto U and r is one–one. For any $s \in {}^nW$ define $g(s) \in {}^nU$ by

$$g(s)_i = \begin{cases} t(s_i) & \text{if } i = 0 \\ r(s_i) & \text{if } i \neq 0. \end{cases}$$

Define $h : A' \longrightarrow \mathcal{P}({}^nW)$ by

$$h(x) = \{s \in {}^nW : g(s) \in x\}, \quad \text{for all } \ x \in A'.$$

Now $h(R) = \underset{i<n}{\times} W_i$ by $g(s)_i \not\equiv s_i \implies i \neq 0, g(s)_i \in U_n$ (by a similar argument to the ones in the previous proofs). Also, $x \neq 0$ implies $h(x) \neq 0$ by $g(s) \equiv s$ for all $s \in {}^nU$, where $\equiv$ is the equivalence relation defined in the proof of Claim 5. Assume $0 \notin \{j, l\}$. Then $s_j = s_l$ iff $r(s_j) = r(s_l)$ iff $g(s)_j = g(s)_l$, hence h is a homomorphism w.r.t. d_{jl}, by Lemma 4.

Let $i < n$. We want to show that h is a homomorphism w.r.t. c_i. This follows from the following easy observation.

$$g(s)(i/u) = g(s(i/w)) \ \text{ if } \ i = 0, u = t(w) \ \text{ or } \ i \neq 0, u = r(w).$$

QED(Claim 6)

We turn to showing how to eliminate the assumption $\ell = 0$. Let us define the algebra $\mathfrak{B}$ as follows. Intuitively, $\mathfrak{B}$ is the same algebra as $\mathfrak{A}$ (in this proof) except that we interchange the indices 0 and ℓ. I.e., the universe of $\mathfrak{B}$ is the

same as that of $\mathfrak{A}$, and all the operations of $\mathfrak{B}$ are the same as those of $\mathfrak{A}$ except that $c_0^{\mathfrak{B}} = c_\ell^{\mathfrak{A}}$, $c_\ell^{\mathfrak{B}} = c_0^{\mathfrak{A}}$, $d_{0i}^{\mathfrak{B}} = d_{i0}^{\mathfrak{B}} = d_{0\ell}^{\mathfrak{A}}$, $d_{\ell i}^{\mathfrak{B}} = d_{i\ell}^{\mathfrak{B}} = d_{0i}^{\mathfrak{A}}$, for $i \in n \setminus \{0, \ell\}$. (This algebra is denoted in [HMT71] by $\mathfrak{Rd}^\xi\mathfrak{A}$ where ξ is the permutation of n interchanging 0 and ℓ and fixing the other elements of n.) Now, by using the corresponding properties of $\mathfrak{A}$, it is easy to see that $\mathfrak{B} \notin SNr_nCA_{n+2}$, $\mathfrak{B}$ satisfies (i), (iii) as stated at the beginning of the proof of Theorem 3, and also $\mathfrak{B}$ satisfies (ii) if we replace 0 in it by ℓ. This proves Theorem 3 by the argument given at the beginning of the proof of Theorem 3.
QED(Theorem 3)

THEOREM 4. *Let Σ be a set of quantifier–free formulas axiomatizing RCA_n, $2 < n < \omega$. Let k be any natural number and let $\ell < n$. Then Σ contains infinitely many formulas that contain all the cylindrifications $c_0, ..., c_{n-1}$, contain a diagonal constant with index ℓ and contain more than k variables. The same holds for $\mathcal{K}$ in place of RCA_n if ${}_\infty Cs_n \subseteq \mathcal{K} \subseteq RCA_n$.*

REMARK 4. We want to prove Theorem 4 with an argument similar to the one in the proof of Theorem 3. I.e. we want to show nonrepresentable algebras $\mathfrak{A}$ which e.g. can be represented whenever we omit one of the cylindrifications. To this end, however, we have to modify our construction used in the proof of Theorem 1 for $n < \omega$, because in that construction the "cause of nonrepresentability" was contained entirely in the indices $0, 1, 2$, i.e. those algebras cannot be represented in such a way that c_i, d_{ij} for $i, j < 3$ would be "real". To prove Theorem 4, we will "merge" the constructions (used in the proof of Theorem 1) for $n \geq \omega$ and for $n < \omega$. ■

Proof of Theorem 4: <u>PLAN:</u> Let n, Σ, k and ℓ be as in the statement of Theorem 4. We will construct an algebra $\mathfrak{A}$ with the following properties:

(i) $\mathfrak{A} \notin RCA_n$.
(ii) For any $i < n$ there is a representation of $\mathfrak{A}$ as a ${}_\infty Cs_n$ in which all the operations are the natural ones except c_i.
(iii) There is a representation of $\mathfrak{A}$ as a ${}_\infty Cs_n$ in which all the operations are the natural ones except d_{ij} with $\ell \in \{i, j\}$.
(iv) Every k–generated subalgebra of $\mathfrak{A}$ is in $\mathbf{I}{}_\infty Cs_n$.

Let Σ^c denote the set of all quantifier–free formulas valid in $\mathcal{K}$ which do not contain all the cylindrifications $c_0, ...c_{n-1}$, let Σ^d denote the set of all quantifier–free formulas valid in $\mathcal{K}$ in which no d_{ij} with $\ell \in \{i,j\}$ occurs and let Σ^v denote the set of all quantifier–free formulas valid in $\mathcal{K}$ in which at most k variables occur. Then $\mathfrak{A} \models \Sigma^c \cup \Sigma^d \cup \Sigma^v$ by (ii)–(iv) above and by ${}_\infty Cs_n \subseteq \mathcal{K}$. However, $\mathfrak{A} \not\models \Sigma$ by (i) and $\mathcal{K} \subseteq RCA_n$, hence $\Sigma \nsubseteq \Sigma^c \cup \Sigma^d \cup \Sigma^v$ which means that Σ contains a formula in which all the cylindrifications $c_0, ..., c_{n-1}$ occur, in which some d_{ij} with $\ell \in \{i,j\}$ occurs and in which more than k variables occur.

If Σ contains at least one formula as described for all $k < \omega$, then Σ contains infinitely many such formulas for all k: Assume that we already have $\varphi_1, \ldots, \varphi_m$ of the desired form for k. Ket K be bigger than the number of variables occurring in $\varphi_1 \wedge \cdots \wedge \varphi_m$, and let φ_{m+1} be a formula of the described form containing more than K variables. There is at least one such, and this φ_{m+1} will be different from $\varphi_1, \ldots, \varphi_m$ and will contain more than k variables.

Just as in the proof of Theorem 3, we may assume $\ell = 0$.

<u>CONSTRUCTION OF $\mathfrak{A}$:</u> Let $K < \omega$ be such that $2^k \leq K \cdot (n-1)$ and let $m = K \cdot (n-1)$. Let $\langle U_i : i \leq n \rangle$ be a system of disjoint sets such that

$$|U_0| = m \quad \text{and} \quad |U_i| \geq \omega \ \text{ for all } \ 0 < i \leq n.$$

Let $f : U_0 \rightarrowtail\!\!\!\!\rightarrow U_0$ be a bijection such that all orbits of f have cardinality K. E.g. we can choose $U_0 = K \times (n-1)$ and define for $i < K,\ j < n-1$
$f(i,j) = (i + 1 (mod\ K), j)$. Let

$$U = \bigcup \{U_i : i \leq n\},$$
$$R = \mathop{\times}_{i<n} U_i,$$
$$F = \{s \in {}^nU : s_0, s_1 \in U_0 \text{ and } s_1 = f(s_0)\}, \quad \text{and let}$$
$\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U \rangle_{i,j<n}$ generated by R, F.

Now R is an atom of $\mathfrak{A}'$, this can be seen exactly as in the previous proofs, i.e. R satisfies condition (*) in Lemma 1. Let $\mathfrak{A}$ be the algebra we obtain from $\mathfrak{A}'$ by splitting R into $m+1$ new atoms $R_j,\ j \leq m$.

CLAIM 7. $\mathfrak{A} \notin RCA_n$.

Proof: For any j define

$$F_1 = F$$
$$F_{j+1} = c_2(s_2^1 F_j \cap s_2^0 F), \text{ and}$$
$$E = F_1 \cup ... \cup F_K.$$

Then $F_1, ..., F_K, E \in A'$ and it is easy to check that the following hold:

$$F_j = \{s \in {}^nU : s_0, s_1 \in U_0,\ s_1 = f^j(s_0)\},$$
$$E = \{s \in {}^nU : s_0 \text{ and } s_1 \text{ are in the same orbit of } f\}.$$

Therefore the following equations are easily seen to hold in $\mathfrak{A}'$ for all $i, j < n, i \neq j$ ((1) and (2),(3) below express that "E is the union of K functions", "E is an equivalence relation with $< n$ blocks on U_0"):

$$(1) \qquad F_i \cap s_2^1 F_i \subseteq d_{12}, \quad E = F_1 \cup ... \cup F_K, \quad F_i = c_2 F_i.$$

$$(2) \qquad c_2(s_2^1 E \cap s_2^0 E) \subseteq E, \quad E = s_0^2 s_1^0 s_2^1 c_2 E, \quad d_{01} \cap c_1 E \subseteq E.$$

$$(3) \qquad \bigcap_{i<n} s_i^0 c_1 E \cap \bigcap_{i<j<n} - s_i^0 s_j^1 E = 0, \quad c_1 E = c_1 ... c_{n-1} R.$$

By $\mathfrak{A}' \subseteq \mathfrak{A}$, (1)–(3) hold in $\mathfrak{A}$, too.

Assume that $\mathfrak{A}$ is represented somehow. Then there is a homomorphism $h : \mathfrak{A} \to \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$ such that $h(R) \neq 0$. We will derive a contradiction. By $h(R) \neq 0$ there is $s \in h(R)$. By $R \leq c_0 R_i$ we have $h(R) \subseteq c_0 h(R_i)$ so there is w_i such that $s(0/w_i) \in h(R_i)$, for all $i \leq m$. These w_i's are different from each other since the R_i's are disjoint from each other and so the $h(R_i)$'s are disjoint from each other. Let

$$H = \{w_i : i \leq m\} \text{ and}$$
$$G = \{(u, v) \in H \times H : s(0/u, 1/v) \in h(E)\}.$$

Then $|H| = m+1$, by the above. Also, G is an equivalence relation on H such that each block of G is smaller than $K+1$, this can be inferred from (1)–(2) as follows: If $(u, v), (v, w) \in G$, then $s(0/u, 1/w, 2/v) \in s_2^1 h(E) \cap s_2^0 h(E)$, thus $s(0/u, 1/w) \in c_2(s_2^1 h(E) \cap s_2^0 h(E)) \subseteq h(E)$, thus $(u, w) \in G$. This shows that G is transitive. If $(u, v) \in G$, then $s(0/u, 1/v) \in h(E)$, therefore $s(0/u, 2/v) \in s_2^1 c_2 h(E)$, thus $s(1/u, 2/v) \in s_1^0 s_2^1 c_2 h(E)$. Then $s(1/u, 0/v) \in s_0^2 s_1^0 s_2^1 c_2 h(E) = h(E)$, so $(v, u) \in G$. This shows that G is symmetric. Finally, G is reflexive because $s(0/w_i) \in h(R_i) \subseteq h(R)$, therefore $s(0/w_i, 1/w_i) \in d_{01} \cap c_1 h(R) \subseteq d_{01} \cap c_1 h(E) \subseteq h(E)$, thus $(u, u) \in G$ for all $u \in H$. We have seen that G is an equivalence relation.

Assume that $(u, v_i) \in G$ for all $i \leq K$. We will show that $v_i = v_j$ for some $i < j < K$. By $(u, v_i) \in G$ we have $s(0/u, 1/v_i) \in h(E) = h(F_1) \cup ... \cup h(F_K)$. So, for each $i \leq K$ let pi be such that $s(0/u, 1/v_i) \in h(F_{pi})$. Then $pi = pj$ for some $i < j \leq K$. But then $s(0/u, 1/v_i, 2/v_j) \in h(F_{pi}) \cap s_2^1 h(F_{pi}) \subseteq d_{12}$, thus $v_i = v_j$. This shows that each block of G is smaller than $K+1$.

We have seen that G is an equivalence relation on H such that each block of G is smaller than $K+1$. By $|H| = m+1 > K\cdot(n-1)$, then G has at least n blocks. Let $v_0, ..., v_{n-1}$ be elements of H in n different blocks and let

$$z = \langle v_0, v_1, ..., v_{n-1}\rangle.$$

Then $z \in \bigcap_{i<j<n} - s^0_i s^1_j h(E)$, by $(v_i, v_j) \notin G$ for $i < j < n$. But also $z \in \bigcap_{i<n} s^0_i c_1 ... c_{n-1} h(R)$ by $v_i \in H$, i.e. by $s(0/v_i) \in h(R)$. These contradict (3). **QED**(Claim 7)

Let $W_0 \supsetneqq U_0$ be disjoint from $U \smallsetminus U_0$, $|W_0| = K\cdot n$, let $W_i = U_i$ for $0 < i \leq n$ and let $W = \bigcup\{W_i : i \leq n\}$.

CLAIM 8. *Let $\gamma < n$. There is an embedding $h : A' \rightarrowtail \langle \mathfrak{P}(^n W), c^W_i, d^W_{ij}\rangle_{i,j<n}$ such that h is a homomorphism w.r.t. all operations of $\mathfrak{A}'$ except for c_γ, and such that $h(R) = \times_{i<n} W_i$.*

Proof: Let us extend our earlier permutation f of U_0 from U_0 to a permutation of W such that f permutes W_0, all orbits of $f \restriction W_0$ are of size K, and f is the identity on $W \smallsetminus W_0$. We will denote this extension with f, too. Let e denote the equivalence relation on W_0 with blocks the orbits of f, i.e.

$$e = \bigcup\{f^l : 1 \leq l \leq K\} \cap (W_0 \times W_0).$$

Let S' be a binary relation on the blocks of e satisfying the same conditions as in the proof of Claim 5, i.e. S' contains the identity relation and each block is in relation with exactly $n-1$ blocks (since we have n blocks, this means that to each block there is exactly one, other, block not related to it). It will be more convenient to view S' as a relation on W_0. Thus we formulate the above conditions as follows. Let $S \subseteq W_0 \times W_0$ be a relation with the following properties for all $u, v, w \in W_0$:

(i) There is z such that $(z, w) \notin S$.
(ii) $v \not S w$, $u \not S w$ imply $u\ e\ v$ and $v \not e w$.
(iii) $v\ S\ w$, $u\ e\ v$ imply $u\ S\ w$.

In other words, the relation S is such that (I)-(III) below are true for it.

(I) $(-S)^{-1}|(W_0 \times W_0) = W_0 \times W_0$
(II) $(-S)|(-S)^{-1} \subseteq e \subseteq S$
(III) $e|S \subseteq S$.

Such a relation S clearly exists.

We will need an equivalence relation on sequences of perhaps different length. Let s, z be two sequences. We define $s \equiv z$ to hold iff the following are satisfied:

$$Dom(s) = Dom(z) \text{ and } Rng(s), Rng(z) \subseteq W. \text{ Let } H = Dom(s).$$
$$s_i \in W_j \text{ iff } z_i \in W_j \quad \text{for all } i \in H,\ j \leq n$$
$$s_l = f^j s_i \text{ iff } z_l = f^j z_i \quad \text{for all } i, l \in H,\ 1 \leq j \leq K.$$

We note that by $f^K = Id_W = \{(w, w) : w \in W\}$, $s \equiv z$ implies $ker(s) = ker(z)$. It is easy to see that for $s, z \in {}^nU$ we have $s \equiv z$ iff $s = \pi \circ z$ for some permutation π of U fixing R and F, therefore

(1) $s \equiv z$ implies $s \in x$ iff $z \in x$ for all $x \in A'$.

We now define a function $g : {}^nW \longrightarrow {}^nU$. Let $s \in {}^nW$ be arbitrary. Let $\Omega_s = \Omega = \{i < n : s_i \in W_0\}$. Let $I = n \smallsetminus \{\gamma\}$.

Assume $\gamma \notin \Omega$, i.e. $\Omega \subseteq I$. Let $s' \in {}^{\Omega}U_0$ be such that $s' \equiv s \restriction \Omega$. Such s' exists by $|\Omega| < n$. We define

$$g(s) = s[\Omega/s'].$$

Assume $\gamma \in \Omega$, i.e. $\Omega \nsubseteq I$. Let $\Omega' = \{i \in \Omega : s_i\ S\ s_\gamma\}$, $\Omega'' = \Omega \smallsetminus \Omega'$. Let $s' \in {}^{\Omega'}U_0$ be such that $s' \equiv s \restriction \Omega'$ and let $s'' \in {}^{\Omega''}(U_n \smallsetminus Rng\ s)$ be such that $ker(s'') = ker(s \restriction \Omega'')$. Such s', s'' exist by $|\{s_i : i \in \Omega'\}| \leq n - 1$, $|U_n| \geq \omega$. We define

$$g(s) = s[\Omega'/s'][\Omega''/s''].$$

This function g is defined analogously to the function in the proof of Theorem 3, and has analogous properties. E.g. it satisfies the following (G1)–(G5) (this is easy to check). Let $s \in {}^nW$, $\Omega = \{i < n : s_i \in W_0\}$, and $u \in W$. Let $\mathbb{Z}$ denote the set of all integers.

(G1) $ker(g(s)) = ker(s)$, and $[s_i = f^l s_j$ iff $g(s)_i = f^l g(s)_j]$ whenever $s_i \equiv g(s)_i$, for all $i, j \in n, l \in \mathbb{Z}$.
(G2) $g(s) \equiv s$ if $\gamma \notin \Omega$.
(G3) $g(s) \restriction (n \smallsetminus \{i\}) \equiv g(s(i/u)) \restriction (n \smallsetminus \{i\})$ if $i \neq \gamma$.
(G4) $g(s)_i \not\equiv s_i$ implies $g(s)_i \in U_n$, $s_i, s_\gamma \in W_0$, $s_i \not S\ s_\gamma$ and $i \neq \gamma$.
(G5) $s_i, s_\gamma \in W_0$, $s_i \not S\ s_\gamma$, $i \neq \gamma$ imply $g(s)_i \in U_n$.

We include verification of the second part of condition (G1). If $\gamma \notin \Omega$, then we are done by (G2). So assume $\gamma \in \Omega$.

To check direction $\rightarrow$, assume $s_i \equiv g(s)_i$ and $s_i = f^l s_j$. If $i \notin \Omega$, then $j \notin \Omega$ by $s_i = f^l s_j$, and so $g(s)_i = s_i$ and $g(s)_j = s_j$ and we are done. If $i \in \Omega$, then $i \in \Omega'$ by $s_i \equiv g(s)_i$, and thus $j \in \Omega'$ by $s_i = f^l s_j$ and condition (iii) on S. Then $g(s)_i = s'_i$ and $g(s)_j = s'_j$ and so we are done by $s' \equiv s \restriction \Omega'$.

To show direction $\leftarrow$, assume $s_i \equiv g(s)_i$ and $g(s)_i = f^l g(s)_j$. If $i \notin \Omega$, then $j \notin \Omega$ by $g(s)_i = f^l g(s)_j$, and so $g(s)_i = s_i$, $g(s)_j = s_j$ and we are done. If $i \in \Omega$, then $i \in \Omega'$ by $s_i \equiv g(s)_i$. Then $j \in \Omega'$ by $g(s)_i = f^l g(s)_j$, because $g(s)_i \in W_0$ by $i \in \Omega'$ and $g(s)_j \notin W_0$ if $j \notin \Omega'$. Then $g(s)_i = s'_i$ and $g(s)_j = s'_j$ and so we are done by $s' \equiv s \restriction \Omega'$.

We define $h : A' \longrightarrow \mathcal{P}({}^nW)$ by

$$h(x) = \{s \in {}^nW : g(s) \in x\}, \quad \text{for all } x \in A'.$$

We begin with showing $h(R) = \times_{i<n} W_i$. Now $s \in h(R)$ iff $g(s) \in R$ iff $g(s)_i \in U_i$ for all $i < n$ iff(by (G4)) $s_i \in W_i$ for all $i < n$ iff $s \in \times_{i<n} W_i$.

Next we show that h is one–one. Let $x \in A'$, $x \neq 0$. Let $s \in x$ be arbitrary and $\Omega = \{i < n : s_i \in U_0\}$. If $\gamma \notin \Omega$ then $g(s) \equiv s$ by (G2), hence $g(s) \in x$ by (1), i.e. $s \in h(x)$. Assume $\gamma \in \Omega$. Let $w \in U_0$ be arbitrary and let $s' \in {}^{\Omega}(\{v \in W_0 : v\ S\ w\})$ be such that $s'_\gamma = w$ and $s' \equiv s \restriction \Omega$. Such an s' exists by $s \in {}^nU$. Let $z = s[\Omega/s']$. Then $g(z) \equiv s$, hence $g(z) \in x$ by (1), i.e. $z \in h(x)$.

By Lemma 4 and (G1), h is a Boolean homomorphism preserving d_{ij} for all $i, j < n$.

It remains to show that h is a homomorphism w.r.t. c_i, $i \neq \gamma$. Let $i < n$, $i \neq \gamma$. Let $s \in {}^nW$ be arbitrary. Let $\Omega = \{j < n : s_j \in W_0\}$. Let $u \in U$, $w \in W$. Consider conditions (2),(3) below.

(2) $[s_j \equiv g(s)_j,\ u = f^l g(s)_j,\ w = f^l s_j]$ or $[s_j \not\equiv g(s)_j,\ w = f^l s_j,\ u \in U_n$ and $(\forall k \in n \smallsetminus \{i\})(u = g(s)_k \text{ iff } w = s_k)]$, for some $1 \leq l \leq K$ and $j \in n \smallsetminus \{i\}$.

(3) $u \notin \{f^l g(s)_j : 1 \leq l \leq K,\ j \in n \smallsetminus \{i\}\}$, $w \notin \{f^l s_j : 1 \leq l \leq K,\ j \in n \smallsetminus \{i\}\}$, and
either $u \equiv w$, $(w \in W_0, \Omega \not\subseteq I \implies w\ S\ s_\gamma)$,
or $u \in U_n$, $w \in W_0$, $\Omega \not\subseteq I$, $w \not S\ s_\gamma$.

Then it is easy to adapt the argument in the proof of Theorem 3 to show that $g(s)(i/u) \equiv g(s(i/w))$ whenever (2) or (3) holds, as follows.

Let $p = g(s)(i/u)$, $q = g(s(i/w))$, and $z = s(i/w)$. By $i \in I$ and (G3) we have

(+) $p \restriction (n \smallsetminus \{i\}) \equiv q \restriction (n \smallsetminus \{i\})$.

Thus we have to show

(*) $p_i \equiv q_i$ and
(**) $(p_i = f^r p_k$ iff $q_i = f^r q_k)$, for $k \in n \smallsetminus \{i\}, r \in \mathbb{Z}$.

Assume that (2) holds with $s_j \equiv g(s)_j$. Then $z_j \equiv g(z)_j$ because by $s_j \equiv g(s)_j$ we have that it is not the case that $s_j, s_\gamma \in W_0$ and $s_j \not S s_\gamma$, and $z_j = s_j$, $z_\gamma = s_\gamma$. Thus $z_i = w = f^l s_j = f^l z_j$ implies, by (G1), that $q_i = f^l q_j$. Also $p_i = u = f^l p_j$. By $j \neq i$ and (+) we have $p_j \equiv q_j$. Thus $p_i = f^l p_j \equiv p_j \equiv q_j \equiv f^l q_j = q_i$, hence $p_i \equiv q_i$. Let $k \in n \smallsetminus \{i\}$, $r \in \mathbb{Z}$ be arbitrary. Now $p_i = f^r p_k$ iff $f^l p_j = f^r p_k$ iff $p_j = f^{(r-l)} p_k$ iff(by (+)) $q_j = f^{(r-l)} q_k$ iff $f^l q_j = f^r q_k$ iff $q_i = f^r q_k$.

Assume now that (2) holds with $s_j \not\equiv g(s)_j$. Then $s_\gamma, s_j \in W_0$, $s_j \not S s_\gamma$, and $g(s)_j \in U_n$ by (G4). By $z_i = w = f^l s_j$ and condition (iii) on S then $w \not S s_\gamma = z_\gamma$, so $q_i = g(z)_i \in U_n$. Thus $q_i \equiv p_i = u \in U_n$. This verifies (*). To show (**), let $k \in n \smallsetminus \{i\}$. Then $p_i = f^l p_k$ iff(by $p_i = u \in U_n$) $u = p_k$ iff $u = g(s)_k$ iff(by our assumption) $w = s_k$ iff $z_i = z_k$ iff(by (G1), $q = g(z)$) $q_i = q_k$ iff(by $q_i \in U_n$) $q_i = f^l q_k$.

Assume now that (3) holds. To show (**), let $k \in n \smallsetminus \{i\}$ and $r \in \mathbb{Z}$ be arbitrary. Then $p_i = u \neq f^r g(s)_k = f^r p_k$ and $z_i = w \neq f^r s_k = f^r z_k$. We show that $z_i \equiv g(z)_i$ or $z_k \equiv g(z)_k$, so $z_i \neq f^r z_k$ implies $g(z)_i \neq f^r g(z)_k$ by (G1), i.e. $q_i \neq f^r q_k$. Assume the contrary, i.e. assume that $z_i \not\equiv g(z)_i$ and $z_k \not\equiv g(z)_k$. Then $z_i, z_k, z_\gamma \in W_0$ and $z_i \not S z_\gamma$, $z_k \not S z_\gamma$ by (G5). Thus $z_i\ e\ z_k$ by condition (ii) on S. Hence $w\ e\ s_k$ by $w = z_i$, $s_k = z_k$, contradicting our hypothesis (3).

To show (*), notice that $p_i = u$ and $z_i = w$. Assume first that $\Omega \subseteq I$, i.e. $\gamma \notin \Omega$. Then $u \equiv w$ and $z_i \equiv q_i$ by (G2), so $p_i \equiv q_i$. Assume now that $\Omega \not\subseteq I$. Assume $u \equiv w$ and $(w \in W_0 \Longrightarrow w\ S\ s_\gamma)$. By $z_\gamma = s_\gamma$, if $w \in W_0$ then $w = z_i\ S\ z_\gamma$, so $q_i \equiv z_i$ by (G4) and hence $p_i \equiv q_i$. Assume $u \in U_n, w \in W_0, w \not S s_\gamma$. Then by (G5) we have $q_i \in U_n$, so $q_i \equiv u = p_i$.

Also, it is easy to modify the argument in the proof of Thm.3 to show that for every $u \in U$ there is $w \in W$ and for every $w \in W$ there is $u \in U$ such that u, w satisfy (2) or (3). We include the modified argument here.

Let $u \in U$ be arbitrary. Assume that $u = f^l g(s)_j$ for some $j \neq i$ and $l \in \mathbb{Z}$. If $s_j \equiv g(s)_j$ then let $w = f^l s_j$. If $s_j \not\equiv g(s)_j$, then $u = g(s)_j \in U_n$ and let $w = s_j$. Then (2) is satisfied, because $u = g(s)_k$ iff $g(s)_j = g(s)_k$ iff $s_j = s_k$ iff $w = s_k$. Assume that $u \notin \{f^l g(s)_j : j \in n \smallsetminus \{k\}, l \in \mathbb{Z}\}$. If $u \notin U_0$ then there is

$w \equiv u$ with $w \notin \{f^l s_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$ by $|U_k| \geq \omega$, $f^l \restriction U_k \subseteq Id$ for all $k > 0$. This w will satisfy (3).

Assume now $u \in U_0$. Assume further $\Omega \subseteq I$, i.e. $s_\gamma \notin W_0$. Then $|U_0 \cap \{f^l s_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}| = |U_0 \cap \{f^l s_j : j \in n \smallsetminus \{i, \gamma\}, l \in \mathbb{Z}\}| \leq K \cdot (n-2)$. Thus there is $w \in U_0 \smallsetminus \{f^l s_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$ and this w will satisfy (3).

Assume now $\Omega \not\subseteq I$. Then $g(s)_j \in U_0$ iff $s_j \; S \; s_\gamma$, for all $j \in n$. By $u \in U_0 \smallsetminus \{f^l g(s)_j : j \in n \smallsetminus \{i\}\}$ and $|U_0| = K \cdot (n-1)$ we have $|\{g(s)_j/e : j \in n \smallsetminus \{i\}, g(s)_j \in U_0\}| \leq n-2$. Then $\{g(s)_j/e : s_j \; S \; s_\gamma\}| \leq n-2$, hence by $\{w/e : w \; S \; s_\gamma\}| = n-1$ there is $w \in W_0 \smallsetminus \{f^l s_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$ with $w \; S \; s_\gamma$. This w satisfies (3).

Conversely, let $w \in W_0$ be arbitrary. Assume that $w = f^l s_j$ for some $j \in n \smallsetminus \{i\}, l \in \mathbb{Z}$. If $s_j \equiv g(s)_j$, then let $u = f^l g(s)_j$. Assume $s_j \not\equiv g(s)_j$. If $w = s_k$ for some $k \in n \smallsetminus \{i\}$, then let $u = g(s)_k$. By $w = f^l s_j$ then $s_k \; e \; s_j$, so $g(s)_k \in U_n$ by condition (iii) on S. If $w = s_r$ for some $r \in n \smallsetminus \{i,k\}$, then $s_r = s_k$, so $u = g(s)_k = g(s)_r$. Thus u satisfies (2). Assume that $w \neq s_k$ for all $k \in n \smallsetminus \{i\}$. Then let $u \in U_n \smallsetminus \{g(s)_k : k \in n\}$ be arbitrary. Then (2) is satisfied.

Assume that $w \notin \{f^l s_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$. If $w \notin W_0$ then there is $u \equiv w$ with $u \notin \{f^l g(s)_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$ by $|U_k| \geq \omega$ for all $k > 0$. Assume $w \in W_0$. Assume further $\Omega \subseteq I$. Then by $g(s)_j \in U_0 \to s_j \in W_0$ for all j, we have $\{g(s)_j \in U_0 : j \in n \smallsetminus \{i\}\} \subseteq \{s_j \in W_0 : j \in n \smallsetminus \{i\}\} = \{s_j \in W_0 : j \in n \smallsetminus \{i,\gamma\}\}$, thus $|\{g(s)_j \in U_0 : j \in n \smallsetminus \{i\}\}| \leq n-2$. Then by $|U_0| = K \cdot (n-1)$ there is $u \in U_0 \smallsetminus \{f^l g(s)_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$. This w will satisfy (3).

Assume now $\Omega \not\subseteq I$, $w \; S \; s_\gamma$. Then $\{j \in n \smallsetminus \{i\} : g(s)_j \in U_0\} \subseteq \{j \in n \smallsetminus \{i\} : s_j \; S \; s_\gamma\}$. So by $w \notin \{f^l s_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$, $w \; S \; s_\gamma$ we have $|\{s_j/e : s_j \; S \; s_\gamma\}| \leq n-2$, so $|\{g(s)_j/e : g(s)_j \in U_0, j \in n \smallsetminus \{i\}\}| \leq n-2$. Thus there is $u \in U_0 \smallsetminus \{f^l g(s)_j : j \in n \smallsetminus \{i\}, l \in \mathbb{Z}\}$. This u satisfies (3).

Assume finally $\Omega \not\subseteq I$, $w \;\not\!S\; s_\gamma$. Then let $u \in U_n \smallsetminus \{g(s)_j : j \in n \smallsetminus \{i\}\}$ be arbitrary. Such a u exists by $|U_j| \geq \omega$, and this u satisfies (3). **QED**(Claim 8)

CLAIM 9. *There is an embedding $h : A' \rightarrowtail \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$ such that $h(R) = \times_{i<n} W_i$ and h is a homomorphism w.r.t. all operations of $\mathfrak{A}'$ except for d_{0i}, d_{i0}, $i < n$.*

The proof of Claim 9 is exactly the same as that of Claim 6, therefore we omit it. Now (ii),(iii) at the beginning of the proof of Theorem 4 follow from Claims 8,9 and Lemma 3(5).

CLAIM 10. *Every k–generated subalgebra of $\mathfrak{A}$ is in $\mathbf{I}_\infty Cs_n$.*

Proof: This follows from Lemma 3(3), $2^k \leq K \cdot (n-1) = |U_0|$ and from Lemma 2. **QED**(Claim 10)

QED(Theorem 4)

Let U be a set and $f : \mathcal{P}({}^nU) \longrightarrow \mathcal{P}({}^nU)$ be a unary function on n–ary relations over U. We say that f is additive (or distributes over join) if

$$f(R \cup S) = f(R) \cup f(S) \quad \text{for all} \quad R, S \subseteq {}^nU.$$

Let $S(U)$ denote the set of all permutations of U. For any $\pi \in S(U)$ and n–ary relation $R \subseteq {}^nU$, $\pi(R)$ denotes the image of R under the permutation π, i.e.

$$\pi(R) = \{\pi \circ s : s \in R\}.$$

We say that f is permutation–invariant if for all permutation π of U and for all n–ary relation $R \subseteq {}^nU$ we have

$$f(\pi R) = \pi(fR).$$

Let α, U be sets, $i, j \in \alpha$. Then $[i,j]^{(\alpha)} \in S(\alpha)$ denotes the permutation of α which interchanges i and j and leaves all other elements fixed. Let $\sigma \in S(\alpha)$ and $\tau : \alpha \longrightarrow \alpha$. Then $p^U_\sigma, \mathsf{s}^U_\tau : \mathcal{P}({}^\alpha U) \longrightarrow \mathcal{P}({}^\alpha U)$ are defined as follows. For any $x \subseteq {}^\alpha U$

$$\begin{aligned}
&p^U_\sigma(x) = \{s \circ \sigma : s \in x\},\\
&\mathsf{s}^U_\tau(x) = \{s \in {}^\alpha U : s \circ \tau \in x\},\\
&p^U_{ij} = p^U_{[i,j]} \quad \text{and}\\
&\mathsf{s}^U_{ij}(x) = \{s \in {}^\alpha U : s(i/s_j) \in x\},
\end{aligned}$$

i.e. $\mathsf{s}^U_{ij} = \mathsf{s}^U_{[i/j]}$, where $[i/j] : \alpha \longrightarrow \alpha$ is identity everywhere except on i where its value is j. We often omit the upper indices.

We note that $c^U_i, p^U_\sigma, \mathsf{s}^U_\tau$ are all additive, permutation–invariant, unary operations and $\mathsf{s}_{ij}(x) = s^i_j(x)$, and $\mathsf{s}_{ij}(x) = c_i(d_{ij} \cap x)$ if $i \neq j$.

The next theorem states that no unary, additive, permutation invariant functions help in finitely axiomatizing RCA_n for $3 \leq n < \omega$. Theorem 5 below complements Sain[87a]'s result and extends Biró[89]'s result from first–order to "beyond first–order". For more detail on this see Remark 5 after Theorem 5.

THEOREM 5. *Let $3 \leq n < \omega$. For any set U let $f^U_1, ..., f^U_r$ be at most unary, additive, permutation–invariant functions on $\mathcal{P}({}^nU)$. Let*

$$RA^+_n = \mathbf{S}\{\langle \mathfrak{P}({}^nU), c^U_i, d^U_{ij}, f^U_1, ..., f^U_r\rangle_{i,j<n} : U \text{ is a set}\}.$$

Let Σ be any set of quantifier–free formulas valid in RA_n^+ and containing only finitely many variables. Then Σ does not axiomatize the set of equations valid in RCA_n, i.e. there is an equation valid in RCA_n which does not follow from Σ. Moreover, there is an equation e with $RCA_3 \models e$ but $\Sigma \not\models e$.

Proof: PLAN: Let $k < \omega$ be arbitrary. We will construct an algebra $\mathfrak{A}$ with the following properties.

a) $\mathfrak{A} \notin RA_n^+$, moreover $\mathfrak{A} \not\models e$ for an equation e valid in RCA_3.
b) Every k–generated subalgebra of $\mathfrak{A}$ is in RA_n^+.

This will prove the theorem because of the following. Let Σ be as in the statement of the theorem, let Σ contain k variables. Let $\mathfrak{A}, e$ satisfy a),b). Then $\mathfrak{A} \models \Sigma$ by b), and $\mathfrak{A} \not\models e$ by a), hence $\Sigma \not\models e$ and e is an equation valid in RCA_3.

CONSTRUCTION OF $\mathfrak{A}$: Let $m \geq 2^{(1+k \cdot n!)}$, $m < \omega$. Let $U_0 = m = \{i : i < m\}$, $U_i = S(m) \times \{i\}$ for all $0 < i < n$ and let $U = \bigcup\{U_i : i < n\}$. Let

$R = \underset{i<n}{\times} U_i$,
$F = \{s \in {}^nU : s_0, s_1 \in U_0, s_1 = s_0 + 1 (mod\ m)\}$, and let
$\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U, p_{ij}^U \rangle_{i,j<n}$ generated by R, F.

Then R is an atom of $\mathfrak{A}'$, this can be seen by checking that R satisfies (*) in Lemma 1 and that Lemma 1 remains true if we add any permutation–invariant functions to our algebras. (Now we added p_{ij}.)

Our algebra $\mathfrak{A}$ will be obtained from $\mathfrak{A}'$ basically by splitting R into $m+1$ parts, $R_j, j \leq m$. Then $\mathfrak{A}$ will be nonrepresentable by the arguments used so far. However, we have to define the operations $f_1, ..., f_r$ on the new atoms $R_j, j \leq m$ in such a way that the k–generated subalgebras stay representable. This is now difficult because we know almost nothing about the operations $f_1, ..., f_r$. Therefore first we will split R in $\mathfrak{A}'$ into m parts $R'_j, j < m$ only, but in a "good way". Then we will observe how the "real" operations $f_1^U, ..., f_r^U$ behave on $R'_j, j < m$, and then try to copy this behaviour to the abstract atoms $R_j, j \leq m$.

CLAIM 11. *There is a partition $\langle R_j : j < m \rangle$ of R satisfying the following properties (R1)–(R3) for all $j < m$, $0 < i < n$ and $s \in R$.*

(R1) $|\{u \in U : s(0/u) \in R_j\}| = 1$.
(R2) $|\{u \in U : s(i/u) \in R_j\}| \geq 2n$.
(R3) *For every $\rho \in S(m)$ there is $\pi \in S(U)$ such that $\pi(R_j) = R_{\rho(j)}$ and $\pi(a) = a$ for all $j < m$ and $a \in \mathfrak{A}'$.*

Proof: Recall that $U_0 = m = \{i : i < m\}$ and $U_i = S(U_0) \times \{i\}$ for all $0 < i < n$. For any $j < m$ define

$$R_j = \{\langle u, (\sigma_1, 1), ..., (\sigma_{n-1}, n-1)\rangle : u \in U_0, \sigma_1, ..., \sigma_{n-1} \in S(U_0), \sigma_1 ... \sigma_{n-1}(u) = j\}.$$

Now $\langle R_j : j < m\rangle$ is a partition of R satisfying (R1).

Let $0 < i < n$ and $s = \langle u, (\sigma_1, 1), ..., (\sigma_{n-1}, n-1)\rangle \in R$ be arbitrary. Let $\sigma = \sigma_1 \circ \sigma_2 \circ ... \circ \sigma_{i-1}$ and $v = \sigma_{i+1} ... \sigma_{n-1}(u)$. Let $\delta \in S(U_0)$ be such that $\delta(v) = \sigma^{-1}(j)$. There are $(m-1)! \geq 2n$ many such choices for δ. Let $z = s(i/(\delta, i))$. Then $z \in R_j$ by $\sigma_1\sigma_2...\sigma_{i-1}\delta\sigma_{i+1}...\sigma_{n-1}(u) = \sigma\delta(v) = j$. Thus (R2) is satisfied.

To prove (R3), let $\rho \in S(m)$ be arbitrary. Let $\pi \in S(U)$ be defined by

$$\pi \restriction (U \smallsetminus U_1) \subseteq Id \ \text{ and } \ \pi((\sigma, 1)) = (\rho \circ \sigma, 1) \ \text{ for all } \sigma \in S(U_0).$$

Let $j < m$. We show that $\pi R_j = R_{\rho(j)}$. Let $s \in R$. Then $s = \langle u, (\sigma_1, 1), ..., (\sigma_{n-1}, n-1)\rangle$ for some $u \in U_0$ and $\sigma_1, ..., \sigma_{n-1} \in S(U_0)$ and $\pi s = \langle u, (\rho \circ \sigma_1, 1), ..., (\sigma_{n-1}, n-1)\rangle$. Now $s \in R_j$ iff $\sigma_1 ... \sigma_{n-1}(u) = j$ iff $\rho\sigma_1 ... \sigma_{n-1}(u) = \rho j$ iff $(\pi \circ s) \in R_{\rho(j)}$. Thus $\pi(R_j) = R_{\rho(j)}$. Clearly, $\pi R = R$ and $\pi F = F$, thus $\pi a = a$ for all $a \in A'$ since A' is generated by R, F. **QED**(Claim 11)

Let $R_0, ..., R_{m-1}$ be a partition of R satisfying (R1)–(R3). Let A'' be the universe of the subalgebra of $\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U, p_{ij}^U\rangle_{i,j<n}$ generated by $\{R_0, ..., R_{m-1}, F\}$. We first show that A'' is closed under $f_1^U, ..., f_r^U$ whatever they may be.

CLAIM 12. *A'' is closed under all unary, additive, permutation–invariant functions.*

Proof: To begin, we show that $\mathfrak{A}'$ is closed under all permutation–invariant functions. Recall the notation $Fix(R, F)$ from Lemma 1. Then $Fix(R, F) = \{\pi \in S(U) : \pi^*(U_i) = U_i$ for all $i < n$ and there is $l \in U_0$ such that $\pi(u) = u + l(mod\ m)$ for all $u \in U_0\}$. From this it is not difficult to check that

$$\{\pi \circ s : \pi \in Fix(R, F)\} \in A' \ \text{ for all } s \in {}^nU.$$

This, together with $\pi(a) = a$ for all $a \in A'$ and $\pi \in Fix(R, F)$ implies that

$$A' = \{a \subseteq {}^nU : \pi(a) = a \text{ for all } \pi \in Fix(R, F)\}.$$

This immediately implies that

(1) $\quad A'$ is closed under all permutation-invariant functions.

Let $\mathcal{R} = \{p_\sigma R_j : \sigma \in S(n), j < m\}$. Next we show that

$$A'' = \{a + \sum X : a \in A', X \subseteq \mathcal{R}\}. \tag{2}$$

Indeed, let $H = \{a + \sum X : a \in A', X \subseteq \mathcal{R}\}$, $i, j < n$ and $\sigma \in S(n)$. Clearly, $H \subseteq A''$ and H is closed under the Boolean operations. By (R1),(R2) we have that

$$c_i p_\sigma R_j = c_i p_\sigma R \quad \text{for all} \quad j < m, \sigma \in S(n),$$

thus H is closed under c_i. Since p_σ is additive and both A' and $\mathcal{R}$ are closed under p_σ, we have that H is closed under p_σ. Finally, $d_{ij} \in A' \subseteq H$. We have proved (2). By (2) we have that every element of $\mathcal{R}$ is an atom of A''.

Let f be any unary, additive, permutation–invariant function on $\mathcal{P}({}^nU)$. We turn to proving that A'' is closed under f. Since f is additive and $\mathcal{R}$ is finite, by (1) it is enough to show that $f(p_\sigma R_j) \in A''$ for all $\sigma \in S(n)$, $j < m$. Let $\sigma \in S(n)$, $j < m$, $a = p_\sigma R_j$. Since ${}^nU$ is finite, A'' is also finite, thus every element of A'' is a finite sum of atoms. Let y be any atom of A''. We will show that $y \cap f(a) \neq 0$ implies $y \subseteq f(a)$. This will imply that $f(a)$ is a union of atoms of A'', hence $f(a) \in A''$.

Let $s \in y \cap f(a)$ and $z \in y$ be arbitrary. By finiteness of ${}^nU$ and additivity of f, $s \in f(a)$ implies that $s \in f(\{q\})$ for some $q \in a$. We will construct a $\pi \in S(U)$ such that $\pi \circ q \in a$ and $\pi \circ s = z$. This will suffice because then $z = \pi \circ s \in \pi f\{q\} = f\{\pi \circ q\} \subseteq f(a)$.

Let $V = Rng\ q \smallsetminus Rng\ s$. If $V = \emptyset$ then $s = q \circ \delta$ for some $\delta \in S(n)$, because q is repetition–free. Then $s \in p_\delta a$, hence $z \in p_\delta a$ because s, z are contained in the same atom y. Let $q' = z \circ \delta^{-1}$. Then $z = q' \circ \delta$ and $q' \in a$ by $z \in p_\delta a$. Let $\pi \in S(U)$ be such that $\pi \circ q = q'$. Such a π exists because q, q' are repetition–free. Now $\pi \circ s = z$ by $s = q \circ \delta, z = q' \circ \delta$ and we are done.

Assume $V \neq \emptyset$. Let $V = \{q_{p0}, ..., q_{pt}\}$ for some t such that $q_{p0} \in U_0$ if $U_0 \cap V \neq \emptyset$ and the sequence $q_{p0}, ..., q_{pt}$ is without repetitions. By $q \in p_\sigma R_j$ we have $q_i \in U_{\sigma(i)}$ for all $i < n$. Let $i < n$ be such that $q_i \notin V$. Then $q_i = s_l$ for some $l < n$. Let us define

$$q'_i = z_l.$$

This definition is sound by $ker(s) = ker(z)$. Also, $q'_i \in U_{\sigma(i)}$ because s, z are contained in the same atom of A'' and $q_i = s_l \in U_{\sigma(i)}$. Now for any $i \leq t$ choose $q'_{pi} \in U_{\sigma(pi)} \smallsetminus Rng\ z$ such that for all $l \leq t$ $q'_{pi} \neq q'_{pl}$ if $pi \neq pl$. Such a choice is possible because $|U_{\sigma(pi)}| \geq 2n$. By the above we defined q'_i for all $i < n$. Let $q' = \langle q'_i : i < n \rangle$. Then $q' \in p_\sigma R$ and $q_i = s_l \Longleftrightarrow q'_i = z_l$ for all $i, l < n$.

Let $i = pt$. We will show that there is $u \in U_{\sigma(i)} \setminus Rng\ z$ such that $q'(i/u) \in a$. If $\sigma(i) \neq 0$ then by (R2) we can choose such a u. Assume therefore $\sigma(i) = 0$. Then by (R1), the $u \in U_0$ for which $q'(i/u) \in a$ is unique. We will show that $u \notin Rng\ z$. By $\sigma(i) = 0$ we have $q_i \in U_0$. Then $t = 0$ and $V = \{q_i\}$ by our hypothesis on the order of enumerating V (and since $|V \cap U_0| \leq 1$). Also, $s = q(i/v) \circ \delta$ and $z = q'(i/v') \circ \delta$ for some $\delta \in S(n)$ and $v, v' \in U$. By $\sigma(i) = 0$ and (R1), there is only one w for which $q(i/w) \in a$. By $v \neq q_i$ and $q \in a$ then $q(i/v) \notin a$. Thus $s \notin p_\delta a$, hence $z \notin p_\delta a$ since s and z are contained in the same atom of A'', and thus $q'(i/v') \notin a$. Hence $u \neq v'$ by $q'(i/u) \in a$. By $z = q'(i/v') \circ \delta$ then $u \notin Rng\ z$ and we are done. Let $q'' = q'(i/u)$. Then $q'' \in a$.

Now there is $\pi \in S(U)$ such that $\pi \circ q = q''$ and $\pi \circ s = z$ because q, q'' are repetition–free, $ker(s) = ker(z)$ and $q''_i = z_l$ iff $q_i = s_l$ for all $i, l < n$.
QED(Claim 12)

We define

$$\mathfrak{A}'' = \langle A'', \cup, \setminus, c_i^U, d_{ij}^U, f_1^U, ..., f_r^U \rangle_{i,j<n}.$$

Clearly, $\mathfrak{A}'' \in RA_n^+$.

We will define our algebra $\mathfrak{A} \notin RA_n^+$ by modifying $\mathfrak{A}'' \in RA_n^+$. The idea is that we split R into more than m parts, but otherwise we let the construction be "the same". In order to be able to do this, we prove that the operations $f_1^U, ..., f_r^U$ do not "distinguish" the R_j's, moreover they are defined on the R_j's according to some scheme that can be applied to an "imaginary R_j ", too. Here we will use condition (R3) in Claim 11. Then we will prove that the k–generated subalgebras of $\mathfrak{A}$ are isomorphic to subalgebras of $\mathfrak{A}''$. We shall need the second statement of Claim 13 below when showing this.

CLAIM 13. *Let f be a unary, additive, permutation–invariant function on $\mathcal{P}({}^nU)$ and let $\sigma \in S(n)$. Then there are terms $\tau_\sigma(x), \tau'_\sigma(x)$ using the operations $+, -, c_i$, $d_{ij}, p_\delta,\ i, j < n, \delta \in S(n)$ and some constants from A' (i.e. $\tau_\sigma, \tau'_\sigma$ are terms of the language of $\mathfrak{P} = \langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U, p_\delta^U, a \rangle_{i,j<n,\delta\in S(n),a\in A'}$) such that (i)–(ii) below hold.*

(i) *$f(p_\sigma^U R_j) = \tau_\sigma(R_j)$ for all $j < m$ (in $\mathfrak{P}$).*

(ii) *Let $\mathfrak{C}$ be any algebra similar to $\mathfrak{P}$ and such that $(C, +^{\mathfrak{C}}, -^{\mathfrak{C}})$ is a Boolean algebra and the $p_\delta^{\mathfrak{C}}$'s are additive. Let $|J| \geq 2$, J finite, and let $a_j, j \in J$ be disjoint elements of C. Then $\sum\{\tau_\sigma(a_j) : j \in J\} = \tau'_\sigma(\sum\{a_j : j \in J\})$ in $\mathfrak{C}$.*

Proof: Let f, σ be as in the statement of Claim 13. For any permutation $\delta \in S(n)$ let

$$F_\delta = f(p_\sigma R_0) \cap p_\delta R$$

and define the term $\varepsilon_\delta(x)$ as

$$\varepsilon_\delta(x) = \begin{cases} 1 & \text{if } F_\delta = p_\delta R \\ x & \text{if } F_\delta = p_\delta R_0 \\ 0 & \text{if } F_\delta = 0 \\ -x & \text{otherwise.} \end{cases}$$

Let $b = f(p_\sigma R_0) - \sum\{p_\delta R : \delta \in S(n)\}$ and define the term $\tau_\sigma(x)$ as

$$\tau_\sigma(x) = \sum\{p_\delta \varepsilon_\delta(x) \cdot p_\delta R : \delta \in S(n)\} + b.$$

We note that $b \in A'$ by Claim 12 and (2) in the proof of Claim 12. We now will use (R3) to prove that $\tau_\sigma(x)$ has the desired property. We will also use that p_δ is permutation–invariant for all δ. Let $j < n$. We want to show $f(p_\sigma R_j) = \tau_\sigma(R_j)$.

From (R3) we prove the following statements (3)–(6).

(3)

$$p_\delta R_i \subseteq f(p_\sigma R_j) \text{ for some } i < n, i \neq j \quad \text{implies} \quad p_\delta R_i \subseteq f(p_\sigma R_j) \text{ for all } i < n, i \neq j.$$

Indeed, let $i, l \neq j$. Assume $p_\delta R_i \subseteq f(p_\sigma R_j)$. Let $\pi \in S(U)$ be such that $\pi(R_i) = R_l$ and $\pi(R_j) = R_j$. Then $p_\delta R_l = p_\delta \pi R_i = \pi p_\delta R_i \subseteq \pi f(p_\sigma R_j) = f(\pi p_\sigma R_j) = f(p_\sigma(\pi R_j)) = f(p_\sigma R_j)$.

By (3) we have that $f(p_\sigma R_j) \cap p_\delta R \in \{p_\delta R, p_\delta R_j, p_\delta R - p_\delta R_j, 0\}$.

$$(4) \qquad p_\delta R_j \subseteq f(p_\sigma R_j) \quad \text{implies} \quad p_\delta R_i \subseteq f(p_\sigma R_i).$$

Indeed, let $\pi \in S(U)$ be such that $\pi R_j = R_i$. Then $p_\delta R_i = p_\delta \pi R_j = \pi p_\delta R_j \subseteq \pi f(p_\sigma R_j) = f(\pi p_\sigma R_j) = f(p_\sigma \pi R_j) = f(p_\sigma R_i)$.

$$(5) \qquad p_\delta R - p_\delta R_j \subseteq f(p_\sigma R_j) \quad \text{implies} \quad p_\delta R - p_\delta R_i \subseteq f(p_\sigma R_i).$$

Indeed, let $l \in n \smallsetminus \{i, j\}$ and let $\pi \in S(U)$ be such that $\pi(R_j) = R_i$ and $\pi(R_l) = R_l$. Then $p_\delta R_l = \pi p_\delta R_l \subseteq \pi f(p_\sigma R_j) = f(p_\sigma R_i)$, and we are done by (3).

By (3)–(5) we have $f(p_\sigma R_j) \cap p_\delta R = \tau_\sigma(R_j) \cap p_\delta R$, for all $\delta \in S(n)$ and $j < m$. Let $X = -\sum\{p_\delta R : \delta \in S(n)\}$.

$$(6) \qquad f(p_\sigma R_j) \cap X = f(p_\sigma R_i) \cap X \quad \text{for all } i, j < m.$$

By (2) we have that $X \in A'$ and $f(p_\sigma R_i) \cap X \in A'$. Let $\pi \in S(U)$ be such that $\pi R_i = R_j$ and $\pi a = a$ for all $a \in A'$. Then $f(p_\sigma R_j) \cap X = f(p_\sigma \pi R_i) \cap \pi X = \pi f(p_\sigma R_i) \cap \pi X = \pi(f(p_\sigma R_i) \cap X) = f(p_\sigma R_i) \cap X$.

We have proved $f(p_\sigma R_j) = \tau_\sigma(R_j)$ for all $j < m$. To prove the second statement of Claim 13, let us define for all $\delta \in S(n)$

$$\varepsilon'_\delta(x) = \begin{cases} 1 & \text{if } \varepsilon_\delta(x) = -x \\ \varepsilon_\delta(x) & \text{otherwise,} \end{cases}$$

and

$$\tau'_\sigma(x) = \sum\{p_\delta \varepsilon'_\delta(x) \cdot p_\delta R : \delta \in S(n)\} + b.$$

Now it is easy to check that τ'_σ has the desired property. **QED**(Claim 13)

We are ready to define our final algebra $\mathfrak{A}$. For any $\sigma \in S(n)$ and $j \leq m$ let $X_{\sigma j}$ be new elements, different for different pairs $\langle \sigma, j \rangle$. We will write X_j for $X_{Id\, j}$. Let $\mathcal{X} = \{X_{\sigma j} : \sigma \in S(n), j \leq m\}$, $\mathcal{Y} = At\mathfrak{A}' \setminus \{p_\sigma R : \sigma \in S(n)\}$. For $f_1^U, ..., f_r^U$ and $\sigma \in S(n)$ let the terms $\tau_\sigma^1, ..., \tau_\sigma^r$ be as in Claim 13. Below, we will define the auxiliary functions $p_\delta^{\mathfrak{A}}$, too. We will need them when defining $f_i^{\mathfrak{A}}$. We define $\mathfrak{A} = \langle A, +, -, c_i^{\mathfrak{A}}, d_{ij}^{\mathfrak{A}}, f_1^{\mathfrak{A}}, ..., f_r^{\mathfrak{A}} \rangle$ as follows. Let $i < n$, $\delta, \sigma \in S(n)$, $j \leq m$.

> $\langle A, +, - \rangle$ is a Boolean algebra with atoms $\mathcal{Y} \cup \mathcal{X}$.
> $\mathfrak{A}' \subseteq \langle A, +, -, c_i^{\mathfrak{A}}, d_{ij}^{\mathfrak{A}} \rangle_{i,j<n}$, $\quad R = \sum\{X_j : j \leq m\}$.
> the operations $c_i^{\mathfrak{A}}, p_\delta^{\mathfrak{A}}, f_1^{\mathfrak{A}}, ..., f_r^{\mathfrak{A}}$ are additive,
> $c_i^{\mathfrak{A}} X_{\sigma j} = c_i^U p_\sigma^U R$, $\quad p_\delta^{\mathfrak{A}} X_{\sigma j} = X_{\sigma \circ \delta j}$, $\quad p_\delta^{\mathfrak{A}} \restriction A' = p_\delta^U \restriction A'$ and
> $f_1^{\mathfrak{A}}(X_{\sigma j}) = \tau_\sigma^{1\mathfrak{A}}(X_j), ..., f_r^{\mathfrak{A}}(X_{\sigma j}) = \tau_\sigma^{r\mathfrak{A}}(X_j)$,
> $f_1^{\mathfrak{A}} \restriction A' = f_1^U \restriction A', ..., f_r^{\mathfrak{A}} \restriction A' = f_r^U \restriction A'$.

CLAIM 14. *There is an equation e valid in RCA_n such that $\mathfrak{A} \not\models e$. Moreover, $RCA_3 \models e$.*

Proof: Our algebra $\mathfrak{A}$ is very similar to the one used in the proof of Theorem 1 for finite n; and the proof of Claim 14 is practically the same as that of Claim 3. Now we indicate how the equation e exhibited in Remark 2 fails in $\mathfrak{A}$. Let $x, x_0, ..., x_m, y_0, ... y_m$ be variables and let e denote the following equation

$$\prod_{j \leq m} c_0(x \cdot x_j \prod_{j \neq k \leq m} -x_k) \;\leq\; c_0 c_1 c_2 \prod_{j<m} c_1 x \cdot s_1^0 c_1 x - [c_2 y_j - c_2(s_2^1 c_2 y_j - d_{12})].$$

Now $\mathfrak{A} \not\models e$ can be seen by evaluating the variable x to R, the variables x_j to $X_j \in A$ and evaluating the variables y_j to $F_j \in A'$. Then the value of the term

on the left–hand side of $\leq$ in e is $c_0R \neq 0$ while the value of the term on the right–hand side of $\leq$ is 0. Next we show that $RCA_n \models e$. Indeed, this equation e is the same as ε_m from Remark 2, and we showed in the proof of Thm.2 that $SNr_nCA_{n+2} \models \varepsilon_m$. By $RCA_n \subseteq SNr_nCA_{n+2}$ this implies $RCA_n \models e$.
QED(Claim 14)

CLAIM 15. *Every k–generated subalgebra of $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{A}''$.*

Proof: Let $G \subseteq A, |G| \leq k$. We define the equivalence relation $\equiv$ on $m+1$ by

$$i \equiv j \quad \text{iff} \quad (\forall g \in G)(\forall \sigma \in S(n))[X_i \leq p_\sigma g \iff X_j \leq p_\sigma g].$$

Let p be the number of blocks of $\equiv$. Then $p \leq 2^{k \cdot n!}$ because $|\{p_\sigma g : g \in G, \sigma \in S(n)\}| \leq k \cdot n!$. Define

$$B = \{a \in A : (\forall i, j \leq m)(\forall \sigma \in S(n))[i \equiv j \text{ and } X_i \leq p_\sigma a \text{ imply } X_j \leq p_\sigma a]\}.$$

We now show that B is closed under the operations of $\mathfrak{A}$. Let $b \in A$ be arbitrary. Then for any $\sigma \in S(n)$ there are sets $H_\sigma \subseteq m+1$ and there is $a \in A'$ such that $a \cdot p_\sigma R = 0$ for all $\sigma \in S(n)$ and

$$b = a + \sum\{X_{\sigma j} : \sigma \in S(n), j \in H_\sigma\}.$$

It is not difficult to check that $A' \subseteq B$ and

$$b \in B \quad \text{iff} \quad \text{for every } \sigma \in S(n) \;\; H_\sigma \text{ is a union of blocks of } \equiv.$$

This immediately implies that B is closed under the Boolean operations. By $c_iX_{\sigma j} = c_ip_\sigma R \in A'$ we have $c_ib \in A'$ for all $b \in A$, hence B is closed under c_i. Let $\delta \in S(n)$. Then

$$p_\delta b = p_\delta a + \sum\{X_{\sigma \circ \delta j} : \sigma \in S(n), j \in H_\sigma\},$$

and hence $p_\delta b \in B$ by $p_\delta a \in A'$. Finally, we show that B is closed under the functions $f_1, ..., f_r$. Let f be one of $f_1, ..., f_r$ and let $\tau_\sigma, \tau'_\sigma$ be the terms using $+, -, c_i, d_{ij}, p_\delta$ and constants from A' belonging to f according to Claim 13. By additivity of f,

$$f(b) = f(a) + \sum\{f(\sum\{X_{\sigma j} : j \in H_\sigma\}) : \sigma \in S(n)\}.$$

Now $f(a) \in A' \subseteq B$ by (1) in the proof of Claim 12. Let $\sigma \in S(n)$ be such that $H_\sigma \neq \emptyset$. Then $f(\sum\{X_{\sigma j} : j \in H_\sigma\}) = \sum\{f(X_{\sigma j}) : j \in H_\sigma\} = \sum\{\tau_\sigma(X_j) : j \in$

$H_\sigma\} = \tau''_\sigma(\sum\{X_j : j \in H_\sigma\})$, where τ''_σ is τ_σ if $|H_\sigma| = 1$, otherwise τ''_σ is τ'_σ. Now $x = \sum\{X_j : j \in H_\sigma\} \in B$ since H_σ is a union of blocks of $\equiv$, hence $\tau''_\sigma(x) \in B$ because B is closed under $+, -, c_i, d_{ij}, p_\delta$ and $A' \subseteq B$. We have seen that $f(b) \in B$ for all $b \in B$.

Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ with universe B. Clearly, $G \subseteq B$, hence the subalgebra of $\mathfrak{A}$ generated by G is contained in $\mathfrak{B}$. Therefore it is enough to show that $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}''$.

Let us recall that $\equiv$ contains $\leq 2^{k\cdot n!}$ blocks and $m \geq 2^{1+k\cdot n!} = 2 \cdot 2^{k\cdot n!}$. Let $\langle E_j : j < p\rangle$ be the partition of $m+1$ belonging to $\equiv$. Then there is a partition $\langle G_j : j < p\rangle$ of m with the following property:

$$|E_j| > 1 \quad \text{iff} \quad |G_j| > 1, \qquad \text{for all } j < p.$$

For any $H \subseteq m+1$ define

$$\gamma(H) = \bigcup\{G_k : k < p, E_k \subseteq H\}.$$

We define $h : B \longrightarrow A''$ as follows. For $a \in A'$ and $\langle H_\sigma : \sigma \in S(n)\rangle$ with $H_\sigma \subseteq m+1$ for all $\sigma \in S(n)$,

$$h(a + \sum\{X_{\sigma j} : \sigma \in S(n), j \in H_\sigma\}) = a + \sum\{p_\sigma R_j : \sigma \in S(n), j \in \gamma(H_\sigma)\}.$$

In more explicit form, the definition of h is: For any $b \in B$

$$\begin{aligned} &h(b) = b - (\textstyle\sum\{X_{\sigma j} : \sigma \in S(n), j \leq m\}) \\ &+ \textstyle\sum\{p_\sigma R_j : \sigma \in S(n), (\exists k, i)[j \in G_k, i \in E_k, X_{\sigma i} \leq b]\}. \end{aligned}$$

We want to show that h is a one–one homomorphism. It is easy to check that h is a Boolean homomorphism and h is one–one. Also, it is not difficult to check that h is a homomorphism w.r.t. c_i and p_δ for $i < n, \delta \in S(n)$, and that h is the identity on A'. Let f be one of the operations $f_1^{\mathfrak{A}}, ..., f_r^{\mathfrak{A}}$. We check that h is a homomorphism w.r.t. f.

Let $b \in B$. Then $b = a + \sum\{X_{\sigma j} : \sigma \in S(n), j \in H_\sigma\}$ for some $a \in A'$ and sets $H_\sigma \subseteq m+1$ such that each H_σ is a union of blocks of $\equiv$. Thus $b = a + \sum\{b_{\sigma j} : \sigma \in S(n), j \in P_\sigma\}$ where $P_\sigma \subseteq p$ and $b_{\sigma j} = \sum\{X_{\sigma k} : k \in E_j\}$. Consider $b_{\sigma j}$ for $j \in P_\sigma$. Then

$$f(b_{\sigma j}) = \begin{cases} \tau_\sigma(\sum\{X_k : k \in E_j\}) & \text{if } |E_j| = 1 \\ \tau'_\sigma(\sum\{X_k : k \in E_j\}) & \text{if } |E_j| > 1. \end{cases}$$

and

$$h(b_{\sigma j}) = \sum\{p_\sigma R_k : k \in G_j\},$$
$$fh(b_{\sigma j}) = \begin{cases} \tau_\sigma(\sum\{R_k : k \in G_j\}) & \text{if } |G_j| = 1 \\ \tau'_\sigma(\sum\{R_k : k \in G_j\}) & \text{if } |G_j| > 1. \end{cases}$$

By h being a homomorphism w.r.t. $+, -, c_i, d_{ij}, p_\delta$ and being identity on A' then

$$hf(b_{\sigma j}) = \begin{cases} \tau_\sigma(\sum\{R_k : k \in G_j\}) & \text{if } |E_j| = 1 \\ \tau'_\sigma(\sum\{R_k : k \in G_j\}) & \text{if } |E_j| > 1. \end{cases}$$

By $|E_j| > 1$ iff $|G_j| > 1$ we then have $hf(b_{\sigma j}) = fh(b_{\sigma j})$. Now $h(fb) = h(f(a) + \sum\{f(b_{\sigma j}) : \sigma \in S(n), j \in P_\sigma\}) = f(ha) + \sum\{fh(b_{\sigma j}) : \sigma \in S(n), j \in P_\sigma\} = f(h(a) + \sum\{h(b_{\sigma j}) : \sigma \in S(n), j \in P_\sigma\}) = fh(a + \sum\{b_{\sigma j} : \sigma \in S(n), j \in P_\sigma\}) = fh(b)$. We have seen that h is a one–one homomorphism, therefore $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}''$. **QED**(Claim 15)

QED(Theorem 5)

REMARK 5. (i) In Sain[87a] it is proved that if $n \geq \omega$ then there are unary, additive, permutation invariant functions $f_1^U, ..., f_5^U$ such that the variety generated by $\{\langle \mathfrak{P}({}^nU), c_0^U, f_1^U, ..., f_5^U\rangle : U \text{ is a set}\}$ is finitely axiomatizable (in fact, axiomatizable over Boolean algebras by using only one variable) and the operations $c_i (i < n)$ are all term-definable. Thus our Theorem 5 shows that the condition $n \geq \omega$ and dropping the diagonal constants in Sain's result are essential: without these there is a very strong negative result.

(ii) The condition $n < \omega$ cannot be omitted from Theorem 5, because of the following. Assume $n \geq \omega$ and let H, G be a partition of n into two parts of size $|n|$. Let f, g, h, k be functions mapping n into n with the following properties:

$$f : n \rightarrowtail\!\!\!\!\rightarrow H, \quad g \supseteq f^{-1},$$
$$h : n \rightarrowtail\!\!\!\!\rightarrow G, \quad k \supseteq h^{-1}.$$

For any set U, $\Gamma \subseteq n$ and $X \subseteq {}^nU$ define

$$c^U_{(\Gamma)}X = \{s \in {}^nU : s \upharpoonright \Gamma \subseteq z \text{ for some } z \in X\}.$$

We claim that the equational theory of

$$RA_n^+ = \{\langle \mathfrak{P}({}^nU), c_i, d_{ij}, \mathsf{s}_f, \mathsf{s}_g, \mathsf{s}_h, \mathsf{s}_k, c_{(H)}, c_{(G)}\rangle_{i,j<n} : U \text{ is a set}\}$$

follows from the set of all equations containing only one variable and valid in RA^+_n. The reason for this is that the extra operations can be used to "code together two variables" as follows. Let

$$\tau(x,y) = \mathsf{s}_f x \cap \mathsf{s}_h y.$$

Then it is not difficult to check that

$$\mathsf{s}_g c_{(G)}\tau(x,y) = x \quad \text{if} \quad y \neq 0 \quad \text{and}$$
$$\mathsf{s}_g c_{(G)}\tau(x,y) = y \quad \text{if} \quad x \neq 0.$$

(iii) In Biró[89] it is proved that no "first–order definable" functions help in finitely axiomatizing $RCA_n, n < \omega$, in the following sense. Let $f_1^U, ..., f_r^U$ be any first–order definable functions. Then the variety generated by $\{\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U, f_1^U, ..., f_r^U\rangle_{i,j<n} : U \text{ is a set}\}$ is not finitely axiomatizable. Thus our Theorem 5 extends Biró's result beyond first–order definable, in the case the extra operations are unary and additive. We do not know whether "unary, additive" can be omitted[15] in Theorem 5. That "permutation invariant" cannot be omitted from Theorem 5 is proved both in Biró[89] and in Maddux[89b]. We note that permutation invariance of the operations of RA^+_n is very desirable. More on this can be read in Németi[90].

(iv) It is proved in Andréka[90d] that Theorem 5 remains true if we replace the condition "f_i is additive" with "f_i is an exotic quantifier", where this latter is defined as follows. We say that $\langle f^U : U \text{ is a set}\rangle$ is an exotic quantifier, if for all U there is a subset $Q_U \subseteq \mathcal{P}(U)$ of the powerset of U such that for all $X \subseteq {}^nU$

$$f^U(X) = \{s \in {}^nU : \{u : s(0/u) \in X\} \in Q_U\}.$$

Exotic quantifiers are not additive, and are not first–order definable, in general. ■

Let $n < \omega$. Then RPA_n denotes the variety generated by

$$\{\langle \mathfrak{P}({}^nU), c_i^U, \mathsf{s}_{ij}^U, p_{ij}^U\rangle_{i,j<n} : U \text{ is a set}\}$$

and $RPEA_n$ denotes the variety generated by

$$\{\langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U, \mathsf{s}_{ij}^U, p_{ij}^U\rangle_{i,j<n} : U \text{ is a set}\}.$$

[15] Added in proof: The condition "at most unary" was shown superfluous in Theorem 5 by Madarász. J. and Németi, I. For further improvements of Theorem 5 see the 1997 version of Németi[90], and Madarász-Németi-Sági[97].

(RPA_n and $RPEA_n$ are called in the literature the classes of all representable polyadic and representable polyadic equality algebras, respectively.[16]) Johnson[69] proved that none of RPA_n and $RPEA_n$ is finitely axiomatizable if $3 \leq n < \omega$, and he asked whether the diagonal constants contribute to nonfinite axiomatizability of $RPEA_n$, i.e. whether $RPEA_n$ is finitely axiomatizable over RPA_n. This is Problem 2a in Johnson[69]. Theorem 6 below gives a negative solution to this problem. We note that Problem 2a in Johnson[69] is equivalent to asking whether $RPEA_n$ has an equational axiomatization in which the diagonal constants occur in finitely many equations only.

For $n \geq \omega$, the analogous algebras are called representable quasi-polyadic equality algebras, and their class is denoted by $RQPEA_n$. We do not know whether Theorem 6 remains true if we drop the condition $n < \omega$ and replace $RPEA_n$ with $RQPEA_n$ in it.

We note that for $n \geq \omega$, the algebras in $RPEA_n$ as defined in the literature have much more operations, e.g. they have all the s_τ's where $\tau : n \to n$ is any map. It is proved in Németi–Sági[96] that the set of equations valid in $RPEA_\omega$ is at least Π^1_1-hard, hence, in particular, is not recursively enumerable. There are more theorems in Németi–Sági[96] supporting the claim that there is no schema-axiomatization of $RPEA_\omega$.

THEOREM 6. *Let Σ be a set of quantifier–free formulas axiomatizing $RPEA_n$, $3 \leq n < \omega$. Let $k < \omega$ be any number. Then Σ contains a formula with more than k variables in which some diagonal constant occurs. Thus the diagonal constants occur in infinitely many elements of Σ.*

Proof: We will use the proof of Theorem 5 with letting $\{f_1^U, ..., f_r^U\} = \{p_{ij}^U, \mathsf{s}_{ij}^U : i, j < n\}$. Take the algebra $\mathfrak{A}$ constructed in the proof of Theorem 5. We choose for p_{ij} the terms of Claim 13 to be $\tau_\sigma = \tau'_\sigma = p_{\sigma\circ[i,j]}(x)$. We proved in the proof of Theorem 5 that

(i) $\mathfrak{A} \notin RPEA_n$
(ii) every k–generated subalgebra of $\mathfrak{A}$ is in $RPEA_n$.

[16] Polyadic algebras were introduced and extensively studied by Halmos (see Halmos[54,60,62] and [HMT85]section 5.4). Sometimes they are called Halmos algebras, cf. e.g. Plotkin[88]. Usually, more basic operations are present in polyadic algebras than the ones we use, but these are all term–definable from our basic operations if $n < \omega$. The connections between the two kinds of definitions of polyadic algebras are investigated in Sain–Thompson[88].

So to prove Theorem 6, it is enough to prove that

(iii) there is a representation of $\mathfrak{A}$ in which all operations are the natural ones except for the diagonal constants.

Let U be as in the proof of Theorem 5, and let $W \supsetneqq U$ be any set.

CLAIM 16. *There is a one-one mapping* $h : A \rightarrowtail \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W, \mathsf{s}_{ij}^W, p_{ij}^W \rangle_{i,j<n}$ *such that h is a homomorphism w.r.t. all operations of $\mathfrak{A}$ except for d_{ij}, $i, j < n$.*

Proof: Recall $U_i, i < n$ and R from the proof of Theorem 5. Let $W_0 = U_0 \cup (W \setminus U)$, $W_i = U_i$ for $0 < i < n$. First we define a function $h : \mathcal{P}({}^nU) \longrightarrow \mathcal{P}({}^nW)$ with the desired properties satisfying in addition $h(R) = \underset{i<n}{\times} W_i$.

Let $t : W \twoheadrightarrow U$ be a surjective function which is the identity on U and which maps W_0 to U_0, i.e. $t \restriction U = Id_U$ and $t^* W_0 \subseteq U_0$. Define $g : {}^nW \longrightarrow {}^nU$ by $g(s) = t \circ s$ for all $s \in {}^nW$, and for all $x \subseteq {}^nU$ define

$$h(x) = \{s \in {}^nW : g(s) \in x\}.$$

Clearly, $h(R) = h(\underset{i<n}{\times} U_i) = \underset{i<n}{\times} W_i$, and h is a one–one homomorphism w.r.t. all operations of $\langle \mathfrak{P}({}^nU), c_i, \mathsf{s}_{ij}, p_{ij} \rangle_{i,j<n}$ by Lemma 4(iv).

Recall from the proof of Theorem 5 that $|U_0| = m$, and $|U_i| > m$ for all $0 < i < n$. Therefore $|W_i| \geq m + 1$ for all $i < n$ and hence, by Lemma 2, there is a partition $\langle R'_j : j \leq m \rangle$ of $R' = \underset{i<n}{\times} W_i$ such that $c_i^W R'_j = c_i^W R'$ for all $i < n$ and $j \leq m$. Then $\langle p_\sigma R'_j : j \leq m \rangle$ is an analogous partition of $p_\sigma R'$, for all $\sigma \in S(n)$. We define $\overline{h} : A \longrightarrow \mathcal{P}({}^nW)$ by

$$\begin{aligned} &\overline{h}(a) = h(a) \quad \text{if } a \in A' \\ &\overline{h}(X_{\sigma j}) = p_\sigma R'_j \quad \text{if } \sigma \in S(n), j \leq m \quad \text{and} \\ &\overline{h}(x + y) = \overline{h}(x) + \overline{h}(y) \quad \text{for } x, y \in A. \end{aligned}$$

Now it is easy to check that $\overline{h}$ is a one–one homomorphism w.r.t. the operations $+, -, c_i, p_{ij}$, $i, j < n$. Let $i, j < n, i \neq j$. We are going to check homomorphism w.r.t. s_{ij}.

The quantifier–free formula $x \leq -d_{ij} \rightarrow \mathsf{s}_{ij}(x) = 0$ is valid in RA_n^+, hence it is valid in $\mathfrak{A}$ since the k–generated subalgebras of $\mathfrak{A}$ are in RA_n^+ and we may assume $k \geq 1$. Let $\sigma \in S(n), l \leq m$. Then $\mathsf{s}_{ij}(X_{\sigma l}) = 0$ in $\mathfrak{A}$. Now $\overline{h}(\mathsf{s}_{ij}(X_{\sigma l})) = \overline{h}(0) = 0 = \mathsf{s}_{ij} p_\sigma R'_l = \mathsf{s}_{ij} \overline{h}(X_{\sigma l})$. Assume that $a \in A'$. Then $\overline{h}(\mathsf{s}_{ij} a) = h(\mathsf{s}_{ij} a) = \mathsf{s}_{ij} h(a) = \mathsf{s}_{ij} \overline{h}(a)$ since A' is closed under s_{ij} and h is a homomorphism w.r.t. s_{ij}. Since both $\overline{h}$ and s_{ij} are additive, we are done. **QED**(Claim 16)

QED(Theorem 6)

Problem 2.16 in Henkin–Monk–Tarski[HMT71] asks if RCA_n can be axiomatized with a set of equations in which complementation occurs in only finitely many equations. One reason for asking this was that the "perfect extension" of an RCA_n is an RCA_n again, and the natural condition for equations to be preserved under "perfect extensions" is that complementation $-$ does not occur in them. Theorem 7 below gives a negative answer to this problem.

Subalgebras of complementation–free reducts of RCA_n's arise in a natural way in the study of databases, cf. Comer[89], Cosmadakis[87], Imielinski–Lipski[84], Düntsch[90,90a,93]. Comer asked in the problem session of the 1988 Algebraic Logic Conference in Budapest whether the class RCA_n^- of all these subreducts is a variety or not. This was Problem 11 in Maddux[88]. Comer proved that RCA_n^- is a quasi–variety which is not finitely axiomatizable. Theorem 7 below states that RCA_n^- is not a variety, thus giving a negative answer to Comer's question.

Theorem 7 below states that not only complementation, but also the other Boolean operations, $+$ or $\cdot$, have to occur infinitely many times in any axiomatization of RCA_n.

For any algebra $\mathfrak{C}$ similar to RCA_n's, the complementation–free reduct $\mathfrak{C}^-$ of $\mathfrak{C}$ is defined as

$$\mathfrak{C}^- = \langle C, +^{\mathfrak{C}}, \cdot^{\mathfrak{C}}, 0^{\mathfrak{C}}, 1^{\mathfrak{C}}, c_i^{\mathfrak{C}}, d_{ij}^{\mathfrak{C}} \rangle_{i,j<n}.$$

Then RCA_n^- is the class of all subalgebras of $\mathfrak{C}^-$ for some $\mathfrak{C} \in RCA_n$.

THEOREM 7. *(i) Let Σ be a set of equations axiomatizing RCA_n, $n \geq 3$. Let $\ell < n$, and $k < n$, $k' < \omega$ be natural numbers. Then Σ contains infinitely many equations in which $-$ occurs, one of $+$ or $\cdot$ occurs, a diagonal constant with index ℓ occurs, more than k cylindrifications and more than k' variables occur. The same holds for any $\mathcal{K}$ in place of RCA_n such that ${}_\infty Cs_n \subseteq \mathcal{K} \subseteq RCA_n$.*
(ii) RCA_n^- is not a variety.

Proof: To prove Theorem 7, we shall use the constructions of the proofs of Theorem 3, Theorem 4. We proved earlier that several reducts of these algebras $\mathfrak{A}$ are representable. Here we shall prove that

(a) the complementation–free reduct $\mathfrak{A}^-$ of $\mathfrak{A}$ is a homomorphic image of a subalgebra $\mathfrak{C}$ of the complementation free reduct $\mathfrak{P}^-$ of a $\mathfrak{P} \in {}_\infty Cs_n$.
(b) $\mathfrak{A}^- \notin RCA_n^-$.
(c) $\mathfrak{A}$ can be represented as a ${}_\infty Cs_n$ such that every operation of $\mathfrak{A}$ except for "$\cup$" and "$\cap$" are the natural ones.

These will prove Theorem 7 as follows. Let Σ^- denote the set of all equations valid in ${}_\infty Cs_n$ in which $-$ does not occur. Let $\mathfrak{C}, \mathfrak{P}$ be as in (a). Then $\mathfrak{P} \models \Sigma^-$ because $\mathfrak{P} \in {}_\infty Cs_n$. Then $\mathfrak{P}^- \models \Sigma^-$, because $-$ does not occur in Σ^-. Then $\mathfrak{C} \models \Sigma^-$ by $\mathfrak{C} \subseteq \mathfrak{P}^-$, and then $\mathfrak{A}^- \models \Sigma^-$ since $\mathfrak{A}^-$ is a homomorphic image of $\mathfrak{C}$ and Σ^- consists of equations. Then $\mathfrak{A} \models \Sigma^-$. This, together with our previous arguments, proves (i) of Thm.7. To prove (ii) of Thm.7, notice that (a) implies that $\mathfrak{A}^-$ is a homomorphic image of a member of RCA_n^-, namely of $\mathfrak{C}$, and thus (b) implies that RCA_n^- is not closed under taking homomorphic images, showing that RCA_n^- is not a variety.

First we prove (a),(b) for the case $n \geq \omega$. Let $\langle W_i : i \leq n\rangle, m$ be as in the proof of Theorem 3, i.e. let $w \notin U$, and $W_0 = U_0 \cup \{w\}$, $|U_0| = m$, $W_i = U_i$ if $0 < i \leq n$ and let $W = \bigcup\{W_i : i \leq n\} = U \cup \{w\}$.

Let $R' = \underset{i<n}{\times} W_i$ and let $R'_j, j \leq m$ be a partition of R' such that $c_i R'_j = c_i R'$ for all $i < n, j \leq m$. Such a partition exists by $|W_i| \geq m+1$ for all $i < n$ and by Lemma 2.

Let $u \in U_0$ be fixed. Let $[w/u] : W \longrightarrow W$ be the mapping of W that sends w to u and leaves all other points of W fixed, and let $[w,u]$ be the permutation of W that interchanges w and u and leaves all other elements of W fixed. For any $s \in {}^nW$ let $s(w/u) = [w/u] \circ s$, i.e.

$$s(w/u)_i = \begin{cases} s_i & \text{if } s_i \neq w, \\ u & \text{if } s_i = w. \end{cases}$$

Let $\pi = [u,w]$. Let G denote the set of all permutations of U which leave each U_i $(i \leq n)$ fixed. For $a \subseteq {}^nU$ let

$$Ga = \{g \circ s : s \in a, g \in G\}.$$

Let

$D = \{s \in {}^nW : \{u,w\} \subseteq Rng\ s\}$,
$B' = \{x \subseteq {}^nW : x = \pi x, (x \cap {}^nU) = G(x \cap {}^nU),$ and $(\forall s \in x \cap D) s(w/u) \in x\}$,
$B = \{\bigcup\{R'_j : j \in J\} \cup x \quad : \quad J \subseteq m+1, x \in B'\}$.

We are going to show that B is closed under $\cup, \cap, c_i^W$ and $\{\emptyset, {}^nW, d_{ij}^W\} \subseteq B' \subseteq B$ for all $i, j < n$. It is not difficult to see that B' is closed under $\cup, \cap$ and $\emptyset, {}^nW, d_{ij}^W \in B'$. Next we show that B' is closed under c_i^W. Let $i < n$ and $x \in B'$. Clearly, $c_i x = \pi c_i x$ by $x = \pi x$. Let $s \in c_i x \cap {}^nU$ and $g \in G$. Then $z = s(i/v) \in x$ for some v. If $v \in U$, then $z \in x \cap {}^nU$, and then $g \circ z \in x$. But $g \circ s$ and $g \circ z$ differ at most at place i, so $g \circ s \in c_i x$ and we are done. Assume therefore $v \notin U$, i.e. $v = w$. Then i is the only element of n for which $z_i = w$, hence $z(w/u)$ differs from s only at place i. So it is enough to show $z(w/u) \in x$. If $z \in D$, then this is immediate by $x \in B'$. If $z \notin D$, then $u \notin Rng\ z$, and hence $z(w/u) = \pi \circ z \in x$. We have seen $(c_i x \cap {}^nU) = G(c_i x \cap {}^nU)$. To verify the last condition for $c_i x$, let $s \in D, s \in c_i x$. Then $z = s(i/v) \in x$ for some v. If $z \in D$ then $z(w/u) \in x$ by $x \in B'$, and therefore $s(w/u) \in c_i x$ because $s(w/u) = z(w/u)(i/v')$ for some v'. Assume therefore $z \notin D$. Then $s_i \in \{u, w\}$ and $s_i \notin Rng\ z$ by $s \in D$. If $s_i = w$ then $s(w/u) = s(i/u) \in c_i x$ and we are done. Assume $s_i = u$. Then $z(w/u) = [w, u] \circ z$ by $u \notin Rng\ z$, hence $z(w/u) \in x$ by $x \in B'$, $z \in x$. Then $s(w/u) \in c_i x$ as in the previous case and we are done with showing $c_i x \in B'$.

Now we are going to show that B is also closed under $\cup, \cap$, and c_i^W. It is clear that B is closed under $\cup$. To show closure of B under $\cap$, notice first that $R' \in B'$, and $R' \subseteq x$ whenever $x \cap R' \neq \emptyset$ and $x \in B'$. (To show this latter, we use $x \cap {}^nU = G(x \cap {}^nU)$ and $x = \pi x$.) Thus $x \cap R'_j = \emptyset$ or $x \cap R'_j = R'_j$ for all $x \in B'$ and $j \leq m$. This implies that B is closed under $\cap$. To show closure of B under c_i, let $b = r + x$ where $r = \bigcup\{R'_j : j \in J\}$ for some $J \subseteq m + 1$ and $x \in B'$. Then $c_i b = c_i r + c_i x$. By $c_i R'_j = c_i R'$ then $c_i r = c_i R'$ or $c_i r = \emptyset$. Thus $c_i r, c_i x \in B'$ by $R', \emptyset, x \in B'$, hence $c_i b \in B' \subseteq B$.

We have seen that B is closed under $\cup, \cap, c_i^W$ and $\emptyset, {}^nW, d_{ij}^W \in B$. Let $\mathfrak{B} = \langle B, \cup, \cap, \emptyset, {}^nW, c_i^W, d_{ij}^W \rangle_{i,j<n}$. Then $\mathfrak{B}$ is a subalgebra of the complementation–free reduct $\mathfrak{P}^-$ of $\mathfrak{P} = \langle \mathfrak{P}({}^nW), c_i^W, d_{ij}^W \rangle_{i,j<n}$.

We are going to show that the complementation–free reduct $\mathfrak{A}^-$ of $\mathfrak{A}$ is embeddable into a homomorphic image of $\mathfrak{B}$. To this end, first we define an algebra $\mathfrak{C}$ and a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$. Let

$$V = {}^nW \smallsetminus D,$$
$$C = \mathcal{P}(V), \quad c_i^{\mathfrak{C}}(x) = (c_i^W x) \cap V, \quad d_{ij}^{\mathfrak{C}} = d_{ij}^W \cap V,$$
$$\mathfrak{C} = \langle C, \cup, \cap, \emptyset, V, c_i^{\mathfrak{C}}, d_{ij}^{\mathfrak{C}} \rangle_{i,j<n}.$$

Let $g : B \longrightarrow C$ be defined by $g(x) = x \cap V$ for all $x \in B$. We are going to show that $g : \mathfrak{B} \longrightarrow \mathfrak{C}$ is a homomorphism. It is clear that g is a homomorphism w.r.t. $+, \cdot, 0, 1, d_{ij},\ i, j < n$.

Let $i < n$ and $x \in B$. We want to show that $g(c_i^{\mathfrak{B}} x) = c_i^{\mathfrak{C}} g(x)$, i.e. that $(c_i^W x) \cap V = c_i^W(x \cap V) \cap V$. It is enough to check this for $x \in \{R'_j : j \leq m\} \cup B'$.

Now $R' \cap D = \emptyset$ because if $s \in R' = \underset{i<n}{\times} W_i$, then the only i for which $s_i \in W_0$ is 0, i.e. $(Rng(s)) \cap W_0 = \{s_0\}$. Then if $s_0 = u$ then $w \notin Rng(s)$, and if $s_0 = w$ then $u \notin Rng(s)$ by $u \in U_0$. So, if $x = R'_j$ for some $j \leq m$ then $x \cap V = x$ and we are done. Assume $x \in B'$. The inclusion $\supseteq$ is clear. Let $s \in c_i^W x,\ s \notin D$. Then $z = s(i/v) \in x$ for some v. If $z \notin D$ then $z \in x \cap V$ and we are done. Assume $z \in D$. Then $v \in \{u, w\}$ and $v \notin Rng\ s$. Assume $v = w$. By $z \in x \cap D,\ x \in B'$ then $z(w/u) = s(i/u) \in x$, and $s(i/u) \in V$ by $w \notin Rng\ s$. Thus $s(i/u) \in x \cap V$ and we are done. The case $v = u$ is similar: By $z \in x \cap D$ we have $z(u/w) = \pi \circ z(w/u) \in x$. By $u \notin Rng(s)$ then $z(u/w) = s(i/w) \in x \cap V$ and we are done with showing that g is a homomorphism w.r.t. c_i.

Let $M = \{x \cap V : x \in B\}$ and $\mathfrak{M} = \langle M, \cup, \cap, \emptyset, V, c_i^{\mathfrak{C}}, d_{ij}^{\mathfrak{C}}\rangle_{i,j<n}$. Then $\mathfrak{M}$ is a homomorphic image of $\mathfrak{B}$.

We now show that $\mathfrak{A}^-$ is embeddable into $\mathfrak{M}$. Recall that any element of A is of form $\sum\{R_j : j \in J\} + a$ where $J \subseteq m+1$ and $a \in A'$, $a \cap R = 0$, because $\mathfrak{A}$ is obtained from $\mathfrak{A}'$ by splitting R into $m+1$ parts $R_j, j \leq m$. Let $\pi = [u, w]$. We define $h : A \longrightarrow M$ as follows. For any $J \subseteq m+1$ and $a \in A'$, $a \cap R = 0$ we define

$$h(\sum\{R_j : j \in J\} + a) = \bigcup\{R'_j : j \in J\} \cup a \cup \pi a.$$

First we check that h is well defined, h is one–to–one, and $h(a) \in M$ for all $a \in A$. Now, h is well defined because if $\sum\{R_j : j \in J\} + a = \sum\{R_j : j \in J'\} + a'$ $a, a' \in A'$, $a \cdot R = a' \cdot R = 0$, then $J = J'$ and $a = a'$. h is one-to-one because if $a \neq a'$, $a, a' \in A'$, $a \cdot R = a' \cdot R = 0$, and $s \in a - a'$, then $s \notin R' \cup a' \cup \pi a'$. Finally, let $y = \sum\{R_j : j \in J\} + a$, $a \in A'$, $a \cdot R = 0$. We want to show that $h(y) \in M$. Let $x = a \cup \pi a$. Then $x = \pi x$ by $\pi \circ \pi = Id$. Also, $x \cap D = \emptyset$ since if $s \in a$ then $w \notin Rng(s)$ by $a \subseteq {}^nU$, $w \notin U$, and thus $u \notin Rng(\pi \circ s)$. Finally, $x \cap {}^nU = a$ because if $s \in a$, then $\pi \circ s \in {}^nU$ only if $u \notin Rng\ s$, and then $\pi \circ s = s$. By $a \in A', R = G(R)$, and since A' is generated by R, we have that $a = G(a)$. Thus $x \cap {}^nU = G(x \cap {}^nU)$. This shows $x \in B'$ and $x \subseteq V$. We have already seen that $R' \subseteq V$. Thus $h(y) \in B$ and $h(y) \subseteq V$, i.e. $h(y) \in M$.

We are going to check that h is a homomorphism on $\mathfrak{A}^-$. It is easy to see that h preserves $+, \cdot, \emptyset, d_{ij}$ and $h({}^nU) = V$. (When checking $\cdot$, we use $a \cap (\pi b) \subseteq a \cap b$ if $a, b \subseteq {}^nU$.) It remains to show that h preserves c_i. Let $a \in A'$ be arbitrary. First we show that

(*) $h(c_i^U a) = c_i^U a \cup \pi(c_i^U a)$.

Indeed, if $R \cap c_i^U a = \emptyset$, then (*) is immediate by the definition of h. Assume $R \cap c_i^U a \neq \emptyset$. Then $R \subseteq c_i^U a$ by $a \in A'$. Now by using $R' = R \cup \pi R$ and $R \subseteq c_i^U a$ we get

$$\begin{array}{rcl} h(c_i^U a) & = & R' \cup (c_i^U a \smallsetminus R) \cup \pi(c_i^U a \smallsetminus R) \\ & = & R \cup \pi R \cup (c_i^U a \smallsetminus R) \cup (\pi c_i^U a \smallsetminus \pi R) \\ & = & R \cup (c_i^U a \smallsetminus R) \cup \pi R \cup (\pi c_i^U a \smallsetminus \pi R) \\ & = & c_i^U a \cup \pi c_i^U a. \end{array}$$

Next, we will prove that

(**) $c_i^U a \cup \pi(c_i^U a) = c_i^W(a \cup \pi a) \cap V.$

Indeed, the inclusion $\subseteq$ is clear. Let $s \in c_i^W(a \cup \pi a) \cap V$. Then $s(i/v) \in a \cup \pi a$ for some $v \in W$. Assume $s_i \notin \{u, w\}$. Then $s \in c_i^U a \cup \pi c_i^U a$. Assume $s_i = w$. Then $u \notin Rng\ s$ by $s \in V$. If $s(i/v) \in a$, then $s_j \neq w$ for all $j \neq i$, hence $\pi \circ s \in c_i^U a$. If $s(i/v) \in \pi a$, then $\pi \circ s \in c_i^U a$. Assume $s_i = u$. Then $w \notin Rng\ s$ by $s \in V$. If $s(i/v) \in a$, then $s \in c_i^U a$. If $\pi \circ s(i/v) \in a$, then $s_j \neq u$ for all $j \neq i$, hence s and $\pi \circ s(i/v)$ differ only at i, so $s \in c_i^U a$.

We are ready to prove that h preserves c_i. Let $i < n$, $j \leq m$, and $a \in A'$, $a \cap R = \emptyset$ be arbitrary. Now $h(c_i^{\mathfrak{A}} R_j) = h(c_i^U R) = c_i^W(R \cup \pi R) \cap V = (c_i^W R') \cap V = (c_i^W R_j') \cap V = c_i^{\mathfrak{C}} h(R_j)$. Also, $h(c_i^{\mathfrak{A}} a) = h(c_i^U a) = c_i^W(a \cup \pi a) \cap V = c_i^{\mathfrak{C}} h(a)$. Since h preserves $+$, we are done.

We showed that $\mathfrak{A}^-$ is embeddable into $\mathfrak{M}$, and $\mathfrak{M}$ is a homomorphic image of $\mathfrak{B} \subseteq \mathfrak{P}^-$. Thus, by basic universal algebraic facts we have that $\mathfrak{A}^-$ is a homomorphic image of a subalgebra of $\mathfrak{P}^-$. We proved (a).

To show (b), let q denote the following formula

$$\varepsilon \wedge \chi \quad \to \quad \prod_{i \leq m} c_0 x_i = 0$$

where ε, χ are the following formulas respectively

$$\prod_{i \leq m} s_i^0 c_1 \ldots c_m x \ \leq \ \sum_{i < j \leq m} d_{ij}$$

$$\bigwedge_{i,k \leq m+1, i \neq k} x_i \cdot x_k = 0 \ \wedge \ x_i \leq x.$$

Then it is not difficult to check that $RCA_n \models q$ and $\mathfrak{A}^- \not\models q$. One way of doing this is noticing that we basically did this in the proof of Claim 1, since our formula ε is equivalent to $\tau(x) = 0$ for the $\tau(x)$ used in the proof of Claim 1. This proves that $\mathfrak{A}^- \notin RCA_n^-$.

Now we show how to modify the above proofs of (a)-(b) for the case $n < \omega$. Let $W, \langle W_i : i \leq n\rangle, f : W \rightarrowtail\!\!\!\!\!\twoheadrightarrow W$ be as in the proof of Theorem 4 (cf. the proof of Claim 8). Let e be the smallest equivalence relation containing f. Fix some $u \in U_0, w \in W \smallsetminus U$ and let $w/e = \{v : v\ e\ w\}, u/e = \{v : v\ e\ u\}$. Let $\delta : w/e \rightarrowtail\!\!\!\!\!\twoheadrightarrow u/e$ be such that δ preserves f, and let

$$\sigma = \delta \cup Id \restriction U_0,$$
$$\pi = \delta \cup \delta^{-1} \cup Id \restriction (U_0 \smallsetminus u/e).$$

Then $\sigma : W_0 \longrightarrow U_0$ and $\pi : W_0 \rightarrowtail\!\!\!\!\!\twoheadrightarrow W_0$. Define $s(w/u) = \sigma \circ s$. Let $R' = \underset{i<n}{\times} W_i$ and let $R'_j, j \leq n$ be a partition of R' such that $c_i R'_j = c_i R'$ for all $i < n, j \leq m$. Let G be the set of all permutations of U that leave R and F (in the definition of $\mathfrak{A}$) fixed. Let

$$D = \{s \in {}^nW : u/e \cap Rng(s) \neq \emptyset, w/e \cap Rng(s) \neq \emptyset\},$$
$$B' = \{x \subseteq {}^nW : x = \pi x, (x \cap {}^nU) = G(x \cap {}^nU), \text{ and } (\forall s \in x \cap D) s(w/u) \in x\},$$
$$B = \{\bigcup\{R'_j : j \in J\} \cup x \quad : \quad J \subseteq m+1, x \in B'\}.$$

Then B is closed under the operations of $\mathfrak{P}^- = \langle \mathcal{P}({}^nW), \cup, \cap, \emptyset, {}^nW, c_i^W, d_{ij}^W\rangle_{i,j<n}$. Let $\mathfrak{B} \subseteq \mathfrak{P}^-$ with universe B. Let V, C and g be as in the previous case $(n \geq \omega)$. Then $g : \mathfrak{B} \longrightarrow \mathfrak{C}$ is a homomorphism, and the function h defined as in the previous proof embeds $\mathfrak{A}^-$ into the image of $\mathfrak{B}$ along g.

This shows (a). To prove $\mathfrak{A}^- \notin RCA_n^-$ let q' denote the following formula

$$\varphi \wedge \varepsilon \wedge \delta \wedge \chi \rightarrow \prod_{i \leq m} c_0 x_i = 0$$

where $m = K \cdot (n-1)$ and $\varphi, \varepsilon, \delta, \chi$ are the following formulas respectively

$$\bigwedge_{i<K} y_i \cdot s_2^1 y_i \leq d_{12} \ \wedge \ y_i = c_2 y_i \ \wedge \ z = \sum_{i<K} y_i$$
$$c_2(s_2^1 z \cdot s_2^0 z) \leq z \ \wedge \ z = s_0^2 s_1^0 s_2^1 c_2 z \ \wedge \ d_{01} \cdot c_1 z \leq z$$
$$\bigwedge_{i<n} s_i^0 c_1 z \cdot \bigwedge_{i<j<n} - s_i^0 s_j^1 z = 0$$
$$\bigwedge_{i<j\leq m} x_i \cdot x_j = 0 \ \wedge \ c_1 \ldots c_{n-1} \sum_{i\leq m} x_i = c_1 z.$$

Then what we did in Claim 7 was to show that $RCA_n \models q'$ while $\mathfrak{A} \not\models q'$. Since complementation does not occur in q', this shows that $\mathfrak{A}^- \notin RCA_n^-$.

Finally we show how to represent the "$\cup$" and "$\cap$"–free reducts of our algebras. We will treat the cases $n \geq \omega$ and $n < \omega$ together.

Let $m, U, \langle U_i : i \leq n\rangle, R, \mathfrak{A}', R_j, j \leq m$ and $\mathfrak{A}$ be as in the proofs of Theorem 3, Theorem 4. We may assume that $m \geq 3$. Let $R'_j, j < m$ be a partition of R such that $c_i R'_j = c_i R$ for all $i < n, j < m$. Let $z \in R \smallsetminus (R'_0 \cup R'_1)$ be fixed and let $\eta : \{J \subseteq m+1 : 0 \in J\} \rightarrowtail \{G \subseteq R \smallsetminus (R'_0 \cup R'_1) : z \in G\}$ be an arbitrary injection. We are going to define a function $h : A \longrightarrow \mathcal{P}({}^nU)$. Recall that every element of A is of form $\sum\{R_j : j \in J\} + a$ where $J \subseteq m+1$ and $a \in A' \subseteq \mathcal{P}({}^nU)$. Let $J \subseteq m+1$. Then we define

$$h(\sum\{R_j : j \in J\}) = \begin{cases} R'_0 \cup \eta(J) & \text{if } 0 \in J, J \neq m+1 \\ R \smallsetminus (R'_0 \cup \eta((m+1) \smallsetminus J)) & \text{if } 0 \notin J, J \neq \emptyset \\ 0 & \text{if } J = \emptyset \\ R & \text{if } J = m+1. \end{cases}$$

Let $a \in A$ be arbitrary. Then we define

$$h(a) = (a \smallsetminus R) \cup h(a \cap R).$$

It is easy to see that $h : \mathfrak{A} \rightarrowtail \langle \mathfrak{P}({}^nU), c_i^U, d_{ij}^U\rangle_{i,j<n}$ is a one–one homomorphism w.r.t. $-, \emptyset, {}^nU, c_i^U, d_{ij}^U$.
QED(Theorem 7)

ON ALGEBRAS OF BINARY RELATIONS.

Besides the class RCA_n of algebras of n–ary relations, Tarski proposed another class for generalizing Boolean algebras, the class RRA of algebras of binary relations. This class RRA differs from RCA_2 in having one additional binary operation, namely the composition of binary relations and in having also p_{01}.

This new operation makes the algebras very "strong": RRA is not finitely axiomatizable (Monk[64]), though RCA_2 and $RPEA_2$ are finitely axiomatizable (a result of Henkin and Tarski[17]). The theory of RRA is very similar to that of RCA_n, $n \geq 3$, and it is a general practice that theorems proved for RCA_n can be

[17] For proofs see [HMT85]3.2.65, 5.4.33.

proved for RRA, and vice versa (though this "transfer" is not trivial or mechanical at all). Each of these two kinds of algebras of relations has its own advantage and disadvantage over the other one. The classes RCA_n are more convenient in that they have only unary operations in addition to the Boolean ones (unary operations are much easier to handle than binary ones), and the connection between equations of RCA_n and first–order formulas is very close, almost trivial. On the other hand, algebras in RRA are very familiar in the general mathematical practice, their definition is very simple and easy to grasp. The strength of RRA comes from the operation of composition of relations, which forms a semigroup. So, basically, RRA's are Boolean algebras endowed with a semigroup structure.

There are, however, differences between RCA_n's and RRA's. We saw that the "strength of RCA_n" is distributed among the operations of RCA_n quite evenly. We shall see that all the strength of RRA is concentrated in relation composition: this operation is so strong that relative to it, all the other non–Boolean operations are finitely axiomatizable.

Let U be a set, and R, S be binary relations on U, i.e. $R, S \subseteq U \times U$. Then we define

$$
\begin{aligned}
&R \mid S = \{(u,v) : (u,w) \in R \text{ and } (w,v) \in S \text{ for some } w \in U\},\\
&R^{-1} = \{(v,u) : (u,v) \in R\},\\
&Id_U = \{(u,u) : u \in U\},\\
&\mathfrak{Re}(U) = \langle \mathfrak{P}(U \times U), |, {}^{-1}, Id_U \rangle,\\
&Ra_2 = \{\mathfrak{Re}(U) : U \text{ is a set }\}.
\end{aligned}
$$

$\mathfrak{Re}(U)$ is called the full relation set algebra on U. The class of all subalgebras of Ra_2 is called the class Rs of all relation set algebras, or of all proper relation algebras (this is the analogon of Cs_n). The variety generated by Ra_2 is the class of all subdirect products of Rs's and is called RRA, the class of all representable relation algebras. We note that $\mathbf{I}Rs$, the class of isomorphic copies of elements of Rs, is axiomatizable by quantifier–free formulas. We are going to show that every axiomatization of Rs (or of RRA) (by quantifier–free formulas) must contain infinitely many formulas in which all of the Boolean operations and also relation composition occur. This answers, in the negative, a problem raised in Jónsson[59].

The operations $|, {}^{-1}$ and Id_U in (proper) relation algebras are often denoted by $;, {}^{\smile}$ and $1'$, respectively. We often write $\breve{x}$ in place of $x^{\smile}$ (or ${}^{\smile}(x)$). Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\smile}, 1' \rangle$ be an algebra similar to Rs's. We say that $\mathfrak{A}$ is a relation algebra if it satisfies the following finite set of equations (introduced in Chin–Tarski[51], see also Henkin–Monk–Tarski[HMT85]5.3.1 or Jónsson[82]Def.2.1).

$\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra,

$\langle A, ;, \breve{}, 1' \rangle$ is an involuted monoid,

; and $\breve{}$ distribute over join,

$\breve{x}; [-(x; y)] \leq -y$.

A relation algebra $\mathfrak{A}$ is called weakly representable (a $wRRA$) if it is representable as an algebra of binary relations where all the operations except perhaps $+$ and $-$ have their natural set theoretic meanings (i.e. $\cdot, 0, 1, ;, \breve{}, 1'$ denote set theoretic intersection, empty set, biggest set, relation composition, inverse or converse, and identity relation respectively but $+$ and $-$ do not necessarily denote union and complementation). This notion was introduced in Jónsson[59], where an infinite set of quasi–equations was given to axiomatize the class of all weakly representable relation algebras. Problem 3 in Jónsson[59] asks if every $wRRA$ is representable such that every operation including $+$ and $-$ is standard (is an RRA) or not. This amounts to asking whether there is a cause of nonrepresentability that can be attributed to "union and complementation" solely. In this sense, the subject belongs to the investigation of reducts of relation algebras, a survey paper on which is Schein[88]. The infinite set Σ of quasi–equations given in Jónsson[59] and characterizing $wRRA$ is such that $+$ and $-$ occur only in finitely many formulas in it. Therefore our theorem stating that RRA cannot be axiomatized with such sets (Theorem 7 below) implies that $wRRA \neq RRA$, thus giving a negative answer to Problem 3 in Jónsson[59]. (We will state a stronger theorem.)

We note that it is proved in Haiman[87] that $wRRA$ is not axiomatizable with a finite set of quantifier–free formulas (answering the first part of Problem 1 in Jónsson[59]).

THEOREM 8. *Let Σ be a set of quantifier–free formulas axiomatizing RRA over $wRRA$, i.e. such that $RRA = Mod(\Sigma) \cap wRRA$. Assume that no formula in Σ contains both $+$ and $\cdot$. Then there are infinitely many formulas in Σ in which all of $;, -$ and one of $+, \cdot$ occur. The same holds for $\mathbf{I}Rs$ in place of RRA.*

The proof of Theorem 8 can be found in Andréka[94].

REMARK 7. The conditions of Theorem 8 are best possible because of the following. The operation $\cdot$ is term–expressible with $-, +$ and $+$ is term–expressible with $-, \cdot$ in Boolean algebras (e.g. $x \cdot y = -(-x + -y)$), thus Theorem 7 becomes false if we replace "one of $+, \cdot$" in it with "$+$" or with "$\cdot$". Also, $-$ is expressible with $+$ and $\cdot$, namely $x \cdot y = 0 \wedge x + y = 1 \rightarrow y = -x$ holds in Boolean algebras. Hence the condition "no formula in Σ contains both $+$ and $\cdot$" cannot be omitted in Theorem 8. We note that this condition can be omitted if we replace "quantifier–free formulas" with "equations" in Theorem 8. This is proved in Andréka[90b] by using relation algebras belonging to finite projective geometries, i.e. using the so called Lyndon algebras $\mathfrak{L}_n$ for $n < \omega$. It is proved in Andréka[90b] that the complementation–free reduct $\mathfrak{L}_n^-$ of $\mathfrak{L}_n$ is a homomorphic image of a subalgebra of $\mathfrak{L}_m^-$ if $m \geq n$. Since $\mathfrak{L}_m \in RRA$ for infinitely many m, this implies that all the equations valid in RRA that do not contain $-$ are valid in all Lyndon algebras. This argument replaces the one in the proof of Theorem 8 for showing that $\mathfrak{B}(n, \omega)$ is weakly representable. The rest of the proof in Andréka[90b] is very similar to that of Theorem 8. Jónsson[84] proved by using the above mentioned Lyndon algebras that RRA cannot be axiomatized with a set of quantifier–free formulas using finitely many variables. This result is strengthened in Andréka[90b] to the statement that in any equational axiomatization of RRA, for any $k < \omega$, there are infinitely many equations containing more than k variables, and containing at the same time $;$, $-$ and one of $+, \cdot$. We also note that $\breve{}, 1'$ do not necessarily have to occur infinitely many times in an axiomatization of RRA.: There is an equational axiomatization of RRA in which $\breve{}$ and $1'$ occur in finitely many formulas only. This is proved in Andréka–Németi[96]. ■

REMARK 8. Subreducts of RRA are extensively investigated, a survey paper on this is Schein[88]. There are still many interesting open problems in this area.

Clearly, $|$ is characterised as a semigroup, $|, \subseteq$ was characterized by Zaretskij[59], and $|, \cap$ was characterized by Bredikhin–Schein[78]. Characterizations for $|, \subseteq, Id$ or for $|, \cap, Id$ are not known.

The characterization of $|, \cap$ is very simple: Any semilattice–ordered semigroup is isomorphic to a set of binary relations with the operations $|, \cap$. The corresponding question for $|, \cup$ was investigated since at least 1962. It was conjectured that every distributive semilattice–ordered semigroup is representable with binary relations, $|, \cup$. It was also not known whether the class

$$\mathcal{K} = \mathbf{I}\{\langle A, |, \cup\rangle : A \text{ is a set of binary relations closed under } |, \cup\}$$

is a variety or not. It is proved in Andréka[89] that the answer is in the negative:

THEOREM. *$\mathcal{K}$ is not axiomatizable with finitely many variables, and $\mathcal{K}$ is not a variety.*

In the proof of the above theorem, the following set of quasi–equations witnessing non–finite axiomatizability of $\mathcal{K}$ is exhibited: Let $m < \omega$ and let q_m denote the following quasi–equation

$$\bigwedge_{i<m} [x \subseteq x_i' \cup x_i'' \ \wedge \ y \subseteq y_i' \cup y_i''] \quad \rightarrow$$
$$x \mid y \quad \subseteq \quad (x \mid y_0') \cup \bigcup_{i<n-1} (x_i' \mid y_i'') \cup (x_i'' \mid y_{i+1}') \cup (x_{n-1}' \mid y_{n-1}'') \cup (x_{n-1} \mid y).$$

Then it is proved in Andréka[89] that $\mathcal{K} \models \{q_m : m < \omega\}$ but any set Σ of universal formulas containing finitely many variables, or any finite set of first–order formulas, can imply only finitely many of q_m, $m < \omega$. We know that $\{q_m : m < \omega\}$ does not axiomatize $\mathcal{K}$. To find a (relatively simple) axiom system for the quasi–equational theory of $\mathcal{K}$ is still an open problem.

It is proved in Andréka[91] Theorem 1, that a distributive semilattice–ordered semigroup is isomorphic to an algebra of binary relations iff it can be embedded into such a structure where the semilattice part is distributive in the lattice–theoretical sense, i.e. if

$$a \leq b \cup c \quad \text{implies} \quad a = b' \cup c' \ \text{ for some } b' \leq b, c' \leq c.$$

This theorem might help in finding an axiomatization of $\mathcal{K}$.

It is proved in Andréka[89] and in Andréka[91] together, that no distinguished familiar operations on binary relations help in finitely axiomatizing $|, \cup$ in the following sense. Let * denote the operation of forming transitive closure of a binary relation and let M be a set of operations on binary relations such that $\{|, \cup\} \subseteq M \subseteq \{|, \cup, \cap, -, \emptyset, Id, ^{-1}, ^*\}$. Let

$$\mathcal{K}(M) = \mathbf{I}\{\langle A, f\rangle_{f \in M} : A \text{ is a set of binary relations closed under all } f \in M\}.$$

Then $\mathcal{K}(M)$ is not finitely axiomatizable, moreover if $\cap \notin M$ then the quasi–equational theory of $\mathcal{K}(M)$ is not axiomatizable by using finitely many variables. It would be interesting to know whether there are operations definable in RRA which together with $|, \cup$ can be finitely axiomatized.

The operation * is investigated in Kleene–algebras, and the above theorem applies to Kleene–algebras (studied in computer science), too. The operations of Kleene–algebras are $|, \cup, \emptyset, ^{-1}, ^*, Id$. Redko[64] proved that the equational theory of the $^{-1}$–free Kleene–algebras is not finitely axiomatizable, and Kozen[81] proved that the equational theory of the *–free Kleene–algebras is finitely axiomatizable. By our theorem above, the quasi–equational theory of any of these reducts is not finitely axiomatizable. ■

References

Andréka,H. [88], *On taking subalgebras of relativized relation algebras*, Algebra Universalis **25** (1988), 96–100.

Andréka,H. [89], *On the union–relation composition reducts of relation algebras*, Abstracts of Amer. Mat. Soc. **10,2** (March 1989), 174. (*89T–08–21). Preprint of Math. Inst., Budapest, Sept. 1988.

Andréka,H. [90], *One variable is not enough for defining relation algebras but two are*, Algebra Universalis **28** (1991), 274-279.

Andréka,H. [90a], *Weakly representable but not representable relation algebras*, Algebra Universalis **32** (1994), 31-43.

Andréka,H. [90b], *Positive reducts of representable relation algebras is not closed under homomorphisms.* Preprint, Budapest, 1990.

Andréka,H. [90c], *Finite axiomatizability of SNr_nCA_{n+1} and non–finite axiomatizability of SNr_nCA_{n+2}.* Lecture notes for a series of lectures given at Algebraic Logic Meeting, Oakland, CA, June 1990.

Andréka,H. [90d], *The completeness problem of logics with finitely many variables, tools from beyond first–order logic, algebraic logic.* Manuscript, 1990. Material presented at Sixth Southeastern Logic Symposium, Miami Beach, USA, March 1990.

Andréka,H. [91], *Representations of distributive lattice–ordered semigroups with binary relations*, Algebra Universalis **28** (1991), 12–25.

Andréka,H. and Bredikhin,D. [95], *The equational theory of union-free algebras of relations*, Algebra Universalis **33,4** (1995), 516-532.

Andréka,H. Comer,S.D. and Németi,I. [83], *Epimorphisms in cylindric algebras.* Preprint, Math. Inst. Budapest.

Andréka,H. Comer,S.D. and Németi,I. [85], *Clones of operations on relations.* In: Universal algebra and lattice theory (Proc. Conf. Charleston 1985) Lecture Notes in Mathematics Vol. 1149, Springer–Verlag, Berlin, 1985. pp.17–21.

Andréka,H. Givant,S.R. Mikulás,Sz. Németi,I. and Simon,A. [96], *Notions of density that imply representability in algebraic logic*, Annals of Pure and Applied Logic. To appear.

Andréka,H. Givant,S.R. and Németi,I. [94], *The lattice of varieties of representable relation algebras*, Journal of Symbolic Logic **59,2** (1994), 631-661.

Andréka,H. Guessarian,I. and Németi,I. [87], *A unifying theorem for algebraic semantics and dynamic logics*, Information and Computation **72,1** (1987), 31–45.

Andréka,H. Jónsson,B. and Németi,I. [88], *Free algebras in discriminator varieties*, Algebra Universalis **28** (1991), 401-447.

Andréka,H. Maddux,R.D. and Németi,I. [87], *Splitting in relation algebras*, Proc. Amer. Math. Soc **111,4** (1991), 1085-1093.

Andréka,H. Monk,J.D. and Németi,I.(ed.s) [88], *Algebraic Logic.* Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991.

Andréka,H. and Németi,I. [75], *A simple purely algebraic proof of the completeness of some first–order logics*, Algebra Universalis **5** (1975), 8–15.

Andréka,H. and Németi,I. [80], *On systems of varieties definable by schemes of equations*, Algebra Universalis **11** (1980), 105–116.

Andréka,H. and Németi,I. [81], *Dimension complemented and locally finite dimensional cylindric algebras are elementarily equivalent*, Algebra Universalis **13** (1981), 157–163.

Andréka,H. and Németi,I. [96], *Axiomatization of identity-free equations valid in relation algebras*, Algebra Universalis **35** (1996), 256-264.

Andréka,H. Németi,I. and Sain,I. [93], *Applying algebraic logic to logic.* In: Algebraic methodology and software technology (AMAST'93), Proc. Twente, The Netherlands, June 1993), Nivat.M. Rattray,C. Rus,T. and Scollo,G. (eds.), Springer-Verlag, London, 1994. pp.7-28. Full version available as Preprint, Math. Inst. Hung. Acad. Sci. 1995.

Andréka,H. Ryan,M. and Schobbens,P-Y. [95], *Operators and laws for combining preferential relations (Extended abstract).* In: Information Systems: Correctness and Reusability (Selected papers), eds. R.J. Wieringa and R.B. Feenstra, World Scientific Publishing Co, 1995. pp.191-206.

Anellis,I.H. Houser,N., *The nineteenth century roots of universal algebra and algebraic logic: A critical-bibliographical guide for the contemporary logician.*. In: Algebraic Logic (Proc. Conf. Budapest 1988) Colloq. Math. Soc. J. Bolyai Vol 54, North-Holland, Amsterdam, 1991. pp.1–36.

Bednarek,A.R. and Ulam,S.M. [77], *Some remarks on relational composition in computational theory and practice.* In: Fundamentals of Computation Theory'77, Lecture Notes in Computer Science Vol. 56, Springer–Verlag, Berlin, 1977. pp. 33–38.

van Benthem,J. [89], *Modal logic and relational algebra.* Manuscript, University of Amsterdam, ILLC, 1989.

van Benthem,J. [90], *General dynamics.* In: Theoretical Linguistics (ed. Ph. A. Luelsdorff), 1991. pp.151–201.

van Benthem,J. [90a], *Temporal logic.* In: Handbook of Logic in Artificial Intelligence and Logic Programming (ed.s: D. Gabbai, Chr. Hogger and J. Robinson), Vol. 4, Oxford University Press, 1993. pp.241–350.

Berghammer,R. Kempf,P. Schmidt,G. and Ströhlein,T. [88], *Relation algebra and logic of programs.* In: Algebraic Logic , Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991. pp.37-58.

Biró,B. [89], *Non–finite–axiomatizability results in algebraic logic*, J. of Symbolic Logic **57,3** (1992), 832-843.

Biró,B. [89a], *Isomorphic but not lower base–isomorphic cylindric algebras of finite dimension*, Notre Dame J. of Formal Logic **30,2** (1989), 262–267.

Biró,B. and Shelah,S. [88], *Isomorphic but not lower–base–isomorphic cylindric set algebras*, J. of Symbolic Logic **53,3** (1988), 846–853.

Blok,W.J. and Pigozzi,D. [89], *Algebraizable logics*, Memoirs of Amer. Math. Soc. **77** (1989), vi+78pp.

Börner,F. [86], *One–generated clones of operations on binary relations*, Beitrage Alg. u. Geom. **23** (1986), 73–84.

Bredikhin,D.A. [77], *Abstract characteristics of some relation algebras (in Russian)*, Algebra and Theory of Numbers (Nalchik) **2** (1977), 3–19.

Bredikhin,D.A. [77a], *Multiplicative algebra of relations (in Russian).* Dissertation, Saratov State University, Saratov, USSR, 1977.

Bredikhin,D.A. and Schein,B.M. [78], *Representations of ordered semigroups and lattices by binary relations*, Colloq. Math. **39** (1978), 1–12.

Cirulis,J. [88], *Relational algebras in databases and generalized cylindric algebras (in Russian)*, Theoretical Questions of Programming, Latvian University, Riga (1988), 100–107.

Comer,S.D. [84], *Galois theory of cylindric algebras and its applications*, Trans. Amer. Math. Soc. **286** (1984), 771–785.

Comer,S.D. [85], *Combinatorial types.* In: Algebra, Combinatorics and Logics in Computer Science (Proc. Györ 1983) Colloq. Math. Soc. J. Bolyai Vol. 42, North–Holland, Amsterdam, 1985. pp.267–284.

Comer,S.D. [88], *The representation of dimension three cylindric algebras.* In: Algebraic Logic, Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991. pp.147-172.

Comer,S.D. [89], *A remark on representable positive cylindric algebras*, Algebra Universalis **28** (1991), 150–151.

Comer,S.D. [90], *An algebraic approach to the approximation of information*, Fundamenta Informaticae **XIV** (1991), 492-503.

Comer,S.D. [93], *On connections between information systems, rought sets, and algebraic logic.* In: Algebraic Methods in Logic and Computer Science, C. Rauszer (ed), Banach Center Publications Vol. 28, Warsaw, 1993. pp.117-124.

Cosmadakis,S.S. [87], *Database theory and cylindric lattices. (Extended abstract).* In: IEEE 28th Annual Symposium on Foundations of Computer Science, 1987. pp. 411-420.

Craig,W. [74], *Logic in algebraic form.* North–Holland, Amsterdam, 1974. viii+204pp.

Craig,W. [74a], *Unification and abstraction in algebraic logic*, Studies in Algebraic Logic, Math. Assoc. Amer. (1974), 6–57.

Daigneault,A. [69], *Lawvere's elementary theories and polyadic and cylindric algebras*, Fund. Math. **66** (1969), 307–328.

Daneczki,R. [84], *Nondeterministic dynamic logic with intersection is decidable.* In: Computation Theory (Proc. Zaborov, Poland, 1984), Lecture Notes in Computer Science Vol. 208, Springer–Verlag, Berlin, 1984. pp.34–53.

Demaree,D. [72], *Copeland algebras*, J. of Symbolic Logic **37** (1972), 646–656.

De Morgan, A. [1860], *On the syllogism: IV, and on the logic of relations.* Read before the Cambridge Philosophical Society on April 23, 1860. Transactions of the Cambridge Philosophical Society 10 (1964). 331–358.

Diamond,A.H. and McKinsey,J.C.C. [47], *Algebras and their subalgebras*, Bull. Amer. Math. Soc. **53** (1947), 147–154.

Düntsch,I. [90], *Cylindric lattices.* Preprint, Universiti of Brunei, Darussalam, Negara Brunei, 1990.

Düntsch,I. [90a], *Join dependencies and cylindric algebras.* Preprint, Universiti of Brunei, Darussalam, Negara Brunei, 1990.

Düntsch,I. [93], *A note on cylindric lattices.* In: Algebraic Methods in Logic and Computer Science, C. Rauszer (ed), Banach Center Publications Vol. 28, Warsaw, 1993. pp.231-238.

Düntsch,I. [94], *Rough relation algebras*, Fundamenta Informaticae **21** (1994), 321-331.

Feldman,N. [89], *Cylindric algebras with terms*, J. of Symbolic Logic **55,2** (1990), 865–866.

Ferenczi,M. [90], *On inducing homomorphisms between relation set algebras*, Algebra Universalis **27,4** (1990), 474–479.

Ferenczi,M. [92], *On representability of cylindric algebras*, Abstracts of papers presented to the American Math. Soc. **336/13/3** (1992). full version submitted.

Fischer,M.J. and Ladner,R.E. [79], *Propositional dynamic logic of regular programs*, J. of Computation and System Sciences **18,2** (1979), 194–211.

Givant,S.R. [94], *The structure of relation algebras generated by relativizations.* Contemporary Mathematics Vol. 156, Amer. Math. Soc., 1994. xv+134 pages.

Goldblatt,R.I. [90], *Varieties of complex algebras*, Annals of Pure and Applied Logic **44** (1990), 173–242.

Greenwood,R. and Gleason,A. [55], *Combinatorial relations and chromatic graphs*, Can. J. Math. **7** (1955), 1–7.

Haiman,M.D. [87], *Arguesian lattices which are not linear*, Bull. Amer. Math. Soc. **16,1** (1987), 121–123.

Halmos,P.R. [54], *Polyadic Boolean algebras*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 296–301.

Halmos,P.R. [60], *Polyadic algebras.* Summaries of talks presented at the Summer Institute for Symbolic Logic, Cornell University, 1957, second edition, Communications Research Division, Institute for Defense Analyses, 1960. pp.252–255.

Halmos,P.R. [62], *Algebraic Logic.* Chelsea Publ. Co., New York, 1962. 271pp.

Hansen,B. [92], *Representation theorems of reducts of cylindric agebras. In: Abstracts of the 1992 European Summer Meeting of the Assoc. Symb. Logic*, J. Symbolic Logic **58,3** (1993), 1125.

Hansen,B. [95], *Finitizability questions for some reducts of cylindric algebras.* In: Logic Colloquium'92, Csirmaz,L. Gabbay,D.M. and de Rijke,M. (eds), CSLI Publications, Stanford, 1995. pp.115-134.

Henkin,L. [55], *The representation theorem for cylindric algebras.* Mathematical interpretation of formal systems, North–Holland, Amsterdam, 1955. pp.85–97.

Henkin,L. [70], *Extending Boolean operations*, Pac. J. Math. **22** (1970), 723–752.

Henkin,L. and Monk,J.D. [74], *Cylindric algebras and related structures.* In: Proc. Tarski Symposium, Amer. Math. Soc., 1974. pp.105–121.

Henkin,L. Monk,J.D. and Tarski,A. [HMT71], *Cylindric Algebras, Part I.* North–Holland, Amsterdam, 1971, 1985.

Henkin,L. Monk,J.D. and Tarski,A. [81], *Cylindric set algebras and related structures.* In: Lecture Notes in Mathematics Vol 883, North–Holland, Amsterdam, 1981. pp.1–130.

Henkin,L. Monk,J.D. and Tarski,A. [HMT85], *Cylindric Algebras, Part II.* North–Holland, Amsterdam, 1985.

Henkin,L. Monk,J.D. Tarski,A. Andréka,H. and Németi,I. [81], *Cylindric set algebras.* Lecture Notes in Mathematics Vol. 883, Springer–Verlag, Berlin, 1981.

Henkin,L. and Tarski,A. [61], *Cylindric algebras.* In: Proc. of symposia in pure math. 2, Amer. Math. Soc., Providence, Rhode Island, 1961. pp.83–113.

Hennessey,M. [80], *A proof system for the first–order relational calculus*, J. Computer and System Sciences **20** (1980), 96–110.

Hirsch,R. and Hodkinson,I. [95], *Step-by-step - building representations in algebraic logic*, J. Symbolic Logic. to appear.

Howard,C.M. [65], *An approach to algebraic logic.* Doctoral Dissertation, University of California, Berkeley, 1965. 83pp.

Imielinski,T. and Lipski,W. [84], *The relational model of data and cylindric algebras*, J. of Computer and System Sciences **28** (1984), 80–102.

Immerman,N. and Kozen,D. [89], *Definability with bounded number of bound variables*, Information and Computation **83** (1989), 121–139.

Jipsen,P. Jónsson,B. and Rafter,J. [95], *Adjoining units to residuated Boolean algebras*, Algebra Universalis **34** (1995), 118–127.

Johnson,J.S. [69], *Nonfinitizability of classes of representable polyadic algebras*, J. Symbolic Logic **34** (1969), 344–352.

Jónsson,B. [59], *Representation of modular lattices and of relation algebras*, Trans. Amer. Math. Soc. **92** (1959), 449–464.

Jónsson,B. [82], *Varieties of relation algebras*, Algebra Universalis **15** (1982), 273–298.

Jónsson,B. [84], *The theory of binary relations.* In: Algebraic Logic, Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991. pp. 245-292.

Jónsson,B. [84a], *Maximal algebras of binary relations.* In: Contributions to Group Theory: Papers published in honor of Roger Lyndon on his sixty–fifth birthday, Contemporary Mathematics Vol. 33, Amer. Mat. Soc., Providence, 1984. pp.299–307.

Jónsson,B. [90], *Program specification as Boolean operators.* Preprint, Vanderbilt University, circulated at the Jónsson Conference, Iceland, 1990.

Kozen,D. [79], *On the duality of dynamic algebras and Kripke models.* In: Proc. Workshop on Logic of Programs, Lecture Notes in Computer Science Vol. 125, Springer–Verlag, Berlin, 1979. pp.1–11.

Kozen,D. [80], *A representation theorem for models of *–free PDL.* In: Proc. 7–th Colloq. on Automata, Languages and Programming, July 1980. pp.351–362.

Kozen,D. [81], *On induction vs. *–continuity.* Proc. Workshop on Logics of Programs 1981, Lecture Notes in Computer Science Vol 131, Springer– Verlag, Berlin, 1981. pp.167–176.

Kozen,D. [90], *On Kleene algebras and closed semirings.* In: Mathematical Foundations of Computer Science'90, Lecture Notes in Computer Science Vol. 452, Springer–Verlag, Berlin, 1990. pp.26–47.

Lyndon,R.C. [56], *The representation of relation algebras II*, Ann. of Math. **63** (1956), 294–307.

Lyndon,R. [61], *Relation algebras and projective geometries*, Michigan Math. J. **8** (1961), 21–28.

Madarász,J.X.Q. Németi,I. and Sági,G. [97], *On the finitization problem of relation algebras. Extended abstract.* Mathematical Institute, Budapest. Preprint, 1997. On ftp://circle.math-inst.hu-/pub/algebraic-logic.

Maddux,R.D. [78], *Topics in relation algebras.* Dissertation, University of California, Berkeley, 1978.

Maddux,R.D. [88], *Problems posed during the Conference on Algebraic Logic.* Preprint, October 6, 1988. In: Andréka-Monk-Németi[88], pp.727-746.

Maddux,R.D. [89], *Nonfinite axiomatizability results for cylindric and relation algebras*, J. Symbolic Logic **54,3** (1989), 951–974.

Maddux,R.D. [89a], *Canonical relativized cylindric set algebras*, Proc. Amer. Math. Soc. **107,2** (1989), 465–478.

Maddux,R.D. [89b], *Finitary algebraic logic*, Z. Math. Logic. Grundl. Math. **35** (1989), 321–332.

Maddux,R.D. [89c], *A collection of research problems on relation algebras.* Preprint, Iowa State University, Ames, Iowa, 1989.

Maddux,R.D. [90], *The neat embedding problem and the number of variables required in proofs*, Proc. Amer. Math. Soc. **112** (1991), 195–202.

Maddux,R.D. [90a], *The origin of relation algebras in the development and axiomatization of the calculus of relations*, Studia Logica **50,3-4** (1991), 421-456.

Makkai,M. [87], *Stone duality for first–order logic*, Advances in Mathematics **65,2** (1987), 97–170.

Marx,M. Pólos,L. and Masuch,M. (eds) [96], *Arrow Logic and Multi-Modal Logic.* CSLI Publications, Stanford, 1996. vi+247pp.

Marx,M. [95], *Algebraic relativization and arrow logic.* PhD Dissertation, University of Amsterdam, Amsterdam, 1995.

Mikulás,Sz. [95], *Taming Logic.* PhD Dissertation, University of Amsterdam, Amsterdam, 1995.

Mikulás,Sz. Sain,I. and Simon,A. [90], *On the complexity of the equational theory of relation algebras with standard projection elements.* Bulletin of Section of Logic 21,3 (1992), 103-111.

Monk,J.D. [61], *On the representation theory for cylindric algebras*, Pacific J. Math. **11** (1961), 1447–1457.

Monk,J.D. [64], *On representable relation algebras*, Michigan Math. J. **11** (1964), 207–210.

Monk,J.D. [69], *Nonfinitizability of classes of representable cylindric algebras*, J. Symbolic Logic **34** (1969), 331–343.

Monk,J.D. [70], *On an algebra of sets of finite sequences*, J. Symbolic Logic **35** (1970), 19–28.

Monk,J.D. [74], *Connections between combinatorial theory and algebraic logic.* In: Studies in algebraic logic, MAA Studies in Mathematics Vol. 9, The Mathematical Association of America, 1974. pp.58–91.

Németi,I. [80], *Some constructions of cylindric algebra theory applied to dynamic algebras of programs*, Comput. Linguist. Comput. Lang. **14** (1980), 43–65.

Németi,I. [81], *Dynamic algebras of programs.* In: Fundamentals of Computation Theory'81, Lecture Notes in Computer Science Vol. 117, Springer–Verlag, Berlin, 1981. pp.281–290.

Németi,I. [82], *Every free algebra in the variety generated by the representable dynamic algebras is separable and representable*, Theoretical Computer Science **17** (1982), 343–347.

Németi,I. [83], *Combinatorial problem for algebraic logic*, Abstracts of Amer. Math. Soc., **4,5** (1983), p.389. *83T–05–330.

Németi,I. [87], *On varieties of cylindric algebras with aplications to logic*, Annals of Pure and Applied Logic **36** (1987), 235–277.

Németi,I. [90], *Algebraizations of quantifier logics: an introductory overview.* Version 12, Preprint No 13/1996, Mathematical Institute, Budapest. On ftp:/circle.math-inst.hu/pub-/algebraic-logic. Shortened version appeared as: Studia Logica 50/3-4(1991)458-569.

Németi,I. [90a], *On cylindric algebraic model theory.* In: Algebraic Logic and Universal Algebra in Computer Science, Lecture Notes in Computer Science Vol. 425, Springer–Verlag, Berlin, 1990. pp.37–76.

Németi,I. [96], *Ontology can turn negative results to positive (An overview of recent results.)*, Bulletin of Section of Logic **25,1** (1996), 29-40.

Németi,I. and Andréka,H. [94], *General algebraic logic: a perspective on "what is logic".* In: What is a logical system, Gabbay,D.M. (ed), Clarendron Press, Oxford, 1994. pp.393-444.

Németi,I. and Kurucz,Á. [97], *Representability of pairing relation algebras depends on your ontology*, Journal of Symbolic Logic. To appear.

Németi,I. and Sági,G. [96], *On the equational theory of representable polyadic equality algebras.* Preprint, Mathematical Institute, Budapest, 1996. On ftp:/circle.math-inst.hu/pub-/algebraic-logic.

Németi,I. and Simon,A. [95], *Weakly higher order cylindric algebras and finite schema axiomatization of the representables.* Preprint, Mathematical Institute, Budapest, 1995.

Ng,K.C. [84], *Relation algebras with transitive closure.* Dissertation, University of California, Berkeley, 1984. iv+157 pp.

Orlowska,E. [89], *Dynamic logic with program specifications and its relational proof system.* Preprint, Polish Acad. Sci., Warsaw, Poland, 1989.

Orlowska,E. [90], *Nonclassical relation and cylindric algebras.* Preprint, Polish Acad. Sci., Warsaw, Poland, 1990.

Pálfy,P.P. [85], *On the chromatic number of certain highly symmetric graphs*, Discrete Math. **54,3** (1985), 31–38.

Peirce,C.S. [1870], *Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic*, Memoirs of the Amer. Acad. Sci. **9** (1870), 317–378.

Pinter,C. [73], *Cylindric algebras and algebras of substitutions*, Trans. Amer. Math. Soc. **175** (1973), 167–179.

Plotkin,B.I. [88], *Halmos (polyadic) algebras in the database theory.* In: Algebraic Logic, Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991. pp.505-518.

Plotkin,B.I. [94], *Universal algebra, algebraic logic and data–bases.* Kluwer Academic Publishers, 1994.

Pratt,V.R. [79], *Dynamic algebras: Examples, Constructions, Applications.* Technical Report MIT/LCS/TM–138, MIT Laboratory for Computer Science, July 1979.

Pratt,V.R. [90], *Dynamic algebras as a well behaved fragment of relation algebras.* In: Algebraic logic and universal algebra in computer science, Lecture Notes in Computer Science Vol. 425, Springer–Verlag, Berlin, 1990. pp.77–110.

Pratt,V.R. [91], *Dynamic algebras*, Studia Logica **50,3-4** (1991), 57-607.

Pratt,V.R. [92], *Origins of the calculus of binary relations.* In: Proc. 7th Annual IEEE Symposium on Logic in Computer Science, Santa Cruz, CA, June 1992. pp.248-254.

Preller,A. [70], *Substitution algebras in their relation to cylindric algebras*, Arch. Math. Logik Grundl. **30** (1970), 91–96.

Redko,V.N. [64], *On defining relations for the algebra of regular events (in Russian)*, Ukrain. Mat. Z. **16** (1964), 120–126.

Sain,I. [87], *Positive results related to the Jónsson, Tarski–Givant representation problem.* Expanded version of Preprint Math. Inst. Hungar. Acad. Sci., 1987.

Sain,I. [87a], *Searching for a finitizable algebraization of first order logic,* Annals of Pure and Applied Logic. To appear.

Sain,I. [92], *On the problem of finitizing first order logic and its algebraic counterpart. (A survey of results and methods.).* In: Logic Colloquium'92, Csirmaz,L. Gabbay,D.M. and de Rijke,J. (eds), CSLI Publications, Stanford, 1995. pp.243-292.

Sain,I. and Gyuris,V. [96], *Finite schematizable algebraic logic,* Journ. IGPL. In press.

Sain,I. and Németi,I. [97], *Fork algebras in usual and in non-well-founded set theories (an overview).* Studia Logica Library (a special volume dedicated to the memory of Helena Rasiowa), to appear.

Sain,I. and Thompson,R.J. [88], *Strictly finite schema axiomatization of quasi–polyadic algebras.* In: Algebraic Logic, Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991. pp.539-572.

Schein,B.M. [70], *Relation algebras and function semigroups,* Semigroup Forum **1** (1970), 1–62.

Schein,B.M. [88], *Representation of reducts of Tarski relation algebras.* In: Algebraic Logic, Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, Amsterdam, 1991. pp.621-636.

Schmidt,G. and Ströhlein,T. [89], *Relationen und Graphen.* Series Mathematik für Informatiker, Springer–Verlag, Berlin, 1989.

Schönfeld,W. [77], *Applications of relation algebras in computer science,* Notices Amer. Math. Soc. **24** (1977), A–254.

Schröder,E. [1895], *Vorlesungen über die Algebra der Logik (exacte Logik).* Volume 3, "Algebra und Logik der Relative", Part I, Leipzig, 1895.

Serény,Gy. [85], *Compact cylindric set algebras,* Bull. Section of Logic **14,2** (Warsaw–Lodz, 1985), 57–64.

Simon,A. [90], *A finite schema completeness result for algebraic logic.* In: Algebraic Logic, Colloq. Math. Soc. J. Bolyai Vol. 54, North–Holland, 1991. pp.665-670.

Tarski,A. [55], *Contributions to the theory of models, III.* Nederl. Akad. Wetensch. Proc. Ser. A, vol.57 (Indag. Math., Vol. 17) (1955), 56–64.

Tarski,A. and Givant,S. [87], *A formalization of set theory without variables.* AMS Colloquium Publications Vol. 41, Providence, Rhode Island, 1987.

Thompson,R.J. [81], *Transformational structure of algebraic logics.* Dissertation, University of California, Berkeley, 1981.

Thompson,R.J. [87], *Semigroups of finite transformations and cylindric algebras.* Preprint No 45/1987, Mathematical Institute, Budapest, 1987.

Thompson,R.J. [93], *Complete description of substitutions in cylindric algebras and other algebraic logic.* In: Algebraic Methods in Logic and in Computer Science. Banach Center Publications Vol 28. Polish Academy of Science, Warszawa, 1993. pp.327–342.

Trnková,V. and Reiterman,J. [87], *Dynamic algebras with test,* J. Computer System Sciences **35,2** (1987), 229–242.

Venema,Y. [89], *Two–dimensional modal logics for relational algebras and temporal logic of intervals.* Preprint, University of Amsterdam, 1989.

Venema,Y. [90], *Cylindric modal logic,* Journal of Symbolic Logic **60,2** (1995), 591-623.

Volkov,N.D. [86], *The transition from a relation algebra to a Halmos algebra.* In: Algebra and Discrete Mathematics: Theoretical Foundations of Software, (in Russian), Latv. Gos. Univ., Riga, 1986.

Wagner,V.V. [56], *Representation of ordered semigroups (in Russian),* Mat. Sbornik **38** (1956), 203–240. English Transl.: Amer. Math. Soc. Transl. (2) 36 (1964), 295–336.

Zaretskij,K.A. [59], *Representations of ordered semigroups with binary relations (in Russian)*, Matematika **6** (1959), 48–50.

Zierer,H. [83], *Relationale Semantik.* Diplomarbeit, Institut für Informatik, Technische Universitat, München, 1983.

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