5 The “speed of light free” part of our axiomatic theories

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In earlier parts (e.g. §§ 3.4.2, 4.3–4.5) we started to remove the speed of light related axioms from our theories, **Basax** and **Newbasax**, obtaining **Bax**, **Reich**(**Bax**), $\mathbf{Bax}^-$ etc. Below we will remove the remaining “photon axioms” and will identify a fragment **Relnoph** of our theories which does not mention photons at all.[554] For simplicity, the present chapter is based on **Basax** (as opposed to e.g. **Newbasax**).

One of the motivations for introducing **Relnoph** comes from “conceptual analysis” i.e. from the desire to find out how far one can get in developing relativity theory without mentioning photons. Or, equivalently, we want to find out how strong the “photon-free principles” of our formalized relativity theories are.

The name **Relnoph** refers to “a variant of special relativity with no photons”.

Relnoph is intended to be a set of natural and convincing principles, such that despite of the fact that **Relnoph** does not mention photons, the interesting predictions of special relativity (like the twin paradox, or our paradigmatic effects) would be derivable from **Relnoph** (perhaps under some simple extra assumption).

In items 5.0.46, 5.0.48, 5.2.4, 5.2.5, 5.2.6, 5.2.8, 5.2.10, 5.2.14, 5.2.15, 5.2.16, way below, we will see that **Relnoph** can be considered as an adequate axiomatization of special relativity in the above outlined sense. To make this statement more tangible, we use the theory **Flxspecrel**, introduced in §4.1, as our formalized representative of “usual special relativity”.[555] Among others, we will see that, under some mild conditions, **Flxspecrel**+**Relnoph** becomes what is called in logic a “conservative extension” of **Relnoph**. I.e. all photon-free[556] theorems of **Flxspecrel** are derivable from (**Relnoph**+“some natural conditions”). We will see other (e.g. model-theoretical or semantical) results pointing in the direction that **Relnoph** is strong enough to “re-capture special relativity”. We will also indicate open possi-

[554] Acknowledgement: Gyula Dávid [71], (at Dept. Physics ELTE Univ.) developed a speed of light free approach to the special theory of relativity. It remains a task for future research to compare the one in the present chapter and the one developed by Gy. D. and eventually to combine them. Further; the present chapter incorporates ideas due to Gyula Dávid which we learned from him during discussions as well as during his lectures on the seminar of our department.

[555] At this point, we would like to note that **Flxspecrel** is a strong theory, e.g. it proves the twin paradox and all of our paradigmatic effects under assuming that the speed of light is not infinite.

[556] By a photon-free formula we understand one that does not involve the predicate symbol *Ph*.

bilities and promising directions in which the present "**Relnoph**-based" approach could be improved and in which new, interesting theorems could be found. (E.g. Lemma 5.0.59 could be used to prove stronger, more general results in the line of Thm.5.0.46 below.)

To illustrate that **Flxspecrel** is close to "standard special relativity" we note the following. If $\mathfrak{M} \models$ **Flxspecrel** + $(c < \infty)$ then all world-view transformations f_{mk} coming from $\mathfrak{M}$ are generalized *Poincaré transformations* in the sense of Definition 5.0.67 way below, cf. Prop.5.0.69.

On the structure of the present chapter: We start with the definition of **Relnoph** and after that we continue with some "warm-up exercises" (i.e. getting familiar with **Relnoph** and with subject matter of the present chapter). The first two main results (of §5) are Theorem 5.0.46 (p.711) and Conjecture 5.0.48 (p.712). The proof of Theorem 5.0.46 is relatively long, together with preparations it involves pp. 713–743. After the proof of Thm.5.0.46 in §5.1 "Compositions of transformations can be drawn" (pp. 744-750) we discuss a "side-issue" which comes up during the proof and which turns out to be very useful for obtaining insights to how our models work. Further main results come later, in §5.2 "Corollaries and further results" beginning with p.751. At the end of this chapter in §5.3 "Some further potential axioms" we discuss an axiom of continuity **Ax(cont)** which often shows up in relativity books (e.g. in Rindler [224]) as a tacit, implicit assumption.[557]

Let us turn to introducing **Relnoph**. We will build up the theory **Relnoph** in two stages. The first stage is $\mathbf{Relnoph_0}$ below.

$$\mathbf{Relnoph_0} \stackrel{\text{def}}{=} (\mathbf{Ax1}\text{–}\mathbf{Ax4})^{558}+\mathbf{Ax6}+\mathbf{Ax}\Box\mathbf{1}+\mathbf{Ax}\triangle\mathbf{1}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax}(\mathit{Triv})+\mathbf{Ax}(\|).$$

First we prove a few "warm-up exercises" like Propositions 5.0.34, 5.0.35, 5.0.38 and Theorem 5.0.39 below, and somewhat later, beginning with Theorem 5.0.46 (p.711) we turn to formulating our main results.

Ax(group⁻)-t kicserélni **Ax(group)**-ra. Régi **Ax(group)**-ot kidobni, és szólni Csabának!

PROPOSITION 5.0.34

(i) $\mathbf{Relnoph_0} \models \mathbf{Ax(group^+)}$, *hence* $\{\, \mathsf{f}_{mk} : m,k \in \mathit{Obs}\,\}$ *is a group w.r.t.* $\circ$ *etc. Moreover;*

[557] At this point we would like to remind the reader of the folklore-slogan often heard in physics classes "in physics all functions are continuous and differentiable".

[558] **Ax1**, **Ax2**, **Ax3**, **Ax4**. It is true that **Ax2** mentions *Ph* but this has no consequences in the present context. I.e. nothing changes if we remove *Ph* from **Ax2**.

(ii) $\{\mathbf{Ax6}, \mathbf{Ax}(\mathit{Triv}), \mathbf{Ax}\Box\mathbf{1}\} \models \mathbf{Ax(group^+)}$.

(iii) $\mathbf{Relnoph_0} \models v_m(k) = v_k(m)$.

(iv) $\mathbf{Relnoph_0} \models \mathbf{Ax}(\omega) + \mathbf{Ax(symm)}$.

Proof:
Proof of (i), (ii): Assume $\mathbf{Ax6}, \mathbf{Ax}(\mathit{Triv}), \mathbf{Ax}\Box\mathbf{1}$. By $\mathbf{Ax6}$ and $\mathbf{Ax}(\mathit{Triv})$, we have that

$$(\forall m, k \in \mathit{Obs})\ [\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF \text{ is a bijection}].$$

To prove $\mathbf{Ax(group)^+}$ let $m, k, m', k' \in \mathit{Obs}$. We want to prove that $\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'}$ is a world-view transformation f_{mh}, for some $h \in \mathit{Obs}$. Let $h \in \mathit{Obs}$ be such that $\mathsf{f}_{m'k'} = \mathsf{f}_{kh}$. Such an h exists by $\mathbf{Ax}\Box\mathbf{1}$. Now, $\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{mk} \circ \mathsf{f}_{kh} = \mathsf{f}_{mh}$, and this is what we wanted to prove.

szolni Csabának, hogy group fejeztben ezt az erosebb alakot mondja ki, es bizonyitsa

Proof of (iii): Item (iii) follows easily by $\mathbf{Ax}\triangle\mathbf{1}$+$\mathbf{Ax}(\|)$. We leave the details to the reader.

Proof of (iv): Assume $\mathbf{Relnoph_0}$. $\mathbf{Ax}\Box\mathbf{2}$ follows by $\mathbf{Ax}\Box\mathbf{1}$+$\mathbf{Ax}(\|)$, $\mathbf{Ax}\triangle\mathbf{2}$ follows by $\mathbf{Ax}\triangle\mathbf{1}$+$\mathbf{Ax}(\|)$. To prove $\mathbf{Ax(symm)}$, let $m, k \in \mathit{Obs}$. Let $k' \in \mathit{Obs}$ be such that $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$ and

$$tr_k(k') = \bar{t}.$$

Such a k' exists by $\mathbf{Ax}\triangle\mathbf{1}$. Thus $tr_m(k) = tr_{k'}(m)$. Let $m' \in \mathit{Obs}$ be such that

$$\mathsf{f}_{mk} = \mathsf{f}_{k'm'}.$$

Such an m' exists by $\mathbf{Ax}\Box\mathbf{1}$. Thus $tr_m(k) = tr_{k'}(m')$. This and $tr_m(k) = tr_{k'}(m)$ imply that

$$tr_m(m') = \bar{t}.$$

Therefore (by $\mathbf{Ax}(\|)$) $\mathbf{Ax(symm)}$ holds for m and k. ∎

Next we introduce a further axiom specific for the present chapter. In our earlier theories, existence of "slow observers" was ensured by $\mathbf{Ax5_{Obs}}$ which mentions photons. Therefore we have to replace $\mathbf{Ax5_{Obs}}$ with a variant of it which uses observers instead of photons. Roughly speaking the old axiom $\mathbf{Ax5_{Obs}}$ said that speeds slower than a speed of a photon are realized by some observers. In the new axiom $\mathbf{Ax(5nop)}$ we will say the same but with an observer in place of a photon.

$\mathbf{Ax(5nop)}$ $\forall m, k\ (\forall \lambda \in {}^+F)[\lambda < v_m(k)\ \Rightarrow\ \exists k'\ (v_m(k') = \lambda)]$.

The intuitive idea of $\mathbf{Ax(5nop)}$ is that if a certain speed is realized by some observer then the *smaller speeds* are also realized by some observers.

Beginning with item 5.2.20 way below, we will experiment with replacing **Ax(5nop)** with a weaker[559] version $\mathbf{Ax(5nop)^-}$.

PROPOSITION 5.0.35 $\mathbf{Relnoph_0}(2) \not\models \mathbf{Ax(5nop)}$.

Idea of proof: Assume $n = 2$. Start out with a model of **NewtK+Relnoph**. Restrict the set of possible speeds to a small subgroup of $\mathbf{F}$, e.g. keep only those observers whose speed is an integer. Then in the so obtained model $\mathbf{Relnoph_0}$ is true while **Ax(5nop)** fails. ∎

QUESTION 5.0.36 *Assume $n > 2$. Is then* $\mathbf{Relnoph_0} \models \mathbf{Ax(5nop)}$ *true?*

◁

Definition 5.0.37 We define,

$$\mathbf{Relnoph} \stackrel{\text{def}}{=} \mathbf{Relnoph_0} + \mathbf{Ax(5nop)}.$$

◁

PROPOSITION 5.0.38 *Assume* **Relnoph**. *Then*

$$\forall m' \, \forall \ell \, [ang^2(\ell) < v_m(k) \;\Rightarrow\; \exists k' \, (tr_{m'}(k') = \ell)].$$

Proof: The proof uses **Ax□1**, **Ax(5nop)** and $\mathbf{Ax(}\mathit{Triv}\mathbf{)}$. We use an idea which can be formulated intuitively by saying that **Ax□1** copies traces of observers from one world-view into another one (actually it can copy from any world-view into any other world-view). The details of the proof are left to the reader. ∎

THEOREM 5.0.39 *Assume* **Relnoph**. *Then the* f_{mk}*'s are affine transformations.*

Proof: Assume **Relnoph**. Assume $m, k \in Obs$ are such that $v_m(k) \neq 0$. If there are no such m, k then, by $\mathbf{Ax(\|)}$, we are done. Let $\lambda := v_m(k)$. By Prop.5.0.38, we have that $\forall m' \; \forall \ell \; [ang^2(\ell) < \lambda \;\; \Rightarrow \;\; \exists k' \; (tr_{m'}(k') = \ell)]$. Now, the proof of Thm.3.1.1, saying that in **Basax** the f_{mk}'s are bijective collineations, can be pushed through for the present case if we replace $\mathsf{SlowEucl}$ in the proof of Thm.3.1.1 by $\mathsf{SlowEucl}^\lambda := \{\ell \in \mathsf{Eucl} \,:\, ang^2(\ell) < \lambda\}$. By this we get that the f_{mk}'s are bijective collineations. Hence, by Lemma 3.1.6, every f_{mk} can be obtained as a composition of an affine transformation and a map $\widetilde{\varphi}$ induced by a field automorphism φ. By Prop.5.0.34(iv), **Ax△2** holds. Now, we use **Ax△2** and $\mathbf{Ax(}\sqrt{\;}\mathbf{)}$ to exclude the (nontrivial) field automorphisms from the f_{mk}'s, i.e. to prove that the f_{mk}'s are affine transformation. This can be done exactly as in the proof of Lemma 3.9.9, p.353.

∎

[559] To be precise $\mathbf{Ax(5nop)^-}$ is provable in $\mathbf{Relnoph} + (\exists m, k)\, v_m(k) \neq 0$.

QUESTION 5.0.40 *Does* **Relnoph$_0$** *imply that the* f_{mk}*'s are affine transformations?*

We note that in **Relnoph$_0$** *models if an* f_{mk} *is a collineation, then it is also an affine transformation, cf. the proof of Thm.5.0.39.*

$\triangleleft$

As we already said in the introduction of the present chapter we will use the theory **Flxspecrel** introduced in §4.1 as our formalized representative of "usual special relativity". In comparing models of **Relnoph** with models of **Flxspecrel**, on the *intuitive* level we think of **Relnoph** *as if* the predicate Ph was not present in its language. (Formally, Ph is present but we did not use it hence we can throw it away without causing any change in things which are essential.) This intuition motivates the following definition.

Definition 5.0.41 Throughout this definition $\mathfrak{N}$ and $\mathfrak{M}$ are frame models.

(i) By the <u>*photon-free reduct*</u> of $\mathfrak{M}$ we understand the model $\mathfrak{M}^-$ obtained from $\mathfrak{M}$ by omitting the relation $Ph^{\mathfrak{M}}$ and throwing away all the photons.

(ii) $\mathfrak{N}$ is a <u>*photon-free sub-reduct*</u> of $\mathfrak{M}$ if the following hold. Let $\mathfrak{N}^-$ and $\mathfrak{M}^-$ be the photon-free reducts of $\mathfrak{N}$ and $\mathfrak{M}$, respectively. Now, our condition says, that $\mathfrak{N}^-$ is a strong submodel of $\mathfrak{M}^-$, and in addition all the universes of $\mathfrak{N}^-$ are the same as those of $\mathfrak{M}^-$ except for universe of sort B.

(iii) <u>*Notation:*</u> Let Th, Th_1 be sets of formulas in our frame language. Then

$$\mathfrak{N} \models \text{“}Th\text{”} \quad \text{iff} \quad \mathfrak{N} \text{ is a photon-free sub-reduct of } \mathfrak{M}, \text{ for some } \mathfrak{M} \models Th;$$
$$Th_1 \models \text{“}Th\text{”} \quad \text{iff} \quad (\forall \mathfrak{M} \in \mathsf{Mod}(Th_1))\ \mathfrak{M} \models \text{“}Th\text{”}.$$

(iv) Let $\mathfrak{N} \subseteq \mathfrak{M}$. Then $\mathfrak{N}$ is called a <u>*B-submodel*</u> of $\mathfrak{M}$ iff all the universes of $\mathfrak{N}$ are the same as those of $\mathfrak{M}$ except for universe of sort B, i.e. $\mathfrak{F}^{\mathfrak{N}} = \mathfrak{F}^{\mathfrak{M}}$ and $G^{\mathfrak{N}} = G^{\mathfrak{M}}$ (and $Obs^{\mathfrak{N}} = Obs^{\mathfrak{M}} \cap B^{\mathfrak{N}}$, $Ph^{\mathfrak{N}} = Ph^{\mathfrak{M}} \cap B^{\mathfrak{N}}$).

$\triangleleft$

We will see that a mild, extra assumption like e.g. $v_m(k) \neq \infty$ or $(*1)$ or $(*2)$ or $(*3)$ below ensures that models of **Relnoph** become photon-free sub-reducts of models of **Flxspecrel**, if we assume that $\mathfrak{F}$ is Archimedean. This will be stated in the *first main result* of this chapter, Theorem 5.0.46. Further main results will be stated much later beginning with Corollary 5.2.3 on p.752 (in §5.2 "Corollaries and further results".)

For formulating principles $(*1)$, $(*2)$, $(*3)$ we will need the notion of strict standard configuration. Standard configuration was defined in Definition 2.3.16 (p.71). The notion of a strict standard configuration introduced below ensures that the two data $v_m(k)$ and the truth-value of $(m \uparrow k)$ completely determine the transformation f_{mk}, assuming **Relnoph**, cf. Theorem 5.3.1. In connection with the intuitive content of the formal part of the definition below, we note that it might be useful to read Remark 5.0.43 below.

Definition 5.0.42 Assume $m, k \in Obs$.

(i) Then m and k are defined to be in *strict standard configuration* if they are in standard configuration, m sees k moving forwards in direction 1_x, k sees m moving backwards in direction 1_x, further
$[v_m(k) = 0 \;\Rightarrow\; \mathsf{f}_{km}(1_t)_t \cdot \mathsf{f}_{km}(1_x)_x > 0]$ and
$[v_m(k) = \infty \;\Rightarrow\; (\mathsf{f}_{km}(1_t)_x > 0 \;\&\; \mathsf{f}_{km}(1_x)_t < 0)]$. Cf. Remark 5.0.43.

(ii) f_{mk} is called a *strict standard world-view transformation* iff m and k are in strict standard configuration. Cf. Figures 247, 248 (pp. 727, 731).

(iii) We say that the world-view transformation f_{mk} is *of nonzero speed* iff $v_m(k) \neq 0$.

$\lhd$

Remark 5.0.43 In connection with Definition 5.0.42 above we note the following. Assume $m, k \in Obs$.

(i) If $v_m(k) = 0$ or $[v_m(k) = \infty$ and $tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})]$ then m sees k moving both forwards and backwards in direction 1_x.

(ii) Assume **Relnoph**. Then

$$(v_m(k) = 0 \;\Rightarrow\; v_k(m) = 0) \quad \text{and} \quad (v_m(k) = \infty \;\Rightarrow\; v_k(m) = \infty)$$

by Proposition 5.0.34(iii). Therefore, if m and k are in pre-standard configuration we have that

$$\begin{aligned} v_m(k) = 0 \quad &\Rightarrow \quad (\mathsf{f}_{km}(1_t) \in \bar{t} \quad \& \quad \mathsf{f}_{km}(1_x) \in \bar{x})^{560} \quad \text{and} \\ v_m(k) = \infty \quad &\Rightarrow \quad (\mathsf{f}_{km}(1_t) \in \bar{x} \quad \& \quad \mathsf{f}_{km}(1_x) \in \bar{t}), \end{aligned}$$

cf. the transformation in Figure 248 (p.731) represented by $1_{t'''}$ and $1_{x'''}$.

$\lhd$

[560] $\mathsf{f}_{km}(1_x) \in \bar{x}$ holds by $\mathbf{Ax}(\|)$.

Next, we introduce some of our "mild assumptions" promised before, which will ensure that **Relnoph** $\models$ **"Flxspecrel"** under assuming "Archimedean-ness".

(∗1) The sum of finitely many small positive velocities is not infinite, formally: Let $j \in \omega$. Let $m_0, \ldots, m_j \in Obs$. Assume m_i, m_{i+1} are in strict standard configuration and $v_{m_i}(m_{i+1}) < 1$, for all $i < j$. Then $v_{m_0}(m_j) \neq \infty$.

(∗2) The sum of finitely many small positive velocities is nonnegative, formally: Let $j \in \omega$. Let $m_0, \ldots, m_j \in Obs$. Assume m_i, m_{i+1} are in strict standard configuration and $v_{m_i}(m_{i+1}) < 1$, for all $i < j$. Then m_0 sees m_j moving forwards in direction 1_x.

(∗3) The sum of finitely many small positive velocities is nonzero, formally: Let $0 < j \in \omega$. Let $m_0, \ldots, m_j \in Obs$. Assume m_i, m_{i+1} are in strict standard configuration and $0 < v_{m_i}(m_{i+1}) < 1$, for all $i < j$. Then $v_{m_0}(m_j) \neq 0$.

Remark 5.0.44 In the above principles (∗1)–(∗3), we formalized the intuitive idea of a small speed as a speed smaller than 1 *only* for simplicity: As we will see in later proofs in the present chapter, we could express a more sophisticated speed limit[561] (like the speed of light was in earlier sections) and we could use that speed limit in place of 1. This perhaps would make the naturalness of the above principles more convincing for some of our readers. But for simplicity we do not go into this here.

We also note that principles (∗1)–(∗3) are not single formulas but instead they are schemas of formulas (like e.g. the schema of induction in Peano's arithmetic).

◁

The following theorem shows that **Relnoph** + (∗1) + (∗2) + (∗3) is a consistent theory. Namely this theory is satisfied by the Minkowski models.

THEOREM 5.0.45 *Let $\mathfrak{F}$ be a Euclidean field. Recall the definition of the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ over $\mathfrak{F}$ from Def.3.8.42. Then*

$$\mathfrak{M}^M_{\mathfrak{F}} \models \textbf{Relnoph} + (*1) + (*2) + (*3).$$

[561]Using notation in Figure 249 this speed limit is $\tan(\alpha)/|\tan(\beta)|$.

Proof: The proof is a straightforward checking of the axioms. Therefore we leave it to the reader. For completeness we note that Prop.3.9.7 (p.352) can be used in proving the present theorem. ∎

Our next theorem points in the direction that **Relnoph** re-captures the essential part of "usual special relativity" (and it does this without mentioning anything like photons). Further results in the same direction will be stated in §5.2 "Corollaries and further results" way below.

We conjecture that instead of **Relnoph** the weaker theory $\mathbf{Relnoph} \setminus \{\mathbf{Ax\triangle 1}\}$ would be sufficient in the theorem below, cf. Conjecture 5.0.48.

THEOREM 5.0.46

(i) *Assume* $\mathfrak{N} \in \mathsf{Mod}(\mathbf{Relnoph} + (*1))$ *and that* $\mathfrak{F}^{\mathfrak{N}}$ *is Archimedean. Then*

$$\mathfrak{N} \models \text{“}\mathbf{Flxspecrel}\text{”}.$$

Moreover;

(ii) *If in addition* $(\exists m, k)\, v_m(k) \neq 0$ *is assumed then the photon-free reduct of* $\mathfrak{N}$ *is the photon-free reduct of some* **Flxspecrel** *model.*

(iii) *Both (i) and (ii) remain true if we replace* $(*1)$ *with any one of* $(*2)$, $(*3)$ *or* $v_m(k) \neq \infty$.

Before outlining an <u>*idea for proof*</u> we formulate a question.

QUESTION 5.0.47 *How much of the axioms in* **Relnoph** *are needed for proving the above theorem? We conjecture that not all are needed.*

In particular is $\mathbf{Ax\triangle 1}$ *needed?*

Does item (i) hold for $\mathbf{Relnoph_0}$ *in place of* **Relnoph***? If no, then what is the answer for* $n > 2$*?*

Assume $n > 2$*. Is then Thm.5.0.46 true for* $\mathbf{Relnoph_0}$ *in place of* **Relnoph***?*

◁

In connection with the above question we include the following conjecture. Cf. also items 5.2.17, 5.2.19.

Conjecture 5.0.48 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Relnoph} \setminus \{\mathbf{Ax}\triangle\mathbf{1}\} + (v_m(k) \neq \infty))$. *Assume* $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$. *Then we strongly conjecture that (i) and (ii) below hold.*

(i) $\mathfrak{M} \models$ **"Flxspecrel"**.

(ii) *If* $(\exists m, k)\, v_m(k) \neq 0$ *then the photon-free reduct of* $\mathfrak{M}$ *is the photon-free reduct of some* **Flxspecrel** *model.*

Possible proof: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Relnoph} \setminus \{\mathbf{Ax}\triangle\mathbf{1}\} + (v_m(k) \neq \infty))$. Assume $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$. Without loss of generality we can assume that $(\exists m, k)\, v_m(k) \neq 0$, since otherwise, by $\mathbf{Ax}(\|)$, the conclusion of the conjecture is true. Let

$$c := \sup\{\, v_m(k) \ :\ m, k \in Obs \,\}.\text{[562]} \tag{313}$$

Clearly, $c > 0$. Let

$$\mathsf{SlowEucl}^c := \{\, \ell \in \mathsf{Eucl} \ :\ ang^2(\ell) < c \,\}.$$

By Prop.5.0.38 (and (313)), we have that

$$(\forall m)(\forall \ell \in \mathsf{SlowEucl}^c)(\exists k)\ tr_m(k) = \ell. \tag{314}$$

By (314) (and **Ax6**), it can be proved that the f_{mk}'s are bijective collineations in $\mathfrak{M}$, cf. the proof of Thm.5.0.39. Therefore

$$(\forall m, k)\ \mathsf{f}_{mk} \in Aftr, \tag{315}$$

since $\mathfrak{R}$ has no nontrivial automorphisms. By (313)–(315), **Ax4** and $\mathfrak{M} \models v_m(k) \neq \infty$, it can be checked that

$$(\forall m, k)\ v_m(k) < c. \tag{316}$$

By (314) and (316), each f_{mk} induces a permutation on the set $\mathsf{SlowEucl}^c$ the natural way. By this and (315), we have

$$(\forall m, k)(\forall \ell)[ang^2(\ell) = c \ \Leftrightarrow\ ang^2(\mathsf{f}_{mk}[\ell]) = c]. \tag{317}$$

By (314), (316) and (317), we have that the photon-free reduct of $\mathfrak{M}$ is the photon-free reduct of an **Flxbasax** model $\mathfrak{N}$ in which the (square of the) speed of light is c. Let this $\mathfrak{N}$ be fixed. Now, to prove the conjecture it remains to see that $\mathfrak{N} \models$ **Flxspecrel**. By $\mathfrak{M} \models \mathbf{Relnoph} \setminus \{\mathbf{Ax}\triangle\mathbf{1}\}$, we have that $\mathfrak{N} \models \mathbf{Relnoph} \setminus \{\mathbf{Ax}\triangle\mathbf{1}\}$. Hence it remains to prove $\mathfrak{N} \models$ **Ax(symm)**. We will prove this by using one of the

[562] As usual $\sup H$ denotes the supremum (i.e. least upper bound) of the set H taken in the ordered set $\mathrm{R} \cup \{\infty\}$.

"median observer" proof methods from §3.9 as follows. Let $m, k \in Obs$. Let h be their median observer. The existence of such an h can proved by using Thm.3.8.25, $\mathfrak{N} \models$ **Flxbasax**+$\mathbf{Ax}(\sqrt{\ })$ and (316). Let $h' \in Obs$ be such that $\mathsf{f}_{hh'} = \sigma_{\bar{t}}$. Such an h' exists by $\mathbf{Ax}(Triv)$. Let $k', m' \in Obs$ be such that $\mathsf{f}_{mh} = \mathsf{f}_{k'h'}$ and $\mathsf{f}_{hk} = \mathsf{f}_{h'm'}$. Such m', k' exist by $\mathbf{Ax}\square\mathbf{1}$. Now, $\mathsf{f}_{mk} = \mathsf{f}_{mh} \circ \mathsf{f}_{hk} = \mathsf{f}_{k'h'} \circ \mathsf{f}_{h'm'} = \mathsf{f}_{k'm'}$. Further $tr_m(m') = tr_k(k') = \bar{t}$ by $\mathsf{f}_{hh'} = \sigma_{\bar{t}}$ and by the choices of m', k'. So $\mathfrak{N} \models \mathbf{Ax(symm_0)}$. **Ax(eqtime)** holds by $\mathbf{Ax}(\|)$. Hence $\mathfrak{N} \models$ **Ax(symm)**. This finishes the possible proof for Conjecture 5.0.48. It remains a future research task to check whether this proof is correct.

$\triangleleft$

We note that the possible proof given for Conjecture 5.0.48 is simpler than the one given for Theorem 5.0.46 below. A price we have to pay for this simplicity is that we have to assume that $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$. This illustrates that the fact that in the present work we do not assume $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ has far reaching consequences. In other words it is an essential feature of the present work that in our theorems we usually assume only that $\mathfrak{F}^{\mathfrak{M}}$ is an ordered field. If we assumed $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ the proofs would become much simpler but the theorems would become much weaker.

Items 5.0.46, 5.0.48 above belong to the main results of this section. Further "main" results and corollaries[563] will be stated beginning with p.751.

For *proving Thm.5.0.46* above we need some theorems, lemmas and definitions. These come below.

In connection with the theorem below we refer to Prop.4.6.7 (p.604) and the discussion preceding it.

THEOREM 5.0.49 *Strict standard configurations work in* **Relnoph**. *That is,* **Relnoph** $\models$

$$\forall m, k\ \exists m', k'\ (\mathsf{f}_{mm'}, \mathsf{f}_{kk'} \in Triv \text{ and } m', k' \text{ are in strict standard configuration}).$$

In the proof of Theorem 5.0.49 we will use Proposition 5.0.50 below. Hence the proof of Thm.5.0.49 comes below Prop.5.0.50.

PROPOSITION 5.0.50 *Pre-standard configurations work in* **Relnoph**.

Proof: This can be easily proved by using $\mathbf{Ax}(Triv)$ and Thm.5.0.39. The proof is left to the reader. ∎

Proof of Thm.5.0.49: Assume **Relnoph**. Throughout the proof we will tacitly use Thm.5.0.39, i.e. that in models of **Relnoph** all the f_{mk}'s are affine transformations.

[563] i.e. results of the same spirit as that of Thm.5.0.46.

Let $m, k \in Obs$ be fixed (for the duration of this proof). We want to prove that there are $m', k' \in Obs$ such that m', k' are in strict standard configuration and $\mathsf{f}_{mm'}, \mathsf{f}_{kk'} \in Triv$. By Proposition 5.0.50, without loss of generality we can assume that m and k are in pre-standard configuration.

Notation **5.0.51**

(i) For every $h \in Obs$ we let $\bar{t}_h$, $\bar{x}_h$, $\bar{y}_h$, $\bar{z}_h$, 1_t^h to be respectively $\mathsf{f}_{hm}[\bar{t}]$, $\mathsf{f}_{hm}[\bar{x}]$, $\mathsf{f}_{hm}[\bar{y}]$, $\mathsf{f}_{hm}[\bar{z}]$, $\mathsf{f}_{hm}(1_t)$. We would like to emphasize that although m is not explicitly indicated in $\bar{t}_h$, $\bar{x}_h$, etc. $\bar{t}_h$ does depend on the choice of m and therefore m is implicitly present in $\bar{t}_h$. We did not indicate m in $\bar{t}_h$ because m is fixed throughout the present proof.

(ii) For every $h \in Obs$ and $f \in Triv$ we define fh to be an observer with the properties $\bar{t}_{fh} = f[\bar{t}_h]$, $\bar{x}_{fh} = f[\bar{x}_h]$, $\bar{y}_{fh} = f[\bar{y}_h]$, $\bar{z}_{fh} = f[\bar{z}_h]$ and $1_t^{fh} = f(1_t^h)$. In Claim 5.0.52 below we will prove that such an observer exists. We note that there may be several observers with this property but that is not a problem because we let fh be an arbitrary but fixed one of these.

Claim **5.0.52** Assume $h \in Obs$ and $f \in Triv$. Then the observer fh, defined in Notation 5.0.51(ii), exists.

Proof: Assume $h \in Obs$ and $f \in Triv$. Let m_1 be a brother of m such that $\mathsf{f}_{mm_1} = f$. Such an m exists by $\mathbf{Ax}(Triv)$. Let $fh \in Obs$ be such that m sees fh so as m_1 sees h, i.e. $\mathsf{f}_{mfh} = \mathsf{f}_{m_1 h}$. Such an fh exists by $\mathbf{Ax}\square\mathbf{1}$, and has the desired properties.

QED (Claim 5.0.52)

To make the idea of the proof easier to see we first present the proof for the case $n = 3$. (The general proof will be a generalization of the 3-dimensional proof.)

Proof for $n = 3$: Assume $n = 3$.

Lemma **5.0.53** Assume $h, h' \in Obs$ are such that $\bar{t}_h = \bar{t}_{h'}$, $\bar{x}_h = \bar{x}_{h'}$. Then $\bar{y}_h = \bar{y}_{h'}$, cf. Figure 239.

Proof: Let $h, h' \in Obs$ such that $\bar{t}_h = \bar{t}_{h'}$ and $\bar{x}_h = \bar{x}_{h'}$. Let us switch over from the world-view of m to the world-view of h. By $\mathbf{Ax}(\|)$, we have that $\mathsf{f}_{hh'}$ is an isometry leaving $\bar{t}$ and $\bar{x}$ fixed.[564] But then $\mathsf{f}_{hh'}$ must leave $\bar{y}$ fixed too, since $\bar{0} \in \mathsf{f}_{hh'}[\bar{y}] \perp_e$

[564]Since by $\bar{t}_h = \bar{t}_{h'}$ and $\bar{x}_h = \bar{x}_{h'}$, we have $\bar{t} = \mathsf{f}_{mh}[\bar{t}_h] = \mathsf{f}_{mh}[\bar{t}_{h'}] = \mathsf{f}_{h'h}[\bar{t}]$ and similarly $\bar{x} = \mathsf{f}_{h'h}[\bar{x}]$.

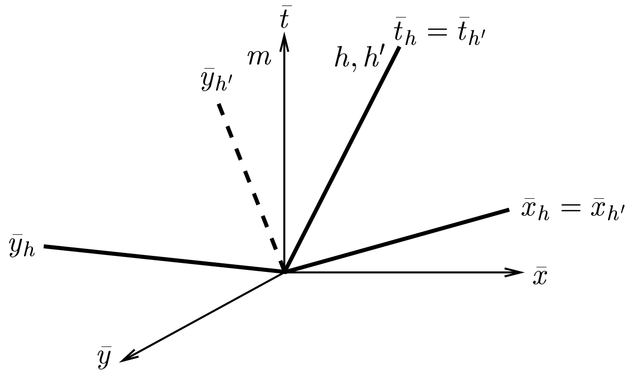


Figure 239: This cannot happen if $n = 3$, by Lemma 5.0.53.

$\mathsf{Plane}(\bar{t}, \bar{x})$.[565] Viewing this from the world-view of m we conclude that $\bar{y}_h = \bar{y}_{h'}$.

QED (Lemma 5.0.53)

Recall that m, k are in pre-standard configuration. Hence,

$$\bar{t}_k, \bar{x}_k \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \quad \text{and} \quad \bar{0} \in \bar{t}_k \cap \bar{x}_k \cap \bar{y}_k,$$

cf. Figure 240.

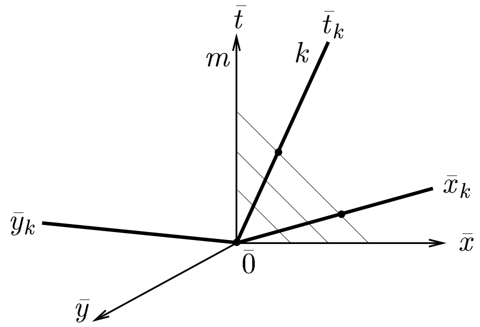


Figure 240: m and k are in pre-standard configuration.

Claim **5.0.54** $\bar{y}_k = \bar{y}$.

[565] $\mathsf{f}_{hh'}[\bar{y}] \perp_e \mathsf{Plane}(\bar{t}, \bar{x})$ holds because of the following: $\bar{y} \perp_e \mathsf{Plane}(\bar{t}, \bar{x})$, $\mathsf{f}_{hh'}$ leaves $\mathsf{Plane}(\bar{t}, \bar{x})$ fixed, and isometries preserve the relation $\perp_e$.

Proof: We will use Lemma 5.0.53 in the proof. Let σ denote the reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$, i.e. let σ denote the linear transformation that takes $1_t, 1_x, 1_y$ respectively to $1_t, 1_x, -1_y$. Since $n = 3$, we will see that it is enough to prove that σ leaves $\bar{y}_k$ fixed. Consider the observer σk, cf. Notation 5.0.51(ii). See Figure 241. k and σk

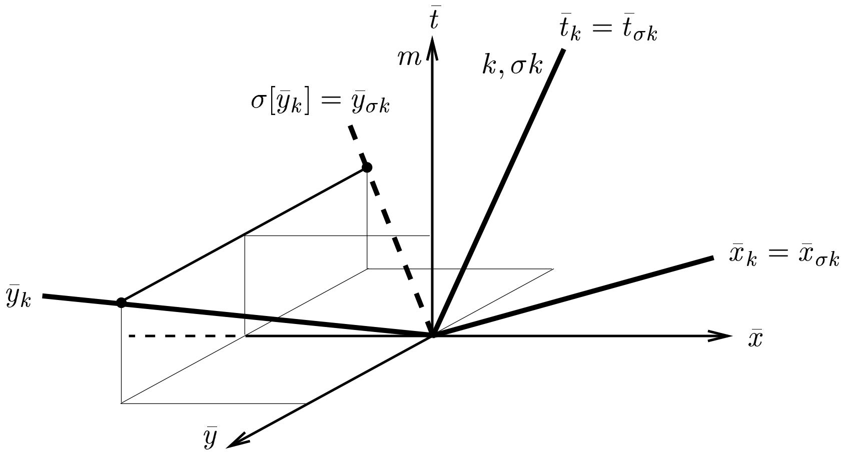


Figure 241: The observer σk.

satisfy the assumptions of Lemma 5.0.53, i.e. $\bar{t}_k = \bar{t}_{\sigma k}$ and $\bar{x}_k = \bar{x}_{\sigma k}$.[566] Applying Lemma 5.0.53 for k and σk, we get that $\bar{y}_k = \bar{y}_{\sigma k} = \sigma[\bar{y}_k]$ (cf. Notation 5.0.51). $\sigma[\bar{y}_k] = \bar{y}_k$ implies that $\bar{y}_k \perp_e \mathsf{Plane}(\bar{t}, \bar{x})$, since $\bar{y}_k \not\subseteq \mathsf{Plane}(\bar{t}_k, \bar{x}_k) = \mathsf{Plane}(\bar{t}, \bar{x})$. Therefore $\bar{y}_k = \bar{y}$ (since $\bar{0} \in \bar{y}_k$), cf. Figure 242.

QED (Claim 5.0.54)

Since m and k are in pre-standard configuration and $\bar{y}_k = \bar{y}$, we have that m and k are almost in standard configuration. Now, for proving the theorem (for $n = 3$) it is enough to improve m and k such that the unit vectors will point in the right direction required for strict standard configuration. This can be done by using **Ax(***Triv***)**, for the idea cf. the pictures on pp. 104, 112–115, 125–126. Hint: This part of the proof involves only such world-view transformations which are trivial and which leave all the coordinate-axes $\bar{x}_i$ together with 1_t fixed. Although this final part of the proof (for $n = 3$) can be easily done for completeness we include a detailed proof which unfortunately is somewhat computational. We think that most of the readers will find it easier to invent the proof on the basis of the above given

[566] This is so because of the following. By the definition of σk and by $\mathsf{Plane}(\bar{t}, \bar{x}) = \mathsf{Plane}(\bar{t}_k, \bar{x}_k)$, we have $\bar{t}_{\sigma k} = \sigma[\bar{t}_k] = t_k$ and $\bar{x}_{\sigma k} = \sigma[\bar{x}_k] = x_k$.

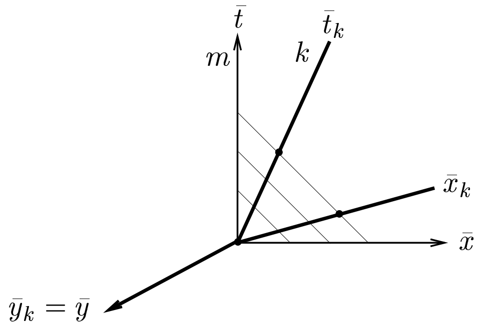


Figure 242: m and k' are almost in standard configuration.

hints as opposed to reading the details given below. For the reader who wants to skip the computational details below we note that the case of $n = 4$ will be treated immediately after the just mentioned details.

We distinguish two cases.

Case 1: Assume $v_m(k) \neq \infty$ and $v_m(k) \neq 0$. Let $i, j \in \{1, -1\}$ be such that

$$\begin{aligned} i = -1 \quad &\Leftrightarrow \quad m \text{ sees } k \text{ moving backwards in direction } 1_x, \quad \text{and} \\ j = -1 \quad &\Leftrightarrow \quad \mathsf{f}_{km}(1_y)_y < 0. \end{aligned}$$

Let m' be a brother of m such that $\mathsf{f}_{mm'} \in \mathit{Triv}_0$ and $\mathsf{f}_{mm'}$ takes $1_t, 1_x, 1_y$ respectively to $1_t, i \cdot 1_x, j \cdot 1_y$. Now, m', k satisfy all the conditions of strict standard configuration except that (318) below may fail.

$$k \text{ sees } m' \text{ moving backwards in direction } 1_x. \tag{318}$$

If (318) fails then let $k' \in \mathit{Obs}$ such that $\mathsf{f}_{kk'}$ is the reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{y})$[567], otherwise let $k' = k$. Now, m', k' are in strict standard configuration.

Case 2: Assume $v_m(k) = 0$ or $v_m(k) = \infty$. See Remark 5.0.43 (p.709). Let $i, j \in \{1, -1\}$ be such that

$$\begin{aligned} i = -1 \quad &\Leftrightarrow \quad \Big([v_m(k) = 0 \ \Rightarrow \ \mathsf{f}_{km}(1_t)_t \cdot \mathsf{f}_{km}(1_x)_x < 0] \ \& \\ &\qquad [v_m(k) = \infty \ \Rightarrow \ \mathsf{f}_{km}(1_t)_x < 0] \Big), \quad \text{and} \\ j = -1 \quad &\Leftrightarrow \quad \mathsf{f}_{km}(1_y)_y < 0. \end{aligned}$$

[567]I.e. $\mathsf{f}_{kk'}$ is a linear transformation taking $1_t, 1_x, 1_y$ respectively to $1_t, -1_x, 1_y$.

Let m' be a brother of m such that $\mathsf{f}_{mm'} \in \mathit{Triv}_0$ and $\mathsf{f}_{mm'}$ takes $1_t, 1_x, 1_y$ respectively to $1_t, i \cdot 1_x, j \cdot 1_y$. Now, by Remark 5.0.43, m', k satisfy all the conditions of strict standard configuration except that (319) below may fail.

(319) $$v_{m'}(k) = \infty \;\Rightarrow\; \mathsf{f}_{km'}(1_x)_t < 0.$$

If (319) fails then let $k' \in \mathit{Obs}$ such that $\mathsf{f}_{kk'}$ is the reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{y})$, otherwise let $k' = k$. Now, by Remark 5.0.43 it can be checked that m', k' are in strict standard configuration. Further $\mathsf{f}_{kk'}, \mathsf{f}_{mm'} \in \mathit{Triv}$. By this Thm.5.0.49 has been proved for $n = 3$.

<u>*Proof for $n = 4$:*</u> Assume $n = 4$.

Lemma 5.0.55 below is a generalization of Lemma 5.0.53 for $n = 4$.

Lemma **5.0.55** Assume $h, h' \in \mathit{Obs}$. Then (i) and (ii) below hold.

(i) Assume $\bar{t}_h = \bar{t}_{h'}$, $\bar{x}_h = \bar{x}_{h'}$, $\bar{y}_{h'} \subseteq (\bar{t}_h, \bar{x}_h, \bar{y}_h$-hyperplane). Then $\bar{y}_h = \bar{y}_{h'}$.

(ii) Assume $\bar{t}_h = \bar{t}_{h'}$, $\bar{x}_h = \bar{x}_{h'}$, $\bar{y}_h = \bar{y}_{h'}$. Then $\bar{z}_h = \bar{z}_{h'}$.

<u>*Proof:*</u> Assume $h, h' \in \mathit{Obs}$ and $\bar{t}_h = \bar{t}_{h'}$, $\bar{x}_h = \bar{x}_{h'}$ and $\bar{y}_{h'} \subseteq (\bar{t}_h, \bar{x}_h, \bar{y}_h$-hyperplane). Clearly h and h' are brothers. Let us switch over from the world-view of m to the world-view of h. Then, by $\mathbf{Ax}(\|)$, $\mathsf{f}_{hh'}$ is an isometry leaving $\bar{t}$ and $\bar{x}$ fixed. Hence $\mathsf{f}_{hh'}[\bar{y}] \perp_e \mathsf{Plane}(\bar{t}, \bar{x})$. Further $\bar{0} \in \mathsf{f}_{hh'}[\bar{y}] \subseteq (\bar{t}, \bar{x}, \bar{y}$-hyperplane).[568] Therefore $\mathsf{f}_{hh'}[\bar{y}] = \bar{y}$. Viewing this from the world-view of m we conclude that $\bar{y}_h = \bar{y}_{h'}$. Hence item (i) has been proved. We omit the proof of item (ii), but cf. the proof of Lemma 5.0.53.

QED (Lemma 5.0.55)

Since we assumed that m and k are in pre-standard configuration, we have that

(320) $$\bar{t}_k, \bar{x}_k \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \quad \text{and} \quad \bar{0} \in \bar{t}_k \cap \bar{x}_k \cap \bar{y}_k \cap \bar{z}_k.$$

Claim 5.0.56 below is the 4-dimensional generalization of Claim 5.0.54.

Claim **5.0.56** $\bar{y}_k, \bar{z}_k \subseteq \mathsf{Plane}(\bar{y}, \bar{z})$, cf. Figure 243.

<u>*Proof:*</u> In the proof we use Lemma 5.0.55 similarly an in the proof of Claim 5.0.54 we used Lemma 5.0.53.

Let σ denote the reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$, i.e. let σ denote the linear transformation that takes $1_t, 1_x, 1_y, 1_z$ respectively to $1_t, 1_x, -1_y, -1_z$. To prove the calim, we will see that it is enough to prove that σ leaves $\bar{y}_k$ and $\bar{z}_k$ fixed. Consider

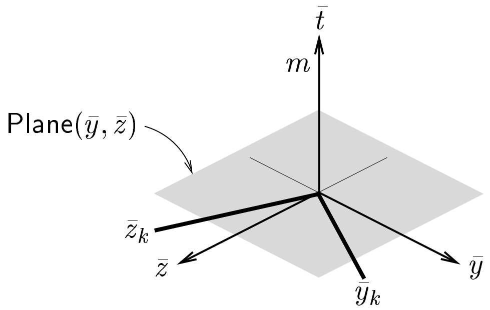


Figure 243: $\bar{y}_k, \bar{z}_k \subseteq \mathsf{Plane}(\bar{y},\bar{z})$.

The $\bar{t},\bar{x},\bar{y}_k$-hyperplane:

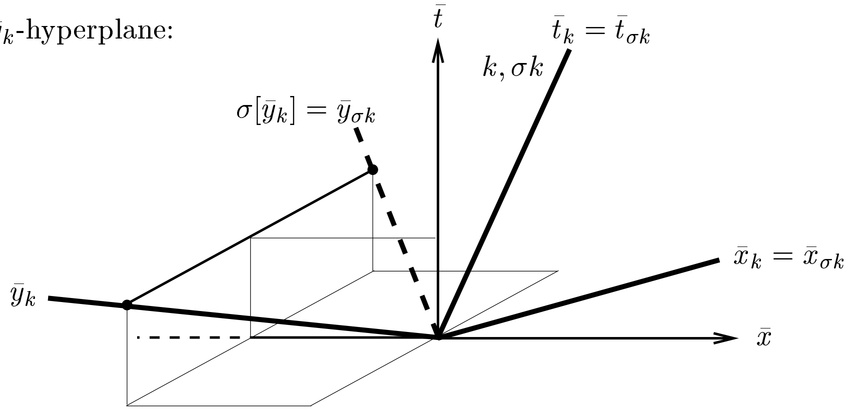


Figure 244: Consider the observer σk. Since $n = 4$, only the $\bar{t},\bar{x},\bar{y}_k$-hyperplane is represented in the picture.

the observer σk, cf. Notation 5.0.51(ii). See Figure 244. Now k and σk satisfy the assumptions of Lemma 5.0.55(i), i.e. $\bar{t}_{\sigma k} = \sigma[\bar{t}_k] = \bar{t}_k$, $\bar{x}_{\sigma k} = \sigma[\bar{x}_k] = \bar{x}_k$ and $\bar{y}_{\sigma k} = \sigma[\bar{y}_k] \subseteq (\bar{t}_k, \bar{x}_k, \bar{y}_k\text{-hyperplane})$[569]. Applying Lemma 5.0.55(i) for k and σk, we get that $\sigma(\bar{y}_k) = \bar{y}_{\sigma k} = \bar{y}_k$. Hence $\bar{y}_k \perp_e \mathsf{Plane}(\bar{t}, \bar{x})$, since $\bar{y}_k \not\subseteq \mathsf{Plane}(\bar{t}_k, \bar{x}_k) = \mathsf{Plane}(\bar{t}, \bar{x})$. This, by $\bar{0} \in \bar{y}_k$, implies that $\bar{y}_k \subseteq \mathsf{Plane}(\bar{y}, \bar{z})$. Cf. Figure 243. Similarly $\bar{z}_k \subseteq \mathsf{Plane}(\bar{y}, \bar{z})$.

QED (Claim 5.0.56)

Let $\varrho \in Triv$ be such that ϱ leaves $\mathsf{Plane}(\bar{t}, \bar{x})$ point-wise fixed and takes $\bar{y}_k$ to $\bar{y}$, cf. Figure 245. Such a ϱ exists by Claim 5.0.56. Let $k' := \varrho k$. See Figure 245. We

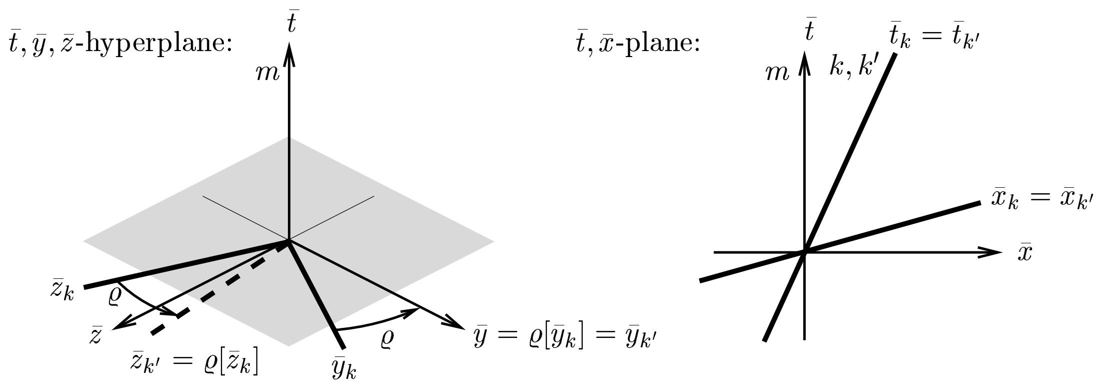


Figure 245: $k' := \varrho k$.

have that $\bar{t}_{k'} = \varrho[\bar{t}_k] = \bar{t}_k$, $\bar{x}_{k'} = \varrho[\bar{x}_k] = \bar{x}_k$, $\bar{y}_{k'} = \varrho[\bar{y}_k] = \bar{y}$, and $1_t^{k'} = \varrho(1_t^k) = 1_t^k$. By $\mathbf{Ax(\|)}$ and $1_t^k = 1_t^{k'}$, we have that

$$\mathsf{f}_{kk'} \in Triv. \tag{321}$$

So far, we have

$$\bar{t}_{k'} = \bar{t}_k, \quad \bar{x}_{k'} = \bar{x}_k, \quad \bar{y}_{k'} = \bar{y}. \tag{322}$$

We will prove that $\bar{z}_{k'} = \bar{z}$. The proof of this will be similar to the proof of $\bar{y}_k = \bar{y}$ for the case $n = 3$ (cf. Claim 5.0.54). Let σ be the reflection w.r.t. the $\bar{t}, \bar{x}, \bar{y}$-hyperplane, i.e. let σ be the linear transformation that takes $1_t, 1_x, 1_y, 1_z$ respectively to $1_t, 1_x, 1_y, -1_z$. Consider the observer $\sigma k'$. By (322), k' and $\sigma k'$ satisfy

[568] $\mathsf{f}_{hh'}[\bar{y}] \subseteq (\bar{t}, \bar{x}, \bar{y}\text{-hyperplane})$ because of the following: $\bar{y}_{h'} \subseteq (\bar{t}_h, \bar{x}_h, \bar{y}_h\text{-hyperplane})$, $\mathsf{f}_{hh'}[\bar{y}] = \mathsf{f}_{mh}[\bar{y}_{h'}]$, $\bar{t} = \mathsf{f}_{mh}[\bar{t}_h]$, $\bar{x} = \mathsf{f}_{mh}[\bar{x}_h]$, $\bar{y} = \mathsf{f}_{mh}[\bar{y}_h]$ and $\mathsf{f}_{mh} \in Aftr$ by Thm.5.0.39.

[569] since $\mathsf{Plane}(\bar{t}, \bar{x}) = \mathsf{Plane}(\bar{t}_k, \bar{x}_k)$ and σ is the reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$.

the assumptions of Lemma 5.0.55(ii) when h and h' are replaced by k' and $\sigma k'$. Applying Lemma 5.0.55(ii) for k' and $\sigma k'$ we get that $\sigma(\bar{z}_{k'}) = \bar{z}_{\sigma k'} = \bar{z}_{k'}$. By $\mathsf{Plane}(\bar{t}, \bar{x}) = \mathsf{Plane}(\bar{t}_k, \bar{y}_k)$ and (322), we have that $\bar{z}_{k'} \not\subseteq (\bar{t}, \bar{x}, \bar{y}$-hyperplane). By this, by $\sigma(\bar{z}_{k'}) = \bar{z}_{k'} \ni \bar{0}$, we conclude that

$$\bar{z}_{k'} = \bar{z}.$$

By this and (322), we have that m and k' are almost in standard configuration (cf. 320), further $\mathsf{f}_{kk'} \in \mathit{Triv}$ (cf. 321). Now, for proving the theorem it is enough to improve m and k' such that the unit vectors will point in the right direction required for strict standard configuration. This can be done exactly as it was done for the case $n = 3$. The details are left to the reader.

The proof for arbitrary $n > 4$ is analogous to the $n = 4$ case. Since we have a convention of not to worry to much about the $n > 4$ cases we do not discuss the details of this case here.

Thm.5.0.49 has been proved. ∎

We note, that at this point we are still in the phase of preparing ourselves for the proof of our first main result Theorem 5.0.46. In the proof of Thm.5.0.46 the key lemmas will be items 5.0.58, 5.0.59, and 5.0.64 below.

ide szerettem volna intuitív szöveget aról hogy mi következik a 2 lemmabol egyutesen.

To formulate Lemma 5.0.59, below we introduce a variant of our axiom $\mathbf{Ax(syt_0)}$.

$\mathbf{Ax(syt)^*}$ $\mathsf{f}_{mk}(\bar{0}) = \bar{0} \;\Rightarrow\; \mathsf{f}_{mk}(1_t)_t = \mathsf{f}_{km}(1_t)_t$.

> $\mathbf{Ax(syt)^*}$ says that when I see that your clock shows 1 is the same time when you see that my clock shows 1. Under mild assumptions this is equivalent with the following. What I see on your clock when my clock shows 1 is the same what you see on my clock when your clock shows 1. A difference between this axiom and $\mathbf{Ax(syt_0)}$ is that $\mathbf{Ax(syt_0)}$ permits me to see your clock running backwards while you see my clock running forwards. If we disregard the simplifying assumption $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ and if we assume that f_{mk} is an affine transformation then $\mathbf{Ax(syt)^*}$ is a stronger version of $\mathbf{Ax(syt_0)}$.

PROPOSITION 5.0.57 $\mathbf{Relnoph} \models \mathbf{Ax(syt)^*}$.

Proof: The proposition follows by $\mathbf{Ax\triangle 1}$+$\mathbf{Ax(\|)}$. We leave the details to the reader. ∎

We note that this implies $\mathbf{Relnoph} \models \mathbf{Ax(syt)}$, too, by $\mathbf{Ax(\mathit{Triv})}$, $\mathbf{Ax(\|)}$ and Thm.5.0.39.

Roughly, Lemma 5.0.59 below says the following, under assuming **Ax(syt)*** and **Ax4**. Assume m, k, h are pairwise in standard configuration, the world-view transformations between them are affine, their life-lines are different, no velocity involved is ∞ and we are in the world-view of m. Then the geometric data 1_t^k, $\bar{x}_k$, $tr_m(h)$ completely determine (in a purely geometrical way) the $\bar{x}$-axis $\bar{x}_h$ of h, moreover $\bar{x}_h$ can be constructed from these data as it is shown in Figure 246. (In some sense this means, that if we know one nontrivial world-view transformation f_{mk} then we have a "handle" on all the others like e.g. f_{mh}). A simpler corollary of Lemma 5.0.59 below is Corollary 5.0.58. Since the corollary is simpler we state the corollary before stating the lemma.

COROLLARY 5.0.58 Assume **Relnoph** and $m, k, h \in Obs$. Assume $v_m(k) \neq 0$ and that m, k, h are pairwise in standard configuration. Let $\bar{x}_k$ and $\bar{x}_h$ denote the $\bar{x}$ axes of k and h respectively as seen by m, i.e. $\bar{x}_k := \mathsf{f}_{km}[\bar{x}]$, $\bar{x}_h := \mathsf{f}_{hm}[\bar{x}]$. Further let 1_t^k denote the time unit vector of k as seen m, i.e. $1_t^k := \mathsf{f}_{km}(1_t)$. Then $\bar{x}_h$ is determined by the data 1_t^k, $\bar{x}_k$, $tr_m(h)$. Moreover these three data determine $\bar{x}_h$ independently of which model we are in. To understand this last statement imagine $\bar{x}_k$ and $tr_m(h)$ as two lines of the "geometry" ${}^n\mathbf{F}$ and 1_t^k as a point of ${}^n\mathbf{F}$. Notice that these three geometrical data are meaningful without fixing the model $\mathfrak{M}$. Now, the claim is if we take a brand new model say $\mathfrak{N}$ but with the same field reduct (and with 1_t^k, $\bar{x}_k$, $tr_m(h)$ the same as before) then $\bar{x}_h$ taken in $\mathfrak{N}$ will be the same as $\bar{x}_h$ in the original model $\mathfrak{M}$ which in turn is completely derermined by the geometric data 1_t^k, $\bar{x}_k$, $tr_m(h)$. I.e. there is a geometrically definable function which from the three geometrical data 1_t^k, $\bar{x}_k$, $tr_m(h)$ uniquelly determines the line $\bar{x}_h$.

Proof: The corollary follows by Lemma 5.0.59 below, **Ax(||)**, Proposition 5.0.34(iii) and Theorem 5.0.39.[570] ∎

LEMMA 5.0.59 *Assume $m, k, h \in Obs$ are such that $\bar{t} \neq tr_m(k) \neq tr_m(h) \neq \bar{t}$ (i.e. the life-lines of m, k, h are pairwise different), $v_m(h), v_k(h) \neq \infty$, and that $(\forall m', m'' \in \{m, k, h\})[m', m''$ are in standard configuration, $\mathsf{f}_{m'm''} \in Aftr$ and m', m'' satisfy both* **Ax(syt)*** *and* **Ax4**$]$. *Let $\bar{x}_k$ and $\bar{x}_h$ denote the $\bar{x}$ axes of k and h respectively as seen by m, i.e. $\bar{x}_k := \mathsf{f}_{km}[\bar{x}]$, $\bar{x}_h := \mathsf{f}_{hm}[\bar{x}]$. Further let 1_t^k denote the time unit vector of k as seen m, i.e. $1_t^k := \mathsf{f}_{km}(1_t)$.*

Then $\bar{x}_h$ can be constructed from[571] *the data 1_t^k, $\bar{x}_k$, $tr_m(h)$ as it is shown in Figure 246 below*[572]*, explained in more detail as follows. Let $p \in tr_m(h)$ such that*

[570] Namely, if $tr_m(h) = \bar{t}$ then $\bar{x}_h = \bar{x}$ by **Ax(||)**. Similarly, if $tr_m(h) = tr_m(k)$ then $\bar{x}_h = \bar{x}_k$. Further, if $v_m(h) = \infty$ then $\bar{x}_h = \bar{t}$ since $\underline{(v_m(h) = \infty \Rightarrow v_h(m) = \infty)}$ by Prop.5.0.34(iii). Similarly, if $v_k(h) = \infty$ then $\bar{x}_h = tr_m(k) = \overline{\bar{0}1_t^k}$.

[571] hence is uniquely determined by

[572] We did not indicate $\bar{t}, \bar{x}, 1_t$ as imput data because we consider them as naturally given things.

$\overline{1^k_t p} \parallel \bar{x}_k$. Such a p exists[573] *and $p \neq \bar{0}$.*[574] *Let $q \in \bar{t}$ such that $\overline{pq} \parallel \bar{x}$. Such a q exists*[575] *and $q \neq \bar{0}$.*[576] *Let $r \in tr_m(k)$ such that $\overline{qr} \parallel \overline{1_t 1^k_t}$. Then $\overline{1_t r} \parallel \bar{x}_h$, that is $\bar{x}_h$ is the line passing through $\bar{0}$ and parallel with $\overline{1_t r}$.*

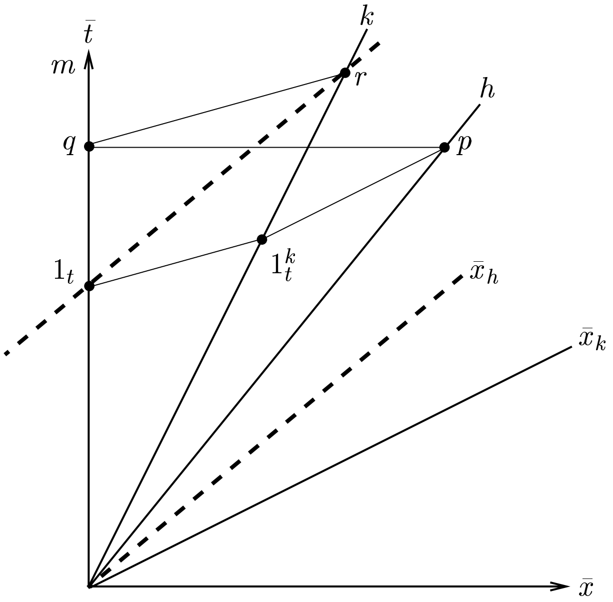


Figure 246: Illustration for Lemma 5.0.59.

Proof: Assume the assumptions of the lemma and that $m, k, h, 1^k_t, \bar{x}_k, p, q, r$ are as in Figure 246. Let $\mathrm{T} := \mathsf{f}_{mh}(p)_t$. Then, by the construction

k	sees at time instant	1	that clock of	h	shows	T	and
m	sees at time instant	q_t	that clock of	h	shows	T.	

By this and, by **Ax(syt)*** (and by $\mathsf{f}_{mh}, \mathsf{f}_{kh} \in Aftr$), we have

h	sees at time instant	1	that clock of	k	shows	T	and
h	sees at time instant	q_t	that clock of	m	shows	T.	

Hence

h	sees at time instant	q_t/T	that clock of	k	shows	q_t	and
h	sees at time instant	q_t/T	that clock of	m	shows	1.	

[573] by $v_k(h) \neq \infty$, because by $v_k(h) \neq \infty$ we have that $\bar{x}_k \not\parallel tr_m(h)$.
[574] If $p = \bar{0}$, then $tr_m(k) = \overline{1^k_t \bar{0}} \parallel \bar{x}_k$ and this contradicts $\mathsf{f}_{mk} \in Aftr$.
[575] Because $p \neq \bar{0}$, and because $p \notin \bar{t}$ by $tr_m(h) \neq \bar{t}$.
[576] $q \neq \bar{0}$ holds by $p \neq \bar{0}$ and $v_m(h) \neq \infty$.

Thus the event where clock of m shows 1 and the event where clock of k shows q_t are simultaneous for h. By the construction[577], clock of k shows q_t in event $w_m(r)$ (i.e. $\mathsf{f}_{mk}(r)_t = q_t$). So the $\bar{x}$-axis $\bar{x}_h$ of h is parallel with $\overline{1_t r}$, and this is what we wanted to prove. ∎

The remark below is a stronger form of Corollary 5.0.58.

Remark 5.0.60 Assume $m, k, h \in Obs$ are such that $v_m(k) \neq 0$ and for any two of m, k, h (i)–(iii) hold. (i) They are in standard configuration. (ii) $\mathbf{Ax(syt)^*}$ and **Ax4** hold. (iii) The world-view transformation between them is affine.[578]

Then the conclusion of Corollary 5.0.58 holds for m, k, h. Moreover $\bar{x}_h$ is already determined by the data 1_t^k, $tr_m(h)$ (i.e. $\bar{x}_k$ is not needed as an input datum in the conclusion of Corollary 5.0.58).

The present remark can be proved by using Lemma 5.0.59, by footnote 570 on p.722,[579] and by the fact that $\bar{x}_k$ is determined by 1_t^k because of (i)–(iii).

◁

Remark 5.0.61 In the present section we obtained some results which are relevant for §4.7 "Which symmetry principles are suitable for our Reichenbachian theories **Reich**(*Th*)?". Namely items 5.0.59, 5.0.60 could be used to conclude that $\mathbf{Ax(syt_0)}$ is a symmetry axiom not adequate for our Reichenbachian theory **Reich**(**Basax**).

◁

LEMMA 5.0.62 *Assume* **Relnoph** *and that m, k are in standard configuration. Then for every $1 < i < n$,*

$$\mathsf{f}_{mk}(1_i) = 1_i.$$

Proof: Assume **Relnoph** and that m, k are in standard configuration. Then by Prop.5.0.34(iii), $\vec{v}_m(k) = \vec{v}_k(m)$ or $\vec{v}_m(k) = -\vec{v}_k(m)$. If $\vec{v}_m(k) = -\vec{v}_k(m)$ then let $k' \in Obs$ be such that $\mathsf{f}_{kk'} = \sigma_{\bar{t}}$, otherwise let $k' = k$. Such a k' exists by $\mathbf{Ax(}$*Triv*$\mathbf{)}$. m, k' are in standard configuration, further $tr_m(k') = tr_{k'}(m)$. Hence by $\mathbf{Ax\Box 2}$ (and Prop.5.0.34(iv)) $\mathsf{f}_{mk'} = \mathsf{f}_{k'm} \circ N$, for some isometry N. Let this N be fixed. Let $1 < i < n$. Clearly, $\mathsf{f}_{mk'}(1_i), \mathsf{f}_{k'm}(1_i) \in \bar{x}_i$ and $\mathsf{f}_{mk'}(1_i)_i, \mathsf{f}_{k'm}(1_i)_i > 0$ by standard configuration (and by Thm.5.0.39). N leaves $\bar{0}$ fixed,[580] and takes $\mathsf{f}_{mk'}(1_i)$ to $\mathsf{f}_{k'm}(1_i)$. Therefore $\mathsf{f}_{mk'}(1_i) = \mathsf{f}_{k'm}(1_i)$. Thus $\mathsf{f}_{mk'}(1_i) = 1_i$, which implies that $\mathsf{f}_{mk}(1_i) = 1_i$. ∎

[577] i.e. by $\overline{1_t 1_t^k} \parallel \overline{qr}$, $q \in \bar{t}$ and $r \in tr_m(k)$

[578] If we formalized $\mathbf{Ax(syt)^*}$ in the style of $\mathbf{Ax(syt_0)}$ (i.e. $\mathsf{f}_{mk}(\bar{0}) = \bar{0} \Rightarrow (\forall p \in {}^nF)\mathsf{f}_{mk}(p)_t = \mathsf{f}_{km}(p)_t$) then "affine transformation" could be replaced by "collineation" in condition (iii).

[579] Instead of $\mathbf{Ax(\parallel)}$ and Prop.5.0.34(iii) in footnote 570 one uses $\mathbf{Ax(syt)^*}$ to conclude the same as in footnote 570.

[580] since both $\mathsf{f}_{mk'}$, $\mathsf{f}_{k'm}$ leave $\bar{0}$ fixed.

Ax(syx)* below is a "spatial" version of our axiom **Ax(syt)***.

Ax(syx)* $(m, k$ are in pre-standard configuration$) \quad \Rightarrow \quad |\mathsf{f}_{mk}(1_x)_x| = |\mathsf{f}_{km}(1_x)_x|$.

PROPOSITION 5.0.63 Relnoph $\models$ **Ax(syx)***.

Proof: The proposition can be proved by using **Ax**(ω)+**Ax**($\|$), cf. Prop.5.0.34(iv). The details are available from J. Madarász. ∎

The following lemma says that, in **Relnoph**, the strict standard f_{mk}'s are completely determined by $\mathsf{f}_{km}[\bar{t}]$, $\mathsf{f}_{km}[\bar{x}]$, and by knowing whether $m \uparrow k$, assuming $v_m(k) \neq \infty$. (We do not need to know even which model we are in.)

LEMMA 5.0.64 *Assume* $\mathfrak{N}, \mathfrak{M} \in \mathsf{Mod}(\mathbf{Relnoph})$ *and* $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{F}^{\mathfrak{N}}$. *Assume* $m, k \in Obs^{\mathfrak{N}}$ *and* $m_1, k_1 \in Obs^{\mathfrak{M}}$ *are such that both observer pairs are in strict standard configuration. Assume further* $v_m(k) \neq \infty$, $tr_m(k) = tr_{m_1}(k_1)$, $\mathsf{f}_{km}[\bar{x}] = \mathsf{f}_{k_1 m_1}[\bar{x}]$ *and* $(m \uparrow k \;\Leftrightarrow\; m_1 \uparrow k_1)$. *Then*

$$\mathsf{f}_{mk} = \mathsf{f}_{m_1 k_1}.$$

Proof: Assume $\mathfrak{N}, \mathfrak{M}$ and m, k, m_1, k_1 are as in the assumptions of the lemma. By Lemma 5.0.62, we have that for every $1 < i < n$, $\mathsf{f}_{mk}(1_i) = \mathsf{f}_{m_1 k_1}(1_i) = 1_i$. Thus to prove that $\mathsf{f}_{mk} = \mathsf{f}_{m_1 k_1}$, it is enough to prove that

(323) $$\mathsf{f}_{km}(1_t) = \mathsf{f}_{k_1 m_1}(1_t) \quad \text{and} \quad \mathsf{f}_{km}(1_x) = \mathsf{f}_{k_1 m_1}(1_x),$$

by Thm.5.0.39, saying that in **Relnoph** models all the f_{mk}'s are affine. By Propositions 5.0.57 and 5.0.63, we have that **Ax(syt)*** and **Ax(syx)*** hold. Let $\bar{t}' := tr_m(k) = tr_{m_1}(k_1)$ and $\bar{x}' := \mathsf{f}_{km}[\bar{x}] = \mathsf{f}_{k_1 m_1}[\bar{x}]$. By $v_m(k) \neq \infty$ and Prop.5.0.34(iii), $v_k(m) \neq \infty$. Hence $\bar{t}' \neq \bar{x}$ and $\bar{x}' \neq \bar{t}$. Now, the lengths of the unit vectors $\mathsf{f}_{km}(1_t)$ and $\mathsf{f}_{km}(1_x)$ are not arbitrary but they are completely determined by the lines $\bar{t}'$ and $\bar{x}'$ (and they do not depend on the rest of the model[581] we are working in). Namely, by **Ax(syt)***+**Ax(syx)*** (or equivalently by **Ax**(ω)+**Ax**($\|$)) we know

(+) $$|\mathsf{f}_{mk}(1_t)_t| = |\mathsf{f}_{km}(1_t)_t|, \text{ and similarly for } 1_x.$$

Just as it was the case of **Basax**+**Ax(symm)** in §2.8, if the lines $\bar{t}'$, $\bar{x}'$ are fixed[582] there is only one possible value for $|\mathsf{f}_{km}(1_t)|$ if we want to satisfy condition (+) above (recall that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$, $\bar{t}' \neq \bar{x}$ and $\bar{x}' \neq \bar{t}$ etc); this fact is shown in Figures 17, 18 of [16]. Therefore

(325) $$|\mathsf{f}_{km}(1_t)| = |\mathsf{f}_{k_1 m_1}(1_t)| \quad \text{and} \quad |\mathsf{f}_{km}(1_x)| = |\mathsf{f}_{k_1 m_1}(1_x)|.$$

[581] except for the obvious dependences like the choices of the operations of $\mathfrak{F}^{\mathfrak{M}}$.

[582] and if the world-view transformations are affine.

Clearly

(326) $\mathsf{f}_{km}(1_t), \mathsf{f}_{k_1m_1}(1_t) \in \bar{t}'$ and $\mathsf{f}_{km}(1_x), \mathsf{f}_{k_1m_1}(1_x) \in \bar{x}'$ and $\bar{0} \in \bar{t}' \cap \bar{x}'$.

By (325), (326), ($m \uparrow k \ \Leftrightarrow\ m_1 \uparrow k_1$) and by strict standard configuration, we conclude that (323) holds. ∎

In the definition below, we define the class of Generalized Minkowski models. We will use generalized Minkowski models e.g. in the proof of Thm.5.0.46. Further we think that generalized Minkowski models are of interest on their own right. Generalized Minkowski models will be similar to Minkowski models defined in §3.8. The main differences between a Minkowski model and a Generalized Minkowski model will be the following: (i) The speed of light in a Minkowski model is 1, while in a Generalized Minkowski model the speed of light can be an arbitrary positive element c of the ordered field corresponding to our model. (ii) In a Minkowski model there are no observers whose clocks run backwards,[583] while in a Generalized Minkowski model each observer will have a brother whose clock runs backwards.

Definition 5.0.65 (Generalized Minkowski models)
Let $\mathfrak{F}$ be Euclidean ($n \geq 2$) and let

$$\mathfrak{M}^M_{\mathfrak{F}} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W\rangle$$

be the Minkowski model corresponding to $\mathfrak{F}$ (and n) as defined in Def.3.8.42. Let $c \in {}^+F$. We are going to define the generalized Minkowski model $\mathfrak{M}^c_{\mathfrak{F}}$ as follows. Intuitively we will change the unit of measurement for time in $\mathfrak{M}^M_{\mathfrak{F}}$ such that for each observer the speed of light will be c, and for every observer we include a new observer whose clock runs backwards. Formally;

$$\begin{aligned}
\mathfrak{M}^c_{\mathfrak{F}} &\stackrel{\text{def}}{=} \langle (B';\ Obs', Ph', Ib'), \mathfrak{F}, G;\ \in, W^c\rangle, \text{ where}\\
B' &\stackrel{\text{def}}{=} B \times \{-1, 1\},\\
Obs' &\stackrel{\text{def}}{=} Obs \times \{-1, 1\},\\
Ph' &\stackrel{\text{def}}{=} Ph \times \{-1, 1\},\\
Ib' &\stackrel{\text{def}}{=} Ib \times \{-1, 1\},\\
W^c &\stackrel{\text{def}}{=} \Big\langle \langle\langle m, i\rangle, p, \langle b, j\rangle\rangle \in Obs' \times {}^nF \times B' \ :\ \langle m, \langle p_0 i\sqrt{c}, p_1, \ldots, p_{n-1}\rangle, b\rangle \in W\Big\rangle.
\end{aligned}$$

Now, the <u>*generalized Minkowski model corresponding to $\mathfrak{F}$ and $c \in {}^+F$*</u> is defined to be $\mathfrak{M}^c_{\mathfrak{F}}$. Cf. Figure 247.

◁

[583] This was an unimportant, arbitrary decision we made when we defined Minkowski models.

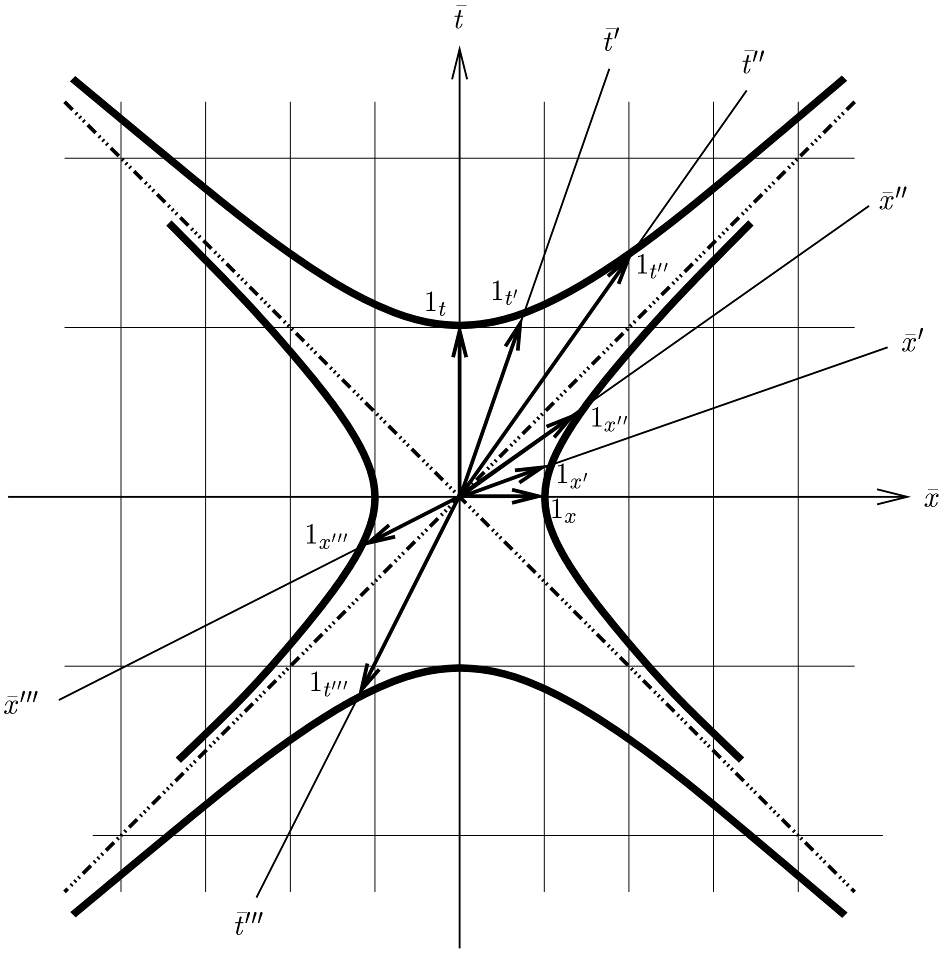


Figure 247: Strict standard world-view transformations in a generalized Minkowski model. The speed of light $\sqrt{c}$ is 2.

PROPOSITION 5.0.66 *Assume $\mathfrak{F}$ is Euclidean and $c \in {}^{+}F$ (and $n \geq 2$). Let $\mathfrak{M}^c_{\mathfrak{F}}$ be the generalized Minkowski model corresponding to $\mathfrak{F}$ and c. Then*

$$\mathfrak{M}^c_{\mathfrak{F}} \models \mathbf{Flxspecrel} + \mathbf{Relnoph} + (*1) + (*2) + (*3).$$

Proof: The proof is a straightforward checking of the axioms. Therefore we leave it to the reader. We note that a proof can be obtained from Thm.5.0.45. ∎

In §2.9 we defined Poincaré transformations. Unfortunately that definition assumed that the speed of light is 1. Therefore below we generalize the notion of Poincaré transformations in a similar way as we generalized the notion of Minkowski models in Def.5.0.65.

Definition 5.0.67 (Generalized Poincaré transformations)
Assume $\mathfrak{F}$ is an ordered field and $c \in {}^{+}F$.

(i) The *square of generalized Minkowski distance* $g^2_c : {}^nF \times {}^nF \longrightarrow F$ is defined as follows. Let $p, q \in {}^nF$. Then

$$g^2_c(p,q) \stackrel{\text{def}}{=} \left| c \cdot (q_0 - p_0)^2 - \left(\sum\nolimits_{0<i\in n} (q_i - p_i)^2\right) \right| .$$

(ii) By a *generalized Poincaré transformation (corresponding to $\mathfrak{F}$ and c)* we understand an $\mathsf{f} \in Aftr$ such that f preserves the square of generalized Minkowski distance, that is

$$(\forall p, q \in {}^nF)\ g^2_c(p,q) = g^2_c(\mathsf{f}(p), \mathsf{f}(q)).$$

◁

The proposition below says that generalized Poincaré transformations are exactly the world-view transformations of generalized Minkowski models.

PROPOSITION 5.0.68 *Assume $\mathfrak{F}$ is Euclidean and $c \in {}^{+}F$. Then f is a generalized Poincaré transformation (corresponding to $\mathfrak{F}$ and c) iff f is a world-view transformation in the generalized Minkowski model $\mathfrak{M}^c_{\mathfrak{F}}$. Cf. Figure 247.*

Proof: We omit the proof. ∎

The following proposition is included to illustrate that **Flxspecrel** is close to "standard special relativity". Namely the proposition says that in a **Flxspecrel** + $(c < \infty)$ model $\mathfrak{M}$ all the world-view transformations are generalized Poincaré transformations corresponding to the "speed of light" c belonging to $\mathfrak{M}$.

PROPOSITION 5.0.69 *Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Flxspecrel} + (c < \infty))$. Then all the world-view transformations f_{mk} of $\mathfrak{M}$ are generalized Poincaré transformations corresponding to $\mathfrak{F}^{\mathfrak{M}}$ and $c \in {}^{+}F$. We note that c is the (square of the) speed of light in $\mathfrak{M}$.*

Proof: We omit the proof. ∎

In the definition below, we define a class of "strange" models that we call rotation models. In these models the f_{mk}'s will be isometries (of the Euclidean geometry). Hence rotations[584] will also occur as world-view transformations (in these models). After the definition, in item 5.0.72 we will see that the rotation models are models of **Relnoph**, and in rotation models over Archimedean ordered fields each one of our principles $(*1)$, $(*2)$, $(*3)$ fails. Failure of $(*1)$ implies failure of $v_m(k) \neq \infty$, trivially. Hence we will not discuss principle $v_m(k) \neq \infty$ too much.

In connection with the style of the definition below we refer to the style of the definition of Minkowski models, Def.3.8.42 (p.331).

Definition 5.0.70 (Rotation models)
Let $\mathfrak{F}$ be Euclidean (and $n \geq 2$).

(i) We define the <u>*rotation model*</u> $\mathfrak{R}_{\mathfrak{F}}$ corresponding to $\mathfrak{F}$ as follows.

$$\begin{aligned}
\mathfrak{R}_{\mathfrak{F}} &\stackrel{\text{def}}{=} \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, G;\, \in, W\rangle, \quad \text{where} \\
Obs &\stackrel{\text{def}}{=} \left\{ f \in {}^{({}^{n}F)n}F \ :\ f \text{ is an isometry} \right\}, \\
B &\stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} Obs, \\
Ph &\stackrel{\text{def}}{=} \emptyset.
\end{aligned}$$

It remains to define W. First, we define a function $w_0 : {}^{n}F \longrightarrow \mathcal{P}(B)$ as follows. For every $p \in {}^{n}F$, let

$$w_0(p) \stackrel{\text{def}}{=} \{\, m \in Obs \ :\ p \in m[\bar{t}\,] \,\}.$$

For every $m \in Obs$, let

$$w_m \stackrel{\text{def}}{=} m \circ w_0.$$

Let

$$W \stackrel{\text{def}}{=} \{\, \langle m, p, b\rangle \in Obs \times {}^{n}F \times B \ :\ b \in w_m(p) \,\}.$$

By this $\mathfrak{R}_{\mathfrak{F}}$ has been defined.

[584] By a rotation we understand a special kind of isometry which is called "rotation" in usual geometry textbooks.

(ii) For every $c \in {}^+F$ we will define the generalized rotation model $\mathfrak{R}^c_{\mathfrak{F}}$ using the rotation model $\mathfrak{R}_{\mathfrak{F}}$ similarly as the generalized Minkowski model $\mathfrak{M}^c_{\mathfrak{F}}$ is defined using the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$.

Let $c \in {}^+F$. Let

$$\mathfrak{R}_{\mathfrak{F}} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle$$

be as in item (i). Intuitively, we obtain $\mathfrak{R}^c_{\mathfrak{F}}$ by slowing down the clocks of the observers in $\mathfrak{R}_{\mathfrak{F}}$ by the factor $\sqrt{c}$. Formally;

$$\begin{aligned} \mathfrak{R}^c_{\mathfrak{F}} &\stackrel{\text{def}}{=} \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W^c \rangle, \quad \text{where} \\ W^c &\stackrel{\text{def}}{=} \left\{ \langle m, p, b \rangle \in Obs \times {}^nF \times B \ :\ \langle m, \langle p_0\sqrt{c}, p_1, p_2, \dots, p_{n-1} \rangle, b \rangle \in W \right\}. \end{aligned}$$

We define the <u>*generalized rotation model corresponding to $\mathfrak{F}$ and c*</u> to be $\mathfrak{R}^c_{\mathfrak{F}}$. Cf. Figure 248.

$\lhd$

Remark 5.0.71 Assume $\mathfrak{F}$ is Euclidean. Then in $\mathfrak{R}_{\mathfrak{F}}$ all the world-view transformations are isometries. Further $\mathfrak{R}^1_{\mathfrak{F}} = \mathfrak{R}_{\mathfrak{F}}$.

$\lhd$

PROPOSITION 5.0.72 *Assume $\mathfrak{F}$ is Euclidean and $c \in {}^+F$. Then (i) and (ii) below hold.*

(i) $\mathfrak{R}^c_{\mathfrak{F}} \models$ **Relnoph**.

(ii) *Assume $\mathfrak{F}$ is Archimedean. Then each one of our principles* $(*1)$, $(*2)$, $(*3)$ *fails in $\mathfrak{R}^c_{\mathfrak{F}}$.*

Proof: The proof of item (i) goes by straightforward checking of the axioms, and it is left to the reader. Item (ii) follows by Lemma 5.0.74 below. ∎

LEMMA 5.0.73 *Assume $\mathfrak{F}$ is Euclidean (and $n \geq 2$). Assume we are in the rotation model $\mathfrak{R}_{\mathfrak{F}}$. Assume $m, k \in Obs$ such that they are in strict standard configuration. Then f_{km} is a rotation with center $\bar{0}$ when restricted to $\mathsf{Plane}(\bar{t}, \bar{x})$ (and leaves the $\bar{x}_i$ axis, for each $1 < i < n$, point-wise fixed).*

Proof: The proof is easy. We omit it. ∎

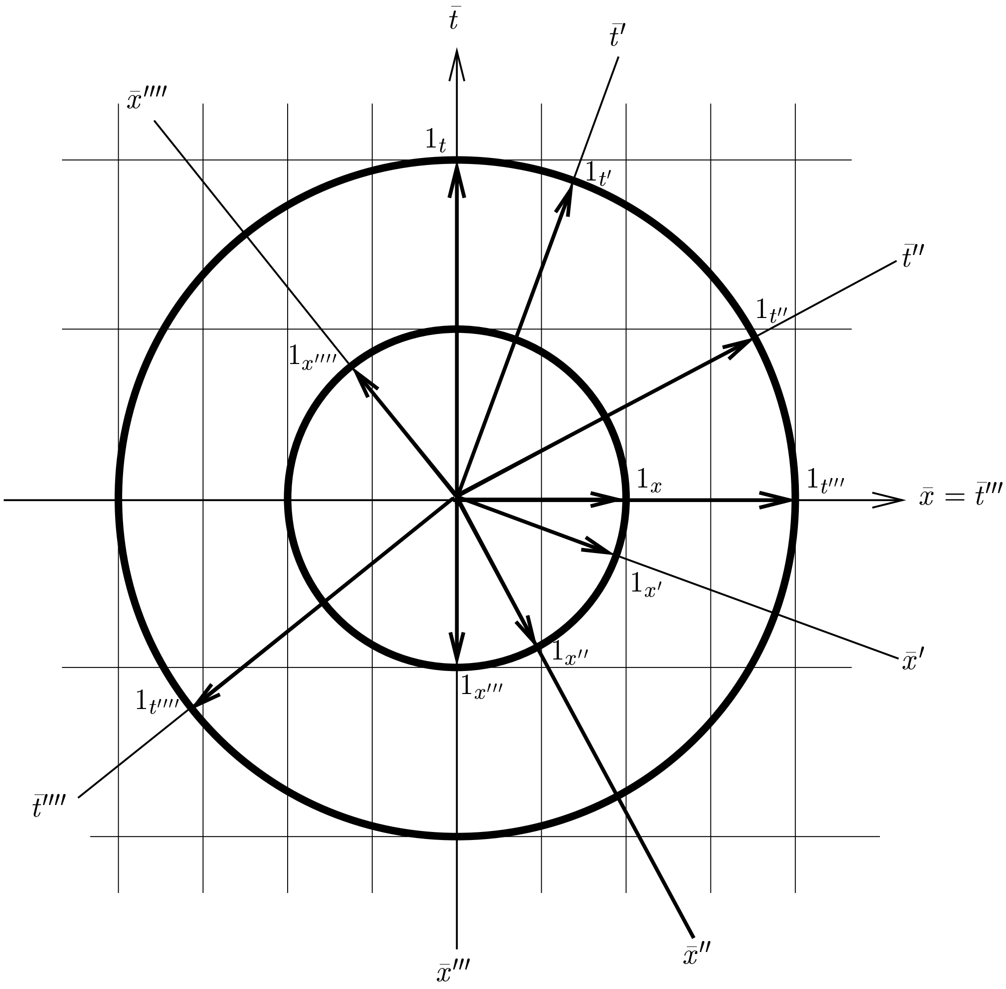


Figure 248: Strict standard world-view transformations in a generalized rotation model. In the picture $\sqrt{c} = 2$.

LEMMA 5.0.74 *Assume $\mathfrak{F}$ is Euclidean and Archimedean. Let $c \in {}^+F$. Assume we are in the generalized rotation model $\mathfrak{R}^c_{\mathfrak{F}}$. Let $\lambda \in {}^+F$. Then (i)–(iii) below hold.*

(i) *There are $j \in \omega$ and $m_0, \ldots, m_j \in Obs$ such that for all $i < j$, m_i and m_{i+1} are in strict standard configuration, $v_{m_i}(m_{i+1}) < \lambda$ and*

$$v_{m_0}(m_j) = \infty.$$

(ii) *There are $j \in \omega$ and $m_0, \ldots, m_j \in Obs$ such that for all $i < j$, m_i and m_{i+1} are in strict standard configuration, $v_{m_i}(m_{i+1}) < \lambda$ and*

m_0 does not see m_j moving forwards in direction 1_x.

(iii) *There are $0 < j \in \omega$ and $m_0, \ldots, m_j \in Obs$ such that for all $i < j$, m_i and m_{i+1} are in strict standard configuration, $0 < v_{m_i}(m_{i+1}) < \lambda$ and*

$$v_{m_0}(m_j) = 0.$$

Proof: Assume $\mathfrak{F}$ is Euclidean and Archimedean. The proof for the special case when $c = 1$, i.e. for the case $\mathfrak{R}_{\mathfrak{F}}$ the lemma follows by Lemma 5.0.73 as follows. Assume we are in the rotation model $\mathfrak{R}_{\mathfrak{F}}$. Let $\lambda \in {}^+F$. By Lemma 5.0.73 and by $\mathfrak{F}$ is Archimedean, there are $m, k \in Obs$ and $j \in \omega$ such that m and k are in strict standard configuration, $v_m(k) < \lambda$, and

$$ang^2((\underbrace{\mathsf{f}_{km} \circ \mathsf{f}_{km} \circ \ldots \circ \mathsf{f}_{km}}_{j\text{-times}})[\bar{t}\,]) = \infty. \tag{327}$$

Let such m, k, j be fixed. Let $m_0 := m$ and for every $i < j$ let $m_{i+1} \in Obs$ be such that $\mathsf{f}_{m_{i+1}m_i} = \mathsf{f}_{km}$. Such m_{i+1}'s exist by[585] **Ax□1**. Then clearly, for every $i < j$, m_i and m_{i+1} are in strict standard configuration. By

$$\underbrace{\mathsf{f}_{km} \circ \mathsf{f}_{km} \circ \ldots \circ \mathsf{f}_{km}}_{j\text{-times}} = \mathsf{f}_{m_j m_{j-1}} \circ \mathsf{f}_{m_{j-1} m_{j-2}} \circ \ldots \circ \mathsf{f}_{m_1 m_0} = \mathsf{f}_{m_j m_0}$$

and (327), we get

$$v_{m_0}(m_j) = \infty.$$

By the above, item (i) of the lemma has been proved for the special case $c = 1$. The proofs for items (ii), (iii) for the special case $c = 1$ are similar and are left to the reader.

[585] and by Prop.5.0.72(i).

Now, the proof for the general case when $c \in {}^+F$ is arbitrary can be obtained by using that the conclusion of the lemma is true for $c = 1$ as follows. Let $c \in {}^+F$. Consider the model $\mathfrak{R}^c_{\mathfrak{F}}$. Let $\lambda \in {}^+F$. Items 1–3 below can be easily proved by using the fact that $\mathfrak{R}^c_{\mathfrak{F}}$ was obtained from $\mathfrak{R}_{\mathfrak{F}}$ by slowing down the clocks of observers in $\mathfrak{R}_{\mathfrak{F}}$ by the factor $\sqrt{c}$, cf. Def.5.0.70.

1. The observers of $\mathfrak{R}^c_{\mathfrak{F}}$ and $\mathfrak{R}_{\mathfrak{F}}$ are the same.
2. Two observers are in strict standard configuration in $\mathfrak{R}^c_{\mathfrak{F}}$ iff they are in strict standard configuration in $\mathfrak{R}_{\mathfrak{F}}$.
3. Let $\vec{v}$ denote the velocity in $\mathfrak{R}_{\mathfrak{F}}$, and let $\vec{v^c}$ denote the velocity in $\mathfrak{R}^c_{\mathfrak{F}}$. Now,

$$(\forall m, k \in Obs)\ \vec{v^c}_m(k) = \sqrt{c} \cdot \vec{v}_m(k).^{586}$$

Now, we use 1–3 above and that the conclusion of the lemma is true for $\mathfrak{R}_{\mathfrak{F}}$ and λ/c to see that the conclusion of the lemma is true for $\mathfrak{R}^c_{\mathfrak{F}}$ and λ. The details are left to the reader. ∎

Outline of proof for Thm 5.0.46:

Assume $\mathfrak{N} \models$ **Relnoph**. Then all the f_{mk}'s are affine transformations by Theorem 5.0.39.

Let $m, k \in Obs$ be fixed (for the duration of this proof) such that they are in *strict standard configuration*, $v_m(k) \neq 0$. If there are no such m, k then, by Thm.5.0.49, $(\forall m, k \in Obs^{\mathfrak{N}}) v_m(k) = 0$ and, by **Ax(||)**, we are done (i.e. $\mathfrak{N} \models$ **"Flxspecrel"**).

We have $\mathsf{f}_{mk}(1_i) = 1_i$, for every $i > 1$ by Lemma 5.0.62.

Therefore, we may restrict attention to drawing the "$\bar{t}, \bar{x}$-part" of our f_{mk} transformations, since nothing happens in the $\bar{y}$ or $\bar{z}$ directions. Let

$$\bar{t}' := tr_m(k) \quad \text{and} \quad \bar{x}' := \mathsf{f}_{km}[\bar{x}].$$

Now, f_{mk} must be one of the three (disjoint) types type A–type C represented in Figure 249 below. More formally, the classification into types A, B, C is defined as follows.

- f_{mk} is of type A iff $(\exists \eta \in {}^+F)\ 1_x + \eta \cdot 1_t \in \bar{x}'$. Intuitively, using the notation in Figure 249, f_{mk} is of type A if $\tan(\beta)$ is positive.
- f_{mk} is of type B iff $\bar{x}' = \bar{x}$.

586 We note that if $v_m(k) = \infty$ then $\sqrt{c} \cdot \vec{v}_m(k) = \vec{v}_m(k)$.

- f_{mk} is of type C iff $\Big([(\exists 0 > \eta \in F)\ 1_x + \eta \cdot 1_t \in \bar{x}'] \text{ or } \bar{x}' = \bar{t}\Big)$. Intuitively, f_{mk} is of type C if $\tan(\beta)$ is negative or infinite.

Let us notice that a type A transformation might look slightly different from the way it is represented in Figure 249 because the clock of k may run backwards. Similarly for types B, C. Cf. the transformation in Figure 247 (p.727) represented by $1_{t'''}$ and $1_{x'''}$ and the transformation in Figure 248 (p.731) represented by $1_{t''''}$ and $1_{x''''}$.

For later use we note that the above classification is intended to be a classification of all strict standard world-view transformations of nonzero speeds into three disjoint types A–C, assuming that they are collineations. (In passing we note that for the above classification we do not have to assume all of **Relnoph**, e.g. it makes sense in $\mathbf{Bax}^-$ too.) We note that in generalized Minkowski models all strict standard world-view transformations (of nonzero speeds) are of type A, in models of $\mathbf{NewtK}^-$ all strict standard world-view transformations (of nonzero speeds) are of type B, and in generalized rotation models all strict standard world-view transformations (of nonzero speeds) are of type C, cf. Figures 247, 248 (pp. 727, 731).[587]

By **Ax(5nop)** and Thm.5.0.49 without loss of generality, *we can assume that* $v_m(k) \neq \infty$. $\mathfrak{N} \models \mathbf{Ax(syt)}^* + \mathbf{Ax(syx)}^*$, by Propositions 5.0.57, 5.0.63.

For completeness let us recall from the proof of Lemma 5.0.64 that in each of the cases type A–C, the lengths of the unit vectors $1_{t'} = \mathsf{f}_{km}(1_t)$, $1_{x'} = \mathsf{f}_{km}(1_x)$ are *not* arbitrary but are completely determined by the lines $\bar{t}'$ and $\bar{x}'$ (and they do not depend on the rest of the model[588] we are working in). Namely, by $\mathbf{Ax(syt)}^* + \mathbf{Ax(syx)}^*$ (or equivalently by $\mathbf{Ax}(\omega)+\mathbf{Ax}(\|)$) we know

$$(+) \qquad |\mathsf{f}_{mk}(1_t)_t| = |\mathsf{f}_{km}(1_t)_t|, \text{ and similarly for } 1_x.$$

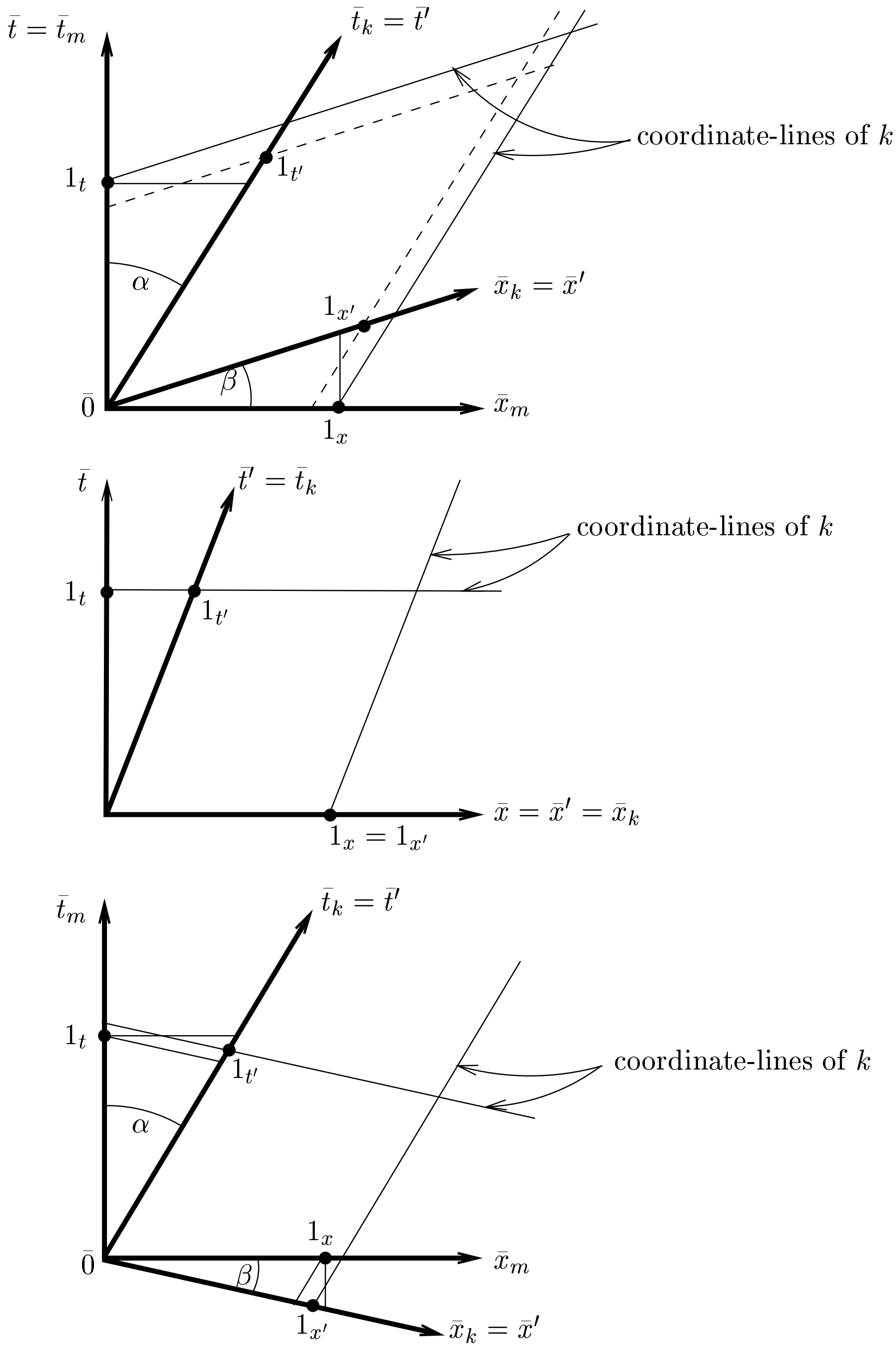
Just as it was the case of **Basax+Ax(symm)** in §2.8, if the lines $\bar{t}'$, $\bar{x}'$ are fixed there is only one possible value for $|\mathsf{f}_{km}(1_t)|$ if we want to satisfy condition (+) above (recall that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ etc); this fact is shown in Figures 17, 18 of [16].

<u>*Proof for Case A*</u>: Assume f_{mk} is of type A.

Intuitive idea of the proof: First we will prove that f_{mk} is a world-view transformation $\mathsf{f}_{m_1k_1}$ in a generalized Minkowski model $\mathfrak{M}$ (for some $m_1, k_1 \in Obs^{\mathfrak{M}}$). In proving this we will use Lemma 5.0.64. Then, using the so obtained $\mathsf{f}_{m_1k_1}$, we will

[587] Roughly speaking one would say that the type A type C distinction distinguishes **Specrel** models from rotation models in the sense that nontrivial strict standard transformations coming from **Specrel** models are of type A while those coming from rotation models are of type C. Now, there is one exception to this rule, namely (strict standard) FTL transformations of infinite speed coming from **Specrel** models turn out to be of type C. Otherwise the just mentioned rule does work.

[588] except for the obvious dependences like the choices of the operations of $\mathfrak{F}^{\mathfrak{M}}$.



f_{mk} of type A

f_{mk} of type B

f_{mk} of type C

Figure 249: f_{mk} must be one of the three (disjoint) types type A–type C.

embed our original model $\mathfrak{N}$ into this Minkowski model $\mathfrak{M}$. The idea of constructing such an embedding $\gamma : \mathfrak{N} \longrightarrow \mathfrak{M}$ goes as follows. For seeing the idea it is enough to concentrate on what γ does with the observers. For m and k we choose their γ-images to be m_1 and k_1 (notice that we already obtained m_1, k_1 as a result of our first step mentioned above). It remains to define γ on the rest of $Obs^{\mathfrak{N}}$. For this let $h \in Obs^{\mathfrak{N}}$ be arbitrary. For simplicity assume that m, h are in strict standard configuration. We choose $\gamma(h) = h_1$ in $\mathfrak{M}$ such that in m's world-view h looks like as h_1 looks like in m_1's world-view (e.g. they have the same life-line orientation of clocks etc). Then we use Corollary 5.0.58 to prove that the $\bar{x}$-axis of h and h_1 coincide (as seen by m and m_1, respectively). Then by Lemma 5.0.64, we will conclude that $\mathsf{f}_{mh} = \mathsf{f}_{m_1 h_1}$. We extend this construction to observers not necessarily in strict standard configuration, by Theorem 5.0.49 saying that strict standard configurations work in **Relnoph**. Summing up $\gamma : Obs^{\mathfrak{N}} \longrightarrow Obs^{\mathfrak{M}}$ has the property $(\forall m', k' \in Obs)\mathsf{f}_{m'k'} = \mathsf{f}_{\gamma(m')\gamma(k')}$. We use this property to show that γ extends to a homomorphism $\gamma^+ : \mathfrak{N} \longrightarrow \mathfrak{M}$, under assuming $\mathfrak{N} \models (B = Obs)$. It is easy to check that γ^+ is actually an embedding, assuming $\mathfrak{N} \models$ **Ax(ext)**. Therefore we conclude $\mathfrak{N} \models$ **"Flxspecrel+**$(c < \infty)$**"**. It is not hard to see that the assumptions we just made on $\mathfrak{N}$ do not restrict generality, i.e. they are relatively easy to eliminate.

Let us turn to the details of the proof. Since $v_m(k) \neq \infty$, we have

$$ang^2(\bar{t}') < ang^2(\bar{x}'), \tag{329}$$

because otherwise m and k will see each other's clocks running in the opposite directions (e.g. $m \uparrow k$ and $k \downarrow m$) and this is excluded by **Ax(syt)***.

We are going to define "the speed of light" c belonging to m, k. Using the notation in Figure 249, intuitively we define $c_{mk} = c := \tan(\alpha)/\tan(\beta)$, formally: Let $\xi, \eta \in F$ be such that $1_t + \xi \cdot 1_x \in \bar{t}'$ and $1_x + \eta \cdot 1_t \in \bar{x}'$. Such ξ, η exist, are unique and are positive, by type A (and strict standard configuration). Intuitively $\xi = \tan(\alpha)$ and $\eta = \tan(\beta)$. Let

$$c := \frac{\xi}{\eta}.$$

By (329), and by our choice of c, it can be checked that

$$ang^2(\bar{t}') < c < ang^2(\bar{x}'). \tag{330}$$

Let $\mathfrak{M} := \mathfrak{M}^c_{\mathfrak{F}^{\mathfrak{N}}}$ be the generalized Minkowski model corresponding to $\mathfrak{F}^{\mathfrak{N}}$ and c.

Claim **5.0.75** Recall that $\mathfrak{M}$ is the generalized Minkowski model corresponding to $\mathfrak{F}^{\mathfrak{N}}$ and c. Let $m_1, k_1 \in Obs^{\mathfrak{M}}$ such that they are in strict standard configuration and

$v_{m_1}(k_1) \neq 0$. Intuitively, if α and β in Figure 249 correspond to m_1, k_1 (instead of m, k) then $c = \tan(\alpha)/\tan(\beta)$, formally: Let $\xi', \eta' \in F$ such that $1_t + \xi' \cdot 1_x \in tr_{m_1}(k_1)$ and $1_x + \eta' \cdot 1_t \in \mathsf{f}_{k_1 m_1}[\bar{x}]$. Then $c = \xi'/\eta'$.

<u>*Proof:*</u> The claim follows by the construction (in Def.5.0.65) of the generalized Minkowski model $\mathfrak{M}$ from the (ordinary) Minkowski model $\mathfrak{M}^M_{\mathfrak{F}^{\mathfrak{N}}}$ and by the fact that "$\tan(\alpha)/\tan(\beta) = 1$" in the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}^{\mathfrak{N}}}$. The details of the proof are left to the reader.

QED (Claim 5.0.75)

The following remark is not needed for checking the proof of Theorem 5.0.46. We include it only for intuitive reasons (i.e. we include it to help the intuition).

Remark **5.0.76** We note that in any **Relnoph** model *"$\tan(\alpha)/\tan(\beta)$" is a constant* in the following sense. There is $c \in F^\infty$ such that for any m', k' in strict standard configuration, with $0 \neq v_{m'}(k') \neq \infty$, the value "$\tan(\alpha)/\tan(\beta)$" associated to $\mathsf{f}_{m'k'}$ is c (cf. Claim 5.0.75). This is a corollary of the proof of Thm.5.0.46. (At the present point of the proof the reader does not have to believe this remark, since for proving the remark we need the whole proof of Thm.5.0.46.)

Claim **5.0.77** Assume $m_1 \in Obs^{\mathfrak{M}}$. Then then there is $k_1 \in Obs^{\mathfrak{M}}$ such that

$$\mathsf{f}_{m_1 k_1} = \mathsf{f}_{mk}.$$

<u>*Proof:*</u> Let $m_1 \in Obs^{\mathfrak{M}}$. Let $k_1 \in Obs^{\mathfrak{M}}$ be such that m_1, k_1 are in strict standard configuration, $tr_{m_1}(k_1) = \bar{t}'$ and $(m_1 \uparrow k_1 \ \Leftrightarrow \ m \uparrow k)$. Such a k_1 exists by (330) (and is unique). By Claim 5.0.75 and by our choice of "the speed of light" c it can be proved that the $\bar{x}$ axis of k_1 as seen by m_1 must be $\bar{x}'$, i.e. that $\mathsf{f}_{k_1 m_1}[\bar{x}] = \bar{x}'$. Now, by Lemma 5.0.64 (and by Prop.5.0.66), we have that $\mathsf{f}_{mk} = \mathsf{f}_{m_1 k_1}$.

QED (Claim 5.0.77)

In the rest of the proof for Case A we will assume

$$\mathfrak{N} \models \mathbf{Ax(ext)} \ + \ (B = Obs).$$

Eliminating this assumption from the proof is easy and is left to the reader.

Claim **5.0.78** $\mathfrak{N}$ is isomorphic with a B-submodel of $\mathfrak{M}$.

<u>*Proof:*</u> To prove this claim it is enough to prove that there is an embedding $\gamma : Obs^{\mathfrak{N}} \longrightarrow Obs^{\mathfrak{M}}$ such that $(\forall m', k' \in Obs^{\mathfrak{N}})\mathsf{f}_{m'k'} = \mathsf{f}_{\gamma(m')\gamma(k')}$. The latter will ensure that γ commutes with the world-view relation W, i.e.

$$(\forall m', k' \in Obs^{\mathfrak{N}})(\forall p \in {}^nF)[W^{\mathfrak{N}}(m', p, k') \ \Leftrightarrow \ W^{\mathfrak{M}}(\gamma(m'), p, \gamma(k'))].$$ [589]

Let $\gamma(m) \in Obs^{\mathfrak{M}}$ be arbitrary, but fixed. From this value we define the rest of γ as follows.

$$\gamma := \left\{ \langle h, h_1 \rangle \in Obs^{\mathfrak{N}} \times Obs^{\mathfrak{M}} \ : \ \mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1} \right\}.^{590}$$

By our assumption $\mathfrak{N} \models \mathbf{Ax(ext)}$, we have that $\gamma : Obs^{\mathfrak{N}} \xrightarrow{\circ} Obs^{\mathfrak{M}}$ is an injective partial function. To prove that γ has the desired properties it is enough to prove that $Dom(\gamma) = Obs^{\mathfrak{N}}$. Clearly $m \in Dom(\gamma)$, and by Claim 5.0.77, $k \in Dom(\gamma)$ too. Next, we prove that

$$(331)\left\{ h \in Obs^{\mathfrak{N}} \ : \ (m, h \text{ are in strict stand. config.}) \text{ and } v_m(h) < c \right\} \subseteq Dom(\gamma).$$

To prove (331), let $h \in Obs^{\mathfrak{N}}$ be such that m, h are in strict standard configuration and $v_m(h) < c$. Let $h_1 \in Obs^{\mathfrak{M}}$ be such that $\gamma(m), h_1$ are in strict standard configuration, $tr_{\gamma(m)}(h_1) = tr_m(h)$ and $(\gamma(m) \uparrow h_1 \ \Leftrightarrow \ m \uparrow h)$. Such an h_1 exists by $v_m(h) < c$ (and is unique). We intend to prove that $\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1}$. Applying Corollary 5.0.58 first for observers m, k, h and then for observers $\gamma(m), \gamma(k), h_1$, by $\mathsf{f}_{mk} = \mathsf{f}_{\gamma(m)\gamma(k)}$ and $tr_m(h) = tr_{\gamma(m)}(h_1)$, we conclude that

$$\mathsf{f}_{hm}[\bar{x}] = \mathsf{f}_{h_1\gamma(m)}[\bar{x}].$$

Thus the assumptions of Lemma 5.0.64 hold for m, h and $\gamma(m), h_1$. Applying Lemma 5.0.64 for m, h and $\gamma(m), h_1$, we get $\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1}$. Hence $h \in Dom(\gamma)$. So (331) has been proved.

We will prove that

$$(332) \qquad (\forall h \in Obs^{\mathfrak{N}})\ v_m(h) < c.$$

Assume that there is $h \in Obs^{\mathfrak{N}}$ such that $v_m(h) \geq c$. Then by **Ax(5nop)** there is $h \in Obs$ such that $v_m(h) = c$. Next, we show that in $\mathfrak{N}$ no observer can have speed c as seen by m.[591] (This is an interesting parallel between **Relnoph** and **Flxbasax**.) Assume $v_m(h) = c$. Let $k' \in Dom(\gamma)$ such that $\bar{t} \neq tr_m(k') \neq tr_m(k)$ and $\bar{0} \in tr_m(k') \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. Such a k' exists by (331), **Ax(5nop)** and Thm.5.0.49. Since $k, k' \in Dom(\gamma)$ and $v_m(h) = c$, we have that, $v_k(h) = v_{k'}(h) = c$.[592] Hence,

[589] This is so because $(\forall m', k' \in Obs)(W(m', p, k') \ \Leftrightarrow \ p \in \mathsf{f}_{k'm'}[\bar{t}])$ holds both in $\mathfrak{N}$ and $\mathfrak{M}$.

[590] It is easy to check that our definition of γ is consistent.

[591] This fact can be shown differently from the method we will use here. Namely we could use Lemma 5.0.59 to prove that if $v_m(h) = c$ then $\mathsf{f}_{hm}[\bar{t}] = \mathsf{f}_{hm}[\bar{x}]$ which in turn contradicts **Ax6**. Therefore no observer of speed c can exists.

[592] This is so because of the following. By $k, k' \in Dom(\gamma)$, f_{mk} and $\mathsf{f}_{mk'}$ are world-view transformations in the generalized Minkowski model $\mathfrak{M}$, hence they take lines with angle c to lines with angle c.

by Proposition 5.0.34(iii), $v_h(m) = v_h(k) = v_h(k') = c$. So h sees three observers (m, k, k') in the same plane and passing through a same point such that their life-lines are pairwise different, and their speeds are the same. This leads to a contradiction. Thus (332) holds. By (331) and (332),

$$\text{(333)} \qquad \big\{\, h \in Obs^{\mathfrak{N}} \;:\; m, h \text{ are in strict stand. config.} \big\} \subseteq Dom(\gamma).$$

We will use (333) to prove that $Dom(\gamma) = Obs^{\mathfrak{N}}$. To see this let $h \in Obs^{\mathfrak{N}}$. By Thm.5.0.49 and **Ax□1**, we have that

$$\mathsf{f}_{mh} = triv \circ \mathsf{f}_{mh'} \circ triv',$$

for some $triv, triv' \in Triv$ and for some $h' \in Obs^{\mathfrak{N}}$ such that m and h' are in strict standard configuration. Let these be fixed. Then $h' \in Dom(\gamma)$ by (333). Hence

$$\mathsf{f}_{mh'} = \mathsf{f}_{\gamma(m)\gamma(h')}.$$

By $\mathfrak{M} \models$ **Ax(*Triv*)** (cf. Prop.5.0.66), $triv = \mathsf{f}_{m_1\gamma(m)}$ and $triv' = \mathsf{f}_{\gamma(h')h_1}$, for some $m_1, h_1 \in Obs^{\mathfrak{M}}$. Let such m_1, h_1 be fixed. Now,
$\mathsf{f}_{mh} = triv \circ \mathsf{f}_{mh'} \circ triv' = \mathsf{f}_{m_1\gamma(m)} \circ \mathsf{f}_{\gamma(m)\gamma(h')} \circ \mathsf{f}_{\gamma(h')h_1} = \mathsf{f}_{m_1h_1}$. So

$$\mathsf{f}_{mh} = \mathsf{f}_{m_1h_1}.$$

By $\mathfrak{M} \models$ **Ax□1** (cf. Prop.5.0.66), we have, $\mathsf{f}_{m_1h_1} = \mathsf{f}_{\gamma(m)h_1'}$, for some $h_1' \in Obs$. For this h_1', we have

$$\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1'},$$

hence $h \in Dom(\gamma)$, and this is what we wanted to prove.

QED (Claim 5.0.78)

By Claim 5.0.78, we have that (for Case A) $\mathfrak{N} \models$ **"Flxspecrel+$(c < \infty)$"**. Hence item (i) of the theorem holds in Case A. To prove that item (ii) of our theorem holds in Case A, by Claim 5.0.78 and by[593] $\mathfrak{N} \models$ **Ax(symm)** it is enough to prove Claim 5.0.79 below.

Claim **5.0.79** Assume that $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean. Then

$$\mathfrak{N} \models (\forall m' \in Obs)[(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < c \;\Rightarrow\; (\exists k' \in Obs)\; tr_{m'}(k') = \ell)].$$

[593] $\mathfrak{N} \models$ **Ax(symm)** holds by Prop.5.0.34(iv).

Proof: We have seen in Claim 5.0.77 that f_{mk} is a strict standard world-view transformation in the generalized Minkowski model $\mathfrak{M}$. But then, by a fact about generalized Minkowski models and by our assumption that $\mathfrak{F}$ is Archimedean, letting $\mathsf{f} = \mathsf{f}_{km}$ we have that

$$(334) \qquad (\forall \lambda \in {}^{+}F)[\lambda < c \;\Rightarrow\; (\exists j \in \omega) ang^2((\underbrace{\mathsf{f} \circ \mathsf{f} \circ \ldots \circ \mathsf{f}}_{j\text{-times}})[\bar{t}]) > \lambda],$$

cf. "Compositions of transformations can be drawn", pp. 744-750. By (334), **Ax(5nop)**, **Ax(group$^+$)**[594] (and by **Ax4**) in $\mathfrak{N}$ arbitrarily large but smaller than c speeds are realized by observers. Using this, together with Prop.5.0.38, we conclude that Claim 5.0.79 is true.

QED (Claim 5.0.79)

Thus our theorem has been proved for Case A.

Proof for Case B: Assume f_{mk} is of type B. Recall that we assumed that $v_m(k) \neq \infty$. Then it is easy to see by **Ax(syt)*** + **Ax(syx)*** (or equivalently by **Ax(ω)**+**Ax($\|$)**) and Thm.5.0.39 that f_{mk} is a generalized Galilean transformation. Let $h \in Obs$ be such that m, h are in strict standard configuration. Without loss of generality we can assume $\bar{t} \neq tr_m(h) \neq tr_m(k)$ because of **Ax($\|$)**. $v_k(h) \neq \infty$ because of the following. Assume $v_k(h) = \infty$. Then, by type B, $v_m(h) = \infty$ too. But then, by Proposition 5.0.34(iii), $v_h(m) = v_h(k) = \infty$ which implies that $tr_m(k) = \bar{t}$. This contradicts $v_m(k) \neq 0$. So $v_k(h) \neq \infty$. Similarly $v_m(h) \neq \infty$. Applying Lemma 5.0.59 for observers m, k, h we get that $\mathsf{f}_{hm}[\bar{x}] = \bar{x}$,[595] cf. Figure 246. But then, by **Ax(syt)*** + **Ax(syx)*** (or equivalently by **Ax(ω)**+**Ax($\|$)**), we conclude that f_{mh} is a generalized Galilean transformation.

Because of **Ax$\Box$1** and Thm.5.0.49 the general case of an arbitrary choice of $m', k' \in Obs$ can be reduced to the just discussed strict standard case f_{mh}. Therefore

$$(\forall m', k' \in Obs)(\mathsf{f}_{m'k'} \text{ is a generalized Galilean transformation}).$$

From this it is easy to prove that $\mathfrak{N} \models$ **"NewtK$^-$"**, hence $\mathfrak{N} \models$ **"Flxspecrel"**. If $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean then the photon-free reduct of $\mathfrak{N}$ coincides with the photon-free reduct of a **NewtK$^-$** model. To prove this, by $\mathfrak{N} \models$ **"NewtK$^-$"** and by[596] $\mathfrak{N} \models$ **Ax(symm)** it is enough to check that

$$(\forall m' \in Obs^{\mathfrak{N}})(\forall \ell \in \mathsf{Eucl})[ang^2(\ell) \neq \infty \;\Rightarrow\; (\exists k' \in Obs)\; tr_{m'}(k') = \ell].$$

[594] **Ax(group$^+$)** holds by Prop.5.0.34(i).
[595] since f_{mk} is a generalized Galilean transformation.
[596] $\mathfrak{N} \models$ **Ax(symm)** holds by Prop.5.0.34(iv).

Checking this is similar to (but easier than) the proof of Claim 5.0.79 and is left to the reader. Hence our theorem has been proved for Case B.

Proof for Case C: Assume that f_{mk} is of type C. We will prove that in this case each one of $(*1)$, $(*2)$, $(*3)$, $v_m(k) \neq \infty$ fails in $\mathfrak{N}$, assuming that $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean. We will prove this in such a way that we will embed $\mathfrak{N}$ into a generalized rotation model, similarly as in Case A we embedded $\mathfrak{N}$ into a generalized Minkowski model. (So in a sense the intuitive idea of the proof for Case C is the same as the one given for Case A).

In passing we note that all the the strict standard world-view transformations (of nonzero speeds) (cf. Def.5.2.1) in $\mathfrak{N}$ are of type C, because otherwise, by the proofs for Cases A and B, $\mathfrak{N} \models$ "**Flxspecrel**" would hold, and this would imply that f_{mk} is of type A or B.

By our assumption $v_m(k) \neq \infty$ and by Prop.5.0.34(iii) we have, $v_k(m) \neq \infty$. Thus $\bar{x}' \neq \bar{t}$, intuitively $\tan(\beta)$ in Figure 249 is not infinite.

Using the notation in Figure 249, intuitively let $c := \tan(\alpha)/(-\tan(\beta))$, formally: Let $\xi, \eta \in F$ be such that $1_t + \xi \cdot 1_x \in \bar{t}'$ and $1_x + \eta \cdot 1_t \in \bar{x}'$. Such ξ, η exist are unique, $\xi > 0$ and $\eta < 0$. Intuitively, $\xi = \tan(\alpha)$ and $\eta = \tan(\beta)$. Let

$$c := \frac{\xi}{-\eta}.$$

Let $\mathfrak{R} := \mathfrak{R}^c_{\mathfrak{F}^{\mathfrak{N}}}$ be the generalized rotation model corresponding to $\mathfrak{F}^{\mathfrak{N}}$ and c.[597]

Claim **5.0.80** Let $m_1, k_1 \in Obs^{\mathfrak{R}}$ such that they are in strict standard configuration and $v_{m_1}(k_1)$ is neither ∞ nor 0. Intuitively, if α and β in Figure 249 correspond to m_1, k_1 (instead of m, k) then $c = \tan(\alpha)/(-\tan(\beta))$, formally: Let $\xi', \eta' \in F$ such that $1_t + \xi' \cdot 1_x \in tr_{m_1}(k_1)$ and $1_x + \eta' \cdot 1_t \in \mathsf{f}_{k_1 m_1}[\bar{x}]$. Then $c = \xi'/(-\eta')$.

The claim follows by the construction (in Def.5.0.70) of the generalized Rotation model $\mathfrak{R}$ from the Rotation model $\mathfrak{R}_{\mathfrak{F}^{\mathfrak{N}}}$ and by the fact that "$\tan(\alpha)/(-\tan(\beta)) = 1$" in the Rotation model $\mathfrak{R}_{\mathfrak{F}^{\mathfrak{N}}}$. The details of the proof are left to the reader.

QED (Claim 5.0.80)

Claim **5.0.81** Assume $m_1 \in Obs^{\mathfrak{R}}$. Then then there is $k_1 \in Obs^{\mathfrak{R}}$ such that

$$\mathsf{f}_{m_1 k_1} = \mathsf{f}_{mk}.$$

[597] We hope it will cause no ambiguity that the same latter $\mathfrak{R}$ denotes both the field of reals and a frame model. The rule is that if $\mathfrak{R}$ is one-sorted then it a field, if it is many-sorted then it is a rotation model.

Proof: Let $m_1 \in Obs^{\mathfrak{R}}$. Let $k_1 \in Obs^{\mathfrak{R}}$ such that m_1, k_1 are in strict standard configuration, $tr_{m_1}(k_1) = \bar{t}'$ and $(m_1 \uparrow k_1 \ \Leftrightarrow \ m \uparrow k)$. Such a k_1 exists (and is unique). By our choice of c and Claim 5.0.80 it can be proved that the $\bar{x}$ axis of k_1 as seen by m_1 must be $\bar{x}'$, i.e. that $\mathsf{f}_{k_1 m_1}[\bar{x}] = \bar{x}'$. Now, by Lemma 5.0.64 (and by Prop.5.0.72), we have that $\mathsf{f}_{mk} = \mathsf{f}_{m_1 k_1}$.

QED (Claim 5.0.81)

In the rest of the proof for Case C we will assume

$$\mathfrak{N} \models \mathbf{Ax(ext)} \ + \ (B = Obs).$$

Eliminating this assumption from the proof is easy and is left to the reader.

Claim **5.0.82** $\mathfrak{N}$ is isomorphic with a B-submodel of $\mathfrak{R}$.

Proof: The proof of this is similar to the proof of Claim 5.0.78. To prove Claim 5.0.82 it is enough to prove that there is an embedding $\gamma : Obs^{\mathfrak{N}} \longrightarrow Obs^{\mathfrak{R}}$ such that $(\forall m', k' \in Obs^{\mathfrak{N}})\mathsf{f}_{m'k'} = \mathsf{f}_{\gamma(m')\gamma(k')}$. The latter will ensure that γ commutes with the world-view relation W.

Let $\gamma(m) \in Obs^{\mathfrak{R}}$ be arbitrary, but fixed. Let

$$\gamma := \left\{ \langle h, h_1 \rangle \in Obs^{\mathfrak{N}} \times Obs^{\mathfrak{R}} \ : \ \mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1} \right\}.$$

By our assumption $\mathfrak{N} \models \mathbf{Ax(ext)}$, we have that $\gamma : Obs^{\mathfrak{N}} \overset{\circ}{\longrightarrow} Obs^{\mathfrak{R}}$ is an injective partial function. To prove that γ has the desired properties it is enough to prove that $Dom(\gamma) = Obs^{\mathfrak{N}}$. Clearly $m \in Dom(\gamma)$, and by Claim 5.0.81, $k \in Dom(\gamma)$ too. Next, we prove that

$$\left\{ h \in Obs^{\mathfrak{N}} \ : \ m, h \text{ are in strict standard configuration} \right\} \subseteq Dom(\gamma). \tag{335}$$

To prove (335), let $h \in Obs^{\mathfrak{N}}$ such that m, h are in strict standard configuration. Let $h_1 \in Obs^{\mathfrak{R}}$ such that $\gamma(m), h_1$ are in strict standard configuration, $tr_{\gamma(m)}(h_1) = tr_m(h)$ and $(\gamma(m) \uparrow h_1 \ \Leftrightarrow \ m \uparrow h)$. Such an h_1 exists (and is unique). We intend to prove that $\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1}$. Applying Corollary 5.0.58 first for observers m, k, h and then for observers $\gamma(m), \gamma(k), h_1$, by $\mathsf{f}_{mk} = \mathsf{f}_{\gamma(m)\gamma(k)}$ and $tr_m(h) = tr_{\gamma(m)}(h_1)$, we conclude that

$$\mathsf{f}_{hm}[\bar{x}] = \mathsf{f}_{h_1\gamma(m)}[\bar{x}]. \tag{336}$$

We distinguish two cases: $v_m(h) \neq \infty$ and $v_m(h) = \infty$.

Case 1: Assume $v_m(h) \neq \infty$. By (336), the assumptions of Lemma 5.0.64 hold for observer pairs m, h and $\gamma(m), h_1$. But then, by Lemma 5.0.64, we have that $\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1}$, hence $h \in Dom(\gamma)$. So (335) has been proved for Case 1.

Case 2: Assume $v_m(h) = \infty$. In this case we cannot apply Lemma 5.0.64 for observer pairs m, h and $\gamma(m), h_1$ to conclude that $\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1}$. But, it can be checked (e.g. by using Remark 5.0.43(ii)) that the assumptions of Lemma 5.0.64 hold for observer pairs k, h and $\gamma(k), h_1$. Thus, by Lemma 5.0.64, we conclude $\mathsf{f}_{kh} = \mathsf{f}_{\gamma(k)h_1}$. This, by $\mathsf{f}_{mk} = \mathsf{f}_{\gamma(m)\gamma(k)}$, implies that $\mathsf{f}_{mh} = \mathsf{f}_{\gamma(m)h_1}$. By this, (335) has been proved.

We use (335) to prove that $Dom(\gamma) = Obs^{\mathfrak{N}}$ in the completely analogous way as (333) was used in the proof of Claim 5.0.78 to prove that $Dom(\gamma) = Obs^{\mathfrak{N}}$. We omit the details of this part of the proof.

QED (Claim 5.0.82)

Now, assume that $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean. To prove the theorem it remains to prove that in this case (Case C) each one of $(*1)$, $(*2)$, $(*3)$, and $v_m(k) \neq \infty$ fails in $\mathfrak{N}$. We will not discuss the condition $v_m(k) \neq \infty$ because failure of $(*1)$ immediately implies failure of $v_m(k) \neq \infty$. For this reason we will often ignore the $v_m(k) \neq \infty$ part of Thm.5.0.46(iii). By Claim 5.0.82, $\mathfrak{N}$ can be embedded into the generalized rotation model $\mathfrak{R}^{c}_{\mathfrak{F}}$, and by Prop.5.0.72(ii), we have that in $\mathfrak{R}^{c}_{\mathfrak{F}}$ each one of $(*1)$, $(*2)$, $(*3)$ fails. We cannot apply Prop.5.0.72(ii) in this situation directly because existential quantifiers need not be preserved under submodels. However, by **Ax(5nop)**, Thm.5.0.49 and **Ax□1**, we can repeat the proof of Lemma 5.0.74 (which was used to prove Prop.5.0.72(ii)) to prove that each one of $(*1)$, $(*2)$, $(*3)$ fails in our model. The details are left to the reader.

By this Thm.5.0.46 has been proved. ∎

By this point, we have proved our first main theorem, Thm.5.0.46. Before discussing further main results (i.e. results like item 5.0.46) we will discuss a useful "side-issue" which came up during the above proof. Namely we will discuss how to draw (i.e. construct geometrically) compositions of world-view transformations which leads to insights to how our models might look like. After doing this we will return to the main subject of the present chapter on p.751, §5.2 "Corollaries and further results".

5.1 Compositions of transformations can be drawn

In connection with Figure 249 representing types A–C; in Figures 250–255 we show how to draw (i.e. construct geometrically) compositions of world-view transformations. For simplicity we will concentrate on strict standard configurations.[598]

We copy the picture of the $\mathsf{f}_{k'k}$ transformation *into* the world-view of k (as seen by m) as represented in the "f_{km}-drawing" in Figure 250. The result is $\mathsf{f}_{k'm} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km}$, see Figures 251, 250. If we choose k' such that $\mathsf{f}_{km} = \mathsf{f}_{k'k}$ then we can draw $(\mathsf{f}_{km})^2 = \mathsf{f}_{km} \circ \mathsf{f}_{km}$ as in Figure 252. To draw compositions of f_{km}'s we do not need to assume $\mathbf{Ax}(\omega)$.[599] This can make the picture simpler and this is illustrated in Figure 253. Picture in Figure 253 shows, that we can construct $(\mathsf{f}_{km})^3 = \mathsf{f}_{km} \circ \mathsf{f}_{km} \circ \mathsf{f}_{km}$. The same way, we can construct $(\mathsf{f}_{km})^j$ for any $j \in \omega$, as in Figure 254. The picture in Figure 254 already suggests that if we started out with a type A f_{mk}, then we will arrive at a situation which is similar to Lorentz transformations, or to **Basax** models. (In the proof of Thm.5.0.46 we see that this similarity is indeed substantial.) The question naturally comes up, where do the iterated transformations $(\mathsf{f}_{km})^j$ converge to if j goes to ω. Assume $v_m(k) \neq 0$, f_{mk} is of type A, and $\alpha + \beta < 90°$,[600] i.e. f_{km} is like as illustrated in Figure 251. Let $\bar{t}_j := (\mathsf{f}_{km})^j[\bar{t}]$ and $\bar{x}_j := (\mathsf{f}_{km})^j[\bar{x}]$. If $\mathfrak{F}$ is Archimedean then $\bar{t}_j$ and $\bar{x}_j$ converge to[601] the same line $\bar{t}_\omega = \bar{x}_\omega$ between them assuming **Relnoph** (or **Basax**). (Here $\bar{t}_\omega$ belongs to the sequence $\langle \bar{t}_j : j \in \omega \rangle$ and $\bar{x}_\omega$ to the sequence $\langle \bar{x}_j : j \in \omega \rangle$.) If $\mathfrak{F}$ is not Archimedean then $\bar{t}_\omega$, $\bar{x}_\omega$ need not coincide as we will explain next. Assume $\mathfrak{F}$ is non-Archimedean. Then a world-view transformation $(\mathsf{f}_{km})^\omega$ together with $\bar{t}_\omega$, $\bar{x}_\omega$ may exist which in some respects behaves like a limit $\lim_{j\to\infty} (\mathsf{f}_{km})^j$, see Figure 255. E.g. $ang^2(\bar{t}_\omega) > ang^2(\bar{t}_j)$ for each $j \in \omega$. Also $\tan(\alpha)/\tan(\beta)$ for $(\mathsf{f}_{km})^\omega$ is the same as $\tan(\alpha)/\tan(\beta)$ for f_{km}. The interesting

[598] Occasionally we have to switch from f_{mk} to f_{km} or vice versa because only one of the two counts as a strict standard world-view transformation. (We treat this as an inessential technical detail and we hope that context will help in these situations.)

[599] We note this because we drew the previous two pictures such that they satisfy $\mathbf{Ax}(\omega)$. Further a part of $\mathbf{Ax}(\omega)$ was assumed in the previous long proof.

[600] α and β associated to an f_{mk} are the angles indicated in Figure 249 on p.735. We remind the reader that beginning with Figure 249 on p.735 we have a ***convention*** that to every (strict standard) f_{mk} there is a pair of associated angles systematically denoted as α and β. Although in α and β it is not indicated to which f_{mk} they belong, we hope context will help. Writing α_{mk} would be more unambiguous, but we choose to write α for sake of simplicity. For completeness we note that we use angles like 90° in the intuitive sense, and similarly α and β etc. are understood in the intuitive way. We hope context will provide sufficient information for formalizing our intuitive sentences involving α, β, 90°, 45° etc.

[601] in some sense

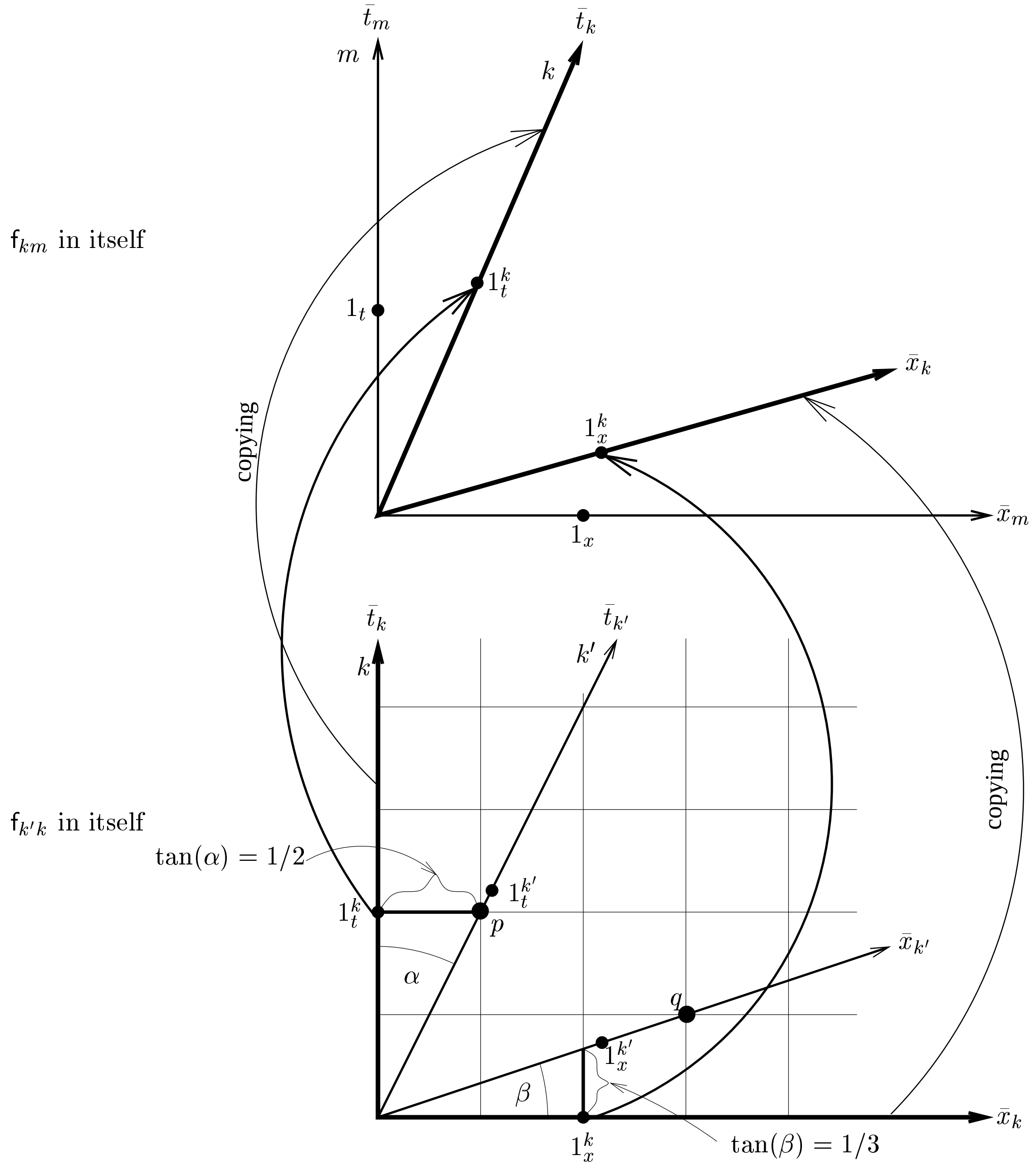


Figure 250: To be continued in Figure 251.

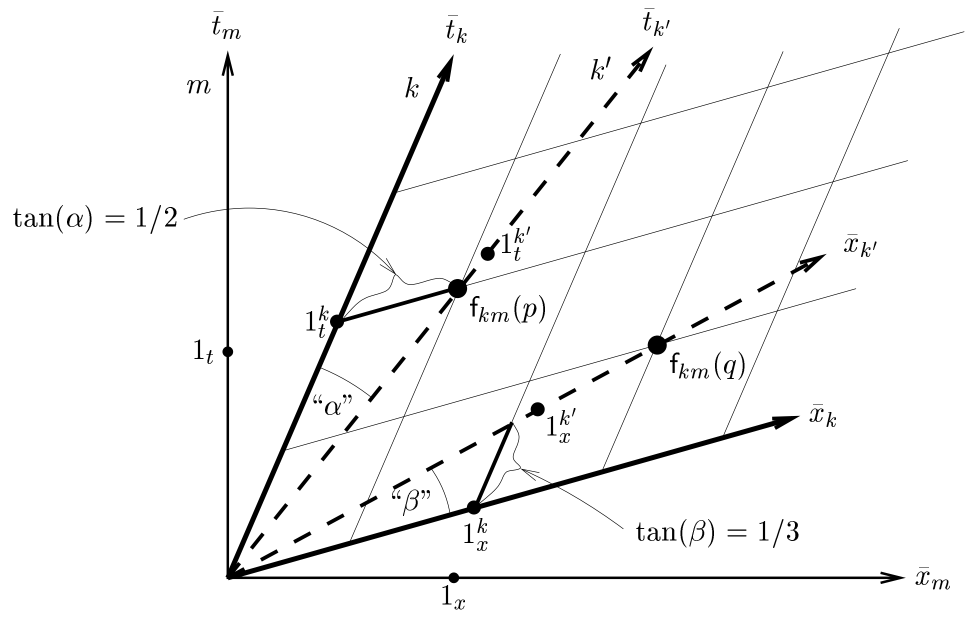


Figure 251: The result is $\mathsf{f}_{k'm} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km}$.

thing is that $(\mathsf{f}_{km})^\omega : {}^nF \longrightarrow {}^nF$ may be a bijection with $\bar{t}_\omega$ strictly slower than $\bar{x}_\omega$, cf. Figure 255. Further $(\mathsf{f}_{km})^\omega$ may belong to some observers, say k_ω and m_ω. All these can happen both in **Relnoph** (with type A) and in **Specrel**. (Let us notice that these cannot happen in the Archimedean case because there $\bar{t}_\omega = \bar{x}_\omega$ [both in **Relnoph** type A and in **Flxbasax**]). Summing up, in the non-Archimedean case a non-trivial $(\mathsf{f}_{km})^\omega$ may exist and then we may define $(\mathsf{f}_{km})^{\omega+1}$, $\bar{t}_{\omega+1}$, $\bar{x}_{\omega+1}$.

All this seems to point in the direction that in the non-Archimedean case we have more interesting possibilities to explore (than in the Archimedean case). These possibilities seem to be connected with the idea of applying non-standard analysis in relativity. In the present work, we do not pursue this direction further.

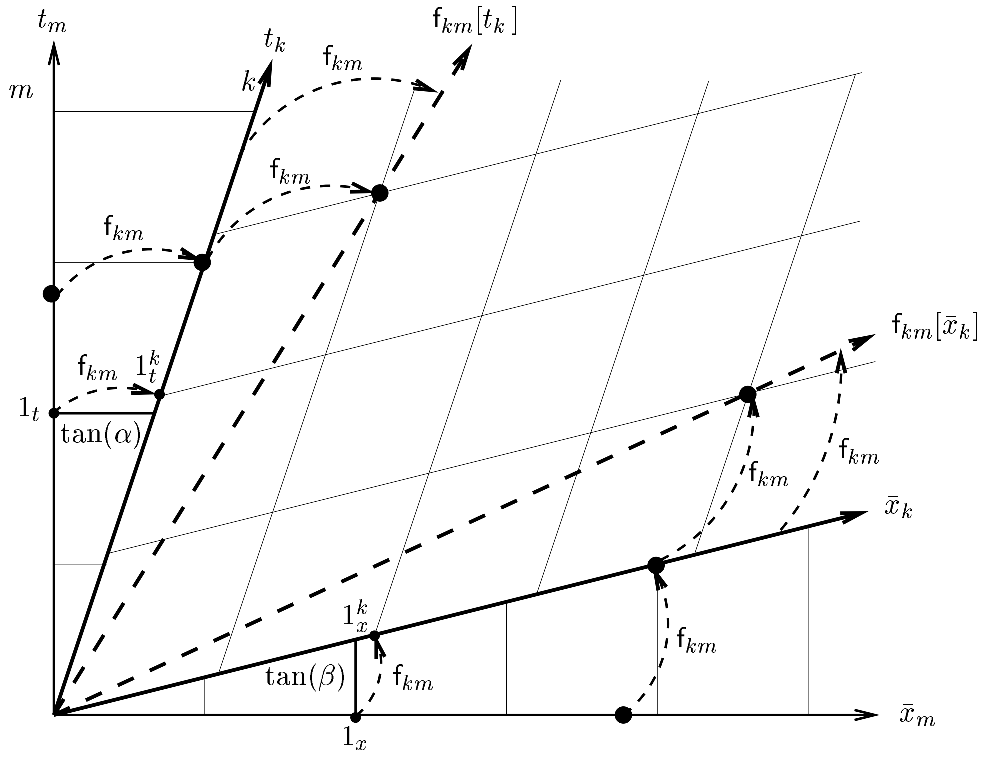


Figure 252: $(\mathsf{f}_{km})^2 = \mathsf{f}_{km} \circ \mathsf{f}_{km}$.

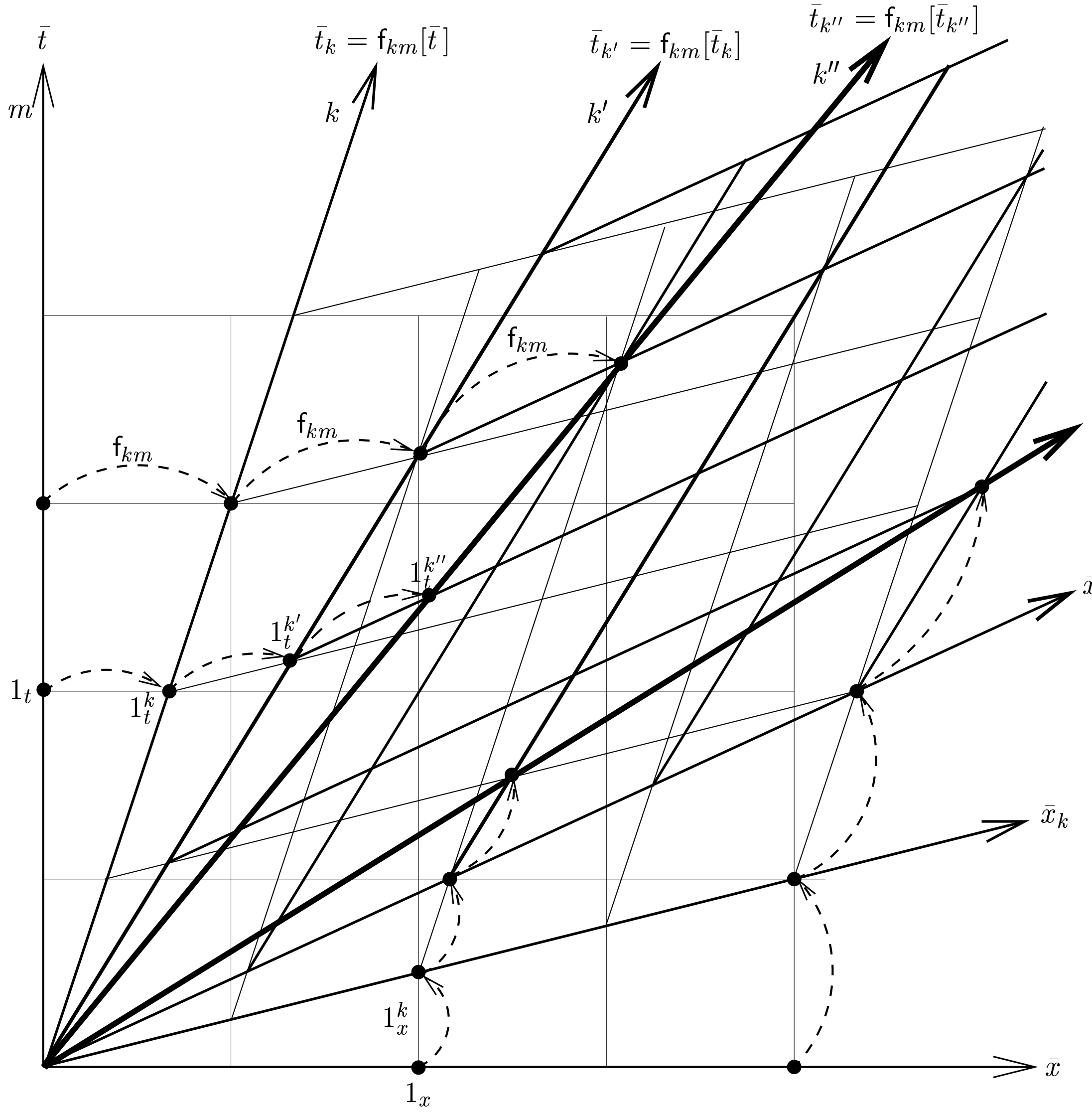


Figure 253: Constructing $(\mathsf{f}_{km})^3 = \mathsf{f}_{km} \circ \mathsf{f}_{km} \circ \mathsf{f}_{km}$.

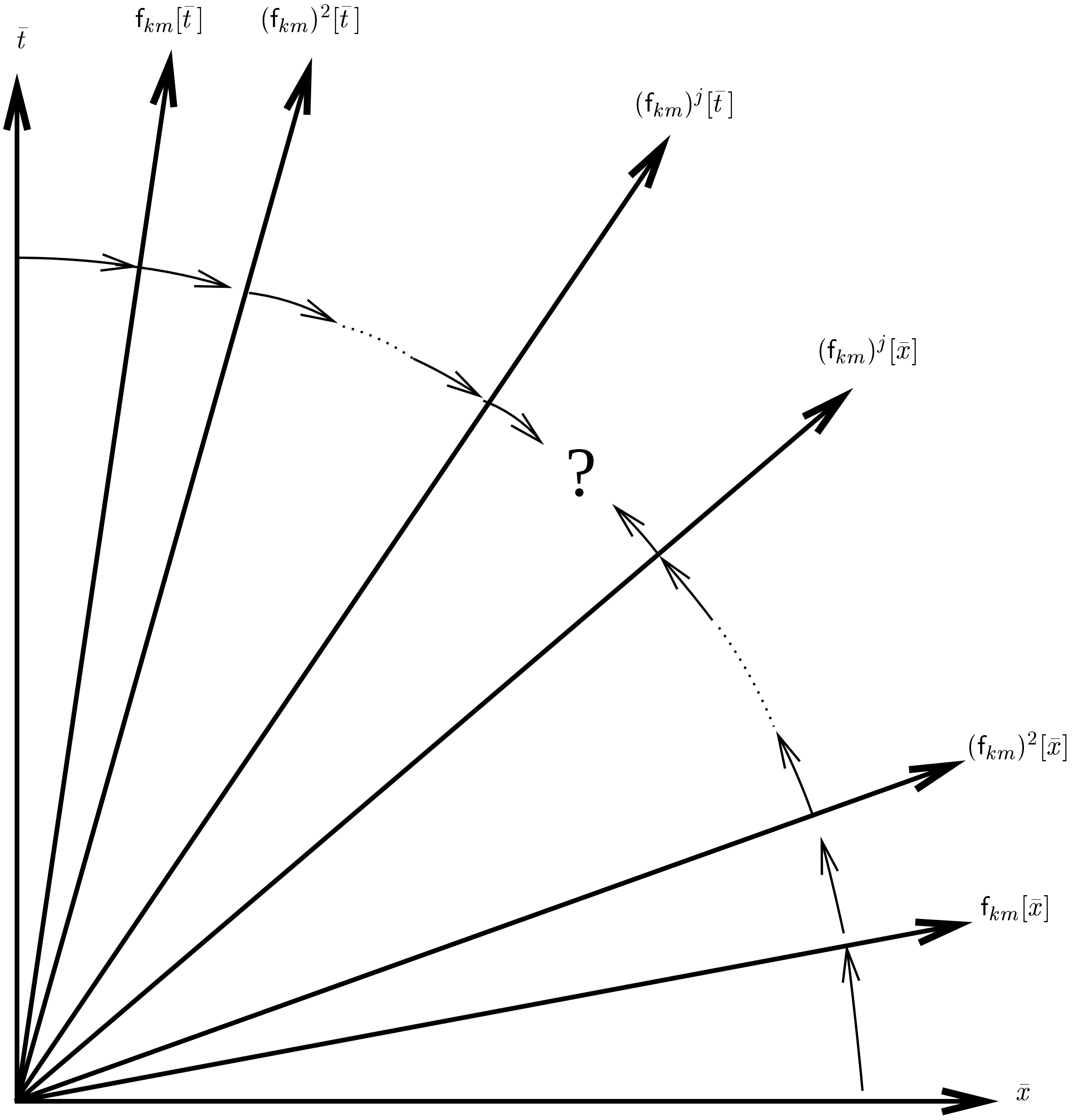


Figure 254: Where do the iterated transformations $(\mathsf{f}_{km})^j$ converge to if j goes to ω?

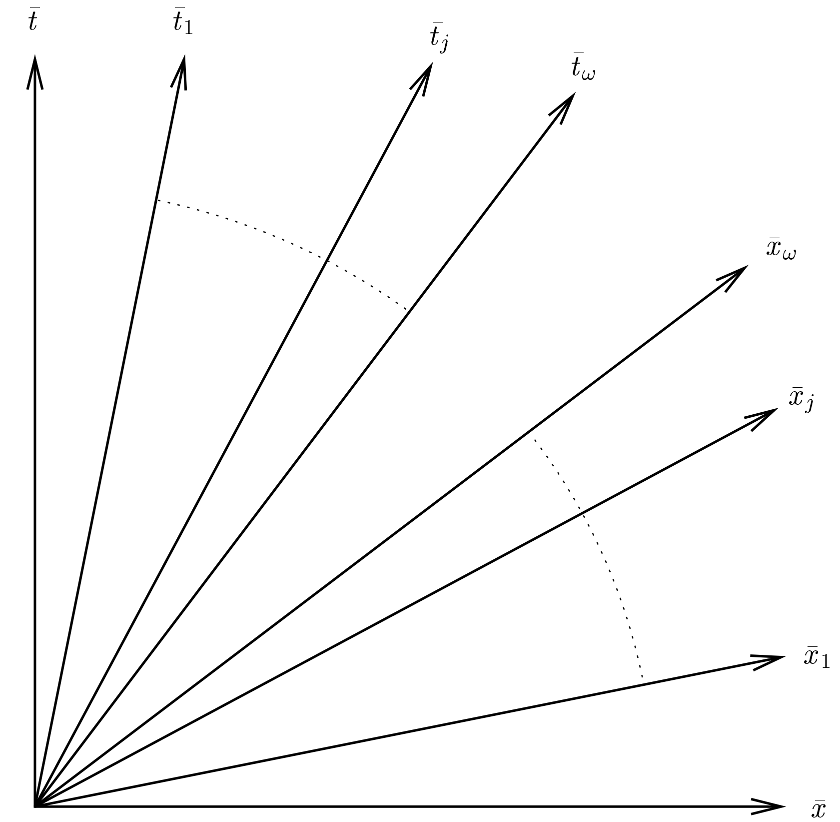


Figure 255: $\bar{t}_\omega$ may be strictly slower than $\bar{x}_\omega$ in the non-Archimedean case.

5.2 Corollaries and further results

Below, we return to the main theme of the present section (§5), i.e. to discussing the adequateness of **Relnoph** for serving as a special relativity theory (without photons). In other words, we return to discussing in what extent **Relnoph** recaptures usual special relativity. (We note that this subject was already addressed in items 5.0.46, 5.0.48.)

In the definition below, assuming **Relnoph**, we classify the world-view transformations into types A, B, C in the sense of Figure 249 (p.735).

Definition 5.2.1

(i) In the proof of Theorem 5.0.46, assuming **Relnoph**, we defined a classification of strict standard world-view transformations (of nonzero speeds) into types A, B, C (cf. p.733, and Figure 249 on p.735).

(ii) We classify the world-view transformations (of nonzero speeds) into types A, B and C by using the classification in item (i) above as follows. Assume **Relnoph**. Let $m, k \in Obs$. We say that f_{mk} is of type A iff there are $m', k' \in Obs$ such that $\mathsf{f}_{mm'}, \mathsf{f}_{kk'} \in Triv$ and $\mathsf{f}_{m'k'}$ is a strict standard world-view transformation of type A. The definition of f_{mk} being of type B or C is completely analogous (i.e. f_{mk} belongs to a type iff $\mathsf{f}_{m'k'}$ belongs to that type). To see that this definition is correct in the sense that each f_{mk} (of nonzero speed) belongs to exactly one type we refer to Fact 5.2.2 below.

$\triangleleft$

FACT 5.2.2 *Assume* **Relnoph**. *Then the world-view transformations are partitioned into four disjoint types: types A, B, C and the ones of speed zero. I.e. each transformation belongs to exactly one of these four types.*

Proof: The part that each f_{mk} belongs to at least one type follows from Theorem 5.0.49 saying that strict standard configurations work in **Relnoph**. The other part saying that each f_{mk} belongs to at most one type follows from some simple properties of **Relnoph**. ∎

The following three items are corollaries of the proof of Theorem 5.0.46.

COROLLARY 5.2.3 Assume $\mathfrak{N} \models$ **Relnoph**. Then all the world-view transformations (of nonzero speeds) in $\mathfrak{N}$ are of the same type (in the sense of Def.5.2.1(iii)).

Proof: This is a corollary of the proof of Thm.5.0.46 (and Fact 5.2.2). ∎

COROLLARY 5.2.4 Assume $\mathfrak{N} \models$ **Relnoph**. Then $\mathfrak{N} \models$ **"Flxspecrel"** or else in $\mathfrak{N}$ all the world-view transformations (of nonzero speeds) are of type C.

Proof: The proof can be recovered from that of Thm.5.0.46 given above. ∎

COROLLARY 5.2.5 Assume $\mathfrak{M} \models$ **Relnoph**. Then there are three possibilities:

(i) All f_{mk}'s (of nonzero speeds) are of type A and $\mathfrak{M} \models$ **"Flxspecrel** $+ (c < \infty)$**"**.

(ii) All f_{mk}'s (of nonzero speeds) are of type B and $\mathfrak{M} \models$ **"NewtK$^-$"**.

(iii) All f_{mk}'s (of nonzero speeds) are of type C and $\mathfrak{M}$ is "basically" embeddable[602] into a generalized rotation model.

Proof: The proof can be recovered from that of Thm.5.0.46 given above. ∎

Let us recall that principles $(*1)$, $(*2)$, $(*3)$ were used in the proof of Thm.5.0.46 above to exclude type C transformations. For brevity, we introduce **Ax(natu)** below.

Ax(natu) $\quad (*1) \vee (*2) \vee (*3)$.

The name **Ax(natu)** refers to the assumption that for many readers at least one of these conditions may look natural.[603] (**Ax(natu)** is a schema of formulas, as we already noted in connection with $(*1)$–$(*3)$.)

However, we emphasize that the present authors do *not* want to "take sides" in this issue, we do not claim anything in the direction that **Ax(natu)** *should* be assumed. We only say that **Ax(natu)** is natural enough for making us interested in studying the case when it is assumed.

In terms of **Ax(natu)** Theorem 5.0.46 way above says the following.

[602]By "basically embeddable" we understand that after making minor modifications on $\mathfrak{M}$ (e.g. making **Ax(ext)** true by collapsing observers, and throwing away bodies which are not observers) $\mathfrak{M}$ becomes embeddable into wherever we wanted to embed it.

[603]E.g. because so far nobody has ever observed a situation where the sum of finitely many small speeds would have been infinite.

COROLLARY 5.2.6 Assume $\mathfrak{N} \models$ **Relnoph** + **Ax(natu)** and $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean. Then (i) and (ii) below hold.

(i) $\mathfrak{N} \models$ **"Flxspecrel"**.

(ii) Assume $(\exists m, k)\, v_m(k) \neq 0$. Then the photon-free reduct of $\mathfrak{N}$ is the photon-free reduct of some **Flxspecrel** model. ∎

Conjecture 5.2.7 *We conjecture that in items 5.2.6, 5.0.46 the condition that $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean is needed. In more detail:*

(i) *There is $\mathfrak{N} \in \mathsf{Mod}(\textbf{Relnoph} + \textbf{Ax(natu)})$ such that $\mathfrak{N} \not\models$* **"Flxspecrel"**. *Moreover:*

(ii) *There is $\mathfrak{N} \in \mathsf{Mod}(\textbf{Relnoph} + (*1) + (*2) + (*3))$ such that $\mathfrak{N} \not\models$* **"Flxspecrel"**.

(iii) *There is $\mathfrak{N} \in \mathsf{Mod}(\textbf{Relnoph} + (*1) + (*2) + (*3) + (\exists m, k)\, v_m(k) \neq 0)$ such that all the world-view transformations (of nonzero speeds) in $\mathfrak{N}$ are of type C.*

Outline of possible proof: Choose a non-Archimedean ordered field $\mathfrak{F}$. Consider the rotation model $\mathfrak{R}_{\mathfrak{F}}$ over $\mathfrak{F}$, cf. Def.5.0.70. Keep only those observers m of $\mathfrak{R}_{\mathfrak{F}}$ for which the angle of the line $m[\bar{t}]$ is an infinitely small element of $\mathfrak{F}$. In such a way we get a B-submodel of $\mathfrak{R}_{\mathfrak{F}}$ which has the desired properties.

$\lhd$

To exclude type C transformations in models of **Relnoph** in non-Archimedean cases too, we introduce an axiom $\textbf{Ax(natu)}^+$ below. $\textbf{Ax(natu)}^+$ will be somewhat less of the "obviously true" kind than **Ax(natu)** is. We note that square-roots are needed in the axiom below, only because $v_m(k)$ was defined to be the square of the usual speed of k as seen by m.

$\textbf{Ax(natu)}^+$ $\quad (\forall m, k, k' \in Obs)\sqrt{v_m(k')} \leq \sqrt{v_m(k)} + \sqrt{v_k(k')}$.

Intuitively, if I see the train moving with 100 km/hours and a child is running on the train with 3 km/hours, then I will see the child moving with speed not greater than 100+3 km/hours.

Some of our readers might accept $\textbf{Ax(natu)}^+$ as a natural axiom, while others might find it less natural. (As far as we know, all experiments made so far agree with $\textbf{Ax(natu)}^+$). Anyway, we do *not* claim that this is an "obviously true" axiom, instead we include it only because it has interesting consequences.

THEOREM 5.2.8 $\quad$ **Relnoph + Ax(natu)$^+$ $\models$ "Flxspecrel".**

Proof: The theorem follows by the proof of Thm.5.0.46, **Ax□1**, **Ax(5nop)**, Thm.5.0.49 and Lemma 5.2.9 below. Namely, by Lemma 5.2.9, in case C in the proof of Thm.5.0.46 we have that **Ax(natu)$^+$** fails. ∎

LEMMA 5.2.9 *Assume $\mathfrak{F}$ is Euclidean and $c \in {}^+F$. Assume we are in the generalized rotation model $\mathfrak{R}^c_{\mathfrak{F}}$. Assume $m, k, k' \in Obs$ are such that $v_m(k) < c$, m and k are in strict standard configuration and $\mathsf{f}_{km} = \mathsf{f}_{k'k}$. Then*

$$\sqrt{v_m(k')} > \sqrt{v_m(k)} + \sqrt{v_k(k')}.$$

Proof: The idea of the proof is illustrated in Figure 256. Let us turn to the details. Assume $\mathfrak{F}$ is Euclidean. First we give a proof for the special case $c = 1$, i.e. for $\mathfrak{R}_{\mathfrak{F}}$. Consider the rotation model $\mathfrak{R}_{\mathfrak{F}}$. Assume $m, k, k' \in Obs$ satisfy the assumptions of the lemma. Let $\mathsf{f} := \mathsf{f}_{km} = \mathsf{f}_{k'k}$. Now, by Lemma 5.0.73, we have that f is a rotation with center $\bar{0}$ when restricted to $\mathsf{Plane}(\bar{t}, \bar{x})$. Further, by $v_m(k) < 1$, we have that $ang^2(\mathsf{f}[\bar{t}\,]) < 1$. Therefore

$$\sqrt{ang^2((\mathsf{f} \circ \mathsf{f})[\bar{t}\,])} > 2 \cdot \sqrt{ang^2(\mathsf{f}[\bar{t}\,])}, \tag{337}$$

see Figure 256.

$\mathsf{f} \circ \mathsf{f} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km} = \mathsf{f}_{k'm}$. Hence, $(\mathsf{f} \circ \mathsf{f})[\bar{t}\,] = tr_m(k')$ and $\mathsf{f}[\bar{t}\,] = tr_m(k) = tr_k(k')$. By these and (337), we have that

$$\sqrt{v_m(k')} > \sqrt{v_m(k)} + \sqrt{v_{k'}(k)}.$$

By this, our lemma has been proved for the special case $c = 1$.

Now, let $c \in {}^+F$ be arbitrary. Consider the generalized rotation model $\mathfrak{R}^c_{\mathfrak{F}}$. Recall that in Def.5.0.70 $\mathfrak{R}^c_{\mathfrak{F}}$ was constructed from $\mathfrak{R}_{\mathfrak{F}}$. By that construction (i) observers of these two models are the same, and (ii) if the speed in $\mathfrak{R}^c_{\mathfrak{F}}$ is denoted by v^c and in $\mathfrak{R}_{\mathfrak{F}}$ is denoted by v, we have that

$$(\forall m, k \in Obs)\ v^c_m(k) = c \cdot v_m(k).$$

By this, and by Def.5.0.70, we conclude that the conclusion of the lemma holds for $\mathfrak{R}^c_{\mathfrak{F}}$, since it holds for $\mathfrak{R}_{\mathfrak{F}}$. Cf. Figure 248 on p.731. The details are left to the reader. ∎

Let us ask ourselves how much of the photon-free part of "usual special relativity" can be proved from **Relnoph**. By usual special relativity we understand more or less **Flxspecrel**.

The following is a corollary of Theorem 5.2.8.

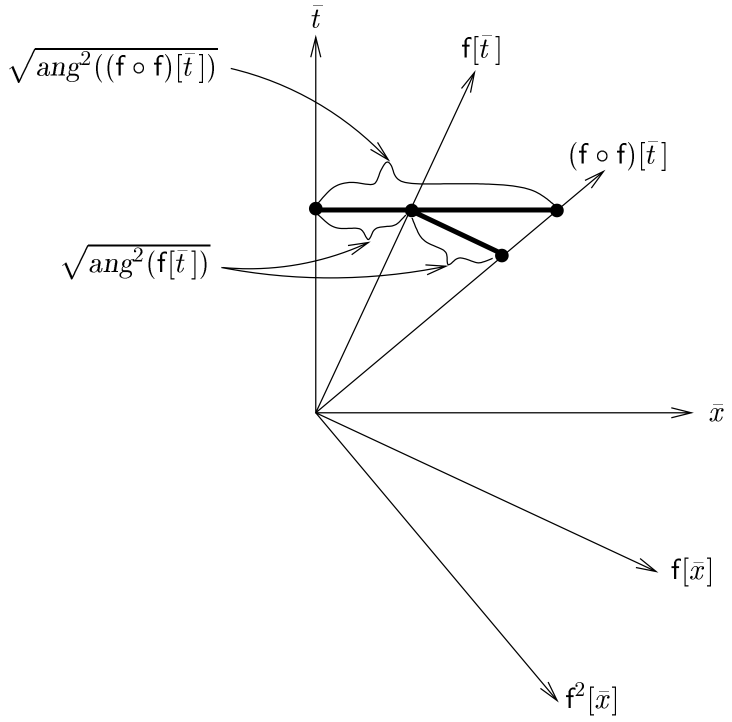


Figure 256: Illustration for the proof of Lemma 5.2.9.

COROLLARY 5.2.10

(i) All the theorems of **Flxspecrel** not involving photons and not containing quantifiers over sort B [604] are provable in **Relnoph+Ax(natu)$^+$**. Formally:

(ii) **Relnoph** + **Ax(natu)$^+$** $\models$ $\{\psi$: (**Flxspecrel** $\models \psi$) and ("*Ph*" does not occur in ψ) and (ψ contains no quantifiers of sort B)[605] $\}$.

Proof: The corollary follows from the following variant ($\star$) of Theorem 5.2.8.

($\star$) Let $\mathfrak{M} \models$ **Relnoph** + **Ax(natu)$^+$**. Let $\mathfrak{M}_0$ be obtained from $\mathfrak{M}$ by omitting the relation $Ph^{\mathfrak{M}}$. Then $\mathfrak{M}_0$ is a B-submodel of the photon-free reduct of some $\mathfrak{N} \in \mathsf{Mod}$(**Flxspecrel**).

A proof of ($\star$) is easily obtainable from Thm.5.2.8 above. From ($\star$) one proves the present corollary by the usual model-theoretic methods connecting submodels with universally quantified formulas, cf. e.g. Chang-Keisler [59]. The details of the proof are available from I. Németi. ∎

In connection with Corollary 5.2.10 above we note the following. If ψ is a formula of the form $\psi(m)$, i.e. ψ contains m as a free variable[606] then

$$Th \models \psi \quad \Leftrightarrow \quad Th \models \forall m\, \forall k\ (\psi)$$

by the definition of the logical consequence relation $\models$. Therefore the theorems ψ of *Th* containing no quantifier over k are *equivalent* with those theorems ψ of *Th* in which k is universally quantified (in prenex normal form) in the sense that ψ is of the form $\forall k\ \varphi(k)$ with φ containing no quantifier over k.

PROPOSITION 5.2.11 *Both the conditions "Ph does not occur in ψ" and "ψ contains no quantifiers of sort B" are needed in Corollary 5.2.10 above. E.g. there is ψ not involving Ph and* **Flxspecrel** $\models \psi$ *such that* **Relnoph** + **Ax(natu)$^+$** $\not\models \psi$.

Proof: E.g. axiom **Ax($\exists$body)** defined below does not involve "*Ph*", **Flxspecrel** $\models$ **Ax($\exists$body)** and **Relnoph** + **Ax(natu)$^+$** $\not\models$ **Ax($\exists$body)**. Further $v_m(ph)$ =

[604] This is equivalent to saying that all the variables of sort B are universally quantified when the formula in question is in prenex normal form.

[605] Like e.g. "($\exists m \in Obs$)".

[606] We use the standard logical convention that $\psi(m)$ denotes the same formula as ψ. The only difference is that if we write $\psi(m)$ then with this we emphasize that m may occur as a free variable in ψ. It is important to note that ψ may have other free variables besides m.

$v_k(ph)$ does not contain a quantifier of sort B, **Flxspecrel** $\models v_m(ph) = v_k(ph)$, but **Relnoph** + **Ax(natu)**$^+ \not\models v_m(ph) = v_k(ph)$. ∎

Our next axiom, **Ax(∃body)** is analogous with axiom (ν) above Prop.4.1.10 in our section on Newtonian Kinematics. This axiom says that certain lines are life-lines of some bodies. We need this axiom if we want to compare our theory **Relnoph** with theories like **Flxspecrel** or **Flxbasax**. The reason for this is that **Flxbasax** states that certain lines are life-lines of photons. (Of course no photon-free theory will imply this naturally. Therefore to remove a purely "administrative" obstacle of comparing **Relnoph** with theories which do involve photons we need to add the innocent axiom **Ax(∃body)**.)

Ax(∃body) $\forall m, k\ (\forall \ell \in \mathsf{Eucl})[(v_m(k) > 0\ \wedge\ \mathsf{f}_{mk}[\ell] = \ell)\ \Rightarrow\ (\exists b \in B) tr_m(b) = \ell]$.

Definition 5.2.12

$$\mathbf{Relnoph}^+ \stackrel{\text{def}}{=} \mathbf{Relnoph} + \mathbf{Ax(natu)} + \mathbf{Ax(\exists body)} + (\exists m, k)\ v_m(k) \neq 0.$$

$\lhd$

Definition 5.2.13 Let Th be a set of formulas in our frame language. Then $\mathsf{Mod}_{\mathsf{Arch}}(Th)$ denote the class of models $\mathfrak{N}$ of Th in which $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean.

$\lhd$

The following is a corollary of item 5.2.6.

COROLLARY 5.2.14

$\mathsf{Mod}_{\mathsf{Arch}}(\mathbf{Relnoph}^+)\ \models\ \{\,\psi : (\mathbf{Flxspecrel} \models \psi)$ and ("Ph" does not occur in ψ) $\}$.

Intuitively, all theorems ψ of **Flxspecrel** not involving photons follow from **Relnoph**$^+$, under assuming $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean.

Proof: To prove the corollary one uses item 5.2.6 together with the definition of **Ax(∃body)** above. Using these two tools one proves that any Archimedean **Relnoph**$^+$ model happens to be an **Flxspecrel** model if we ignore the predicate "Ph". To see that **Ax(∃body)** is enough for our present purposes we note that

$$\mathbf{Flxbasax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) \models \forall ph\, \exists m, k\ [\mathsf{f}_{mk} \text{ leaves } tr_m(ph) \text{ fixed}].$$

We omit the details of the proof. ∎

Corollary 5.2.14 above says, that (under assuming "Archimedean-ness") our theory **Relnoph**$^+$ is a satisfactory photon-free axiomatization of special relativity.

We note that in the section on Newtonian Kinematics (§4.1) we had an investigation analogous to the above one, namely there we formalized a photon-free version (i.e. traditional version) of Newtonian Kinematics, cf. Def.4.1.9 (p.435), and we proved some connections with the non-photon-free version of Newtonian Kinematics, cf. Prop.4.1.10 (p.436). As we already said, the auxiliary axiom (ν) we needed in Prop.4.1.10 is kind of analogous to **Ax(∃body)**.

NewtK^n definicióját megjelőlni: newtonnoph

An indication that **Relnoph** recaptures usual special relativity is that **Relnoph** proves several paradigmatic effects like the ones proved from **Specrel** in §§ 2.8, 2.5, cf. also §4.8. This is the subject matter of our next two theorems. (The fact that the paradigmatic effects remain provable is not surprising in the view of items 5.2.10, 5.2.14.)

THEOREM 5.2.15 *Assume* **Relnoph**. *Assume we are not in a Newtonian world, i.e. there is an* f_{mk} *(of nonzero speed) which is not of type B. Then the following paradigmatic effects hold.*

- **(i)** *Moving clocks change their rates (i.e. they slow down or they speed up), formally: Assume* $m, k \in Obs$, $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ *and* $v_m(k) \neq 0$. *Then* $|\mathsf{f}_{km}(1_t)_t| \neq 1$. *Cf. Thm.2.8.7 (p.131) and effect (E1) in §4.8.*
- **(ii)** *Moving meter rods parallel with the direction of movement change their lengths (i.e. they shrink or grow). Cf. Thm.2.8.8 (p.132) and effect (E2) in §4.8.*
- **(iii)** *Clocks spatially separated in the direction of movement*[607] *get out of synchronism. Cf. Thm.2.5.5(ii) (p.96).*
- **(iv)** *A funny form of the twin paradox holds, namely the clocks of the two twins show different times. We leave the formalization of this to the reader. Cf.* **Ax(TwP)** *on p.140.*
- **(v)** *In the case of type A the effects are the opposite as in the case of type C for items (i), (ii), (iv).*

Proof: Assume $\mathfrak{M} \models$ **Relnoph**, and there is an f_{mk} (of nonzero speed) which is not of type B. Then by Corollary 5.2.5, we have either (I) $\mathfrak{M} \models$ **"Flxspecrel** $+ (c < \infty)$**"** or (II) $\mathfrak{M}$ is "basically" embeddable into a generalized rotation model. In both cases (iii) holds. Further in Case (I) moving clocks slow down, moving meter rods parallel

[607] i.e. one clock is in the nose while the other is in the rear of the spaceship.

with the direction of movement shrink and the twin paradox holds. In Case (II) moving clocks speed up, moving meter rods parallel with the direction of movement grow and the "funny form" of the twin paradox holds, namely the clock of the accelerated brother shows more time than the clock of the inertial brother. ∎

THEOREM 5.2.16 *Assume* **Relnoph**. *Assume some* f_{mk} *(of nonzero speed) is neither of type B nor of type C. Then the following paradigmatic effects hold.*

(i) *Moving clocks slow down, cf. Thm.2.8.7 and effect (E1) in §4.8.*

(ii) *Moving meter rods parallel with the direction of movement shrink, cf. Thm.2.8.8 and effect (E2) in §4.8.*

(iii) *Clocks spatially separated in the direction of movement get out of synchronism, cf. Thm.2.5.5(ii).*

(iv) *The twin paradox holds. I.e. the clock of the accelerated brother shows less time than the clock of the inertial brother, cf. p.140.*

Proof: Assume $\mathfrak{M} \models$ **Relnoph** and that f_{mk} is as in the formulation of the theorem. Then f_{mk} is of type A. Hence $\mathfrak{M} \models$ **"Flxspecrel** $+ (c < \infty)$**"** by Corollary 5.2.5. But then (i)–(iv) hold. ∎

Our theorem below is strongly related to the subject matter of the present section. Namely $\mathbf{Bax}^-$ says so little about photons[608] that we can consider the theory $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax}\square\mathbf{1}$ as an almost photon-free theory. An attractive feature of the theorem below is that we do not have to assume "Archimedean-ness".

THEOREM 5.2.17 *Assume* $n > 2$. *Then*

$$\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax}\square\mathbf{1} + \mathbf{Ax6} \models \mathbf{Flxspecrel}.$$

Outline of proof: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax}\square\mathbf{1} + \mathbf{Ax6})$. By $\mathbf{Ax}(\mathit{Triv})$, $\mathbf{Ax}\square\mathbf{1}$ first one proves that for every $m, k \in \mathit{Obs}$

$$\forall m'\, \forall \ell\, [\mathrm{ang}^2(\ell) = v_m(k) \;\Rightarrow\; \exists k'\ tr_{m'}(k') = \ell].$$

[608]and that little assumption is natural and believable even if one does not know about anything like the Michelson-Morley experiment. I.e. $\mathbf{Bax}^-$ could be considered, in principle, as a pre-relativistic theory.

By this, $\mathbf{Ax5_{Obs}}$ and by the fact that in $\mathbf{Bax}^-$ there are no observers traveling together with a photon (i.e. there are no photons with speed zero) one can check that

$$(\forall m, k)(\forall d, d_1 \in \mathsf{directions})\ \mathsf{c}_m(d) = \mathsf{c}_k(d_1).$$

By this, we have $\mathfrak{M} \models$ **Flxbasax**. To prove $\mathfrak{M} \models$ **Flxspecrel** it remains to prove $\mathfrak{M} \models$ **Ax(symm)**. Since $n > 2$, there are no FTL observers in $\mathfrak{M}$. Now, $\mathfrak{M} \models$ **Ax(symm)** can be proved exactly as $\mathfrak{N} \models$ **Ax(symm)** was proved at the end of the proof of item 5.0.48 on p.712. ∎

In connection with the above theorem cf. Conjecture 5.2.21 on p.762.

Conjecture 5.2.18 *We conjecture that Theorem 5.2.17 remains true if we omit the assumption $n > 2$.*

$\lhd$

The following considerations add interest to the above theorem. $\mathbf{Bax}^-$ contains only two very natural assumptions about photons, which seem to be acceptable *even* if one does know nothing about the Michelson-Morley experiment. These assumptions are: (i) Photons are not like bullets, in the sense that photons moving in the same direction have the same speed (they cannot overtake one another). (ii) If an observer m points his flash-light in a direction d, then the photons emitted by the flash-light will move in direction d (as observed by m of course). I.e. no "ether-wind" can blow the emitted photons side-ways or e.g. backwards. (In this respect light behaves differently from sound).

The point of the above theorem is that these two natural postulates are sufficient for deriving special relativity (if one is willing to use the symmetry principle $\mathbf{Ax\Box 1}$, together with some simple and very convincing axioms).[609] If we add finiteness of speeds of photons as an extra assumption then we can derive all of special relativity from the just discussed assumptions. (But for completeness cf. item 5.2.21, and the discussion above it.) Some of these thoughts are summarized in the following corollary.

[609] This sentence might be slightly over-optimistic because $\mathbf{Ax5_{Obs}}$ (occurring in $\mathbf{Bax}^-$) has a side-effect which amounts to an "extra intuitive assumption" implied by $\mathbf{Bax}^-$ which we did not indicate in the above intuitive discussion of Thm.5.2.17. Cf. item 5.2.21 (p.762) and the discussion preceding it.

COROLLARY 5.2.19

(i) $\mathbf{Relnoph} + \mathbf{Bax}^- \models \mathbf{Flxspecrel}$.

(ii) $\mathbf{Relnoph} + \mathbf{Bax}^- + \mathsf{c}_m(d) < \infty \models \mathbf{Flxspecrel} + (c < \infty)$.[610]

Proof: The corollary follows by the proof of Theorem 5.2.17 and Prop.5.0.34(iv). ∎

We note that in the above corollary parts of **Relnoph** may be omitted, e.g. if $n > 2$ is assumed then $\mathbf{Ax\triangle 1}$ is not needed.

In item 5.2.20 below we will introduce a weaker version $\mathbf{Ax(5nop)}^-$ of $\mathbf{Ax(5nop)}$. Our interest in $\mathbf{Ax(5nop)}^-$ derives from two independent sources. These are (i) and (ii) below. (i) The fact that in the proof of our Theorem 5.2.17 a certain auxiliary aspect of our axiom $\mathbf{Ax5_{Obs}}$ started to play a "disturbingly dominant role", cf. the intuitive text following item 5.2.20 below. (ii) $\mathbf{Ax(5nop)}$ in **Relnoph** is extremely strong.

QUESTION 5.2.20 *Consider the following weaker*[611] *version of* $\mathbf{Ax(5nop)}$.

$\mathbf{Ax(5nop)}^-$ $\forall m\ (\exists c \in {}^+F)(\forall \lambda \in {}^+F)[\lambda < c \ \Rightarrow\ (\exists k) v_m(k) = \lambda]$.

Let $\mathbf{Relnoph}^-$ *be obtained from* **Relnoph** *by replacing* $\mathbf{Ax(5nop)}$ *with* $\mathbf{Ax(5nop)}^-$.

Do our theorems (and other statements) remain true for $\mathbf{Relnoph}^-$ *in place of* **Relnoph**? *In particular we would be interested in the status of items 5.0.46, 5.2.3, 5.2.6, 5.2.8, 5.2.10.*

◁

The new axiom $\mathbf{Ax(5nop)}^-$ introduced in Question 5.2.20 above has a significance which goes far beyond studying **Relnoph**. Namely in most of our theories like e.g. in $\mathbf{Bax}^-$ studied so far we assumed $\mathbf{Ax5_{Obs}}$ or a variant of it. Although the purpose of $\mathbf{Ax5_{Obs}}$ is to states that slow observers exist, it has a nontrivial side-effect. Namely the way $\mathbf{Ax5_{Obs}}$ is formulated now the velocities of observers are implicitly tied to the velocities of photons. It is not quite obvious that one wants to assume this connection in all of our theories. Since it is important to see what happens if we remove the above discussed side-effect built into $\mathbf{Ax5_{Obs}}$, we define

[610]We note that $\mathbf{Flxspecrel} + (c < \infty)$ is a very natural generalization of **Specrel** (obtained by abstracting away the precise numerical value of c).

[611]To be precise $\mathbf{Ax(5nop)}^-$ is provable in $\mathbf{Relnoph} + (\exists m, k)\, v_m(k) \neq 0$.

a "side-effect-free" version $\mathbf{Bax}^-_{\mathbf{nobs}}$ of $\mathbf{Bax}^-$ below, as a part of our conceptual analysis of relativity.

$$\mathbf{Bax}^-_{\mathbf{nobs}} \stackrel{\text{def}}{=} \mathbf{Bax}^- \setminus \{\mathbf{Ax5_{Obs}}\} + \mathbf{Ax(5nop)}^-,$$

cf. Figure 257.

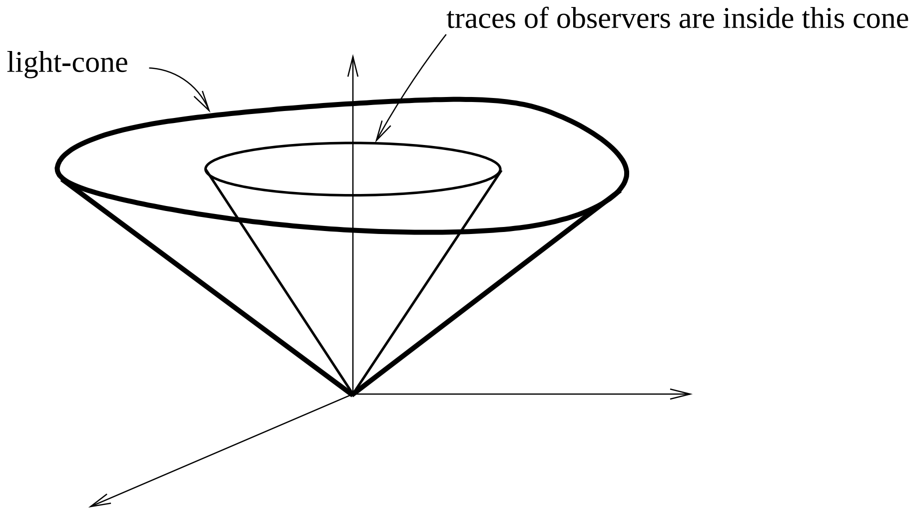


Figure 257: This can be a world-view of an observer in a $\mathbf{Bax}^-_{\mathbf{nobs}}$ model.

Németi írjon eből a témából alfejezetet §3 vagy §4 környékére

Conjecture 5.2.21 *Assume $n > 2$. Then (i)–(iii) hold.*

(i) *Theorem 5.2.17 and Corollary 5.2.19 become false if we replace* $\mathbf{Bax}^-$ *in them with* $\mathbf{Bax}^-_{\mathbf{nobs}}$*, i.e.* $\mathbf{Bax}^-_{\mathbf{nobs}} + \mathbf{Relnoph} \not\models \mathbf{Flxspecrel}$.

(ii) *Moreover,* $\mathbf{Bax}^-_{\mathbf{nobs}} + \mathbf{Relnoph} \not\models$ **"Flxspecrel"**.

(iii) *As a contrast,* $\mathsf{Mod}_{\mathsf{Arch}}(\mathbf{Bax}^-_{\mathbf{nobs}} + \mathbf{Relnoph}) \models$ **"Flxspecrel"**.

Outline of possible proof:

Possible proof for (i), (ii): Choose a non-Archimedean ordered field $\mathfrak{F}$. Consider the rotation model $\mathfrak{R}_{\mathfrak{F}}$ over $\mathfrak{F}$, cf. Def.5.0.70. Keep only those observers m of $\mathfrak{R}_{\mathfrak{F}}$ for which the angle of the line $m[\bar{t}]$ is an infinitely small element of $\mathfrak{F}$. In such a way we get a B-submodel $\mathfrak{M}$ of $\mathfrak{R}_{\mathfrak{F}}$. Extend this $\mathfrak{M}$ by photons, e.g. the following way.

Let $Ph := \mathsf{PhtEucl}$ and m be a fixed observer of $\mathfrak{M}$. Now, for every $k \in Obs$ and $ph \in Ph$, let $tr_k(ph) := \mathsf{f}_{mk}(ph)$. Let $\mathfrak{M}^+$ denote the so obtained extended version of $\mathfrak{M}$. Now, $\mathfrak{M}^+ \models \mathbf{Bax}^-_{\mathbf{nobs}} + \mathbf{Relnoph}$ but $\mathfrak{M}^+ \not\models$ **"Flxspecrel"**.

Possible proof for (iii): Let $\mathfrak{N} \in \mathsf{Mod}_{\mathsf{Arch}}(\mathbf{Bax}^-_{\mathbf{nobs}} + \mathbf{Relnoph})$. Then by Corollary 5.2.5 either $\mathfrak{N} \models$ **"Flxspecrel"** or $\mathfrak{N}$ is "basically" embeddable into a generalized rotation model. Assume that $\mathfrak{N}$ is "basically" embeddable into a generalized rotation model. Then, since $\mathfrak{F}^{\mathfrak{N}}$ is Archimedean and $\mathfrak{N} \models \mathbf{Relnoph}$, we have

$$\mathfrak{N} \models \forall m\, \forall \ell\, \exists k\ (tr_m(k) = \ell).$$

But this contradicts $\mathfrak{N} \models \mathbf{Bax}^-_{\mathbf{nobs}}$, since in a $\mathbf{Bax}^-_{\mathbf{nobs}}$ model for any observer if a straight line is a life-line of a photon then this straight line cannot be a life-line of an observer etc. Hence $\mathfrak{N} \models$ **"Flxspecrel"**.

$\triangleleft$

Future research task 5.2.22 Throughout this item, instead of $\mathbf{Ax(5nop)^-}$ we understand its stronger version $\mathbf{Ax(5nop)^{-+}}$ defined below.

$\mathbf{Ax(5nop)^{-+}}$ $(\forall m)(\exists c \in {}^+F)(\forall \ell \in \mathsf{Eucl})[ang^2(\ell) < c \Rightarrow (\exists k \in Obs)tr_m(k) = \ell]$.

It would be interesting to go over our main theories like **Specrel**, **Basax**, **Bax**, **Reich**(**Bax**) and replace the $\mathbf{Bax}^-$ part the theory in question with $\mathbf{Bax}^-_{\mathbf{nobs}}$ and investigate how the behavior of the new theory differs from the old one. E.g. do our main theorems (in the present work) remain true? Do the paradigmatic effects remain true? Do the f_{mk} transformations remain similar to the old ones etc?

In this connection we note that it is our impression that for our main theories as indicated above the just outlined change (i.e. eliminating the side-effect caused by $\mathbf{Ax5_{Obs}}$) will cause no essential difference in the pattern of our main results. We have this impression despite of items (i) and (ii) of Conjecture 5.2.21 above. In more detail we conjecture that there will be no essential change in the case of theories not weaker than **Reich**(**Bax**). On the other hand in the case of extremely weak theories switching from $\mathbf{Ax5_{Obs}}$ to $\mathbf{Ax(5nop)^-}$ might cause a difference.

$\triangleleft$

Although in **Relnoph** there are no photons and therefore from the formal point of view speaking about FTL observers does not seem to make sense; we already have seen that in **Relnoph** in the case of type A the idea of a certain speed of light does naturally come up and therefore it becomes meaningful to ask whether FTL observers exist. Next we briefly discuss the status of FTL observers in **Relnoph**. Since in **Relnoph** there are no photons we have to clarify what we mean by an

FTL observer. Observer k is defined to be an FTL observer w.r.t. m iff there is k_1 such that f_{mk_1} is a strict standard type A transformation and the "speed of light" $c = \tan(\alpha)/\tan(\beta)$ associated to f_{mk_1} in the proof of Theorem 5.0.46 is strictly "slower" than $v_m(k)$. Now,

$$\textbf{Relnoph} \models \text{“}\nexists \textbf{ FTL observers”}$$

because of e.g. **Ax(5nop)**. (To see this we note that **Bax**$^-$+**Ax(5nop)** already excludes FTL observers because of the following. If an FTL observer existed then by **Ax(5nop)** an observer would travel together with a photon. This leads to a contradiction, since in **Bax**$^-$ there are no photons with speed zero cf. Corollary 5.2.5(i) and (332) on p.738 in the proof of Thm.5.0.46 in this connection.) This motivates our looking at **Relnoph**$^-$. (**Relnoph**$^-$ was defined in Question 5.2.20.)

$$\textbf{Relnoph}^- \models \text{“}\nexists \textbf{ FTL observers”}.$$

This is caused by **Ax**$\triangle$**1** exactly the same way as we saw in §2, cf. Corollary 2.7.6. This motivates our looking at **Relnoph**$^{--}$. We define

$$\textbf{Relnoph}^{--} \stackrel{\text{def}}{=} \textbf{Relnoph}^- \setminus \{\textbf{Ax}\triangle\textbf{1}\} + \textbf{Ax(symm)}.$$

Recall that **Basax** + **Ax**$\Box$**1** + **Ax(symm)**, for $n = 2$, is consistent with FTL observers, while **Basax**+**Ax**$\triangle$**1** is not, cf. Cor.2.7.6 (p.119), Thm.2.8.2 (p.127) and Thm.3.9.8 (p.352).

Questions for future research 5.2.23

(i) Assume $n > 2$. Is then

$$\textbf{Relnoph}^{--} \models \text{“}\nexists \textbf{ FTL observers”}?$$

Relphax definicióját megjelölni "Relphax-def"-el

(ii) Recall that our axiom system **Relphax** (cf. Def.3.4.21) is consistent with FTL observers, for $n > 2$, too. Is **Relphax**+**Ax**$\Box$**1** or **Relphax**+**Ax(symm)** consistent with FTL observers, for $n > 2$?

(iii) Investigate **Relnoph**$^{--}$.

(iv) Do our present theorems, conjectures etc. generalize to **Relnoph**$^{--}$?

$\triangleleft$

atgondolni, hogy ezzel mi legyen

At this point we stop discussing the idea of a photon-free approach to the logical analysis of special relativity.

We note that some of the ideas in the present section can be traced back to Ignatowsky [140] and to Pars [210]. Actually the contents of the present chapter can be viewed the following way: Here, we collect the implicit and tacit assumptions used in the above two works (as well as e.g. in Rindler [224, §2.17].) Then, we make them explicit in the form of axioms in pure first-order logic. Besides making them explicit we generalize them in order to obtain stronger (i.e. more general) statements. An example for such a generalization is our replacing $\mathfrak{R}$ with an arbitrary Euclidean field $\mathfrak{F}^{\mathfrak{N}}$.

As a result of this activity we obtain the set of axioms **Relnoph**.

We consider it as a task for future research to "streamline" **Relnoph** and to figure out how much of it is needed to derive the theorems we formulated above.

5.3 Some further potential axioms

In the present sub-section we discuss a potential axiom **Ax(cont)** which often shows up in relativity books either as an explicit assumption or as a tacit assumption, cf. e.g. Rindler [224, p.31 line 9 bottom up].

The following theorem says that, under reasonable assumptions, the world-view transformation f_{mk} is determined by the velocity $\vec{v}_m(k)$.

THEOREM 5.3.1 *Assume* **Relnoph**. *Assume that m, k and m', k' are both in strict standard configuration. Assume further $(m \uparrow k \;\Leftrightarrow\; m' \uparrow k')$. Then*
$v_m(k) = v_{m'}(k') \;\Rightarrow\; \mathsf{f}_{mk} = \mathsf{f}_{m'k'}$.

Proof: The proof can be recovered from the proof of Thm.5.0.46. Namely, by the proof of Thm.5.0.46 each **Relnoph** model $\mathfrak{N}$ can be "basically" embedded into a generalized Minkowski model or to a generalized rotation model or $\mathfrak{N} \models$ **"NewtK$^-$"**. (Cf. Figures 247, 248.) It is easy to check that in all three cases the conclusion of the theorem holds. We note that a direct proof can be obtained by using **Ax**(ω)+**Ax**$(\|)$ (cf. Prop.5.0.34(iv)). ∎

Intuitively, the above theorem implies that the world-view transformation f_{mk} is a *function of*[612] $v_m(k)$, under assuming strict standard configuration, $m \uparrow k$ and **Relnoph**. Roughly speaking our axiom **Ax(cont)** below will say that this function $v_m(k) \mapsto \mathsf{f}_{mk}$ is not an arbitrary function in the sense that it is "continuous".[613] We state **Ax(cont)** in an informal, mathematical style instead of using the first-order language of $\mathfrak{M}$. Therefore we will mention the model $\mathfrak{M}$ explicitly in the formulation of this axiom. Recall that $1_0 = 1_t, \ldots, 1_3 = 1_z$.

Ax(cont) Let $i < n$. The relation h_i defined below is <u>*continuous*</u>[614] whenever it is a (partial) function.

$$h_i \stackrel{\text{def}}{=} \{ \langle v_m(k), \mathsf{f}_{mk}(1_i) \rangle \ : \ m, k \in Obs^{\mathfrak{M}} \text{ are in strict standard configuration and } m \uparrow k \}.$$

Instead of translating **Ax(cont)** to our frame language we indicate only the idea of such translation. First we note that the relation h_i is definable in our frame language as follows. Let $\psi_i(\lambda, p)$ be the formula saying $\lambda \in F$, $p \in {}^nF$ and $(\exists m, k)$ $(v_m(k) = \lambda,\ \mathsf{f}_{mk}(1_i) = p,\ m$ and k are in strict standard configuration and $m \uparrow k)$. Now, we use the formalization of continuity given in §4.4 (p.536), and then the formalized version of **Ax(cont)** will say that whenever the relation defined by $\psi_i(\lambda, p)$ is a function, than this function is continuous.

FACT 5.3.2

$$\begin{array}{rcl} \textbf{Basax} & \not\models & (h_i \textit{ defined in } \textbf{Ax(cont)} \textit{ is a function}), \quad \textit{but} \\ \textbf{Flxspecrel} & \models & (h_i \textit{ defined in } \textbf{Ax(cont)} \textit{ is a function}). \end{array}$$

$\triangleleft$

The following conjecture is related to the idea of a possible reformulation of **Ax(cont)** such that instead of the functions h_i it would use a single function h^+.

[612]i.e. is determined by

[613]It would be interesting to see whether "strong continuity" as introduced in §4.4 (p.536) can be adapted to the present situation. (The reason why this might be useful is that we are working over arbitrary Euclidean fields $\mathfrak{F}$.)

[614]For the translation of continuity to our frame language we refer to §4.4, p.536.

Conjecture 5.3.3 *Assume* $(\forall m,k)\mathsf{f}_{mk} \in Aftr$ *and* **Ax(cont)**. *Assume the relation* h^+ *below is a (partial) function. Then* h^+ *is continuous in both of its arguments* p *and* $v_m(k)$.

$$h^+ \stackrel{\text{def}}{=} \Big\{ \Big\langle \langle p, v_m(k)\rangle, \mathsf{f}_{mk}(p) \Big\rangle \ :\ m,k \in Obs^{\mathfrak{M}} \textit{ are in strict standard configuration and } m \uparrow k \Big\}.$$

◁

In connection with the above conjecture we note that whenever h^+ is a (partial) function then

$$h^+ : {}^nF \times F \xrightarrow{\circ} {}^nF,$$

where h^+ is defined in the above conjecture.

Ax(cont)$^+$ Let the relation h^+ be defined as in Conjecture 5.3.3. Now, if h^+ is a (partial) function then it is continuous in both of its arguments.

In connection with the above conjecture and axiom we note a possible reformulation (e.g. of **Ax(cont)$^+$**) which might look more intuitive.

First we define a topology on the set of functions ${}^{({}^nF)n}F$. Assume h^+ (defined in Conjecture 5.3.3) is a (partial) function. Then we define

$$h : \text{speeds} \longrightarrow {}^{({}^nF)n}F$$

such that $h(v_m(k))(p) = h^+(p, v_m(k)) = \mathsf{f}_{mk}(p)$ assuming m,k are in strict standard configuration etc. Then we require that this h is continuous. The mathematical content of this new statement would be the same as that of the conclusion of Conjecture 5.3.3 or equivalently **Ax(cont)$^+$**.

The following axiom makes **Ax(cont)** (or **Ax(cont)$^+$**) slightly stronger if we add it to **Ax(cont)**.

Ax($\exists \uparrow$) $\forall m,k\ \exists k'\ [v_m(k) \neq \infty \ \Rightarrow\ (tr_m(k) = tr_m(k') \ \ \wedge \ \ m \uparrow k')]$.

Intuitively, for every observer if a line is realized by another observer, then it is also realized by an observer whose clock runs forwards.

THEOREM 5.3.4

(i) **Relnoph** $\models$ **Ax(cont)** + **Ax(cont)**$^+$.

(ii) **Flxspecrel** $\models$ **Ax(cont)** + **Ax(cont)**$^+$.

Proof: We omit the proof. ∎

It would be interesting to know whether **Basax** $\models$ **Ax(cont)** holds. In this connection we note that in many **Basax** models h_i is not a function (i.e. $v_m(k)$ does not determine f_{mk}) and therefore **Ax(cont)** is vacuously (i.e. trivially) true in them.

Ax(*Triv*) nélkül nem érdemes ezt a kérdét nézni!

In passing, we note that it would be interesting to go over our theories **Specrel**, **Flxspecrel**, ..., **Bax**, ...etc. and decide in which ones do strict standard configurations work. Similarly for standard configurations (and their variants). Cf. e.g. items 5.0.49, 5.0.50 in this chapter.

Remark 5.3.5 Theorem 5.3.1 way above says that, assuming **Relnoph**, velocities determine world-view transformations in a certain sense. From the point of view of usual relativity books this is a very interesting and natural postulate. Let us formalize the conclusion of Theorem 5.3.1 as a potential axiom **Ax(fun)** as follows. (We note that fun refers to functionality.)

Ax(fun) Assume that m, k and m', k' are both in strict standard configuration. Assume further $(m \uparrow k \;\Leftrightarrow\; m' \uparrow k')$. Then $v_m(k) = v_{m'}(k') \;\Rightarrow\; \mathsf{f}_{mk} = \mathsf{f}_{m'k'}$.

Now, we have

$$\begin{array}{rcll} \textbf{Relnoph} & \models & \textbf{Ax(fun)} & \text{but} \\ \textbf{Basax} + \textbf{Ax}(\mathit{Triv}) & \not\models & \textbf{Ax(fun)} & \text{on the other hand} \\ \textbf{Basax} + \textbf{Ax}(\omega) & \models & \textbf{Ax(fun)}. & \end{array}$$

Ax(fun) says something that sounds very natural: It says that the properties of a world-view transformation f_{mk} are, basically, determined by the kind of relative motion (i.e. by $\vec{v}_m(k)$) of the two observers m and k involved in f_{mk}.

Therefore **Ax(fun)** could be used and discussed as a potential natural axiom which could be added to **Basax** and to its variants. We note that in many respects **Ax(fun)** behaves similarly to the symmetry axioms discussed in §3.9. Actually, a variant of **Ax(fun)** appears as potential axiom $(\star\star\star)$ in Remark 3.9.10 in that section. We leave a systematical discussion of **Ax(fun)** as research task to the future.

For completeness we note that

$$\mathbf{Ax(cont)} + \mathbf{Ax(fun)}$$

is a potential axiom which is strong and natural in the context of **Basax** and its variants. In passing we note that **Basax** + **Ax(fun)** + $\mathbf{Ax}(\exists\uparrow)$ + $\mathbf{Ax}(\mathit{Triv})$ + $\mathbf{Ax}(\sqrt{\ })$ $\models$ **Ax(cont)**, i.e. continuity follows already from a symmetry principle in **Basax**. Of course **Ax(fun)** $\not\models$ **Ax(cont)**.

$\triangleleft$