Decidability, undecidability, and Gödel incompleteness in relativity theories

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Abstract

In this paper we investigate the logical decidability and undecidability properties of relativity theories. To this end, we need to recall versions of relativity theory which are built up in a logical framework i.e. which are theories in the sense of mathematical logic (section 2). In Part I (section 3) we investigate decidability properties of versions of relativity. We will find that the answer whether decidable or not depends on how rich version we study. We also study applicability of Gödel's incompleteness theorems to relativity. In Part I we study applicability of the conclusion of Gödel's first incompleteness theorem to the theory in question. In Part II (section 4) we study whether Gödel's second incompleteness theorem applies to the theory of relativity in question (we mean whether the conclusion of Gödel's theorem applies). In Part III (section 5) we study unprovability of consistency. The same investigation leads up to asking whether the theory is Π^0_k hard. This leads up to statements (possible predictions) in the language of relativity which are independent of ZF set theory.

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Part I: Gödel First Incompleteness in relativity theories
Part II: Gödel's Second Incompleteness in relativity
Part III: Π^0_k-hardness and ZFC independence in relativity

1 Introduction

In this paper we investigate the logical decidability and undecidability properties of relativity theories. (Relativity comes in many stages, e.g., special, general, or cosmological relativity.) To this end, we need to recall versions of relativity theory which are built up in a logical framework i.e. which are theories in the sense of mathematical logic. Such are available in the literature, e.g., *Specrel* of [2] or [4] or [18], or *Genrel* of [18] or [3], to mention a few. We start this paper by recalling such logical forms of relativity theory (section 2). In Part I (section 3) we investigate decidability properties of versions of relativity. We will find that the answer whether decidable or not depends on how rich version we study. Also in Part I we study applicability of the conclusion of Gödel's first incompleteness theorem to the theory in question. In Part II (section 4) we study whether Gödel's second incompleteness theorem applies to the theory of relativity in question (we mean whether the conclusion of Gödel's theorem applies). In Part III (section 5) we study unprovability of consistency. In more detail, we discuss the property of a theory T saying that consistency of T, $\mathsf{Con}(T)$, is formalizable in T but not provable from T. We look at the question which versions of relativity have this property. The same investigation leads up to asking whether the theory is Π^0_k hard. (We could call this "third Gödel incompleteness" property.) Roughly speaking, for all three questions we will find that the most sparing versions of relativity have the property (e.g., are decidable) while the richer versions of the same theory have the negative properties (e.g., undecidable). This leads up to statements (possible predictions) in the language of relativity T which are independent of ZF set theory. Independence of set theory motivates our arguing that relativity theory should not aim for proving such statements.

2 Basic axiom system

2.1 The frame language

We introduce the first-order logic language, which we will use for formalizing (first special) relativity, with an eye open for the subsequent generalization of the theory. We want to talk about *motion of bodies.*[1] What is motion? It is changing location in time. Therefore we will talk about *bodies*, *time*, *space*, and about a *location-function* which tells us which body is where at a given time. We want to talk about *relativity* theories; therefore these location functions will depend on *observers*; different observers may see the same motion differently. (The location function determined by an observer m will be called the world-view function w_m of observer m.) We will treat observers as special bodies whose motion will be represented exactly the same way as that of the rest of the bodies. These observers are often called, in the literature, *reference frames.*[2]

It will be convenient for us to be flexible about the dimension of space: we will not only treat 3-dimensional space, but 1 or 2, or higher-dimensional spaces as well. We will treat time as a special dimension of *space-time*. n will denote the dimension of our space-time.[3] Thus, usually $n = 4$ (3 space-dimensions and 1 time-dimension), but we will consider also $n = 2, 3$ or $n > 4$. Our bodies will be idealized, pointlike.

The vocabulary of our language is the following: unary relations

B (bodies)

Obs (observers)

Ph (photons)

Q or F (quantities used for giving location and "measuring time");

an $n + 2$-ary relation, the *location-* or *world-view* relation

W (world-view relation, $W(m, b, t, s_1, \ldots, s_{n-1})$ intends to mean that according to observer (or reference-frame) m, the body b is present at time t and location $\langle s_1, \ldots, s_{n-1}\rangle$);

for dealing with quantities, we will have two binary functions, and a binary relation:

$+, \quad \cdot, \quad \leq.$

[1]In this paper we concentrate only on kinematics; the same kind of investigations can be carried out concerning mass, forces, energy etc., cf. [5]. However, if a theorem can be proved without referring to these extra notions, we consider that a virtue.

[2]This difference is only a matter of linguistic convention.

[3]Recent generalizations of general relativity in the literature (e.g. M-theory) indicate that it might be useful to leave n as a variable.

In our theories, we will always assume the following:

- observers and photons are bodies,
- $W(m, b, t, s_1, \dots, s_{n-1})$ implies that m is an observer, b is a body, and $t, s_1, \dots, s_{n-1}$ are quantities,
- $\langle Q, +, \cdot, \leq \rangle$ is a Euclidean linearly ordered field[4].

We found that the simplest way of treating these assumptions is to use a 2-sorted first-order language, where

B, F are sorts or universes,

Obs, Ph are unary relations of sort B,

W is an $n+2$-ary relation of sort $B \times B \times F \times F \times \dots \times F$,

$+, \cdot$ and $\leq$ are operations and relation of sort F.

Let

$$\mathfrak{M} = \langle B^{\mathfrak{M}}, Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}; F^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, \leq^{\mathfrak{M}}; W^{\mathfrak{M}} \rangle$$

be a model of our two-sorted language. This means that $B^{\mathfrak{M}}$ and $F^{\mathfrak{M}}$ are sets, they are called the *universes of sort B and F* respectively, $Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$ etc. We will omit the superscripts $^{\mathfrak{M}}$. We call $\mathfrak{M}$ a *frame-model* if $\mathcal{F}_{\mathfrak{M}} := \langle F, +, \cdot, \leq \rangle$ is a Euclidean linearly ordered field and $W \subseteq Obs \times B \times F \times \dots \times F$. $\models$ denotes the usual semantical consequence relation *induced by frame-models*, i.e. $Th \models \varphi$ means that for every *frame-model* $\mathfrak{M}$, if $\mathfrak{M} \models Th$, then $\mathfrak{M} \models \varphi$.

Next we introduce some terminology in connection with arbitrary frame-models $\mathfrak{M} = \langle B, Obs, Ph; F, +, \cdot, \leq; W \rangle$.

The essence, the "heart" of a frame-model is the world-view relation W. Since $W \subseteq Obs \times B \times {}^nF$, for every observer $m \in Obs$ it induces a function $w_m : {}^nF \to \{X : X \subseteq B\}$ as follows: for every $p \in {}^nF$

$$w_m(p) := \{b \in B \ : \ W(m, b, p)\}.$$

Thus $w_m(p)$ is the set of bodies present at space-time location p for m. We call a set of bodies an *event*, and w_m is called the *world-view function* of m,

[4]This is why we denote quantities also with F. An ordered field is called *Euclidean* if every positive element has a square root in it, i.e. if $(\forall x > 0)(\exists y) x = y \cdot y$ is valid in it.

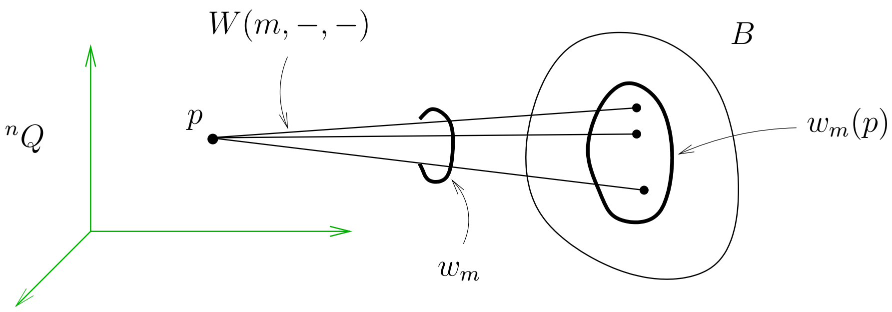

Figure 1: The world-view function w_m.

which to each space-time location p tells us what event observer m observes or "sees happening" at location p. "Seeing" has nothing to do with photons here, it really means "coordinatizing" only.

The *trace* or *world-line* of a body b according to an observer m is the set of space-time locations where m sees b, i.e.

$$tr_m(b) := \{p \in {}^nF : W(m,b,p)\}.$$

The world-view function w_m can be recovered from the family of traces of all bodies (from $\langle tr_m(b) : b \in B\rangle$), and the world-view-relation W can be recovered from all the world-view functions (from $\langle w_m : m \in Obs\rangle$). Thus we can "represent" the function w_m by the *world-view* of m, which is just the indexed family $\langle tr_m(b) : b \in B\rangle$, and which, in turn, we represent by drawing the traces of bodies that we are interested in. See Figure 2.

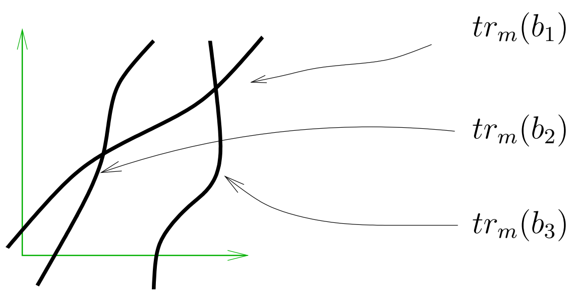

Figure 2: World-view of m.

Since $\mathcal{F} = \langle F, +, \cdot \rangle$ is a field, we can define n-dimensional straight lines as follows (these will be the world-lines of "inertial bodies"). We will use the vector-space structure of ${}^n\mathcal{F}$, i.e. if $p, q \in {}^nF$ and $\lambda \in F$ then $p+q, p-q, \lambda p \in {}^nF$ and $\overline{0}$ denotes the *origin*, i.e. $\overline{0} = \langle 0, \ldots, 0 \rangle$, where 0 is the zero-element of the field. Let $\ell \subseteq {}^nF$. We say that ℓ is a *straight line* iff there are $p, \alpha \in {}^nF$ such that $\alpha \neq \overline{0}$ and

$$\ell = \{p + r \cdot \alpha \ : \ r \in F\}.$$

Lines denotes the set of all straight lines (of dimension n). $\bar{t}$ denotes the *time axis*,

$$\bar{t} := \{\langle r, 0, \ldots, 0 \rangle \ : \ r \in F\}.$$

$\bar{t}$ is a straight line. If $\ell \in Lines$, then $ang(\ell)$, defined below, represents the *angle*[5] between ℓ and $\bar{t}$ (where $\alpha = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ is associated to ℓ as before):

$$ang(\ell) := \tfrac{\alpha_1^2 + \ldots + \alpha_{n-1}^2}{\alpha_0^2} \quad \text{if } \alpha_0 \neq 0, \text{ and}$$

$$ang(\ell) := \infty \quad \text{if } \alpha_0 = 0. \text{ Here } \infty \text{ is any element not in } F.$$

$ang(\ell) = 1$ means intuitively that the angle between ℓ and $\bar{t}$ is 45^o. (See Figure 3.) Assume that $tr_m(k) = \ell$ is a straight line. Then $ang(\ell)$ represents the *velocity*[6] of k as seen by m:

$$v_m(k) := ang(tr_m(k)), \quad \text{if } tr_m(k) \in Lines.$$

E.g., $v_m(k) = 0$ means that $tr_m(k)$ is parallel with $\bar{t}$, i.e. k's location does not change with time, i.e. k is *at rest* w.r.t. m. The bigger $v_m(k)$ is, the bigger distance k travels in a unit time (as seen by m).

2.2 Basic axioms of special relativity

A *plurality* of "competing" axiom systems (or "relativity theories") is an essential feature of a logical analysis of relativity as developed in e.g. [4]. In this section we recall one of these axiom systems and will call it $Specrel_0$. It consists of five axioms. In the following axioms, m, k stand for arbitrary observers, h for an arbitrary body, ℓ for an arbitrary straight line (i.e. element

[5]Actually, $ang(\ell)$ is the square of the tangent of the angle between ℓ and $\bar{t}$.

[6]Instead of "velocity", the precise expression would be "speed", since $v_m(k)$ is a scalar and not a vector.

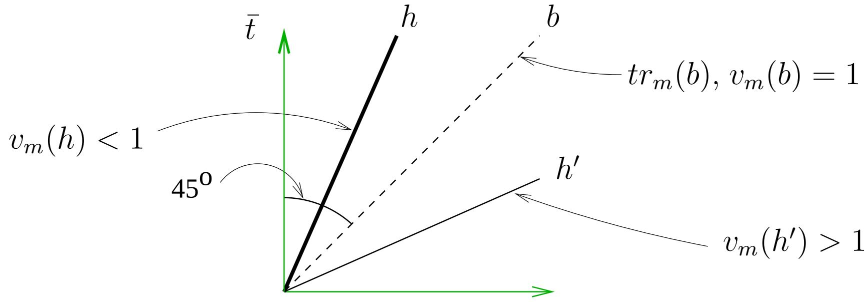

Figure 3: Velocities.

of *Lines*), and *ph* for an arbitrary photon. We use the standard custom in logic that free variables should be understood as universally quantified, e.g., the axiom $tr_m(m) = \bar{t}$ means $(\forall m \in Obs) tr_m(m) = \bar{t}$.

Our first axiom says that the traces of observers and photons, as seen by any observer, are straight lines:

AxLine $tr_m(h) \in Lines \quad$ for $h \in Obs \cup Ph$.

Since translating our intuitive statements to first-order formulas in the language of our frame-models ($\mathfrak{M}$'s) will be straightforward, we will not give these translations, we will only give the intuitive forms.

The second axiom says that any observer sees himself at rest in the origin:

AxSelf $tr_m(m) = \bar{t}$.

The third axiom says that we have the tools for thought-experiments: on any appropriate straight line we can assume there is a potential observer; and the same for photons:[7]

AxPot $ang(\ell) < 1 \quad \Rightarrow \quad (\exists k \in Obs) \ell = tr_m(k)$, and
$\quad ang(\ell) = 1 \quad \Rightarrow \quad (\exists ph \in Ph) \ell = tr_m(ph)$.

[7]This axiom can be "tamed" by using modal logic, such that space-time does not get crowded with k's and ph's, cf. [4].

The fourth axiom says that all observers "see" the same events (possibly at different space-time locations):[8] For a function f, its *range* is $Rng(f) := \{y : \exists x(f(x) = y)\}$.

AxEvents $Rng(w_m) = Rng(w_k)$.

The last axiom says that the velocity of a photon is 1, for each observer:

AxPh $v_m(ph) = 1 \quad (\text{and} \quad tr_m(ph) \in Lines)$.

Our choice for a "first possible" axiom system for special relativity is:

$Specrel_0 :=$ {**AxLine, AxSelf, AxPot, AxEvents, AxPh**}.

When we want to indicate explicitly the number of dimensions, we will write $Specrel_0(n)$ in place of $Specrel_0$. We note that **AxPh** together with the photon part of **AxPot** is the relativistic part of $Specrel_0$. (The rest are true in Newtonian Mechanics.) In [4] we denoted $Specrel_0$ with $Basax(n)$ for "basic axioms".

Let $n > 2$. In [4],[2] we show that $Specrel_0(n)$ is consistent, it is not independent, and it forbids faster than light observers but permits faster than light bodies.[9] In this paper we show that $Specrel_0$ generates an *undecidable* first-order theory but we can strengthen it so that it becomes decidable (moreover categorical); and also we can strengthen it so that it becomes hereditarily undecidable, further both of Gödel's incompleteness properties hold for this strengthened version. We will see that both kinds of extension of $Specrel_0$ are natural.

Now we are going to introduce seven extra natural axioms that will make $Specrel_0$ categorical over any field. The theory $Specrel_0$ extended with these seven axioms (and with any decidable theory of fields) is decidable. We will see that if we leave out any one of six of these axioms, the theory will become undecidable, and such that it can be extended to a hereditarily undecidable theory where both Gödel's incompleteness theorems hold.

[8]This will have to be considerably weakened, when preparing for a generalization of our axiom systems like $Specrel_0$ towards general relativity, cf. [4].

[9]The point in proving things like $Specrel_0 \models$ *no FTL observer* is in the small number of axioms and concepts needed. Actually in [4] we show that a much weaker version of $Specrel_0$ is enough for proving this conclusion. A more refined version of the theorem says that FTL observers "lose most of their meter rods", cf. [4].

2.3 A principle of relativity

The *world-view transformation* f_{mk} between two observers m, k is defined as

$$f_{mk} := \{\langle p, q\rangle : w_m(p) = w_k(q) \text{ and } w_k(q) \neq \emptyset\} .$$

From our previous axioms it follows that f_{mk} is a transformation of ${}^nF$ (and not only an arbitrary binary relation) if m, k are observers.[10] Therefore we will use f_{mk} as a function. Then $f_{mk}(p)$ is the "place" where k sees the same event that m sees at p, i.e.

$$w_m(p) = w_k(f_{mk}(p)) .$$

When $p = \langle p_0, \ldots, p_{n-1}\rangle \in {}^nF$, we will denote p_0 by p_t in order to emphasize that p_t is the "*time component*" of p. Let $p, q \in {}^nF$. Then $p_t - q_t$ is the time passed between the events $w_m(p)$ and $w_m(q)$ as seen by m and $f_{mk}(p)_t - f_{mk}(q)_t$ is the time passed between the same two events as seen by k. Hence $\|(f_{mk}(p)_t - f_{mk}(q)_t)/(p_t - q_t)\|$ is the rate with which k's clock runs slow or fast as seen by m. Here, $\|a\|$ denotes the *absolute value* of a when $a \in F$, i.e. $\|a\| \in \{a, -a\}$ and $\|a\| \geq 0$.

AxSym All observers see each other's clocks run slow to the same extent,

$$\|f_{mk}(p)_t - f_{mk}(q)_t\| = \|f_{km}(p)_t - f_{km}(q)_t\|, \text{ when } m, k \in Obs \text{ and } p, q \in \bar{t}.$$

AxSym states only that any two observers "see" each other's clocks "change" the same way. In principle, this allows the clocks run fast, be right, or run slow. In the Newtonian world **AxSym** is true because there each observer sees that the other's clocks are right. In models of $Specrel_0$, **AxSym** can be true only in the way that any observer sees that the clock of any other observer not at rest with respect to it *runs slow.* Figure 7 in the proof of Thm.2.1 shows how it is possible in models of $Specrel_0$ that *both* observers "see" the clock of the other run slow.

On the choice of our symmetry axiom **AxSym**.: Under mild extra assumptions, $Specrel_0$ implies that **AxSym** is equivalent with an instance of Einstein's *special principle of relativity SPR* as it was formalized in [4],[2]. The principle SPR goes back to Galileo, intuitively it says that the "laws of nature" are the same for all inertial observers. See also Friedman [8, p.153]. We note that in models of $Specrel_0$, **AxSym** is equivalent to the potential

[10]This is a typical example of a property of special relativity which is relaxed in the process of localization (towards general relativity).

axiom requiring that, in space, in the direction orthogonal (in the Euclidean sense) to the direction of the motion there is no relativistic distortion, i.e. there is no length-contraction. Other equivalent formalizations of **AxSym** can be found in [4, §3.9].

2.4 Axioms making $Specrel_0$ categorical

Here we introduce six more axioms that will make $Specrel_0$ categorical (over any given field). As in section 2.2, in the following m, k stand for observers, ℓ for a straight line, ph_i for photons; and free variables in the axioms should be understood as universally quantified.

The first two axioms deal with the direction of flow of time. We define for any two observers m, k

$$m \uparrow k \quad \text{iff} \quad (f_{km}(1_t) - f_{km}(\bar{0}))_t > 0.$$

Intuitively this means that time flows in the same direction for m and k, see Figure 4.

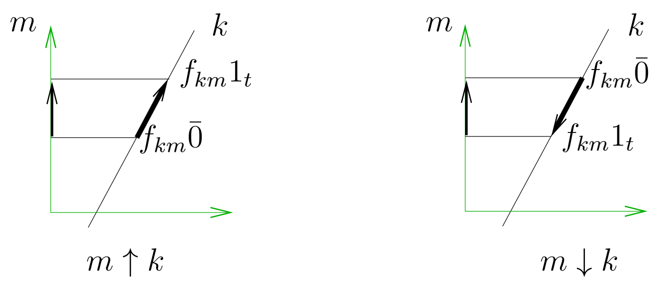

Figure 4: $m \uparrow k$ means that time flows in the same direction for m and k.

Our first axiom is a stronger version of part of **AxPot**, it says that every appropriate straight line is the life-line of an observer whose time flows "forwards".

AxPot$^+$ $ang(\ell) < 1 \quad \Rightarrow \quad (\exists k \in Obs)[\ell = tr_m(k) \text{ and } m \uparrow k]$.

The next axiom says that time flows in the same direction for any observers at rest in the origin.

Ax↑ $tr_m(k) = \bar{t} \quad \Rightarrow \quad m \uparrow k.$

The next axiom says that every observer can "re-coordinatize" his world-view with a so-called Galilean transformation. To formalize the next axiom, first we single out special transformations, that we will call Galilean transformations. A mapping $f : {}^nF \to {}^nF$ is called a *Galilean transformation* if it preserves Euclidean distance and $f(1_t) - f(\overline{0}) = 1_t$ where $1_t = \langle 1, 0, 0, \ldots \rangle$ and 1 denotes the unit element of the field $\mathcal{F}$. In other words, a Galilean transformation is a congruence transformation which is the identity map on $\bar{t}$, composed with a translation. See Figure 5. It is known that a Galilean transformation is a linear transformation composed with a translation, hence the next axiom is a first-order logic one.

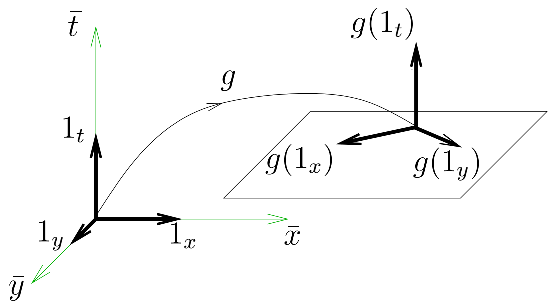

Figure 5: A Galilean transformation takes the unit vectors into pairwise orthogonal vectors of length 1, and does not change the direction of the time-unit vector.

AxGal $G(\overline{0}) \in \bar{t} \;\Rightarrow\; (\exists k \in Obs) f_{mk} = G,$ for every Galilean transformation G.

The next two axioms say, intuitively, that of each kind of observers and photons we have only one copy (or in other words, according to Leibniz's principle, if we cannot distinguish two observers or photons via some observable properties, then we treat them as equal).[11] In other words, these are so-called *extensionality axioms*. Id denotes the identity mapping.

AxExt$_1$ $f_{mk} = Id \quad \Rightarrow \quad m = k.$

[11]We could have named these axioms after Occam, too.

AxExt$_2$ $tr_m(ph_1) = tr_m(ph_2) \quad \Rightarrow \quad ph_1 = ph_2$.

The last axiom says that every body is an observer or photon.

AxNoBody $B = Obs \cup Ph$.

$Compl :=$
$\{$**AxSym, AxPot$^+$, AxGal, AxExt$_1$, AxExt$_2$, AxNoBody**$\}$,

$Specrel := Specrel_0 \cup \{$**AxSym**$\}$,

$Specrel^+ := Specrel \cup Compl \cup \{\mathrm{Ax}\uparrow\}$.

In the terminology of e.g. Malament and Hogarth, $Specrel_0, Specrel$ and $Specrel^+$ correspond to *causal space-time* (or metric-free space-time), *space-time*, and *time-oriented space-time*. $Specrel_0$ is also strongly connected to the "conformal structure of space-time". When we write "causal space-time", we have in mind the *symmetrized* version of the strict "causality relation" $\ll$. (Sometimes "metric-free space-time", "space-time", "time-oriented space-time" are used.)[12]

We did not include **Ax**$\uparrow$ into *Compl* because, as we will see, its effects are different from those of the the elements of *Compl*.[13]

THEOREM 2.1 *Let*[14] $n > 2$ *and let* $\mathcal{F} = \langle F, +, \cdot, \leq \rangle$ *be any Euclidean field.*

(i) There are exactly two models of $Specrel \cup Compl$ *with field-reduct* $\mathcal{F}$*, up to isomorphism.*

(ii) There is a unique model of $Specrel^+$ *with field-reduct* $\mathcal{F}$*, up to isomorphism.*

On the proof. We illustrate that in any model of *Specrel*, all the worldview transformations are so-called Poincaré-transformations (i.e. Lorentz-transformations composed with translations), and this is the most important part of the proof of Theorem 2.1.

[12]The terminology varies with different authors, but what we wanted to point out is that the levels of abstraction corresponding to $Specrel_0, Specrel$ and $Specrel^+$ seem to be generally distinguished levels of abstraction in the literature of relativity.

[13]Intuitively, **Ax**$\uparrow$ excludes only one model of two choices, while the rest exclude an infinite number of possibilities, cf. Thm.s 2.1-3.6.

[14]We exclude the case $n = 2$ for simplicity only.

Let m, k be observers in a model of *Specrel*, we will investigate the world-view transformation $f := f_{mk}$. It is easy to see that $f : {}^nF \to {}^nF$ is a *bijection*. It is a *collineation* by the Alexandrov-Zeeman theorem in case $n > 2$, and by [2, Thm.2] in case $n = 2$. By **AxPh**, f takes light-lines onto light-lines, and this implies that f takes the unit vectors into vectors of the *same length* and *Minkowski-orthogonal* to each other. Figure 6 illustrates the idea of the proof of this part.

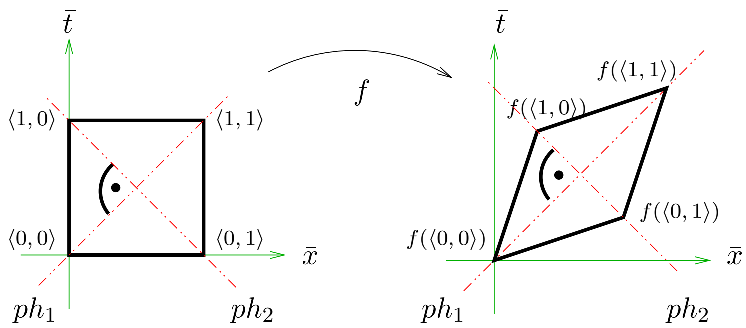

Figure 6: World-view transformations in models of $Specrel_0$ take the unit vectors to vectors Minkowski-orthogonal to each other and of the same length.

Finally, **AxSym** implies that the length of the unit vectors is fixed, as follows. We write out this part of the proof in more detail, because e.g. it shows how it is possible that both observers see each other's clocks run slow.

Let $1_t = \langle 1, 0, 0, \dots\rangle$, and let us see where $e := f_{km}(1_t)$ is on $tr_m(k)$. Let a, b and a' be as in Figure 7; i.e. they are the points on $tr_m(k)$ and on $\bar{t}$ such that the straight line connecting 1_t and a is parallel with $\overline{x}$, and the straight lines connecting 1_t and b and connecting a and a' are parallel with $f_{km}[\overline{x}]$. See Figure 7. If $e = a$, then m sees that k's clock shows 1 just when his clock shows 1, because 1_t and a are simultaneous for m. But k will see that m's clock shows $a' < 1$ when his clock shows 1, because for k, $e = a$ and a' are simultaneous. So k will think that m's clocks run slow, but m will think that k's clocks are right. Analogously, m thinks that k's clocks are right (run slow or fast, respectively) iff $e = b$ ($> b$ or $< b$ respectively). And, k thinks that m's clocks are right (run slow or fast, respectively) iff $e = a$ ($< a$ or $> a$ respectively). Thus both think that the other's clocks run slow iff $b < e < a$. The rate of "slowness" is the same for them at a unique point in between a and b, because the change of rate is a continuous and strictly monotonic function (of the "number" $\|e\|$). Now, *Minkowski-distance* is defined so that the Minkowski-distance is 1 between $\overline{0}$ and this unique point (where the rates of "running slow" are the same for m and k). Figure 8 shows the points whose Minkowski-distance from $\overline{0}$ is 1, i.e. it shows Minkowski-circle with radius 1 and center $\overline{0}$.

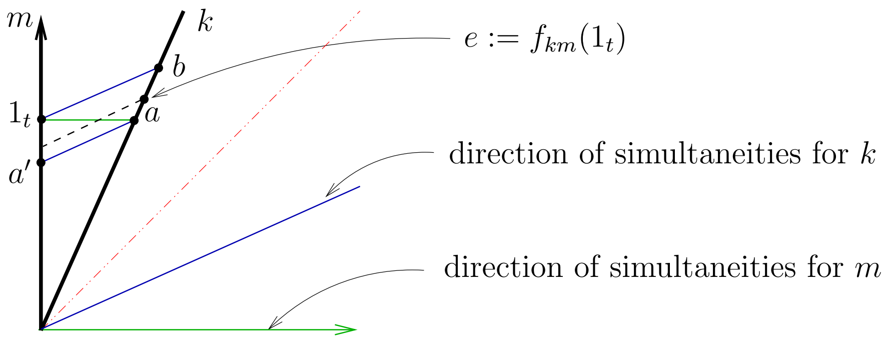

Figure 7: Both m and k think that the other's clocks run slow iff $f_{mk}(1_t)$ is in between a and b. The rates of "running slow" will be equal at a unique point.

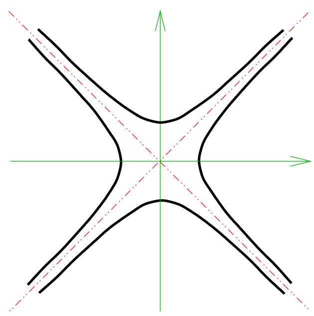

Figure 8: Minkowski-distance 1.

It is known that any collineation is an affine transformation composed with a field-automorphism-induced transformation. Using that the above line of thought is valid for any $p \in \bar{t}$ in place of 1_t, one can show that the world-view transformations are actually *affine* transformations. Summing up: in models of *Specrel*, the world-view transformations take the unit vectors into pairwise Minkowski-orthogonal vectors of Minkowski-length 1. These kinds of affine transformations are called in the literature *Poincaré-transformations*. **QED**

3 Decidability and undecidability, Gödel's First Incompleteness in relativity

We now turn to decidability questions. Let $n > 0$ be fixed. It is known that the first-order logic theory of Euclidean geometries (of dimension n) over real-closed fields is decidable (a result of Tarski), cf. e.g. Schwabhäuser et al. [17]. Similarly, the first-order theory of Minkowski geometries (of some fixed dimension n) over real-closed fields is also decidable (a result of Goldblatt), cf. Goldblatt [9, Appendix A, pp.168-169]. This leads naturally to the question whether our relativity theories $Th \supseteq$ [*Specrel* + *theory of real-closed fields*] are decidable. [15]

We will see that the answer depends on what extra "simplifying assumptions" we make (in Th) on the logical sort "Bodies" (and does not depend too much on what "typically relativity theoretic" assumptions we make on observers, photons and world-view transformations such as **Ax↑**, **AxGal**, etc. [16])

We start by recalling the definition of real-closed fields and by recalling some facts from the literature.

An ordered field $\mathcal{F}$ is real-closed if it is Euclidean (i.e. every positive element has a square root), and if every polynomial of odd degree has zero as a value. This last requirement can be expressed with the infinite set $\{\phi_{2n+1} : n \in \omega\}$ of first-order formulas, where for every $n \in \omega$, ϕ_n denotes the following sentence

$$\forall x_0 \ldots \forall x_n \exists y[x_n \neq 0 \rightarrow (x_0 + x_1 \cdot y + \ldots + x_n \cdot y^n = 0)].$$

[15]The question remains interesting even if we do not insist on $Th \supseteq$ *Specrel*, but we will see that the assumption $Th \supseteq$ [*theory of real-closed fields*] is needed (in some sense) in order to keep the question interesting.

[16]E.g. **Ax↑** says that if the traces of two observers coincide, then their time flows in the same direction, or we can state in an axiom that if two observers are at rest relative to each other, then there is no relativistic distortion between their world-views.

By a *theory* we will understand an arbitrary set of first-order formulas (i.e. we will not assume that it is closed under semantical consequence). We call a theory *Th decidable* (or *undecidable* respectively) if the set of all first-order semantical consequences of *Th* is decidable (or undecidable respectively). We call *Th complete* if it implies either ϕ or $\neg\phi$ for each first-order formula ϕ (of its language). Propositions 3.1,3.2 below are known in the literature. Prop.3.1 is a corollary of Tarski's famous elimination of quantifiers for real-closed fields.

PROPOSITION 3.1 *The theory of real-closed fields is decidable and complete.*

PROPOSITION 3.2 *The theories of ordered fields and Euclidean fields are undecidable.* [17]

CONJECTURE 3.3 *Any finitely axiomatizable consistent theory of ordered fields is undecidable.*

For more on this subject we refer to the book of van den Dries [7]. (Works of Alexander Prestel [University Konstanz] and Martin Ziegler [University of Freiburg] might also be relevant here.)

COROLLARY 3.4 $Specrel_0$, $Specrel$ and $Specrel^+$ are undecidable.

Proof. This is a corollary of Prop.3.2, and the theorem that for any Euclidean field $\mathcal{F}$ there is a model of $Specrel^+$ with $\mathcal{F}$ as the field reduct (Theorem 2.1).: Let ϕ be any field-theoretic first-order formula written by using variables of our quantity sort. Then ϕ is valid in a frame-model $\mathfrak{M}$ with field reduct $\mathcal{F}$ iff ϕ is valid in $\mathcal{F}$. Thus ϕ is valid in the class of Euclidean fields iff ϕ is true in all models of $Specrel^+$. Since the first-order theory of the Euclidean fields is undecidable by Prop.3.2, the first-order consequences of $Specrel^+$ is undecidable, too. Since this is a finite theory, then any subset of it is undecidable, too. **QED**

The above suggests that if we want to obtain interesting decision-theoretic results, we have to concentrate on real-closed fields; or at least include a decidable theory of field-axioms into our theories. Let Φ denote the theory of real-closed fields.

[17] Note that if a finitely (or more generally, recursively) axiomatizable theory is undecidable, then it is not complete.

THEOREM 3.5 *Let $n > 2$.*

(i) $Specrel_0 \cup Compl \cup \Phi$ is decidable.

(ii) $Specrel_0 \cup Compl \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$ is decidable and complete.

(iii) $Specrel_0 \cup (Compl \setminus \{\mathbf{Ax}\}) \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$ is undecidable, for any axiom $\mathbf{Ax} \in Compl$.

Proof. We show that (i) and (ii) are corollaries of Theorem 2.1, we sketch the proof of (ii). Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of $Specrel_0 \cup Compl \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$. We cannot apply Theorem 2.1 yet, because the field-reducts $\mathcal{F}$ and $\mathcal{F}'$ of $\mathfrak{M}$ and $\mathfrak{M}'$ respectively may not be the same. But they are elementarily equivalent, because Φ is complete, so by the Keisler-Shelah isomorphic ultrapowers theorem they have isomorphic ultrapowers, say $\mathcal{F}_1$ and $\mathcal{F}_1'$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_1'$ be the ultrapowers of $\mathfrak{M}$ and $\mathfrak{M}'$ respectively, taken by the same ultrafilter. Then the field-reducts of these are $\mathcal{F}_1$ and $\mathcal{F}_1'$ respectively. Now we can apply Theorem 2.1 to $\mathfrak{M}_1$ and $\mathfrak{M}_1'$ because $\mathcal{F}_1$ and $\mathcal{F}_1'$ are isomorphic, getting that $\mathfrak{M}_1$ and $\mathfrak{M}_1'$ are isomorphic, so elementarily equivalent. But then $\mathfrak{M}$ and $\mathfrak{M}'$ are elementarily equivalent, too, since the former two models are ultrapowers of these. This finishes the proof of (ii). (iii) is a corollary of the next theorem; we included it here because it nicely contrasts (i) and (ii). **QED**

We now turn to the analog of Gödel's first incompleteness theorem.

THEOREM 3.6 *Let $n > 1$ and let $\mathbf{Ax}$ be any member of $Compl$. There is a formula ν (in our frame-language) such that*

(i) ν is consistent with $Specrel_0 \cup (Compl \setminus \{\mathbf{Ax}\}) \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi$

and for any theory Th consistent with ν

(ii) Th is hereditarily undecidable in the sense that no consistent extension of Th is decidable.

(iii) The conclusion of Gödel's first incompleteness theorem applies to the theory Th, i.e. no consistent recursively enumerable extension of Th is complete; moreover there is an algorithm that to each consistent, recursively enumerable extension Th' of Th yields a formula ϕ such that $Th' \not\models \phi$ and $Th' \not\models \neg\phi$.

Proof. The idea of the proof is to show that absence of any member of *Compl* allows us to interpret Robinson's Arithmetic into our theory. We sketch this for the case **Ax** = **AxNoBody**. We will see that in this case ν will be quite natural: it will state the existence of a periodically moving body. Consider the following formulas (with free variables m, b and t):

$$
\begin{aligned}
I(t) \; &:= \; I(m,b,t) \; := \; W(m,b,t,\overline{0}), \quad \text{and} \\
\nu \; &:= \; I(0) \; \wedge \; (\forall t, s) \\
&\quad ([t < 1 \;\wedge\; t \neq 0] \rightarrow \neg I(t) \quad \wedge \\
&\quad t \geq 0 \rightarrow [I(t) \leftrightarrow I(t+1)] \quad \wedge \\
&\quad [I(t) \wedge I(s)] \rightarrow [I(t+s) \wedge I(t \cdot s)]).
\end{aligned}
$$

Add, for a moment, m and b as constants to our language. Then t remains the only free variable of I which then specifies a subset of the field-reduct in any frame-model: the set of time-points where the observer m sees the body b at the origin. Now the formula ν requires that this subset behaves like the set of integers: it is a discrete periodic subset containing $0, 1$ and closed under $+, \cdot$. Since the field-reduct of a frame-model is a field, then Robinson's arithmetic will be true in the field-reduct restricted to the subset defined by I. In other words, I is an *interpretation of Robinson's Arithmetic in* $Th \cup \{\nu\}$, whenever ν is consistent with Th. For definition of Robinson's Arithmetic and (semantical) interpretation see e.g. Monk [12, Def.14.17, Def.11.43]. Thus, Robinson's Arithmetic can be interpreted in $Th \cup \{\nu\}$. Then $Th \cup \{\nu\}$ is inseparable (which is a strong version of undecidability) by Thm.16.1 and Prop.15.6 in [12]; and thus (ii) and (iii) of our Theorem hold by Monk [12, Thm.s 15.9 and 15.8]. Finally, if we omit the constants m, b, then semantical consequence does not change, so (ii) and (iii) will hold for the original language (set of formulas not containing the constants m or b), too (in (iii) a further little argument is needed).

To show (i), we have to construct a model of $Specrel_0 \cup \{\mathbf{Ax}\!\uparrow\} \cup \Phi \cup \{\nu\} \cup (Compl \setminus \{\mathbf{AxNoBody}\})$. This is not difficult as ν basically states the existence of a periodically moving body; see Figure 9.

Take a "standard" model with minimum set of observers and photons; and add one periodically moving body. We omit the details of the definition of this model.

The proofs for the other cases are analogous; we only give different interpretations of Robinson's arithmetic. This means that we give a different

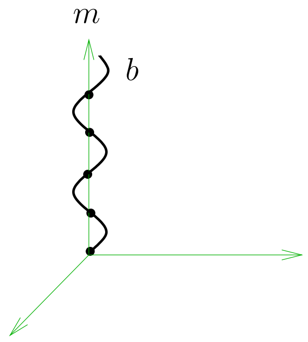

Figure 9: b is a periodically moving body in m's world-view.

formula I, but ν will be the same (speaking about I), and then we only have to show that $Th \cup \{\nu\}$ is consistent, where Th is the theory in (i). To give a flavor, we give this new interpretation I for the case when $\mathbf{Ax} = \mathbf{AxPot}^+$.

$$I(m,t) := (\forall \ell)[ang(\ell) = \tfrac{1}{t} \quad \Rightarrow \quad (\exists k)(tr_m(k) = \ell \wedge m \uparrow k)] \text{ or } t = 0, 1.$$

This finishes the proofidea of Theorem 3.6. **QED**

Since Minkowski geometries over real-closed fields do have a decidable theory (for any fixed n), Theorems 3.4-3.6 above seem to point in the direction that our relativity theories, such as *Specrel*, are essentially richer than the theories of the corresponding Minkowski-style geometries.

REMARK If we add *Compl*, and especially **AxNoBody** to our relativity theory (say $Specrel_0$), then this can be interpreted by saying that we want to abstract away from some aspects of relativity theory and want to concentrate on some very basic aspects only (namely those involving inertial observers and photons). At a certain stage of development, one might even say that this aspect is the "heart" of the theory (from a certain point of view) and for a while one might want to concentrate on the heart only. Some people might even argue that this "heart" part is the one which contains the so called "laws of nature" and therefore immunity of this part to Gödel's incompleteness arguments (cf. Theorem 3.5 above) might point in the direction that perhaps TOE is, after all, possible despite of Gödel's incompleteness proofs because if we figure out carefully which formulas of our frame language count as potential "laws of nature" and distinguish them from the rest of the formulas, then laws of nature conceived in this special way might eventually turn out to admit a complete axiomatization.

We plan to come back to discussing such (and related) kinds of ideas much later, after accelerated observers and other things (in the direction of general relativity) will be incorporated in a variant of our first-order logic approach.

Let us turn, for a second, to the possible "only the heart" approaches, mentioned at the beginning of this remark. In this connection, we would like to emphasize that herein we intend to include into our formalized (and axiomatized) relativity theories not only the heart of the theory but rather "the whole story" (in some sense). In this sense the purposes of this work are different from the purposes of possible works which would want to provide an axiomatic foundation for special relativity by axiomatizing (only) Minkowski geometry in first-order logic.[18] We would call such an approach an "only the heart" approach. (With this we do not want to diminish the importance of such possible approaches. They are legitimate and they are useful. But they do not make our research superfluous, either.)
END OF REMARK

For current research directions in logic started by Gödel's incompleteness theorems we refer to Hájek and Pudlák [10]. In later work we plan to look into the logical structure of general relativistic space-times permitting *closed time-like loops* (which can be regarded as causing a kind of self-reference[19]). In Lewis [11, pp.67-80, pp.212-3] it is pointed out that these causal loops do not imply logical contradictions or even logical paradoxes. They simply have more complex logical structures than "linear causation". We plan to extend the mathematical logic based approach to further analyzing these and related possibilities thoroughly and carefully.

In the rest of the paper we will write *Basax* in place of $Specrel_0$.

[18] E.g., such theories would be decidable while our typical theories such as *Specrel* are not.

[19] such as the ones in Gödel's incompleteness proof, Tarski's proof of undefinability of truth, or Barwise and Etchemendy's book on the "Liar".

4 Gödel's Second Incompleteness in relativity

In Theorem 3.6 above we established that the conclusions of Gödel's first incompleteness theorem apply, among others, to our relativity theory $Basax(n) + \nu$ (where ν was a natural extra axiom which is valid e.g. in the "standard" (or intended) models). In other words, we expanded our theory $Basax(n)$ with a natural extra axiom ν (whose truth has never been doubted by anyone), and we saw that Gödel's first incompleteness theorem became applicable to the so expanded version of $Basax(n)$.

Next, we will see that $Basax(n)$ can be expanded with similarly natural axioms $Ax(ind)$ which are similarly true in the "standard models" such that even Gödel's famous second incompleteness theorem becomes applicable to the so expanded version of $Basax(n)$.

THEOREM 4.1 *Let $n > 1$. Then the following hold.*

(i) *$Basax(n)$ has a consistent extension $Basax(n)^+$ which is obtained by adding a finite schema $Ax(sch)$ of axioms to $Basax(n)$ such that in $Basax(n)^+$ its own consistency $\mathsf{Con}(Basax(n)^+)$ can be formulated by a single formula such that*

$$Basax(n)^+ \nvdash \mathsf{Con}(Basax(n)^+).$$

(ii) *The above remains true for any consistent extension Th of $Basax(n)^+$ if Th is obtained by adding a finite number of axioms (to $Basax(n)^+$).*

Proof. Let $n > 1$. Throughout, we "pretend" that an $\mathfrak{M} \in \mathsf{Mod}(Basax(n))$ has been fixed. Let $m \in Obs$ and $b \in B$. Then let $Z_b^m := \bar{t} \cap tr_m(b)$. We identify Z_b^m with $\{a \in F : \langle a, \bar{0}\rangle \in Z_b^m\}$. We call b m-periodic iff (i)-(v) below hold.

(i) $Z_b^m \neq \emptyset$.

(ii) Z_b^m is discrete in F, i.e.,

$$(\forall x \in Z_b^m)(\exists z \in Z_b^m)(\forall y \in F)(x < z < y \to z \notin Z_b^m).$$

We denote this y by $\mathsf{suc}(x)$.

(iii) Z_b^m has no greatest or smallest element.

(iv) $0 \in Z_b^m$ and $1 = \mathsf{suc}(0)$.

(v) $(\forall x \in Z_b^m)\mathsf{suc}(x) - x = 1$.

The axiom ν^+ says the following.

(ν_1) There are m, b such that b is m-periodic.

(ν_2) $(\forall m, b, m', b')(b$ is m-periodic and b' is m'-periodic $\Rightarrow Z_b^m = Z_{b'}^{m'})$.

(ν_3) b is m-periodic $\Rightarrow Z_b^m$ is closed under $+$ and $\cdot$, i.e., $\langle Z_b^m, 0, 1, +, \cdot \rangle \subseteq \mathcal{F}$ is a subring of our field $\mathcal{F}$.

$\nu^+ := (\nu_1) + (\nu_2) + (\nu_3)$.

Clearly, $Basax(n) + \nu^+$ is consistent (we are in ZF set theory). From now on we assume

(1) $$\mathfrak{M} \models Basax(n) + \nu^+.$$

By $\nu_1 + \nu_2$ we may define $Z := Z_b^m$ for some m-periodic body b. (Z is well defined by ν^+.) Moreover, Z is defined in the language of $Basax(n) + \nu^+$ without parameters in the style $Z = \{x \in F : \varphi_Z(x)$ holds $\}$ where the formula φ_Z has no free variable other than x. This means that the structure $\mathcal{Z} := \mathcal{Z}_{\mathfrak{M}} := \langle Z, 0, 1, +, \cdot \rangle$ is definable in $\mathfrak{M}$ without using parameters. It is not hard to prove that

(2) Robinson's arithmetic Q is true in $\mathcal{Z}_{\mathfrak{M}}$;

but anyway, since Q is finite, we could add Q to ν^+ if Q was not automatically true. (This would leave $Basax(n) + \nu^+ + \mathsf{Q}$ consistent.)

Now, we can comfortably interpret Robinson's arithmetic Q in our theory $Basax(n) + \nu^+$, and this way we can prove all those parts of Gödel's incompleteness theorems (together with the related theorems like Rosser's) which hold for Q. But now we want to do more: we want to establish those stronger incompleteness results which hold for Peano's Arithmetic PA (and we want to prove these for $[Basax(n)+$"some natural assumptions"$]$ in place of PA). To this end we introduce an axiom schema $Ax(ind)$ which postulates a natural induction principle for our models $\mathfrak{M}$. For this, we can pretend that $\mathcal{Z}$ is part of our language, since we defined $\mathcal{Z}$ explicitly by a formula φ_Z.

(3) Let $\psi(x, \overline{u})$ be a formula with free variables x and $\overline{u} = \langle u_0, ..., u_k \rangle$ such that x is of sort F while u_i are of arbitrary (but fixed) sort.

Then the new formula $\mathsf{ind}(\psi, x)$ is defined as follows. (To understand the formula $\mathsf{ind}(\varphi, x)$ we have to recall that $\mathsf{suc}(x)$ was defined in the first-order language of $\mathfrak{M}$.) $\mathsf{ind}(\psi, x)$ is defined to be

$$\forall \overline{u}((\psi(0, \overline{u}) \wedge (\forall x \in Z)[\psi(x, \overline{u}) \to \psi(\mathsf{suc}(x), \overline{u})]) \Rightarrow (\forall x \in Z)\psi(x, \overline{u})).$$

Now,

$Ax(ind) := \{\mathsf{ind}(\psi, x) : \psi(x, \overline{u})$ is a formula in our frame language as specified in (3) above $\}$.

We define $Basax(n)^\star$ as follows:

$$Basax(n)^\star := Basax(n) + \nu^+ + Ax(ind).$$

Now

(4) $Basax(n)^\star$ is an extension of $Basax(n)$ by a finite schema of axioms, it is consistent and it is valid in the "standard" models (or intended models) of $Basax(n)$. E.g., if $\mathcal{F}_\mathfrak{M}$ is Archimedean and $\mathfrak{M} \models (Basax(n) +$ there are periodic bodies), then $\mathfrak{M} \models Basax(n)^\star$.

Let us recall that in Peano's arithmetic PA the consistency of PA can be formalized with a single formula $\mathsf{Con}(PA)$ such that $\mathsf{PA} \not\vdash \mathsf{Con}(PA)$. In the following claim we state a completely analogous result about our relativity theory $Basax(n)^\star$.

Claim 4.2 *There is a formula $\mathsf{Con}(Basax(n)^\star)$ of our frame language which in each model $\mathfrak{M} \models Basax(n)^\star$ expresses the consistency of $Basax(n)^\star$ the same way as $\mathsf{Con}(PA)$ expresses consistency of PA in models of PA. (Cf. e.g., Hájek-Pudlák [10] or Monk [12].) Further,*

$Basax(n)^\star \not\vdash \mathsf{Con}(Basax(n)^\star)$ *and*

$Basax(n)^\star \not\vdash \neg\mathsf{Con}(Basax(n)^\star)$.

To prove this claim we observe two things:

(i) PA can be interpreted in $Basax(n)^\star$ because $\mathcal{Z}$ is definable in $Basax(n)^\star$ and the axioms of PA are derivable (for $\mathcal{Z}$) in $Basax(n)^\star$. This is very easy to check because the axioms of $Basax(n)^\star$ were selected in such a way as to make this true.

(ii) The axiom system $Basax(n)^\star$ is given by a finite schema, completely analogous with the axiom system of PA. Therefore, the axiom system

$Basax(n)^\star$ can be formalized in PA exactly the same way as PA was formalized in PA, e.g., in [10]. Therefore in PA there is a formula $\mathsf{pr}(x, y)$ expressing that x is the Gödel number of a proof from $Basax(n)^\star$ of a formula φ of our frame language whose Gödel number is y. Since PA is interpreted in $Basax(n)^\star$, the same formula $\mathsf{pr}(x, y)$ is available in $Basax(n)^\star$, too. Now, $\exists x \mathsf{pr}(x, y)$ is a provability formula $\pi(y)$ which in $Basax(n)^\star$ expresses that y is the Gödel number of a frame formula provable in $Basax(n)^\star$. Further, one can easily check that the Löb conditions (as presented, e.g., in [10, Def.2.16, p.163]) are satisfied by $\pi(y)$ and by $Basax(n)^\star$. Now, we choose $\mathsf{Con}(Basax(n)^\star)$ to be $\neg\pi(False)$.

The rest of the proof of the "$\nvdash$ $\mathsf{Con}(\ldots)$" part of Claim 4.2 goes exactly the same way as the proof of Thm's 2.21-2.22 (on p.164) in [10]. The "$\nvdash$ $\neg\mathsf{Con}(\ldots)$" part is relatively easy to check since $Basax(n)^\star$ is consistent and we defined $\mathsf{pr}(x, y)$ in an appropriate way. This completes the proof of the theorem for the theory $Basax(n)^+$.

The generalization for (consistent) extensions of $Basax(n)^+$ with finitely many new axioms goes the usual way, e.g., one can use (an appropriately adapted version of) Thm.2.22 of [10]. (Hint: if we have a Σ_1 definition of the Gödel numbers of the axioms of $Basax(n)^\star$ then we can extend this Σ_1-definition to "$Basax(n)^+$ + an extra (concrete) axiom, say φ", since φ has a concrete Gödel number $\ulcorner\varphi\urcorner$ etc.) **QED**

5 Π^0_k-hardness and ZFC independence in relativity

In the above investigations we saw that Gödel's incompleteness theorems apply to some versions, e.g., $Basax(n)+\nu$, for $n > 1$ of our formalized relativity theory. This implies, among others, that all extensions of $Basax(n) + \nu$ are undecidable, moreover, if they are complete then their theorems are not even recursively enumerable.

However, we will see soon that there are stronger (than non-recursively enumerability) negative properties for theories (sets of formulas), and if we are not careful enough then our relativity theories can be "infected" with these very strong negative properties, too.

If a set of formulas is not recursively enumerable that makes life hard, but not impossible. In the theory of computability (or rather non-computability) there is a so called hierarchy of unsolvability which measures how impossible it is to describe certain sets (of formulas or of natural numbers). This hierarchy is also known as the "degree of unsolvability". There is an infinite sequence $\Pi^0_1, \Pi^0_2, ..., \Pi^0_k, ...$ of harder and harder non-computable sets. (The idea is that Π^0_{k+1} is even less computable than Π^0_k.)

As an illustration, let $\underline{\omega} := \langle \omega, +, \cdot, 0, 1 \rangle$ be the standard model of arithmetic. Then it is known that the full first-order theory $\mathsf{Th}(\underline{\omega})$ of $\underline{\omega}$ is harder than Π^0_k, for any $k \in \omega$.

We will see that if we are careless in defining our class $\mathsf{Intmod} \subseteq \mathsf{Mod}(Basax(n))$ of intended models of relativity, then $\mathsf{Th}(\mathsf{Intmod})$ can become at least as hard as $\mathsf{Th}(\underline{\omega})$, i.e., harder than Π^0_k for any k. Roughly speaking, this means that there is a computable function $\mathsf{tr} : \mathsf{Th}(\underline{\omega}) \to \mathsf{Th}(\mathsf{Intmod})$ mapping $\mathsf{Th}(\underline{\omega})$ onto $\mathsf{Th}(\mathsf{Intmod})$ such that if some magic device (usually called an "oracle") could compute (e.g., enumerate) $\mathsf{Th}(\mathsf{Intmod})$, then via tr this would also yield a computation (enumeration) of $\mathsf{Th}(\underline{\omega})$. For the precise definition of Π^0_k-hardness and being "as hard as" (say, $\mathsf{Th}(\underline{\omega})$) the reader is referred to Odifreddi [14].

Next we turn to preparing ourselves for stating this Π^0_k-hardness theorem (and also some statements about being independent from ZFC set theory). Let us recall that the "potential axiom" ν in our frame language was introduced around our extension of Gödel's first incompleteness theorem to our relativity theories, cf. theorem 3.6. Intuitively, ν says that there exists a periodically moving body. Next we define a harmless looking class $\mathsf{Mod}(\mathsf{Arch}, n)$ of models which we will call Archimedean models of relativity. Recall that a field $\mathcal{F}$ is called Archimedean if to each $r \in F$ there is $k \in F$ such that $r < k$ and k is a finite integer, i.e. $k \in \{1, 1+1, 1+1+1, ...\}$.

Definition 5.1 Let $n > 1$ be arbitrary.

$$\mathsf{Mod}(\mathsf{Arch}, n) := \{\mathfrak{M} \in \mathsf{Mod}(Basax(n)) : \mathfrak{M} \models \nu \text{ and } \mathcal{F}_{\mathfrak{M}} \text{ is Archimedean}\}.$$

THEOREM 5.2 *Let $n > 1$. Then the following hold.*

(i) *The first-order theory $\mathsf{Th}(\mathsf{Mod}(\mathsf{Arch}, n))$ is harder than Π^0_k, for any $k \in \omega$. Further,*

(ii) $\mathsf{Th}(\mathsf{Mod}(\mathsf{Arch}, n))$ *is at least as hard (i.e., as uncomputable or undefinable) as the full first-order theory $\mathsf{Th}(\underline{\omega})$ of the standard model $\underline{\omega}$ of arithmetic.*

On the proof. Assume $\mathfrak{M} \models Basax(n)+\nu$ and assume $\mathcal{F}_{\mathfrak{M}}$ is Archimedean. Using the ideas of our earlier "Gödel-oriented" proofs, since ν is assumed, we can define an isomorphic copy of some structure $\mathcal{A}$ similar to $\underline{\omega}$ in our model $\mathfrak{M}$. But since $\mathcal{F}_{\mathfrak{M}}$ is Archimedean, $\mathcal{A}$ will be isomorphic with $\underline{\omega}$. This way we obtained an interpretation tr of $\mathsf{Th}(\underline{\omega})$ in the theory $\mathsf{Th}(\mathfrak{M})$ of $\mathfrak{M}$. A little checking reveals that this interpretation is the same for all choices of $\mathfrak{M}$. This proves interpretability of $\mathsf{Th}(\underline{\omega})$ in $\mathsf{Th}(\mathsf{Mod}(\mathsf{Arch}, n))$. The rest follows from this (since tr is clearly Turing-computable). **QED**

The above theorem can be interpreted as implying that we really should not require $\mathcal{F}_{\mathfrak{M}}$ to be Archimedean in our relativity theories because this requirement would smuggle in very hard meta-mathematical issues into our formalized relativity theory which issues are probably irrelevant to the original subject matter of relativity. (Cf. in this connection van Benthem [6] and Németi and Sain [13].)

THEOREM 5.3 *Let $n > 1$. Then there is a formula φ in our frame language for relativity theory such that truth of statement (i) below is independent of ZFC Set Theory.*

(i)
$$\mathsf{Mod}(\mathsf{Arch}, n) \models \varphi$$

On the proof. We saw in the proof of Thm.4.1 that the theory $\mathsf{Th}(\underline{\omega})$ of full first-order arithmetic can be interpreted (or reconstructed) in our "relativity theory" $\mathsf{Th}(\mathsf{Mod}(\mathsf{Arch}, n))$, if $n > 1$. But in $\mathsf{Th}(\underline{\omega})$ there do exist formulas, e.g., ψ, such that the statement $\underline{\omega} \models \psi$" is independent of ZFC (assuming ZFC is consistent). Such a ψ is the Gödelian formula $\mathsf{Con}(ZF)$. Therefore, if tr is our translation function (from $\mathsf{Th}(\underline{\omega})$ into our frame language) then "$tr(\psi) \in \mathsf{Th}(\mathsf{Mod}(\mathsf{Arch}, n))$" or equivalently "$\mathsf{Mod}(\mathsf{Arch}, n) \models tr(\psi)$" is a statement about our (potential) relativity theory whose truth is independent from ZFC. This implies, roughly speaking, that in some models of ZFC $\mathsf{Mod}(\mathsf{Arch}, n) \models tr(\psi)$ is true while in other models of ZFC the same is false. **QED**

The $tr(\psi)$ in the above proof is a formula in our language for relativity theory whose truth in the potential relativity theory $\mathsf{ThMod}(\mathsf{Arch}, n)$ is unknowable in some meta-mathematical sense. To our minds, this implies that $\mathsf{ThMod}(\mathsf{Arch}, n)$ would be a bad choice for being our relativity theory. Also, $\mathsf{Mod}(\mathsf{Arch}, n)$ would be a bad choice for being our intended class of models of relativity theory. For completeness, we note that besides $\mathsf{Con}(ZF)$, there are infinitely many different formulas $\psi_1, \psi_2, \ldots$ in the language of $\underline{\omega}$ whose validity in $\underline{\omega}$ is independent from ZFC. Such an example is $\mathsf{Con}(\mathsf{Con}(ZF) + ZF)$, but there are also formulas with the same independence property but of different spirit.

The above theorem can be interpreted as saying that $\mathsf{Mod}(\mathsf{Arch}, n)$ is much less adequate for studying relativity theory than, e.g., $\mathsf{Mod}(Th)$ for some of the purely first-order choices Th (such as e.g., *Specrel* for relativity theory for the following reason. Unlike e.g. the situation with *Specrel*, the theorems of $\mathsf{Th}(\mathsf{Mod}(\mathsf{Arch}, n))$ do not depend so much on our choice of explicit assumptions about relativity, but rather they depend on the properties of the set theoretical universe in which "we are doing our mathematics". This may not sound so bad, but further considerations reveal that this can lead to extremely misleading results, and roughly speaking it can contribute to something which is usually called an "artifact", cf. [13], Sain [15]. Because of Thm.5.3, when we will want to have something like the Archimedean property in our relativistic models $\mathfrak{M}$, then instead of simply assuming that $\mathcal{F}_{\mathfrak{M}}$ is Archimedean (which would produce undesirable side effects), we will follow the non-standard analysis-like methodology elaborated in, e.g., Sain [16], Andréka-Goranko-Mikulás-Németi-Sain [1] for temporal logics of actions and for nonstandard dynamic logics. This nonstandard methodology is based on adding extra sorts to $\mathfrak{M}$ representing possibly nonstandard integers and functions mapping these integers into $\mathcal{F}$.

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