6.1 Introduction (to the present chapter on geometries)

In this chapter we will see how observer-independent structures can be found in our frame models of relativity theory, i.e. we will show that there is an *observer-independent* "geometric" structure $\mathfrak{G}_{\mathfrak{M}}$ inside every model $\mathfrak{M}$ of our frame language. We will consider the observer independent geometric structures $\mathfrak{G}_{\mathfrak{M}}$ (associated to "observer oriented" models $\mathfrak{M}$) as representing so called "<u>theoretical</u>" concepts, while we will consider the original $\mathfrak{M}$'s as representing so called more "<u>observational</u>" concepts. Here, the expressions "observational", "theoretical" are technical terms explained and used in the relativity books Reichenbach [223], Friedman [90][616], cf. also §1.1(IX) on p.11 of the present work for a brief explanation and motivation for the observational/theoretical distinction.

The key idea is that in some situations or at some level of the development of our scientific theories, certain concepts can be considered more observational while others can be regarded as being more on the theoretical side. For a more careful description of this distinction (and its justification etc.) we refer to Reichenbach [223]. We are aware of the fact that the observational/theoretical distinction is not absolute[617], it may change during the development of our scientific theories, etc. but, as Friedman [90] writes on p.4 and on p.31, if we are aware of its limitations and its "tentativeness", then it can be used rather fruitfully.[618]

Next we discuss a methodology about the role of theoretical/observational concepts in scientific theories. The methodology and ideas we are going to sketch below originate from an intensive and fruitful interaction between the originators of relativity theory, e.g. Einstein, on the one side, and the logical positivists[619], e.g.

[616] The words "observational" etc. come from Friedman [90]. Reichenbach used other expressions with basically the same meaning. E.g. he writes about theory formation: "... it is <u>advantageous</u> to approach the axiomatization in a different fashion. It is possible to start with the <u>observable</u> facts and to end with the abstract conceptualization, cf. [223, pp. 4–5]. Later he writes "... start an axiomatization with so-called empirical facts", also "... this investigation starts with <u>elementary</u> facts as axioms...", cf. [223, p.8]. Cf. also p.174 in Loose [166].

[617] e.g. a concept which is observational in one situation may appear as theoretical in another situation

[618] Actually, Friedman [90, p.24, first 30 lines] writes that the birth of the <u>modern</u> form of the observational/theoretical distinction can be credited to Einstein's fundamental 1916 paper [81, p.117].

[619] We are neither supporting nor attacking positivism, we simply want to use those of their ideas which proved useful while avoiding their mistakes e.g. oversimplification.

Reichenbach, on the other side. The key ideas go back to Kant[620] (1724-1804), Leibniz (1646–1716) and Occam (1295–1349).[621] Cf. e.g. Friedman [90, §I (pp.3-31)]. (These ideas are also strongly related to ontology, i.e. to the field of research studying the question of which ones of our theoretical entities exist and in what sense they exist.)[622]

The methodology (of the above origin) is the following:[624] Assume, we want to study a part (or aspect) of the physical world. Then, first we build models, like $\mathfrak{M}$ in the present work, which involve observational concepts only. I.e. we try to keep the "ingredients" of $\mathfrak{M}$ to be on the observational side as much as possible. Then we study $\mathfrak{M}$ and develop a theory Th in the language of $\mathfrak{M}$ with $\mathfrak{M} \in \mathsf{Mod}(Th)$. After having studied Th and $\mathsf{Mod}(Th)$ long enough, we begin to see what kind of new, theoretical concepts would be useful for understanding Th, $\mathfrak{M}$ etc. even better. The methodology of introducing such new theoretical concepts is the following.

By the principles of parsimony[625] (i.e. refinements of Occam's razor), we require the new, theoretical concepts to be definable by means of first-order logic, over $\mathfrak{M}$ (or more generally over $\mathsf{Mod}(Th)$). Cf. §6.3 way below for the theory of definability. (The importance of definability is emphasized in the relativity book Reichenbach

[620]For the (positive) role of Kant cf. e.g. Friedman [90] p.7 lines 8–25 and p.18 lines 14–20.

[621]E.g. we mention Leibniz's principle of identity of indistinguishable concepts, and what became popularly known as Occam's razor, c.f. e.g. Friedman [90], Hodges [136, pp. 9, 21]. Roughly, Occam's razor says: do not assume the existence of unnecessary theoretical entities. In passing we note that Leibniz's principle appears as axiom C_7 in algebraic logic (called there Leibniz rule) cf. Henkin-Monk-Tarski [129, Part I, p.172] and Andréka et al. [30]. C_7 is the algebraic counterpart of an axiom of first-order logic. (William Occam was a 14-th century logician from England. His razor is usually summarized as "Do not assume the existence of more entities than you have to".)

[622]It is of interest to note how much philosophy influenced the development of relativity. E.g. Mach's philosophy influenced Einstein in developing general relativity, cf. e.g. Barbour [39] or Friedman [90]. Further, Gödel proved very interesting things about the so obtained general relativity. Gödel's main motivation came from Kant's philosophy, he wanted to justify Kant's views on the nature of time.[623] Gödel's results lead up to one of the most exciting parts of modern relativity, namely to the theory of rotating black holes (closed time-like curves, i.e. "time travel"), at least in some sense. Further, (in a different direction) Gödel's results show that Einstein's equations do not imply Mach's principle, after all (for seeing this in full form one uses Ozsvath's and Schücking's 1969 paper [209]), cf. also Friedman [90, pp. 209–211]. This does not prove that Mach's principle would not be true, instead it proves only that it is not implied by Einstein's axioms for general relativity. Cf. e.g. Gödel's collected works [103, pp. 189–217], and [104, pp. 202–289], and Dawson [73]. Cf. also footnote 637 on p.781. See Figure 355 for a visual representation of Gödel's model satisfyng Einstein's equations but not Mach's principle. This model usually is called Gödel's rotating universe.

[623]In this connection we recommend that the reader reads footnote 861 on p.912.

[624]We present it, here, only in a simplified form.

[625]"principle of parsimony" = "economy of explanation", cf. footnote 621

[223] e.g. on p.3, pp.7-13.) Then we expand our observational model $\mathfrak{M}$ with the defined concepts obtaining something like $\mathfrak{M}^+ = \langle \mathfrak{M}, \text{defined concepts} \rangle$ in the hope that the theory of $\mathfrak{M}^+$ will be more "streamlined", more elegant and more illuminating (than that of $\mathfrak{M}$) in various ways. Indeed, in the present chapter, we will define a streamlined theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ over the model $\mathfrak{M}$, and we will call $\mathfrak{G}_{\mathfrak{M}}$ the "observer independent geometry" associated to $\mathfrak{M}$.[626] First we will identify what desirable theoretical entities we would like to put into $\mathfrak{G}_{\mathfrak{M}}$, and then comes the "hard work" of checking that these new entities are indeed first-order logic definable over $\mathfrak{M}$, cf. §§ 6.2.2, 6.2.6, 6.3, and Theorems 6.3.22–6.3.24 (p.961) for the definability investigations, while considerations on what should go into $\mathfrak{G}_{\mathfrak{M}}$ are in §6.2.3 (but cf. also §§ 6.2.1–6.2.5).

Having defined, over $\mathfrak{M}$, our structure $\mathfrak{G}_{\mathfrak{M}}$ of theoretical entities, we expand our observational structure $\mathfrak{M}$ with these theoretical entities obtaining the richer structure $\mathfrak{M}^+ = \langle \mathfrak{M}, \mathfrak{G}_{\mathfrak{M}} \rangle$. Our theoretically enriched structure $\mathfrak{M}^+$ corresponds to the structure $\mathcal{A}$ in Friedman [90, p.236] while our observational $\mathfrak{M}$ corresponds to the sub-reduct $\mathcal{B}$ of $\mathcal{A}$ on the same page in [90]. In the language of our enriched structure $\mathfrak{M}^+$ we have both theoretical and observational concepts, so we could go on indefinitely studying the theory of our $\mathfrak{M}^+$. However, this is not what we do, because, so to speak, we become greedy in connection with improving our language, our concepts. Namely, *if we are lucky*, then we will find that (not only $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{M}$ but) also our observational structure $\mathfrak{M}$ is definable over the theoretical $\mathfrak{G}_{\mathfrak{M}}$. *If* this is the case, then we may forget our original observational structure $\mathfrak{M}$, and may stick with the more streamlined, elegant theoretical structure $\mathfrak{G}_{\mathfrak{M}}$.[627] (The reason for this is that if $\mathfrak{M}$ is definable over $\mathfrak{G}_{\mathfrak{M}}$, then in a rather concrete sense $\mathfrak{M}$ is "present" [or available] in $\mathfrak{G}_{\mathfrak{M}}$, e.g. all questions about $\mathfrak{M}$ can be translated to questions about $\mathfrak{G}_{\mathfrak{M}}$.) Very probably, if we are permitted to concentrate on $\mathfrak{G}_{\mathfrak{M}}$ and to forget about $\mathfrak{M}$, then our investigations of the theory of $\mathfrak{G}_{\mathfrak{M}}$ will be more efficient, we will be able to reach deeper insights in a shorter time etc. Motivated by these considerations, in the present work we will prove various results to the effect that the observational "world" $\mathfrak{M}$ is indeed first-order logic definable over the theoretical world $\mathfrak{G}_{\mathfrak{M}}$. This will be one of the main themes of §6.6.

We will extend these definability results from individual models to axiomatizable

[626] At the beginning of this chapter it will not be very obvious why we think that $\mathfrak{G}_{\mathfrak{M}}$ is much more streamlined than $\mathfrak{M}$, but around the end of this chapter, in §6.7, we will see that $\mathfrak{G}_{\mathfrak{M}}$ admits rather streamlined reformulations. Cf. e.g. Theorems 6.7.20 (p.1157), 6.7.30 (p.1164), 6.7.37 (p.1167).

[627] In the above sentence we want to refer to a kind of "tension" which regards $\mathfrak{M}$ as being too close "to the original thing being modeled", detail oriented or "mosaic-like" or coordinate systems oriented while $\mathfrak{G}_{\mathfrak{M}}$ is regarded to be more "whole oriented" or more "essence oriented". Cf. item (7) on p.852.

classes of models. E.g. if $\mathsf{Mod}(Th)$ is an axiomatizable class of observational models, then we will write $\mathsf{Ge}(Th)$ for the corresponding class of theoretical models, i.e. the corresponding class of geometries. Then we will prove that $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$, and in the other direction, $\mathsf{Mod}(Th)$, too, is definable over $\mathsf{Ge}(Th)$.

Actually, we will do more than this in two respects:

(i) We will prove that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent[628] which in some sense means that they are different "linguistic representations" of the same theory. (Cf. Thm.6.6.13 on p.1031.[629])

(ii) We will also elaborate a so called duality theory between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ which is analogous with the various duality theories (adjoint situations, etc.) playing important roles all over mathematics.[630] We will make the connections explicit with several distinguished duality theories in §§ 6.6.5–6.6.7, cf. also pp. 1014–1027, pp. 1096–1107. One of the uses of these duality theories is that they establish strong connections between seemingly distant parts of mathematics, and they help us to solve problems in one area by using the methods of a completely different kind of area (where the solution for this particular problem might be drastically easier).

In §6.7 we will use the methods of definability theory for streamlining our "theoretical structure" $\mathfrak{G}_{\mathfrak{M}}$ in the spirit outlined way above. Since definability theory plays such a central role in our investigations (as well as in other parts of relativity, cf. e.g. Reichenbach [223], Friedman [90]), we devoted §6.3 to recalling and further elaborating this theory.

Potential laws of nature, characterization of symmetry principles:

Our theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ can also be used in identifying potential laws of nature and in characterization of symmetry principles, as follows. Recall from the introduction of §3.9 and from §2.8 that some of our axioms like **Ax(symm)** or **Ax(ω)** were called symmetry principles (and were regarded as special instances of Einstein's SPR). In earlier parts (e.g. in §3.9) we experimented with giving logical or model theoretic characterizations for symmetry principles. Cf. the first theorem in §3.9 which is based on Def.3.8.2 (p.298). The intuitive idea was, roughly, that symmetry principles say that inertial observers cannot be distinguished from each other by laws of nature. (An equivalent formulation says that the same laws of nature hold for m and k if m, k are inertial observers.) So if $\varphi(x)$ is a potential law

[628] Cf. §6.3 (p.969) for definitional equivalence.

[629] Cf. also Remark 6.3.31 on p.973 and the intuitive text above that remark.

[630] This duality theory works under weaker conditions needed for (i) above. (Note that definitional equivalence between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ automatically implies a very strong form of duality. [Actually what we will call weak definitional equivalence is sufficient for this.] However duality in general does not imply definitional equivalence.)

of nature then $\mathfrak{M} \models$ "symmetry principles" iff $[\mathfrak{M} \models \varphi(m) \leftrightarrow \varphi(k)$, for all inertial observers m and k of $\mathfrak{M}]$. The problem with carrying this programme through was that we did not know which formulas of our frame language $Fm(\mathfrak{M})$ count as potential laws of nature and which formulas are of an "accidental" (or contingent) character (e.g. making some random statement about the state of affairs on the life-line of m, say, at the event where m sees the origin). For the distinction between "accidental" statements and potential laws cf. e.g. the entry "lawlike generalization" in the Cambridge dictionary of philosophy [34]. So, the problem was to provide a logical or model theoretic distinction between those formulas in $Fm(\mathfrak{M})$ which are regarded as potential laws from those formulas which count only as potential "accidental facts"[631]. In §3.9 we avoided this dilemma by assuming so many axioms (called $\mathbf{BaCo}^-$) on $\mathfrak{M}$ that these axioms ensured that all the remaining possible statements (about observers in $\mathfrak{M}$) can be regarded as potential laws. This way, it became possible to give a model theoretic characterization of **Ax(symm)** (in Thm.3.9.2, p.348) but the price was that we had to make strong assumptions.

In the present chapter, we will be in a better situation. By associating an observer independent theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ to each model $\mathfrak{M}$, we will be in a better position for doing both things mentioned above, namely

(i) to characterize **Ax(symm)** model theoretically by looking at the more abstract (than $\mathfrak{M}$) model $\mathfrak{G}_{\mathfrak{M}}$ and

(ii) to distinguish those elements of $Fm(\mathfrak{M})$ which are closer to being potential laws.

We will turn to doing (ii) in §6.6.8 (p.1107). There our intuition is the following. If a formula φ talks about theoretical concepts only, then the chances are better for φ to be a potential law. In this chapter, from our observation-oriented model $\mathfrak{M}$ we define a theoretical super-structure[632] $\mathfrak{G}_{\mathfrak{M}}$ built up from more theoretical concepts (than the parts of $\mathfrak{M}$). As a first approximation to characterizing potential laws, we will postulate that a formula φ is a potential law if φ talks about theoretical concepts only, i.e. only the $\mathfrak{G}_{\mathfrak{M}}$-part of $\mathfrak{M}$. This makes sense because $\mathfrak{G}_{\mathfrak{M}}$ will be definable over $\mathfrak{M}$, which implies that the ingredients (or basic concepts) of $\mathfrak{G}_{\mathfrak{M}}$ can be regarded as derived notions of $\mathfrak{M}$. §6.6.8 is devoted to implementing and elaborating the just outlined ideas. For more detail on these ideas we refer to §6.6.8 (p.1107).

[631] Like, "the number of non-inertial bodies present at the origin is smaller than that at coordinates $\langle 1, 0, 0, 0\rangle$".

[632] Cf. Friedman [90] § VI.3 (p.236) under the title "Theoretical Structure and Theoretical Unification".

The goal (i) of characterizing symmetry principles will be addressed in Theorems 6.2.106, 6.2.108 in §6.2.8 (cf. also Prop.6.2.61 on p.865) which will state, roughly, that the symmetry principle $\mathbf{Ax}(\omega)$ [633] is equivalent with stating that the world-view transformations coincide (in some sense) with the automorphisms of the geometry $\mathfrak{G}_{\mathfrak{M}}$, under some assumptions. In more detail, we will state that

(⋆) $\mathfrak{M} \models \mathbf{Ax}(\omega) \iff (\forall m, k \in Obs)$ [$(w_m^{-1} \circ w_k)$ induces an automorphism on $\mathfrak{G}_{\mathfrak{M}}$ the natural way],[634]

under some assumptions[635] on $\mathfrak{M}$. Under the same assumptions the following stronger form of (⋆) will also be stated:

(⋆⋆) $\mathfrak{M} \models \mathbf{Ax}(\omega) \iff$ the nice automorphisms of $\mathfrak{G}_{\mathfrak{M}}$ <u>are exactly</u> the mappings induced by some $w_m^{-1} \circ w_k$, with $m, k \in Obs$.

Here an automorphism of $\mathfrak{G}_{\mathfrak{M}}$ is called nice if it leaves the elements of F fixed.[636] Summing up, (⋆⋆) says that our symmetry axiom is equivalent with saying that the world-view transformations are exactly the automorphisms of the geometry $\mathfrak{G}_{\mathfrak{M}}$.

In this connection we invite the reader to explore possibilities of extending (⋆) and (⋆⋆) to e.g. the Reichenbachian versions of our relativity theories, where $\mathfrak{G}_{\mathfrak{M}}$ gets replaced with the Reichenbachian geometry $\mathfrak{G}^R_{\mathfrak{M}}$, and $\mathbf{Ax}(\omega)$ with a symmetry principle adequate for **Reich**(*Th*), cf. §4.7 in particular pp. 607, 611. This might yield a method for finding new symmetry principles adequate for **Reich**(*Th*).

§4.7-ből utalni majd ide!

Let us return to explaining in what sense we regard $\mathfrak{G}_{\mathfrak{M}}$ as an observer independent geometry (sitting in $\mathfrak{M}$). Originally, in the Newtonian world view, there was a common "outside reality" for all observers. In our models $\mathfrak{M}$, each observer has a "kind of private world", namely his world view (determined by $w_m : {}^nF \longrightarrow \mathcal{P}(B)$). The f_{mk} transformations tell us how these worlds are connected. However, they do not tell us which of these worlds is the "real one". Moreover, by Einstein's SPR these worlds are of equal status. (Of course, one can live with this arrangement forever, there is nothing wrong with it.) The question comes up naturally: Can one find a single "monolithic" (or "fundamental") reality behind all these "pluralistic"

[633] $\mathbf{Ax}(\omega)$ and $\mathbf{Ax(symm)}$ are symmetry principles of the same kind and they are very strongly related. Therefore in the intuitive text we sometime use them interchangeably (i.e. as if they were synonyms).

[634] The points in the geometry $\mathfrak{G}_{\mathfrak{M}}$ will be the events of $\mathfrak{M}$ hence $w_m^{-1} \circ w_k$ is a function on the universe (set of points) of $\mathfrak{G}_{\mathfrak{M}}$.

[635] like $\mathbf{Bax}^{-\oplus} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|)$

[636] We note that F will be a sort of $\mathfrak{G}_{\mathfrak{M}}$ (similarly to F's being a sort of $\mathfrak{M}$).

personal worlds? If yes, then we could call this "monolithic" reality the outside reality (behind our experiences). All the personalized worlds (i.e. the world-views) could be regarded as different projections of this single outside reality. The situation is analogous with descriptive geometry, where we have a spatial body (or figure) which has a "front-view", a "side-view", etc., i.e. which can be viewed from all possible angles or directions. That spatial figure corresponds to our observer in-

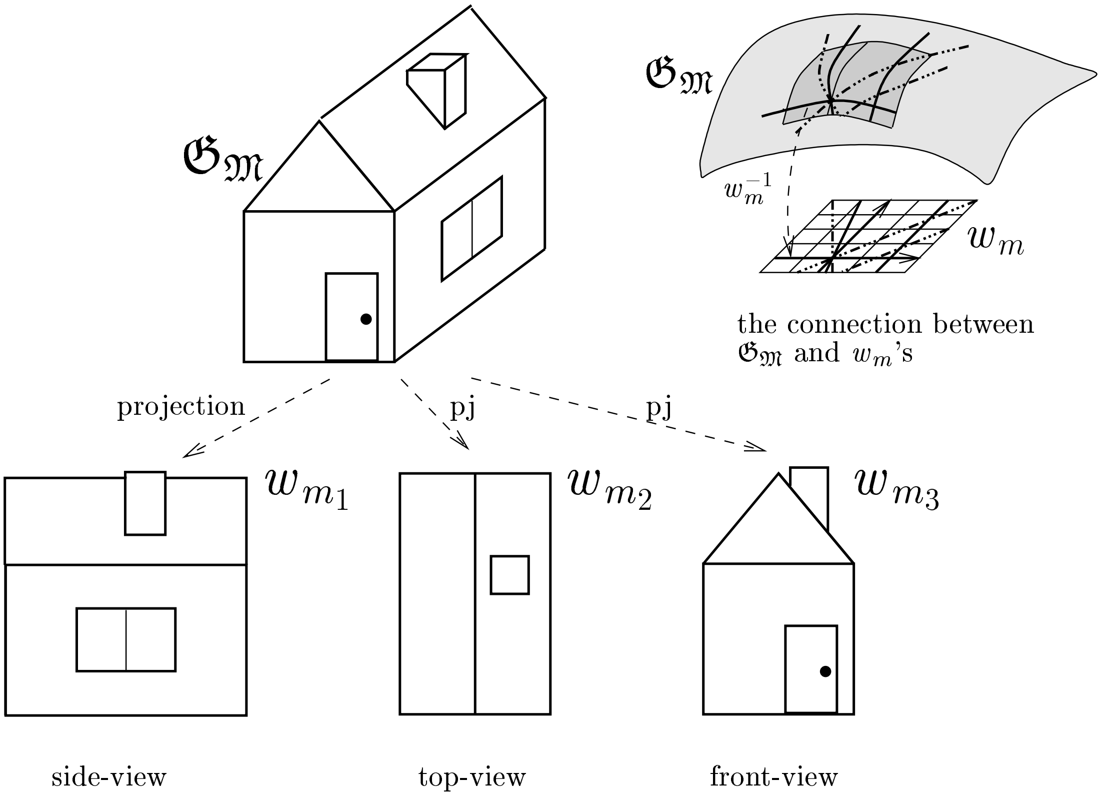

Figure 259: Descriptive geometry put into analogy with the connection between the unique $\mathfrak{G}_{\mathfrak{M}}$ and the many world-views w_m in $\mathfrak{M}$.

dependent geometry $\mathfrak{G}_{\mathfrak{M}}$ while the views (or projections) of that body from possible directions correspond to the "personalized" world-views of our observers $m \in Obs^{\mathfrak{M}}$. Cf. Fig.259 for descriptive geometry put into analogy with the connections between our single observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ and the many personalized worlds, i.e. the w_m's in $\mathfrak{M}$.

Our observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$ is intended to serve as such a monolithic outside reality. Indeed, in our duality-theory section (§6.6) we will see that the different personalized world-views (of form $w_m : {}^nF \longrightarrow \mathcal{P}(B)$) can be recovered from the single geometry $\mathfrak{G}_{\mathfrak{M}}$, cf. e.g. the definition of functor $\mathcal{M}$ on p.1054.[637]

For further introductory thoughts on why we "celebrate" the observer independent character of our geometries $\mathfrak{G}_{\mathfrak{M}}$ we refer to items 1–10 on pp. 851–853. (That is in sub-section "On the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$".)

On the contents of some of the sections (in this chapter). §6.2 contains the definition of the observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$. §§ 6.3, 6.4 contain the basics of definability theory we will need. §6.6 contains our duality theories between observation oriented models $\mathfrak{M}$ and observer independent geometries $\mathfrak{G}_{\mathfrak{M}}$. More precisely, the duality theories act between axiomatizable classes of frame models and of geometries. §6.7 studies interdefinability of the ingredients (sorts, relations, functions etc) of $\mathfrak{G}_{\mathfrak{M}}$, and via this it aims at simplifying and streamlining $\mathfrak{G}_{\mathfrak{M}}$ as a mathematical structure. §6.8 defines and discusses geodesics of $\mathfrak{G}_{\mathfrak{M}}$. Geodesics play an essential role in the theory of accelerated observers (to be discussed in the continuation of the present work on accelerated observers [23])[638] and in the theory of general relativity.

The figure representing Gödel's rotating universe (proving e.g. that Einstein's equations do not imply Mach's principle), mentioned on p.775, is postponed to the section on geodesics (Figure 355, p.1208) because the notion of a geodesic is essential for understanding the picture.

[637]There seems to be an analogy here with Kantian philosophy: Namely, $\mathfrak{G}_{\mathfrak{M}}$ corresponds to the outside world in itself ("ding an sich") and each observer creates his "own" world of phenomena via perceiving (in the Kantian sense) the outside world, where Kant emphasizes that each observer contributes to the creation of the world of phenomena (and not only the outside world contributes), cf. Kant [151, 152]. In our case, the contribution of observer m is his *coordinatization* of $\mathfrak{G}_{\mathfrak{M}}$. Cf. Friedman [90, pp.286-287], Reichenbach [222]. In passing we also note that Kant's philosophy of science was continued by the logical positivists, e.g. Carnap [57], Reichenbach [222]. Logical positivism began as a neo-Kantian movement whose central preoccupation was the content/form distinction where the "content" is supplied by the outside world while the "form" is supplied by the observer's mind (e.g. by his logic). [Here, phenomenon = (content + form).] In Carnap's works, the "form" part or the part supplied by the mind is logic. (In this respect, our present approach is positively related to those of Carnap and Reichenbach.) Reichenbach emphasizes that the content part, e.g. the basic definitions of the concepts of a theory do change during the evolution (or development) of the theory in question. In agreement with Reichenbach, we think that this is in agreement with the modern view of Kant-oriented philosophy of science. Cf. also footnote 622 on p.775.

[638]cf. also the "Accelerated observers" chapter of [24]

For the case the reader would not have time for reading the whole of the present chapter the following is a possible minimalist reading of the backbone or most essential parts. (A possibility is first reading the backbone listed below and then reading those of the remaining parts which interest the reader.) §6.2.1, §6.2.2 (without proofs), §6.2.3 first 5 pages only i.e. up to Prop.6.2.48, (perhaps §6.2.4 and §6.2.8), §6.3, §6.4, the introduction of §6.6, §6.6.1, §6.6.2, §6.6.3, §6.6.4, Remark 6.6.61 (pp. 1078–1080), §6.6.6, §6.6.8 (perhaps §6.6.9), §6.7.3 (perhaps the introduction of §6.7, item (4) of §6.7.1 in particular Figures 340–343, and §6.7.2 [for this §6.5 might be needed]), §6.8, §6.9. See Figure 260, where these sub-sections are boxed in. Figure 260 shows the structure of Chapter 6. I.e. the figure shows which (sub-)sections are needed for reading a given (sub-)section.

For the reader, who does not have time even for the above outlined minimalist reading, the following is an even more radically shortened possible first reading of a sample of the present chapter: §6.2.1, first 5 pages of §6.2.3, §6.3, (§6.4 without proofs) the introduction of §6.6, §§ 6.6.1–6.6.3, (perhaps §6.6.6), §6.6.8, item (4) of §6.7.1, §6.7.3 [for this some of §6.7.2 without proofs might be needed], §6.8, (perhaps §6.9). For the readers convenience, we note that this last shortened reading involves approximately only 182 pages.

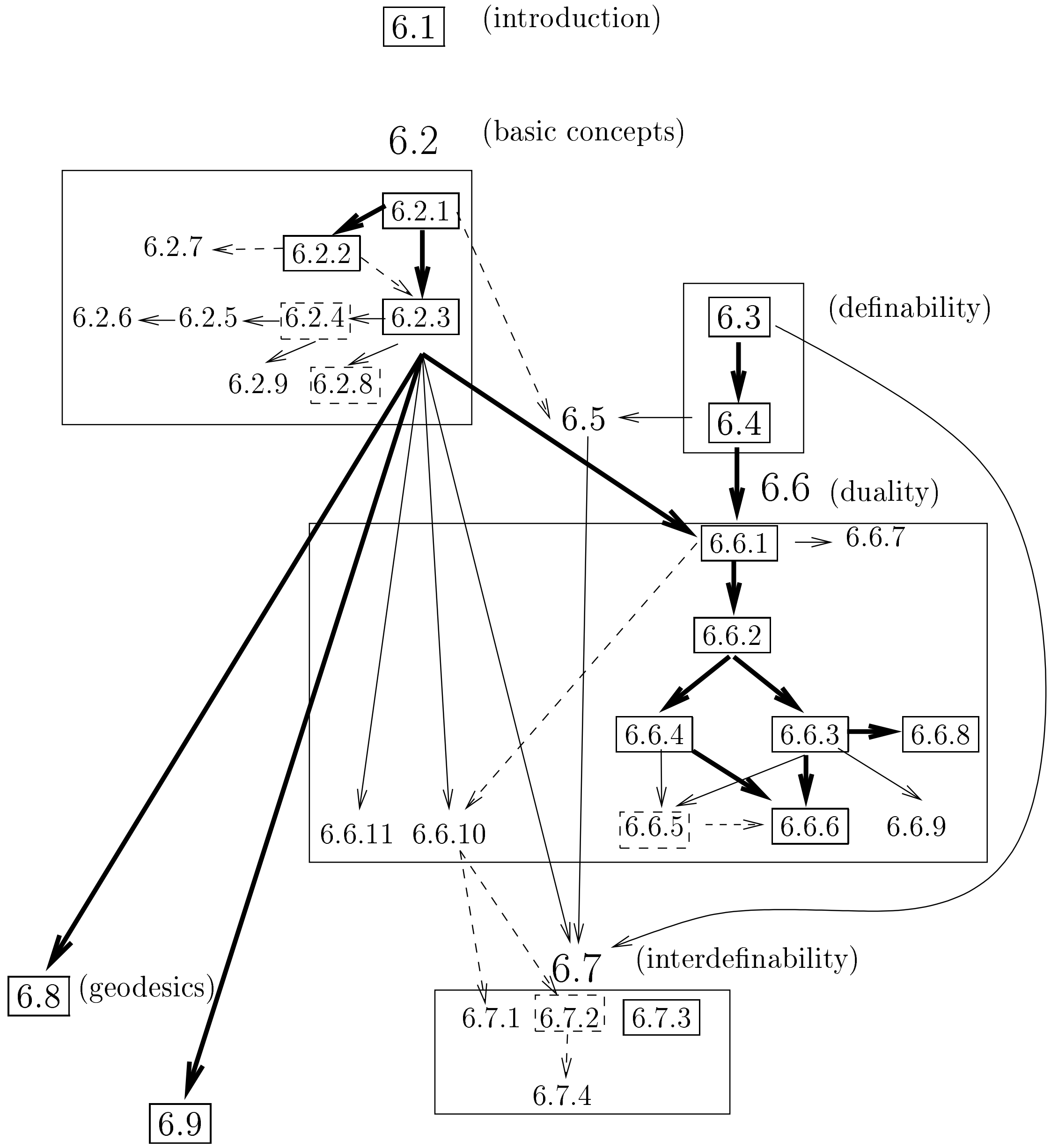

Figure 260: a→b means that reading (sub-)section “a” is a prerequisite for “b”. Further, the dashed (broken) arrows a→b mean that leafing through the definitions and main ideas in (sub-)section “a” is desirable before reading “b”.

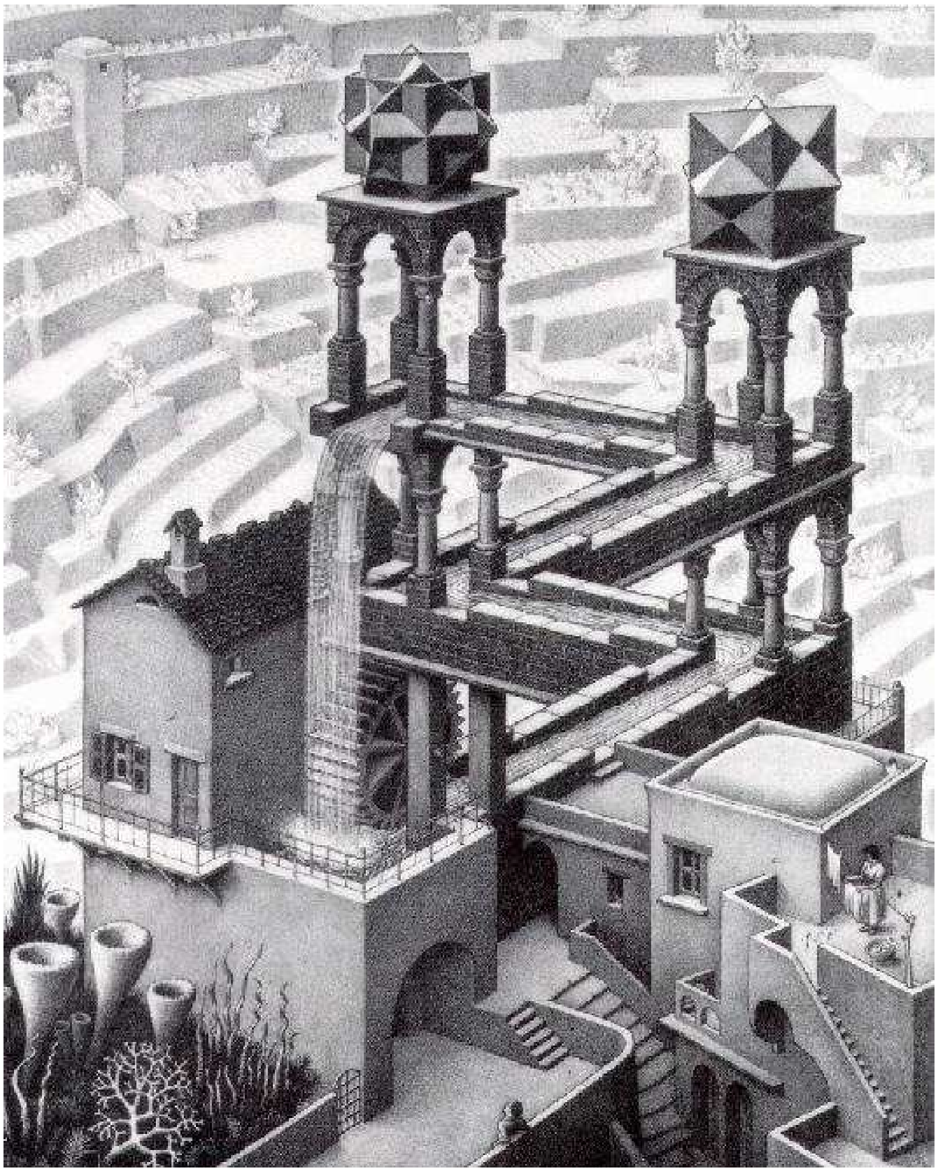

Figure 261: Paving the road toward general relativity by gluing together locally well behaved geometries yielding something globally strange, cf. §6.2.5 for gluing geometries. This Escher picture shows a “paradigm for general relativity” which locally behaves like special relativity: On the picture locally everything is normal, globally it is like time travel via a rotating black hole, cf. Thorne [258] or O’Neil [208] for the latter. Cf. also Fig.258 on p.770.

6.2 Basic concepts

In this section we show how observer-independent structures can be found in our frame models of relativity theory, i.e. we will show that there is an *observer-independent* "geometric" structure $\mathfrak{G}_{\mathfrak{M}}$ inside every model $\mathfrak{M}$ of our frame language. We will also define the Reichenbachian version $\mathfrak{G}^R_{\mathfrak{M}}$ of the geometric structure corresponding to a frame model $\mathfrak{M}$ of relativity theory.[639]

Conventions, clarifying possible ambiguities:

The symbol $\perp$ for orthogonality will be used in the present work in an ambiguous way. Sometimes it denotes Euclidean orthogonality (as defined in §3.1) and sometimes it denotes *relativistic orthogonality* as will be the case in the middle of Definition 6.2.2 below. We hope that context will help. If somewhere we want to emphasize the difference then we will write $\perp_e$ and $\perp_r$ (for the Euclidean and the relativistic version, respectively). We note that the so called Minkowskian orthogonality is a special case of our relativistic orthogonality $\perp_r$. A further source of ambiguity is the following (issue about where exactly our geometry lives). For a second let $Mn := \mathcal{P}(B)$, where B is the set of bodies for our model $\mathfrak{M}$. (Later we will slightly change this convention but that is beside the point now.) In §2 we had *"Lines"* $\subseteq \mathcal{P}({}^nF)$ while in the present chapter we will have *"Lines"* $\subseteq \mathcal{P}(Mn)$. That is, now lines are understood on the set Mn of events, while at the beginning (when we defined frame models) lines were understood on the vector-space ${}^n\mathbf{F}$. We hope, context will help.

For a class K of models $\mathbf{I}\mathsf{K}$ denotes the class of isomorphic copies of members of K.

Warning 6.2.1 The word <u>algebra</u> is used in 3 different senses, both here and in the literature. These are:

(i) Algebra is a branch of mathematics.

(ii) An algebra[640] is a structure $\mathfrak{A} = \langle A;\ f_i\rangle_{i\in I}$ in the sense of universal algebra.[641]

(iii) An algebra <u>over a field</u> $\mathbf{F}$ is a vector space over $\mathbf{F}$ with an extra binary operation "$\cdot$" as indicated in footnote 1105, p.1101 (§6.6.6, sub-title "On ... omnipresence ..." item (2)).[642] $\lhd$

[639] Cf. §4.5 for what we call the Reichenbachian approach to relativity.

[640] or equivalently an algebraic structure

[641] Here A is an arbitrary set and $f_i : {}^nA \longrightarrow A$ is arbitrary too (for some $n \in \omega$).

[642] The literature often writes simply <u>"an algebra"</u> for an algebra over a field.

6.2.1 Definition of the observer-independent (or relativistic) geometry $\mathfrak{G}_{\mathfrak{M}}$

643 lábjegyzetben emlegetett $\mathsf{Mog}(TH) = \mathbf{I}Geom(Th)$ tipusú tétel nincs, legyen valahol remark

The reader may find that $\mathfrak{G}_{\mathfrak{M}}$ defined below has too many components; however there is no need to worry, our theory will be not as complicated as suggested by the number of these components as it will be explained in §6.2.9 (Some reducts ...). The reader is asked not to be disturbed by the complexity (or size) of the geometry $\mathfrak{G}_{\mathfrak{M}}$. We include here the whole of $\mathfrak{G}_{\mathfrak{M}}$ *only* for completeness: We will almost never study the whole $\mathfrak{G}_{\mathfrak{M}}$. Most of the time, we will study simpler geometries, e.g. $\mathbf{G}_{\mathfrak{M}}$ defined in the fifth line of Def.6.2.2 or some even simpler variant of this simpler geometry like $\langle Mn, L; \in, eq\rangle$ or $\langle Mn, L; \in, \perp\rangle$ or the streamlined time-like metric structure $\langle Mn, \mathbf{F_1}; g^{\prec}\rangle$ on p.1170 (§6.7.3). Cf. also the beginning of §6.2.9 (p.923).

We would like to emphasize that we want to treat relativistic geometries as abstract structures. An abstract structure is determined only up to isomorphism.[643] Therefore it is important to emphasize that relativistic geometries are defined up to isomorphism only, cf. Def.6.2.2(III) (p.798). That is, any isomorphic copy of the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ counts as "the geometric counterpart" of the frame model $\mathfrak{M}$ (where recall, that $\mathfrak{G}_{\mathfrak{M}}$ is the observer-independent geometry associated to $\mathfrak{M}$). In still other words this means that when studying $\mathfrak{G}_{\mathfrak{M}}$ we will concentrate on its *isomorphism invariant* properties only (as is usual in the structuralist branches of mathematics like algebra). The reason why this is impor-

[643]By an ***abstract structure*** we understand a class K of structures such that $(\forall \mathfrak{A} \in \mathsf{K})\, \mathsf{K} = \mathbf{I}\{\mathfrak{A}\}$. Similarly an ***abstract class of structures*** is one which is closed under isomorphisms. As a contrast, a concrete class is usually not closed under $\mathbf{I}$. An example for the abstract/concrete distinction is provided by Stone duality on pp. 1015, 1019. The class BA of Boolean algebras is an abstract class (since $\mathsf{BA} = \mathbf{I}\mathsf{BA}$). The class BSA of Boolean set algebras, i.e. algebras whose operations are the real, set theoretic $\cup, \cap, -$ is a concrete class of structures because if we know the universe A of an algebra $\mathfrak{A} \in \mathsf{BSA}$ then from A the rest of $\mathfrak{A}$ is recoverable.[644] Accordingly $\mathsf{BSA} \neq \mathbf{I}\mathsf{BSA}$ $(= \mathsf{BA})$. Stone's representation theorem says that every member of the abstract class BA is representable by (i.e. is isomorphic to) a member of the concrete class BSA. Cf. also p.799, Remark 6.6.87 ("On representation theorems ...") on p.1106. E.g. on p.799 $Geom(Th)$ will be a concrete class while $\mathsf{Ge}(Th) = \mathbf{I}Geom(Th)$ will be an abstract class. The theorems later saying that for an axiomatizable class $\mathsf{Mog}(TH)$ of geometries $\mathsf{Mog}(TH) = \mathbf{I}Geom(Th)$ are typical representation theorems. Cf. e.g. items 6.6.57, 6.6.67, 6.6.70, 6.6.71 (pp. 1073–1083). Though these items are not exactly of the desired form "$\mathsf{Mog}(TH) = \mathbf{I}Geom(Th)$" they (and the "tools" scattered around them) can be used for obtaining theorems of the desired form. We leave it as a useful exercise for the reader to carry this through. Cf. for more on "concrete", "abstract", "axiomatic-abstract" classes and their connections with representation theorems in Remark 6.6.87 and also Németi [206].

[644]For the notion of concrete classes of algebras and for the importance of the concrete/abstract distinction cf. Németi [206].

tant is explained in Remark 6.2.4 (p.801). Treating $\mathfrak{G}_{\mathfrak{M}}$ as abstract structure will make some of our results, e.g. the duality theory, stronger. The fact that we treat isomorphic geometries as identical is important for the philosophy of the present chapter, cf. Remark 6.2.4 (p.801).

More motivation for the definition below will come in §6.2.3 ("On the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$"), we would like to, particularly, emphasize Remark 6.2.46 on p.853.

hosszútávon legyen $\mathfrak{G}_{\mathfrak{M}}$ helyett $\mathfrak{G}^*_{\mathfrak{M}}$, cf. §6.6.9, tehát mindenütt minden inerciális életút (body-k is) egyenesnek számít majd.

Definition 6.2.2
(Observer-independent, relativistic geometry and related definitions)
Let $\mathfrak{M}$ be a frame model.

(I) Then the *observer-independent geometry* $\mathfrak{G}_{\mathfrak{M}}$ is a *three-sorted* structure to be defined below.[645] But cf. also the improved geometry $\mathfrak{G}^*_{\mathfrak{M}}$ in §6.6.9 (p.1111). (The "simplified" geometries $\mathbf{G}_{\mathfrak{M}}$ and $\mathsf{G}_{\mathfrak{M}}$ will be only two-sorted.)

$$\mathfrak{G}_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq, g, \mathcal{T}\,\rangle, \text{ and}$$

$\mathbf{G}_{\mathfrak{M}}$ is the $(g, L^S, \mathcal{T})$-free reduct

$$\mathbf{G}_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ L^T, L^{Ph}, \in, \prec, Bw, \perp, eq\rangle$$

of $\mathfrak{G}_{\mathfrak{M}}$, and $\mathsf{G}_{\mathfrak{M}}$ is the $(L^T, L^{Ph}, \prec, Bw, eq)$-free reduct

$$\mathsf{G}_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ \in, \perp\rangle$$

of $\mathbf{G}_{\mathfrak{M}}$; where:

1. The universes (or sorts) of $\mathfrak{G}_{\mathfrak{M}}$ are Mn, F (= universe of the structure $\mathbf{F_1}$) and L, while the rest are the relations of $\mathfrak{G}_{\mathfrak{M}}$.[646]

2. $\perp$ denotes $\perp_r$ to be defined in item 11 way below.

3. $Mn :\stackrel{\text{def}}{=} \bigcup\{\, Rng(w_m) \ :\ m \in Obs\,\}$ $(\subseteq \mathcal{P}(B))$. Intuitively Mn is the set of all *events* in our relativistic model $\mathfrak{M}$. Mn is the *set of points* of our geometry

[645] Later, in Remark 6.2.4 we will define the geometric counterpart of the model $\mathfrak{M}$ to be $\mathbf{I}\mathfrak{G}_{\mathfrak{M}}$.

[646] The statuses of all the relations $\in$, $\prec$ etc. should be clear with the possible exception of the topology $\mathcal{T}$. We can declare that $\mathcal{T}$ is a so called second-order relation on Mn. Equivalently, we could declare that $\mathcal{T}$ is the 4[th] *sort* (or universe) of $\mathfrak{G}_{\mathfrak{M}}$, and use the set theoretic membership relation $\in_{Mn,\mathcal{T}} \subseteq Mn \times \mathcal{T}$ to connect $\mathcal{T}$ with the remaining sorts. Cf. §6.3 for more detail on this.

$\mathfrak{G}_{\mathfrak{M}}$. We also call Mn *space-time*, cf. Convention 6.2.5 (p.802). The acronym Mn abbreviates the word *manifold*.[647]

4. $\mathbf{F_1} :\stackrel{\text{def}}{=} \langle F;\ 0, 1, +, \leq\rangle$ is the ordered group reduct $\langle F;\ 0, +, \leq\rangle$ of the ordered field $\mathfrak{F}^{\mathfrak{M}}$ expanded with the constant 1, where 0 and 1 are the usual zero and one of the field $\mathfrak{F}^{\mathfrak{M}}$.

5.
$$\begin{aligned} L^T &:\stackrel{\text{def}}{=} \{\{e \in Mn : m \in e\} : m \in Obs \cap Ib\}.^{648} \\ L^{Ph} &:\stackrel{\text{def}}{=} \{\{e \in Mn : ph \in e\} : ph \in Ph\}. \end{aligned}$$

I.e. L^T, called the set of *time-like lines*, is the set of life-lines of *inertial observers*, and similarly L^{Ph}, called the set of *photon-like lines*, is the set of life-lines of *photons*. Here, life-lines are understood as subsets of Mn ($\subseteq \mathcal{P}(B)$), while in earlier parts of this work they were understood as subsets of ${}^nF$.

L^S consists of the *space-like lines* of $\mathfrak{M}$ defined as follows:

$$L^S :\stackrel{\text{def}}{=} \{ w_m[\bar{x}_i] : m \in Obs \cap Ib,\ 0 < i \in n\}.$$

[647]We do not need the manifold structure on Mn yet. So, the reader may safely skip the following. Mn is only the universe of a manifold $\mathbf{Mn}$. Assume Mn comes from $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax}(\sqrt{\ }))$. Then $\mathbf{Mn} := \langle Mn, \mathfrak{F};\ w_m\rangle_{m \in Obs}$. (Here the w_m's are called the maps and $\{w_m : m \in Obs\}$ is called the atlas of $\mathbf{Mn}$.) This structure looks like a manifold except that $\mathfrak{F}$ may be different from $\mathfrak{R}$ and the topology induced on Mn by the coordinatizations $\{w_m : m \in Obs\}$ may be of an uncountable base. In a *generalized manifold* we allow the base set to be uncountable but otherwise we do require all the remaining usual properties. So if $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ and $\mathfrak{M}$ satisfies some natural conditions then $\mathbf{Mn}$ is a generalized manifold. In later generalizations to accelerated observers and towards general relativity we will have to generalize $\mathbf{Mn}$ further, e.g. there will be maps much more general than world-views of inertial observers. (Cf. also footnote 75 on p.55 where we indicated generalizations such that w_m becomes a partial function $w_m : {}^nF \xrightarrow{\circ} Mn$, i.e. w_m coordinatizes a subset of Mn with only a subset of ${}^nF$. Cf. also Fig.3 on p.34, together with the sentence on p.37 containing [reference to] footnote 50. Cf. also footnote 198 on p.188 and Fig.64 on p.191. This is of course only a first step in the direction of generalization we are discussing.)

[648]In our theories studied so far we always assumed $Obs \subseteq Ib$. The latter is implied by $\mathbf{Ax2}$. Therefore instead of inertial observers i.e. members of $Obs \cap Ib$ we usually talked about simply observers[649], Obs only, for simplicity (since we knew that $Obs = Obs \cap Ib$ was the case). However, later when studying accelerated observers and other generalizations towards general relativity we will need to pay special attention to $Obs \cap Ib$, since the Obs-part of $\mathbf{Ax2}$ (i.e. $Obs = Obs \cap Ib$) will not be assumed any more. This is why at the present point we start to pay attention to the distinction between Obs and $Obs \cap Ib$. (In some sense, in general relativity $Obs \cap Ib$ will be a kind of "backbone" of our theory.) Cf. in connection with these ideas Remark 6.2.46 on p.853.

[649]since we knew that they were inertial anyway.

I.e. L^S consists of the w_m-images of the spatial coordinate axes (like $\bar{x}$, $\bar{y}$, $\bar{z}$) of ${}^nF$ for inertial m's.

We note that, assuming $\mathbf{Ax4} + \mathbf{Ax6_{00}}$,

$$L^T = \{\, w_m[\bar{t}\,] \;:\; m \in Obs \cap Ib \,\},$$

i.e. L^T consists of the w_m-images of the time axis for inertial m's.[650]

The set L of all *lines* of $\mathfrak{G}_{\mathfrak{M}}$ is defined as

$$L :\stackrel{\text{def}}{=} L^T \cup L^{Ph} \cup L^S.$$

Cf. Figure 288 on p.885 for the spirit of working in Mn instead of ${}^nF$ and for the connections of the two.

6. $\in$ is the set theoretic membership relation between Mn and L.[651] In other words $\in$ is the usual *incidence relation* of our geometry $\langle Mn, \ldots, L; \ldots \rangle$.

$\prec$ nevét meggondolni

7. We define the binary relation, called *causality pre-ordering*,[652] $\prec$ on Mn as follows.[653] Let $e, e_1 \in Mn$. Intuitively, $e \prec e_1$ holds if there is an inertial observer who is present both in e and e_1 (i.e. e and e_1 are on his life-line) and sees that event e precedes event e_1 in time; formally:

$$e \prec e_1$$
$$\stackrel{\text{def}}{\iff}$$
$$(\exists m \in Obs \cap Ib)\ (m \in e \cap e_1 \ \wedge\ (\exists p \in w_m^{-1}(e))(\exists q \in w_m^{-1}(e_1))\ p_t < q_t).^{654}$$

[650] The difference between the style of definitions of L^T and L^S is connected to the fact (emphasized e.g. by Reichenbach) that in relativity theory L^S is somewhat less tangible than L^T, cf. §4.5 (and the definition of "Reichenbachian" geometries on p.799).

[651] In our many-sorted approach we encounter several situations where members of one sort U_1 act as sets of members of another sort, say U_2. In such situations we use the set theoretical symbol "$\in$" as the relation connecting U_2 and U_1, i.e. "$\in$" $\subseteq U_2 \times U_1$. We can add the names of the sorts involved as indices of $\in$ like $\in_{Mn,L}$ but for simplicity we often omit these indices.

[652] The word "causality" in "causality pre-ordering" here is used only because we want to be consistent with the literature. We emphasize that with this word we do not mean to imply that we would have a theory of real causality around at this point. Cf. Remark 6.7.22 on p.1158.

[653] Under very mild assumptions on $\mathfrak{M}$, $\prec$ becomes a so called irreflexive pre-ordering, i.e. $\prec \cup \mathrm{Id}$ is a pre-ordering, i.e. is transitive and reflexive. Note that $e \prec e_1 \Rightarrow e, e_1 \in \ell \in L^T$, for some ℓ.

[654] We defined $\prec$ in the "existential" style. The universal version $\prec^u$ of $\prec$ is defined as follows. $(\exists m \in Obs \cap Ib)\, m \in e \cap e_1 \wedge (\forall m \in Obs \cap Ib)[m \in e \cap e_1 \Rightarrow (\exists p \in w_m^{-1}(e))(\exists q \in w_m^{-1}(e_1))\, p_t < q_t]$. Under mild assumptions $\prec^u$ is a (strict) partial ordering, moreover $\prec^u$ is the antisymmetric part of $\prec$ (i.e. $x \prec^u y \Leftrightarrow [x \prec y \wedge y \not\prec x]$).

8. The *relation* $Bw \subseteq Mn \times Mn \times Mn$ *of betweenness* is a ternary relation defined as follows: Let $e, e_1, e_2 \in Mn$. Intuitively, $Bw(e, e_1, e_2)$ holds if there is an inertial observer who thinks that event e_1 is between events e and e_2; formally:

$$Bw(e,e_1,e_2) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \Big((\exists m \in Obs \cap Ib)(\exists p,q,r \in {}^nF)$$
$$[w_m(p) = e \ \wedge \ w_m(r) = e_1 \ \wedge \ w_m(q) = e_2 \ \wedge \ \mathsf{Betw}(p,r,q)]\Big),$$

where, we recall from footnote 405 on p.492 that, $\mathsf{Betw}(p,r,q)$ means that p, r, q are collinear points of ${}^nF$ and r is strictly in between p and q, formally: $r \neq p, q$ and $r = p + \lambda \cdot (q - p)$ for some $0 < \lambda < 1$.

9. In analogy with our notation "$\mathfrak{G}_{\mathfrak{M}}$", if we want to indicate that Mn or L comes from $\mathfrak{G}_{\mathfrak{M}}$ then we will write $Mn_{\mathfrak{M}}$, $L_{\mathfrak{M}}$ etc.[655]

10. Next we define the derived relation of parallelism in our observer-independent geometries $\mathfrak{G} = \langle Mn, L; \in, Bw\rangle \cong \langle Mn_{\mathfrak{M}}, L_{\mathfrak{M}}; \in, Bw_{\mathfrak{M}}\rangle$, where "$\cong$" is the usual relation of isomorphism between structures (cf. Conventions 3.1.2, 3.8.4). Let $\ell, \ell_1 \in L$. Intuitively, ℓ and ℓ_1 are *$\mathfrak{G}$-parallel* iff each inertial observer who sees them thinks, they are parallel; formally:

$$\ell \parallel_{\mathfrak{G}} \ell_1$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(\forall a,b,c \in Mn)\Big((\, Bw(a,b,c) \ \wedge \ a \in \ell \ \wedge \ a \notin \ell_1 \ \wedge \ c \in \ell_1 \,) \ \Rightarrow$$
$$[\, (\exists d \in \ell)(\exists e \in \ell_1)(\, Bw(d,b,e) \ \wedge \ d \neq a \ \wedge$$
$$(\nexists f \in \ell_1)(Bw(a,d,f) \ \vee \ Bw(a,f,d) \ \vee \ Bw(d,a,f)\,)\,]\Big)^{656}$$

see Figure 262.

11. For the definition of (relativistic) *orthogonality* $\perp = \perp_r$ we need first an auxiliary definition.[657]

Alternative (shorter) definitions of relativistic orthogonality ($\perp_r$) are available in Definitions 6.2.9, 6.2.17 (pp. 810, 821) below, cf. Remark 6.2.8, too.

[655]Formally, $Mn^{\mathfrak{G}_{\mathfrak{M}}}$ would be the standard model theoretic notation. However, it is too complicated.

[656]The formal definition became so long because we have to take into account lines which are present in several windows, for "windows" cf. the intuitive text above Thm.3.3.12 on p.196 (think of photon-like lines).

[657]We would like to mention that on p.821 we will give an alternative definition ($\perp_r^{\omega}$) for relativistic orthogonality which is just as natural as the present one and is shorter. The only disadvantage of $\perp_r^{\omega}$ is that it "works" only for $n > 2$.

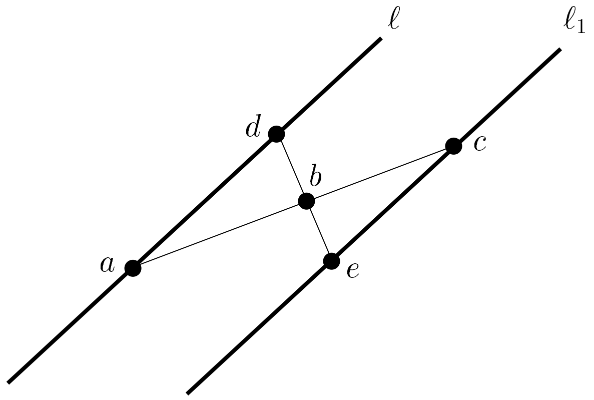

Figure 262: $\ell \parallel_{\mathfrak{S}} \ell_1$.

Ordinals denotes the class of ordinal numbers in the usual set theoretic sense, cf. e.g. Handbook of Mathematical Logic [42].

The definition of convergence given below agrees with what one would intuitively expect, cf. Figure 263.

Definition: Let $\alpha \in$ Ordinals. Let $\mathcal{S} \in {}^{\alpha}L$ (i.e. $\mathcal{S}$ is an α-sequence of lines) and $\ell \in L$. Then we say that $\mathcal{S}$ *converges to* ℓ iff

$$(\exists p \in Mn)\Big[(\forall i \in \alpha)(p \in \mathcal{S}(i) \cap \ell) \ \wedge$$
$$(\exists \ell' \in L)\Big(p \notin \ell' \ \wedge \ (\exists q \in {}^{\alpha}Mn)(\forall i \in \alpha)\ [\, q(i) \in \mathcal{S}(i) \cap \ell' \ \wedge$$
$$(q \text{ converges to some } q^+ \in \ell' \cap \ell \text{ w.r.t. } Bw)\ ^{658}]\Big)\Big], \text{ see Figure 263.}$$

First we define *basic orthogonality* $\perp_0 \subseteq L \times L$. Intuitively, two lines are $\perp_0$-orthogonal if there is an inertial observer who thinks that these two lines coincide with two distinct coordinate axes; formally: Let $\ell, \ell' \in L$. Then

$$\ell \perp_0 \ell' \ \stackrel{\text{def}}{\Longleftrightarrow}\ \Big((\exists m \in Obs{\cap}Ib)(\exists i, j \in n)(i \neq j \wedge \ell = w_m[\bar{x}_i] \wedge \ell' = w_m[\bar{x}_j])\Big),$$

see Figure 264.

[658] I.e., $(\exists q^+ \in \ell \cap \ell')(\forall a, b \in \ell')\ [Bw(a, q^+, b) \ \Rightarrow\ (\exists \beta \in \alpha)(\forall i \in (\alpha \setminus \beta))\ Bw(a, q(i), b)]$.

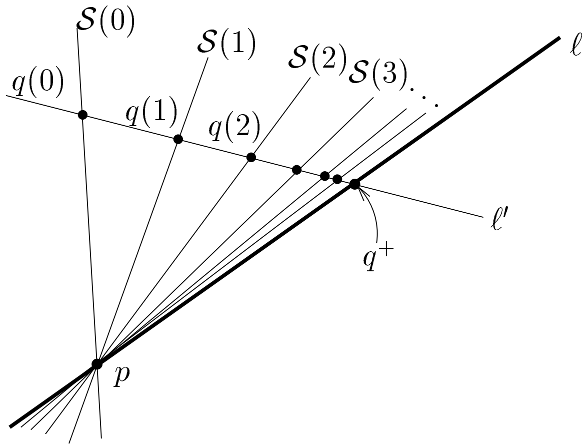

Figure 263: $\mathcal{S} \in {}^{\alpha}L$ converges to $\ell \in L$.

The relation of *relativistic orthogonality* $\perp = \perp_r$ is defined to be the smallest subset of $L \times L$ containing $\perp_0$ and closed under taking limits and parallelism, i.e. $\perp_r$ is the smallest subset of $L \times L$ having properties (i)–(iii) below.

(i) $\perp_0 \subseteq \perp_r$, i.e. $\ell \perp_0 \ell' \ \Rightarrow\ \ell \perp_r \ell'$.

(ii) $\perp_r$ is closed under taking limits, i.e.

$$\Big((\exists \alpha \in \mathsf{Ordinals})(\exists \mathcal{S}, \mathcal{S}' \in {}^{\alpha}L)(\forall i \in \alpha)(\mathcal{S}(i) \perp_r \mathcal{S}'(i) \quad \wedge$$
$$\mathcal{S} \text{ and } \mathcal{S}' \text{ converge to } \ell \text{ and } \ell' \text{ respectively})\Big) \quad \Rightarrow \quad \ell \perp_r \ell',$$

see Figure 265.[659] We note that this property (i.e. that $\perp_r$ is closed under taking limits) can be formulated in the first-order language of the structure $\langle Mn, L; \in, Bw, \perp_r\rangle$, cf. axiom $\mathbf{L_{10}}$ on p.1077.

(iii) $\perp_r$ is closed under parallelism, i.e.

$$(\ell \perp_r \ell_1 \ \wedge\ \ell' \parallel_{\mathfrak{G}} \ell \ \wedge\ \ell'_1 \parallel_{\mathfrak{G}} \ell_1) \ \Rightarrow\ \ell' \perp_r \ell'_1.$$

[659]Figure 265 is understood in the world-view of an observer, under assuming $\mathbf{Bax}^-$+$\mathbf{Ax}(\sqrt{\ })$. The 8 pictures represent all the possibilities as new "orthogonal pairs" ($\perp_r$-pairs) can be generated by "old orthogonal pairs" ($\perp_0$-pairs) by taking limits as described above. For that possible reader who wants to see the "intuitive counterparts" of these pictures in, say, **Basax** + $\mathbf{Ax}(\sqrt{\ })$ models we suggest concentrating on figures (a), (b), (c). We note that we do not claim that all these 8 possibilities are realized in, say, **Basax** models. (Though, in passing we note that, (a), (b), (c), (d), (f), (h) do occur, and we did not check with the rest.)

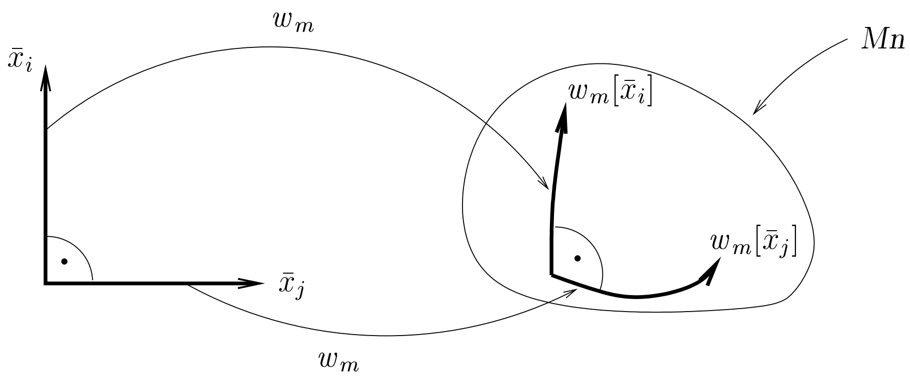

Figure 264: Illustration for the definition of $\perp_0$.

In connection with Figure 265, it might be useful to have a look also at Remark 6.2.6 (pp.802–805) and Figure 267 in that remark.

We refer to Remark 6.2.6 (p.802) at the end of §6.2.1 for intuitive motivation (and considerations) for our using limits in the definition of $\perp_r$. That remark might also help in improving our intuitive picture of $\perp_r$ (and perhaps other parts of $\mathfrak{G}_{\mathfrak{M}}$).

12. The _relation_ $eq \subseteq {}^4Mn$ _of equidistance_ is a 4-ary relation defined as follows. Intuitively, $eq(a,b,c,d)$ will mean that segments $\langle a,b\rangle$ and $\langle c,d\rangle$ are of equal length (in some sense). First we define the relation eq_0 of basic equidistance. Let $e, e_1, e_2, e_3 \in Mn$. Then

$$eq_0(e,e_1,e_2,e_3)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs \cap Ib)(\exists i,j \in n)(\exists p,q \in \bar{x}_i)(\exists r,s \in \bar{x}_j)$$
$$\Big(|p-q| = |r-s| \wedge w_m(p) = e \wedge w_m(q) = e_1 \wedge w_m(r) = e_2 \wedge w_m(s) = e_3\Big).[660]$$

[660]We could "improve" the definition of eq_0 by adding $eq_0(e,e_1,e,e_1)$. This would perhaps simplify some of our upcoming statements, but we did not explore this. Further, analogously to the definition of $\perp_r$, if we closed eq_0 under taking limits then perhaps the new eq_0 would behave better (i.e. if e,e_1,e_2,e_3 are on a photon-like line then $eq_0(e,e_1,e_2,e_3)$ would be the case).

(a) (b) (c) (d) (e) (f) (g) (h)

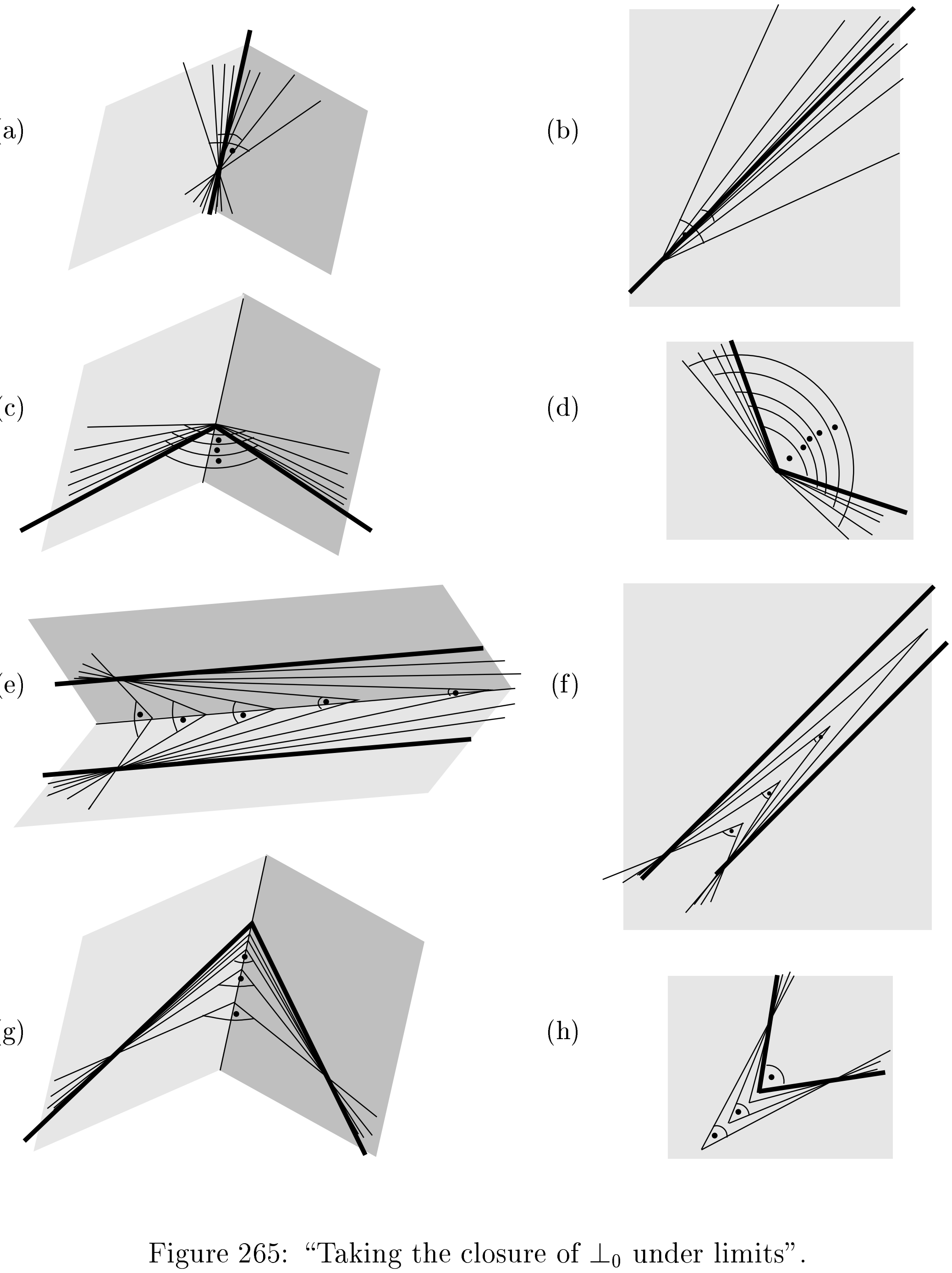

Figure 265: "Taking the closure of $\perp_0$ under limits".

Intuitively, segments $\langle e, e_1 \rangle$ and $\langle e_2, e_3 \rangle$ are eq_0-related if there is an inertial observer m who "thinks" that the distance between e and e_1 is the same as the distance between e_2 and e_3 (and sees the segments $\langle e, e_1 \rangle$ and $\langle e_2, e_3 \rangle$ on some [perhaps different] coordinate axes).

Now we define eq to be the transitive closure of eq_0 understood as a binary relation between pairs of points (cf. Figure 266); in more detail: First for every $i \in \omega$ we define eq_{i+1} as follows.

$$eq_{i+1} :\stackrel{\text{def}}{=} \{ \langle a, b, c, d \rangle \in {}^4Mn \,:\, (\exists e, f \in Mn)\, \langle a, b, e, f \rangle, \langle e, f, c, d \rangle \in eq_i \}.$$[661]

Now,

$$eq :\stackrel{\text{def}}{=} \bigcup \{ eq_i \,:\, i \in \omega \}, \quad \text{see Figure 266.}$$

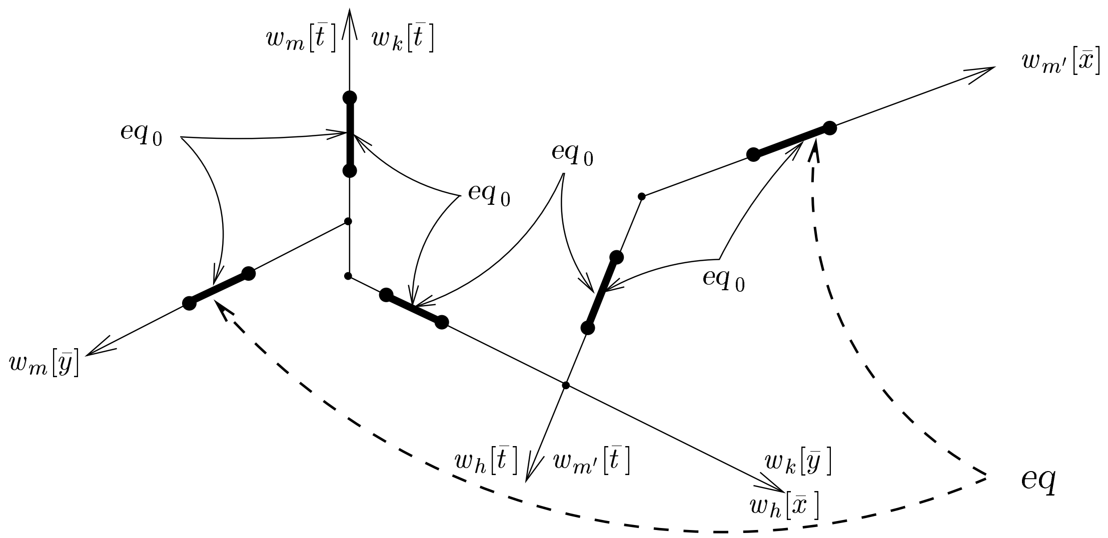

Figure 266: eq is defined to be the transitive closure of eq_0.

Instead of $eq(a, b, c, d)$ sometimes we write $\langle a, b \rangle \;\; eq \;\; \langle c, d \rangle$. Similarly for eq_0, eq_1, etc.

We note that eq is an equivalence relation (when understood on pairs of points) on the set $\{ \langle a, b \rangle \in Mn \times Mn \,:\, (\exists m \in Obs \cap Ib)(\exists i \in n)\, a, b \in w_m[\bar{x}_i] \}$.

hosszútávon jó lenne "max" az ikerparadoxon miatt

13. $g : Mn \times Mn \overset{\circ}{\longrightarrow} F$ is a partial function defined as follows. Let $e, e_1 \in Mn$.

Intuitively, the distance between events e and e_1 as measured by an inertial observer, call it m, is λ (where $0 \le \lambda \in F$) iff m sees both e and e_1 happening on the same coordinate axis $\bar{x}_i$ with coordinate distance λ. Further, the distance between events e and e_1 as measured by a photon, call it ph, is λ iff $\lambda = 0$ and ph is present both in e end e_1.[662] Now,

$$g(e, e_1) :\overset{\text{def}}{=} \min\{\, \lambda \in F \;:\; (\exists h \;\in\; (Obs \cap Ib) \cup Ph\,) \\ \text{(the distance between } e \text{ and } e_1 \text{ as measured by } h \text{ is } \lambda\,)\,\};^{663}$$

Formally:

$$g(e, e_1) :\overset{\text{def}}{=} \min\{\, \lambda \in F \;:\; (\exists ph \in Ph)\,[\, ph \in e \cap e_1 \;\wedge\; \lambda = 0\,] \quad \text{or} \\ (\exists m \in Obs \cap Ib)(\exists i \in n)(\exists p, q \in \bar{x}_i)\,[w_m(p) = e \wedge w_m(q) = e_1 \wedge \lambda = |p - q|\,]\,\},$$

if this min[665] exists, *otherwise* $g(e, e_1)$ is undefined.

Under mild assumptions, the "min" part of the definition of $g(e, e_1)$ can be omitted. (More precisely, the essential occurrence of "min" could be omitted.[666]) An example for such sufficient assumptions is the axiom of equimeasure **Ax(eqm)** below. Intuitively, **Ax(eqm)** says that all inertial observers agree on distances (which they can measure).

equimeasure neve meg lett változtatva equimeasure ra!

Ax(eqm) §3.9-ben módosítandó lesz.

Ax(eqm) $(\forall m, k \in Obs \cap Ib)(\forall i, j \in n)(\forall p, q \in \bar{x}_i)(\forall p', q' \in \bar{x}_j)$
$$\Big([w_m(p) = w_k(p') \;\wedge\; w_m(q) = w_k(q')] \;\Rightarrow\; |p - q| = |p' - q'| \Big).^{667}$$

[661] We note that $(\forall i \in \omega)\ eq_i \subseteq eq_{i+1}$ since eq_0 is "reflexive", i.e. $[\,\langle a, b, c, d\rangle \in eq_0 \;\Rightarrow\; \langle a, b, a, b\rangle \in eq_0]$.

[662] We could have achieved the "photons measure zero distance" effect by first using inertial observers only and then closing the concept of a distance under taking limits like we did in the definition of $\perp_r$ (from $\perp_0$).

[663] It is important in the definition of g that we required "$h \in Ib$ (unless $h \in Ph$)" i.e. we used only inertial observers in measuring distances, because of the twin paradox cf. §2.8.4 (p.139). Namely by the twin paradox to time-like separated events e, e_1 we can have accelerated observers who see e and e_1 closer and closer[664] (and therefore $g(e, e_1)$ would not be defined etc).

[664] This closeness would be not a property of e and e_1 instead it would only represent the extent of acceleration of the "measuring observer".

[665] As usual, $\min H$ denotes the minimal element (or smallest element) of the set H taken in the ordered set $\langle F, \le\rangle$. Note that $\min H$ need not exist (even if $\mathfrak{F}$ is complete).

[666] I.e. before trying to remove min we would reformulate the definition of g according to the following pattern. $g(e, e_1) = 0$ if e, e_1 are on a photon-like line, otherwise $g(e, e_1) = \min\{\, \lambda \in F : (\exists m \in Obs \cap Ib) \ldots \}$.

Let us note that $g(e, e_1)$ can easily become undefined, since either (i) there may exist no inertial observer m who sees e and e_1 on the same coordinate axis and no photon ph who is present both in e and e_1 or (ii) there may exist an infinity of inertial observers who measure smaller and smaller distances between e and e_1.

We will call g the *pseudo-metric*[668] of $\mathfrak{G}_{\mathfrak{M}}$ because it remotely does resemble a metric (of a geometry) and because the elements of L will turn out to be so called "geodesics" (cf. Def.6.8.2 on p.1179) w.r.t. g, under very mild assumptions (like e.g. $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqm)}$). It is important to note that a pseudo-metric g is usually *not* a metric because e.g. the "triangle inequality axiom of metrics" fails for g.[669]

14. $\mathcal{T}$ is the *topology*[670] on Mn determined by pseudo-metric g. In more detail:

    Let $e \in Mn$, $\varepsilon \in {}^{+}F$. The *ε-neighborhood* of e is defined as

    $$S(e, \varepsilon) \stackrel{\text{def}}{:=} \{\, e_1 \in Mn \ : \ g(e, e_1) < \varepsilon \,\}.^{671}$$

    See Figures 340–343 (pp. 1149–1152) for how such neighborhoods can look like (there we use the word g-circle instead of neighborhood). Cf. also Fig.29 on p.88.

    Now, the topology $\mathcal{T} \subseteq \mathcal{P}(Mn)$ is the one generated by[672]

    $$T_0 \stackrel{\text{def}}{:=} \left\{\, S(e, \varepsilon) \ : \ e \in Mn, \ \varepsilon \in {}^{+}F \,\right\},$$

[667] Cf. footnote 663 on p.796.

[668] In the relativity book Rindler [224] p.62 footnote 1 the expression *"pseudometric"* is used the same way as we use it here. For completeness we note that several other relativity works use a slightly different terminology. Namely, our $g : Mn \times Mn \xrightarrow{\circ} F$ is a *variant* of what, in certain relativity works, is called a Lorentzian metric cf. e.g. Naber [201, p.83, line 8] or Wald [274, p.23, line 20] or Hawking-Ellis [126] (where a "metric" is really a bilinear function on a vector-space, like our ${}^{n}\mathbf{F}$; however this difference does not effect what is important for the present work).

[669] Under very mild assumptions on $\mathfrak{M}$, our g does satisfy the axioms $g(a, a) = 0$ and $g(a, b) = g(b, a)$ but it does not satisfy the remaining axioms usually required from metrics cf. e.g. James & James [141, p.232]. One of the axioms which fail for g is the triangle inequality $g(a, b) + g(b, c) \geq g(a, c)$, another one is $g(a, b) = 0 \ \Rightarrow \ a = b$.

[670] i.e. $\langle Mn, \mathcal{T} \rangle$ forms a topological space in the usual sense, cf. p.870 for a definition

[671] Note that, by our convention on equations involving partial functions, $g(e, e_1) < \varepsilon \ \Rightarrow \ (\, g(e, e_1)$ is defined$\,)$. Cf. Convention 2.3.10 on p.61.

[672] Where, "topology generated by T_0" means taking finite intersections first, and then infinite unions as usual. So $\mathcal{T} := \{\, \bigcup Y \ : \ Y \subseteq \{\bigcap X \ : \ X \text{ is a finite subset of } T_0\}\,\}$.

i.e. T_0 is a <u>*subbase*</u>[673] for the topology $\mathcal{T}$.[674]

Alternative definitions for the topology part of $\mathfrak{G}_{\mathfrak{M}}$ are available in Definition 6.2.31 (p.838).

(II) Structures with the same similarity type[675] as that of $\mathfrak{G}_{\mathfrak{M}}$ are called structures <u>*similar*</u> to $\mathfrak{G}_{\mathfrak{M}}$. By an *isomorphism* between $\mathfrak{G}_{\mathfrak{M}}$ and $\mathfrak{G}_{\mathfrak{N}}$ we understand an isomorphism in the usual sense which is a homeomorphism[676] w.r.t. the topologies $\mathcal{T}_{\mathfrak{M}}$ and $\mathcal{T}_{\mathfrak{N}}$.[677] Since $\mathfrak{G}_{\mathfrak{M}}$ is a *three-sorted* structure (with sorts Mn, F and L) an isomorphism is a usual three sorted isomorphism, i.e. it consists of three functions, one defined on Mn, one on F, and one on L, cf. Convention 3.8.4 on p.298. The definition of an isomorphism for structures similar to $\mathfrak{G}_{\mathfrak{M}}$, is the same, but as we will see in Convention 6.2.3 (p.801) the membership relations $\in$ of our structures similar to $\mathfrak{G}_{\mathfrak{M}}$ always have to coincide with the standard, set theoretic membership relation.[678]

(III) By a <u>*relativistic geometry*</u> we understand an <u>*isomorphic copy*</u> of $\mathfrak{G}_{\mathfrak{M}}$, for some frame model $\mathfrak{M}$.[679]

Let Th be a set of formulas in our frame language for relativity theory. Then the classes of relativistic geometries $Geom(Th)$ and $\mathsf{Ge}(Th)$ associated to Th are defined as follows.

Recall that, for a class K of models $\mathbf{I}\mathsf{K}$ denotes the class of isomorphic copies of members of K.

Now,

$$\begin{aligned} Geom(Th) &\overset{\text{def}}{:=} \{\, \mathfrak{G}_{\mathfrak{M}} \,:\, \mathfrak{M} \in \mathsf{Mod}(Th)\,\}, \text{ and} \\ \mathsf{Ge}(Th) &\overset{\text{def}}{:=} \mathbf{I}Geom(Th), \text{ i.e.} \\ \mathsf{Ge}(Th) &= \{\, \mathfrak{G} \,:\, (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}\,\}. \end{aligned}$$

[673]For the definition of a subbase for a topology cf. footnote 1004 on p.1016.

[674]In the "standard" literature the members of $\mathcal{T}$ are called the open sets of the topology $\mathcal{T}$. Cf. p.870.

[675]Recall that similarity type = vocabulary.

[676]We note that a homeomorphism between topologies is what the category theorist (or a naive outsider) would call an isomorphism.

[677]Because of the presence of g, the homeomorphism condition is automatically satisfied, but in reducts from which g has been omitted this condition will become nontrivial.

[678]When looking at structures similar to $\mathfrak{G}_{\mathfrak{M}}$ we always assume that they satisfy the axiom of extensionality for $\in$.

[679]Therefore a relativistic geometry is nothing but the observer-independent geometry of some model $\mathfrak{M}$.

We will use the just introduced notation $\mathsf{Ge}(Th)$ in the spirit of Convention 6.2.3 (p.801) below.

In the terminology of algebraic logic $Geom(Th)$ is a *concrete class* while $\mathsf{Ge}(Th)$ is an *abstract class* of structures. The distinction between the two becomes important in duality theories (cf. §6.6 way below) and "representation theorems". Cf. Remark 6.6.87 (p.1106), footnote 643 on p.786, and e.g. Andréka-Németi-Sain [30] the Remark below Def.42.

By a *Th geometry* we understand a member of $\mathsf{Ge}(Th)$. E.g. we will talk about **Basax** geometries. In the same spirit when in a theorem we discuss relativistic geometries then by writing "assume Th" we mean that the geometries in question are in $\mathsf{Ge}(Th)$.

(IV) $\mathfrak{G}_{\mathfrak{M}}$ is *definitionally equivalent* (in first-order logic)[680] with its expansion

$$\mathfrak{G}_{\mathfrak{M}}^{\equiv} :\stackrel{\text{def}}{=} \langle \mathfrak{G}_{\mathfrak{M}};\ \equiv^T, \equiv^{Ph}, \equiv^S \rangle,$$

where we define the relations $\equiv^T, \equiv^{Ph}, \equiv^S \subseteq Mn \times Mn$ as follows.

$$\begin{array}{lcll} e \equiv^T e_1 & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists \ell \in L^T) & e, e_1 \in \ell. \\ e \equiv^{Ph} e_1 & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists \ell \in L^{Ph}) & e, e_1 \in \ell. \\ e \equiv^S e_1 & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists \ell \in L^S) & e, e_1 \in \ell. \end{array}$$

Intuitively: $e \equiv^T e_1$, that is e and e_1 are *time-like separated*, iff there is an inertial observer m which is present both in e and e_1. $e \equiv^{Ph} e_1$, that is e and e_1 are *photon-like separated*[681] iff there is a photon ph which is present both in e and e_1. Further, e and e_1 are called *space-like separated* iff $(\exists m \in Obs \cap Ib)$ m thinks that e and e_1 are simultaneous.[682]

Since $\mathfrak{G}_{\mathfrak{M}}$ and $\mathfrak{G}_{\mathfrak{M}}^{\equiv}$ are definitionally equivalent, we will not distinguish between them (except when explicitly stated). E.g. we will say that $\langle Mn, \equiv^T \rangle$ is a reduct of $\mathfrak{G}_{\mathfrak{M}}$.

(V) Let $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Then the relation $\parallel_{\mathfrak{G}}$ of parallelism in $\mathfrak{G}$ is defined in item (I).10 above (p.790).

(VI) We define the *Reichenbachian version* of the geometric structure corresponding to $\mathfrak{M}$ as follows:[683]

$$\mathfrak{G}_{\mathfrak{M}}^{R} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L^R;\ L^T, L^{Ph}, \in, \prec, Bw, g^R, \mathcal{T}^R \rangle,$$

[680] Cf. §6.3 (p.969) for definitional equivalence.

[681] In the literature this is often called *null-separated*. The word null comes from the fact that (if $e \neq e_1$ then) $e \equiv^{\mathrm{Ph}} e_1 \ \Leftrightarrow\ g(e, e_1) = 0$.

[682] The connection between $\equiv^S$ and space-like separatedness will be discussed in Prop.6.2.56(ii), p.858.

[683] Cf. §4.5 for motivation.

where $Mn, \mathbf{F_1}, L^T, L^{Ph}, \prec, Bw$ are as defined in item (I), $\in$ is the set theoretic membership relation between Mn and $L^R := L^T \cup L^{Ph}$ and $g^R, \mathcal{T}^R$ are defined in items 1 and 2 below.

1. $g^R(e,e_1) :\stackrel{\text{def}}{=} \min\{\, \lambda \in F \,:\, (\exists ph \in Ph)\,[\, ph \in e \cap e_1 \ \wedge\ \lambda = 0] \quad \text{or}$
$(\exists m \in Obs{\cap}Ib)(\exists p,q \in \bar{t}\,)\,[\, w_m(p) = e \wedge w_m(q) = e_1 \wedge \lambda = |p-q|\,]\,\}$,
if this min exists, *otherwise* $g^R(e,e_1)$ is undefined.

2. $\mathcal{T}^R$ is the topology on Mn determined by the pseudo-metric g^R. In more detail: Let $e \in Mn$ and $\varepsilon \in {}^+F$. Then
$$S^R(e,\varepsilon) :\stackrel{\text{def}}{=} \{\, e_1 \in Mn \,:\, g^R(e,e_1) < \varepsilon,\ e \neq e_1 \,\}\,.$$
Now, the topology $\mathcal{T}^R \subseteq \mathcal{P}(Mn)$ is the one generated by
$$T_0^R :\stackrel{\text{def}}{=} \{\, S^R(e,\varepsilon) \,:\, e \in Mn,\ \varepsilon \in {}^+F \,\}\,,$$
i.e. T_0^R is a subbase for the topology $\mathcal{T}^R$.[684]

We define the expansion,
$$(\mathfrak{G}^R_{\mathfrak{M}})^{\equiv} :\stackrel{\text{def}}{=} \langle \mathfrak{G}^R_{\mathfrak{M}};\ \equiv^T, \equiv^{Ph}\rangle$$
of $\mathfrak{G}^R_{\mathfrak{M}}$, where $\equiv^T$ and $\equiv^{Ph}$ are defined in item (IV) above.

We note that a somewhat richer, improved version of the geometry $\mathfrak{G}_{\mathfrak{M}}$ will be defined in §6.6.9 on p.1111, it will be denoted as $\mathfrak{G}^*_{\mathfrak{M}}$.

END OF DEFINITION OF $\mathfrak{G}_{\mathfrak{M}}$ AND RELATED DEFINITIONS.

$\lhd$

A discussion of the *intuitive meaning* of (parts of) $\mathfrak{G}_{\mathfrak{M}}$ will be given on pp. 848–851. Connections with the literature will be discussed beginning with p.923 (§6.2.9).[685]

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[684] We note that under some assumptions the topology $\mathcal{T}^R$ agrees with the usual Euclidean topology. E.g. $\mathbf{Reich(Basax)} + \mathbf{R(sym)} + \mathbf{Ax}(\mathit{Triv})$ is enough for this. Further we note that the alternative topologies $\mathcal{T}', \mathcal{T}''$ (basically equivalent with $\mathcal{T}$) defined in Def.6.2.31 (p.838) (see Fig.279, p.839) can be used here in the Reichenbachian approach too as alternative possibilities for defining $\mathfrak{G}^R_{\mathfrak{M}}$.

[685] In passing, we note that Busemann [56] obtains very attractive results by using a geometric structure similar to the following version $\mathfrak{G}^B_{\mathfrak{M}}$ of our $\mathfrak{G}^R_{\mathfrak{M}}$. $\mathfrak{G}^B_{\mathfrak{M}} := \langle Mn, \mathbf{F_1}, L^T, L^{Ph};\ \in, \prec, g^R, \mathcal{T}^R\rangle$. More precisely, but still very roughly speaking, Busemann uses only (a version of) the $\mathbf{G}^B_{\mathfrak{M}} := \langle Mn, \mathbf{F_1};\ \prec, g^R, \mathcal{T}^R\rangle$ reduct of $\mathfrak{G}^B_{\mathfrak{M}}$ and recovers L^T as geodesics in the sense of §6.8 way below. (L^{Ph} is definable in Busemann's structures $\mathfrak{G}^B_{\mathfrak{M}}$.) Then he introduces the local version of $\mathbf{G}^B_{\mathfrak{M}}$ with which he obtains very attractive insights into the problem of obtaining transparent axiomatizations of (aspects of) general relativity. Cf. §6.7.3 (p.1169).

The following convention is only a matter of convenience and does not have far reaching consequences. It is motivated by the fact that the axiom of extensionality holds in $\mathfrak{G}_{\mathfrak{M}}$ (for $\in$ connecting Mn and L), therefore it holds in any isomorphic copy $\mathfrak{G}'_{\mathfrak{M}}$ of $\mathfrak{G}_{\mathfrak{M}}$. Therefore we do not loose generality if we assume that "$\in$" is the real set theoretic membership in $\mathfrak{G}'_{\mathfrak{M}}$, too. To make $\in$ the real one in $\mathfrak{G}'_{\mathfrak{M}}$ the only change we have to make is renaming the lines. Cf. Convention 2.1.2 (p.35), footnote 651 on p.789 and the text below $\mathbf{Ax}_{\mathrm{G}}$ on p.31.

CONVENTION 6.2.3 Let $\mathfrak{G} := \langle Mn, \ldots, L; \ldots, \in, \ldots \rangle \in Geom(\emptyset)$. By an isomorphic copy $\mathfrak{G}'$ of $\mathfrak{G}$ we understand an isomorphic copy in the usual sense as it was explained in item (II) of Def.6.2.2, but with the ***restriction*** that $\in'$ of $\mathfrak{G}'$ is the real, set theoretic membership relation,[686] cf. Convention 2.1.2 on p.35.

The definition of $\mathsf{Ge}(Th)$ is understood accordingly. Hence $\mathsf{Ge}(Th) = \{\, \mathfrak{G}' : (\exists \mathfrak{G} \in Geom(Th))\, \mathfrak{G} \cong \mathfrak{G}'$ and $\in'$ of $\mathfrak{G}'$ is the real set theoretic membership relation $\}$. Throughout we understand isomorphism closeness of classes of structures in this sense. In this chapter we concentrate on isomorphism closed classes of structures (with the above restriction on $\in$). It is important to emphasize that isomorphism closed classes of models are more important for us than the rest. We also emphasize that the restriction on $\in$ does not contradict our philosophy of concentrating on isomorphism closed classes. In particular we consider $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ as being closed under isomorphisms. I.e. we consider $\mathsf{Mod}(Th) = \mathbf{I}\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th) = \mathbf{I}\mathsf{Ge}(Th)$.

Our conventions concerning the symbol $\in^{\mathfrak{G}}$ (or $\in$ of $\mathfrak{G}$) can be summarized and clarified by postulating that we are working in an extremely weak version of higher-order logic where $\in$ is considered as a logical symbol. This applies to our frame language as well as to our various geometric languages. For more detail on this (i.e. "$\in$" and higher-order logic reduced to many-sorted one etc.) we refer to the Appendix ("Why first-order logic?"). In order to keep things simple we leave it to the reader to elaborate the logical machinery of treating our various $\in$ symbols as logical symbols. Cf. e.g. $\mathfrak{M}^+$ on p.807 for a case when there are more than one incarnations of the $\in$ symbol in the same language. In such cases one uses $\in$ with appropriate subscripts.

$\triangleleft$

Remark 6.2.4 Our convention that we regard isomorphic relativistic geometries as *identical* is important for the philosophy of the present chapter. Without this

[686] We note that this does not cause loss of generality.

convention it would be easy to cheat. Namely, it would be too easy to "geometrize" relativity (or any other theory, for that matter) so that from the geometry say $\mathfrak{G}^{+}_{\mathfrak{M}}$ associated to an observational model $\mathfrak{M}$ all properties of the original model $\mathfrak{M}$ would be recoverable. E.g. let $\mathfrak{G}^{+}_{\mathfrak{M}}$ be obtained from $\mathfrak{G}_{\mathfrak{M}}$ (in Def.6.2.2) by replacing the universe $Mn = Points^{\mathfrak{G}_{\mathfrak{M}}}$ of $\mathfrak{G}_{\mathfrak{M}}$ with $Mn \times \{\mathfrak{M}\}$ in such a way that $\mathfrak{G}^{+}_{\mathfrak{M}} \cong \mathfrak{G}_{\mathfrak{M}}$ holds. But now, it is a trivial matter to recover $\mathfrak{M}$ from the "concrete" geometry $\mathfrak{G}^{+}_{\mathfrak{M}}$ since $\mathfrak{M}$ is sitting inside the elements of $\mathfrak{G}^{+}_{\mathfrak{M}}$ (we can find it if we are willing to "dig" deep enough along the set theoretic membership relation). Actually $\mathfrak{M}$ can be obtained by applying the projection function pj_1 to any element $e \in Points^{\mathfrak{G}^{+}_{\mathfrak{M}}}(= Mn \times \{\mathfrak{M}\})$. Further[687] $\mathfrak{M} = \bigcup pj_1[Points^{\mathfrak{G}^{+}_{\mathfrak{M}}}]$, since $\bigcup\{x\} = x$, where pj_1 is the projection function $\langle a, b\rangle \mapsto b$ associating the 1st member of a sequence to the sequence (cf. p.947 for pj_1).

But, if we *define*[688] $\mathbf{I}\mathfrak{G}_{\mathfrak{M}}$ (or equivalently $\mathbf{I}\mathfrak{G}^{+}_{\mathfrak{M}}$) to be the *geometric counterpart* of $\mathfrak{M}$ [689] then the above trick does not work (for reconstructing $\mathfrak{M}$ from its geometric counterpart in a cheap way). Defining the geometric counterpart this way is the same as defining the geometry of $\mathfrak{M}$ only up to isomorphism.

The presently discussed convention makes some of our theorems in the duality theory section stronger. Moreover, it provides formal justification for Einstein's remark to the effect that it is interesting that relativistic physics can be fully geometrized, cf. e.g. Misner-Thorne-Wheeler [196].[690]

$\triangleleft$

CONVENTION 6.2.5 Throughout, by the *space-time* of a model $\mathfrak{M}$ we mean either the geometry $\mathfrak{G}_{\mathfrak{M}}$ or a reduct of $\mathfrak{G}_{\mathfrak{M}}$ like e.g. $\mathsf{G}_{\mathfrak{M}}$ in Def.6.2.2.(I) (p.787).

$\triangleleft$

Remark 6.2.6 (Intuitive motivation for our definition of $\perp_r$)[691]

For simplicity in the present discussion we are assuming $\mathbf{Bax}^{\oplus}(4) + \mathbf{Ax}(Triv_t) + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ })$ but most of these assumptions are *not* essential (i.e. they could be eliminated on the expense of making the text longer). In particular c_m is the speed of light for m. Also, throughout the present remark we assume $n = 4$ (i.e.

[687] It is not necessary to understand this formula, for understanding the rest of this work.

[688] For any structure $\mathfrak{G}$, $\mathbf{I}\mathfrak{G} := \mathbf{I}\{\mathfrak{G}\}$ is the class of isomorphic copies of $\mathfrak{G}$.

[689] i.e. we insist on "geometric counterpart of $\mathfrak{M}$" = $\mathbf{I}$"geometric counterpart of $\mathfrak{M}$".

[690] Geometrizability of relativity (or any other theory) would be vacuously true without the condition "geometric counterpart of $\mathfrak{M}$" = $\mathbf{I}$"geometric counterpart of $\mathfrak{M}$".

[691] We note that for the case $n = 2$ Goldblatt [108] defines $\perp_r$ practically the same way as we do, and he provides intuitive motivation which is also similar to ours (e.g. uses limits) cf. Goldblatt [108] p.6 lines 9–6 bottom up and p.8 first 6 lines.

we are in 4-dimensions) but, when it *does not matter*, we often talk as if we were in 3-dimensions e.g. this is what we do in the pictures.

We would like to base our definition of $\perp_r$ on the intuitive observer-oriented notions in $\mathfrak{M}$. So, what corresponds to orthogonality in $\mathfrak{M}$? Well, two coordinate axes of any observer are considered "orthogonal". So, we would like to say that two lines are $\perp_r$-orthogonal if some observer thinks that they are parallel with two of his coordinate axes. The problem with this is that then no photon-like line will be orthogonal to *any* line because photon-like lines are not parallel with any coordinate axis of any observer. The reason for this is, roughly, that no observer can move with the speed of light, i.e. $v_m(k) \neq c_m$ for any m, k. But this can be circumnavigated because we can have observers whose speed is arbitrarily close to c_m, i.e. we can have a sequence $k_0, k_1, \ldots \in Obs$ with $\lim_{i\mapsto\infty} v_m(k_i) = c_m$.[692] Cf. the picture on p.816. Let such $m, k_0, k_1, \ldots$ be fixed. Assume

$$\forall i \,(\, m \text{ and } k_i \text{ are in strict standard configuration} \quad \text{and} \quad m \uparrow k_i \,).$$

Now, we are working in the world-view of m. To ensure existence of limits let us work with F^∞ instead of F. We can try to construct an <u>*imaginary observer*</u> k_∞ as the limit of the sequence $k_0, k_1, \ldots, k_i, \ldots$ $(i \in \omega)$ of "real" observers, in some sense.[693] So the intuitive idea is to "define" $k_\infty :=$ "$\lim_{i\mapsto\infty}(k_i)$" and $\mathsf{f}_{mk_\infty} :=$ "$\lim_{i\mapsto\infty}(\mathsf{f}_{mk_i})$". We did *not* define what we mean by $\lim_{i\mapsto\infty}(k_i)$, but we can define at least "parts" of this imaginary observer k_∞ $(= \lim_{i\mapsto\infty}(k_i))$. Cf. the picture on p.816. E.g. we can choose the coordinate axes of k_∞ to be

$$\bar{t}_\infty := \lim_{i\mapsto\infty}(\bar{t}_i) \ (\in L^{Ph})^{694}, \quad \bar{x}_\infty := \lim_{i\mapsto\infty}(\bar{x}_i), \quad \bar{y}_\infty := \lim_{i\mapsto\infty}(\bar{y}_i), \quad \bar{z}_\infty := \lim_{i\mapsto\infty}(\bar{z}_i),$$

where we use the notation $\bar{t}_i = \mathsf{f}_{k_i m}(\bar{t})$ and $1_t^i = \mathsf{f}_{k_i m}(1_t)$, and similarly for $\bar{x}_i, \bar{y}_i, \bar{z}_i$ and for $1_x^i, 1_y^i, 1_z^i$. To ensure existence of the time unit vector 1_t^∞ of the imaginary observer k_∞, we define the limit of a growing sequence like $\langle 1, 2, 3, \ldots, i, \ldots\rangle$ of members of F to be ∞. Further for the sake of (nice behavior e.g. convergence of) the unit vectors we assume $\mathbf{Ax(symm)}^\dagger$. However, at the same time we would like to emphasize, that for the present argument about $\perp_r$ we do *not* need the unit vectors, hence $\mathbf{Ax(symm)}^\dagger$ is not really needed here (we assumed it only for making our "picture prettier"). Then we can define

$$1_t^\infty := \lim_{i\mapsto\infty}(1_t^i), \quad 1_x^\infty := \lim_{i\mapsto\infty}(1_x^i), \quad 1_y^\infty := \lim_{i\mapsto\infty}(1_y^i), \quad 1_z^\infty := \lim_{i\mapsto\infty}(1_z^i).$$

[692]It is possible that we need sequences longer than ω for this limit to exist but that does not change anything essential.

[693]For a similar train of thought (or construction) cf. Figure 254 on p.749 and §5.1 (pp. 744–750).

[694]For simplicity we write L^{Ph} for $\{\, tr_m(ph) \,:\, ph \in Ph \,\}$, in the present remark.

This way we will obtain

$\bar{t}_\infty = \bar{x}_\infty \in L^{Ph}$, $\qquad \bar{y}_\infty = \bar{y}, \quad \bar{z}_\infty = \bar{z},$

$\qquad 1^\infty_y = 1_y, \quad 1^\infty_z = 1_z;$ further

$1^\infty_t = 1^\infty_x =$ "the infinitely long vector pointing in the photon-like direction $\bar{t}_\infty$".

More formally,

$1^\infty_t = 1^\infty_x = \langle \infty, \infty, 0, 0 \rangle$.

See Figure 267. As the figure shows our *imaginary observer* k_∞ has some exotic features. E.g. its space $Space_{k_\infty} \subseteq {}^n(F^\infty)$ is a Robb hyper-plane[695], i.e. $Space_{k_\infty}$ is a hyper-plane tangent to the light-cone. It contains $\bar{y}$, $\bar{z}$ and a photon-like line $\bar{t}_\infty = \bar{x}_\infty$. Though k_∞ is only an imaginary observer, studying its mathematical

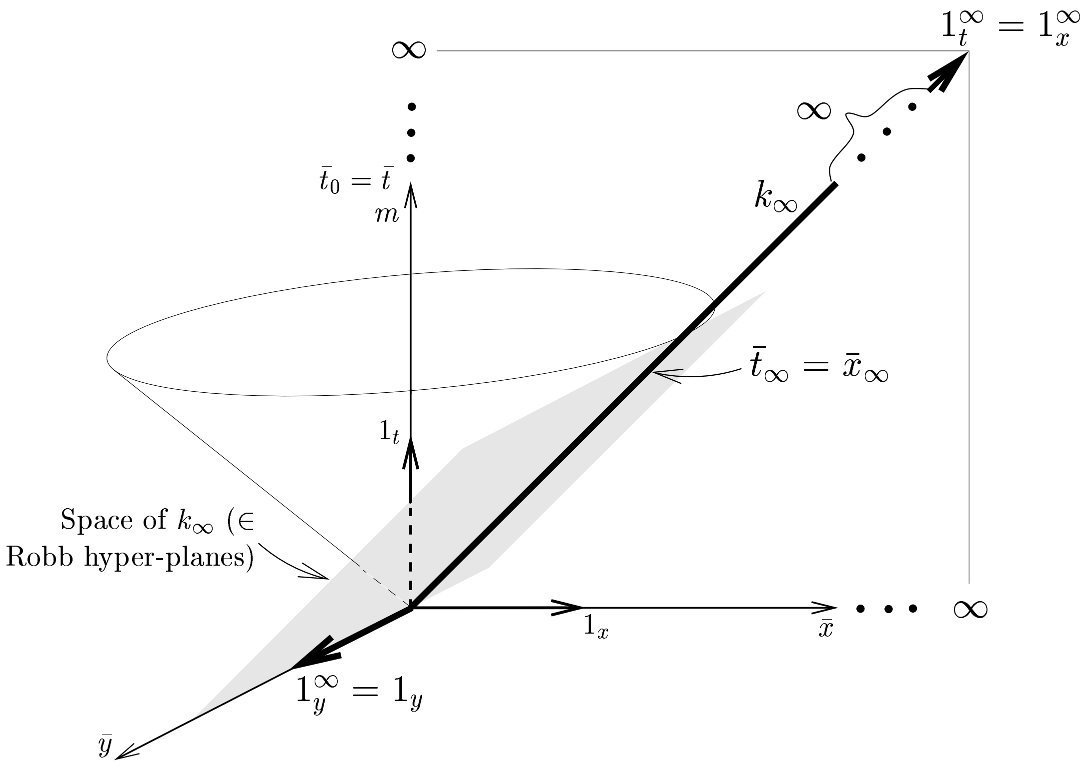


Figure 267:

structure can give us insight e.g. to the structure of $\mathfrak{G}_{\mathfrak{M}}$. k_∞ does not satisfy our axiom $\mathbf{Ax6_{00}}$, i.e. k_∞ does not see most of the events m sees, *but* k_∞ does see

[695] Cf. e.g. Robb [225] or Goldblatt [108] for the Robb hyper-plane (called in [108] Robb threefold) cf. also p.1163 herein.

the events on $\mathsf{Plane}(\bar{y}, \bar{z})$ since 1^∞_y and 1^∞_z are finite (and agree with 1_y and 1_z, respectively). In the direction $\langle 1, 1, 0, 0\rangle$ however k_∞ is "blind": of the events on his life-line $\bar{t}_\infty = tr_m(k_\infty)$ he sees only the event at the origin $\bar{0}$ because k_∞'s unit vectors (1^∞_t, 1^∞_x) in this direction[696] are too long.

Let us return to relativistic orthogonality $\perp_r$. Our k_∞ thinks that his axes $\bar{t}_\infty$, $\bar{x}_\infty$, $\bar{y}_\infty$, $\bar{z}_\infty$ are orthogonal, therefore according to our philosophy for defining $\perp_r$ it is natural that we wish to have

$$\bar{t}_\infty = \bar{x}_\infty \perp_r \bar{t}_\infty \perp_r \bar{y} \perp_r \bar{z} \qquad \text{etc.}$$

How could we achieve this (e.g. $\bar{t}_\infty \perp_r \bar{y}$) in a natural way? Well, k_∞ was obtained from real observers k_i by taking a limit. Parts of k_∞ (e.g. the coordinate axes of k_∞) were also obtained by the same limit procedure. Therefore, all this suggests that we should close our relativistic orthogonality $\perp_0$ up under taking limits and then probably this will yield for us those orthogonal pairs (like $\bar{t}_\infty \perp_r \bar{t}_\infty$, $\bar{t}_\infty \perp_r \bar{y}$ etc.) which are coordinate axes of imaginary observers which in turn were obtained by a limit procedure analogous to the one with which we obtained k_∞.

In passing, we also note the following. k_∞ thinks that his time axis $\bar{t}_\infty$ is orthogonal to his space, $Space_{k_\infty}$, which in turn is the hyper-plane generated by $\{\bar{x}_\infty = \bar{t}_\infty, \bar{y}, \bar{z}\}$. Hence k_∞ will think that $\bar{t}_\infty$ is orthogonal to *any* line in this hyper-plane.[697] Thus, any photon-like line in a Robb hyper-plane is expected to be $\perp_r$-orthogonal to all lines in that hyper-plane.

Summing up, on a very-very informal level we could say the following. Of course speed-of-light observers cannot exist. But *if* they existed they would behave like k_∞ does.[698] In claiming this we are relying on the "rule of thumb" that in physics everything is continuous (i.e. is preserved under taking limits). We emphasize that the above train of thought is *not* a precise mathematical argument, and it should not be taken too seriously[699], it only serves to help the intuition about some parts of $\mathfrak{G}_\mathfrak{M}$ (especially about $\perp_r$).

Concerning the above intuitive remark we also refer to Goldblatt [108] in the middle of page 13 for an analogous argument.

$\lhd$

Besides discussing definability issues and alternative definitions, the next subsection can also serve to improve our intuitive understanding of certain parts of $\mathfrak{G}_\mathfrak{M}$.

[696] i.e. in the direction of $\langle 1, 1, 0, 0\rangle$

[697] Cf. Proposition 6.2.51 (p.856).

[698] Practically the same argument is found in Goldblatt [108] p.8 lines 4–7.

[699] e.g. it uses "rules of thumb" which are *not* axioms in our theories.

6.2.2 On first-order definability of observer-independent geometry over observational concepts; and alternative definitions for $\perp_r$, eq, $\mathcal{T}$

The parts of our observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ can be considered as "theoretical" concepts as opposed to the parts of $\mathfrak{M}$ which in turn can be considered as "observational". Here we use the observational/theoretical distinction as introduced and discussed e.g. in Friedman [90]. The observational/theoretical distinction is known to be relative, hence we are aware of the fact that someone might challenge the observational status of $\mathfrak{M}$, but let us consider observational-ness of $\mathfrak{M}$ as a working hypothesis only. There is a long tradition (going back e.g. to Mach, Carnap) in theoretical physics where people try to restrict attention to such theoretical concepts which are *definable* in terms of observational ones[700], cf. the introduction to the present chapter (§6.1, p.774), cf. also e.g. Friedman [90]. This (among other things) motivates our asking ourselves[701] whether parts of $\mathfrak{G}_{\mathfrak{M}}$ are definable in first-order logic over $\mathfrak{M}$, and more generally whether $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$. Indeed, e.g. in Theorem 6.2.44 (p.847) we will see results in the direction that $\mathsf{Ge}(Th)$ is first-order definable over $\mathsf{Mod}(Th)$, under mild assumptions. Of course, we begin studying definability of $\mathfrak{G}_{\mathfrak{M}}$ (over $\mathfrak{M}$) by discussing definability of parts of $\mathfrak{G}_{\mathfrak{M}}$ over $\mathfrak{M}$. In passing, we also note that the above sketched ideas serve as part of the motivation for our section 6.3 on definability (and for our concern for definability issues throughout the present §6).

We will use the notion of (first-order logic) ***definability*** of a new structure say $\mathfrak{N}^+$ in[702] an "old" structure, say, $\mathfrak{N}$. Here $\mathfrak{N}^+$ is an expansion of $\mathfrak{N}$ possibly both with new sorts and new relations. Intuitively, $\mathfrak{N}^+$ looks like

$$\mathfrak{N}^+ = \langle \mathfrak{N}, U_1^{new}, \ldots, U_j^{new};\ R_1^{new}, \ldots, R_l^{new} \rangle$$

where U_i^{new} are new sorts and R_i^{new} are new relations. Such a definition of $\mathfrak{N}^+$ in $\mathfrak{N}$ induces an ***interpretation*** of the language $Fm(\mathfrak{N}^+)$ of $\mathfrak{N}^+$ in the language $Fm(\mathfrak{N})$ of $\mathfrak{N}$,[703] like

$$\text{intrp} : Fm(\mathfrak{N}^+) \longrightarrow Fm(\mathfrak{N}),$$

cf. Theorems 6.3.26 (p.962) and 6.3.27 (p.965) (in those theorems we will write "Tr" instead of "intrp"). In more detail, the basic concepts of "definability theory" (also

[700] in our opinion "definable" should almost always mean definable in the language of (many-sorted) first-order logic.
[701] cf. p.775
[702] "Definable in" means the same as "definable over".
[703] $Fm(\mathfrak{N})$ is the set of formulas in the "language" of $\mathfrak{N}$, cf. Convention 6.3.25 (p.962).

called "the theory of definability") elaborated for the case of many-sorted first-order logic (i.e. definability of a new sort) will be discussed in §6.3 beginning with p.928 way below. For even further details (for a level of generality somewhere in between the traditional one-sorted case and our present many-sorted case) we refer the interested reader to the logic textbook Hodges [136, Chapter 5] ("Interpretations"), but we note that §6.3 herein is sufficient for understanding the present work (i.e. the present work is intended to be self contained in connection with "definability theory").[704]

CONVENTION 6.2.7 By definability we automatically mean explicit definability throughout the present work, cf. §6.3.2. Similarly first-order logic definability also means explicit definability. The adjective "first-order logic" is there only to emphasize that our explicit definitions of new relations will be formulas of first-order logic as one would expect. Similarly "definition" means explicit definition in the sense of §6.3.2.

$\triangleleft$

Remark 6.2.8 (On first-order definability of $\mathfrak{G}_{\mathfrak{M}}$ in $\mathfrak{M}$.)
Let $\mathfrak{M}$ be a frame model and let $\mathfrak{G}_{\mathfrak{M}}$ be the observer-independent geometry corresponding to $\mathfrak{M}$ (defined in Def.6.2.2(I) above). We will look into the question whether the ingredients of $\mathfrak{G}_{\mathfrak{M}}$ are definable in $\mathfrak{M}$ using first-order logic (i.e. using our first-order frame language) or, more boldly, whether $\mathfrak{G}_{\mathfrak{M}}$ is first-order definable in $\mathfrak{M}$. In the present remark, for simplicity, instead of $\mathfrak{M}$ we will use its expansion

$$\mathfrak{M}^+ := \langle \mathfrak{M}, Mn, L;\ \in_{Mn}, \in_L \rangle$$

where events Mn and lines L are new sorts as defined in items 3, 5 of Def.6.2.2(I) and $\in_{Mn}, \in_L$ are the restrictions of the usual set theoretic membership relation $\in$ to $B \times Mn$ and $Mn \times L$, respectively. We will see in Proposition 6.3.18 (p.957) that

($\star$) $\mathfrak{M}^+$ is first-order definable[705] in $\mathfrak{M}$, moreover this definition is uniform for the whole class FM of frame models.

This justifies our decision of studying definability in $\mathfrak{M}^+$ instead of definability in $\mathfrak{M}$. Whenever below we say that something is definable in $\mathfrak{M}^+$ then by ($\star$) above this *automatically* means that that thing is *also* definable in $\mathfrak{M}$. So, we ask ourselves which parts (ingredients) of the geometry $\mathfrak{G}_{\mathfrak{M}}$ are first-order definable in

[704] We will return to defining (but only informally) interpretations on p.984 Fig.306 and on p.1023. Cf. also p.968 and footnote 936 (on p.968). [The above intrp : $Fm(\mathfrak{N}^+) \longrightarrow Fm(\mathfrak{N})$ is only a special kind of interpretations (involving only one of the many possible ways of using this concept).]

[705] I.e. $\mathfrak{M}^+$ is rigidly definable over $\mathfrak{M}$ in the sense of §6.3.2.

the expanded frame model $\mathfrak{M}^+$. Let us notice that the definitions of all ingredients of $\mathfrak{G}_{\mathfrak{M}}$, given in Def.6.2.2(I) above, <u>*are*</u> indeed first-order definitions in $\mathfrak{M}^+$ <u>*except*</u> for the relation $\perp_r$ of relativistic orthogonality, the relation eq of equidistance, and the topology $\mathcal{T}$. Therefore, it is sufficient to discuss here definability of $\perp_r$, eq and $\mathcal{T}$ in $\mathfrak{M}^+$. Let us turn to doing this.

On $\perp_r$: The relation $\perp_0$ <u>is</u> first-order definable. This gives us a promising start (for checking definability of $\perp_r$), but disappointingly, the definition of *relativistic orthogonality* $\perp_r$ (item 11 of Def.6.2.2.(I) on p.790) involves closing $\perp_0$ up under taking limits, then closing up under parallelism, and then iterating this two step procedure arbitrarily many times. Clearly this definition in its present form is not a first-order one. As we indicated we can translate the step of closing up under limits and the step of closing up under parallelism to our first-order frame language, cf. pp. 792, 1077, but it is not completely obvious how to translate iteration to first-order logic. (The iteration comes into the picture when we say that $\perp_r$ is the smallest set with certain properties [this happens above item (i) in the definition of $\perp_r$].)[706] In Definitions 6.2.9, 6.2.17 below we give three alternative definitions for $\perp_r$, which are (i) in the (first-order) language of $\mathfrak{M}^+$, and (ii) they are equivalent with the original definition of $\perp_r$, under some assumptions on $\mathfrak{M}$, like e.g. $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax6}$ (cf. Theorems 6.2.10, 6.2.19). Therefore $\perp_r$ *becomes first-order definable* in $\mathfrak{M}^+$, under some assumptions on $\mathfrak{M}$. Another use of exploring alternatives for $\perp_r$ (and proving equivalence) is that we obtain some insights into "how $\perp_r$ works".

On *eq*: The relation eq of equidistance (item 12 of Def.6.2.2.(I) on p.793) was defined to be the transitive closure of the relation eq_0 of basic equidistance, so it uses the set of natural numbers ω. Being a natural number is usually not first-order definable in $\mathfrak{M}^+$. Hence[707] the definition of eq is not a first-order definition in $\mathfrak{M}^+$. Let us recall that for every $i \in \omega$ eq_i was defined to be the "i-long-transitive closure" of eq_0. Let us notice that the definition of each one of our relations eq_i is indeed a first-order definition in $\mathfrak{M}^+$. In Theorems 6.2.22, 6.2.23 we will see that eq_2 coincides with eq under some assumptions on $\mathfrak{M}$, like e.g. $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\|) + \mathbf{Ax}(\sqrt{\ })$. Hence *the relation eq of equidistance*

[706] In theory it is possible that one could prove that the above mentioned iteration (of taking limits and parallels) terminates in a bounded finite number of steps, under certain assumptions. If that is the case then the original definition of $\perp_r$ will get translated to our first-order frame language. However we did not have time to think about this direction. Instead of pursuing this direction (i.e. checking whether iteration stops) we explore alternative definitions for relativistic orthogonality.

[707] Anyway, transitive closure is a typical example of (usually) not first-order definable concepts.

becomes first-order definable in $\mathfrak{M}^+$, under some assumptions on $\mathfrak{M}$.[708]

On $\mathcal{T}$: Let us recall that the topology $\mathcal{T}$ was defined from the subbase

$$T_0 = \left\{ S(e,\varepsilon) \,:\, e \in Mn,\ \varepsilon \in {}^+F \right\}.$$

We will see in Proposition 6.3.19 (p.959) that

($\star\star$) the subbase T_0 for $\mathcal{T}$ is first-order definable[709] in $\mathfrak{M}$, and that this definition is uniform for the whole class FM of frame models.

By ($\star\star$), we consider the topology $\mathcal{T}$ as first-order definable in $\mathfrak{M}$, however *we do not discuss* here which basic concepts of topology are first-order definable, e.g. we do not discuss whether the set of open subsets of $\mathcal{T}$ (i.e. $\mathcal{T}$ itself) is first-order definable.

($\star\star\star$) To be honest, we should call $\mathcal{T}$ first-order definable only if a *base*[710] , say T, for $\mathcal{T}$ is first-order definable. (This is so because then standard notions of topology like e.g. continuity would become expressible by using T which in turn is definable.) To pursue this direction we should investigate the question, under what conditions (axioms) does definability of a subbase T_0 imply definability of a base T. However, in the present work we do not want to investigate this direction. Therefore (perhaps slightly misleadingly) we call $\mathcal{T}$ definable if a subbase T_0 for $\mathcal{T}$ is definable. Investigating the question of under what assumptions is a base T for $\mathcal{T}$ definable (over FM) remains a task for future research. For a similar notion of explicit definability of a topology $\mathcal{T}$ we refer to the model theory book Barwise-Feferman [43, p.567, lines 5-8, §3.3 (Definability) of Chap.XV].

We will introduce alternative versions $\mathcal{T}'$ and $\mathcal{T}''$ for the definition of the topology part of our observer-independent geometry in Definition 6.2.31 (p.838). From the point of view of first-order definability over $\mathfrak{M}$, $\mathcal{T}''$ will behave just as nicely as $\mathcal{T}$ does (cf. Prop.6.3.20, p.960) while to ensure nice behavior of $\mathcal{T}'$ we will assume $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$ (cf. Prop.6.3.21, p.961).

As a corollary of ($\star$), ($\star\star$) above and Theorems 6.2.10, 6.2.19, 6.2.22, 6.2.23 below we obtain that under reasonably mild assumptions Th on our models $\mathfrak{M} \in \mathsf{Mod}(Th)$, the *geometry* $\mathfrak{G}_{\mathfrak{M}}$ is *definable in first-order logic* in the structure $\mathfrak{M}$. Moreover this

[708]If we make no assumptions then eq becomes undefinable in some frame models $\mathfrak{M}$, cf. Thm.6.2.24 (p.830).

[709]More precisely T_0 together with "$\in$-relation" acting between Mn and T_0 are first-order definable, where definable here means rigidly definable in the sense of §6.3.2.

[710]A set $T \subseteq \mathcal{T}$ is called a base for topology $\mathcal{T}$ iff each member (i.e. "open set") of $\mathcal{T}$ can be obtained as a (possibly infinite) union of sets from T.

definition is uniform for the whole class $\mathsf{Mod}(Th)$, see Theorems 6.3.22 (p.961) and 6.3.23, cf. also Theorems 6.3.24 and 6.2.44. Hence $\underline{\mathsf{Ge}(Th)\text{ is}}$ uniformly $\underline{\text{definable}}$ $\underline{\text{over }\mathsf{Mod}(Th)}$.

As we have already said, more on definability theory can be found in §6.3 way below.

$\triangleleft$

In Definition 6.2.9 below we give two alternative definitions $\perp_r'$ and $\perp_r''$ for the relation $\perp_r$ of relativistic orthogonality. The advantages of the definitions of $\perp_r'$ and $\perp_r''$ over the definition of $\perp_r$ will be that (i) they will be first-order definitions (in the expanded frame models $\mathfrak{M}^+$ defined in Remark 6.2.8, p.807) and (ii) they will be easier to understand. However, we consider the definitions of $\perp_r'$ and $\perp_r''$ less natural than that of $\perp_r$, because they (i.e. the definitions of $\perp_r'$ and $\perp_r''$) use case-distinctions, i.e. they distinguish photon-like lines from the rest of the lines, cf. items (iii) and (iv)' of Def.6.2.9 below. In Definition 6.2.17 (p.821) way below we give two further alternative definitions $\perp_r'''$ and $\perp_r^\omega$ (for relativistic orthogonality) which we consider just *as natural as* the definition of $\perp_r$ is. The definition of $\perp_r'''$ will be a first-order one (in the expanded frame models $\mathfrak{M}^+$). In Theorems 6.2.10, 6.2.18 and 6.2.19 below we will see that, under some assumptions, all versions of relativistic orthogonality, i.e. $\perp_r, \perp_r', \perp_r'', \perp_r''', \perp_r^\omega$, coincide, cf. Corollary 6.2.20. These theorems imply that $\perp_r$ is first-order definable (in the expanded frame models $\mathfrak{M}^+$ mentioned above), under certain conditions.

Definition 6.2.9 (Alternatives $\perp_r'$, $\perp_r''$ for relativistic orthogonality $\perp_r$) Let $\mathfrak{M}$ be a frame model. L, L^{Ph} and the relation of parallelism ($\|_{\mathfrak{G}}$) on L are defined in items 5, 10 of Def.6.2.2.(I) (pp. 787–790). We define $\perp_r' \subseteq L \times L$ and $\perp_r'' \subseteq L \times L$ as follows.

Intuitively, two lines are $\perp_r'$-orthogonal if they are parallel photon-like lines or there is an inertial observer who thinks that one of the lines coincides with a coordinate axis, call it $\bar{x}_i$, and the other line lies in the subspace determined by two (possibly coinciding) coordinate axes different from $\bar{x}_i$, see the left-hand side of Figure 268. Formally: Let $\ell, \ell' \in L$. Then

$$\ell \perp_r' \ell' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \text{(one of (i)–(iii) below holds)}, \quad \text{cf. Figure 268.}$$

In the formula in item (i) below, if $j = l$ then $\mathsf{Plane}(\bar{x}_j, \bar{x}_l)$ denotes the coordinate axis $\bar{x}_j$ (i.e. $\mathsf{Plane}(\bar{x}_j, \bar{x}_j) = \bar{x}_j$, as one would expect).

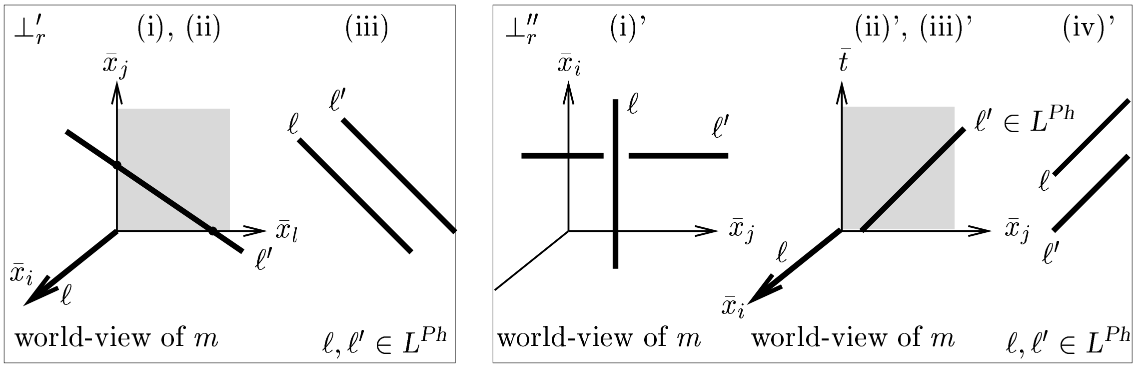


Figure 268: Illustration for the definitions of $\perp'_r$ and $\perp''_r$.

(i) $(\exists m \in Obs \cap Ib)(\exists i, j, l \in n)$
$$\Big(j \neq i \neq l \quad \wedge \quad \ell = w_m[\bar{x}_i] \quad \wedge \quad \ell' \subseteq w_m[\mathsf{Plane}(\bar{x}_j, \bar{x}_l)] \Big).$$

(ii) The same as (i) but with ℓ, ℓ' interchanged.

(iii) $(\ell, \ell' \in L^{Ph} \quad \wedge \quad \ell \parallel_{\mathfrak{G}} \ell')$.

Now we turn to defining $\perp''_r$. Intuitively, two lines are $\perp''_r$-orthogonal if they are parallel photon-like lines or there is an inertial observer, call it m, who thinks that the two lines are parallel with two different coordinate axes or m thinks that one of the lines coincides with a spatial coordinate axis, call it $\bar{x}_i$, and the other line is the trace of a photon and this photon moves in the (spatial) direction determined by a spatial coordinate axis different from $\bar{x}_i$, see the right-hand side of Figure 268. Formally: Let $\ell, \ell' \in L$.

$$\ell \perp''_r \ell' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \text{(one of (i)'–(iv)' below holds),} \quad \text{cf. Figure 268.}$$

(i)' $(\exists m \in Obs \cap Ib)(\exists i, j \in n)$
$$\Big(i \neq j \quad \wedge \quad w_m[\bar{x}_i], w_m[\bar{x}_j] \in L \text{ [711]} \quad \wedge \quad \ell \parallel_{\mathfrak{G}} w_m[\bar{x}_i] \quad \wedge \quad \ell' \parallel_{\mathfrak{G}} w_m[\bar{x}_j] \Big).$$

(ii)' $\ell' \in L^{Ph} \quad \wedge \quad (\exists m \in Obs \cap Ib)(\exists i, j \in n)$
$$\Big(0 \neq i \neq j \neq 0 \ \wedge \ \ell = w_m[\bar{x}_i] \ \wedge \ \ell' \subseteq w_m[\mathsf{Plane}(\bar{t}, \bar{x}_j)] \Big).$$

(iii)' The same as (ii)' but with ℓ, ℓ' interchanged.

[711] We note that, assuming $\mathbf{Ax4} + \mathbf{Ax6_{00}}$, $(\forall m \in Obs)(\forall i \in n)\ w_m[\bar{x}_i] \in L$.

(iv)' $(\ell, \ell' \in L^{Ph} \quad \wedge \quad \ell \parallel_{\mathfrak{G}} \ell')$.

$\lhd$

For stating the next theorem we introduce two new axioms. The first one is called axiom of disjoint windows (**Ax(diswind)**) formulated below. We note that there is a model of **Newbasax** in which **Ax(diswind)** fails (see Figure 307 on p.1001 or Figure 289 on p.887).

Ax(diswind) $(\forall m, k \in Obs \cap Ib)\,[(m \overset{\odot}{\to} ph \ \wedge\ k \overset{\odot}{\to} ph) \ \Rightarrow\ m \overset{\odot}{\to} k]$.

The intuitive meaning of **Ax(diswind)** is the following. In models of $\mathbf{Bax}^-$ the visibility relation $\overset{\odot}{\to}$ is an equivalence relation on the set of (inertial) observers,[712] cf. Theorem 4.3.11 (p.481) and the intuitive text above Theorem 3.3.12 (p.196). The "windows" correspond to the equivalence classes of $\overset{\odot}{\to}$. Now, **Ax(diswind)** says that there is no photon connecting the windows. Cf. also Figure 289 on p.887 for the intuitive idea of a window. Very roughly, one could say that the window of an observer m is that part of space-time which "unquestionably exists" for m.[713]

The second new axiom is the auxiliary axiom $\mathbf{Ax}(\mathit{Triv}_t)^-$ which is a weakened version of $\mathbf{Ax}(\mathit{Triv}_t)$ (p.135). The advantage of the axiom $\mathbf{Ax}(\mathit{Triv}_t)^-$ over $\mathbf{Ax}(\mathit{Triv}_t)$ is that $\mathbf{Ax}(\mathit{Triv}_t)^-$ can survive the transition from special relativity to general relativity, while $\mathbf{Ax}(\mathit{Triv}_t)$ might probably not survive this transition. Recall from p.135, that $\mathbf{Ax}(\mathit{Triv}_t)$ postulates the existence of certain very simple f_{mk} transformations *not involving motion* (or even changing the time axis $\bar{t}$). Recall that

$$\mathit{Triv} = \{\, f \ :\ f \text{ is an isometry of } {}^{n}\mathbf{F} \text{ and } f(1_t) - f(\bar{0}) = 1_t \,\}\,.$$

Our new, weaker axiom will say less than $\mathbf{Ax}(\mathit{Triv}_t)$, namely it will prescribe only what the required f_{mk}'s do with the spatial coordinate axes, i.e. what they do with *spatial directions* $\bar{x}, \bar{y}, \ldots$ but it will not prescribe what they do with e.g. the *lengths* of the unit vectors $1_t, 1_x, 1_y, \ldots$.

hosszútávon $\mathbf{Ax}(\mathit{Triv}_t)^-$-ba beletenni $\mathbf{Ax}(\sqrt{\ })$-et, vagy megnézni hogy tételekben $\mathbf{Ax}(\sqrt{\ })$ kell-e

$\mathbf{Ax}(\mathit{Triv}_t)^-$ $(\forall m \in Obs)(\forall f \in \mathit{Triv})\,[\, f[\bar{t}\,] = \bar{t} \quad \Rightarrow$
$(\exists k \in Obs)(\forall i \in n)\ (\,\mathsf{f}_{km}[\bar{x}_i] = f[\bar{x}_i] \quad \wedge \quad m \uparrow k\,)\,]$.

That is, assume we are given an observer m and a *Triv* transformation f that leaves the time-axis fixed. Then m has a brother, call it k, such that m thinks that (i) the coordinate axes of k are the f-images of the original coordinate axes $\bar{x}_i$, and (ii) the clock of k runs forwards.

[712]We note that in $\mathbf{Bax}^-$ all the observers are inertial ones.

[713]One could also say that a window is such a part of *Mn* which can be obtained in the form $\mathit{Rng}(w_m)$, for some m (i.e. which can be "coordinatized" by m).

Intuitive motivation explaining *why* we will need $\mathbf{Ax}(\mathit{Triv}_t)^-$ often can be found on p.848 (beginning of §6.2.3).

THEOREM 6.2.10 *$\perp_r$ coincides with both $\perp_r'$ and $\perp_r''$,*[714] *therefore $\perp_r$ is first-order definable*[715]*, assuming $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax(diswind)} + \mathbf{Ax}(\sqrt{\ })$. (The assumption is needed for all parts of the statement, e.g. for $\perp_r = \perp_r'$, of course.)*

We will give two proofs for this theorem (which appeal to slightly different "tastes").

First proof: Assume $n > 2$. A sketch of the idea of the proof is illustrated in Figure 269.

Let $\mathfrak{M}, \mathfrak{M}_1, \ldots, \mathfrak{M}_5$ be as in the figure. We start out with $\mathfrak{M}$ and by gradually changing it we arrive at $\mathfrak{M}_5$.[716] What is invariant during this process is that the

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_0, \perp_r, \perp_r', \perp_r'' \rangle$$

generalized reducts of geometries corresponding to all these models[717] are isomorphic (actually with the exception of $\mathfrak{M}_5$ these reducts are identical). It can be proved[718] that

$$\mathfrak{M}_5 \models \mathbf{Newbasax} + \mathbf{Compl} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}.$$

Therefore $\mathfrak{M}_5$ is a disjoint union[719] of models of $\mathbf{BaCo}+\mathbf{Ax}(\sqrt{\ })$, i.e. it is a disjoint union of Minkowski models.[720]

Since the above indicated "geometry reducts" of $\mathfrak{M}$ and $\mathfrak{M}_5$ are isomorphic, to prove the theorem, it is enough to prove its conclusion for Minkowski models. I.e. it

[714]Recall that $\perp_r$ is defined in Def.6.2.2 (p.790) and $\perp_r'$ and $\perp_r''$ are defined in Definition 6.2.9 above.

[715]we mean, definable over $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, of course. First one defines $\mathfrak{M}^+ = \langle \mathfrak{M}, Mn, L;\ \in \rangle$ over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then $\perp_r$ over $\mathfrak{M}^+$. (The point is that for defining $\perp_r$ first we need to have lines.)

[716]See Figure 269. Step $\mathfrak{M} \mapsto \mathfrak{M}_1$ goes exactly as step $\mathfrak{M} \mapsto \mathfrak{N}$ in the proof of item 6.2.89 on p.895. Step $\mathfrak{M}_1 \mapsto \mathfrak{M}_2$ goes as follows: Assume $\mathfrak{M}_1 = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle$. Let $m_0 \in Obs$ be arbitrary, but fixed. For every $k \in Obs$ let $\varphi_k \in Aut(\mathbf{F})$ be such that $\mathsf{f}_{m_0 k} = f \circ \widetilde{\varphi}_k$, for some $f \in Aftr$. Such φ_k's exist by Fact 4.7.7 (p.617). Let $W' := \{ \langle k, p, b \rangle \in Obs \times {}^nF \times B\ :\ W(k, \widetilde{\varphi}_k(p), b) \}$ and $\mathfrak{M}_2 := \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W' \rangle$. For the step $\mathfrak{M}_2 \mapsto \mathfrak{M}_3$ we refer the reader to Figures 96, 97 (p.324) and to the intuitive model construction on pp. 322–325. In step $\mathfrak{M}_3 \mapsto \mathfrak{M}_4$ we change the direction of flow of time for some observers so that $\mathbf{Ax}(\uparrow) + \mathbf{Ax5}^+$ becomes true, and in step $\mathfrak{M}_4 \mapsto \mathfrak{M}_5$ we throw away some bodies so that $\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}$ becomes true.

[717]$\mathfrak{M}, \ldots, \mathfrak{M}_5$

[718]by Thm.3.3.12 (p.196), by Prop.2.8.15 (p.136), by noticing that $\mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}_t)^- \models \mathbf{Ax}(\mathit{Triv}_t)$ and by Thm.2.8.17 (p.138)

[719]For disjoint unions of models cf. pp. 868-869.

[720]Cf. Def.3.8.42 (p.331) for Minkowski models.

$$\mathfrak{M} \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

$$\Big\downarrow \ \text{changing the units of measurement for time}$$

$$\mathfrak{M}_1 \models \boxed{\mathbf{Newbasax}} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

$$\Big\downarrow \ \text{getting rid of field automorphisms from the } \mathsf{f}_{mk}\text{'s}$$

$$\mathfrak{M}_2 \models \boxed{\mathsf{f}_{mk} \in \mathit{Aftr}} + \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

$$\Big\downarrow \ \text{changing the units of measurement}$$

$$\mathfrak{M}_3 \models \boxed{\mathbf{Ax(syt_0)}} + \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

$$\Big\downarrow \ \text{changing the direction of flow of time for some observers}$$

$$\mathfrak{M}_4 \models \mathbf{Ax(syt_0)} + \boxed{\mathbf{Ax}(\uparrow) + \mathbf{Ax5}^{+}} + \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

$$\Big\downarrow \ \text{throwing away superfluous bodies}$$

$$\mathfrak{M}_5 \models \mathbf{Ax(syt_0)} + \mathbf{Ax}(\uparrow) + \mathbf{Ax5}^{+} + \boxed{\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}} + \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

Figure 269: By gradually changing $\mathfrak{M}$ we arrive at $\mathfrak{M}_5$. For explanation cf. footnote 716.

is enough to prove that in Minkowski models $\perp_r$, $\perp_r'$, $\perp_r''$ coincide. We leave checking this to the reader; but we note that a generalized version of this will be proved as Claim 6.2.11 in the second proof.

Assume $n = 2$. Then we use the first part of Figure 269 involving $\mathfrak{M}, \ldots, \mathfrak{M}_3$. For this part the proof is the same as in the $n > 2$ case (e.g. we use the same geometry reduct). It is not hard to prove that $\mathfrak{M}_3$ is a disjoint union of models of $(\mathbf{Basax}+\mathbf{Ax(syt)}+\mathbf{Ax(}Triv_t\mathbf{)}+\mathbf{Ax}(\sqrt{\ }))$, cf. footnote 718. Then it is enough to prove the conclusion for $(\mathbf{Basax}+\mathbf{Ax(syt)}+\mathbf{Ax(}Triv_t\mathbf{)}+\mathbf{Ax}(\sqrt{\ }))$. Since $n = 2$ this is not too hard. We leave this step to the reader; but we note that a generalized version of this will be proved as Claim 6.2.11 in the second proof. ∎

Second proof: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}+\mathbf{Ax(}Triv_t\mathbf{)}^{-}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(diswind)})$. Let $\mathfrak{N}$ be a model of **Newbasax** obtained from $\mathfrak{M}$ by changing the units of measurement for time, i.e. $\mathfrak{N}$ is obtained from $\mathfrak{M}$ exactly the same way as in the proof of item 6.2.89 on p.896. (This corresponds to the first step in Figure 269.) Then, the generalized geometry reducts

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_0, \perp_r, \perp_r', \perp_r''\rangle$$

of $\mathfrak{M}$ and $\mathfrak{N}$ coincide. Further,

$$\mathfrak{N} \models \mathbf{Newbasax} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}.$$

Therefore $\mathfrak{N}$ is a photon-disjoint union[721] of models of $(\mathbf{Basax} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }))$. Since the above indicated "geometry reducts" of $\mathfrak{M}$ and $\mathfrak{N}$ coincide, to prove the theorem, it is enough to prove its conclusion for $(\mathbf{Basax} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }))$ models. This is proved as Claim 6.2.11 below.

Claim **6.2.11** Assume $\mathbf{Basax}+\mathbf{Ax(}Triv_t\mathbf{)}^{-}+\mathbf{Ax}(\sqrt{\ })$. Then $\perp_r$, $\perp_r'$, $\perp_r''$ coincide.

<u>*Proof:*</u> In the proof we will use Minkowskian orthogonality $\perp_\mu \subseteq \mathsf{Eucl} \times \mathsf{Eucl}$ which will be introduced in Def.6.2.58 (p.859). We will prove items (a)–(d) formulated below. Then, by (a)–(d) below, it is clear that $\perp_r$, $\perp_r'$, $\perp_r''$ coincide. Namely by (a) and (b), $\perp_r' = \perp_r'' \subseteq \perp_r$, and by (c) and (d), $\perp_r \subseteq \perp_r'$. For every $m \in Obs$ and $\ell \in L$, let

$$\ell_m :\stackrel{\text{def}}{=} w_m^{-1}[\ell].$$

Let $\ell, \ell' \in L$. Then (a)–(d) below hold. (The proofs of (a)–(d) will be given below (a)–(d).)

(a) $\quad \ell \perp_r'' \ell' \quad \Rightarrow \quad \ell \perp_r \ell'.$

[721]For disjoint and photon-disjoint unions of models cf. item 1 on p.868.

(b) $\ell \perp_r' \ell' \quad \Leftrightarrow \quad \ell \perp_r'' \ell'$.

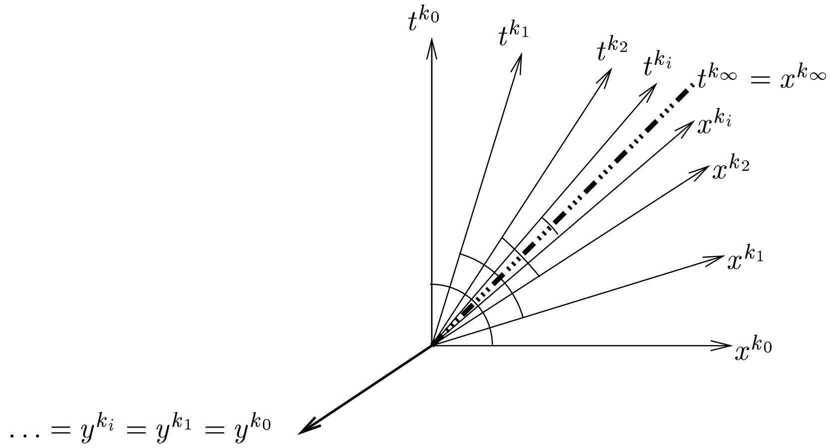
(c) $\ell \perp_r \ell' \quad \Rightarrow \quad (\forall m)\, \ell_m \perp_\mu \ell'_m$.

(d) $\ell \perp_r' \ell' \quad \Leftarrow \quad (\forall m)\, \ell_m \perp_\mu \ell'_m$.

Proof of (a): Assume $\ell \perp_r'' \ell'$. Then one of (i)'–(iv)' in the def. of $\perp_r''$ hold, cf. the right-hand side of Figure 268 (p.811). For the case (i)' it is clear that $\ell \perp_r \ell'$. So we can assume that ℓ and ℓ' are parallel photon-like lines or there is an observer who thinks that ℓ coincides with spatial coordinate axis $\bar{x}_i$ and ℓ' is a trace of a photon, and this photon moves along a spatial coordinate axis $\bar{x}_j$ different from $\bar{x}_i$, cf. the right-hand side of Figure 268. We note that at this point we are discussing cases (ii)'–(iv)'. To see that $\ell \perp_r \ell'$ we include the following intuitive explanation:

Intuitive explanation for the photon-like part of $\perp_r''$, i.e. to our saying among others that any two parallel photon-like lines are $\perp_r$-orthogonal:

Consider a sequence $k_0, k_1, k_2, \ldots, k_i, \ldots$ of faster and faster observers converging to the speed of light. Cf. Figure 267 on p.804 and Remark 6.2.6 on pp. 802–805.



Clearly $t^{k_0} \perp_0 x^{k_0}, \ldots, t^{k_i} \perp_0 x^{k_i}$ etc. So, the faster t^{k_i} moves the "closer" it gets to x^{k_i} while $t^{k_i} \perp_0 x^{k_i}$. Now, if we stretch our imagination and imagine, for a second only, that there is an observer k_∞ moving with the speed of light (which is impossible of course, but at the present point is an useful "metaphor") then we find $t^{k_\infty} = x^{k_\infty}$ and at the same time $t^{k_\infty} \perp_r x^{k_\infty}$, hence $t^{k_\infty} \perp_r t^{k_\infty}$.

This gives us intuitive motivation for declaring that photon-like lines are self orthogonal. Since $\perp_r$ was obtained from $\perp_0$ by closing up under taking limits and parallelism, this finishes the idea of proof for case (iv)'. When elaborating this proof one may need limits "longer" than ω. Similarly, we arrive at the further conclusion that $t^{k_\infty} \perp_r y^{k_\infty} = y$, yielding a proof for cases (ii)', (iii)'.

Proof of (b): Direction "$\Leftarrow$" can be easily checked by $\mathbf{Ax}(\mathit{Triv}_t)^-$+**Ax5** and by the fact that in **Basax** models "parallel" observers[722] agree on simultaneity and on Euclidean orthogonality. The proof of direction "$\Rightarrow$" uses $\mathbf{Ax}(\mathit{Triv}_t)^-$ and some basic properties of **Basax** (e.g. clocks orthogonal to movement do not get out of synchronism and remain orthogonal to movement). E.g. assume item (i) in def. of $\perp_r'$ holds for ℓ, ℓ'. Now, if $\ell' \in L^{Ph}$ then (ii)' in def. of $\perp_r''$ holds for ℓ, ℓ', while if $\ell' \in L^T \cup L^S$ then item (i)' holds for ℓ, ℓ', cf. Figure 268. The details are left to the reader.

Proof of (c): Item (c) follows from items 1–4 below. Item 1 follows by Thm.6.2.63 on p.866 (saying that in **Basax** models the f_{mk}'s preserve Minkowskian orthogonality) and by the fact that any two distinct coordinate axes are Minkowski-orthogonal (and from the def. of $\perp_0$); item 2 follows by items 1f and 2a of Prop.6.2.79 (p.884); item 3 follows by item 5a of Prop.6.2.79; and item 4 is easy to check by the def. of $\perp_\mu$.

1. $(\forall \ell, \ell' \in L)\, (\ell \perp_0 \ell' \;\Rightarrow\; (\forall m)\ell_m \perp_\mu \ell'_m)$.
2. Let $\alpha \in \mathsf{Ordinals}$, $\mathcal{S} \in {}^{\alpha}L$ and $\ell \in L$. Then

   $$(\mathcal{S} \text{ converges to } \ell) \quad \Leftrightarrow \quad (\forall m)(\langle\, \mathcal{S}(i)_m : i \in \alpha\,\rangle \text{ converges to } \ell_m \text{ w.r.t. } \mathsf{Betw}),$$

   cf. Figure 263 (p.792).
3. $(\forall \ell, \ell' \in L)\, (\, \ell \parallel_{\mathfrak{G}} \ell' \;\Leftrightarrow\; (\forall m)\ell_m \parallel \ell'_m\,)$.
4. Minkowskian orthogonality is "closed under taking limits and parallelism" (and $\perp_r$ was obtained from $\perp_0$ by closing up under these two).

Proof of (d): Assume that $(\forall m \in \mathit{Obs})\, \ell_m \perp_\mu \ell'_m$.

First, assume that both ℓ and ℓ' are photon-like lines. Then, by the def. of $\perp_\mu$, it can be checked that

$$(\forall m \in \mathit{Obs})\, (\ell_m, \ell'_m \in \mathsf{PhtEucl} \quad \text{and} \quad \ell_m \parallel \ell'_m).$$

But then $\ell \parallel_{\mathfrak{G}} \ell'$ by item 5a of Prop.6.2.79 (p.889). Hence $\ell \perp_r' \ell'$.

Now, assume that one of ℓ, ℓ' is space-like or time-like, e.g. assume that $\ell \in L^S \cup L^T$. Let $m \in \mathit{Obs}$ such that m sees ℓ on some coordinate axis, i.e. $\ell = w_m[\bar{x}_i]$, for some $i \in n$.[723] Fix this $\bar{x}_i$. Then $\bar{x}_i$ and ℓ'_m are Minkowski-orthogonal, i.e.

[722] Observers m and k are called parallel if their life-lines are parallel, i.e. $tr_m(k) \parallel \bar{t}$.

[723] Such an m exists since we assumed that ℓ is space-like or time-like.

$\bar{x}_i \perp_\mu \ell'_m$. By the definition of $\perp_\mu$, it can be checked that if a coordinate axis $\bar{x}_i$ is Minkowski-orthogonal to a straight line, then $\bar{x}_i$ is orthogonal to that straight line in the Euclidean sense, too. Hence, $\bar{x}_i \perp_e \ell'_m$. But then, by $\mathbf{Ax(}Triv_t\mathbf{)}^-$+$\mathbf{Ax5}$,[724] there is an observer m' such that $w_{m'}[\bar{x}_i] = w_m[\bar{x}_i] = \ell$ and item (i) in definition of $\perp'_r$ holds for m', ℓ, ℓ', i, cf. the left-hand side of Figure 268. So $\ell \perp'_r \ell'$. This completes the proof of Claim 6.2.11 and the proof of Theorem 6.2.10. ∎

Before defining the third and the fourth versions $\perp'''_r$ and $\perp^\omega_r$ of our relativistic orthogonality relation we need the definition of the "plane generated by a set of points $H \subseteq Mn$". To explain certain technicalities in this definition we include Proposition 6.2.14 below. To improve readability we will use the following abbreviations.

Notation 6.2.12 Let $\mathfrak{G}$ be a relativistic geometry.

(i) We define the binary relation $\sim$ of *connectedness* on points Mn as follows.[725] Let $e, e_1 \in Mn$. Then

$$e \sim e_1 \quad \stackrel{\text{def}}{\iff} \quad \Big(e = e_1 \ \vee \ (\exists e_2 \in Mn)\, Bw(e, e_1, e_2)\Big).$$

(ii) Let $a, b, c \in Mn$. Then

$$\begin{aligned} coll(a,b,c) \quad \stackrel{\text{def}}{\iff} \quad & \Big(a \sim b \sim c \sim a \quad \wedge \\ & (\, Bw(a,b,c) \ \vee \ Bw(a,c,b) \ \vee \ Bw(b,a,c) \ \vee \ a=b \ \vee \ b=c \ \vee \ a=c\,)\Big). \end{aligned}$$

"$coll(a,b,c)$" abbreviates "a, b, c are collinear".

Warning: The "real" collinearity relation of our relativistic geometries $\mathfrak{G}$ (to be denoted as Col) will be defined later by using the set L of lines and (e.g. in $\mathsf{Ge}(\mathbf{Bax}^-)$) it will not necessarily coincide with the recently defined $coll$. However, the two collinearity relations ($coll$ and Col) will coincide in the geometries of models of $\mathbf{Bax} + \mathbf{Ax(}Triv_t\mathbf{)}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. More generally, $coll \subseteq Col$, assuming $\mathbf{Pax} + \mathbf{Ax(diswind)}$, cf. Item 6.6.39 on p.1052.

◁

[724] and since "parallel" observers agree on simultaneity and Euclidean orthogonality

[725] We note that the present notion of connectedness is a completely different thing than the topological notion of connectedness.

Remark 6.2.13 Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$. Then two events $e, e_1 \in Mn_{\mathfrak{M}}$ are connected iff there is an observer who sees both of them. I.e.

$$e \sim e_1 \quad \Longleftrightarrow \quad (\exists m \in Obs)\, e, e_1 \in Rng(w_m).$$ [726]

$\lhd$

PROPOSITION 6.2.14 *Assume* $\mathbf{Bax}^-$. *Then* $(\forall a, b, c \in Mn)$

$$coll(a,b,c)$$
$$\Updownarrow$$
(*there is an observer who sees that events* a, b, c *are collinear*)[727]
$$\Updownarrow$$
(*each observer who sees events* a, b, c *"thinks" that they are collinear and some observer sees all of* a, b, c)[728].

Proof: The proposition follows by Thm.4.3.11 (p.481) (and by the definition of Bw). ∎

We note that the directions "$\Uparrow$" in the above proposition hold for any frame model (i.e. the assumption $\mathbf{Bax}^-$ is not needed for these directions).

In Def.6.2.15 below, the first definition we give for $Plane(H)$ is short, but is not in the first-order language of our geometry $\langle Mn_{\mathfrak{M}};\, Bw_{\mathfrak{M}}\rangle$. This is why, still in Def.6.2.15 we continue discussing alternative definitions for $Plane(H)$. A similar remark applies to Def.6.2.17 (the definition of $\perp_r^\omega$).

Definition 6.2.15 Let $\mathfrak{M}$ be a frame model. Let $\langle Mn;\, Bw\rangle \cong \langle Mn_{\mathfrak{M}};\, Bw_{\mathfrak{M}}\rangle$.[729]
Let $H \subseteq Mn$.

(i) By $Plane(H)$ we denote the "plane generated by H", i.e. $Plane(H)$ is the smallest subset of Mn having properties 1 and 2 below.[730]

1. $H \subseteq Plane(H)$.
2. $(\, a, b \in Plane(H) \ \wedge\ coll(a,b,c)\,) \ \Rightarrow\ c \in Plane(H)$.

[726] This holds by the definitions of Bw and $\sim$.
[727] Formally: $(\exists m)\,[\, a,b,c \in Rng(w_m) \ \wedge\ (w_m^{-1}(a),\ w_m^{-1}(b),\ w_m^{-1}(c)$ are collinear$)\,]$.
[728] Formally: $(\forall m)\,[\, a,b,c \in Rng(w_m) \ \Rightarrow\ (w_m^{-1}(a),\ w_m^{-1}(b),\ w_m^{-1}(c)$ are collinear$\,)\,] \ \wedge\ (\exists m)\, a,b,c \in Rng(w_m)$.
[729] $Mn_{\mathfrak{M}}, Bw_{\mathfrak{M}}$ are defined in items 3, 8 of Definition 6.2.2.(I).
[730] What we denote by $Plane(H)$, is usually denoted as $Span(H)$, in the literature.

As we already said, the above definition of $Plane(H)$ is <u>not</u> formulated in first-order logic. (In passing we note, that actually, it is in second-order logic.) Next we prepare for making the definition of $Plane(H)$ a first-order logic one (under some mild assumptions). An equivalent definition for $Plane(H)$ is the following. First, for every $i \in \omega$ we define $Plane^i(H)$ as follows.

$$\begin{aligned} Plane^0(H) &\stackrel{\text{def}}{:=} H, \\ Plane^{i+1}(H) &\stackrel{\text{def}}{:=} \left\{ c \in Mn \,:\, (\exists a, b \in Plane^i(H))\ coll(a,b,c) \right\}. \end{aligned}$$

We note that

$$Plane^i(H) \subseteq Plane^{i+1}(H), \quad \text{for any } i \in \omega.$$

Now we observe, that

$$Plane(H) = \bigcup \left\{ Plane^i(H) \,:\, i \in \omega \right\}.^{731}$$

(ii) Below we introduce the "first-order version" $Plane'(H)$ of $Plane(H)$ which will be defined in the first-order language of the structure $\langle Mn;\, H, Bw\rangle$. Let us recall that $n > 1$ is the dimension of our space-time. We define

$$Plane'(H) \stackrel{\text{def}}{:=} Plane^n(H).$$

We note that $Plane'(H) = Plane(H)$, assuming $\mathfrak{M} \models \mathbf{Bax}^-$, cf. Prop.6.2.16.

(iii) We write $Plane(\ell_1, \ldots, \ell_i)$ for $Plane(\ell_1 \cup \ldots \cup \ell_i)$, where $\ell_1, \ldots, \ell_i \in L$.

$\triangleleft$

PROPOSITION 6.2.16 *Assume* $\mathbf{Bax}^-$. *Then* $Plane(H) = Plane'(H)$.

On the proof: A proof can be obtained by items 1g (p.884) and 3b (p.889) of Proposition 6.2.79 way below, cf. also Prop.6.2.14 (p.819). ∎

Now, we are ready for defining our third and fourth versions $\perp_r'''$ and $\perp_r^\omega$ of relativistic orthogonality.

[731] We leave the very simple proof to the reader.

Definition 6.2.17 (Alternatives $\perp_r'''$, $\perp_r^\omega$ for relativistic orthogonality $\perp_r$) Let $\mathfrak{M}$ be a frame model. Mn, L, $\in$, Bw, $\|_{\mathfrak{G}}$, and the basic relation $\perp_0 \subseteq L \times L$ of orthogonality are defined in Def.6.2.2.(I). In the present definition we define two alternatives $\perp_r^\omega$ and $\perp_r'''$ for the relativistic orthogonality. The definition of $\perp_r'''$ will be a first-order one over $\langle Mn, L; \in, Bw\rangle$[732] while that of $\perp_r^\omega$ will not be such.

(i) $\perp_r^\omega$ is defined to be the smallest subset of $L \times L$ having properties 1–4 below.

1. $\perp_0 \subseteq \perp_r^\omega$, i.e. $\ell \perp_0 \ell' \Rightarrow \ell \perp_r^\omega \ell'$.
2. $\perp_r^\omega$ is a symmetric relation, i.e. $\ell \perp_r^\omega \ell' \Rightarrow \ell' \perp_r^\omega \ell$. [733]
3. If lines ℓ, ℓ_1, ℓ_2 concur at point e, with $\ell_1 \neq \ell_2$ and ℓ is $\perp_r^\omega$-orthogonal to both ℓ_1 and ℓ_2, then ℓ is $\perp_r^\omega$-orthogonal to every line through e in the plane determined by ℓ_1 and ℓ_2, see Figure 270;[734] formally: Let $e \in Mn$ and $\ell, \ell_1, \ell_2, \ell' \in L$. Then

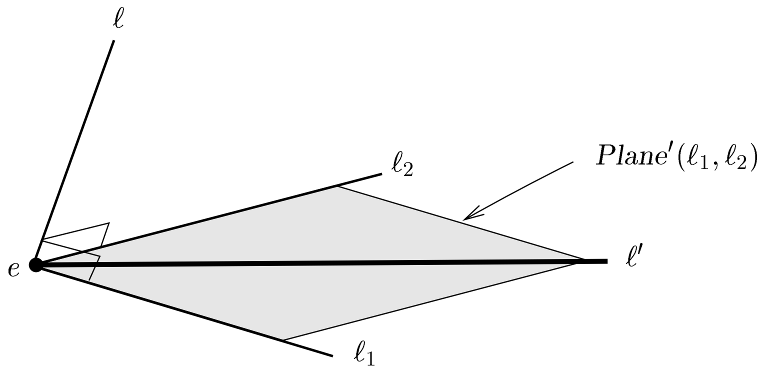


Figure 270: [735]

$$[\, e \in \ell \cap \ell_1 \cap \ell_2 \cap \ell' \quad \wedge \quad \ell_1 \neq \ell_2 \quad \wedge$$

$$\ell \perp_r^\omega \ell_1 \quad \wedge \quad \ell \perp_r^\omega \ell_2 \quad \wedge \quad \ell' \subseteq Plane'(\ell_1, \ell_2)\,] \qquad \Rightarrow \qquad \ell \perp_r^\omega \ell'.$$

In such situations we may also say that ℓ is $\perp_r^\omega$-orthogonal to the plane $Plane'(\ell_1, \ell_2)$.

[732] Recall that the relation of parallelism $\|_{\mathfrak{G}}$ was first-order defined over $\langle Mn, L; \in, Bw\rangle$.
[733] We note that in Goldblatt [108, p.115] this is an axiom for a metric affine space called OS1.
[734] We note that in Goldblatt [108, p.115] this is an axiom for a metric affine space called OS4.
[735] In Figure 270, the little "diamonds" around point e indicate that lines ℓ and ℓ_i are orthogonal, for $i \in \{1, 2\}$.

4. $\perp_r^\omega$ is closed under parallelism, i.e.

$$(\,\ell \perp_r^\omega \ell_1 \;\wedge\; \ell_1 \parallel_{\mathfrak{G}} \ell_2\,) \quad\Rightarrow\quad \ell \perp_r^\omega \ell_2.$$ [736]

Next we prepare for making the definition of $\perp_r^\omega$ a first-order logic one (under some assumptions). An equivalent definition for $\perp_r^\omega$ is the following. First, for every $i \in \omega$ we define $\perp_r^i \subseteq L \times L$ as follows. For easier readability, we note that the formulas ψ_2^i, ψ_3^i, ψ_4^i below correspond to "taking the closure of $\perp_r^i$ in one step[737]" to properties 2, 3, 4 above, respectively.

$$\begin{aligned}
\perp_r^0 &\stackrel{\text{def}}{:=} \perp_0,\\
\perp_r^{i+1} &\stackrel{\text{def}}{:=} \{\, \langle \ell, \ell' \rangle \in L \times L \;:\; \psi_2^i \vee \psi_3^i \vee \psi_4^i \,\}, \quad \text{where}\\
\psi_2^i &:= \ell' \perp_r^i \ell,\\
\psi_3^i &:= (\exists \ell_1, \ell_2)(\exists e)\,[e \in \ell \cap \ell_1 \cap \ell_2 \cap \ell' \;\wedge\; \ell_1 \neq \ell_2 \;\wedge\\
&\qquad \ell \perp_r^i \ell_1 \;\wedge\; \ell \perp_r^i \ell_2 \;\wedge\; \ell' \subseteq \mathit{Plane}'(\ell_1, \ell_2)\,],\\
\psi_4^i &:= (\exists \ell_1)\,(\ell \perp_r^i \ell_1 \;\wedge\; \ell_1 \parallel_{\mathfrak{G}} \ell'\,).
\end{aligned}$$

We note that $\perp_r^i \subseteq \perp_r^{i+1}$, for all $i \in \omega$.

Now we observe, that

$$\perp_r^\omega = \bigcup \{\, \perp_r^i \;:\; i \in \omega \,\}.$$ [738]

Let us notice that, for every $i \in \omega$, $\perp_r^i$ is a first-order definition in $\langle Mn, L; \in, Bw, \perp_0 \rangle$.

(ii) $\perp_r''' \stackrel{\text{def}}{:=} \perp_r^4$. So the definition of $\perp_r'''$ is a first-order definition in $\langle Mn, L; \in, Bw, \perp_0 \rangle$. Therefore the definition of $\perp_r'''$ is a first-order one in the expanded frame model $\mathfrak{M}^+$ defined in Remark 6.2.8 on p.807.

$\lhd$

THEOREM 6.2.18 *Assume* $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. *Then* $\perp_r^\omega = \perp_r'''$, *therefore* $\perp_r^\omega$ *is first-order definable*[739].

[736] This property is called axiom OS5 (for metric affine space) in Goldblatt [108, p.116].

[737] i.e. to "making one step only in the process of taking the closure".

[738] We leave the very simple proof to the reader.

[739] we mean, definable over $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, of course. First one defines L over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then $\perp_r$ over $\mathfrak{M}$ and L.

Proof: The proof for the case $n = 2$ is easy. Namely, if $n = 2$ then both $\perp_r^\omega$ and $\perp_r'''$ coincide with $\perp_0$. For the case $n > 2$, the theorem follows by the proof of Theorem 6.2.19 below. Namely, in the proof of Thm.6.2.19 we will prove (a) $\perp_r' \subseteq \perp_r'''$ and (b) 1–4 of Def.6.2.17 hold for $\perp_r'$, i.e. 1–4 hold when $\perp_r^\omega$ is replaced by $\perp_r'$ in them. Since $\perp_r''' \subseteq \perp_r^\omega$ and $\perp_r^\omega$ is the smallest subset of $L \times L$ having properties 1–4, (a) and (b) imply that $\perp_r' = \perp_r''' = \perp_r^\omega$. ∎

THEOREM 6.2.19 *Assume* $n > 2$ *and* $\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. *Then* $\perp_r$ *and* $\perp_r'''$ *coincide, therefore* $\perp_r$ *is first-order definable*[739].

The proof will be given below item 6.2.20.

The following is an immediate corollary of Theorems 6.2.10 (p.813), 6.2.18 and 6.2.19.

COROLLARY 6.2.20 Assume $n > 2$ and $\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$. Then $\perp_r$, $\perp_r'$, $\perp_r''$, $\perp_r'''$, $\perp_r^\omega$ coincide (and are definable[739]). ∎

Proof of Thm.6.2.19: Assume $n > 2$.

Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. By Thm.6.2.10 (which says that $\perp_r, \perp_r', \perp_r''$ coincide) it is enough to prove (a) and (b) below, as it was shown in the proof of Thm.6.2.18.

(a) $\perp_r' \subseteq \perp_r'''$, i.e. $\ell \perp_r' \ell' \;\Rightarrow\; \ell \perp_r''' \ell'$.

(b) $\perp_r'$ has the properties 1–4 in Def.6.2.17, i.e. 1–4 in Def.6.2.17 hold when $\perp_r^\omega$ is replaced with $\perp_r'$ in them.

Let $\mathfrak{N}$ be a model of **Newbasax** obtained from $\mathfrak{M}$ by changing the units of measurement for time, i.e. $\mathfrak{N}$ is obtained from $\mathfrak{M}$ exactly the same way as in the proof of item 6.2.89 on p.896. The generalized geometry reducts

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r', \perp_r''' \rangle$$

of $\mathfrak{M}$ and $\mathfrak{N}$ coincide. Further,

$$\mathfrak{N} \models \mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$$

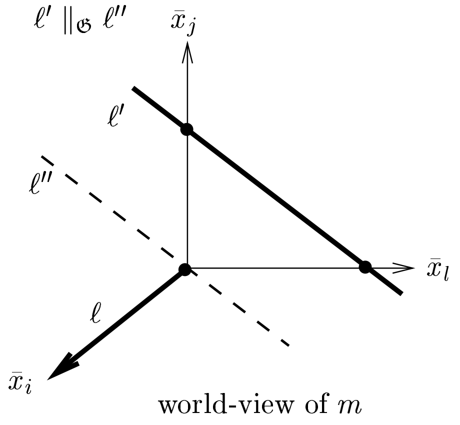
Therefore $\mathfrak{N}$ is a photon-disjoint union[740] of models of ($\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$). Since the above indicated "geometry reducts" of $\mathfrak{M}$ and $\mathfrak{N}$ coincide, it

[740]For disjoint and photon-disjoint union of models cf. item 1 on p.868.

is enough to prove (a) and (b) for ($\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$)-models. Now, we turn to proving these two statements.

<u>*Proof of (a):*</u> Recall that $\perp_r^i$ ($i \in \omega$) is the "i-long closure" of $\perp_0$ to properties 2, 3, 4 in Def.6.2.17 and that $\perp_r'''=\perp_r^4$, cf. Def.6.2.17 (p.821).

Assume $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$. Let $\ell, \ell' \in L$ be such that $\ell \perp_r' \ell'$, see the left-hand side of Figure 268 (p.811). Then one of (i)–(iii) in the definition of $\perp_r'$ on p.810 hold for ℓ, ℓ'. By $\mathbf{Ax}(\mathit{Triv}_t)^-$,[741] in cases (i) and (ii) $\ell \perp_r''' \ell'$ holds, cf. Figure 268. Actually, in case (i) $\ell \perp_r^2 \ell'$ and in case (ii) $\ell \perp_r^3 \ell'$, see Figure 271.



$\ell \perp_r^0 w_m[\bar{x}_l]$, $\ell \perp_r^0 w_m[\bar{x}_j]$ by 1 in Def.6.2.17

$\ell \perp_r^1 \ell''$ by 3 in Def.6.2.17 [742]

$\ell \perp_r^2 \ell'$ by 4 in Def.6.2.17 [743]

Figure 271: In case (i) $\ell \perp_r^2 \ell'$.

So it remains to prove (a) for case (iii).

Assume (iii) holds for ℓ, ℓ', i.e. $\ell, \ell' \in L^{Ph}$ and $\ell \parallel_{\mathfrak{G}} \ell'$, cf. Figure 268. To prove $\ell \perp_r''' \ell'$ it is enough to prove $\ell \perp_r^3 \ell$ because $\ell \perp_r^3 \ell$ and $\ell \parallel_{\mathfrak{G}} \ell'$ imply $\ell \perp_r^4 \ell'$, i.e. $\ell \perp_r''' \ell'$.

Now, we turn to proving $\ell \perp_r^3 \ell$. There is $m \in \mathit{Obs}$ and $ph \in \mathit{Ph}$ such that

$$w_m[tr_m(ph)] = \ell. \text{ [744]}$$

Fix such m and ph. Without loss of generality we can assume that $\bar{0} \in tr_m(ph) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ because of $\mathbf{Ax}(\mathit{Triv}_t)^-$ and $\mathbf{Ax5}$. Throughout the remaining part of the proof of (a) the reader is advised to consult Figure 272. We are in the world-view

[741] and some basic properties of **Basax**
[742] and by 5b of Prop.6.2.79
[743] and by 5a of Prop.6.2.79
[744] Cf. items 1c and 2a of Prop.6.2.79 (p.884).

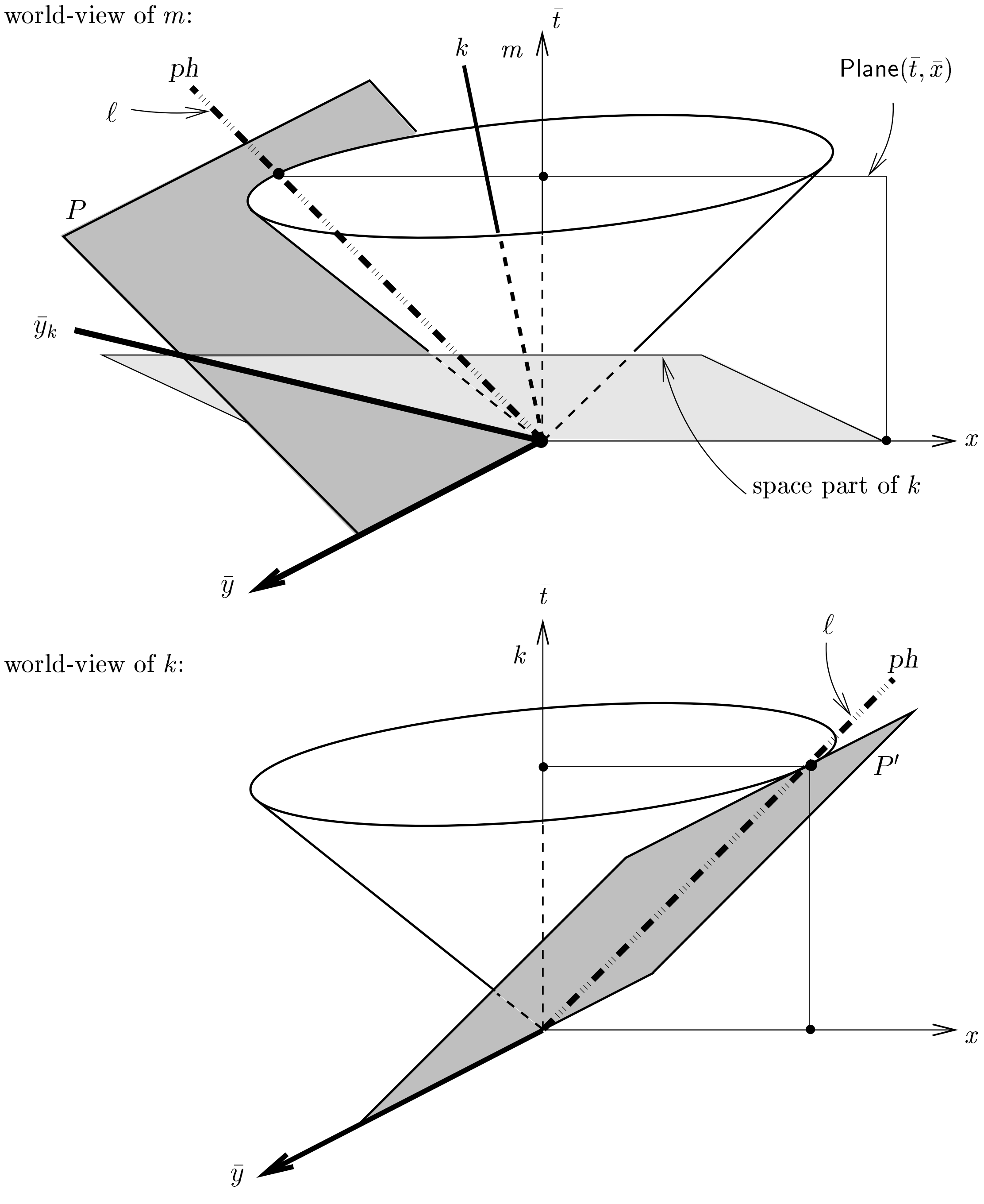


Figure 272: Illustration for the proof of Thm.6.2.19.