3 Special Relativity continued

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Contents of Chapter 3

Introduction to Chapter 3

Symmetry axiomsok-ról írni a bevezetőben!

In this chapter we investigate **Basax** itself as well as variants of **Basax** designed for various purposes. Further, in chapter §4 we will study a hierarchy of weak subtheories of **Basax** and of related theories. More precisely we will do things listed in items (i)–(iv) below.

(i) In §§ 3.1, 3.2, 3.4.1, 3.5–3.7 we continue the investigation of our axiomatic (special) relativity theory **Basax** started in §2. E.g. we will prove theorems announced but not proved in §2. As a contrast with §2, the emphasis here will be on the case $n > 2$, but the $n = 2$ case will not be ignored either.

(ii) In §3.8 we will continue experimenting with making **Basax** ***stronger*** (by adding e.g. further symmetry principles to it), which was started in §2.8, and eventually extending it to a complete theory ($\mathbf{BaCo} + \mathbf{Ax(rc)}$) axiomatized by a finite schema of axioms. This complete version $\mathbf{BaCo} + \mathbf{Ax(rc)}$ of our axiomatic theory will agree with the most traditional version of Einstein's (special) theory of relativity, and its models will be seen to be the same as the usual Minkowski models over real-closed fields (cf. p.331 for Minkowski models).

(iii) In §3.3 we will use our experience (obtained so far) with **Basax** for developing a new, *refined*, more flexible version **Newbasax** of **Basax**.[163] Roughly, the motivation for **Newbasax** is twofold: (1) To prepare ourselves for developing a the-

[163] This procedure of studying a "logical approach", say **Basax**, for relativity for a while, and then *using* the experience so gained to *develop* a more refined, more advanced and (in some sense) more subtle *new* logical approach (e.g. **Newbasax**) to the same subject is a typical part of the methodology (or philosophy) of this work, and will be repeatedly applied. E.g. after studying **Basax** and its variants (e.g. its Reichenbachian variant) in chapters 2, 3, we will *use again* the so obtained experience (and intuition) to develop an even further logical theory, more precisely a further logical "angle" to the subject matter which in §6 will be called "observer independent geometry" (and "duality theory" etc). This "stepwise refinements" architecture (or style) of the present work gives it a kind of "bottom-up" character and is strongly tied up with the goals connected to the observational/theoretical duality in items (I), (X) of §1.1. At this point we also note the following, which builds on item (IX) of §1.1.

By *"bottom up"* character we mean that we do not want to start out with the full and finalized mathematical machinery from which everything can be computed efficiently, and which then would have to be accepted by the reader as an article of faith (and which then would necessarily be on the theoretical side of the observational/theoretical duality), but instead we want to reach that theory as a result of a series of insights and (hopefully) deep understandings (of something) which series begins with a relatively simple theory on the observational side. Certainly, a top down approach, beginning with the mathematical apparatus of the final theory would give us *knowledge* (perhaps more quickly than the present bottom up approach). However, in the present work we are seeking *understanding* as opposed to mere knowledge.

ory in which accelerated observers (hence a kind of gravity etc.) are permitted[164], and via this (and similar improvements) to prepare our intuition for understanding, eventually, some of the basic ideas of general relativity. (2) To improve **Basax** from a certain logical point of view (which might help us achieving some of the goals in item (III) of §1.1).

(iv) In §1.1 ("Introduction")[165] we mentioned several goals, all of which require "breaking up" our theory **Basax** (and also **Basax+Ax(symm)**) into smaller, more flexible sub-theories (and study how they relate to each other, which "paradigmatic effect" of relativity is the consequence of which sub-theory etc.). Among these goals was a continuation of the "conceptual analysis" of relativity started by e.g. Reichenbach and Grünbaum, Friedman [90], Gyula Dávid and others. §§3.4.2, 3.4.3, 4 will be devoted to these goals. E.g. we will see that some of the exotic predictions of relativity remain true even in extremely weak versions of **Basax** (and we will also see what axioms to change if we "desperately" want to get rid of some of these predictions cf. pp. 222–226). Also, in §4.5 we will formalize the often discussed (e.g. in the philosophy of science) Reichenbachian version of relativity theory, and we will see that its connections with the Einsteinian version can be made quite explicit by means of logic (parts of this comparison are postponed to §6, because there we will have the right tools for doing them). As we already mentioned in §4 we study a hierarchy of weak theories (opening up new perspectives the exploration of which remains a promising opportunity for future research). We will also look into modifying **Basax** such that it will admit Newtonian Kinematics as a special case. We will also look into a possibility of deriving special relativity without mentioning photons (or the speed of light).

* * *

Section 3.1 below is a direct continuation of §§ 2.3, 2.9. It contains proofs of results stated in §2 as well as some further considerations related to these results. The reason why this material is in §3 and not in §2 is that in §2 we wanted to include things which are intuitively interesting (from the point of view of relativity) without slowing down the reader with checking too much of the "technical detail", while, from this point on, we want to include the "details" too. Further, the material in §3.1 will be needed in later parts of §3 in the form in which it is presented in §3.1.

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[164] Such a first-order theory called **Acc** will be elaborated in §8 ("Accelerated observers").
[165] in items (V), (X), (III) cf. pp. 8, 12, 7.

3.1 Some properties of the world-view transformation

In this section we prove that **Basax** implies that every world-view transformation f_{mk} takes all straight lines to straight lines. I.e. that the f_{mk}'s are collineations. We will do this in Thm.3.1.1 below. Later we will see that the f_{mk}'s will be provable to be collineations in axiom systems weaker than **Basax** too, e.g. in **Newbasax**, **Bax** and $\mathbf{Bax}^-$ to be introduced in §§ 3.3, 3.4.2, 4.3. We note that $\mathbf{Bax}^-$ is <u>much</u> weaker than **Basax** (under assuming $\mathbf{Ax}(\sqrt{\ })$). Actually we will introduce in §4.5 an axiomatization **Reich**(**Basax**) of the Reichenbachian version of relativity theory, cf. Friedman [90, §IV.7]. This Reichenbachian version has some non-negligible philosophical significance cf. e.g. [90] and Szabó [244]. We mention this because **Reich**(**Basax**) will be weaker than **Basax**; and $\mathbf{Bax}^-$ will be a common weakening of **Reich**(**Basax**) and **Basax** (all these are understood under assuming $\mathbf{Ax}(\sqrt{\ })$), cf. Figure 180 on p.552. Hence the f_{mk}'s will be collineations in e.g. **Reich**(**Basax**) too.

After proving Thm.3.1.1 we discuss some corollaries of the proof and also some further properties of the world-view transformations.

On the philosophical significance of these "f_{mk}-theorems", in connection with e.g. items (III) and the second half of (I) of §1.1, we will write in Remark 3.1.5 below.

THEOREM 3.1.1

$$\begin{array}{lrcl} \textbf{(i)} & \mathbf{Basax} & \models & (\forall m,k \in Obs)(\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}\,. \\ \textbf{(ii)} & \{\mathbf{Ax1},\mathbf{Ax2},\mathbf{Ax3},\mathbf{Ax5},\mathbf{Ax6}\} & \models & (\forall m,k \in Obs)(\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}\,. \end{array}$$

We will give the **proof** of Thm.3.1.1 on p.169. We will start working on the preparations for the proof of Thm.3.1.1 below Lemma 3.1.6 on p.163.

Before proving Thm.3.1.1 we state one of its corollaries in Thm.3.1.4 below. But first we include Conventions 3.1.2 and 3.1.3 below.

CONVENTION 3.1.2 Throughout we use the unifying terminology (and notation) of <u>*universal algebra*</u> (and model theory). E.g. a subalgebra of a vector space is automatically a vector space,[166] a subalgebra of a group is automatically a group, and we need not call it a subgroup, etc. Homomorphisms, automorphisms etc. are defined for any kind of algebraic structures in a uniform way. For example let $\mathfrak{A}$ be any algebraic structure or model. Then

[166]Instead of calling it a sub-vector space or subspace we call it a subalgebra; we emphasize that our using the expression "subalgebra" has nothing to do with the special algebraic structures called "algebras" in classical and linear algebra.

$Aut(\mathfrak{A})$ denotes the set of automorphisms of the structure $\mathfrak{A}$.

Another example for universal algebraic notational conventions which we will use is the following. For any one-sorted structure denoted as $\mathfrak{A}$ or $\mathbf{A}$, its universe is denoted by the corresponding "plain" capital A (we emphasize that this does not apply to many-sorted structures, for the obvious reason). Returning briefly to subalgebras, assume $\mathfrak{M}$ and $\mathfrak{N}$ are two similar[167] (possibly many sorted) algebraic structures, i.e. models. Then

$\mathfrak{M} \subseteq \mathfrak{N}$ means that $\mathfrak{M}$ is a strong submodel (or a strong substructure[168]) of $\mathfrak{N}$,

i.e. all relations and sorts of $\mathfrak{M}$ are restricted to the universe[169] of $\mathfrak{N}$, cf. e.g. Chang-Keisler [59, p.21] ([59] writes submodel for strong submodel). So, to *simplify* the discussion, we rely on the fact that universal algebra *unifies* the terminology of a large part of mathematics (including practically all parts of algebra). Cf. e.g. [54]. The reader not familiar with the terminology and "spirit" of universal algebra is asked to look briefly into a universal algebra book e.g. Burris-Sankappanavar [54] or Henkin-Monk-Tarski [129, Part I, Chapter 0] or McKenzie-McNulty-Taylor [192] or Grätzer [112].[170]

Cf. also Convention 3.8.4 on p.298.

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CONVENTION 3.1.3 Usually "$\models$" is used with a first-order formula on its right hand side. A typical example is

$$\mathbf{Basax} \models (\mathsf{f}_{mk} \text{ is a bijection}).$$

Sometimes however we use "$\models$" in a more general way, as follows. Whenever A is a meaningful statement about frame models then

$$\mathbf{Th} \models A \qquad \text{means that } (\forall \mathfrak{M} \in \mathsf{Mod}(\mathbf{Th}))\ [\text{statement } A \text{ is true for } \mathfrak{M}],$$

[167]Two structures are similar if they have the same language, cf. the Index.

[168]or a subalgebra if there are no relation symbols involved

[169]We want to define the universe of many sorted structure $\mathfrak{N}$. Recall that for each sort s of $\mathfrak{N}$ there is a universe U_{s} of $\mathfrak{N}$. Now, the universe of $\mathfrak{N}$ is defined to be the union $\bigcup \{ U_{\mathsf{s}} : \mathsf{s} \text{ is a sort of } \mathfrak{N} \}$. In the case of frame models the universe of $\mathfrak{N}$ is $B^{\mathfrak{N}} \cup F^{\mathfrak{N}} \cup G^{\mathfrak{N}}$.

[170]It *is* (easily) possible to understand this material without familiarity with the language of universal algebra, but that familiarity can be achieved in a very short time, and then it will "pay back" to the reader in rendering the present material simpler and more transparent for him.

where **Th** is any axiom system (like e.g. **Basax**) in our frame language. An example is Thm.3.1.4 below.[171] We will use this convention throughout the present work, and not only for frame models, e.g. in the geometry chapter (§6) we will use it for the structures (e.g. geometries) occurring there.

$\triangleleft$

The following theorem states some useful properties of world-view transformations in **Basax** models. (Later we will see that these properties remain true in axiom systems weaker than **Basax**, too).

THEOREM 3.1.4

(i) $\mathbf{Basax} \models (\forall m, k \in Obs)\Big(\mathsf{f}_{mk} = \widetilde{\varphi} \circ \mathsf{f},\ \ \textit{for some } \mathsf{f} \in Aftr \textit{ and } \varphi \in Aut(\mathbf{F})\Big).$

(ii) $\mathbf{Basax} \models (\forall m, k \in Obs)\Big(\mathsf{f}_{mk} = \mathsf{f} \circ \widetilde{\varphi},\ \ \textit{for some } \mathsf{f} \in Aftr \textit{ and } \varphi \in Aut(\mathbf{F})\Big).$

(iii) $\mathbf{Basax} \models (\forall m, k \in Obs)\Big(\mathsf{f}_{mk} = \mathsf{f} \circ \tau,\ \ \textit{for some } \mathsf{f} \in Aut({}^{n}\mathbf{F}_2) \textit{ and } \tau \in Tran\Big).$

Proof: The theorem is a corollary of Thm.3.1.1 and Lemma 3.1.6 below. ∎

Remark 3.1.5 (Philosophical significance of the "f_{mk}"-theorems.)
Throughout this remark by an f_{mk} we understand a $\bar{0}$-preserving world-view transformation f_{mk}, for simplicity.

The above theorem states that the f_{mk}'s are very close to linear transformations. Later under some extra conditions we will see that they are actually linear transformations (cf. Prop.3.8.35 on p.317). In this connection, we would like to point out some methodological considerations which were hinted at in §1.1 e.g. in items (III) ("Searching for insight") and the second half of (I). Namely, one could formulate a potential axiom saying that the f_{mk}'s are linear. However such an axiom would be undesirable because it is not clear what the intuitive meaning of the axiom is, why would a "child" believe in it. Actually, we feel that our metaphorical child might not even be able to grasp what such an axiom really means. In some sense, the physical content of such an axiom would not be concrete enough. (On the other hand having it as a theorem i.e. a consequence of "intuitively convincing" axioms

[171] In most of the cases these statements A will be translatable to our frame language, but there may be exceptions. In the case of Thm.3.1.4 the formulas standing on the right hand side of "$\models$" are translatable to our frame language, but one needs a little extra effort for translating them. A non-translatable example (for A) would be something like this:
$\mathfrak{M} \models$ ($\mathfrak{F}$ has no nontrivial automorphism).

is very nice and completely fits our philosophy.) Line preserving does not sound as "bad" as linear[172], but even that is such that we are happy that we do not have to state it as an *axiom* and are happy to have it as a theorem, instead (cf. Thm.3.1.1).

If we are already at this topic then we also note the following. If for some reason we had to postulate that the f_{mk}'s are close to being linear, we would be more willing to say that they are continuous, for example. So, continuity sounds a little bit more "intuitively convincing" cf. item (III) of §1.1. Saying that the f_{mk}'s are betweenness preserving sounds even more acceptable as a potential axiom.[173] (Fortunately, we will have that too, as a theorem, cf. Prop.6.6.5 on p.1028.)

$\lhd$

The following lemma is known from algebra and geometry.Among others, it says that a function $\mathsf{f} : {}^nF \longrightarrow {}^nF$ is a bijection taking straight lines to straight lines iff it is an affine transformation composed with a map induced by a field automorphism.

LEMMA 3.1.6 *Assume* $\mathsf{f} : {}^nF \longrightarrow {}^nF$ *is a function. Then (i)–(iv) below are* equivalent.

(i) f *is a bijective collineations.*

(ii) $\mathsf{f} = \widetilde{\varphi} \circ \mathsf{g}$, *for some* $\mathsf{g} \in \mathit{Aftr}$ *and* $\varphi \in \mathit{Aut}(\mathbf{F})$.

(iii) $\mathsf{f} = \mathsf{g} \circ \widetilde{\varphi}$, *for some* $\mathsf{g} \in \mathit{Aftr}$ *and* $\varphi \in \mathit{Aut}(\mathbf{F})$.

(iv) $\mathsf{f} = \mathsf{g} \circ \tau$, *for some* $\mathsf{g} \in \mathit{Aut}({}^n\mathbf{F}_2)$ *and* $\tau \in \mathit{Tran}$.

On the proof: A proof for (i) $\Rightarrow$ (ii) (or analogously for (i) $\Rightarrow$ (iii)) can be obtained using the coordinatization procedure described in Goldblatt [108, pp.23–27] or Hilbert [134, §24] (cf. also §6.6 of the present work). The parts (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (i) are obvious. We guess that a full proof of this lemma is available in the literature. ∎

In connection with Lemma 3.1.6 cf. Remark 2.3.13.

Now, we turn to proving Thm.3.1.1. In the proof of Thm.3.1.1 we will use Prop.2.3.3(v),(viii) (p.58) and two elementary propositions from "Euclidean geometry" (cf. Prop.3.1.12, Prop.3.1.13 below) and Lemmas 3.1.9, 3.1.10 below. For completeness, let us recall that Prop.2.3.3(v) states that $\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF$ is a bijection, and Prop.2.3.3(viii) states that f_{mk} takes slow-lines to straight lines, i.e. if $\ell \in \mathsf{SlowEucl}$ then $\mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$ (and both (v) and (viii) of Prop.2.3.3 assume **Basax**). First, we need some notation and definitions.

[172]We mean stating line preserving as an axiom versus stating linearity as an axiom.

[173]We note that in **Basax** models if the f_{mk}'s are betweenness preserving then they are continuous too. (This can be seen by Thm.3.1.4.) This remains true if we replace **Basax** with one of the weaker theories like **Bax**$^-$ to be studied in §4.3.

Notation 3.1.7 Let $p, q \in {}^nF$ such that $p \neq q$. By $\overline{pq}$ we denote the Euclidean straight line which contains both p and q. I.e. $p, q \in \overline{pq} \in \mathsf{Eucl}$. Whenever we write $\overline{pq}$, we assume that $p, q \in {}^nF$ and $p \neq q$. (This will be slightly different in the geometry chapter, §6.)

$\lhd$

Definition 3.1.8 Let $\mathbf{F}$ be a field.

1. Let $j \leq n$. We say that P is a *j-dimensional plane* iff there is a j-dimensional subspace[174] $\mathbf{W}$ of ${}^n\mathbf{F}$ and a vector $p \in {}^nF$ such that $P = W + p$, where $W + p \stackrel{\text{def}}{=} \{\, w + p \,:\, w \in W \,\}$.[175]
   By a *plane* we understand a 2-dimensional plane.
   By a *hyper-plane* we understand an $n-1$-dimensional plane.

2. Let $\ell_1, \ell_2 \in \mathsf{Eucl}$.
   (i) We say that ℓ_1 and ℓ_2 are in the same plane if there is a 2-dimensional plane P such that $\ell_1, \ell_2 \subseteq P$.[176]
   (ii) If there is a unique 2–dimensional plane P such that $\ell_1, \ell_2 \subseteq P$, then we denote this unique P by
   $$\mathsf{Plane}(\ell_1, \ell_2)\,.$$
   E.g. $\mathsf{Plane}(\bar{t}, \bar{x}) = F \times F \times {}^{n-2}\{0\}$ and
   $\mathsf{Plane}(\bar{t}, \bar{y}) = F \times \{0\} \times F \times {}^{n-3}\{0\}$.
   (iii) We say that ℓ_1 and ℓ_2 are *parallel*, in symbols $\ell_1 \parallel \ell_2$, iff ℓ_1 and ℓ_2 are in the same plane and $\ell_1 \cap \ell_2 = \emptyset$ or $\ell_1 = \ell_2$.
   (iv) Whenever we write $\ell \parallel \ell'$ and we do not indicate what kinds of objects ℓ and ℓ' are, then the symbol $\ell \parallel \ell'$ abbreviates the formula ($\ell \parallel \ell'$ and $\ell, \ell' \in \mathsf{Eucl}$).[177] (This will be slightly different in the geometry chapter, §6.) $\lhd$

[174] Let us recall from the literature that by a *subspace* of the vector space ${}^n\mathbf{F}$ we understand a subalgebra (in the universal algebraic sense, cf. Conv.3.1.2) $\mathbf{W} \subseteq {}^n\mathbf{F}_1$ of the one-sorted vector space ${}^n\mathbf{F}_1$. Further a one-sorted vector space $\mathbf{W}$ is *j-dimensional* iff there is a j-element minimal generator system $G \subseteq W$, i.e. G generates $\mathbf{W}$ but no proper subset of G generates $\mathbf{W}$. (G *generates* $\mathbf{W}$ if no proper subalgebra of $\mathbf{W}$ contains G.)

[175] We use the universal algebraic convention that $\mathbf{W}$ denotes the algebra (vector space) and W denotes its universe. (We also note that by a plane one understands a set of form $W + p$, where $\mathbf{W}$ is 2-dimensional.)

[176] The standard geometry literature uses the expression "ℓ_1 and ℓ_2 are coplanar" for this.

[177] For completeness, we note that there will be situations when e.g. ℓ' is a plane (in the formula $\ell \parallel \ell'$) but then this will be indicated explicitly.

We note that P is a j-dimensional plane for some $j \leq n$ iff $\overline{pq} \subseteq P$ whenever $p, q \in P$.

The following lemma states that **Basax** implies that f_{mk} takes parallel slow-lines to parallel straight lines.

LEMMA 3.1.9 (parallelism is preserved)

Basax $\models (\forall m, k \in Obs)(\forall \ell_1, \ell_2 \in \mathsf{SlowEucl})(\ell_1 \parallel \ell_2 \;\Rightarrow\; \mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2])$.

Proof: Let $\ell_1, \ell_2 \in \mathsf{SlowEucl}$ with $\ell_1 \parallel \ell_2$. We have to prove that $\mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]$. We may assume that $\ell_1 \neq \ell_2$. Let $\ell_3, \ell_4 \in \mathsf{SlowEucl}$ such that $\ell_1 \cap \ell_3 = \{p\}$, $\ell_1 \cap \ell_4 = \{q\}$, $\ell_2 \cap \ell_4 = \{r\}$, $\ell_2 \cap \ell_3 = \{s\}$, $\ell_3 \cap \ell_4 = \{t\}$, for some distinct $p, q, r, s, t \in {}^nF$ (see Figure 51). Let such p, q, r, s, t be fixed.

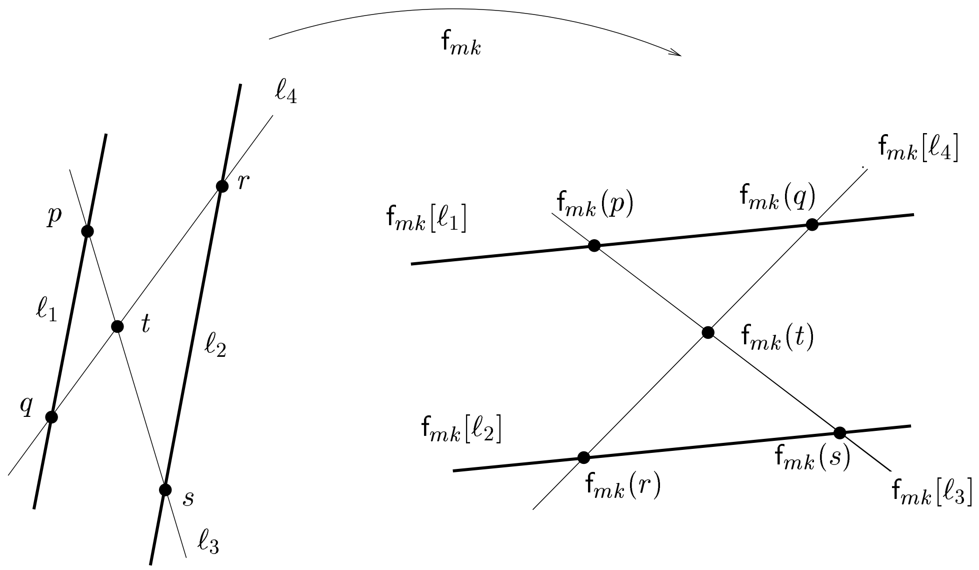


Figure 51: Illustration for the proof of Lemma 3.1.9.

Obviously such ℓ_3 and ℓ_4 exist. Now by the above construction and by f_{mk} being a bijection taking slow-lines to straight lines (cf. Prop.2.3.3(v),(viii)), we have that (10)–(13) below hold (see Figure 51).

$$\begin{aligned}
\mathsf{f}_{mk}[\ell_1] \cap \mathsf{f}_{mk}[\ell_3] &= \{\mathsf{f}_{mk}(p)\}\,, \\
\mathsf{f}_{mk}[\ell_1] \cap \mathsf{f}_{mk}[\ell_4] &= \{\mathsf{f}_{mk}(q)\}\,,
\end{aligned}$$

$$(10) \qquad \begin{aligned} \mathsf{f}_{mk}[\ell_2] \cap \mathsf{f}_{mk}[\ell_4] &= \{\mathsf{f}_{mk}(r)\}\,, \\ \mathsf{f}_{mk}[\ell_2] \cap \mathsf{f}_{mk}[\ell_3] &= \{\mathsf{f}_{mk}(s)\}\,, \text{ and} \\ \mathsf{f}_{mk}[\ell_3] \cap \mathsf{f}_{mk}[\ell_4] &= \{\mathsf{f}_{mk}(t)\}\,. \end{aligned}$$

$$(11) \qquad \mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r), \mathsf{f}_{mk}(s), \mathsf{f}_{mk}(t) \text{ are distinct points.}$$

$$(12) \qquad \mathsf{f}_{mk}[\ell_1], \mathsf{f}_{mk}[\ell_2], \mathsf{f}_{mk}[\ell_3], \mathsf{f}_{mk}[\ell_4] \in \mathsf{Eucl}\,.$$

$$(13) \qquad \mathsf{f}_{mk}[\ell_1] \cap \mathsf{f}_{mk}[\ell_2] = \emptyset\,.$$

By (10)–(12), we have that $\mathsf{f}_{mk}[\ell_1]$ and $\mathsf{f}_{mk}[\ell_2]$ are in the same plane. This and (13) imply that $\mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]$. ∎

Let $p, q \in {}^nF$. Then, by the ***segment*** pq we understand the ordered pair $\langle p, q\rangle$. Intuitively the segment pq is that part of the line $\overline{pq}$ which is between p and q. The ***midpoint*** of a segment pq is that point $r \in \overline{pq}$ which is halfway between p and q, i.e. $r = \frac{1}{2}(p+q) = \frac{p+q}{2}$. [178]

The following lemma states that if $\overline{pq}$ is a slow-line then f_{mk} takes the midpoint of segment pq to the midpoint of segment $\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(q)$.

LEMMA 3.1.10 (midpoint goes to midpoint)
Basax $\models (\forall m, k \in Obs)(\forall p, q \in {}^nF)\Big(\overline{pq} \in \mathsf{SlowEucl} \quad \Rightarrow$

$$\mathsf{f}_{mk}\left(\frac{p+q}{2}\right) = \frac{\mathsf{f}_{mk}(p) + \mathsf{f}_{mk}(q)}{2}\Big)\,.$$

(See Figure 52.)

We will give the **proof** below Prop.3.1.12.

For the proof of Lemma 3.1.10 we will use Proposition 3.1.12 below which is a proposition in "elementary Euclidean geometry" and it says that the diagonals of a parallelogram bisect each other.

[178] Since p is a vector and $\frac{1}{2} \in F$ $\;\frac{1}{2} \cdot p = \frac{p}{2}$ is defined. (Similarly for $p+q$ in place of p.)

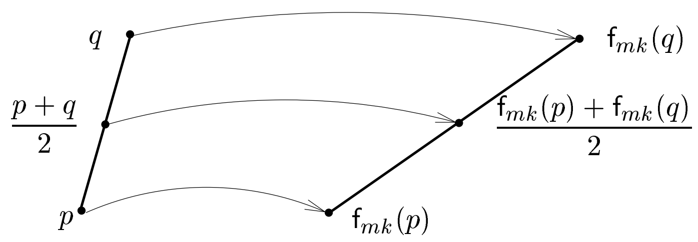


Figure 52: Illustration for Lemma 3.1.10.

Definition 3.1.11 Let $q, r, p, s \in {}^nF$. We say that $\langle q, r, p, s\rangle$ is a <u>*parallelogram*</u> iff the following hold. No three of q, r, p, s are on the same straight line (i.e. collinear), $\overline{qr} \parallel \overline{sp}$, and $\overline{qs} \parallel \overline{rp}$. (Cf. Figure 53.) We write $qrps$ for $\langle q, r, p, s\rangle$.

$\lhd$

PROPOSITION 3.1.12 *Assume $q, r, p, s \in {}^nF$ such that $qrps$ is a parallelogram. Then the diagonals of parallelogram $qrps$ bisect*[179] *each other, that is $\overline{pq} \cap \overline{rs} = \{\frac{p+q}{2}\} = \{\frac{r+s}{2}\}$ (see Figure 53).*

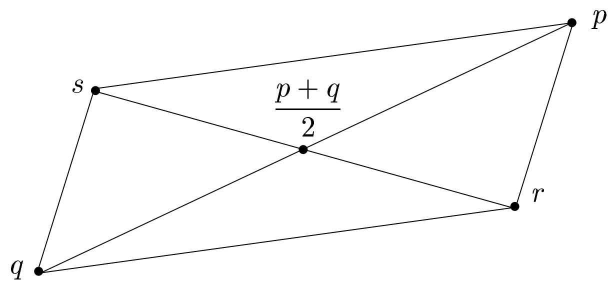


Figure 53: The diagonals of a parallelogram bisect each other.

Proof: The proof known from geometry uses only those properties of our geometry $\langle {}^nF, \mathsf{Eucl}, \ldots\rangle$ which follow from the fact that it is a "usual", Cartesian style geometry over an arbitrary ordered field $\mathfrak{F}$. The reader is invited to check the details. ∎

[179]We say that two segments bisect each other if they intersect and the intersection point is the midpoint of both segments.

Proof of Lemma 3.1.10: Let $p, q \in {}^nF$ with $\overline{pq} \in \mathsf{SlowEucl}$. We will show that $\mathsf{f}_{mk}(\frac{p+q}{2}) = \frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}$. To prove this let $r, s \in {}^nF$ such that $qrps$ is a parallelogram and $\overline{rs}, \overline{qr}, \overline{sp}, \overline{qs}, \overline{rp} \in \mathsf{SlowEucl}$ (see Figure 54). (It is easy to see that such r and s exist because of the following. Choose $\ell \in \mathsf{SlowEucl}$ such that $\frac{p+q}{2} \in \ell$ and $\overline{pq} \neq \ell$. Choose $r, s \in \ell$ such that r and s are "near to" $\frac{p+q}{2}$, then the lines $\overline{rs}, \overline{qr}, \overline{sp}, \overline{qs}, \overline{rp} \in \mathsf{SlowEucl}$, since $\overline{pq} \in \mathsf{SlowEucl}$.) Then by applying Lemma 3.1.9, since $qrsp$ is a parallelogram, $\mathsf{f}_{mk}(q)\mathsf{f}_{mk}(r)\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(s)$ will be a parallelogram (see Figure 54).

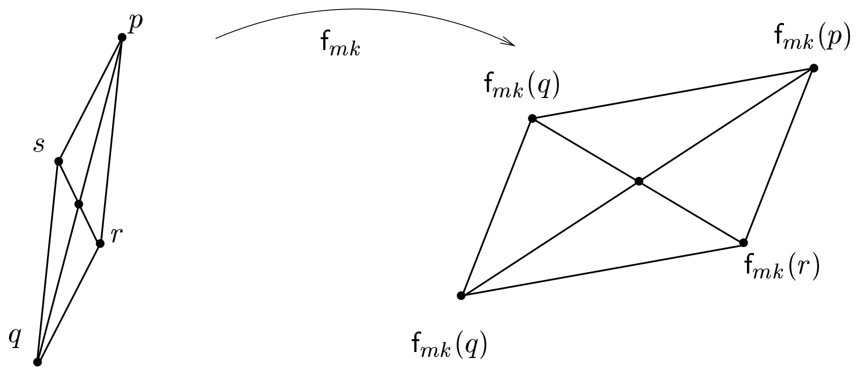


Figure 54: Illustration for the proof of Lemma 3.1.10.

By Prop.3.1.12 the intersection of the diagonals of the parallelograms $qrps$ and $\mathsf{f}_{mk}(q)\mathsf{f}_{mk}(r)\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(s)$ are $\frac{p+q}{2}$, $\frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}$, respectively. Since f_{mk} is a bijection taking slow-lines to straight lines (and since $\overline{pq}, \overline{rs}, \overline{qr}, \overline{ps}, \overline{qs}, \overline{rp} \in \mathsf{SlowEucl}$) f_{mk} will take the intersection $(\frac{p+q}{2})$ of the diagonals of the parallelogram $qrps$ to the intersection $\left(\frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}\right)$ of the diagonals of the parallelogram $\mathsf{f}_{mk}(q)\mathsf{f}_{mk}(r)\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(s)$. Hence

$$\mathsf{f}_{mk}\left(\frac{p+q}{2}\right) = \frac{\mathsf{f}_{mk}(p) + \mathsf{f}_{mk}(q)}{2}. \quad \blacksquare$$

The following proposition is a proposition of "elementary Euclidean geometry".

PROPOSITION 3.1.13 *Assume $p, q, r, s, t, u, v \in {}^nF$ are distinct points such that $\overline{ps} \neq \overline{pt}$, $u \in \overline{ps}$, $v \in \overline{pt}$, $r \in \overline{uv}$, $\overline{st} \parallel \overline{uv}$, and q is the midpoint of segment st (see Figure 55). Then*

$$(r \textit{ is the midpoint of segment } uv) \iff (p, q, r \textit{ are collinear })^{180}.$$

[180] I.e. there is a straight line containing p, q, r.

(See Figure 55).

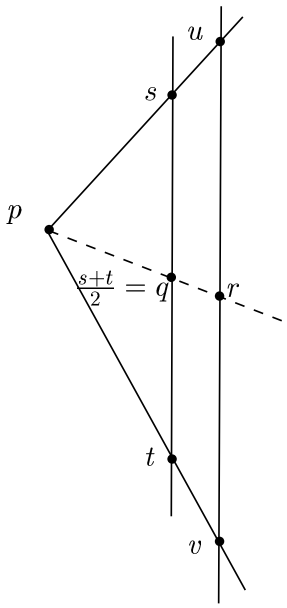


Figure 55: Illustration for Prop.3.1.13 and for the proof of Thm.3.1.1.

Proof: The proof known from geometry uses only those properties of our geometry $\langle {}^nF, \mathsf{Eucl}, \ldots\rangle$ which follow from the fact that it is a "usual", Cartesian style geometry over an arbitrary ordered field $\mathfrak{F}$. The reader is invited to check the details. ∎

Proof of Thm.3.1.1: We will present the proof only for (i) but it will be clear that we never use **Ax4** and **AxE**.

Let $p, q, r \in {}^nF$ be collinear and distinct. To prove Thm.3.1.1, since f_{mk} is a bijection, it is enough to prove that $\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r)$ are collinear. Let $s, t, u, v \in {}^nF$ such that p, q, r, s, t, u, v satisfy the conditions of Prop.3.1.13 (i.e. p, q, r, s, t, u, v are distinct, $\overline{ps} \neq \overline{pt}$, $u \in \overline{ps}$, $v \in \overline{pt}$, $r \in \overline{uv}$, $\overline{st} \parallel \overline{uv}$, and $q = \frac{s+t}{2}$) and they satisfy an extra condition which is $\overline{ps}, \overline{pt}, \overline{st}, \overline{uv} \in \mathsf{SlowEucl}$ (see Figure 55). (It is easy to see that such s, t, u, v exist because of the following. Choose $\overline{st}, \overline{uv}$ to be parallel with the time axis, and choose s and t "very far" from q. Then clearly $\overline{st}, \overline{uv}, \overline{ps}, \overline{st}$ will be slow-lines). Then by direction "$\Leftarrow$" of Prop.3.1.13, we have that r is the midpoint of segment uv. Since $\overline{st}, \overline{uv} \in \mathsf{SlowEucl}$ and q and r are the midpoints of segments st and uv, respectively, by Lemma 3.1.10, we have that (14) and (15) below hold.

(14) $\quad \mathsf{f}_{mk}(q)$ is the midpoint of segment $\mathsf{f}_{mk}(s)\mathsf{f}_{mk}(t)$.

(15) $\quad \mathsf{f}_{mk}(r)$ is the midpoint of segment $\mathsf{f}_{mk}(u)\mathsf{f}_{mk}(v)$.

Now by the above construction, by f_{mk} being a bijection taking slow-lines to straight lines, and by (14), we have that $\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r), \mathsf{f}_{mk}(s), \mathsf{f}_{mk}(t), \mathsf{f}_{mk}(u), \mathsf{f}_{mk}(v)$ satisfy the conditions of Prop.3.1.13 (see Figure 56).

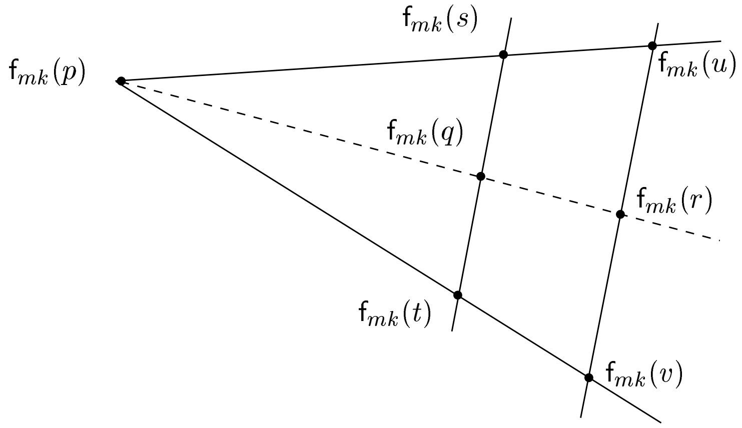


Figure 56: Illustration for the proof of Thm.3.1.1.

But $\mathsf{f}_{mk}(r)$ is the midpoint of segment $\mathsf{f}_{mk}(u)\mathsf{f}_{mk}(v)$ by (15), hence by Prop.3.1.13, we have that $\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r)$ are collinear. By this,
Thm.3.1.1 is proved. ∎

Remark 3.1.14 In connection with the proof of Thm.3.1.1 which says that

$$(\star) \qquad \mathbf{Basax} \models (\text{the } \mathsf{f}_{mk}\text{'s are collineations})$$

we note the following two things.

(i) There exists a different proof using Desargues Theorem.[181] For the idea of that proof we refer to Figure 344 on p.1162.

(ii) For $n > 2$, a proof for

$$(\star\star) \qquad (\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })) \models (\text{the } \mathsf{f}_{mk}\text{'s are collineations})$$

can be obtained from the proof of the celebrated Alexandrov-Zeeman Theorem, as presented in Goldblatt [108, Appendix B], cf. also Thm.6.7.23 on

[181] Cf. e.g. Hilbert [134] for Desargues Theorem.

p.1159 of the present work. This proof for $(\star\star)$ (for $n > 2$) can be recovered from the proof of Theorem 2 (i) in [16] (and uses only axioms **Ax1**, **Ax5**, **Ax6**, **AxE**, **Ax**$(\sqrt{\ })$ from **Basax** + **Ax**$(\sqrt{\ })$). Let us notice that $(\star\star)$ is a slightly weaker form of Thm.3.1.1(i), i.e. of $(\star)$. There are two important differences, which we would like to emphasize, between the two proofs. (1) In the proof of Thm.3.1.1 we do not use **AxE** at all (cf. Thm.3.1.1(ii)), while the use of the Alexandrov-Zeeman theorem does require **AxE**. (2) The proof given in the present section goes through (with the obvious modifications) for very weak axiom systems to be studied in this work, like e.g. for **Bax**$^-$ (cf. Thm.4.3.11, p.481), while we do not see how the proof using the Alexandrov-Zeeman theorem could be generalized to prove the desired properties of these weak axiom systems, too. For completeness, we note that the just outlined proof for $(\star\star)$ via the Alexandrov-Zeeman theorem might go through without assuming **Ax**$(\sqrt{\ })$. This would yield a proof of $(\star)$ via the Alexandrov-Zeeman theorem. We did not have time to check this.

$\lhd$

COROLLARY 3.1.15 Assume $\mathsf{f} : {}^nF \longrightarrow {}^nF$ is a bijection such that $(\forall \ell \in \mathsf{SlowEucl})\mathsf{f}[\ell] \in \mathsf{Eucl}$. Then $(\forall \ell \in \mathsf{Eucl})\mathsf{f}[\ell] \in \mathsf{Eucl}$.

Proof: This is a corollary of the proof of Thm.3.1.1. (It can be checked that we did not use more than the present conditions). ∎

COROLLARY 3.1.16 Assume $j \leq n$, and assume P is a j-dimensional plane. Then

$$\textbf{Basax} \models (\forall m, k \in Obs)(\mathsf{f}_{mk}[P] \text{ is a } j\text{-dimensional plane}).$$

Proof: This is a corollary of Thm.3.1.1. ∎

For completeness, in Items 3.1.17, 3.1.21, 3.1.22 below we formalize and prove some of the properties of the world-view transformations which were already studied in §2.3, but here we discuss them with a slightly different emphasis. (The main reason for reformulating them here is that later we will use them in their present form.)

PROPOSITION 3.1.17
$\textbf{Basax} \models (\forall m, k \in Obs)(\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \iff \mathsf{f}_{mk}[\ell] \in \mathsf{PhtEucl})$.

Proof: The proof is straightforward, but for completeness we mention that the proposition follows by **Ax5** and **AxE**. Cf. also Prop.2.3.3(ix), (v). ∎

For Prop.3.1.21 below we will need the notion of a rhombus, and also the notion of a subset of ${}^nF$ being the mirror image of another subset (w.r.t. a line). But for these, first we need to define (Euclidean) orthogonality $\perp_e$. This comes next.

Definition 3.1.18

(i) Assume $p, q \in {}^nF$. Then we say that *p is orthogonal to q* (in the Euclidean sense), in symbols $p \perp_e q$, iff $p_0q_0 + p_1q_1 + \cdots + p_{n-1}q_{n-1} = 0$.[182]

(ii) Assume $\ell = \{\, r + a \cdot s \,:\, a \in F \,\}$, $\ell' = \{\, r' + a \cdot s' \,:\, a \in F \,\} \in \mathsf{Eucl}$, for some $r, r' \in {}^nF$ and $s, s' \in {}^nF \setminus \{\bar{0}\}$. Then we say that *$\ell$ is orthogonal to ℓ'* (in the Euclidean sense), in symbols $\ell \perp_e \ell'$, iff $s \perp_e s'$.

◁

Definition 3.1.19 Assume $q, r, p, s \in {}^nF$ and assume that $qrps$ is a parallelogram. We say that $qrps$ is a *rhombus* iff $\overline{qp} \perp_e \overline{rs}$. See Figure 57.

◁

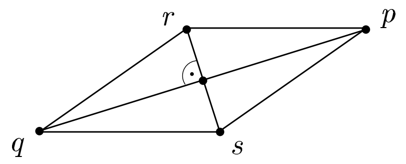


Figure 57: Illustration for Def.3.1.19 (rhombus).

Let us look at Figures 57, 58. Intuitively, we say that in Figure 57, points p and q are mirror images of each other w.r.t. the line $\overline{rs}$. Similarly, in Figure 58, $\mathsf{f}[\bar{t}\,]$ and $\mathsf{f}[\bar{x}]$ are mirror images of each other w.r.t. line ℓ_1. Below, we formalize the definition of this intuitive idea of mirror images w.r.t. a line.

[182]For completeness we note that this is Euclidean orthogonality which is different from Minkowski orthogonality (and also from the third kind of orthogonality [called relativistic orthogonality] to be introduced in the geometry chapter [§6]). This is the reason why we use the subscript "e" in the notation $\perp_e$.

Definition 3.1.20 Let $\mathbf{F}$ be a field, and $n \geq 2$.
Assume $\ell \in \mathsf{Eucl}$.

(i) We define $\sigma_\ell : {}^nF \longrightarrow {}^nF$ to be the reflection w.r.t. line ℓ, i.e.

$$\sigma_\ell \stackrel{\text{def}}{=} \{\, \langle p, q\rangle \in {}^nF \times {}^nF \ : \ \overline{pq} \perp_e \ell \ \wedge \ (\text{the midpoint of segment } pq \text{ is on } \ell)\,\}.$$

(ii) Assume $p,\ q \in {}^nF$. Then we say that p and q are mirror images of each other w.r.t. ℓ iff $\sigma_\ell(p) = q$. (E.g. in Figure 57, p and q are mirror images of each other w.r.t. the line $\overline{rs}$.) Similarly for subsets of ${}^nF$: Assume $P,\ Q \subseteq {}^nF$. Then we say that P and Q are mirror images of each other w.r.t. ℓ iff $\sigma_\ell[P] = Q$.

$\lhd$

PROPOSITION 3.1.21 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(2))$. *Let* $m, k \in Obs$ *and let* $\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{mk}$. *Then (i)–(iii) below hold.*

(i) $\mathsf{f}(1_t)$ *and* $\mathsf{f}(1_x)$ *are mirror images of each other w.r.t. a line* ℓ *with* $\ell \in \mathsf{PhtEucl}$ *and* $\ell \ni \mathsf{f}(\bar{0})$.

(ii) $\mathsf{f}[\bar{t}]$ *and* $\mathsf{f}[\bar{x}]$ *are mirror images of each other w.r.t. both of the lines* $\ell_1, \ell_2 \in \mathsf{PhtEucl}$ *with* $\ell_1, \ell_2 \ni \mathsf{f}(\bar{0})$ *and* $\ell_1 \neq \ell_2$.

(iii) $\mathsf{f}(\bar{0})\mathsf{f}(1_t)\mathsf{f}(\langle 1,1\rangle)\mathsf{f}(1_x)$ *is a rhombus such that* $\overline{\mathsf{f}(\bar{0})\mathsf{f}(\langle 1,1\rangle)},\ \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)} \in \mathsf{PhtEucl}$.

See Figure 58.

Proof of Prop.3.1.21: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(2))$, $m, k \in Obs$ and $\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{mk}$. Throughout the proof the reader is asked to consult Figure 58.

Throughout the proof we will use that f is a bijection (cf. Prop.2.3.3(v)).

By Thm.3.1.1 and Prop.3.1.17 we have that (18) and (19) below hold.

(18) $\qquad (\forall \ell \in \mathsf{Eucl}) \qquad \mathsf{f}[\ell] \in \mathsf{Eucl}\,.$

(19) $\qquad (\forall \ell \in \mathsf{Eucl}) \qquad (\ell \in \mathsf{PhtEucl} \ \Leftrightarrow\ \mathsf{f}[\ell] \in \mathsf{PhtEucl})\,.$

Now

(20) $\qquad \mathsf{f}(\bar{0})\mathsf{f}(1_t)\mathsf{f}(\langle 1,1\rangle)\mathsf{f}(1_x) \quad$ is a parallelogram,

since f is a bijection satisfying (18) above and since "$\bar{0}1_t\langle 1,1\rangle 1_x$" is a parallelogram.

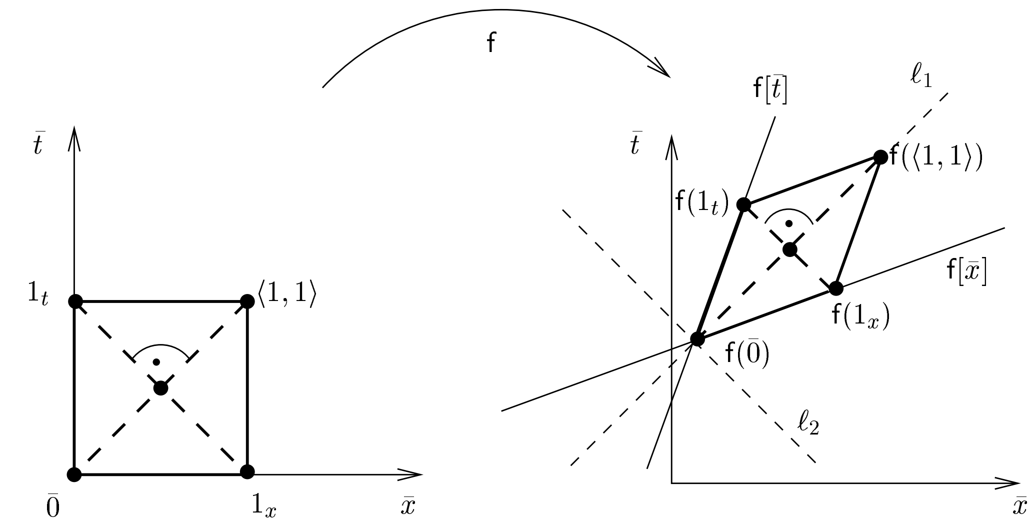


Figure 58: Illustration for Prop.3.1.21.

We have

$$\overline{\mathsf{f}(\bar 0)\mathsf{f}(\langle 1,1\rangle)}\,,\ \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)} \in \mathsf{PhtEucl}\,, \tag{21}$$

since $\overline{\bar 0\langle 1,1\rangle}\,,\ \overline{1_t 1_x} \in \mathsf{PhtEucl}$ and since f is a bijection satisfying (19) above.

Now by (20), we have that $\overline{\mathsf{f}(\bar 0)\mathsf{f}(\langle 1,1\rangle)} \neq \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)}$ and $\overline{\mathsf{f}(\bar 0)\mathsf{f}(\langle 1,1\rangle)} \cap \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)} \neq \emptyset$. By this, by $n = 2$ and by (21), we have

$$\overline{\mathsf{f}(\bar 0)\mathsf{f}(\langle 1,1\rangle)} \perp_e \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)}\,. \tag{22}$$

(20), (21) and (22) completes the proof of item (iii) of Prop.3.1.21.

Item (i) of Prop.3.1.21 follows from item (iii) for the choice $\ell = \overline{\mathsf{f}(\bar 0)\mathsf{f}(\langle 1,1\rangle)}$.

Item (ii) of Prop.3.1.21 follows from item (iii) of Prop.3.1.21 and from (18) above for the choice $\ell_1 = \overline{\mathsf{f}(\bar 0)\mathsf{f}(\langle 1,1\rangle)}$ and $\ell_2 \in \mathsf{PhtEucl}$ with $\ell_2 \ni \mathsf{f}(\bar 0)$ and $\ell_2 \neq \ell_1$. Clearly such an ℓ_2 exists and is unique by $n = 2$.

This completes the proof of Prop.3.1.21. ∎

COROLLARY 3.1.22 Assume $\mathsf{f} : {}^2F \longrightarrow {}^2F$ is a bijection such that

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl}\ \wedge\ (\mathsf{f}[\ell] \in \mathsf{PhtEucl}\ \Leftrightarrow\ \ell \in \mathsf{PhtEucl})\Big)\,.$$

Then $\mathsf{f}[\bar{t}]$ and $\mathsf{f}[\bar{x}]$ are mirror images of each other w.r.t. both of the lines $\ell_1, \ell_2 \in \mathsf{PhtEucl}$ with $\ell_1, \ell_2 \ni \mathsf{f}(\bar{0})$ and $\ell_1 \neq \ell_2$.

Proof: This is a corollary of the proof of Prop.3.1.21(ii). (It can be checked that we did not use more than the present conditions.) ∎

Remark 3.1.23 At this point one could formulate a generalized version of Thm.2.3.12 (p.65) which was a *characterization* of the f_{mk}'s in **Basax**(2) models. The generalization would consist of replacing $n = 2$ with "$n \geq 2$ is arbitrary". Such a characterization (of the f_{mk}'s in **Basax** models) will be stated as Thm.3.6.16 on p.273.

◁

3.2 Intuitive outline of proof for consistency of Basax(3)

Remark 3.2.1 Of course, consistency of **Basax** is very far from being new. Further, a "computational" proof for the consistency of **Basax**(n) is available e.g. by taking the well known Minkowskian geometries recalled (from the literature) in §6 ("Observer independent geometry") and translating them to models the way described in that chapter. The purpose of the present intuitive outline is different: It wants to give some intuitive, introductory, ideas to the non-specialist about why **Basax**(3) has models, and how one can visualize these models, easily.

$\lhd$

We will use the following (well known) definition and lemma from geometry. These are understood in a Euclidean geometry over some Euclidean field $\mathfrak{F}$.[183]

Definition 3.2.2 By an *ellipse* we understand a subset C of ${}^nF$ such that C is the image $\mathsf{f}[C_1]$ of a circle[184] C_1 by some bijective linear transformation f.

$\lhd$

Notation: In the present section r stands for elements of F despite of the fact that in the rest of the present work r usually stands for elements of ${}^nF$.

LEMMA 3.2.3 *We are in three dimensions, i.e. in ${}^3F$. Assume C is a "closed curve" obtainable as an intersection of a cone and a plane.*[185] *Then C is an ellipse. Actually, then C is an intersection of a plane and a cylinder.*

On the proof: Since the proof is available in geometry textbooks (cf. e.g. Hajós [120, Chapt.5, §41]); we indicate only the key ideas.

(1) Assume C is a "closed cone-slice" (as in the lemma). Then

(∗) C is a curve specifiable by two "focal-points" F_1, F_2 and the length λ of a "string" the usual way, i.e. C is the set of those points on the plane of C whose distances from F_1 and F_2 sum up to λ.

The proof of this is illustrated in Figure 59 below.[186]

---

[183] For our present purposes, it is sufficient to concentrate on the "standard" case $\mathfrak{F} = \mathfrak{R}$.

[184] A set C_1 of points is called a *circle* if there are a plane P a point $p \in P$ and $0 \le r \in F$ such that $C_1 = \{ q \in P : \|p - q\| = r \}$.

[185] In this context, by a closed curve we mean something like a circle or an ellipse as opposed to a parabola or a hyperbola.

[186] For completeness, we note that Hajós [120] around Figure 395 therein contains a fully detailed proof of (∗). However, we hope, our Figure 59 will satisfy the reader.

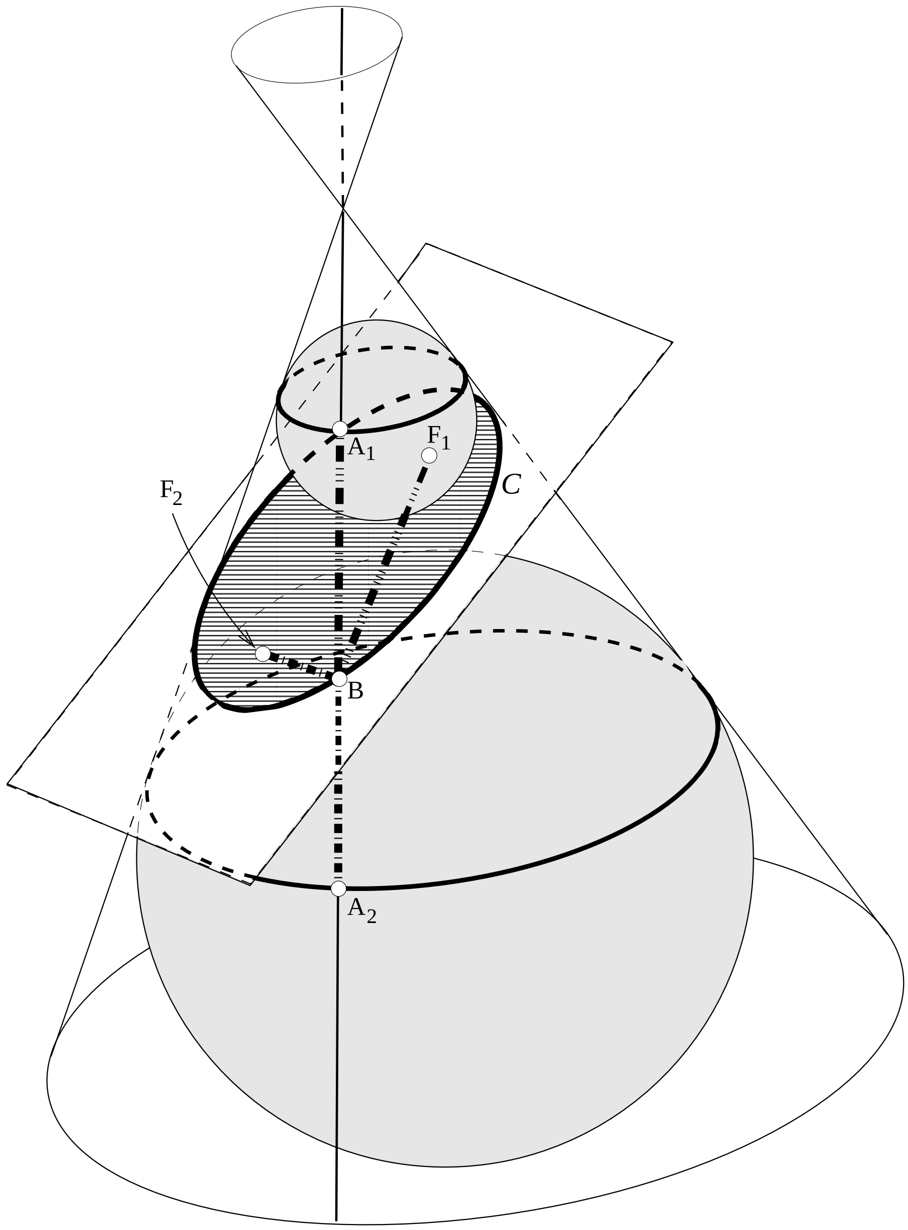
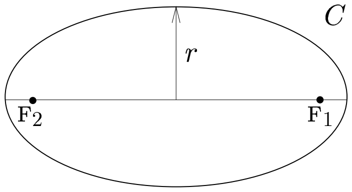


Figure 59: The two spheres touch the plane in points F_1 and F_2 respectively, and they also touch the cone. Segments BF_1 and BA_1 are of equal length because both are tangent to the small sphere. Similarly for segments BF_2 and BA_2. Thus $|\mathrm{F}_1\mathrm{B}|+|\mathrm{BF}_2| = |\mathrm{A}_1\mathrm{A}_2|$; where for any $p, q \in {}^3F$, $|pq| := |p - q|$.

(2) Assume $(*)$ holds. I.e. C is specified by two focal-points $\mathtt{F}_1$, $\mathtt{F}_2$ and we know the "shortest radius" r of C. From this one shows that C is the intersection of a cylinder and a plane as follows.



We will construct a variant of Figure 59 above but with the two spheres having the _same_ size. Actually the spheres will have radius r. As illustrated in Figure 61 (p.179) let one sphere touch the plane of C at point $\mathtt{F}_1$ and the other at $\mathtt{F}_2$. A side-view of the situation is in Figure 60 below. In connection with constructing Figure 61 we note that if we have a plane and point $\mathtt{F}_2$ in the plane and we have a ball then we can put the ball on the plane such that the ball touches the plane exactly at point $\mathtt{F}_2$.

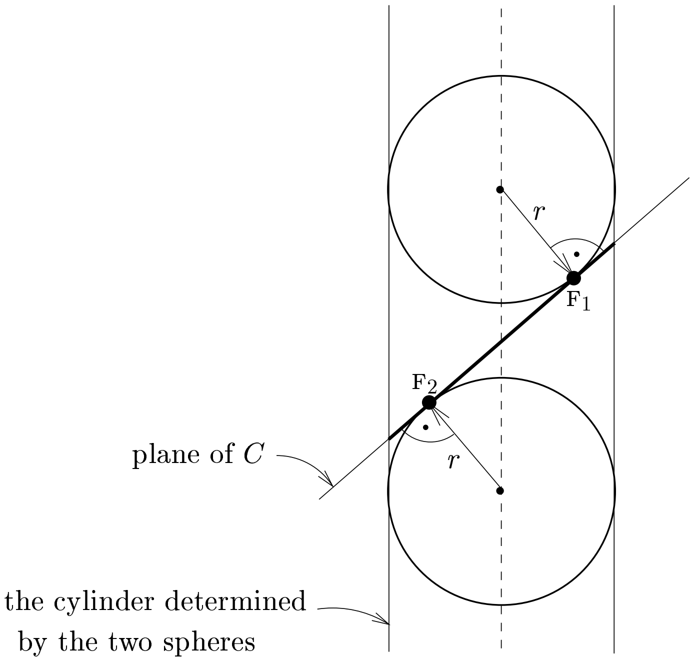


Figure 60: Side-view of Figure 61 below.

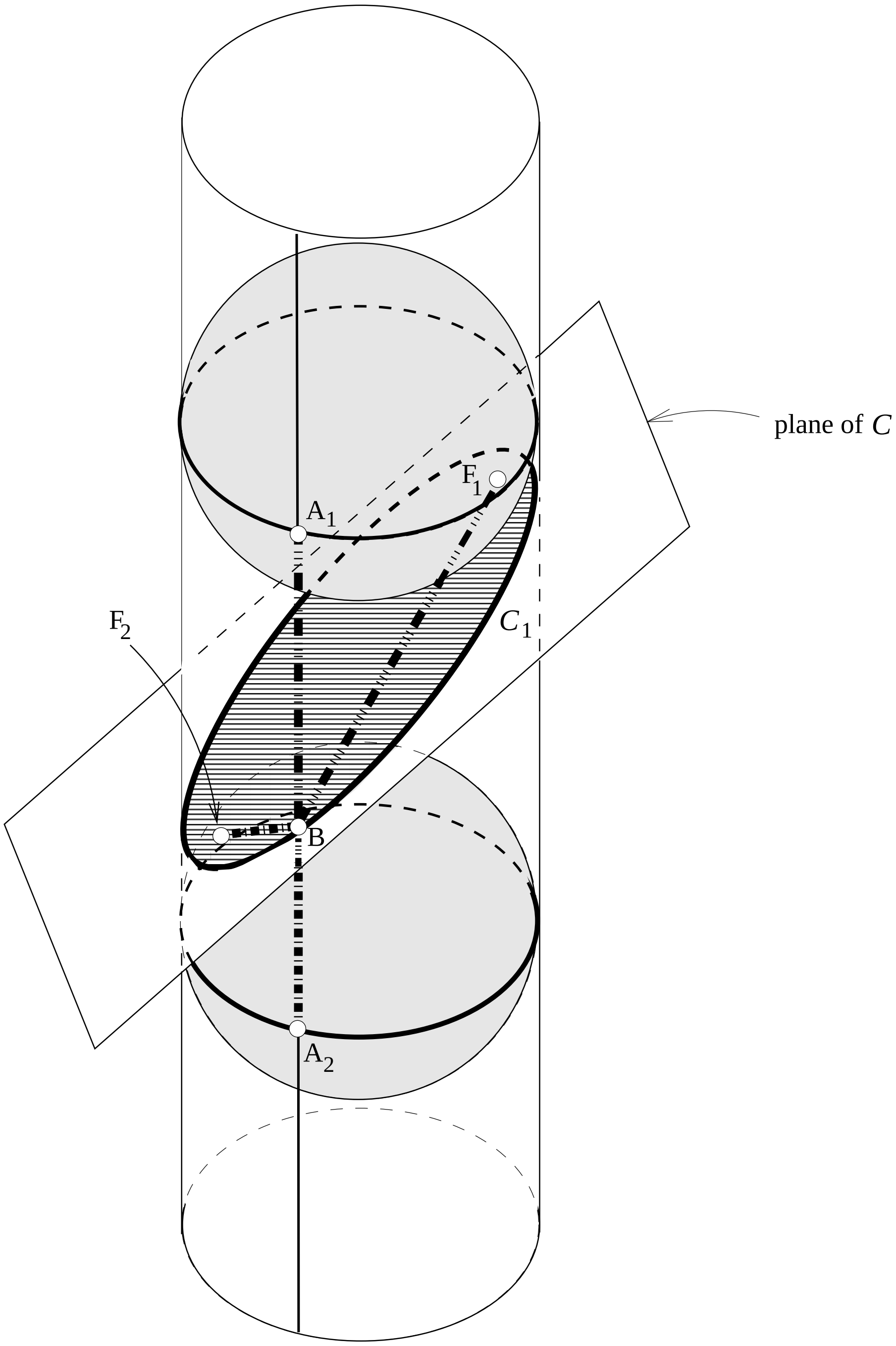


Figure 61: Proof of $(*) \Rightarrow$ (C is an ellipse).

Now, the two spheres determine a cylinder, which by the proof in our step (1) above[187] intersects with the plane of C in a curve C_1 satisfying $(*)$ above. But the focal-points of C_1 are the same as those of C. Further the "smallest radius" of C_1 is r. Hence $C_1 = C$.

To prove the lemma, it remains to see that the intersection C of a plane and a cylinder satisfies Def.3.2.2, i.e. that it is a bijective linear image of some circle. This is easy to see if one visualizes the situation (and thinks about it a little bit).

END of PROOF-IDEA for Lemma 3.2.3. ∎

Now, we turn to proving the consistency of **Basax**(3). The structure of the proof will be similar to that given for **Basax**(2) in §2.4.

(I) Assume, we are given a "partial model"

$$\mathfrak{M} = \langle (B;\ \{m\}, Ph, Ib), \mathfrak{F}, G;\ \in, W\rangle\,,$$

which satisfies all the axioms in **Basax** *except* for the observer-part of **Ax5**. (In §2.4 m was called m_0.) Let us use the notation **Ax5** = **Ax5(Obs)** + **Ax5(Ph)**. Then

$$\mathfrak{M} \models (\mathbf{Ax1}\text{–}\mathbf{Ax4}, \mathbf{Ax5(Ph)}, \mathbf{Ax6}, \mathbf{AxE})\,.$$

Assume further $\mathfrak{F} = \mathfrak{R}$, and that

$$(\forall \ell \in G)(\exists b \in Ib)\ \ell = tr_m(b)\,.$$

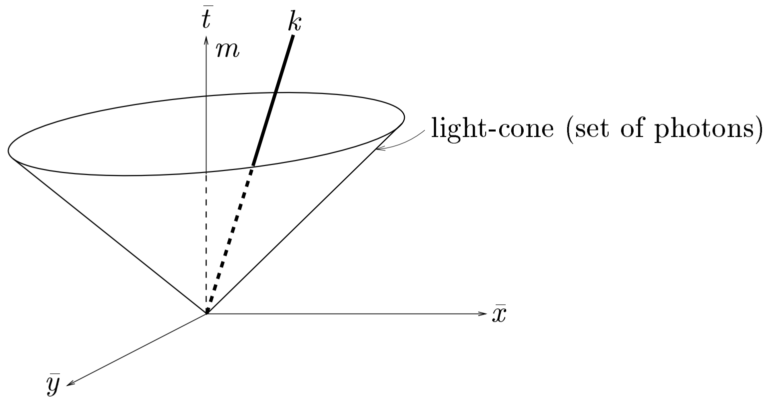
Constructing such a partial model is easy, and is left to the reader.

(II) Next, we would like to add new observers to $\mathfrak{M}$ so that eventually **Ax5(Obs)** would become true without destroying validity of the other axioms (hence **Basax** would become true).

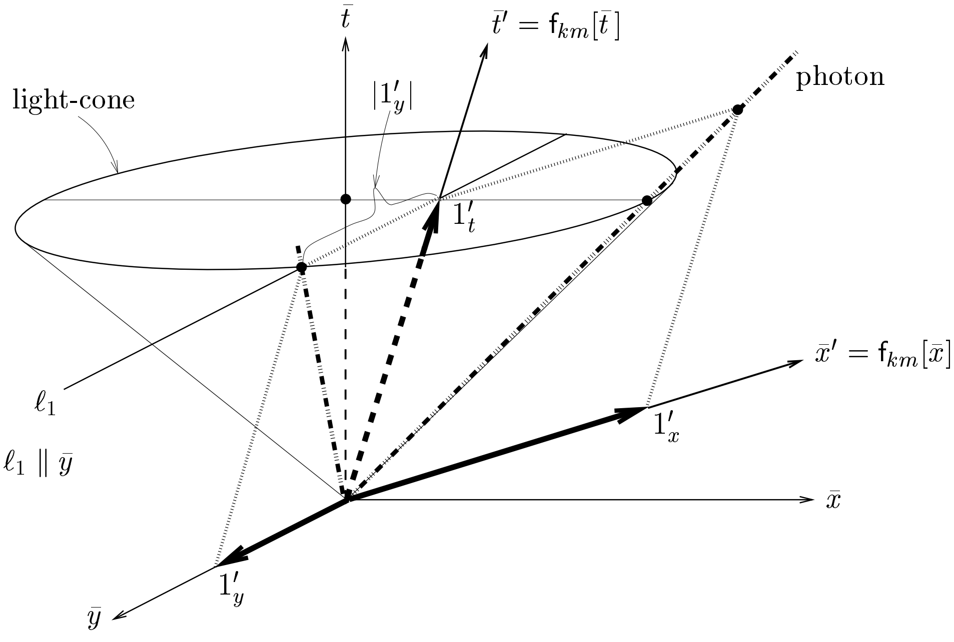
Clearly, in $\mathfrak{M}$ we do have a world-view function $w_m : {}^3\mathrm{R} \longrightarrow \mathcal{P}(B)$, to begin with. From this world-view function we will construct world-views for new observers. Let us pick randomly $k \in Ib$ such that $v_m(k) < 1$. Now, we would like to raise k to the level of being an observer. For simplicity, let us assume that m sees k in $\mathsf{Plane}(\bar{t}, \bar{x})$ passing through $\bar{0}$.[188] Then, m will see this:

[187] The proof of step (1) goes through for a cylinder in place of the cone, without any change.

[188] For the case when k is not in $\mathsf{Plane}(\bar{t}, \bar{x})$ or is not passing through $\bar{0}$ the present construction can be easily generalized by adding to observer m another one m' such that m' does not move relative to m and that m' sees k in $\mathsf{Plane}(\bar{t}, \bar{x})$ passing through $\bar{0}$.



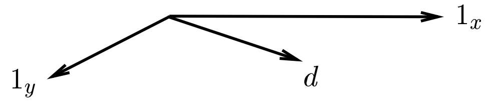
Our task is to choose the world-view $w_k : {}^3\mathrm{R} \longrightarrow \mathcal{P}(B)$ of k such that $\mathsf{f}_{mk} = w_m \circ w_k^{-1}$ preserves all photon-lines (i.e. that **AxE** holds for k too). In other words, we want to ensure that k observes that the speed of light is one in all directions. For simplicity, let us choose $\mathsf{f}_{mk} \in Linb$. Throughout this proof by definition $\mathsf{f}_{km} := \mathsf{f}_{mk}^{-1}$. To determine f_{mk}, it is enough to choose the k-unit-vectors $\mathsf{f}_{km}(1_t)$, $\mathsf{f}_{km}(1_x)$, $\mathsf{f}_{km}(1_y)$ which we will denote as $1'_t$, $1'_x$, $1'_y$, respectively.[189] Let $1'_t \in tr_m(k)$ be fixed (but arbitrary) such that $1'_t \neq \bar{0}$. It is easy to choose the (rest of the) k-unit-vectors such that k observes the speed of light correctly in the following directions: 1_x, -1_x, 1_y and -1_y. This is represented in the picture below:



[189] $1'_t \ldots 1'_y$ will be denoted as $1^k_t \ldots 1^k_y$ on p.325 (above Def.3.8.38); and they will be denoted as $x_{k,0} \ldots x_{k,2}$ on p.255.

I.e. we choose $1'_x$ to match $1'_t$ in the usual style of our rhombus transformations (cf. §2 Figures 15–20 on pp. 63–70).[190] This choice of $1'_x$ will ensure that k sees the speed of light correctly in directions 1_x and -1_x. We choose $1'_y$ to be in the $\bar{y}$ axis, i.e. $1'_y \in \bar{y}$. Further we choose the length $|1'_y|$ of $1'_y$ such that k will see the speed of light in direction 1_y as desired (i.e. to be one). (This choice will ensure that the speed of light will be right in the direction -1_y, too.)

But, with these choices, f_{mk} is completely determined. Therefore if we choose an arbitrary spatial direction, call it d, like this



then the question whether or not k sees the *speed of light in direction d* correctly has already been decided. I.e. we cannot choose any new parameter freely to ensure that k sees the d-photons correctly. So, it remains to check whether we were lucky enough with our choice of f_{mk} from the point of view of d-photons.

First, let us notice that a *simultaneity* of k is a plane parallel with $\mathsf{Plane}(\bar{y}', \bar{x}') = \mathsf{f}_{km}[\mathsf{Plane}(\bar{y}, \bar{x})] = \mathsf{Plane}(\bar{y}, \bar{x}')$ where $\bar{x}' = \mathsf{f}_{km}[\bar{x}]$ and $\bar{y}' = \mathsf{f}_{km}[\bar{y}] = \bar{y}$.

Consider the photons coming from the origin $\bar{0}$ and observed by k at time instance 1 (and all this represented in the world-view of m); see Figure 62. These "photon-instances" form the "curve" C_k obtained as the intersection of the k-simultaneity-plane S_k and the light-cone, as represented in Figure 62.[191] By our Lemma 3.2.3 way above,[192] C_k is an *ellipse*. For our purposes, it is enough to prove that *k "thinks"* that the ellipse C_k is actually a circle with center $1'_t$. In other words, we want to prove that in k's world-view C_k is a circle with center $1'_t$. (Formally this means that $\mathsf{f}_{mk}[C_k]$ is a circle with center 1_t.) We already know that k thinks that C_k is an ellipse (formally that $\mathsf{f}_{mk}[C_k]$ is an ellipse) because by our definition of an ellipse (Def.3.2.2), we have that any bijective linear image of an ellipse is an ellipse.

Figure 63 below represents the same world-view of m as Figure 62 did, but with points A, B, C, D and $q := 1'_t$ added. In Figure 63, the line $\overline{\mathrm{AC}}$ is in $\mathsf{Plane}(\bar{t}, \bar{x})$ while $\overline{\mathrm{BD}}$ is in $\mathsf{Plane}(\bar{y}, \bar{t}')$ where $\bar{t}' = tr_m(k)$. In the present proof, we write [AC] for the *segment* AC and similarly [BD] for the segment BD. Thus in particular $[\mathrm{AC}] \subseteq \overline{\mathrm{AC}}$.

[190] I.e. we choose $1'_x$ to be the mirror image (w.r.t. an appropriate photon-line) of $1'_x$. For the computationally minded reader, details are on p.252 (item 3.5.5 [case $n = 3$ sub-item (i)]).

[191] Figure 62 still represents the world-view of m with the life-line of k and a simultaneity S_k of k as observed by m.

[192] C_k is a closed curve because of the choice of the unit-vectors $1'_x$, $1'_y$ on Figure 62 (i.e. the angle between $\bar{t}$ and $1'_x$ is bigger than the angle of $\bar{t}$ and photons.)

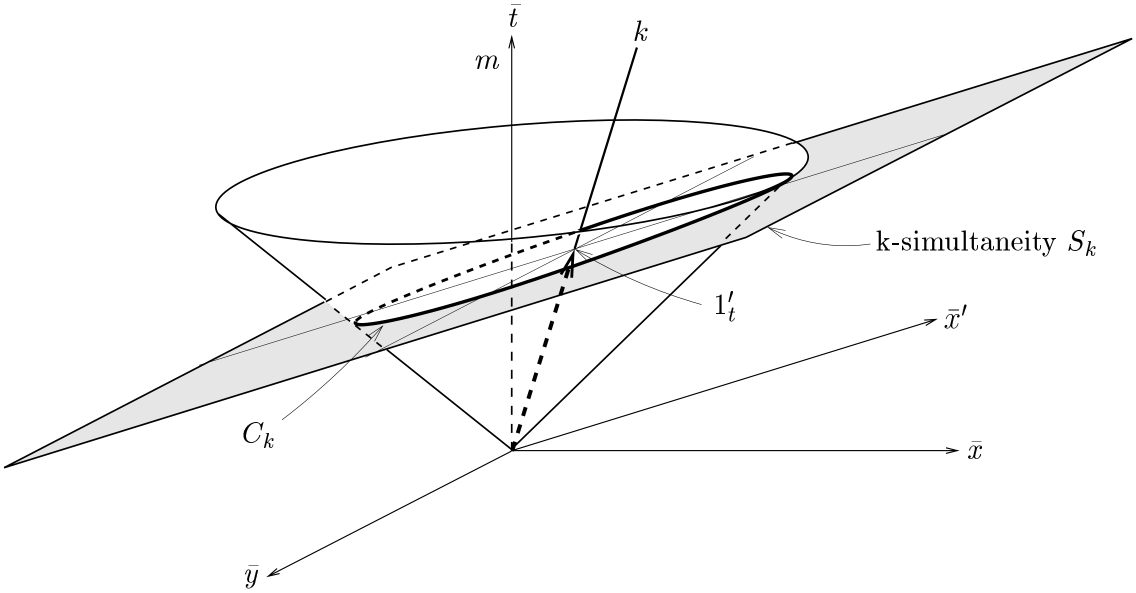


Figure 62:

Further, throughout this proof, $\mathsf{f} := \mathsf{f}_{mk}$ and we write $\mathsf{f}\textsc{a}$ for $\mathsf{f}(\textsc{a})$. We will use the notion of parallelism and the symbol $\parallel$ not only for lines, but also for planes in the usual way.

The line <u>*tangent*</u> to C_k at point A is denoted as $\ell_{\textsc{a}}$, and similarly for $\ell_{\textsc{b}}$, $\ell_{\textsc{c}}$, $\ell_{\textsc{d}}$.

Now,

$(\star)$ [AC] and [BD] are diameters of the ellipse C_k such that $[\textsc{ac}] \parallel \ell_{\textsc{b}} \parallel \ell_{\textsc{d}}$ and $[\textsc{bd}] \parallel \ell_{\textsc{a}} \parallel \ell_{\textsc{c}}$;

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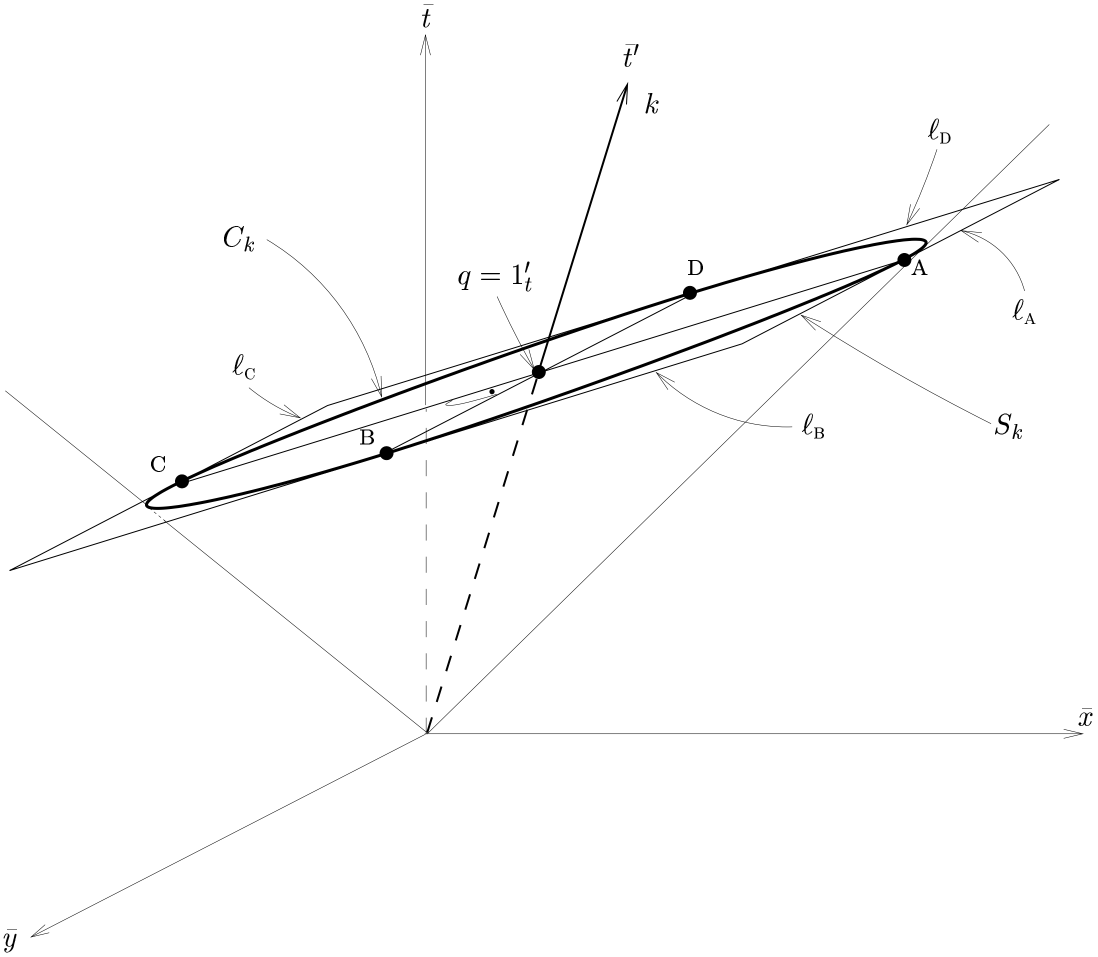
hold because of the following. $[\textsc{bd}] \parallel \bar{y}$, because (by $S_k \parallel \mathsf{Plane}(\bar{y}, \bar{x}')$) $S_k \parallel \bar{y}$ which by $\overline{\textsc{bd}} = S_k \cap \mathsf{Plane}(\bar{y}, \bar{t}')$ yields that $\overline{\textsc{bd}} \parallel \bar{y}$. The light-cone as-well-as S_k are symmetrical w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$ hence C_k is also symmetrical w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$. Since C_k is symmetrical w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$ we conclude that $\ell_{\textsc{a}} \parallel \bar{y}$ hence $\ell_{\textsc{a}} \parallel [\textsc{bd}]$. Similary $\ell_{\textsc{c}} \parallel [\textsc{bd}]$. Since C_k is symmetrical w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$ we have that [AC] is one of the symmetry-axes of C_k. q is a midpoint of both [AC] and [BD] since k sees the speed of light to be the same in the spatial directions 1_x, -1_x, 1_y and -1_y. Further, $\overline{\textsc{ac}} \perp_e \overline{\textsc{bd}}$ because $\overline{\textsc{bd}} \parallel \bar{y}$ and $\overline{\textsc{ac}} \parallel \mathsf{Plane}(\bar{t}, \bar{x})$ (and of course $\bar{y} \perp_e \mathsf{Plane}(\bar{t}, \bar{x})$). Since $[\textsc{bd}] \perp_e [\textsc{ac}]$ and they bisect each other, the other symmetry-axis of the ellipse C_k is [BD]. Therefore $\ell_{\textsc{b}} \parallel \ell_{\textsc{d}} \parallel \overline{\textsc{ac}}$.

A pair of diameters of an ellipse C_k satisfying $(\star)$ above is called a



$\bar{t}$
$\bar{t}'$
k
ℓ_{D}
C_k
D
A
$q = 1'_t$
ℓ_{A}
ℓ_{C}
ℓ_{B}
S_k
B
C
$\bar{x}$
$\bar{y}$

Figure 63:

conjugated pair of diameters. A usefulness of this property is in that it is preserved under linear transformations, i.e. for any linear transformation h, if $(\star)$ holds for C, A, B, C, D then $(\star)$ will also hold for $\mathsf{h}[C]$, $\mathsf{h}(\mathrm{A})$, $\mathsf{h}(\mathrm{B})$, $\mathsf{h}(\mathrm{C})$, $\mathsf{h}(\mathrm{D})$.[193] Therefore, k will also think that $(\star)$ is true (in k's world-view), i.e. k will also *"think"* that [AC] and [BD] are conjugated diameters of ellipse C_k.[194]

Recall that we have $[\mathrm{AC}] \perp_e [\mathrm{BD}]$. Now,

$(\star\star)$ k *will also think* that $[\mathrm{AC}] \perp_e [\mathrm{BD}]$

because f_{mk} preserves both $\bar{y}$ and $\mathsf{Plane}(\bar{t}, \bar{x})$.

We have choosen the unit vectors $1'_y$, $1'_t$, $1'_x$ such that

$(\star\star\star)$ k will think that the lengths of [AC] and [BD] are equal

(because $|\mathsf{f}\mathrm{B} - \mathsf{f}q| = 1 = |\mathsf{f}\mathrm{A} - \mathsf{f}q|$ and because q is the midpoint of both [AC] and [BD].) From $(\star)$, $(\star\star)$, $(\star\star\star)$ we want to infer that k thinks that C_k is a circle. For this we recall the following easy lemma from elementary geometry.

LEMMA 3.2.4 (geometry)
We are in two-dimensional Euclidean geometry over an arbitrary Euclidean field $\mathfrak{F}$. Assume C is an ellipse with a pair [AC], [BD] *of conjugated diameters of equal length. Assume further* $[\mathrm{AC}] \perp_e [\mathrm{BD}]$.

Then C is a circle.

Proofs of this are available in geometry textbooks, but for completeness, we include one in a footnote.[195] ∎

[193]This is true because the only basic concepts involved in $(\star)$ are parallelism, incidence, and the center of an ellipse; and these are obviously preserved.

[194]Formally this means that $[\mathsf{f}\mathrm{A}, \mathsf{f}\mathrm{C}]$ and $[\mathsf{f}\mathrm{B}, \mathsf{f}\mathrm{D}]$ are conjugated diameters of ellipse $\mathsf{f}[C_k]$. Throughout this proof we use this figurative way of speaking; e.g. by saying that k thinks that A, B, and C have a certain property "Prop", we mean to say that $\mathsf{f}(\mathrm{A})$, $\mathsf{f}(\mathrm{B})$, $\mathsf{f}[C]$ have property "Prop". The intuitive reason for this is that in reality, instead of A, B and C we are thinking about the *events* $w_m(\mathrm{A})$, $w_m(\mathrm{B})$ and $w_m[C]$. For k, these events appear in k's coordinate system as $w_k^{-1}(w_m(\mathrm{A})) = \mathsf{f}(\mathrm{A})$, ..., $w_k^{-1}[w_m[C]] = \mathsf{f}[C]$.

[195]The proof consists of (I)–(III) below. (I) The fact that by definition, any ellipse C is an image of a *circle* C_1 by some bijective linear transformation h. (II) Conjugated diameters bisect each other and are preserved under such transformations, therefore in particular under h^{-1}. Therefore the h^{-1} images of diameters [AC] and [BD] are orthogonal, bisect each other and are of equal length (they are diameters of circle C_1). (III) Any two conjugated diameters of a circle are orthogonal (and of equal length of course). Since [BD] and [AC] (in Lemma 3.2.4) are orthogonal and of equal length (and bisect each other), h is a linear transformation taking a pair of bisecting orthogonal segments (which are diameters of C_1) of equal length to a pair of orthogonal and bisecting segments of equal length. Therefore h takes the circle C_1 into a circle. This circle is C.

By $(\star)$, $(\star\star)$, $(\star\star\star)$ the conditions of Lemma 3.2.4 are satisfied by $\mathsf{f}[C_k]$, $\mathsf{f}\mathrm{A}$, $\mathsf{f}\mathrm{B}$, $\mathsf{f}\mathrm{C}$, $\mathsf{f}\mathrm{D}$. (I.e. $[\mathsf{f}\mathrm{A}, \mathsf{f}\mathrm{C}]$ is a diameter of ellipse $\mathsf{f}[C_k]$ etc.). Therefore by Lemma 3.2.4, $\mathsf{f}[C_k]$ is a *circle*. I.e. k thinks that C_k is a circle, hence k thinks that the *speed of light is the same in all directions*. But this is what we wanted to prove.

Now, we choose

$$w_k \stackrel{\text{def}}{=} \mathsf{f}_{km} \circ w_m \,.$$

From this point on, the rest of the proof is easy and goes exactly as in the case of §2.4 (pp. 80–84), but for completeness we include it below.

One can check that for the extended model

$$\mathfrak{M}' := \langle (B;\ \{m,k\}, Ph, Ib), \mathfrak{F}, G;\ \in, W^+ \rangle$$

we have **Ax1**–**Ax4**, **Ax5(Ph)**, **Ax6**, **AxE** still valid. Here, W^+ denotes the extension of W with the world-view function w_k of the *new* observer k.

To complete the "intuitive" proof, one does the above extension *not only* with a single $k \in Ib$ but with the class $K = \{\, k \in Ib : v_m(k) < 1 \,\}$ of all potential candidates for being an observer. This will make **Ax5(Obs)** true.

More computational model constructions for **Basax**(n) will be given in §§ 3.5, 3.6. (Actually the construction in §3.6 will have several parameters such that by appropriately choosing these parameters [basically] all possible models of **Basax** will be "constructible".)

As we indicated in §2 whenever we investigate **Basax** from some point of view then it is desirable to investigate **Basax** + **Ax(symm)** from the same point of view. Therefore we note that a consistency proof for **Basax**(n) + **Ax(symm)** will be given in §3.8.2 ("Model construction ...").

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On rhombus, and Lorentz transformations

In the above intuitive proof of the consistency of **Basax**(3) we also constructed an example f_{km} of a 3-dimensional *rhombus* transformation. The reader is invited to consult the sequence of figures beginning with the ones on p.181, and "meditate" over their connections with the definition of rhombus transformations (Def.2.3.18, p.72) and with Remark 2.3.19. Further if we choose the length $1'_t$ according to the construction which will be given in §3.8 (cf. Figures 99, 100 on pp. 332–333) then the above construction of f_{km} yields an example of a *standard Lorentz* transformation (and hence also of a *Poincaré* one).

3.3 Allowing different observers see different events: The refined theory Newbasax

In this section we use our experience obtained so far with **Basax**, for elaborating a *refined* version **Newbasax** of **Basax**. Among others we show that **Newbasax** is strictly weaker than **Basax** and we characterize the models of **Newbasax** in terms of models of **Basax**: the models of **Newbasax** are, roughly speaking, unions of models of **Basax**.

Let us turn to discussing the motivation of introducing **Newbasax** as a refinement of **Basax** (in this discussion we will also see how **Newbasax** should differ from **Basax**). A disadvantage of **Basax** is that **Ax6** in it is too strong. Namely, **Ax6** says that all observers see the same events.

(i) This sounds all right for beginning special relativity, but in our progress toward preparing the road to general relativity[196] we will find **Ax6** too strong. For example we will see in §8 ("Accelerated observers"), that if we want to extend (a variant of) **Basax** to the case when accelerated observers are permitted, then we will need to allow the existence of events which exist for observer, call it, m but not for another observer, call it, k. Therefore we will have $Rng(w_m) \neq Rng(w_k)$ for some observers m and k.

In the case of approximating general relativity with "pushing accelerated observers to the extreme" like is done in the middle part of Rindler [224, §7.4, pp.114–125], it will be even more apparent that not all observers see the same events. Finally, in the case of general relativity it is the typical case that $Rng(w_m) \neq Rng(w_k)$. Namely, in general relativity the notion of an atlas plays a central role. Now, an atlas is by definition a set of maps (like in geography where the atlas of the Earth is a set of overlapping maps of parts of the globe). One such map corresponds, very roughly, to what we call the world-view, say w_m, of one observer, say m. The part of space-time coordinatized by map w_m is basically $Rng(w_m)$.[197] It is a key

[196]With this we do ***not*** mean that we could obtain general relativity as, say, a variant of **Basax**. What we mean is that by studying more-and-more refined versions of **Basax** allowing accelerated observers, rotating observers etc, we can prepare our intuition for eventually formalizing general relativity in first-order logic, but maybe in a drastically different language (and different framework) from our present frame language.

[197]Cf. Figure 3 on p.34 for a general idea of a collection $\{ w_m : m \in Obs \}$ of maps coordinatizing various parts of space-time which in that figure coincided with $\mathcal{P}(B)$. Cf. also approximately the last 10 lines above "Summing up ..." on p.33 about the difference between the coordinate system (which is used for mapping) and space-time (which is being mapped). We also refer to Convention 6.2.5 in the geometry chapter (§6) for a definition of space-time (of a model $\mathfrak{M}$).

"thing" in general relativity that these "mapped areas" are different, i.e. that typically $Rng(w_m) \neq Rng(w_k)$. Imagine for example m being an observer living inside (the event horizon of) a huge black hole while observer k is on the outside, far away from the black hole. Then there are events seen by m which need not exist for k (actually they will not exist if we use the most natural coordinate system for k). Similarly, if k_2 is an inertial observer starting life outside the event horizon and then falling into the black hole, then the event when k_2 enters the event horizon does exist for k_2 but it may not exist for k. Summing up, sooner or later we will have to refine axiom **Ax6** such that it will not state that all observers see the same events. Indeed, this will happen in defining **Newbasax** below, where we will replace **Ax6** with two weaker axioms $\mathbf{Ax6_{00}}$ and $\mathbf{Ax6_{01}}$.[198]

(ii) There is another (less important) respect in which **Newbasax** will be an improvement of **Basax**. Namely, in **Basax** the proof theoretical power distributed among the axioms is uneven, **Ax6** is extremely strong, the rest are relatively weak therefore it is hard to fine-tune **Basax** by omitting one of the axioms (i.e. if we omit **Ax6** than almost all power vanishes, while if we omit something else then the change will be relatively small). The axioms in **Newbasax** will carry a more balanced (more even) distribution of proof theoretic power. To explain why this is useful, recall from the Introduction that we wanted to do the following: when we see an intriguing prediction of relativity theory we planned to ask ourselves "which one of the axioms is responsible for this prediction?". If one axiom like **Ax6** is too strong then the answer to this question will be almost always "it is **Ax6**" which, after all, is not too informative. In other words, if we omit **Ax6** from **Basax** then suddenly the remaining part becomes almost hopelessly weaker (than **Basax**), e.g. the f_{mk}'s can be almost "anything". As a contrast if we omit an axiom from **Newbasax** then we may get a substantially weaker axiom system but it will not be absurdly weak (i.e. not all of the proof theoretic power goes away when omitting a single axiom).[199]

[198] A further refinement analogous with the **Basax** $\mapsto$ **Newbasax** change, and also pointing in the direction of general relativity is the following. In the spirit of footnote 75 on p.55, let $w_m^- := \{ \langle p, e\rangle \in w_m \,:\, e \neq \emptyset \}$, for every $m \in Obs$. Now, in **Basax** we had $Rng(w_m^-) = Rng(w_k^-)$ as well as $Dom(w_m^-) = {}^nF$. Sometime on the road towards general relativity we will have to replace $Dom(w_m^-) = {}^nF$ with $Dom(w_m^-) \subseteq {}^nF$. That is, presently **Ax5** says that at every point of our coordinate system ${}^nF$ observer m sees bodies (e.g. photons). But in general relativity there may be points of ${}^nF$ (i.e. coordinate-values) where the world-view w_m^- of m is simply not defined (hence m does not see photons there). However we will not implement this change soon; we will mention it next in §3.4 ("FTL observers") and in §8 ("Accelerated observers").

[199] In Thm.3.3.10 below we will see that

(∗) "visibility is an equivalence relation when restricted to the set Obs^+ of those observers which are seen by some observer"

is provable both from **Newbasax** and from **Basax**. In the case of **Newbasax** (∗) is not provable

(iii) A further advantage of **Newbasax** is that it is weaker, hence more flexible than **Basax**, while we will see that it proves all the paradigmatic predictions of relativity which we proved from **Basax**. I.e. it is a step in the direction formulated in item (X) of the Introduction, i.e. try to prove the paradigmatic predictions from as few assumptions as possible. **Newbasax** is our first axiom system obtained as a refinement of **Basax**. In later sections (especially §§ 3.4.2, 4.4, 4.5) we will introduce even more flexible refinements of **Newbasax** some of which will have kind of philosophical significance too.

* * *

The essential step in obtaining **Newbasax** from **Basax** is the following: As we said, we replace **Ax6** (saying that all observers see the same events) with much milder assumptions $\mathbf{Ax6_{00}}$ and $\mathbf{Ax6_{01}}$. (The rest of the changes are only adjustments to this one.)

Notation 3.3.1 Assume $\mathfrak{F}$ is an ordered field.

(i) $^{+}F \stackrel{\text{def}}{=} \{ x \in F \ : \ x > 0 \}$. I.e. ^{+}F is the set of positive elements of $\mathfrak{F}$.

(ii) Let $p \in {}^{n}F$. Then the square $\|p\|$ of the Euclidean length of the vector p is defined as follows.[200]

$$\|p\| \stackrel{\text{def}}{=} p_0^2 + p_1^2 + \ldots + p_{n-1}^2 \,.$$

(iii) Let $p \in {}^{n}F$ and $\varepsilon \in {}^{+}F$. Then by $S(p,\varepsilon)$ we denote the *ε-neighborhood of p* defined as follows.[201]

$$S(p,\varepsilon) \stackrel{\text{def}}{=} \{ q \in {}^{n}F \ : \ \|q-p\| < \varepsilon \} \,.$$

from $\mathbf{Ax6_0} := \mathbf{Ax6_{00}} + \mathbf{Ax6_{01}}$ alone (we need the other axioms too for deriving $(*)$). As a contrast, in the case of **Basax**, $(*)$ is provable from **Ax6** in itself.

Similarly, let $(**)$ be the conclusion of Thm.3.3.8. Then, $\mathbf{Ax6_0}$ does not imply $(**)$, though **Newbasax** does; while **Ax6** in itself implies $(**)$.

We hope, that the above simple examples already illustrate our feeling that, in some sense, in **Basax** the axiom **Ax6** is "superfluously" strong and that this is not the case with **Newbasax** and $\mathbf{Ax6_0}$ (in place of **Basax** and **Ax6**). The reader is invited to browse through the proofs of theorems proved from **Basax**, and try to collect more convincing examples.

[200]We use the square of the length instead of the length itself because we did not assume that $\mathfrak{F}$ is Euclidean.

[201]In the notation $S(p,\varepsilon)$ the letter S refers to the word "sphere". Further we note that there is a slight danger of confusion because S will denote the space-part of our coordinate-system ${}^{n}F$. We hope context will help.

(iv) Let $H \subseteq {}^nF$. We say that H is an <u>*open set*</u> iff

$$(\forall q \in H)(\exists \varepsilon \in {}^+F)\ S(q, \varepsilon) \subseteq H\,.$$

The set of open subsets of ${}^nF$ is denoted by $Open = Open(n, \mathfrak{F})$.[202]

$\triangleleft$

As it is indicated at the beginning of §2.2, we introduce a more refined system of axioms, **Newbasax**. This comes next. Recall from §2.2 (Definition 2.2.3) the axioms **Ax1**–**AxE** and the set **Basax**.

Below we postulate axioms $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$, $\mathbf{Ax3_0}$, $\mathbf{AxE_0}$.

$\mathbf{Ax6_{00}}$ $(\forall m, k \in Obs)\ w_m[tr_m(k)] \subseteq Rng(w_k)$.

Intuitively, observer k sees all those events which are seen by another observer m on k's life-line. Even more intuitively, if someone sees k participating in an event then k should not be allowed to deny that that event happened at all.

$\mathbf{Ax6_{01}}$ $(\forall m, k \in Obs)\ Dom(\mathsf{f}_{mk}) \in Open$.

Intuitively, if observers m and k see event $e \subseteq B$ then k sees all those events which m sees "very close" to e. $\mathbf{Ax6_{01}}$ wants to express our intuition that the "world" of any observer k is endless: No matter how far you travel from the Earth, you can always go a little further (in any direction), cf. Figure 64.[203] What is the world of k? It is $Rng(w_k)$. So, we want to say that the set $Rng(w_k)$ has no "edge". This could be expressed by saying that $Rng(w_k)$ is an open subset of space-time, say $\mathcal{P}(B)$. But on $\mathcal{P}(B)$ we do not yet have a topology. Therefore instead of saying that the world of k is an open subset in $\mathcal{P}(B)$, we <u>*pull*</u> the whole situation <u>*back*</u> along all the possible w_m's saying that $w_m^{-1}[Rng(w_k)]$ is open in ${}^nF$. Since $w_m^{-1}[Rng(w_k)] = Dom(\mathsf{f}_{mk})$, this is exactly what $\mathbf{Ax6_{01}}$ says.

[202] The set *Open* is of course not definable (at least not as an entity on its own right) in our frame language, but if we have a definable subset like $Dom(\mathsf{f}_{mk})$ then $Dom(\mathsf{f}_{mk}) \in Open$ counts as a first-order formula of our frame language, i.e. is translatable to a formula of our frame language. A similar remark applies e.g. to Eucl, *Linb*, *Rhomb* etc, and we will not repeat this remark each time we introduce an abbreviation like *Open*, Eucl, *Linb* etc. Summing up, one could say that *Open*, Eucl etc. are definable only as "predicate symbols" (but not necessarily as individual objects or entities.)

[203] This explanation emphasizes the space aspect only, by it can be generalized to time etc.

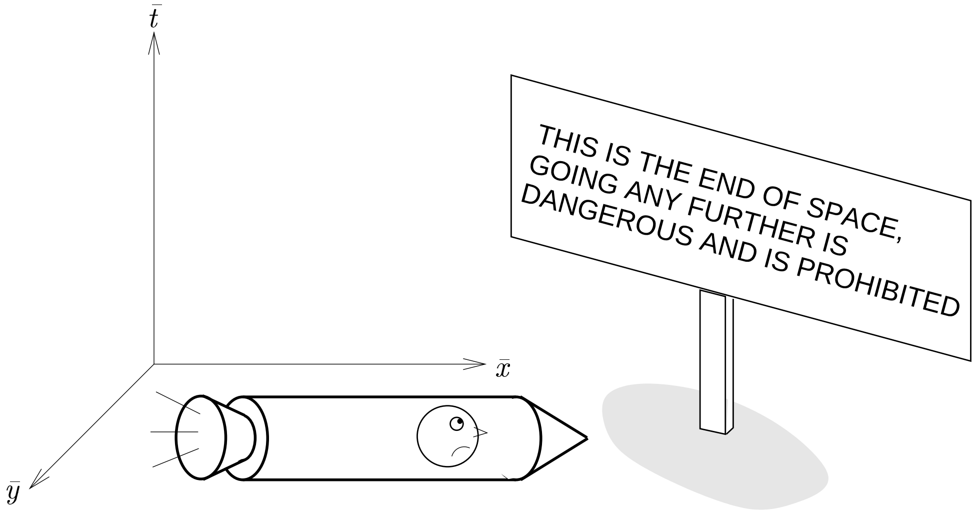

Figure 64: Illustration for $\mathbf{Ax6_{01}}$.

$\mathbf{Ax3_0}$ $(\forall h \in Ib)\ (tr_m(h) \in G \cup \{\emptyset\} \quad \wedge \quad (\exists k \in Obs) tr_k(h) \neq \emptyset)$.[204]

That is, the life-line of any inertial body h as seen by any observer m must be a line or the empty-set, and there is an observer k such that the life-line of h for k is not the empty-set. $\mathbf{Ax3_0}$ differs from **Ax3** in that the life-line of an inertial body seen by an observer can be the empty-set.

$\mathbf{AxE_0}$ $(\forall m \in Obs)(\forall ph \in Ph)(tr_m(ph) \neq \emptyset \quad \Rightarrow \quad v_m(ph) = 1)$.

That is, if the life-line of photon ph is not the empty-set for observer m, then the speed of ph for observer m is 1.

Definition 3.3.2 (Newbasax)
We define

$$\mathbf{Newbasax} \stackrel{\text{def}}{=} (\mathbf{Basax} \setminus \{\mathbf{Ax6}, \mathbf{Ax3}, \mathbf{AxE}\}) \cup \{\mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{Ax3_0}, \mathbf{AxE_0}\},$$

where the axioms $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$, $\mathbf{Ax3_0}$, $\mathbf{AxE_0}$ were defined above.

$\lhd$

[204]The part "$(\exists k \in Obs) tr_k(h) \neq \emptyset$" of $\mathbf{Ax3_0}$ is needed for technical reasons only: If we had omitted this part from $\mathbf{Ax3_0}$ then the formulation of Thm.3.3.12 on p.196 would have been more complicated than it is in its present form.

The reason for changing **Ax6** in **Newbasax** was explained in detail in the introduction to the present section (§3.3). Namely the main reason is a physical intuition (connected to general relativity) saying that there may be observers m, k who do not see the same events. The reason for changing **AxE** to $\mathbf{AxE_0}$ is basically the same intuition, namely if m is inside a black hole and k is far away then there may be a photon observed by m but not observed by k. A similar reason applies to replacement of **Ax3** to $\mathbf{Ax3_0}$. (There is a technical difference though: we will soon see theorems to the effect that we cannot weaken **Ax6** without weakening **Ax3**, while we could weaken **Ax6** without changing **AxE**.)

Remark 3.3.3 In §2.3 e.g. on p.75 we announced that the axioms of **Basax** are *independent* of each other (in the logical sense). In a future version we plan to include the proof of this. Further, in a future version we plan to investigate *whether* **Newbasax** *is an independent* axiom system in the same sense. In this connection we strongly *conjecture* the following. Let $\mathbf{Ax6_0} := \mathbf{Ax6_{00}} + \mathbf{Ax6_{01}}$. Then, in the following form

$$\mathbf{Newbasax}' := (\mathbf{Basax} \setminus \{\mathbf{Ax6}, \mathbf{Ax3}, \mathbf{AxE}\}) \cup \{\mathbf{Ax6_0}, \mathbf{Ax3_0}, \mathbf{AxE_0}\}$$

our new axiom system will turn out to be independent (for $n > 1$). (We did not have time to think about this, though).

At this point, the reader is invited to decide whether in its original form (having $\mathbf{Ax6_{00}}$ and $\mathbf{Ax6_{01}}$ as *separate* axioms) **Newbasax**(3) is independent.[205]

$\lhd$

Remark 3.3.4 We note that **Basax** $\models$ **Newbasax**.

$\lhd$

In definition below we will define a binary relation $\overset{\odot}{\to}$ between observers and bodies. The intuitive meaning of $m \overset{\odot}{\to} b$ is that observer m "sees" body b, that is the life-line of b seen by m is not the empty-set.

Definition 3.3.5 Let $\mathfrak{M}$ be a frame model. We define the binary relation $\overset{\odot}{\to} \subseteq Obs \times B$ as follows.

$$(\forall m \in Obs)(\forall b \in B)(m \overset{\odot}{\to} b \overset{\text{def}}{\iff} tr_m(b) \neq \emptyset)\,.$$

We call the relation $\overset{\odot}{\to}$ *visibility relation*. $\lhd$

[205] We are inclined to conjecture that $\mathbf{Newbasax} \setminus \{\mathbf{Ax6_{00}}\} \not\models \mathbf{Ax6_{00}}$, but we did not have time to think about the other direction (i.e. the same thing but now about $\mathbf{Ax6_{01}}$ in place of $\mathbf{Ax6_{00}}$).

Thm.3.3.7 below states that **Newbasax** together with **Ax3** is equivalent with **Basax**. Even if we had not had other (philosophical) reasons for replacing **Ax3** with $\mathbf{Ax3_0}$, this fact alone would have forced us to do it. (However, as we said, we had other reasons.) As a contrast, we will see that **Newbasax** + **AxE** is strictly weaker than **Basax**, cf. Thm.3.3.13 on p.197.

Notation 3.3.6 Let $\mathbf{Th}_1$, $\mathbf{Th}_2$ be sets of first-order formulas. Then

$$\mathbf{Th}_1 \mathrel{=\!\!|\!\!\models} \mathbf{Th}_2$$

abbreviates the following longer statement

$$(\mathbf{Th}_1 \models \mathbf{Th}_2 \quad \text{and} \quad \mathbf{Th}_2 \models \mathbf{Th}_1)\,.$$

That is, $\mathbf{Th}_1 \mathrel{=\!\!|\!\!\models} \mathbf{Th}_2$ means that that $\mathbf{Th}_1$ and $\mathbf{Th}_2$ imply the same theorems (i.e. formulas).

$\lhd$

THEOREM 3.3.7 *(i) and (ii) below hold.*

(i) $\mathbf{Newbasax} + \mathbf{Ax3} \;\mathrel{=\!\!|\!\!\models}\; \mathbf{Basax}$.

(ii) $\mathbf{Newbasax} + \{(\forall m,k \in Obs)\; m \overset{\odot}{\rightarrow} k\} \;\mathrel{=\!\!|\!\!\models}\; \mathbf{Basax}$.

We will give the **proof** after Thm.3.3.10 below.

By the above, if in **Basax** we replace **Ax6** and **AxE** with $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$ and $\mathbf{AxE_0}$, we still could prove exactly the same theorems (as from "old" **Basax**). It would be interesting to know if this generalizes to the case when we replace **Ax6** with $\mathbf{Ax6_{00}}$ only (in **Basax**, of course), i.e. if $(\mathbf{Basax} \setminus \{\mathbf{Ax6}\}) + \mathbf{Ax6_{00}} \models \mathbf{Basax}$. Cf. Remark 3.3.3 and footnote 205 in it.

Items 3.3.4 and 3.3.7 concern inter-derivability issues between **Newbasax** and **Basax**. We will return to this inter-derivability subject on p.197 (cf. Thm's 3.3.13, 3.3.14).

The following theorem states that two observers either see the same events or they see completely different events. Using the terminology of the introduction to the present section (p.187) this means, roughly, that the mapped areas in our atlas are either disjoint or they coincide. When further generalizing our theory in the direction of general relativity, we will have to allow these maps to overlap cf. e.g. the chapter on accelerated observers (i.e. the theory **Acc**).

THEOREM 3.3.8

Newbasax $\models (\forall m,k \in Obs)(Rng(w_m) = Rng(w_k) \quad \vee \quad Rng(w_m) \cap Rng(w_k) = \emptyset)$.

We will give the **proof** after the proof of Lemma 3.3.16 way below. We will start working on the preparations for the proof of Thm.3.3.8 below Thm.3.3.14 on p.198. Between the present point and p.198 we will use Thm.3.3.8 as a "black box" in the proofs of results stated before p.198. (We will be careful to avoid circularity, of course.)

Intuitively, the next theorem says that if an observer sees another then they map (or in other words they coordinatize) the same parts of space-time.

THEOREM 3.3.9

$$\textbf{Newbasax} \models (\forall m,k \in Obs)(m \overset{\odot}{\to} k \ \Leftrightarrow \ Rng(w_m) = Rng(w_k)) .$$

Proof: Let $\mathfrak{M} \in \mathsf{Mod}(\textbf{Newbasax})$. Let $m,k \in Obs$. Then $(Rng(w_m) = Rng(w_k) \ \Rightarrow \ m \overset{\odot}{\to} k)$ is obvious (by **Ax4**), hence we will concentrate on the other direction. To prove $(m \overset{\odot}{\to} k \ \Rightarrow \ Rng(w_m) = Rng(w_k))$ assume that $m \overset{\odot}{\to} k$. $m \overset{\odot}{\to} k$ and $\mathbf{Ax6_{00}}$ imply that $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. By this and by Thm.3.3.8, we have that $Rng(w_m) = Rng(w_k)$. ∎

THEOREM 3.3.10
Newbasax $\models$ "$\overset{\odot}{\to}$ *is an equivalence relation when restricted to Obs*".

Proof: The proof easily follows from Thm.3.3.9.

Poof of Thm.3.3.7: Direction $\Dashv$ is obvious. Hence it remains to prove direction $\models$.

Let us notice that $\{\mathbf{Ax2}, \mathbf{Ax3}\} \models (\forall m,k \in Obs) m \overset{\odot}{\to} k$. Hence it is enough to prove $\textbf{Newbasax} \cup \{(\forall m,k \in Obs)\ m \overset{\odot}{\to} k\} \models \textbf{Basax}$. Next we turn to prove this. By Thm.3.3.9, we have $\textbf{Newbasax} \cup \{(\forall m,k \in Obs)\ m \overset{\odot}{\to} k\} \models \mathbf{Ax6}$. Now it is easy to check that $\textbf{Newbasax} \cup \{\mathbf{Ax6}\} \models \{\mathbf{Ax3}, \mathbf{AxE}\}$. ∎

Notation 3.3.11 Let Σ be a set of formulas in the frame language of relativity theory. Then we define $\mathsf{Mod}_{\mathfrak{F}}(\Sigma)$ to be the class of all those frame models which are models of Σ and the ordered field reduct of which coincides with $\mathfrak{F}$, i.e.

$$\mathsf{Mod}_{\mathfrak{F}}(\Sigma) \overset{\text{def}}{=} \{\, \mathfrak{M} \in \mathsf{FM}(n, \mathfrak{F}) \ : \ \mathfrak{M} \models \Sigma \,\} .$$

◁

Thm.3.3.12 below says that $\mathfrak{M}$ is a model of **Newbasax** iff $\mathfrak{M}$ can be obtained by taking a kind of a "disjoint union" of some models of **Basax**. Thus, a study of models of **Basax** will also give a study of models of **Newbasax**, cf. §§3.2, 3.5, 3.6. As we said, Thm.3.3.12 below says that the models of **Newbasax** are certain kinds of unions of models of **Basax**. For simplicity let us consider the union of two models $\mathfrak{N}_0$ and $\mathfrak{N}_1$ of **Basax** which differ only in their body sorts (B_0, B_1). For this see Figure 65. Now, their union will satisfy **Newbasax** if no common body[206]

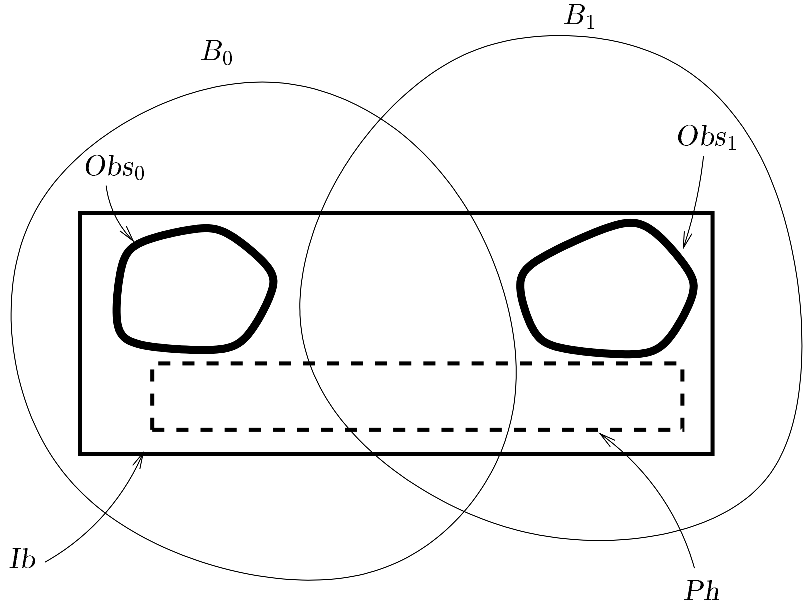

Figure 65: "Gluing" **Basax** models together to obtain **Newbasax** models.

is an observer in either one of the two models, and for each common body the two models agree on
(i) whether it is an inertial body, and
(ii) whether it is a photon.
In the other direction, any **Newbasax** model can be obtained as such a union of some **Basax** models. We suggest consulting Figure 307 (p.1001) at this point. That figure represents a characteristic **Newbasax** model. In connection with that figure we note the following. Intuitively, thinking in terms of world-views a "**Newbasax**-

[206]By a common body we understand an element of $B_0 \cap B_1$.

world" is a disjoint union of "**Basax**-worlds" which may be connected by photons (or by other bodies) but not by observers. The visibility relation $\overset{\odot}{\to}$ is an equivalence relation on the observers (cf. Thm.3.3.10). In the above quoted **Newbasax** picture the "windows" (i.e. the constituent "**Basax**-worlds") correspond to the equivalence classes of this visibility relation. We note, that what we call windows here (and subsequently e.g. in the geometry chapter, §6) are the same intuitive "things" what were called maps or mapped areas in the introduction of the present section cf. e.g. pp. 187, 193.

THEOREM 3.3.12 (characterization of models of Newbasax)

$$\mathfrak{M} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Newbasax})$$

$$\Longleftrightarrow$$

$$(\exists \mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax}))\Big((\forall \mathfrak{N}_0, \mathfrak{N}_1 \in \mathsf{K})(\mathfrak{N}_0 \neq \mathfrak{N}_1 \quad \Rightarrow$$

$$(Obs^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} = \emptyset \quad \& \quad Ph^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} \subseteq Ph^{\mathfrak{N}_1} \quad \& \quad Ib^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} \subseteq Ib^{\mathfrak{N}_1})) \textit{ and}$$

$$\mathfrak{M} = \langle (B^{\mathfrak{M}}, Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, Ib^{\mathfrak{M}}), \mathfrak{F}, G; \mathrm{E}, W^{\mathfrak{M}} \rangle, \textit{ where}$$

$$\begin{aligned}
B^{\mathfrak{M}} &\overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} B^{\mathfrak{N}}, \\
Obs^{\mathfrak{M}} &\overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} Obs^{\mathfrak{N}}, \\
Ib^{\mathfrak{M}} &\overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} Ib^{\mathfrak{N}}, \\
Ph^{\mathfrak{M}} &\overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} Ph^{\mathfrak{N}}, \textit{ and} \\
W^{\mathfrak{M}} &\overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} W^{\mathfrak{N}}\Big).
\end{aligned}$$

(Cf. Figure 65.)

Proof:
(i) Direction $\Longleftarrow$ goes by checking the axioms.

(ii) Direction $\Longrightarrow$ follows from Theorems 3.3.8, 3.3.9, 3.3.10 as follows. Assume $\mathfrak{M} = \langle B, Obs, \ldots\rangle \models$ **Newbasax**. By Thm.3.3.10 there is an equivalence relation $\equiv \subseteq Obs \times Obs$ (induced by $\overset{\odot}{\to}$). For simplicity assume $Obs/\equiv = \{Obs_0, Obs_1\}$. For each Obs_i $(i < 2)$ we construct a **Basax** model $\mathfrak{N}_i$. Let $\mathfrak{N}_i = \langle (B_i, Obs_i, Ph_i, Ib_i), \mathfrak{F}, G; \ \mathrm{E}\ , W_i\rangle$, where

$$\begin{aligned}
Ph_i &\overset{\text{def}}{=} \left\{ ph \in Ph \ : \ (\exists m \in Obs_i)\ m \overset{\odot}{\to} ph \right\}, \\
Ib_i &\overset{\text{def}}{=} \left\{ h \in Ib \ : \ (\exists m \in Obs_i)\ m \overset{\odot}{\to} h \right\}, \\
B_i &\overset{\text{def}}{=} B \setminus Ib_j \cup Ib_i \quad (\{i,j\} = \{0,1\}), \text{ and} \\
W_i &\overset{\text{def}}{=} W \restriction (B_i \times {}^nF \times B_i).
\end{aligned}$$

Now one can check that $\mathfrak{N}_i \models$ **Basax** and $\mathfrak{M}$ is obtained from $\mathfrak{N}_0, \mathfrak{N}_1$ as described in the theorem. ∎

$\not\models$-et átvinni az első előfordulásba!

THEOREM 3.3.13

(i) **Newbasax** $\not\models$ **Basax**, *moreover:*

(ii) **Newbasax** + **AxE** $\not\models$ **Basax**.[207]

Proof: Let $\mathfrak{N}_0, \mathfrak{N}_1 \in \mathsf{Mod}(\mathbf{Basax})$ with $B^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} = \emptyset$. Let $\mathfrak{M}$ be the model which is obtained by $\mathsf{K} = \{\mathfrak{N}_0, \mathfrak{N}_1\}$ as described in Thm.3.3.12. Then $\mathfrak{M} \models$ **Newbasax** by Thm.3.3.12.

It is easy to check that if $m_0 \in Obs^{\mathfrak{N}_0}$ and $m_1 \in Obs^{\mathfrak{N}_1}$ then $tr_{m_0}(m_1) = \emptyset$. Hence $\mathfrak{M} \not\models$ **Ax3**. This completes the proof of item (i).

The proof of item (ii) is similar: First one takes two models, call them $\mathfrak{N}_0$ and $\mathfrak{N}_1$, of **Basax** such that they do not have common observers (i.e. $Obs^{\mathfrak{N}_0} \cap Obs^{\mathfrak{N}_1} \neq \emptyset$) but they agree on the set of photons (i.e. $Ph^{\mathfrak{N}_0} = Ph^{\mathfrak{N}_1}$). Then one forms the union of these two models as described in Thm.3.3.12. The so obtained model will be a model of **Newbasax** + **AxE** but it will not validate **Ax3**. Checking the details is left to the reader. ∎

THEOREM 3.3.14 **Newbasax** + **Ax(symm)** $\not\models$ **Basax**.

Proof: The proof is similar to that of Thm.3.3.13, and we leave it to the reader. ∎

[207] $Th \not\models \psi$ abbreviates that "it is not the case that $Th \models \psi$", as usual.

The philosophical importance of the above theorem is that **Newbasax** <u>*remains*</u> truly more flexible than **Basax** <u>*even*</u> if we assume the natural symmetry principle **Ax(symm)** studied e.g. in §2.8 and to which we will return in §3.8 (**BaCo**).

The reader is invited to do the following. Whenever we introduce a new, refined axiom system (like **Newbasax** was above), check whether adding **Ax(symm)** to this new axiom system makes it equivalent with some old axiom system together with **Ax(symm)**.

Now we turn to the proof of Thm.3.3.8. Lemmas 3.3.15, 3.3.16 are needed for the proof of Thm.3.3.8. Lemma 3.3.15 below is an analogous counterpart of Claim 2.3.6.

LEMMA 3.3.15

$$\mathbf{Newbasax} \models (\forall m, k \in Obs)(m \overset{\odot}{\to} k \;\Rightarrow\; v_m(k) \neq 1)\,.$$

Proof: The proof goes by contradiction. Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Newbasax})$. Let $m, k \in Obs$ such that $m \overset{\odot}{\to} k$ and $v_m(k) = 1$. By **Ax5** we have that there is $ph \in Ph$ such that $tr_m(ph) = tr_m(k)$. Let such a ph be fixed. Let $p, q \in tr_m(k)$ such that $p \neq q$. Now $ph, k \in w_m(p) \cap w_m(q)$ and $\mathbf{Ax6_{00}}$ imply that $w_k(p') = w_m(p)$ and $w_k(q') = w_m(q)$, and $ph, k \in w_k(p') \cap w_k(q')$, for some $p', q' \in {}^nF$. Let such p' and q' be fixed. By $p \neq q$ and Claim 2.3.8(ii), we have that $p' \neq q'$. By $k \in w_k(p') \cap w_k(q')$ and **Ax4** we have $\overline{p'q'} = \bar{t} \overset{\text{def}}{=} F \times {}^{n-1}\{0\}$. Now $\overline{p'q'} = \bar{t}$ and $ph \in w_k(p') \cap w_k(q')$ contradict $\mathbf{AxE_0}$. ∎

The intuitive content of Lemma 3.3.16 below is the following. Assume **Newbasax**. Assume that e and e_1 are events connected by a photon in the world-view of some observer.[208] Then any observer who observes event e observes event e_1, too.[209] We note that the assumption that events e and e_1 are observed by a common observer is needed in the following sense. There are a model of **Newbasax** and events e and e_1 such that there is a photon ph which is present both in e and e_1 but there is no observer who observes both e and e_1. The illustration of this is in Figure 307 on p.1001.

LEMMA 3.3.16 **Newbasax** $\models (\forall m, k \in Obs)(\forall p, q \in {}^nF)$

$$\Big((\mathrm{ang}^2(\overline{pq}) = 1 \;\wedge\; p \in Dom(\mathsf{f}_{mk})) \;\Rightarrow\; q \in Dom(\mathsf{f}_{mk})\Big)\,.$$

[208] Formally, $(\exists m \in Obs)\; e, e_1 \in Rng(w_m) \;\wedge\; (\exists ph \in Ph)\; ph \in e \cap e_1$.
[209] Formally, $(\forall k \in Obs)\; [e \in Rng(w_k) \;\Rightarrow\; e_1 \in Rng(w_k)]$.

Proof: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Newbasax})$. Let $m, k \in Obs$ and $p, q \in {}^nF$ with $ang^2(\overline{pq}) = 1$ and $p \in Dom(\mathsf{f}_{mk})$. We have to prove that $q \in Dom(\mathsf{f}_{mk})$.

By $ang^2(\overline{pq}) = 1$ and **Ax5** we have $tr_m(ph) = \overline{pq}$, for some $ph \in Ph$. Let this ph be fixed. By $\mathbf{Ax6_{01}}$ and $p \in Dom(\mathsf{f}_{mk})$, we have that $S(p, \varepsilon) \subseteq Dom(\mathsf{f}_{mk})$, for some $\varepsilon \in {}^+F$. Let this ε be fixed.

Then it is easy to see that there are $r, s, u, v \in S(p, \varepsilon)$ such that p, q, r, s, u, v are distinct, $q \in \overline{rs}$, $q \in \overline{pu}$, $\overline{pr} \cap \overline{su} = \{v\}$, and $ang^2(\overline{rs}), ang^2(\overline{pr}), ang^2(\overline{su}) < 1$ (see Figure 66). Let such r, s, u, v be fixed.

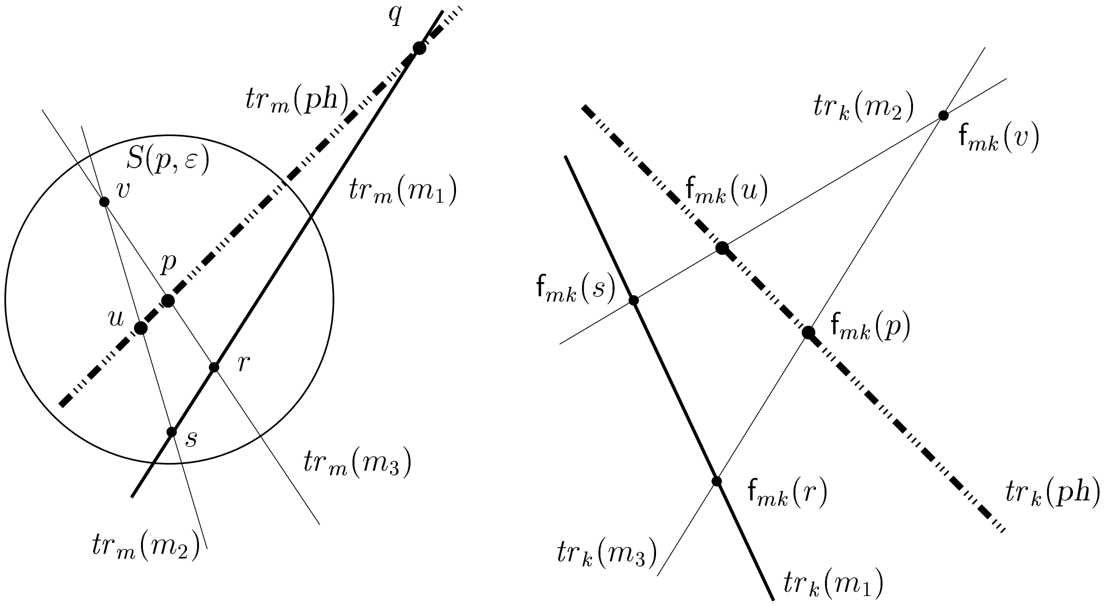

Figure 66: Illustration for the proof of Lemma 3.3.16.

By $ang^2(\overline{rs}), ang^2(\overline{pr}), ang^2(\overline{su}) < 1$ and **Ax5** we have that $tr_m(m_1) = \overline{rs}$, $tr_m(m_2) = \overline{su}$, and $tr_m(m_3) = \overline{pr}$, for some $m_1, m_2, m_3 \in Obs$. Let such m_1, m_2, m_3 be fixed. By the above construction we have that

$$\begin{aligned} tr_m(ph) \cap tr_m(m_3) &= \{p\}, \\ tr_m(m_1) \cap tr_m(m_3) &= \{r\}, \\ tr_m(m_1) \cap tr_m(m_2) &= \{s\}, \\ tr_m(ph) \cap tr_m(m_2) &= \{u\}, \\ tr_m(m_2) \cap tr_m(m_3) &= \{v\}. \end{aligned} \tag{28}$$

Now by $p, r, s, u, v \in S(p, \varepsilon) \subseteq Dom(\mathsf{f}_{mk})$ by $\mathbf{Ax3_0}$, by Claim 2.3.8(ii) and by (28), we have

$$
\begin{aligned}
tr_k(ph) \cap tr_k(m_3) &= \{\mathsf{f}_{mk}(p)\}\,, \\
tr_k(m_1) \cap tr_k(m_3) &= \{\mathsf{f}_{mk}(r)\}\,, \\
tr_k(m_1) \cap tr_k(m_2) &= \{\mathsf{f}_{mk}(s)\}\,, \\
tr_k(ph) \cap tr_k(m_2) &= \{\mathsf{f}_{mk}(u)\}\,, \\
tr_k(m_2) \cap tr_k(m_3) &= \{\mathsf{f}_{mk}(v)\}\,.
\end{aligned} \tag{29}
$$

$\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(r), \mathsf{f}_{mk}(s), \mathsf{f}_{mk}(u), \mathsf{f}_{mk}(v)$ are distinct points by Claim 2.3.8(ii) because p, r, s, u, v were distinct. Hence by (29) and $\mathbf{Ax3_0}$, we have that $tr_k(m_1)$ and $tr_k(ph)$ are in the same plane. Now by this and by Lemma 3.3.15, we have $tr_k(m_1) \cap tr_k(ph) = \{w\}$, for some $w \in {}^nF$. Let this w be fixed. Now

$$
m_1, ph \in w_m(q) \cap w_k(w)\,. \tag{30}
$$

By (30) and by $\mathbf{Ax6_{00}}$, we have that there are $q', w' \in {}^nF$ such that

$$
w_{m_1}(q') = w_m(q) \quad \text{and} \quad w_{m_1}(w') = w_k(w)\,. \tag{31}
$$

Let such q' and w' be fixed. By (30) and by (31), we have that $m_1, ph \in w_{m_1}(q') \cap w_{m_1}(w')$. By this, by $\mathbf{Ax4}$ and by $\mathbf{AxE_0}$, we have $q' = w'$. By $q' = w'$ and by (31), we have $w_m(q) = w_k(w)$. Hence $q \in Dom(\mathsf{f}_{mk})$. ∎

Proof of Thm.3.3.8: The intuitive idea of the proof is based on the following property of **Newbasax**. If $e,\ e_1$ are events "seen" by a common observer, then there is a finite "zig-zag" pass of photons connecting e and e_1. Because of this we can apply Lemma 3.3.16 to prove the theorem (cf. the intuitive text above that lemma). Below comes the formalization of this idea: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Newbasax})$. Let $m, k \in Obs$ with $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. We have to prove that $Rng(w_m) = Rng(w_k)$. To prove this it is enough to prove that $Dom(\mathsf{f}_{mk}) = {}^nF$ and $Dom(\mathsf{f}_{km}) = {}^nF$. By $Rng(w_m) \cap Rng(w_k) \neq \emptyset$, we have that there is $p \in Dom(\mathsf{f}_{mk})$. Let such a p be fixed. Let $q \in {}^nF$. We will prove that $q \in Dom(\mathsf{f}_{mk})$. It is easy to see that

$$
(32)\ (\exists j \in \omega)(\exists r^0, r^1, \ldots, r^j \in {}^nF)\Big((\forall i < j) ang^2(\overline{r^i r^{i+1}}) = 1\ \&\ r^0 = p\ \&\ r^j = q\Big)\,.
$$

Let such j and $r^0, \ldots, r^j$ be fixed. Now by applying Lemma 3.3.16 j times, by (32), we get that $q \in Dom(\mathsf{f}_{mk})$. Hence $Dom(\mathsf{f}_{mk}) = {}^nF$. Analogously $Dom(\mathsf{f}_{km}) = {}^nF$. ∎

Remark 3.3.17 Recall that in §2 we proved so called paradigmatic effects of relativity theory from **Basax**, cf. §§ 2.5, 2.8. We would like to point out that all these paradigmatic effects are provable from the weaker system **Newbasax** too.[210] The interested reader is invited to check that this is true, moreover beginning with §3.4.2 ("Weakening ...") we will introduce even weaker axiom systems and the reader is invited to check whether the just mentioned paradigmatic effects remain provable even from those.

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[210]Some re-formalization of these effects in our frame language might be needed for this. We do not go into more detail about this.

3.4 Faster than light observers

Contents of Section 3.4

In the first sub-section (§3.4.1) of the present section, we show that our relativity theories (**Basax**, **Newbasax**) introduced so far imply the nonexistence of FTL observers, for $n > 2$. (Let us recall that in §2 we saw that in some models of **Basax**(2) FTL observers do exist. I.e. **Basax**(2) permits the existence of FTL observers.) At this point, we would like to emphasize that the main theorem of §3.4.1 is a theorem of purely logical nature and it does not involve concepts like "mass", "force" or "energy". The theorem (and its improvement in §3.4.2) says that a very small number of weak and natural assumptions already implies "logical" impossibility of say "tachyons being observers".

In §3.4.2 we investigate the reason for our "no FTL observer" result, by weakening the axioms of **Newbasax**. We arrive at various rather weak axiom systems, some of which are called **Bax**, **Bax**$^-$, **Relphax**, and **Reich**(**Bax**). To be precise **Bax**$^-$ and **Reich**(**Bax**) will be reached only in the next chapter §4 (which will be a direct continuation of §3.4.2). We will see that most (but not all) of these axiom systems still prove the nonexistence of FTL observers (assuming $n > 2$, of course). On the other hand, e.g. **Relphax** permits FTL observers for arbitrary n. The interesting aspect of this is that **Relphax** is not too weak to be considered as a possible (special) relativity theory.

These weak axiom systems (**Bax**, **Relphax** etc.) will lead us to the subject matter of our next chapter §4 ("Weak, flexible axiom systems for relativity"). The purpose of that chapter §4 is motivated by the literature and is twofold: (i) Friedman [90] started a kind of conceptual analysis[211] of relativity theory which is taken up and is further elaborated in §4 and especially in §§ 4.3, 4.4. (ii) Reichenbach, Grünbaum and others (cf. e.g. Szabó [244]) initiated a variant of relativity theory which differs from Einstein's one in the treatment of simultaneity.[212] In §4.5 of the present work, we will formalize the Reichenbachian versions (**Reich**(**Bax**), **Reich**(**Basax**) etc.)

[211] Cf. also footnote 2 on p.8.
[212] For more on these purposes cf. the introduction of §4.3, p.469 (see items 1–4 there).

of our relativity theories (**Bax**, **Basax**, etc. respectively), in first order logic. Of course, besides formalizing these Reichenbachian versions, we want to study them, among others in order to answer some problems raised in e.g. Friedman [90]. Subsection 3.4.2 contains preparations for this plan. As a continuation of §3.4.2, in the next chapter §4 we introduce the weak theory $\mathbf{Bax}^-$ which will be flexible enough to serve as a basis (or starting point) for developing the Reichenbachian theory **Reich**(**Bax**) (which will be done also in the next chapter).

3.4.1 Main stream investigations

In this sub-section we prove that **Basax** implies that there is no FTL observer if we assume that $n \geq 3$. In §2 we already saw that the assumption $n \geq 3$ is necessary. Here we will see this in a somewhat stronger form. All these are formulated in Thm.3.4.1 below.[213]

pezsgő után megnézni, hogy Fuch féle parciálisan rendezett testekre is átmegy-e a biz?

THEOREM 3.4.1 *Let $n \geq 3$. Then (i)–(iii) below hold.*

(i) $\mathbf{Basax}(n) \models (\forall m, k \in Obs)\ v_m(k) < 1.$

(ii) $\mathbf{Basax}(2) \not\models (\forall m, k \in Obs)\ v_m(k) < 1.$

(iii) *Assume $\mathfrak{F}$ is an arbitrary ordered field. Then*
$\mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax}(2)) \not\models (\forall m, k \in Obs)\ v_m(k) < 1.$

[213] A more general form of Thm.3.4.1(i) below, is in Madarász-Németi [177]. In that paper, we weaken $\mathbf{Basax}(n)$ so that the new $\mathbf{Partial}(\mathbf{Basax}(n))$ will not imply $Dom(w_m) = {}^nF$. Then we prove in that work that FTL observers are not possible even in $\mathbf{Partial}(\mathbf{Basax}(n))$ if $n > 2$. Let us recall from footnote 198 on p.188 that w_m^- is the partial function $w_m^- = \{\langle p, e\rangle \in w_m : e \neq \emptyset\}$. Now, in discussions like the present one whenever we write $Dom(w_m)$ we really mean $Dom(w_m^-)$. That is, from now on, in contexts like $Dom(w_m) \neq {}^nF$ etc. $Dom(w_m)$ denotes $Dom(w_m^-)$. We note that the axioms of $\mathbf{Partial}(\mathbf{Basax}(n))$ imply that $Dom(w_m^-)$ and $Dom(\mathsf{f}_{mk})$ are open connected subsets of ${}^nF$. The theory $\mathbf{Partial}(\mathbf{Basax}(n))$ could also be called "locally" **Basax**, because it deviates from **Basax** in the direction of the "local" spirit of general relativity. Its axioms are of the "flavour" like saying that $(\forall m \in Obs)(\forall p \in Dom(w_m^-))$ (there is a neighborhood of p in which "**Basax** is true"). In this connection we note the following. The method of formalizing the statement that "$Dom(w_m^-)$ is a connected subset of ${}^nF$" [and the same for $Dom(\mathsf{f}_{mk})$] in our first-order frame language is discussed in §8 ("Accelerated observers") in Item **??**.

On the **proof**:[214] We will give the proof later in this sub-section, on p.206. We will start working on the preparations for the proof of Thm.3.4.1 above Lemma 3.4.5 on p.205.

Recall from §3.3 (Def.3.3.2) the axiom system **Newbasax** which is a refined version of **Basax**. *Thm.3.4.2 below says that* **Newbasax** *does not allow FTL observers if* $n > 2$. It is an immediate corollary of Thm.3.4.1 and Thm.3.3.12 (§3.3).

THEOREM 3.4.2 *Let* $n \geq 3$. *Then*

$$\mathbf{Newbasax}(n) \models (\forall m, k \in Obs)\ (m \stackrel{\odot}{\to} k \ \Rightarrow\ v_m(k) < 1)\,.$$

Proof: The theorem follows directly from Thm.3.4.1 and Thm.3.3.12 (§3.3). ∎

Remark 3.4.3 (Are FTL observers possible?) Do Theorems 3.4.1, 3.4.2 mean that there are no FTL observers (if $n \geq 3$)? Of course, they do not. What they do say is that if FTL observers exist then something must be different than is assumed in these theorems. We will return to discussing this several times both in this work and in Madarász-Németi [176]. For a further discussion of Thm's 3.4.1, 3.4.2, let us assume we have two space-dimensions $\bar{x}$ and $\bar{y}$. Then one way for making FTL observers possible is assuming that we have two time-dimensions $\bar{t}_1$ and $\bar{t}_2$, too (orthogonal to each other). Both of $\bar{t}_1$ and $\bar{t}_2$ would be inside the light-cone starting from the origin $\bar{0}$. This idea is suggested by the proofs of our "no FTL" theorems. In general, if the number of space-dimensions is the same as that of time-dimensions then FTL observers could exist. We do not discuss this idea further here (but $n = 2$ is an example of number of space-dimensions = that of time-dimensions). Of course, it is not clear what physical interpretation one should give to the second time-dimension $\bar{t}_2$ (... but some ideas do come to ones mind).

In connection with "making FTL observers possible" we mention two further materials: (i) the axiom system **Relphax** discussed on pp. 222–226 of §3.4.2 ("Weakening the axioms (FTL observers)") and (ii) a future section of the present work entitled "Inner clocks (of bodies)".

◁

Remark 3.4.4 Our "no-FTL-observers" theorems would be relatively easy to prove if we would restrict them to the standard Minkowski models[215] over the ordered field $\mathfrak{R}$ of the real numbers. However, the main point in our theorems is that they are

[214] We note that a stronger theorem will be stated and proved soon, cf. Thm.3.4.19 (p.221) generalizing the present "no FTL" result to the weaker axiom system $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$.

[215] These will be introduced in Def.3.8.42 on p.331.

much more general than this special case e.g. we do not require our ordered field $\mathfrak{F}$ to be Euclidean.[216] Further, in **Basax** we did not state anything to the effect that f_{mk} would be continuous. However, using the no FTL theorem stated above we can prove that the f_{mk}'s are betweenness preserving (cf. Prop.6.6.5, p.1028) and that this implies that they are continuous. Cf. also Remark 3.4.6, in this connection.

$\lhd$

For the proof of Thm.3.4.1 we need Lemmas 3.4.5, 3.4.7 below. First we will state the lemmas, after that we will give the proof of Thm.3.4.1. After the proof of Thm.3.4.1 we will give the proof of the lemmas.

LEMMA 3.4.5 *Assume* $n \geq 3$ *and* $\mathfrak{F}$ *is Euclidean. Assume* $\mathsf{f} : {}^nF \longrightarrow {}^nF$ *is a bijection such that*

$$(\star) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \ \wedge \ (\mathsf{f}[\ell] \in \mathsf{PhtEucl} \ \Leftrightarrow \ \ell \in \mathsf{PhtEucl})\Big).$$

Then $\mathsf{f}[\bar{t}\,] \in \mathsf{SlowEucl}$.

On the **proof**: We will give the proof of Lemma 3.4.5 on p.208.

Remark 3.4.6 We note that the following stronger form of Lemma 3.4.5 is also true:

Assume $n \geq 3$ and that $\mathfrak{F}$ is Euclidean. Assume $\mathsf{f} : {}^nF \longrightarrow {}^nF$ is a bijection such that

$$(\forall \ell)(\mathsf{f}[\ell] \in \mathsf{PhtEucl} \ \Leftrightarrow \ \ell \in \mathsf{PhtEucl}).$$

Then f preserves the set of slow-lines, that is $(\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}[\ell] \in \mathsf{SlowEucl}$.

We note that the proof of this stronger form of Lemma 3.4.5 is based on the proof of the celebrated Alexandrov-Zeeman Theorem, cf. Thm.6.7.23 on p.1159.

[216] Indeed, the most natural ideas that come to mind to prove a no-FTL-observer theorem do not work for the following reason. Let us assume that we wanted to prove our no FTL theorem via Item 3.4.6 below. If we assume that $\mathfrak{F}$ is Euclidean then this strategy will indeed work. However, there exist an ordered field $\mathfrak{F}$ and a photon-preserving collineation f of ${}^nF$ such that f does *not* preserve $\mathsf{SlowEucl}$. (This means that there is $\mathfrak{F}$ such that the reduct $\langle {}^4F, \mathsf{Eucl}, \mathsf{PhtEucl}, \in \rangle$ of the usual Minkowski geometry over $\mathfrak{F}$ [in the sense of Def.6.7.25 on p.1160] admits an automorphism which does *not* preserve $\mathsf{SlowEucl}$. As we will see, this geometry cannot be extended to a model of **Basax**, but one has to prove this).

Further we note that this proof (i.e. the proof of the stronger form of Lemma 3.4.5) is basically the same as the proof of Theorem 3 in [16] which says that $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ })$ does not allow FTL observers, if $n \geq 3$ is assumed.

$\lhd$

LEMMA 3.4.7 *Assume* $\mathsf{f} \in Aftr(n, \mathfrak{F})$ *satisfying* $(\star)$ *in Lemma 3.4.5. Assume* $\mathfrak{F}_*$ *is an ordered field such that* $\mathfrak{F} \subseteq \mathfrak{F}_*$.[217] *Let* $\mathsf{f}_* \in Aftr(n, \mathfrak{F}_*)$ *such that* $\mathsf{f}_* \restriction {}^nF = \mathsf{f}$. *Let us notice that such an* f_* *exists and is unique. Then* f_* *satisfies* $(\star)$ *in Lemma 3.4.5 when* f_* *and* $\mathfrak{F}_*$ *are substituted in place of* f *and* $\mathfrak{F}$, *respectively.*

On the **proof**: We will give the proof of Lemma 3.4.7 on p.213.

Proof of Thm.3.4.1(i): Let $n \geq 3$ and $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(n))$. Let $m, k \in Obs$. We have to prove that $v_m(k) < 1$.

Intuitive idea of the proof: We want to prove that $tr_m(k)$ is "slow". We know that $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, where f is an affine transformation satisfying $(\star)$ of 3.4.5. By 3.4.7 f will continue satisfying $(\star)$ in a larger field $\mathfrak{F}_*$, which, in turn will be Euclidean. Looking at it from $\mathfrak{F}_*$, $\mathsf{f}[\bar{t}\,]$ must be "slow" by 3.4.5. Therefore $\mathsf{f}[\bar{t}\,]$ must be slow in $\mathfrak{F}$, too, and then $tr_m(k) = \mathsf{f}[\bar{t}\,]$ will complete the proof.

Formally: By Theorems 3.1.1, 3.1.4 and Proposition 3.1.17, we have that $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, for some $\varphi \in Aut(\mathbf{F})$ and for some $\mathsf{f} \in Aftr(n, \mathfrak{F})$ satisfying $(\star)$ in Lemma 3.4.5. Let such φ and f be fixed. By $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, by $\widetilde{\varphi}[\bar{t}\,] = \bar{t}$, by $tr_k(k) = \bar{t}$ and by $\mathsf{f}_{km}[tr_k(k)] = tr_m(k)$, we have

$$\mathsf{f}[\bar{t}\,] = tr_m(k)\,. \tag{34}$$

Let $\mathfrak{F}_*$ be an ordered field such that $\mathfrak{F}_*$ is Euclidean and $\mathfrak{F} \subseteq \mathfrak{F}_*$. Such an $\mathfrak{F}_*$ exists, e.g. the real closure[218] of $\mathfrak{F}$ is such. Let $\mathsf{f}_* \in Aftr(n, \mathfrak{F}_*)$ such that $\mathsf{f}_* \restriction {}^nF = \mathsf{f}$. Then by Lemma 3.4.7, f_* satisfies $(\star)$ in Lemma 3.4.5 when f_* and $\mathfrak{F}_*$ are substituted in place of f and $\mathfrak{F}$, respectively. According to our Convention 3.1.2, F_* denotes the universe of $\mathfrak{F}_*$. Let $\bar{t}_* := F_* \times {}^{n-1}\{0\}$. Then $\mathsf{f}_*[\bar{t}_*] \in \mathsf{SlowEucl}(n, \mathfrak{F}_*)$ by Lemma 3.4.5. But $\mathsf{f}[\bar{t}\,] \subseteq \mathsf{f}_*[\bar{t}_*]$ by $\mathsf{f}_* \restriction {}^nF = \mathsf{f}$ and $\bar{t} \subseteq \bar{t}_*$. Hence $\mathsf{f}[\bar{t}\,] \in \mathsf{SlowEucl}$. By this and by (34), we have $v_m(k) < 1$. ∎

The following is a corollary of the proof of Thm.3.4.1(i) above.

[217] According to our Convention 3.1.2, $\mathfrak{F} \subseteq \mathfrak{F}_*$ means that $\mathfrak{F}$ is a strong sub-model of $\mathfrak{F}_*$. In the present context this means that $\mathfrak{F}$ is an ordered subfield of $\mathfrak{F}_*$.

[218] The notion of the real closure of an ordered field can be found e.g. in [92].

COROLLARY 3.4.8 Lemma 3.4.5 remains true if we omit the condition that $\mathfrak{F}$ is Euclidean. ∎

Proof of Thm.3.4.1(iii): Let $\mathfrak{F}$ be arbitrary. Let P be a choice function that to each $\ell \in \mathsf{Eucl}(2, \mathfrak{F})$ associates two distinct points lying on ℓ. Let the model $\mathfrak{M}_1^P$ be defined for $\mathfrak{F}$ as it was defined in §2.4 for the case when $\mathfrak{F}$ was $\mathfrak{R}$, the ordered field of real numbers. There are FTL observers in the frame model $\mathfrak{M}_1^P$. Analogously to Thm.2.4.2 in §2.4 one can prove that $\mathfrak{M}_1^P \models$ **Basax**(2).

Proof of Thm.3.4.1(ii): Item (ii) directly follows from item (iii) of Thm.3.4.1. ∎

Now we turn to the proof of Lemma 3.4.5. For the proof of Lemma 3.4.5 we need the definition of the light-cone of $p \in {}^nF$. This comes next.

Warning: The following definition of a light-cone applies only to models of **Newbasax**. A more general definition of a light-cone will be given in Def.4.4.9 (p.538) in §4.4.

Definition 3.4.9

(i) We define the *light-cone* of $\bar{0} \in {}^nF$ as follows:

$$\mathsf{LightCone}(\bar{0}) \stackrel{\text{def}}{=} \left\{ q \in {}^nF \ : \ q_0^2 = q_1^2 + q_2^2 + \ldots + q_{n-1}^2 \right\} .$$

(ii) We define the *light-cone* of $p \in {}^nF$ as follows:

$$\mathsf{LightCone}(p) \stackrel{\text{def}}{=} \mathsf{LightCone}(\bar{0}) + p \stackrel{\text{def}}{=} \{ q + p \ : \ q \in \mathsf{LightCone}(\bar{0}) \} .$$

◁

Remark 3.4.10 Let us notice that for every $p \in {}^nF$

$$\mathsf{LightCone}(p) = \bigcup \{ \ell \in \mathsf{PhtEucl} \ : \ p \in \ell \} .$$

That is, in the world-view of an observer in a model of **Basax**, $\mathsf{LightCone}(p)$ is the union of traces of photons going through p. This is where the name "light-cone" comes from.

◁

Notation 3.4.11 We let S denote the <u>*"space" part*</u> of our coordinate-system ${}^nF$, i.e.

$$S \stackrel{\text{def}}{=} \{0\} \times {}^{n-1}F\,.$$

$\lhd$

Proof of Lemma 3.4.5: The proof goes by contradiction. Let $\mathsf{f} : {}^nF \longrightarrow {}^nF$ be a bijection satisfying ($\star$). Assume that $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$.

<u>*Intuitive idea of the proof:*</u> We will see that $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$ will imply that $\mathsf{f}[S]$ will contain a straight line "inside" a light-cone. But then the intersection of $\mathsf{f}[S]$ and this light-cone will contain a straight line. Clearly this straight line will be a "photon-line" and its inverse image by f will be contained in S. See Figure 68. This will contradict ($\star$), i.e. the fact that the inverse images of "photon-lines" by f must be "photon-lines".

<u>*Formally:*</u> By ($\star$), we have $\mathsf{f}[\bar{t}\,] \notin \mathsf{PhtEucl}$. Therefore $ang^2(\mathsf{f}[\bar{t}\,]) > 1$. Without loss of generality we can assume (35) and (36) below.

$$\begin{aligned} &(35) \qquad & \mathsf{f}[\mathsf{Plane}(\bar{t},\bar{x})] &= \mathsf{Plane}(\bar{t},\bar{x}) \quad \textit{and} \\ &(36) \qquad & \mathsf{f}(\bar{0}) &= \bar{0}\,. \end{aligned}$$

We will explain at the end of the proof why we can assume (35), (36) above.

By f being a bijection taking straight lines to straight lines and by (36), we have

(37) $\qquad \mathsf{f}[S]$ is an $(n-1)$–dimensional subspace of ${}^n\mathbf{F}$.[219]

Now by the assumption $ang^2(\mathsf{f}[\bar{t}\,]) > 1$ and by (35), (36), we have

(38) $\qquad \mathsf{f}[\bar{t}\,] = \{\, x \cdot \langle a, 1, 0, \dots, 0\rangle \;:\; x \in F\,\}$, for some $a \in F$ with $|a| < 1$.

Let this a be fixed.

Let ℓ be the mirror image of $\mathsf{f}[\bar{t}\,]$ w.r.t. either one of the lines in $\mathsf{PhtEucl} \cap \mathsf{Plane}(\bar{t},\bar{x})$ going through $\bar{0}$, i.e. let

$$\ell \stackrel{\text{def}}{=} \{\, x \cdot \langle 1, a, 0, \dots, 0\rangle \;:\; x \in F\,\}\,.$$

Let us notice that $\ell \in \mathsf{SlowEucl}$ by $|a| < 1$.

Claim 3.4.12 $\ell \subseteq \mathsf{f}[S]$. Actually, $\ell = \mathsf{f}[\bar{x}]$.

[219]For the visually oriented reader we note that $\mathsf{f}[S]$ is a, so called, hyperplane.

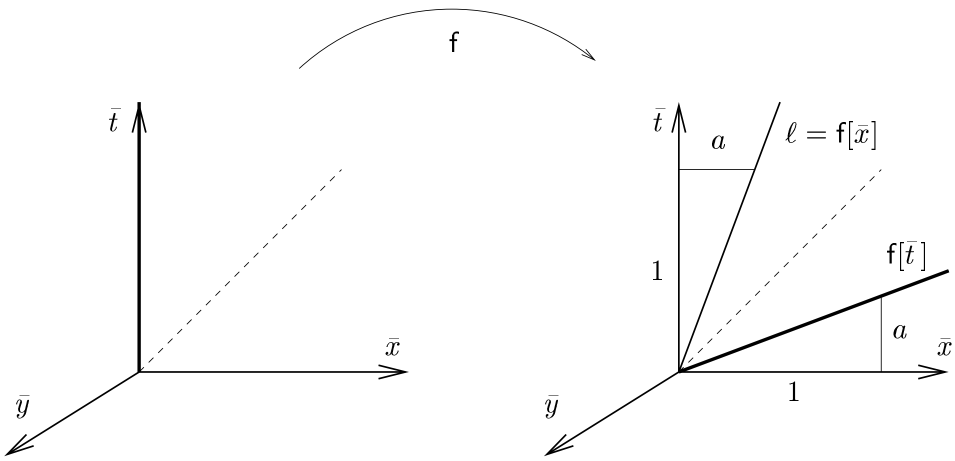

Figure 67: Illustration for Claim 3.4.12.

Proof of Claim 3.4.12: Throughout the proof the reader is asked to consult Figure 67. By (35) we can define $\mathsf{f}_0 : {}^2F \longrightarrow {}^2F$ as follows:

$$(\forall p \in {}^2F)\ \mathsf{f}_0(p_0, p_1) \stackrel{\text{def}}{=} \mathsf{f}(p_0, p_1, 0, \dots, 0)\,.$$

Then $\mathsf{f}_0 : {}^2F \longrightarrow {}^2F$ is a bijection such that

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}_0[\ell] \in \mathsf{Eucl} \quad \wedge \quad (\mathsf{f}_0[\ell] \in \mathsf{PhtEucl} \ \Leftrightarrow\ \ell \in \mathsf{PhtEucl})\Big)\,,$$

because f is a bijection satisfying ($\star$) in Lemma 3.4.5. By Corollary 3.1.22 (§3.1), and by $\mathsf{f}(\bar{0}) = \bar{0}$ (cf. (36)), we have that $\mathsf{f}_0[\bar{t}\,]$ and $\mathsf{f}_0[\bar{x}]$ are mirror images of each other w.r.t. either one of the lines in $\mathsf{PhtEucl}(2)$ containing $\bar{0}$.[220] Using this we conclude $\mathsf{f}[\bar{x}] = \ell$ by the definitions of ℓ and f.

QED (Claim 3.4.12)

Claim 3.4.13 There is $p \in {}^nF$ such that $\bar{0} \neq p \in \mathsf{LightCone}(\bar{0}) \cap \mathsf{f}[S]$. (See Figure 68.)

We will give the *detailed proof* of Claim 3.4.13 very soon. Claim 3.4.13 is true in the case $n = 3$ and $\mathfrak{F} = \mathfrak{R}$, because the plane $\mathsf{f}[S]$ has a point inside the cone $\mathsf{LightCone}(\bar{0})$, namely $\ell \subseteq \mathsf{f}[S]$ and ℓ is inside the cone by $\ell \in \mathsf{SlowEucl}$. Thus the

[220]According to our convention $\mathsf{PhtEucl}(2)$ denotes $\mathsf{PhtEucl}(2, \mathfrak{F})$.

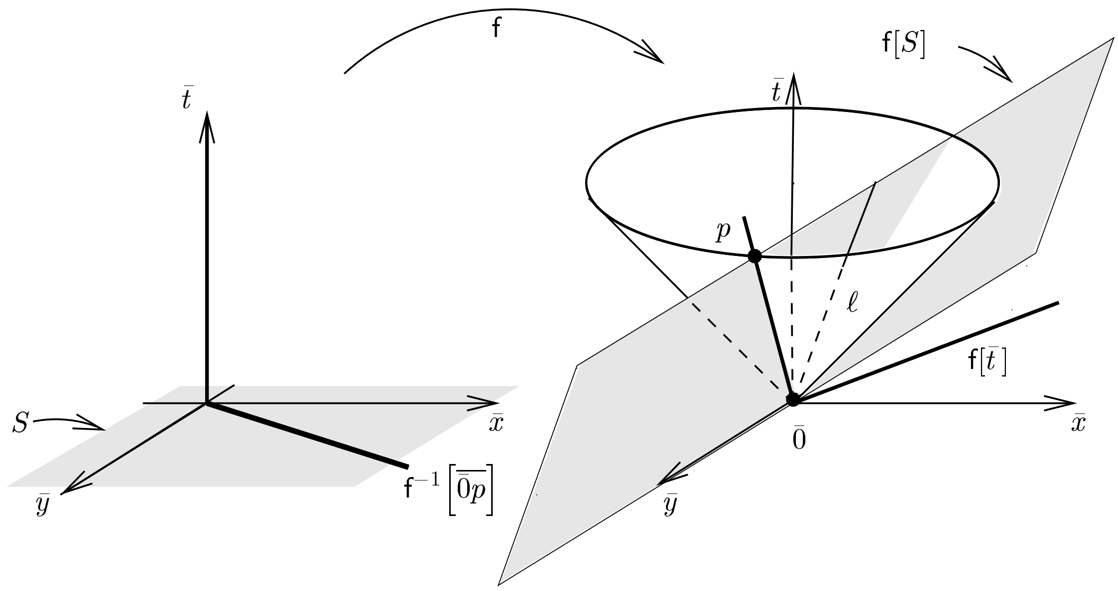

Figure 68: Illustration for the proof of Lemma 3.4.5 and for Claim 3.4.13.

plane $\mathsf{f}[S]$ intersects the cone in a point different from $\bar{0}$. (See Figure 68.) We will give the detailed proof for the general case ($n \geq 3$ and $\mathfrak{F}$ is an arbitrary Euclidean field) very soon. We note that if $\mathfrak{F}$ is the ordered field of rational numbers, (in which $\sqrt{2}$ does not exist), then the intersection of the plane $\mathsf{Plane}(\bar{t}, \overline{\bar{0}\langle 0,1,1\rangle})$ with the cone $\mathsf{LightCone}(\bar{0})$ is the singleton of $\bar{0}$, while the plane clearly contains points that are inside the cone (e.g. $\bar{t}$ is inside the cone). This motivates the use of $\mathbf{Ax}(\sqrt{\ })$ in the proof of Claim 3.4.13.

Now we return to the proof of Lemma 3.4.5. Throughout the proof the reader is asked to consult Figure 68. By Claim 3.4.13, there is $\bar{0} \neq p \in \mathsf{LightCone}(\bar{0}) \cap \mathsf{f}[S]$. Let such a p be fixed. $p \in \mathsf{LightCone}(\bar{0}) \setminus \{\bar{0}\}$ implies that $\overline{\bar{0}p} \in \mathsf{PhtEucl}$. Now we conclude $\overline{\bar{0}p} \subseteq \mathsf{f}[S]$ by $p \in \mathsf{f}[S]$ and by $\mathsf{f}[S]$ being a subspace of ${}^{n}\mathbf{F}$ (cf. (37)). By this we have $\mathsf{f}^{-1}\left[\overline{\bar{0}p}\right] \in S$, hence $\mathsf{f}^{-1}\left[\overline{\bar{0}p}\right] \notin \mathsf{PhtEucl}$. $\overline{\bar{0}p} \in \mathsf{PhtEucl}$ and $\mathsf{f}^{-1}\left[\overline{\bar{0}p}\right] \notin \mathsf{PhtEucl}$ contradict ($\star$) in the formulation of Lemma 3.4.5. Hence $\mathsf{f}[\bar{t}\,] \in \mathsf{SlowEucl}$.

Lemma 3.4.5 is proved modulo Claim 3.4.13, and the explanation of why we can assume (35) and (36) above. Now we turn to prove these.

<u>*Proof of Claim 3.4.13:*</u> Throughout the proof the reader is asked to consult Figure 69. Let ℓ be as defined above Claim 3.4.12, i.e. $\ell \stackrel{\text{def}}{=} \{\, x \cdot \langle 1, a, 0, \ldots, 0\rangle \,:\, x \in F \,\}$.

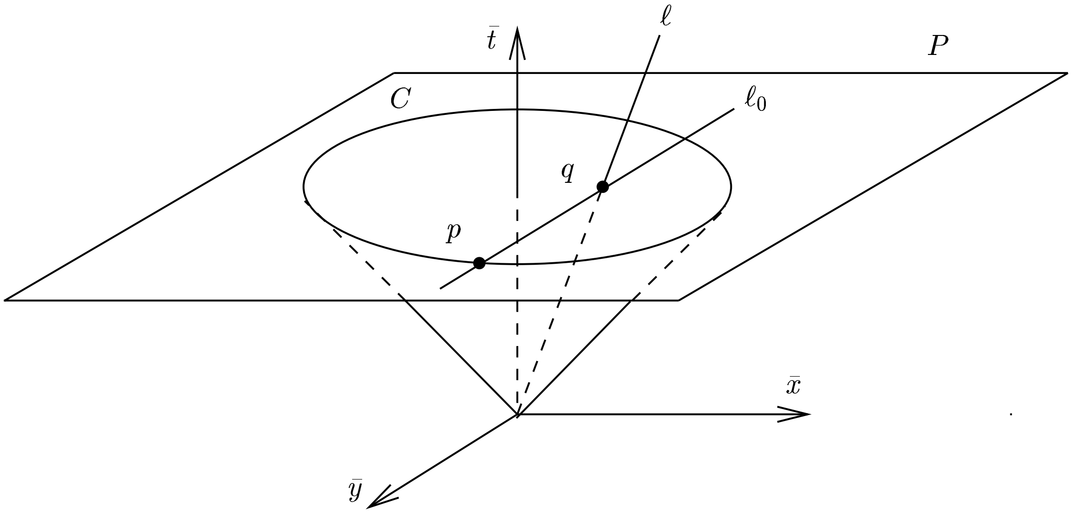

Figure 69: Illustration for the proof of Claim 3.4.13.

Let us recall that $a \in F$ was fixed below (38) and $|a| < 1$. By Claim 3.4.12, we have that $\ell \subseteq \mathsf{f}[S]$. Let P be the plane parallel with $\mathsf{Plane}(\bar{x}, \bar{y})$ and with height 1, i.e. let

$$P \stackrel{\text{def}}{=} \{ \langle 1, x, y, 0, \ldots, 0 \rangle \ : \ x, y \in F \} \ .$$

Let $q \stackrel{\text{def}}{=} \langle 1, a, 0, \ldots, 0 \rangle$. Then $q \in \mathsf{f}[S] \cap P$ by $q \in \ell \subseteq \mathsf{f}[S]$. Since $\mathsf{f}[S]$ is an $(n-1)$–dimensional subspace, $n \geq 3$, and $q \in \mathsf{f}[S] \cap P$, $\mathsf{f}[S] \cap P$ contains a straight line ℓ_0 such that $q \in \ell_0$.[221]

The intersection $\mathsf{LightCone}(\bar{0}) \cap P$ is a circle, i.e.

$$C \stackrel{\text{def}}{=} \mathsf{LightCone}(\bar{0}) \cap P = \left\{ \langle 1, x, y, 0, \ldots, 0 \rangle \ : \ x^2 + y^2 = 1 \right\} \ .$$

The point q is inside this circle by $|a| < 1$. We will show that

$$\ell_0 \cap C \neq \emptyset \, . \tag{39}$$

This is true, because it is known from geometry that if $\mathfrak{F}$ is Euclidean then inside every plane, whenever a straight line has a point inside a circle, it intersects the circle.[222] For completeness, we give here a direct proof of (39), too.

[221] This is known from linear algebra; intersection of a (≥ 2)–dimensional subspace with a plane contains a straight line through each point in the intersection.

[222] Cf. Jones-Morris-Pearson[145] chapter 5 or Szmielew [246]. Actually, (i) and (ii) below are equivalent in every two-dimensional geometry over an ordered field $\mathfrak{F}$:

Since $q \in \ell_0 \subseteq P$, we have that, for some $b \in F$,

$$\ell_0 = \{ \langle 1, x, y, 0, \dots, 0 \rangle \ :\ x = by + a \} \,.$$

Let this b be fixed. Then $\langle 1, x, y, 0, \dots, 0 \rangle \in \ell_0 \cap C$ iff

$$x^2 + y^2 = 1 \quad \text{ and } \quad x = by + a \,.$$

By replacing x with $by + a$ in the first equation we get

$$(b^2 + 1)y^2 + 2bay + (a^2 - 1) = 0 \,.$$

This quadratic equation, by $\mathfrak{F}$ being Euclidean, has a solution iff

$$(2ba)^2 - 4(b^2 + 1)(a^2 - 1) \geq 0 \,.$$[223]

Carrying out some simplifications, this is equivalent to

$$b^2 + (1 - a^2) \geq 0 \,.$$

By $|a| < 1$ this always holds. (39) is proved.
Now, let $p \in \ell_0 \cap C$. Then $p \in \mathsf{f}[S] \cap \mathsf{LightCone}(\bar{0})$ by $\ell_0 \subseteq \mathsf{f}[S]$, $C \subseteq \mathsf{LightCone}(\bar{0})$. Also, $p \neq \bar{0}$ because $p \in P$ while $\bar{0} \notin P$.

QED (Claim 3.4.13)

Explanation of why we can assume (35) and (36) above: Throughout this explanation the reader is asked to consult Figure 70. Let us recall from the beginning of the proof of Lemma 3.4.5 that f is a bijection satisfying $(\star)$ in the formulation of Lemma 3.4.5 and $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$. We also recall assumptions (35) and (36):

$$\begin{aligned} \mathsf{f}[\mathsf{Plane}(\bar{t}, \bar{x})] &= \mathsf{Plane}(\bar{t}, \bar{x}) \quad \textit{and} \\ \mathsf{f}(\bar{0}) &= \bar{0} \,. \end{aligned}$$

Now we will explain why these assumptions can be made.

Let $s := \mathsf{f}(\bar{0})$. Let us recall that τ_{-s} denotes the translation by vector $-s$. Now let $\mathsf{f}_s := \mathsf{f} \circ \tau_{-s}$. We have $\mathsf{f}_s(\bar{0}) = \bar{0}$ and $\mathsf{f}_s[\bar{t}\,] \notin \mathsf{SlowEucl}$ by $\mathsf{f}(\bar{0}) = s$ and $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$.

(i) If a straight line ℓ has a point inside a circle, then ℓ intersects the circle.
(ii) The square root of any $x > 0$ exists.

[223]We note that this means that the so called discriminant of the above quadratic equation is non-negative.

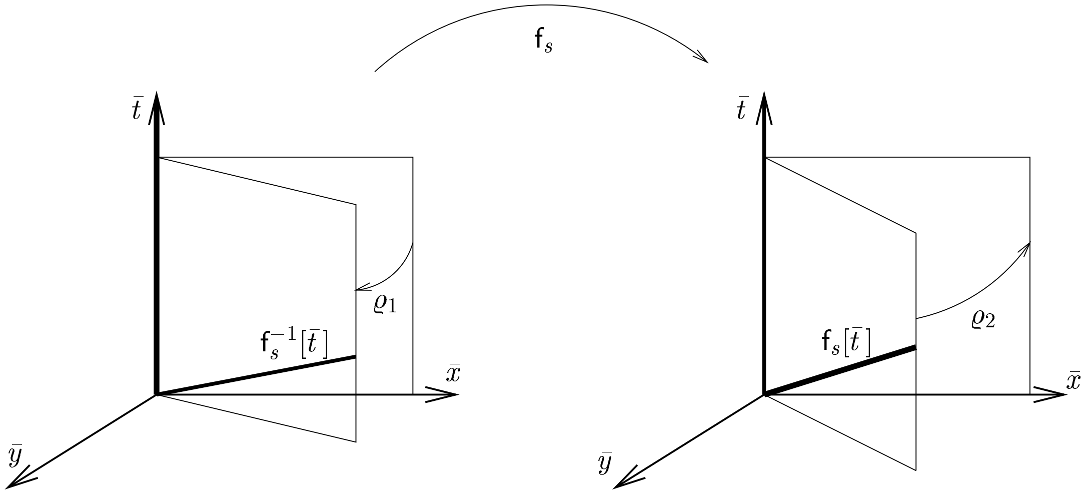

Figure 70: Illustration for explanation of assumptions (35), (36).

Let ϱ_1 and ϱ_2 be linear transformations such that ϱ_1 and ϱ_2 satisfy 1-3 below.

1. ϱ_1 and ϱ_2 leave $\bar{t}$ (the time axis) point-wise fixed.
2. ϱ_1 and ϱ_2 are distance preserving transformations (i.e. congruence transformations).
3. $\varrho_1[\mathsf{Plane}(\bar{t},\bar{x})] = \mathsf{Plane}(\bar{t},\mathsf{f}_s^{-1}[\bar{t}])$ and $\varrho_2[\mathsf{Plane}(\bar{t},\mathsf{f}_s[\bar{t}]) = \mathsf{Plane}(\bar{t},\bar{x})$.

By $\mathfrak{F}$ being Euclidean, it is easy to see that such ϱ_1 and ϱ_2 exist, e.g. rotations around time axis $\bar{t}$ which take $\mathsf{Plane}(\bar{t},\bar{x})$ and $\mathsf{Plane}(\bar{t},\mathsf{f}_s[\bar{t}])$ to $\mathsf{Plane}(\bar{t},\mathsf{f}_s^{-1}[\bar{t}])$ and $\mathsf{Plane}(\bar{t},\bar{x})$, respectively, are such.[224] See Figure 70.

It is easy to check that $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ is a bijection satisfying $(\star)$ in Lemma 3.4.5 when $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ is substituted in place of f, $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2[\bar{t}] \notin \mathsf{SlowEucl}$, and $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ satisfies (35), (36).[225] By this it is clear that assumptions (35), (36) can be made.

∎

Proof of Lemma 3.4.7: The proof below is somewhat "computational". We plan to replace it, in a future version with an intuitive, "structuralist" proof.

A detailed proof will be given only for $n = 3$. After that we will outline how to modify the proof for $n = 3$ to obtain a proof for $n = 4$. For the case $n = 4$,

[224] For a detailed proof cf. the proof of Lemma 3.5.3 in §3.5.

[225] $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ satisfies (35) because both $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ and $(\varrho_1 \circ \mathsf{f}_s \circ \varrho_2)^{-1}$ take $\bar{t}$ into $\mathsf{Plane}(\bar{t},\bar{x})$.

Lemma 3.4.41 in §3.4.3 is a generalization of Lemma 3.4.7. In §3.4.3 we will give a detailed proof of Lemma 3.4.41, therefore we will not give a detailed proof of Lemma 3.4.7 for $n = 4$.

Claim 3.4.14 Let $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in the formulation of Lemma 3.4.7. Then

$$(\star\star) \qquad (\forall \ell \in \mathsf{PhtEucl}(n, \mathfrak{F}_*))\ \mathsf{f}_*[\ell] \in \mathsf{PhtEucl}(n, \mathfrak{F}_*).$$

We will give the *proof* of Claim 3.4.14 very soon. Lemma 3.4.7 follows from Claim 3.4.14 because of the following. Intuitively: If f satisfies $(\star)$ in Lemma 3.4.5 then also f^{-1} satisfies $(\star)$. Then applying Claim 3.4.14 to f and f^{-1}, respectively, we obtain Lemma 3.4.7. More formally: Let $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in the formulation of Lemma 3.4.7. Let $(\mathsf{f}^{-1})_* \in \mathit{Aftr}(n, \mathfrak{F}_*)$ such that $(\mathsf{f}^{-1})_* \restriction {}^nF = \mathsf{f}^{-1}$. Obviously $(\mathsf{f}^{-1})_* = (\mathsf{f}_*)^{-1}$. Now applying Claim 3.4.14 to $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ and $\mathfrak{F}_*, \mathsf{f}^{-1}, (\mathsf{f}^{-1})_*$, respectively, we get that

$$(\forall \ell \in \mathsf{PhtEucl}(n, \mathfrak{F}_*))\ (\mathsf{f}_*[\ell] \in \mathsf{PhtEucl}(n, \mathfrak{F}_*)\ \wedge\ (\mathsf{f}_*)^{-1}[\ell] \in \mathsf{PhtEucl}(n, \mathfrak{F}_*)).$$

Hence f_* satisfies $(\star)$ in Lemma 3.4.5 when f_* and $\mathfrak{F}_*$ are substituted in place of f and $\mathfrak{F}$, respectively.

Proof of Claim 3.4.14: Let $n = 3$ and $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in formulation of Lemma 3.4.7. We have to prove that f_* satisfies $(\star\star)$ in Claim 3.4.14. Without loss of generality we may assume that $\mathsf{f}(\bar{0}) = \bar{0}$.

On the structure of the proof: Item (42) below is a reformulation of saying that f satisfies $(\star\star)$ of 3.4.14. Item (46) says the same for f_*. Therefore our task is to prove (46) from (42). This is done by the linear algebraic considerations given below. For undefined terminology from linear algebra the reader is referred to any linear algebra book, e.g. to Halmos [122]. O.K?

By our assumption that f is a linear transformation we have that

$$(\forall p \in {}^3F)\ \mathsf{f}(p) = \langle p_0a_{00} + p_1a_{10} + p_2a_{20}, p_0a_{01} + p_1a_{11} + p_2a_{21}, p_0a_{02} + p_1a_{12} + p_2a_{22}\rangle\,,$$

for some $a_{ij} \in F$, where $i, j \in 3$. Let these a_{ij}'s be fixed. According to our Convention 3.1.2, F_* denotes the universe of $\mathfrak{F}_*$. By $\mathsf{f}_* \in \mathit{Aftr}(3, \mathfrak{F}_*)$ and $\mathsf{f}_* \restriction {}^3F = \mathsf{f}$, we have

$$(41) \quad (\forall p \in {}^3F_*)\mathsf{f}(p) = \langle p_0a_{00} + p_1a_{10} + p_2a_{20}, p_0a_{01} + p_1a_{11} + p_2a_{21}, p_0a_{02} + p_1a_{12} + p_2a_{22}\rangle\,.$$

By f satisfying $(\star)$, we have

$$(42)\,(\forall p \in {}^3F)\ \Big(p_0^2 = p_1^2 + p_2^2 \ \Rightarrow$$
$$(p_0a_{00} + p_1a_{10} + p_2a_{20})^2 = (p_0a_{01} + p_1a_{11} + p_2a_{21})^2 + (p_0a_{02} + p_1a_{12} + p_2a_{22})^2\Big)\,.$$

Let $b_0 := a_{00}^2 - a_{01}^2 - a_{02}^2$, $b_1 := a_{10}^2 - a_{11}^2 - a_{12}^2$, $b_2 := a_{20}^2 - a_{21}^2 - a_{22}^2$, $b_3 := 2a_{00}a_{10} - 2a_{01}a_{11} - 2a_{02}a_{12}$, $b_4 := 2a_{00}a_{20} - 2a_{01}a_{21} - 2a_{02}a_{22}$, $b_5 := 2a_{10}a_{20} - 2a_{11}a_{21} - 2a_{12}a_{22}$. By this and by (42), we get

$$(43)\qquad (\forall p \in {}^3F)\ \Big(p_0^2 = p_1^2 + p_2^2 \ \Rightarrow$$
$$p_0^2b_0 + p_1^2b_1 + p_2^2b_2 + p_0p_1b_3 + p_0p_2b_4 + p_1p_2b_5 = 0\Big)\,.$$

But (43) is equivalent with (44) below.

(44) $\langle b_0, b_1, b_2, b_3, b_4, b_5\rangle$ is a solution for the system of linear equations
$E := \{\, p_0^2x_0 + p_1^2x_1 + p_2^2x_2 + p_0p_1x_3 + p_0p_2x_4 + p_1p_2x_5 \ :\ p \in {}^3F \ \&\ p_0^2 = p_1^2 + p_2^2 \,\}$.

To prove that f_* satisfies $(\star\star)$ it is enough to prove that

(45) $\langle b_0, b_1, b_2, b_3, b_4, b_5\rangle$ is a solution for the system of linear equations
$E_* := \{\, p_0^2x_0 + p_1^2x_1 + p_2^2x_2 + p_0p_1x_3 + p_0p_2x_4 + p_1p_2x_5 \ :\ p \in {}^3F_* \ \&\ p_0^2 = p_1^2 + p_2^2 \,\}$,

because (45) is equivalent with

$$(46)\,(\forall p \in {}^3F_*)\ \Big(p_0^2 = p_1^2 + p_2^2 \ \Rightarrow$$
$$(p_0a_{00} + p_1a_{10} + p_2a_{20})^2 = (p_0a_{01} + p_1a_{11} + p_2a_{21})^2 + (p_0a_{02} + p_1a_{12} + p_2a_{22})^2\Big)\,,$$

and (41) and (46) imply that f_* satisfies $(\star\star)$. To complete the proof it remains to prove (45) above. It is easy to check that the vectors in

$$B := \Big\{\langle p_0^2, p_1^2, p_2^2, p_0p_1, p_0p_2, p_1p_2\rangle \ :$$
$$p \in \{\,\langle 1,1,0\rangle, \langle 1,-1,0\rangle, \langle 1,0,1\rangle, \langle 1,0,-1\rangle, \langle 5,4,3\rangle\,\}\Big\}$$

are linearly independent.[226] Let $\mathbf{W}_*$ denote the subspace of ${}^6\mathbf{F}_*$ generated by

$$A_* := \{\,\langle p_0^2, p_1^2, p_2^2, p_0p_1, p_0p_2, p_1p_2\rangle \ :\ p \in {}^3F_* \ \&\ p_0^2 = p_1^2 + p_2^2\,\}\ .$$

[226] I.e. no element of B is generated by the others (in the universal algebraic sense) in the one sorted vector space ${}^n\mathbf{F}_1$. In this proof we will use that these vectors are linearly independent in the bigger vector space $({}^n\mathbf{F}_*)_1$ too. If we have a vector space ${}^n\mathbf{F}$ and a bigger one ${}^n\mathbf{F}_*$ (such that $\mathbf{F}$ is a subfield of $\mathbf{F}_*$) and p and q are in the small vector space ${}^n\mathbf{F}$ then p, q are linearly independent in the small vector space iff they are linearly independent in the big one. The same applies to a set of vectors like B above (this is a theorem of linear algebra).

$\mathbf{W}_*$ is at most 5-dimensional because of the "condition" $p_0^2 = p_1^2 + p_2^2$ in the definition of A_*. Hence B is a basis[227] of $\mathbf{W}_*$ by $B \subseteq A_*$. By this, we have that each equation in E_* (cf. (45)) is a linear combination of equations in

$$J := \Big\{ p_0^2 x_0 + p_1^2 x_1 + p_2^2 x_2 + p_0 p_1 x_3 + p_0 p_2 x_4 + p_1 p_2 x_5 \ : \\ p \in \{ \langle 1,1,0\rangle, \langle 1,-1,0\rangle, \langle 1,0,1\rangle, \langle 1,0,-1\rangle, \langle 5,4,3\rangle \} \Big\} .$$

By this, we have that (45) holds because $\langle b_0, b_1, b_2, b_3, b_4, b_5\rangle$ is a solution for the system of equations J by (44) and $J \subseteq E$. This completes the proof for $n = 3$. For $n = 4$ the proof is similar. Analogously to the proof for $n = 3$ it has to be shown that in the set

$$\{ \langle p_0^2, p_1^2, p_2^2, p_3^2, p_0p_1, p_0p_2, p_0p_3, p_1p_2, p_1p_3, p_2p_3\rangle \ : \ p \in {}^4\mathrm{Z} \ \& \ p_0^2 = p_1^2 + p_2^2 + p_3^2 \}$$

can be found 9 linearly independent vectors, where Z denotes the set of integers.[228]

∎

We will return to the possibility of (strongly) improving the result (Thm.3.4.1) "there are no FTL observers" after Thm.3.4.22 at the end of §3.4.2 below.

[227]By a basis of a vector space $\mathbf{V}$ we understand a minimal (i.e. independent) generator set of the one-sorted version of $\mathbf{V}$.

[228]We note that for every ordered field the set Z of the integers is embeddable into the ordered field in a natural way.

3.4.2 Weakening the axioms (FTL observers)

To our minds, Theorems 3.4.1(i) and 3.4.2 are important and strong theorems. They say that from our (deliberately weak) axiom systems of relativity theory it already follows that no observer can move faster than light. Hence it is worth to discuss which axiom is responsible for this. Therefore we have started a series of investigations such that we weaken our axiom system (in several ways), and then we check whether the new, weaker axiom systems allow FTL observers. These investigations also make it possible to understand better why Theorems 3.4.1, 3.4.2 are true.[229]

In this sub-section we present several examples of these investigations. First we present a very weak version (**Bax**) of **Newbasax** and show that this weak version still excludes FTL observers. In the next chapter (§4) we also look at even weaker versions of **Bax**. Then we present another weakened axiom system (**Relphax**), and show that it allows FTL observers. In all these cases we weaken Einstein's axiom, **AxE**. Clearly, if we omitted **AxE** from **Basax**, then the new axiom system would allow FTL observers. Besides the present FTL motivation, we will have other kinds of motivation for investigating weak systems like **Bax**; but these will lead us to the next chapter §4 ("Weak, flexible axiom systems for relativity"), where systems even weaker than **Bax** called e.g. **Reich**(**Bax**), **Bax**$^-$, **Rel**(**noph**) will be discussed. Some of these other motivations will be discussed in the introduction of §4.3 way below.

In the case of **Bax** we relax the condition that the speed of light is the same for all observers (but we retain the condition that each observer sees photons moving in all directions with the same speed). In the case of **Relphax** we relax the condition (that now is built into the language) that all observers perceive the same bodies as photons, i.e. being a photon becomes "relative" to observers.

[229]This kind of research is also about how the universe could look like from the logical point of view.These investigations are also motivated by the existence of Tachyon–Theory. About tachyon–theory we note that tachyons are hypothetical particles which move faster than the speed of light, cf. e.g. [75], [52].

The axiom system Bax (speed of light is observer-dependent) and connection to the literature

Besides the purposes outlined above, investigating **Bax** also serves other purposes e.g. a kind of continuation of the conceptual analysis started by Friedman [90]. We will write more about this beginning with Remark 4.3.40 way below.

Let us recall that Thm.3.4.2 says that **Newbasax** implies that there is no FTL observer, and let us recall from §3.3 that **Newbasax** is a refined version of our basic axiom system **Basax** where we allow that different observers observe different sets of events. We now introduce a new axiom system **Bax** which will be a refined version of **Newbasax**. As we said, we will fine-tune $\mathbf{AxE_0}$ of **Newbasax** in that the speed of light will not be the same for every observer, but for each observer photons moving in different directions will have the same speed. We will change **Ax5** only because $\mathbf{AxE_0}$ will be changed. After that we will state a theorem which says that **Bax** still implies that there is no FTL observer, if $n \geq 3$. More on this subject is in Madarász-Németi [175] and Madarász [172].

We will discuss the connections between **Bax** and the Kennedy-Thorndike experiment (cf. Taylor-Wheeler [256, pp.86–88] for the latter) in Remark 3.4.25 at the end of the present sub-section.

Below we postulate axioms $\mathbf{AxE_{00}}$, $\mathbf{AxE_{01}}$, $\mathbf{Ax5^{Obs}}$, and $\mathbf{Ax5^{Ph}}$. Recall from §3.3 the definition of relation $\overset{\odot}{\rightarrow}$ (Def.3.3.5) and the definition of **Newbasax** (Def.3.3.2).

$\mathbf{AxE_{00}}$ $(\forall m \in Obs)(\forall ph_1, ph_2 \in Ph)$
$$\Big((m \overset{\odot}{\rightarrow} ph_1 \ \wedge \ m \overset{\odot}{\rightarrow} ph_2) \ \Rightarrow \ v_m(ph_1) = v_m(ph_2)\Big).$$

That is, if observer m sees photons ph_1, ph_2 then the speed of ph_1 and ph_2 is same for m.

$\mathbf{AxE_{01}}$ $(\forall m \in Obs)(\forall ph \in Ph)(m \overset{\odot}{\rightarrow} ph \ \Rightarrow \ v_m(ph) \neq 0)$.

That is, there is no photon at rest.

$\mathbf{Ax5^{Obs}}$ $(\forall m \in Obs)(\exists ph \in Ph)(\forall \ell \in G)$
$$\Big(m \overset{\odot}{\rightarrow} ph \quad \wedge \quad [ang^2(\ell) < v_m(ph) \ \Rightarrow \ (\exists k \in Obs) tr_m(k) = \ell]\Big).$$

That is, every observer m sees some photon ph such that on every line slower than this photon there is an observer.

$\mathbf{Ax5^{Ph}}$ $(\forall m \in Obs)(\forall ph \in Ph)(\forall \ell \in G)$
$$(ang^2(\ell) = v_m(ph) \ \Rightarrow \ (\exists ph \in Ph) tr_m(ph) = \ell).$$

The intuitive content of $\mathbf{Ax5^{Ph}}$ will be quite important for us: Assume that observer m sees a photon with speed v. Now if in some other direction speed v can be realized by a line ℓ, then in that direction too there is a photon moving with the same speed v. We will call $\mathbf{Ax5^{Ph}}$ a *Weak Principle of Isotropy* (WPI) because it can be interpred as follows: (i) all directions are alike as far as speed of light is concerned, i.e. speed of light behaves the same way in all directions; (ii) more carefully: Assume $\mathbf{Ax}(\sqrt{\ })$. Now $\mathbf{Ax5^{Ph}}$ says that if observer m sees a photon ph with speed v in some direction, then in every other direction m will see a photon with the same speed v.

Remark 3.4.15 For completeness, we note that, intuitively, a principle of *isotropy* says that, from some point of view, all spatial directions are alike. E.g. saying that for all $m \in Obs$ and $p \in {}^nF$, the speed of light is the same in all spatial directions is an instance of the principle of isotropy. (It might be useful to notice that this is a kind of symmetry principle; for observer m, the "laws of physics" will not change if he rotates his coordinate system ${}^nF$ around his time-axis $\bar{t}$. We note, that we have not included this principle into our axiom system **Basax**.) We will use isotropy only in intuitive discussions, therefore we do not formalize it in our frame language.

$\triangleleft$

Definition 3.4.16
We define

$$\mathbf{Bax} \stackrel{\text{def}}{=} \Big(\mathbf{Newbasax} \setminus \{\mathbf{Ax5}, \mathbf{AxE_0}\}\Big) \cup \{\mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}}\},$$

where $\mathbf{Ax5^{Obs}}$, $\mathbf{Ax5^{Ph}}$, $\mathbf{AxE_{00}}$, $\mathbf{AxE_{01}}$ are defined above. Therefore

$$\mathbf{Bax} = \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}}\},$$

where for completeness we summarize below the axioms.

Ax1 $G = \mathsf{Eucl}(n, \mathbf{F})$.

Ax2 $Obs \cup Ph \subseteq Ib$.

$\mathbf{Ax3_0}$ $tr_m(h) \in G \cup \{\emptyset\} \ \wedge \ (\exists k) tr_k(h) \neq \emptyset$.

Ax4 $tr_m(m) = \bar{t}$.

Ax5$^{\mathbf{Obs}}$ $(\exists ph)(\forall \ell)\Big(m \overset{\odot}{\to} ph \ \wedge\ [ang^2(\ell) < v_m(ph) \ \Rightarrow\ (\exists k)\ell = tr_m(k)]\Big)$.

Ax5$^{\mathbf{Ph}}$ $ang^2(\ell) = v_m(ph) \ \Rightarrow\ (\exists ph)\ell = tr_m(ph)$.

Ax6$_{\mathbf{00}}$ $w_m[tr_m(k)] \subseteq Rng(w_k)$.

Ax6$_{\mathbf{01}}$ $Dom(\mathsf{f}_{mk}) \in Open$.

AxE$_{\mathbf{00}}$ $(m \overset{\odot}{\to} ph_1, ph_2) \ \Rightarrow\ v_m(ph_1) = v_m(ph_2)$.

AxE$_{\mathbf{01}}$ $v_m(ph) \neq 0$. $\lhd$

Remark 3.4.17 We will see later in Prop.4.3.6 that in **Bax** we can replace **AxE$_{\mathbf{00}}$**, saying that photons move with the same speed in each direction, with a weaker axiom **AxP1** saying that in each direction photons can move with only a unique speed (which speed may depend on the direction). This will lead us to investigating connections with well-studied directions in the literature.

$\lhd$

Definition 3.4.18 Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\})$. For every $m \in Obs$ we will define $c_m \in F \cup \{\infty\}$ as follows. By **Ax5$^{\mathbf{Obs}}$**, $m \overset{\odot}{\to} ph$, for some $ph \in Ph$. Let this ph be fixed. We define

$$c_m \overset{\text{def}}{=} v_m(ph)\,.$$

The definition of c_m is unambiguous by **AxE$_{\mathbf{00}}$**. Intuitively, c_m is the speed of light for observer m. **AxE$_{\mathbf{01}}$** then implies that $c_m \neq 0$.

$\lhd$

Recall (from Thm.3.3.12 in §3.3) that the models of **Newbasax** are, roughly speaking, unions of models of **Basax**. Now, the models of **Bax** are like those of **Newbasax**, except that the speed of light can vary from world-view to world-view in the model. **Newbasax** is equivalent then to (**Bax**$\cup\{c_m = 1\}$). **Bax** is consistent, and is still weaker than **Newbasax**, i.e. **Bax** $\not\models$ **Newbasax**. (A model for **Bax** which is not a model for **Newbasax** is given in Madarász [172].)

Next we state an FTL-type theorem, i.e. that **Bax** implies that there is no FTL observer. Clearly **Basax** $\models$ **Newbasax** $\models$ **Bax**. Therefore Thm.3.4.19(i) below implies Theorems 3.4.1(i), 3.4.2.

THEOREM 3.4.19 *Assume $n \geq 3$. Then (i) and (ii) below hold.*

(i) **Bax** $\models (\forall m, k \in Obs)(m \overset{\odot}{\to} k \;\Rightarrow\; v_m(k) < c_m)$.

(ii) $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\} \models (\forall m, k \in Obs)\Big(m \overset{\odot}{\to} k \;\Rightarrow\; (v_m(k) \leq c_m \land$
$(c_m \neq 0 \;\Rightarrow\; v_m(k) < c_m))\Big)$.

On the **proof**: We will give the proof for $n = 3$ in §3.4.3 on p.233, and for $n = 4$ in §3.4.3 on p.242.

For $n = 3$ the proof will be given in the following way. To every model $\mathfrak{M}$ of **Bax** a model $\mathfrak{N}$ of **Newbasax** will be associated, roughly speaking in such a way that $v_m(k) < c_m$ holds in $\mathfrak{M}$ iff $v_m(k) < 1$ holds in $\mathfrak{N}$. Then using Thm.3.4.2, which says that **Newbasax** does not allow FTL observers, we will conclude that **Bax** does not allow FTL observers. The proof for $n = 4$ will be carried out analogously to the proof of Thm.3.4.1(i), recall that the latter theorem says that **Basax** does not allow FTL observers. To do this we will have to formulate and prove analogous counterparts of theorems of §3.1 and §3.3 for **Bax**.

It remains an open question whether the proof for $n = 3$ can be generalized to a proof for $n = 4$.

The theorem above stating that our no FTL theorem is still provable in the weak system **Bax** belongs to exploring the limits of applicability of the no FTL theorem. (At the same time we could interpret the <u>*same*</u> quest as trying to answer the "<u>*why*</u> type question" about this theorem.) In a certain direction we will explore this question in the next item ("The axiom system **Relphax** ..."). However a more direct exploration of this question is to weaken **Bax** even further (and checking whether our no FTL theorem can be proved). This will be done on pp.491–501. Cf. also Items 4.4.14, 4.4.14, 4.4.15.

The axiom system Relphax (being a photon is observer-dependent)[230]

In what follows we will introduce a new, refined version **Relphax** of **Basax**. We will again fine-tune **AxE**, but in a different way as we did in the case of **Bax**. In the new axiom system **Relphax**, being a photon will be relative, and it will depend on the observer.

To refine (or weaken) **AxE**, we will introduce a new variant L^+ for our language of relativity theory.

Definition 3.4.20 The new language L^+ for relativity theory is the same as the old one except that Ph is no more a unary relation. Ph is a binary relation of sort $\langle \mathrm{B}, \mathrm{B} \rangle$.

More precisely B, Q and G are the same sorts as in the old definition of our language, i.e. Def.2.1.1 of §2.1, and $\mathfrak{M}$ is a *model of language L^+* iff

$$\mathfrak{M} = \langle B, F, G;\ Obs, Ph, Ib, +, \cdot, \leq,\ \mathrm{E}\ , W \rangle\ , \text{ where}$$

B, $\langle F, +, \cdot, \leq \rangle$, G, Obs, Ib, E , and W are as in Def.2.1.1 of §2.1, and

- Ph is a binary relation of sort $\langle \mathrm{B}, \mathrm{B} \rangle$.

Now, similarly as in Def.2.1.1 a model $\mathfrak{M}$ of L^+ is a *frame model of L^+* iff

$$\mathfrak{M} \models \mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,h) \to Obs(m)\}, \text{ where}$$

$\mathbf{Ax}_{\mathrm{OF}}$ and $\mathbf{Ax}_{\mathrm{G}}$ were defined in Def.2.1.1. Now $\models^{\mathrm{OFG}}$ and $\mathsf{Mod}_{\mathrm{OFG}}(\Sigma)$ are defined as at the end of Def.2.1.1; and (according to Def.2.1.1) for brevity we will write $\models$ and $\mathsf{Mod}(\Sigma)$ for $\models^{\mathrm{OFG}}$ and $\mathsf{Mod}_{\mathrm{OFG}}(\Sigma)$, respectively.

Let $\mathfrak{M}$ be a frame model of L^+. Let $m \in Obs$. Then we define

$$Ph_m \stackrel{\text{def}}{=} \{b \in B\ :\ Ph(m,b)\}\,.$$

Intuitively, Ph_m is the set of those bodies which appear as photons for m (actually for m, they are photons). $\lhd$

Now $\mathbf{AxE_1}$ and $\mathbf{AxE_2}$ below constitute a weaker version of **AxE**. We will change **Ax5** and **Ax2** to $\mathbf{Ax5_1}$ and $\mathbf{Ax2_1}$, respectively, only to fit our axioms to the new language L^+.

Below we postulate axioms $\mathbf{AxE_1}$, $\mathbf{AxE_2}$, $\mathbf{Ax2_1}$, $\mathbf{Ax5_1}$.

[230] For related investigations concerning the case of $n = 2$ we refer to Dávid [71], [69].

AxE$_1$ $(\forall m \in Obs)(\forall b \in Ph_m)\ v_m(b) = 1.$

Intuitively, for each observer m the speed of light is 1.

AxE$_2$ $(\forall m, k \in Obs)\Big(v_m(k) < 1 \ \Rightarrow\ Ph_m = Ph_k\Big)\,.$

Intuitively, if observer m sees observer k moving slower than the speed of light, then m and k perceive exactly the same bodies as photons.

Ax2$_1$ $Obs \subseteq Ib \ \ \land\ (\forall m \in Obs)\ Ph_m \subseteq Ib\,.$

Ax5$_1$ $$(\forall m \in Obs)(\forall \ell \in G)\Big((ang^2(\ell) < 1 \ \Rightarrow\ (\exists k \in Obs)\ \ell = tr_m(k)) \land (ang^2(\ell) = 1 \ \Rightarrow\ (\exists ph \in Ph_m)\ \ell = tr_m(ph))\Big)\,.$$

Definition 3.4.21 We define

$$\mathbf{Relphax} \stackrel{\text{def}}{=} \Big(\mathbf{Basax} \setminus \{\mathbf{Ax2}, \mathbf{Ax5}, \mathbf{AxE}\}\Big) \cup \{\mathbf{Ax2_1}, \mathbf{Ax5_1}, \mathbf{AxE_1}, \mathbf{AxE_2}\}\,.$$

$\lhd$

The following theorem says that in **Relphax** FTL observers are consistently possible.

THEOREM 3.4.22 *Assume $n \geq 2$. Then (i) and (ii) below hold.*

(i) **Relphax** $\not\models (\forall m, k \in Obs)\ v_m(k) < 1.$

(ii) *Assume $\mathfrak{F}$ is Euclidean or* $\mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax}) \neq \emptyset$. *Then there is* $\mathfrak{M} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Relphax})$ *such that*

$$\mathfrak{M} \models (\exists m, k \in Obs)\ v_m(k) > 1\,.$$

On the proof: The main idea of the proof is illustrated in Figures 71 and 72. Namely, these figures represent possible models of **Relphax**(3) in which FTL observers exist. The key idea is the following. Take two **Basax** models, call them $\mathfrak{M}$ and $\mathfrak{N}$. Now turn $\mathfrak{N}$ around such that its time axis will coincide with the $\bar{x}$ axis of the first model $\mathfrak{M}$. Next, try to glue the two models together in this position, see Figure 71. Figure 71 represents the world-view of an observer m_0 coming from the first model $\mathfrak{M}$. The observers coming from $\mathfrak{M}$ live inside the vertical light-cone, while the observers coming from $\mathfrak{N}$ live inside the horizontal light-cone. The rest of the details of the construction can be found in [175, 172] (available from J. Madarász), we do not include them here. In Figure 72 we glue together three **Basax** models. ∎

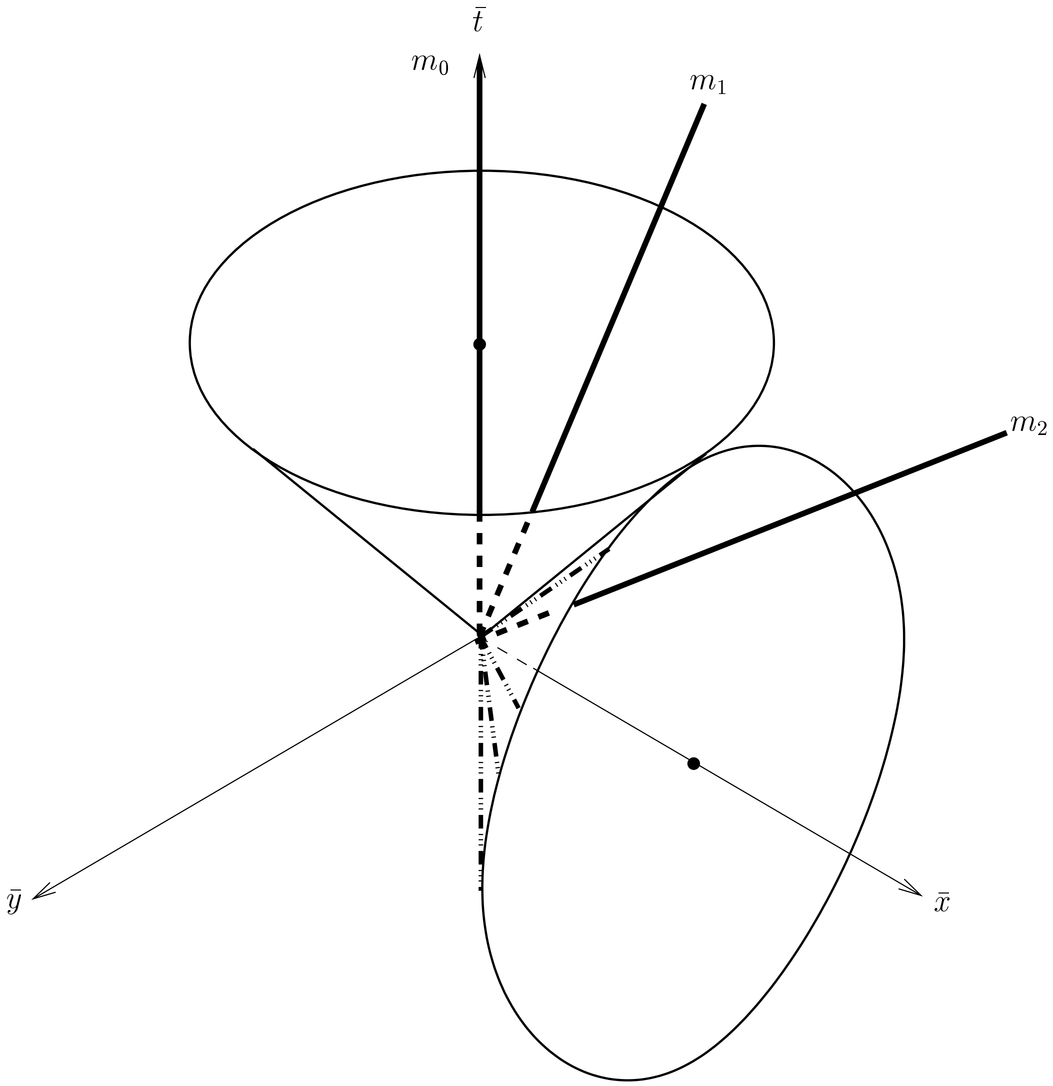

Figure 71: The lines like "$- \cdots - \cdots -$" are tachyons for m_0 and m_1 but are photons for m_2. Observer m_2 moves FTL relative to m_0 and m_1.

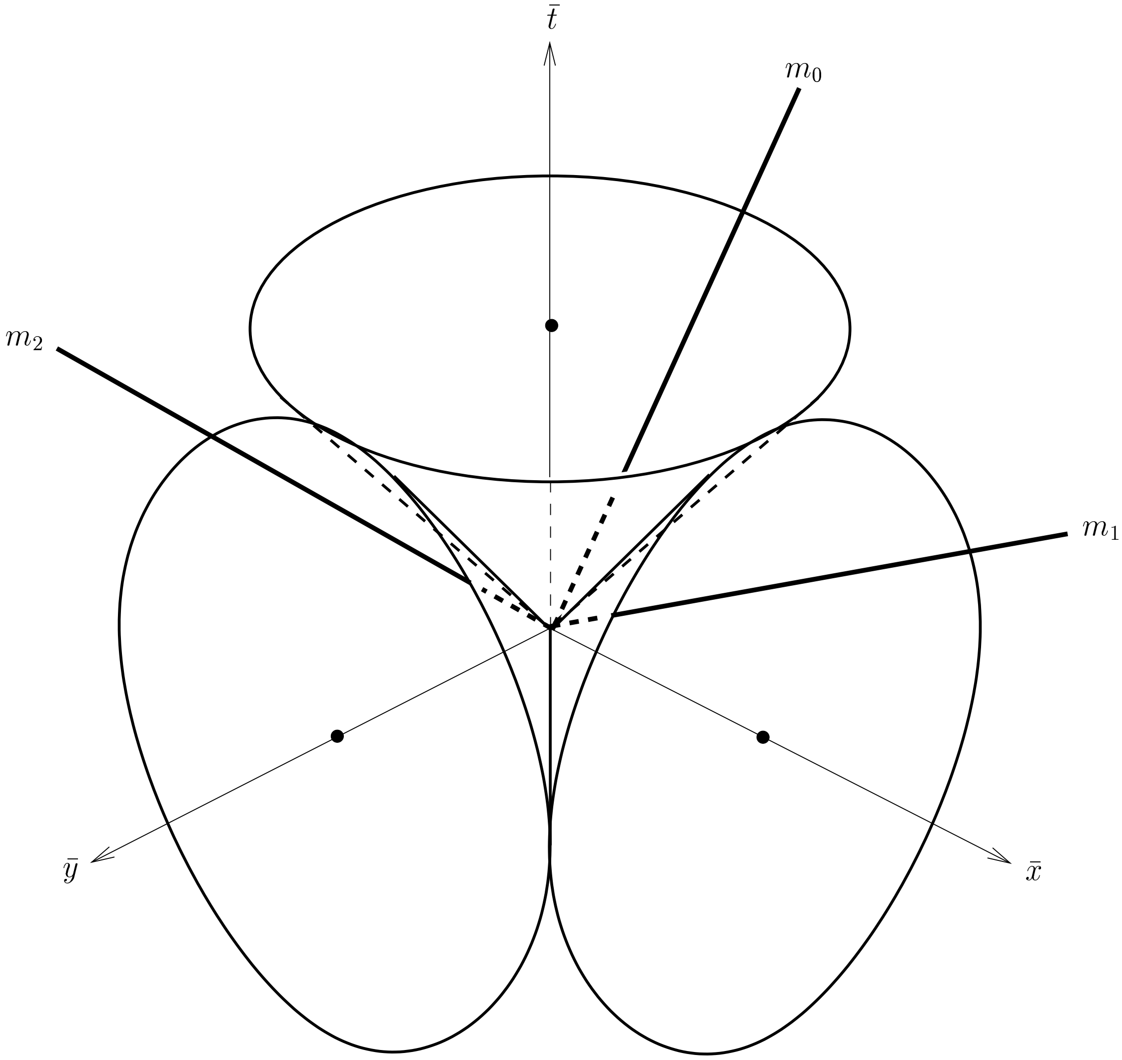

Figure 72: A **Relphax** model with an FTL observer going in the $\bar{x}$ direction and *another* FTL observer in the $\bar{y}$ direction. These (together with a "time-like" observer) are "responsible" for the 3 different light-cones.

We suggest that the reader compare Theorems 3.4.1, 3.4.2, 3.4.19, and Theorem 3.4.22.

We note that one can consider **Relphax** to be a weakened version of **Basax**, namely **Basax** $\dashv\models$ (**Relphax** $\cup \{Ph_m = Ph_k\}$).

The reader is invited to compare **Relphax**(3), **Relphax**(2) and **Basax**(2). Both in **Relphax**(n) and **Basax**(2) FTL observers *are* possible. So, there are some things which **Relphax**(n) and **Basax**(2) have in common. Further, **Basax**(2) and **Relphax**(2) are essentially equivalent, though formally they are not quite equivalent. It would be nice to think over to what extent (and in what sense) **Relphax**(2) and **Basax**(2) are close to each other.

Investigations of FTL phenomena in **Basax**(2) were done in §2.7 as well as in the 1998 early March version of the present work. Cf. also Dávid [71], [69].

A brief return to Bax etc.

Now, we turn to the possibility of improving the "no FTL observers" theorem. Let us discuss briefly how much of our assumption **Basax** is needed for proving the nonexistence of FTL observers, in Thm.3.4.1(i). We will concentrate on the question of how much of **Ax5** was needed.

In the next conjecture we will use the notion of the angle between two lines ℓ, ℓ_1. Though we did not define this, we hope the reader will understand what we mean. Since we will need this notion only in two conjectures, we do not define it.

Conjecture 3.4.23 *Replace the photon part of* **Ax5** *by the following weaker axiom.*

(*) $(\forall \varepsilon \in {}^{+}F)(\forall \ell \in \mathsf{PhtEucl})(\forall p \in {}^{n}F)(\forall m \in Obs)(\exists ph \in Ph)$
$\Big($*the angle between* ℓ *and* $tr_m(ph)$ *is smaller than* ε*, and* $p \in tr_m(ph)\Big)$.

Intuitively, (*) together with **AxE** *means that the traces of photons passing through* p *as seen by any observer are "dense" on* $\mathsf{LightCone}(p)$*, for every* p.

Let **Basax**$_0$ *be obtained from* **Basax** *by replacing the photon part of* **Ax5** *with (*) above. Then we conjecture that for* $n \geq 3$

$$\mathbf{Basax}_0(n) \models \textit{ there are no FTL observers.}$$

$\lhd$

The axiom system **Bax** is introduced in Def.3.4.16.

Conjecture 3.4.24 *We conjecture that Conjecture 3.4.23 remains true if we replace* **Basax** *with* **Bax** *in it. I.e. we obtain* $\mathbf{Bax}_0$ *by replacing* $\mathbf{Ax5^{Ph}}$ *with*
$(\forall\varepsilon \in {}^{+}F)(\forall \ell \in \mathsf{Eucl})(\forall p \in {}^{n}F)(\forall m \in Obs)(\exists ph \in Ph)\Big(ang^2(\ell) = c_m \Rightarrow$
$($*the angle between* ℓ *and* $tr_m(ph)$ *is smaller than* ε*, and* $p \in tr_m(ph))\Big)$.
Then we conjecture that for $n \geq 3$

$$\mathbf{Bax}_0 \models \textit{there are no FTL observers.}$$

◁

Remark 3.4.25 The Kennedy-Thorndike experiment (cf. Taylor-Wheeler [256, pp.86–88]) seems to suggest that the reality (of special relativity) may be closer to **Newbasax** than to **Bax**. All the same, we have motivation for studying **Bax** (and even weaker systems) summarized in (i)–(v) below:

(i) Conceptual analysis (like the one in Friedman [90], cf. also the introduction of this work).

(ii) We do not know what future experiments will say.

(iii) Applicability of the *"theory"* of **Bax** to accelerated observers and other general relativity situations where the flexibility of **Bax** renders it more applicable than **Newbasax**. (E.g. certain structures are "'locally **Bax**" but not "locally **Newbasax**".)

To formulate items (iv) and (v) below we will use Friedman's principles (P1), (P2), (P3) concerning the speed of light which will be recalled in detail in Remark 4.3.40 way below (p.522). The interested reader is suggested to look them up there. Here we only say that (P1–P3) can be considered as "fragments" of Einstein's speed of light axiom **AxE**.

(iv) Friedman [90] p.159 line 6 bottom up – p.160 line 2, writes that Maxwell's (classical) electrodynamics predicts **Bax** but not **Newbasax** (or using Friedman's terminology it predicts (P1 + something) but not (P2)).[231]

[231] This is relevant because of the following. Consider the "dynamics of theories" as outlined e.g. in Andréka-Gergely-Németi-Sain [12], [11], [203] (of which a more accessible continuation is Jánossy et al. [143]). In this "paradigm" one can reconstruct the development of special relativity the following way. We take two pre-relativistic theories, (1) Newton's mechanics and (2) Maxwell's electrodynamics. Then we form the so called amalgamated coproduct of the two theories where the basis of amalgamation is the postulate saying that what is an electromagnetic wave in theory (2) is

(v) A further motivation for looking at **Bax** is that Friedman suggests to study the logical connections between (P1) and (P3). To do this in a logical framework, we look at a logical counterpart called[232] **Bax**$^-$ of (P1) [as opposed to the counterpart **Newbasax** of (P2)] and study the (P3)-style aspects (or properties) of this theory e.g. in the form of the "no FTL observers" theorems of this sub-section.

$\lhd$

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a particular case of inertial body in theory (1). For completeness, we note that the mathematical mechanism (in algebraic form) of forming such amalgamations is studied in Madarász [170], cf. also Andréka-Németi-Sain [28]. If we amalgamate these two pre-relativistic theories (1) and (2), we will arrive at a theory say Th_3 which turns out to be inconsistent. Then we use the usual methodology of logic to weaken the axioms little-by-little until the so obtained version Th_3^- of Th_3 becomes consistent. Then this Th_3^- is called a possible version of special relativity.

Later we repeat this act of amalgamating theories by putting together special relativity and Newton's theory of gravitation. Again the amalgamated theory will be inconsistent, and then again we can try to apply the methods of logic to modify the axioms until it becomes consistent.

[232]In §4 we will refine **Bax** to several weaker, more flexible subsystems. The first of these will be the theory **Bax**$^-$ developed in §4.3.

3.4.3 Proof that Bax does not allow FTL observers

Németi olvassa el!

In this sub-section we will prove Thm.3.4.19 (p.221) which says that **Bax** does not allow FTL observers. In this work we try to make our proofs "structuralist" ones as opposed to "computational". The idea is that a well designed structuralist proof should provide "instant" insight, i.e. it should make the reader see in his minds eye the essence of the proof. The mathematical logic framework adopted for the present approach should help us in achieving this aim (if we work hard enough on it). Two of these examples are §3.2 ("Intuitive ...") and most of the proofs in §3.1 (where we guess we were not very far from achieving this aim). In most parts of this work we have not yet completely succeeded in reaching this aim, but in later versions we hope we will get closer and closer to realizing it. The proof in the present sub-section is an example of exceptional cases when we did not try to achieve the above outlined structuralist aim. I.e. the proof below (especially that of Lemma 3.4.42) is a computational one, we include it only for completeness and we plan to replace it with a more structuralist proof in a later version. The reader who would like to avoid computational proofs may safely skip the proofs of the lemmas in the present sub-section.

In order to prove Thm.3.4.19 for $n = 3$, we need Lemmas 3.4.26, 3.4.27, 3.4.28 below.

LEMMA 3.4.26 *Assume $n \geq 3$. Then (i)-(iv) below hold.*

(i) $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\} \models \Big(Rng(w_m) \cap Rng(w_k) \neq \emptyset \ \Rightarrow \ (c_m = \infty \Leftrightarrow c_k = \infty)\Big).$

(ii) $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\} \models \Big(m \overset{\odot}{\to} k \ \Rightarrow \ (c_m = \infty \Leftrightarrow c_k = \infty)\Big).$

(iii) $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\} \models \Big(Rng(w_m) \cap Rng(w_k) \neq \emptyset \ \Rightarrow \ (c_m = 0 \Leftrightarrow c_k = 0)\Big).$

(iv) $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\} \models \Big(m \overset{\odot}{\to} k \ \Rightarrow \ (c_m = 0 \Leftrightarrow c_k = 0)\Big).$

Proof:
<u>*Proof of (i):*</u> Throughout the proof the reader is asked to consult Figure 73.

Let $\mathfrak{M}$ be a frame model of $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$. Let $m, k \in Obs$ with

$$Rng(w_m) \cap Rng(w_k) \neq \emptyset \quad \text{and} \quad c_m = \infty.$$

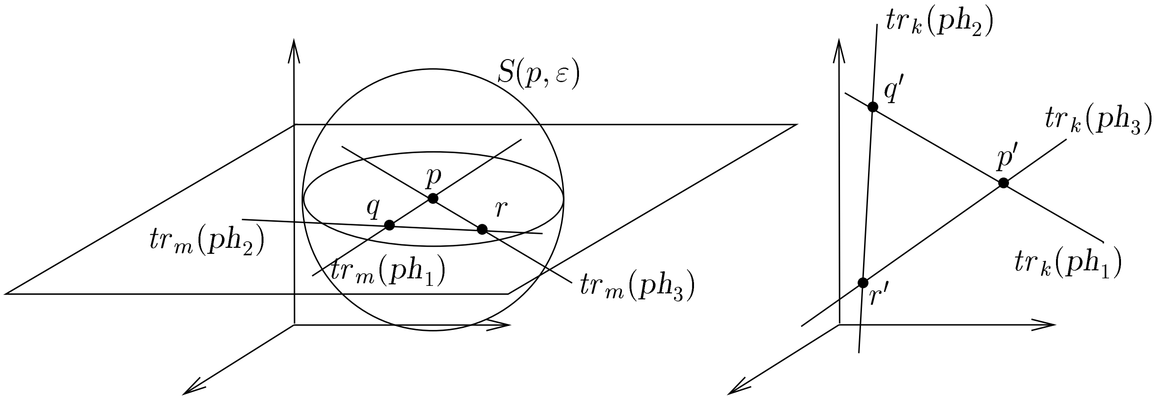


Figure 73: Illustration for the proof of Lemma 3.4.26(i).

We will prove that $c_k = \infty$.

Intuitive idea of the proof: We will see that there is a neighborhood $S(p,\varepsilon) \subseteq {}^nF$ such that k "sees" all those events which m sees in $S(p,\varepsilon)$. By $c_m = \infty$, observer m sees three photons such that they "form a triangle" inside $S(p,\varepsilon)$. See Figure 73. Since k sees all events which m sees in $S(p,\varepsilon)$, those three photons form a triangle in the world-view of k, too. This can only happen if $c_k = \infty$, and this will complete the proof.

Formally: Let $p \in Dom(\mathsf{f}_{mk})$. Such a p exists by $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. Now by $\mathbf{Ax6_{01}}$ there is $\varepsilon \in {}^+F$ such that $S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$. Let such an ε be fixed. Let $q, r \in S(p,\varepsilon)$ such that p, q, r are non-collinear and $ang^2(\overline{pq}) = ang^2(\overline{qr}) = ang^2(\overline{pr}) = \infty$. Such q, r exist by $n \geq 3$. By $\mathbf{Ax5^{Ph}}$ and $c_m = \infty$, there are $ph_1, ph_2, ph_3 \in Ph$ such that $tr_m(ph_1) = \overline{pq}$, $tr_m(ph_2) = \overline{qr}$, $tr_m(ph_3) = \overline{pr}$. Let such ph_1, ph_2, ph_3 be fixed. We have

$$(47) \qquad \begin{aligned} &ph_1 \in w_m(p) \cap w_m(q), \qquad ph_1 \notin w_m(r),\\ &ph_2 \in w_m(q) \cap w_m(r), \qquad ph_2 \notin w_m(p),\\ &ph_3 \in w_m(p) \cap w_m(r), \qquad ph_3 \notin w_m(q). \end{aligned}$$

By $p, q, r \in S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$, there are $p', q', r' \in {}^nF$ such that

$$(48) \qquad w_m(p) = w_k(p'),\ \ w_m(q) = w_k(q') \ \text{ and } \ w_m(r) = w_k(r').$$

By (47), (48) and $\mathbf{Ax3_0}$, we have that p', q', r' are non-collinear and $tr_k(ph_1) = \overline{p'q'}$, $tr_k(ph_2) = \overline{q'r'}$ and $tr_k(ph_3) = \overline{p'r'}$. By this and by $\mathbf{AxE_{00}}$, we have $v_k(ph_1) = v_k(ph_2) = v_k(ph_3) = \infty$. Hence $c_k = \infty$.

Proof of (ii): Item (ii) follows by item (i) because

$$\mathbf{Ax6_{00}} \models (m \xrightarrow{\odot} k \ \Rightarrow Rng(w_m) \cap Rng(w_k) \neq \emptyset).$$

Proof of (iii): Throughout the proof the reader is asked to consult Figure 74.

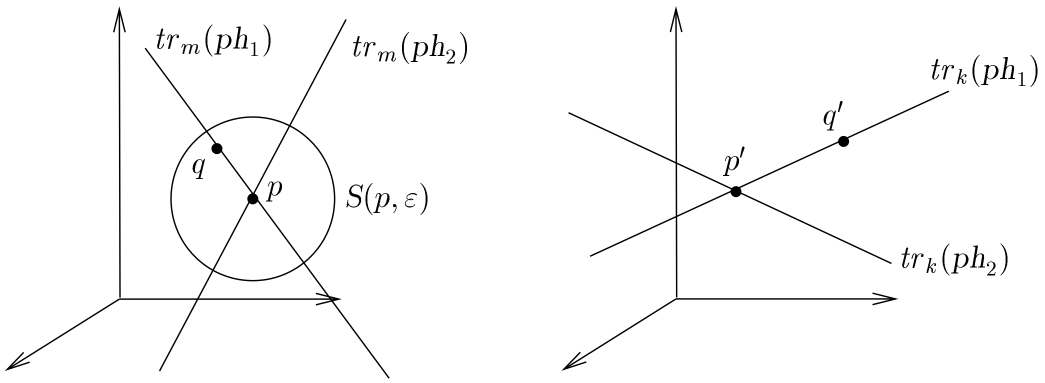


Figure 74: Illustration for the proof of Lemma 3.4.26(iii).

Let $\mathfrak{M}$ be a frame model of $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$. Let $m, k \in Obs$ with

$$Rng(w_m) \cap Rng(w_k) \neq \emptyset \ \text{ and } \ c_m \neq 0.$$

We will prove that $c_k \neq 0$.

Intuitive idea of the proof: We will see that there is a neighborhood $S(p,\varepsilon) \subseteq {}^nF$ such that k "sees" all those events which m sees in $S(p,\varepsilon)$. Now by $c_m \neq 0$, m sees two photons intersecting each other in one point which is in $S(p,\varepsilon)$. See Figure 74. But then k sees these two photons intersecting each other in one point. See Figure 74. But this implies $c_k \neq 0$, and this will complete the proof.

Formally: Let $ph_1, ph_2 \in Ph$ such that $tr_m(ph_1) \neq tr_m(ph_2)$ and $p \in tr_m(ph_1) \cap tr_m(ph_2)$. Such ph_1, ph_2 exist because of the following. By $\mathbf{Ax5^{Obs}}$, we have $m \xrightarrow{\odot} ph$, for some $ph \in Ph$. Let such a ph be fixed. Let A be the linear transformation which takes $1_t, 1_x, 1_y, 1_3, \dots, 1_{n-1}$ to $1_t, -1_y, 1_x, 1_3, \dots, 1_{n-1}$, respectively (for $n = 3$ A is the rotation around $\bar{t}$ axis with 90 degrees). Now let $\ell_1, \ell_2 \in \mathsf{Eucl}$ such that $p \in \ell_1 \cap \ell_2$, $\ell_1 \parallel tr_m(ph)$, $\ell_2 \parallel A[tr_m(ph)]$. Obviously $\ell_1 \neq \ell_2$ and $ang^2(\ell_1) = ang^2(\ell_2) = v_m(ph)$. Hence by $\mathbf{Ax5^{Ph}}$, there are $ph_1, ph_2 \in Ph$ such that $tr_m(ph_1) = \ell_1$ and $tr_m(ph_2) = \ell_2$. For such ph_1 and ph_2, $tr_m(ph_1) \neq tr_m(ph_2)$ and $p \in tr_m(ph_1) \cap tr_m(ph_2)$ hold.

$p \in Dom(\mathsf{f}_{mk})$ and $\mathbf{Ax6_{01}}$ imply that $S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$, for some $\varepsilon \in {}^{+}F$. Let such an ε be fixed. Let $q \in S(p,\varepsilon)$ such that

$$ph_1 \in w_m(q) \ \text{ and } \ ph_2 \notin w_m(q). \tag{49}$$

By $p, q \in S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$ there are $p', q' \in {}^{n}F$ such that

$$w_m(p) = w_k(p') \ \text{ and } \ w_m(q) = w_k(q'). \tag{50}$$

$p \in tr_m(ph_1) \cap tr_m(ph_2)$, (49) and (50) imply that $p' \in tr_k(ph_1) \cap tr_k(ph_2)$ and $tr_k(ph_1) \neq tr_k(ph_2)$. This means that observer k "sees" two photons whose traces are different and contain point p'. But $tr_k(ph_1), tr_k(ph_2) \in \mathsf{Eucl}$ by $\mathbf{Ax1}$, $\mathbf{Ax2}$, $\mathbf{Ax3_0}$. By this, we conclude that $c_k \neq 0$ because there is exactly one $\ell \in \mathsf{Eucl}$ such that $p' \in \ell$ and $ang^2(\ell) = 0$.

Proof of (iv): Item (iv) follows from item (iii) because

$$\mathbf{Ax6_{00}} \models (m \overset{\odot}{\rightarrow} k \ \Rightarrow \ Rng(w_m) \cap Rng(w_k) \neq \emptyset). \blacksquare$$

Prop.3.4.27 below is an analogue of Prop.2.3.3(iv) (§2.3).

PROPOSITION 3.4.27

(i) $\mathbf{Bax} \setminus \mathbf{AxE_{01}} \models (\forall m \in Obs)\Big(c_m \neq 0 \ \Rightarrow \ (w_m \ \textit{is an injection})\Big).$

(ii) $\mathbf{Bax} \qquad \models (\forall m \in Obs)(w_m \ \textit{is an injection}).$

Proof: It is enough to prove item (i) because item (ii) follows from item (i). Let $\mathfrak{M}$ be a frame model of $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$. Let $m \in Obs$ with $c_m \neq 0$. Let $p, q \in {}^{n}F$ with $p \neq q$. We will prove that $w_m(p) \neq w_m(q)$. By $c_m \neq 0$, there is $\ell \in \mathsf{Eucl}$ such that $ang^2(\ell) < c_m$, $p \in \ell$ and $q \notin \ell$. $ang^2(\ell) < c_m$, $\mathbf{Ax5^{Obs}}$ and $\mathbf{AxE_{00}}$ imply that there is $k \in Obs$ with $tr_m(k) = \ell$. For such a k, $k \in w_m(p)$ and $k \notin w_m(q)$. Thus $p \neq q$. $\blacksquare$

Lemma 3.4.28 below is an analogue of Lemma 3.3.15 (§3.3).

LEMMA 3.4.28

(i) $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\} \models (\forall m, k \in Obs)\Big((m \overset{\odot}{\rightarrow} k \ \wedge \ c_m \neq 0) \ \Rightarrow \ v_m(k) \neq c_m\Big).$

(ii) $\mathbf{Bax} \qquad \models (\forall m, k \in Obs)(m \overset{\odot}{\rightarrow} k \ \Rightarrow \ v_m(k) \neq c_m).$

Proof: It is enough to prove item (i) because item (ii) follows from item (i). The proof goes by contradiction. Let $\mathfrak{M}$ be a frame model of $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$. Let $m, k \in Obs$ with $m \xrightarrow{\odot} k$, $c_m \neq 0$ and $v_m(k) = c_m$. By $\mathbf{Ax5^{Ph}}$ and $v_m(k) = c_m$, there is $ph \in Ph$ such that $tr_m(ph) = tr_m(k)$. Let such a ph be fixed. Let $p, q \in tr_m(k)$ such that $p \neq q$. Then $w_m(p) \neq w_m(q)$ by Prop.3.4.27(i). Now by $\mathbf{Ax6_{00}}$ we have that

(51) $$w_m(p) = w_k(p') \text{ and } w_m(q) = w_k(q'),$$

for some $p', q' \in {}^nF$. Let such p', q' be fixed. By $w_m(p) \neq w_m(q)$ and (51), we have $p' \neq q'$. Further $ph, k \in w_k(p') \cap w_k(q')$ because $ph, k \in w_m(p) \cap w_m(q)$. By $k \in w_k(p') \cap w_k(q')$ and $\mathbf{Ax4}$, we have $\overline{p'q'} = \bar{t}$. Now $\overline{p'q'} = \bar{t}$, $ph \in w_k(p') \cap w_k(q')$ and $\mathbf{Ax3_0}$ imply that $tr_k(ph) = \bar{t}$. By this and $\mathbf{AxE_{00}}$, we have that $c_k = 0$. But by Lemma 3.4.26(iv), it follows that $c_k \neq 0$ because $c_m \neq 0$ and $m \xrightarrow{\odot} k$. ∎

Proof of Thm.3.4.19 for n = 3: It is enough to prove item (ii) because item (i) follows from item (ii).

Assume $n = 3$. Let

$$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G; \in, W \rangle \models \mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}.$$

Let $m_0, m_1 \in Obs$ with $m_0 \xrightarrow{\odot} m_1$. We have to prove the following. If $c_{m_0} \neq 0$ then $v_{m_0}(m_1) < c_{m_0}$, and if $c_{m_0} = 0$ then $v_{m_0}(m_1) = 0$.

<u>*Case 1: $c_{m_0} \neq 0$ and $c_{m_0} \neq \infty$.*</u> Throughout the proof for Case 1 the reader is asked to consult Figure 75.

<u>*Intuitive idea of the proof:*</u> For the beginning of the intuitive idea of the proof the reader is referred to the formulation of Thm.3.4.19 in §3.4.2. Recall from there that from the model $\mathfrak{M}$ above we want to construct another model $\mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax})$ such that certain connections between $\mathfrak{M}$ and $\mathfrak{N}$ hold. In particular we will have to check that an observer say m is FTL in $\mathfrak{N}$ iff it is FTL in $\mathfrak{M}$. We will construct $\mathfrak{M}$ in the following way. For each observer m we will change the world-view $w_m^{\mathfrak{M}}$ of m in such a way that the speed of light (for m) becomes 1. See Figure 75. This change will be implemented by using a linear transformation A_m. We note that A_m will leave the time-axis $\bar{t}$ point-wise fixed, will take the vectors $1_x, 1_y$ to two orthogonal vectors $(\langle 0, x, y \rangle, \langle 0, -y, x \rangle)$ of the same length.

<u>*Formally:*</u> Assume $c_{m_0} \neq 0$ and $c_{m_0} \neq \infty$. Then we define a frame model

$$\mathfrak{N} = \langle (B, Obs^{\mathfrak{N}}, Ph^{\mathfrak{N}}, Ib^{\mathfrak{N}}), \mathfrak{F}, G; \in, W^{\mathfrak{N}} \rangle$$

as follows.

$$Obs^{\mathfrak{N}} \stackrel{\text{def}}{=} \{m \in Obs : c_m \neq 0 \;\&\; c_m \neq \infty\},$$

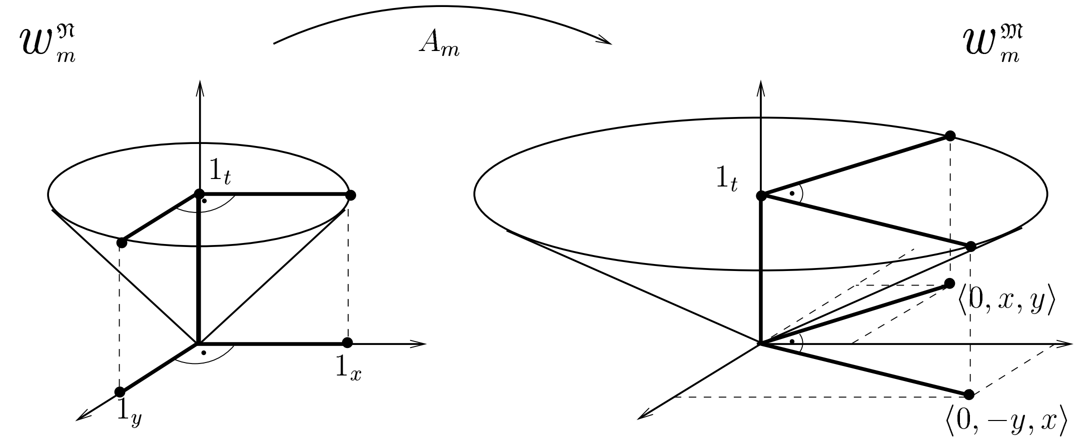


Figure 75: Illustration for the proof of Thm.3.4.19 for $n = 3$.

$$Ph^{\mathfrak{N}} \stackrel{\text{def}}{=} \{ph \in Ph \;:\; (\exists m \in Obs^{\mathfrak{N}})(m \stackrel{\odot}{\to} ph \text{ holds in } \mathfrak{M})\},$$
$$Ib^{\mathfrak{N}} \stackrel{\text{def}}{=} \{b \in Ib \;:\; (\exists m \in Obs^{\mathfrak{N}})(m \stackrel{\odot}{\to} b \text{ holds in } \mathfrak{M})\}.$$

Now we are going to define $W^{\mathfrak{N}}$. For every $m \in Obs^{\mathfrak{N}}$ first we will define a linear transformation A_m of ${}^3\mathbf{F}$ and then we will define $w_m^{\mathfrak{N}}$ as follows. Let $m \in Obs$. By $\mathbf{Ax5^{Obs}}$ and $\mathbf{Ax5^{Ph}}$, $m \stackrel{\odot}{\to} ph$ for some $ph \in Ph$ with $\bar{0} \in tr_m(ph)$. Let such a ph be fixed. By $v_m(ph) = c_m \neq \infty$ there is $\langle 1, x, y\rangle \in tr_m(ph)$. Let this $\langle 1, x, y\rangle$ be fixed. Let A_m be the linear transformation of ${}^3\mathbf{F}$ which takes $1_t, 1_x, 1_y$ to $1_t, \langle 0, x, y\rangle, \langle 0, -y, x\rangle$, respectively. By $c_m \neq 0$, we have that A_m is bijective. Let

$$w_m^{\mathfrak{N}} \stackrel{\text{def}}{=} A_m \circ w_m.$$

Now

$$W^{\mathfrak{N}} \stackrel{\text{def}}{=} \{\langle m, p, b\rangle \;:\; m \in Obs^{\mathfrak{N}} \;\&\; p \in {}^3F \;\&\; b \in w_m^{\mathfrak{N}}(p)\}.$$

By the above $\mathfrak{N}$ is defined.

We will prove that (I)–(III) below hold.

(I) $(\forall m \in Obs^{\mathfrak{N}})(\forall b \in B)\left(tr_m^{\mathfrak{M}}(b) = A_m\left[tr_m^{\mathfrak{N}}(b)\right] \;\wedge\; tr_m^{\mathfrak{N}}(b) = A_m^{-1}\left[tr_m^{\mathfrak{M}}(b)\right]\right).$

(II) $(\forall m \in Obs^{\mathfrak{N}})(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) = 1 \;\Leftrightarrow\; ang^2(A_m[\ell]) = c_m).$

(III) $(\forall m \in Obs^{\mathfrak{N}})(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < 1 \iff ang^2(A_m[\ell]) < c_m)$.

To prove (I) let $m \in Obs^{\mathfrak{N}}$ and $b \in B$. Then

$$\begin{aligned} tr_m^{\mathfrak{N}}(b) &= \{p \in {}^3F : b \in w_m^{\mathfrak{N}}(p)\} \\ &= \{p \in {}^3F : b \in A_m \circ w_m^{\mathfrak{M}}(p)\} \\ &= \{p \in {}^3F : b \in w_m^{\mathfrak{M}}(A_m(p))\} \\ &= A_m^{-1}\left[\{p \in {}^3F : b \in w_m^{\mathfrak{M}}(p)\}\right] \\ &= A_m^{-1}[tr_m^{\mathfrak{M}}(b)]. \end{aligned}$$

Hence $tr_m^{\mathfrak{N}}(b) = A_m^{-1}[tr_m^{\mathfrak{M}}(b)]$ and $tr_m^{\mathfrak{M}}(b) = A_m[tr_m^{\mathfrak{N}}(b)]$. To prove (II) and (III) let $m \in Obs^{\mathfrak{N}}$ and let $\ell \in \mathsf{Eucl}$. We will prove that

$$(ang^2(\ell) = 1 \iff ang^2(A_m[\ell]) = c_m) \text{ and } (ang^2(\ell) < 1 \iff ang^2(A_m[\ell]) < c_m).$$

Without loss of generality we can assume that $\bar{0} \in \ell$ and $ang^2(\ell) \neq \infty$ because A_m takes parallel lines to parallel lines and because $(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) = \infty \iff ang^2(A_m[\ell]) = \infty)$. By $\bar{0} \in \ell$ and $ang^2(\ell) \neq \infty$, we have $\ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle}$, for some $\lambda, \mu \in F$. Let this λ, μ be fixed. By the definition of A_m, A_m takes $1_t, 1_x, 1_y$ to $1_t, \langle 0, x, y\rangle, \langle 0, -y, x\rangle$, for some $x, y \in F$ with $\langle 1, x, y\rangle \in tr_m(ph)$, for some $ph \in Ph$ with $m \overset{\odot}{\to} ph$ and $\bar{0} \in tr_m(ph)$. Let this x, y and ph be fixed. By $\langle 1, x, y\rangle \in tr_m(ph)$, we have $x^2 + y^2 = c_m$. Now

$$\begin{array}{lll} & ang^2(A_m[\ell]) = c_m & \\ \iff & ang^2(\overline{\bar{0}, \langle 1, \lambda x + \mu y, \mu x - \lambda y\rangle}) = c_m & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle} \text{ and by the def. of } A_m) \\ \iff & (\lambda x + \mu y)^2 + (\mu x - \lambda y)^2 = c_m & (\text{by the def. of } ang^2) \\ \iff & (\lambda^2 + \mu^2)(x^2 + y^2) = c_m & (\text{by computation}) \\ \iff & \lambda^2 + \mu^2 = 1 & (\text{by } x^2 + y^2 = c_m) \\ \iff & ang^2(\ell) = 1 & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle}). \end{array}$$

(II) is proved. The proof of (III) is analogous, but for completeness we write down all the details.

$$\begin{array}{lll} & ang^2(A_m[\ell]) < c_m & \\ \iff & ang^2(\overline{\bar{0}, \langle 1, \lambda x + \mu y, \mu x - \lambda y\rangle}) < c_m & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle} \text{ and by the def. of } A_m) \\ \iff & (\lambda x + \mu y)^2 + (\mu x - \lambda y)^2 < c_m & (\text{by the def. of } ang^2) \\ \iff & (\lambda^2 + \mu^2)(x^2 + y^2) < c_m & (\text{by computation}) \\ \iff & \lambda^2 + \mu^2 < 1 & (\text{by } x^2 + y^2 = c_m) \\ \iff & ang^2(\ell) < 1 & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle}). \end{array}$$

(III) is proved.

Now we will prove that $\mathfrak{N} \models$ **Newbasax**.

$\mathfrak{N} \models$ **Ax1** by $\mathfrak{M} \models$ **Ax1**.

$\mathfrak{N} \models$ **Ax2** because of the following. Let $m \in Obs^{\mathfrak{N}}$. By $m \in Ib$ and by $m \overset{\odot}{\to} m$ holds in $\mathfrak{M}$, we have $m \in Ib^{\mathfrak{N}}$. Hence $Obs^{\mathfrak{N}} \subseteq Ib^{\mathfrak{N}}$. $Ph^{\mathfrak{N}} \subseteq Ib^{\mathfrak{N}}$ holds by $Ph \subseteq Ib$ and by the definitions of $Ph^{\mathfrak{N}}$, $Ib^{\mathfrak{N}}$. $\mathfrak{N} \models$ **Ax2** is proved.

$\mathfrak{N} \models$ **Ax3$_0$** because of the following. $(\forall b \in Ib^{\mathfrak{N}})(tr_m^{\mathfrak{N}}(b) \in G \cup \{\emptyset\})$ holds because of (I), $\mathfrak{M} \models$ **Ax3$_0$** and $Ib^{\mathfrak{N}} \subseteq Ib$. By (I), we have that

$$(\forall b \in Ib^{\mathfrak{N}})(\forall m \in Obs^{\mathfrak{N}})\Big((m \overset{\odot}{\to} b \text{ holds in } \mathfrak{M}) \Leftrightarrow (m \overset{\odot}{\to} b \text{ holds in } \mathfrak{N})\Big).$$

Hence we have

$$(\forall b \in Ib^{\mathfrak{N}})(\exists m \in Obs^{\mathfrak{N}})(m \overset{\odot}{\to} b \text{ holds in } \mathfrak{N})$$

by the definition of $Ib^{\mathfrak{N}}$. $\mathfrak{N} \models$ **Ax3$_0$** is proved.

$\mathfrak{N} \models$ **Ax4** because $\mathfrak{M} \models$ **Ax4**, because of (I) and because $(\forall m \in Obs^{\mathfrak{N}})\, A_m^{-1}[\bar{t}\,] = \bar{t}$.

$\mathfrak{N} \models$ **Ax5** because of the following. Let $m \in Obs^{\mathfrak{N}}$ and let $\ell_1, \ell_2 \in \mathsf{Eucl}$ with $ang^2(\ell_1) = 1$ and $ang^2(\ell_2) < 1$. We have to prove that there are $ph \in Ph^{\mathfrak{N}}$ and $k \in Obs^{\mathfrak{N}}$ such that $tr_m(ph) = \ell_1$ and $tr_m(k) = \ell_2$. $ang^2(A_m[\ell_1]) = c_m$ and $ang^2(A_m[\ell_2]) < c_m$ hold by (II) and (III). Thus by $\mathfrak{M} \models \{$**Ax5$^{\mathbf{Obs}}$**, **Ax5$^{\mathbf{Ph}}$**, **AxE$_{\mathbf{00}}$**$\}$, we have that

$$tr_m^{\mathfrak{M}}(ph) = A_m[\ell_1] \ \text{ and } \ tr_m^{\mathfrak{M}}(k) = A_m[\ell_2], \tag{52}$$

for some $ph \in Ph$ and $k \in Obs$. Let such ph and k be fixed. By $m \overset{\odot}{\to} ph$ holds in $\mathfrak{M}$, we have $ph \in Ph^{\mathfrak{N}}$. By $m \in Obs^{\mathfrak{N}}$, we have $c_m \neq 0$ and $c_m \neq \infty$. Hence by $m \overset{\odot}{\to} k$ and Lemma 3.4.26, we have that $c_k \neq 0$ and $c_k \neq \infty$. Therefore $k \in Obs^{\mathfrak{N}}$ by the definition of $Obs^{\mathfrak{N}}$. Now by (52) and (I), we get

$$tr_m^{\mathfrak{N}}(ph) = \ell_1 \ \text{ and } \ tr_m^{\mathfrak{N}}(k) = \ell_2.$$

Hence $\mathfrak{N} \models$ **Ax5**.

$\mathfrak{N} \models$ **Ax6$_{\mathbf{00}}$** because

$$(\forall m, k \in Obs^{\mathfrak{N}})\Big(w_m^{\mathfrak{N}}[tr_m^{\mathfrak{N}}(k)] = w_m^{\mathfrak{M}}[tr_m^{\mathfrak{M}}(k)] \text{ and } Rng(w_k^{\mathfrak{N}}) = Rng(w_k^{\mathfrak{M}})\Big)$$

and because $\mathfrak{M} \models$ **Ax6$_{\mathbf{00}}$**.

$\mathfrak{N} \models$ **Ax6$_{01}$** because $\mathfrak{M} \models$ **Ax6$_{01}$**, because

$$(\forall m, k \in Obs^{\mathfrak{N}}) \left(Dom(\mathsf{f}^{\mathfrak{N}}_{mk}) = A^{-1}_m \left[Dom(\mathsf{f}^{\mathfrak{M}}_{mk}) \right] \right),$$

and because A_m is a continuous function.

$\mathfrak{N} \models$ **AxE$_0$** by $\mathfrak{M} \models$ **AxE$_{00}$**, by the definition of c_m and by (I) and (II).

By the above, $\mathfrak{N} \models$ **Newbasax** is proved.

Since $m_0 \neq 0$, $m_0 \neq \infty$ and $m_0 \overset{\odot}{\to} m_1$, we have $c_{m_1} \neq 0$ and $c_{m_2} \neq \infty$ by Lemma 3.4.26. Hence $m_0, m_1 \in Obs^{\mathfrak{N}}$. By Thm.3.4.2, which says that **Newbasax** does not allow FTL observers, we have $ang^2(tr^{\mathfrak{N}}_{m_0}(m_1)) < 1$. By this and by (III), we get $ang^2(A_{m_0}[tr^{\mathfrak{N}}_{m_0}(m_1)]) < c_{m_0}$. By (I), this is equivalent with $ang^2(tr^{\mathfrak{M}}_{m_0}(m_1)) < c_{m_0}$. Hence $v^{\mathfrak{M}}_{m_0}(m_1) < c_{m_0}$.

Case 2: $c_{m_0} = \infty$. Assume $c_{m_0} = \infty$. Then $v_{m_0}(m_1) < c_{m_0}$ holds by Lemma 3.4.28.

Case 3: $c_{m_0} = 0$. Assume $c_{m_0} = 0$. Then we have to prove that $v_{m_0}(m_1) = 0$. Since $c_{m_0} = 0$ and $m_0 \overset{\odot}{\to} m_1$, we have $c_{m_1} = 0$ by Lemma 3.4.26. Then by **Ax4** and **Ax5$^{\mathbf{Ph}}$**, there is $ph \in Ph$ such that $tr_{m_1}(m_1) = tr_{m_1}(ph)$. Let this ph be fixed. By **Ax6$_{00}$**, we have that $w_{m_0}[tr_{m_0}(m_1)] \subseteq Rng(w_{m_1})$. By this and $tr_{m_1}(m_1) = tr_{m_1}(ph)$, we have $(\forall p \in tr_{m_0}(m_1))\, ph \in w_{m_0}(p)$. Hence $tr_{m_0}(m_1) = tr_{m_0}(ph)$. By this, we have $v_{m_0}(m_1) = 0$ since $v_{m_0}(ph) = c_{m_0} = 0$. ∎

Now we turn to the proof of Thm.3.4.19 for $n = 4$. As we said, to do this we have to formulate and prove analogous counterparts of theorems and statements of §3.1 and §3.3 for **Bax**.

Claim 3.4.29 below is an analogue of Claim 2.3.8(ii) (§2.3).

Claim 3.4.29 **Bax** $\models$ (f_{mk} is a (possibly) partial one-to one function).

Proof: The proof follows by Prop.3.4.27(ii). ∎

Lemma 3.4.30 below is an analogue of Lemma 3.3.16 (§3.3).

LEMMA 3.4.30
Bax $\models (\forall m, k \in Obs)(\forall p, q \in {}^nF)\Big((ang^2(\overline{pq}) = c_m \ \wedge\ p \in Dom(\mathsf{f}_{mk}))$

$$\Rightarrow\ q \in Dom(\mathsf{f}_{mk}) \Big).$$

Proof: The proof is analogous to the proof of Lemma 3.3.16. Checking the details is left to the reader. ∎

Thm.3.4.31 below is an analogue of Thm.3.3.8 (§3.3).

THEOREM 3.4.31

$$\mathbf{Bax} \models (\forall m, k \in Obs)(Rng(w_m) = Rng(w_k) \ \lor \ Rng(w_m) \cap Rng(w_k) = \emptyset).$$

Proof: The proof follows by Lemma 3.4.30 as follows. Let $\mathfrak{M}$ be a frame model of **Bax**. Let $m, k \in Obs$ with $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. We will prove that $Rng(w_m) = Rng(w_k)$. To prove this it is enough to prove that $Dom(\mathsf{f}_{mk}) = {}^nF$ and $Dom(\mathsf{f}_{km}) = {}^nF$. We will prove that $Dom(\mathsf{f}_{mk}) = {}^nF$, the proof of $Dom(\mathsf{f}_{km}) = {}^nF$ is analogous. To prove this let $q \in {}^nF$. We will prove that $q \in Dom(\mathsf{f}_{mk})$.

<u>*Case 1:* $c_m = \infty$.</u> Assume $c_m = \infty$. Then $c_k = \infty$ by Lemma 3.4.26(i). First we will prove that $m \overset{\odot}{\to} k$. Let $p \in Dom(\mathsf{f}_{km})$. Such a p exists by $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. Then there is $r \in \bar{t}$ such that $ang^2(\overline{pr}) = \infty = c_k$. Let this r be fixed. Since $p \in Dom(\mathsf{f}_{km})$ and $ang^2(\overline{pr}) = c_k$, we have that $r \in Dom(\mathsf{f}_{km})$ by Lemma 3.4.30. By $r \in \bar{t} = tr_k(k)$, we have that $k \in w_k(r)$. This and $r \in Dom(\mathsf{f}_{km})$ imply that $k \in w_m(r')$, for some $r' \in {}^nF$. Hence $tr_m(k) \neq \emptyset$, i.e. $m \overset{\odot}{\to} k$. By Lemma 3.4.28, we have $v_m(k) \neq c_m = \infty$. Since $v_m(k) \neq \infty$, there is $s \in tr_m(k)$ with $ang^2(\overline{sq}) = \infty = c_m$. Now we have $s \in Dom(\mathsf{f}_{mk})$ by $s \in tr_m(k)$ and $\mathbf{Ax6_{00}}$. $s \in Dom(\mathsf{f}_{mk})$ and $ang^2(\overline{sq}) = c_m$ imply $q \in Dom(\mathsf{f}_{mk})$ by Lemma 3.4.30.

<u>*Case 2:* $c_m \neq \infty$.</u> Assume $c_m \neq \infty$. Let $p \in Dom(\mathsf{f}_{mk})$. We will show at the end of the proof that

$$(53) \quad (\exists r^0, r^1, \ldots, r^n \in {}^nF)\Big(r^0 = p \ \land \ r^n = q \ \land \ (\forall i \in n) ang^2(\overline{r^i r^{i+1}}) = c_m\Big).$$

Now by (53) and $p \in Dom(\mathsf{f}_{mk})$, by applying Lemma 3.4.30 n times, we get $q \in Dom(\mathsf{f}_{mk})$. Thm.3.4.31 is proved modulo (53). To prove (53) we need Claim 3.4.32 below.

Claim 3.4.32 Let $c \in {}^+F$ such that there are $a_1, a_2, \ldots, a_{n-1} \in F$ with $c = a_1^2 + a_2^2 + \ldots + a_{n-1}^2$. Then the vector-space ${}^n\mathbf{F}$ is generated by

$$\{\langle 1, p_1, p_2, \ldots, p_{n-1}\rangle \ : \ p_1, p_2, \ldots, p_{n-1} \in F \ \& \ c = p_1^2 + p_2^2 + \ldots + p_{n-1}^2\}.$$

<u>*Proof of Claim 3.4.32:*</u> The proof goes via straightforward induction on n. For completeness we write down all the details.

Assume $n = 2$. Let $c \in {}^+F$ such that there is a_1 with $c = a_1^2$. Let such an a_1 be fixed. Then $\langle 1, a_1\rangle, \langle 1, -a_1\rangle$ are linearly independent and

$$\langle 1, a_1\rangle, \langle 1, -a_1\rangle \in \{\langle 1, p_1\rangle \ : \ p_1 \in F \ \& \ c = p_1^2\}.$$

Hence ${}^2\mathbf{F}$ is generated by $\{\langle 1, p_1\rangle \ : \ p_1 \in F \ \& \ c = p_1^2\}$. Thus Claim 3.4.32 holds for $n = 2$.

Assume that Claim 3.4.32 holds for $n = k$, where $k \geq 2$. We will prove that Claim 3.4.32 holds for $n = k+1$. To prove this let $c \in {}^+F$ such that there are $a_1, a_2, \ldots, a_k \in F$ with $c = a_1^2 + a_2^2 + \ldots + a_k^2$. Let such $a_1, a_2, \ldots, a_k$ be fixed. We have to prove that ${}^{k+1}\mathbf{F}$ is generated by

$$A := \{\langle 1, p_1, p_2, \ldots, p_k\rangle \; : \; p_1, p_2, \ldots, p_k \in F \;\&\; c = p_1^2 + p_2^2 + \ldots + p_k^2\}.$$

By $c = a_1^2 + a_2^2 + \ldots + a_k^2$ and $k \geq 2$, there is a_i ($1 \leq i \leq k$) such that $c > a_i^2$. Without loss of generality we can assume that $c > a_k^2$. Then we have $c - a_k^2 = a_1^2 + a_2^2 + \ldots + a_{k-1}^2$ and $c - a_k^2 \in {}^+F$. Then by the assumption that Claim 3.4.32 holds for $n = k$, we have that ${}^k\mathbf{F}$ is generated by

$$\{\langle 1, p_1, p_2, \ldots, p_{k-1}\rangle \; : \; p_1, p_2, \ldots, p_{k-1} \in F \;\&\; c - a_k^2 = p_1^2 + p_2^2 + \ldots + p_{k-1}^2\}.$$

Hence the sub-space of ${}^{k+1}\mathbf{F}$ generated by

$$C := \{\langle 1, p_1, p_2, \ldots, p_{k-1}, a_k\rangle \; : \; p_1, p_2, \ldots, p_{k-1} \in F \;\&\; c - a_k^2 = p_1^2 + p_2^2 + \ldots + p_{k-1}^2\}$$

is k-dimensional.

<u>*Case I: $a_k \neq 0$.*</u> Assume $a_k \neq 0$. Then $\langle 1, a_1, a_2, \ldots, a_{k-1}, -a_k\rangle$ is not an element of the subspace generated by C and it is an element of A. By this, by $C \subseteq A$ and by C generates a k dimensional subspace, we have that A generates a $(k+1)$-dimensional subspace, i.e. ${}^{k+1}\mathbf{F}$ is generated by A.

<u>*Case II: $a_k = 0$.*</u> Assume $a_k = 0$. Then there is a_i ($1 \leq i \leq k-1$) such that $a_i \neq 0$. Without loss of generality we can assume $a_{k-1} \neq 0$. Then $\langle 1, a_1, a_2, \ldots, a_{k-2}, a_k, a_{k-1}\rangle$ is not an element of the subspace generated by C and it an element of A. By this as in Case I, it follows that ${}^{k+1}\mathbf{F}$ is generated by A. This completes the proof of Claim 3.4.32.

QED (Claim 3.4.32)

<u>*Proof of (53):*</u> By $\mathbf{Ax5^{Ph}}$, there is $ph \in Ph$ with $v_m(ph) = c_m$ and $tr_m(ph) \ni \bar{0}$. Let such a ph be fixed. By $v_m(ph) = c_m \neq \infty$ there is $\langle 1, a_1, \ldots, a_{n-1}\rangle \in tr_m(ph)$. Let this $\langle 1, a_1, a_2, \ldots, a_{n-1}\rangle$ be fixed. By $\bar{0} \in tr_m(ph)$ and $v_m(ph) = c_m$, we have $c_m = a_1^2 + a_2^2 + \ldots + a_{n-1}^2$. Then by Claim 3.4.32, ${}^n\mathbf{F}$ is generated by

$$A := \{\langle 1, p_1, p_2, \ldots p_{n-1}\rangle \; : \; p_1, p_2, \ldots, p_{n-1} \in F \;\&\; c_m = p_1^2 + p_2^2 + \ldots + p_{n-1}^2\}.$$

For every $u \in A$ we have $ang^2(\overline{\bar{0}u}) = c_m$. By this and because ${}^n\mathbf{F}$ is generated by A, there is a basis $u^1, u^2, \ldots, u^n$ of ${}^n\mathbf{F}$ such that

$$ang^2(\overline{\bar{0}u^1}) = ang^2(\overline{\bar{0}u^2}) = \ldots = ang^2(\overline{\bar{0}u^n}) = c_m.$$

Let such $u^1, \ldots, u^n$ be fixed. Recall that q is a fixed element of ${}^nF$ and p is a fixed element of $Dom(\mathsf{f}_{mk})$.

$$q - p = \lambda_1 u^1 + \lambda_2 u^2 + \ldots + \lambda_n u^n,$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$. Let such $\lambda_1, \lambda_2, \ldots, \lambda_n$ be fixed. Let

$$r^0 := p \text{ and } (\forall i \in n)\ r^{i+1} := r^i + \lambda_i u^i.$$

Now for $r^0, r^1, \ldots, r^n$ we have

$$r^0 = p,\ \ r^n = q,\ \text{ and } (\forall i \in n) ang^2(\overline{r^i r^{i+1}}) = c_m.$$

Hence (53) above holds, and this completes the proof of Thm.3.4.31. ∎

Thm.3.4.33 below is an analogue of Thm.3.3.9 (§3.3).

THEOREM 3.4.33 **Bax** $\models (\forall m, k \in Obs)(m \stackrel{\odot}{\to} k \ \Leftrightarrow\ Rng(w_m) = Rng(w_k))$.

Proof: The proof follows by Thm.3.4.31. ∎

Thm.3.4.34 below is an analogue of Thm.3.3.10 (§3.3).

THEOREM 3.4.34
Bax $\models$ "$\stackrel{\odot}{\to}$ *is an equivalence relation when restricted to Obs.*"

Proof: The proof follows by Thm.3.4.31. ∎

Proposition 3.4.35 below is an analogue of Proposition 2.3.3(v) (§2.3).

PROPOSITION 3.4.35
Bax $\models (\forall m, k \in Obs)\Big(m \stackrel{\odot}{\to} k \ \Rightarrow\ (\mathsf{f}_{mk} \textit{ is a bijection } \mathsf{f}_{mk} : {}^nF \to {}^nF)\Big)$.

Proof: One could think that this proposition immediately follows from Thm.4.3.11 about **Bax**$^-$. However in **Bax**$^-$ we assumed that there is a photon in every direction. This is not assumed in **Bax**, therefore we include the proof here. The proof follows by Claim 3.4.29 and Thm.3.4.33. ∎

Thm.3.4.36 below is an analogue of Thm.3.1.1 (§3.1) and Thm.4.3.11 (§3.4.2).

THEOREM 3.4.36 **Bax** $\models (\forall m, k \in Obs)(\forall \ell \in \mathsf{Eucl})(m \stackrel{\odot}{\to} k \ \Rightarrow \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl})$.

Proof: One could think that this theorem immediately follows from Thm.4.3.11 about $\mathbf{Bax}^-$. However in $\mathbf{Bax}^-$ we assumed that there is a photon in every direction. This is not assumed in **Bax**, therefore we include the proof here. Let $\mathfrak{M}$ be a frame model of **Bax**. Let $m, k \in Obs$ with $m \overset{\odot}{\to} k$. To prove that f_{mk} takes lines to lines we need Claims 3.4.37, 3.4.38 and 3.4.39 below which are analogs of Prop.2.3.3(viii) (§2.3), Lemma 3.1.9 (§3.1) and Lemma 3.1.10 (§3.1), respectively.

Claim 3.4.37 $(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < c_m \;\Rightarrow\; \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl})$.

Proof of Claim 3.4.37: The proof is analogous to the proof of Prop.2.3.3(viii).

Claim 3.4.38 $(\forall \ell_1, \ell_2 \in \mathsf{Eucl})\Big((ang^2(\ell_1) < c_m \;\wedge\; \ell_1 \parallel \ell_2) \;\Rightarrow\; \mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]\Big)$.

Proof of Claim 3.4.38: The proof is analogous to the proof of Lemma 3.1.9.

Claim 3.4.39 $(\forall p, q \in {}^nF)\left(ang^2(\overline{pq}) < c_m \;\Rightarrow\; \mathsf{f}_{mk}(\frac{p+q}{2}) = \frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}\right)$.

Proof of Claim 3.4.39: The proof is analogous to the proof of Lemma 3.1.10.

Now $(\forall \ell \in \mathsf{Eucl})\mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$ can be proved by Claims 3.4.37, 3.4.38, 3.4.39 as Thm.3.1.1 was proved by Prop.2.3.3(viii) and Lemmas 3.1.9, 3.1.10. ∎

Thm.3.4.40 below is an analogue of Thm.3.1.4 (§3.1).

THEOREM 3.4.40
$\mathbf{Bax} \models (\forall m, k \in Obs)\Big(m \overset{\odot}{\to} k \;\Rightarrow\; (\mathsf{f}_{mk} = \widetilde{\varphi} \circ \mathsf{f},\ \textit{for some}\ \mathsf{f} \in Aftr\ \textit{and}\ \varphi \in Aut(\mathbf{F})\Big)$.

Proof: The proof follows by Prop.3.4.35, Thm.3.4.36 and Lemma 3.1.6 (§3.1). ∎

Lemma 3.4.41 below is a generalization of Lemma 3.4.5 (§3.4.1).

LEMMA 3.4.41 *Assume $n \geq 3$ and $\mathfrak{F}$ is Euclidean. Let $c_1, c_2 \in {}^+F$. Assume $\mathsf{f} : {}^nF \to {}^nF$ is a bijection such that*

$$(\star) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \;\wedge\; (ang^2(\ell) = c_1 \;\Leftrightarrow\; ang^2(\mathsf{f}[\ell]) = c_2)\Big).$$

Then $ang^2(\mathsf{f}[\bar{t}]) < c_2$.

Proof: Let $c_1, c_2 \in {}^+F$ and $\mathsf{f} : {}^nF \to {}^nF$ be a bijection such that $(\star)$ holds. Let $\mathsf{f}_1, \mathsf{f}_2$ be the linear transformations for which $\mathsf{f}_1(1_t) = 1_t$, $(\forall i \in n \setminus \{0\})\mathsf{f}_1(e_i) = \sqrt{c_1} \cdot e_i$, $\mathsf{f}_2(1_t) = 1_t$ and $(\forall i \in n \setminus \{0\})\mathsf{f}_2(e_i) = \sqrt{c_2} \cdot e_i$. Then for $\mathsf{g} := \mathsf{f}_1 \circ \mathsf{f} \circ \mathsf{f}_2^{-1}$ we have

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{g}[\ell] \in \mathsf{Eucl} \;\wedge\; (\mathsf{g}[\ell] \in \mathsf{PhtEucl} \;\Leftrightarrow\; \ell \in \mathsf{PhtEucl})\Big).$$

By Lemma 3.4.5 we have that $\mathsf{g}[\bar{t}\,] \in \mathsf{SlowEucl}$, i.e. $ang^2((\mathsf{f}_1 \circ \mathsf{f} \circ \mathsf{f}_2^{-1})[\bar{t}\,]) < 1$. But this is equvivalent with $ang^2(\mathsf{f}[\bar{t}\,]) < c_2$. ∎

For $n = 4$ Lemma 3.4.42 below is a generalization of Lemma 3.4.7 (§3.4.1).

LEMMA 3.4.42 *Assume $c_1, c_2 \in {}^+F$ such that $c_1 = ang^2(\ell)$, for some $\ell \in \mathsf{Eucl}$. Assume $\mathsf{f} \in Aftr(4, \mathfrak{F})$ satisfying ($\star$) in Lemma 3.4.41 above. Assume $\mathfrak{F}_*$ is an ordered field such that $\mathfrak{F} \subseteq \mathfrak{F}_*$. Let $\mathsf{f}_* \in Aftr(4, \mathfrak{F}_*)$ for which $\mathsf{f}_* \restriction {}^4F = \mathsf{f}$. Then f_* satisfies ($\star$) in Lemma 3.4.41 above when f_* and $\mathfrak{F}_*$ are substituted in place of f and $\mathfrak{F}$, respectively.*

We will give the **proof** of Lemma 3.4.42 after the proof of Thm.3.4.19 for $n = 4$.

Proof of Thm.3.4.19 for $\mathbf{n} = 4$:

<u>*Proof of (i):*</u> Assume $n = 4$. Let $\mathfrak{M}$ be frame model of **Bax**. Let $m, k \in Obs$ with $m \overset{\odot}{\to} k$. We have to prove that $v_m(k) < c_m$.

<u>*Intuitive idea of the proof:*</u> We want to prove that $ang^2(tr_m(k)) \leq c_m$. We will see that $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$ where $\varphi \in Aut(\mathbf{F})$ and f is an affine transformation satisfying ($\star$) in 3.4.41 for $c_1 := \varphi(c_k)$ and $c_2 := c_m$. By 3.4.42 f will continue satisfying ($\star$) in a larger field $\mathfrak{F}_*$, which, in turn will be Euclidean. Looking at it from $\mathfrak{F}_*$, $ang^2(\mathsf{f}[\bar{t}\,]) < c_m$ by 3.4.41. Therefore $ang^2(\mathsf{f}[\bar{t}\,]) < c_m$ in $\mathfrak{F}$, too and then $tr_m(k) = \mathsf{f}[\bar{t}\,]$ will complete the proof.

<u>*Formally:*</u> By Thm.3.4.40, $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$ for some $\mathsf{f} \in Aftr$ and $\varphi \in Aut(\mathbf{F})$. Let this f and φ be fixed. We have $\mathsf{f}_{km}[tr_k(k)] = tr_m(k)$ because f_{mk} is a bijection. By $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, by $\widetilde{\varphi}[\bar{t}\,] = \bar{t}$, by $tr_k(k) = \bar{t}$ and by $\mathsf{f}_{km}[tr_k(k)] = tr_m(k)$, we have

$$\mathsf{f}[\bar{t}\,] = tr_m(k). \tag{55}$$

By $\mathbf{Ax5^{Obs}}$, $\mathbf{Ax5^{Ph}}$, $\mathbf{AxE_{00}}$ and Prop.3.4.35 it is easy to see that

$$(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) = c_k \;\Leftrightarrow\; ang^2(\mathsf{f}_{km}[\ell]) = c_m). \tag{56}$$

By (56) and $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, we have

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \;\;\land\;\; (ang^2(\ell) = \varphi(c_k) \;\Leftrightarrow\; ang^2(\mathsf{f}[\ell]) = c_m)\Big). \tag{57}$$

$c_k = v_k(ph)$, for some $ph \in Ph$. Let such a ph be fixed. Then $\widetilde{\varphi}[tr_k(ph)] \in \mathsf{Eucl}$ and $ang^2(\widetilde{\varphi}[tr_k(ph)]) = \varphi(c_k)$. Thus

$$(\exists \ell \in \mathsf{Eucl}) ang^2(\ell) = \varphi(c_k). \tag{58}$$

Let $\mathfrak{F}_*$ be an ordered field such that $\mathfrak{F}_*$ is Euclidean and $\mathfrak{F} \subseteq \mathfrak{F}_*$. Such an $\mathfrak{F}_*$ exists, e.g. the real closure of $\mathfrak{F}$ is such. Let $\mathsf{f}_* \in Aftr(n, \mathfrak{F}_*)$ such that $\mathsf{f}_* \restriction {}^4F = \mathsf{f}$. By $\mathbf{AxE_{01}}$ and (58), we have $\varphi(c_k), c_m \in {}^+F$. According to our Convention 3.1.2, F_* denotes the universe of $\mathfrak{F}_*$. By this, by (57), by (58) and by Lemma 3.4.42, we have that

$$(59) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}_*[\ell] \in \mathsf{Eucl} \ \wedge \ (ang^2(\ell) = \varphi(c_k) \ \Leftrightarrow \ ang^2(\mathsf{f}_*[\ell]) = c_m)\Big).$$

Let $\bar{t}_* := F_* \times {}^{n-1}\{0\}$. Then $ang^2(\mathsf{f}_*[\bar{t}_*]) < c_m$ by Lemma 3.4.41. Hence $ang^2(\mathsf{f}[\bar{t}\,]) < c_m$. By this and by (55), we have $v_m(k) < c_m$.

<u>*Proof of (ii):*</u> Assume $n = 4$. Let $\mathfrak{M}$ be a a frame model of $\mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$. Let $m_0, m_1 \in Obs$ such that $m_0 \overset{\odot}{\rightarrow} m_1$. We have to prove the following. If $c_{m_0} \neq 0$ then $v_{m_0}(m_1) < c_{m_0}$, and if $c_{m_0} = 0$ then $v_{m_0}(m_1) = 0$. For $c_{m_0} = 0$ or $c_{m_0} = \infty$ the proof is analogous to the proof when $n = 3$. Assume $c_{m_0} \neq 0$ and $c_{m_0} \neq \infty$. We define a frame model

$$\mathfrak{N} = \langle (B, Obs^{\mathfrak{N}}, Ph^{\mathfrak{N}}, Ib^{\mathfrak{N}}), \mathfrak{F}, G; \in, W^{\mathfrak{N}} \rangle$$

as follows.

$$\begin{aligned}
Obs^{\mathfrak{N}} &\overset{\text{def}}{=} \{m \in Obs \ : \ c_m \neq 0 \ \& \ c_m \neq \infty\}, \\
Ph^{\mathfrak{N}} &\overset{\text{def}}{=} \{ph \in Ph \ : \ (\exists m \in Obs^{\mathfrak{N}})(m \overset{\odot}{\rightarrow} ph \text{ holds in } \mathfrak{M})\}, \\
Ib^{\mathfrak{N}} &\overset{\text{def}}{=} \{b \in Ib \ : \ (\exists m \in Obs^{\mathfrak{N}})(m \overset{\odot}{\rightarrow} b \text{ holds in } \mathfrak{M})\}, \\
W^{\mathfrak{N}} &\overset{\text{def}}{=} W \restriction (Obs^{\mathfrak{N}} \times {}^4F \times B).
\end{aligned}$$

It is easy to check that $\mathfrak{N} \models \mathbf{Bax}$ by $\mathfrak{M} \models \mathbf{Bax} \setminus \{\mathbf{AxE_{01}}\}$ and Lemma 3.4.26. Further $m_0, m_1 \in Obs^{\mathfrak{N}}$, $v^{\mathfrak{M}}_{m_0}(m_1) = v^{\mathfrak{N}}_{m_0}(m_1)$ and $c^{\mathfrak{M}}_{m_0} = c^{\mathfrak{N}}_{m_0}$. By item (i) we have $v^{\mathfrak{N}}_{m_0}(m_1) < c^{\mathfrak{N}}_{m_0}$, hence $v^{\mathfrak{M}}_{m_0}(m_1) < c^{\mathfrak{M}}_{m_0}$. ∎

Proof of Lemma 3.4.42:

Claim 3.4.43 Let $c_1, c_2, \mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in the formulation of Lemma 3.4.42. Then

$$(\star\star) \qquad (\forall \ell \in \mathsf{Eucl}(4, \mathfrak{F}_*))(ang^2(\ell) = c_1 \ \Rightarrow \ ang^2(\mathsf{f}_*[\ell]) = c_2).$$

We will prove Claim 3.4.43 very soon. Lemma 3.4.42 follows from Claim 3.4.43 because of the following. Intuitively: If f satisfies $(\star)$ in 3.4.41 then f^{-1} satisfies $(\star)$ when $\mathsf{f}^{-1}, c_2, c_1$ are substituted in place of f, c_1, c_2, respectively. Then applying Claim 3.4.43 to f, c_1, c_2 and to $\mathsf{f}^{-1}, c_2, c_1$, respectively, we obtain Lemma 3.4.42.

More formally: Let $c_1, c_2, \mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ as in the formulation of Lemma 3.4.42. Then by Claim 3.4.43, we have

$$(61) \qquad (\forall \ell \in \mathsf{Eucl}(4, \mathfrak{F}_*))(ang^2(\ell) = c_1 \;\Rightarrow\; ang^2(\mathsf{f}_*[\ell]) = c_2).$$

Let $(\mathsf{f}^{-1})_* \in Aftr(4, \mathfrak{F}_*)$ such that $(\mathsf{f}^{-1})_* \restriction {}^4F = \mathsf{f}^{-1}$. Obviously $(\mathsf{f}^{-1})_* = (\mathsf{f}_*)^{-1}$.

By f satisfying $(\star)$ in Lemma 3.4.41 we have

$$(62) \qquad (\forall \ell \in \mathsf{Eucl}(4, \mathfrak{F})) \qquad (ang^2(\ell) = c_2 \;\Leftrightarrow\; ang^2(\mathsf{f}^{-1}[\ell]) = c_1).$$

$$(63) \qquad (\exists \ell \in \mathsf{Eucl}(4, \mathfrak{F})) \qquad ang^2(\ell) = c_2$$

because there is $\ell \in \mathsf{Eucl}$ such that $ang^2(\ell) = c_1$, and $ang^2(\mathsf{f}[\ell]) = c_2$ for that ℓ by $(\star)$. By (62) and (63), Claim 3.4.43 can be applied to $c_2, c_1, \mathfrak{F}_*, \mathsf{f}^{-1}, (\mathsf{f}^{-1})_*$. So we have

$$(64) \qquad (\forall \ell \in \mathsf{Eucl}(4, \mathfrak{F}_*))(ang^2(\ell) = c_2 \;\Rightarrow\; ang^2((\mathsf{f}^{-1})_*[\ell]) = c_1).$$

By (61), (64) and $(\mathsf{f}^{-1})_* = (\mathsf{f}_*)^{-1}$, we have

$$(\forall \ell \in \mathsf{Eucl}(4, \mathfrak{F}_*))(ang^2(\ell) = c_1 \;\Leftrightarrow\; ang^2(\mathsf{f}_*[\ell]) = c_2).$$

Thus Lemma 3.4.42 follows by Claim 3.4.43.

Proof of Claim 3.4.43: Let $c_1, c_2, \mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in formulation of Lemma 3.4.41. We have to prove that f_* satisfies $(\star\star)$ in Claim 3.4.43. Without loss of generality we may assume that $\mathsf{f}(\bar{0}) = \bar{0}$.

On the structure of the proof: Items (65) an (66) below are reformulations of saying that f and f_* satisfy $(\star\star)$, respectively. Items (69) and (70) below are equivalent forms of (65) and (66), respectively. Hence our task is to prove (70) from (69). This is done by the linear algebraic considerations given below.

By our assumption that f is a linear transformation, we have that

$$(\forall p \in {}^4F)\mathsf{f}(p) = \langle \sum_{i=0}^{3} p_i a_{i0}, \sum_{i=0}^{3} p_i a_{i1}, \sum_{i=0}^{3} p_i a_{i2}, \sum_{i=0}^{3} p_i a_{i3} \rangle,$$

for some $a_{ij} \in F$, where $i, j \in 4$. Let these a_{ij}'s be fixed. By the definition of f_*, we have

$$(\forall p \in {}^4F_*)\mathsf{f}_*(p) = \langle \sum_{i=0}^{3} p_i a_{i0}, \sum_{i=0}^{3} p_i a_{i1}, \sum_{i=0}^{3} p_i a_{i2}, \sum_{i=0}^{3} p_i a_{i3} \rangle.$$

Since f satisfies $(\star)$, we have that (65) below holds, and to prove that f_* satisfies $(\star\star)$ we have to prove (66) below.

$$(65) \quad (\forall p \in {}^4F) \qquad \left(c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \;\Rightarrow\; c_2 \left(\sum_{i=0}^{3} p_i a_{i0} \right)^2 = \sum_{j=1}^{3} \left(\sum_{i=0}^{3} p_i a_{ij} \right)^2 \right).$$

$$(66)\ (\forall p\in {}^4F_*)\qquad \left(c_1p_0^2=p_1^2+p_2^2+p_3^2\ \Rightarrow\ c_2\left(\sum_{i=0}^{3}p_ia_{i0}\right)^2=\sum_{j=1}^{3}\left(\sum_{i=0}^{3}p_ia_{ij}\right)^2\right).$$

Let $(\forall i,j\in 4)\ d_{ij}:=c_2a_{i0}a_{j0}-\sum_{k=1}^{3}a_{ik}a_{jk}$, and let $b_0:=d_{00}$, $b_1:=d_{11}$, $b_2:=d_{22}$, $b_3:=d_{33}$, $b_4:=d_{01}+d_{10}$, $b_5:=d_{02}+d_{20}$, $b_6:=d_{03}+d_{30}$, $b_7:=d_{12}+d_{21}$, $b_8:=d_{13}+d_{31}$, $b_9:=d_{23}+d_{32}$. Then (65) and (66) above are equivalent with (67) and (68) below, respectively.

$$(67)\qquad (\forall p\in {}^4F)\ \Big(c_1p_0^2=p_1^2+p_2^2+p_3^2\ \Rightarrow\ p_0^2b_0+p_1^2b_1+p_2^2b_2+p_3^2b_3+$$
$$+p_0p_1b_4+p_0p_2b_5+p_0p_3b_6+p_1p_2b_7+p_1p_3b_8+p_2p_3b_9=0\Big).$$
$$(68)\qquad (\forall p\in {}^4F_*)\Big(c_1p_0^2=p_1^2+p_2^2+p_3^2\ \Rightarrow\ p_0^2b_0+p_1^2b_1+p_2^2b_2+p_3^2b_3+$$
$$+p_0p_1b_4+p_0p_2b_5+p_0p_3b_6+p_1p_2b_7+p_1p_3b_8+p_2p_3b_9=0\Big).$$

Let E and E_* be the following sets of linear equations.

$$E:=\{p_0^2x_0+p_1^2x_1+p_2^2x_2+p_3^2x_3+p_0p_1x_4+p_0p_2x_5+p_0p_3x_6+p_1p_2x_7+$$
$$+p_1p_3x_8+p_2p_3x_9=0\ :\ p\in {}^4F\ \&\ c_1p_0^2=p_1^2+p_2^2+p_3^2\}.$$
$$E_*:=\{p_0^2x_0+p_1^2x_1+p_2^2x_2+p_3^2x_3+p_0p_1x_4+p_0p_2x_5+p_0p_3x_6+p_1p_2x_7+$$
$$+p_1p_3x_8+p_2p_3x_9=0\ :\ p\in {}^4F_*\ \&\ c_1p_0^2=p_1^2+p_2^2+p_3^2\}.$$

Now (67) and (68) above are equivalent with (69) and (70) below.

(69) $\langle b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9\rangle$ is a solution for the system of equations E.

(70) $\langle b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9\rangle$ is a solution for the system of equations E_*.

Thus to prove Claim 3.4.43 it is enough to prove (69)$\Rightarrow$(70). To prove (69)$\Rightarrow$(70) it is enough to prove that each linear equation from E_* is a linear combination of some equations from E, i.e. that each vector from A_* is a linear combination of some vectors from A, where A and A_* are defined below.

$A:=\{\langle p_0^2,p_1^2,p_2^2,p_3^2,p_0p_1,p_0p_2,p_0p_3,p_1p_2,p_1p_3,p_2p_3\rangle : p\in {}^4F\ \&\ c_1p_0^2=p_1^2+p_2^2+p_3^2\}$.
$A_*:=\{\langle p_0^2,p_1^2,p_2^2,p_3^2,p_0p_1,p_0p_2,p_0p_3,p_1p_2,p_1p_3,p_2p_3\rangle : p\in {}^4F_*\ \&\ c_1p_0^2=p_1^2+p_2^2+p_3^2\}$.

Both A and A_* are at most 9-dimensional because of the "condition" $c_1p_0^2=p_1^2+p_2^2+p_3^2$ in the definitions of A and A_*. Hence to prove that each vector from A_* is a linear combination of some vectors from A it is enough to prove that A generates a 9-dimensional sub-space of ${}^{10}\mathbf{F}$ because $A\subseteq A_*$. Now to prove that subspace generated by A is 9-dimensional it is enough to prove that the subspace $\mathbf{W}$ of ${}^9\mathbf{F}$ generated by

$$C := \{\langle p_1^2, p_2^2, p_3^2, p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3\rangle \,:\, p_1, p_2, p_3 \in F \;\&\; c_1 = p_1^2 + p_2^2 + p_3^2\}$$

is 9-dimensional, i.e. $\mathbf{W} = {}^9\mathbf{F}$. By $(\exists \ell \in \mathsf{Eucl})ang^2(\ell) = c_1$, we have that there is ℓ with $ang^2(\ell) = c_1$ and $\bar{0} \in \ell$. Let such an ℓ be fixed. Then $\langle 1, \lambda, \mu, \nu\rangle \in \ell$, for some $\langle 1, \lambda, \mu, \nu\rangle$. Let this $\langle 1, \lambda, \mu, \nu\rangle$ be fixed. Now $c_1 = \lambda^2 + \mu^2 + \nu^2$ by $ang^2(\ell) = c_1$. We can assume that $0 \neq |\lambda| \neq |\mu| \neq 0$ (we checked that this is true, but we do not include the details here). We have

$$\begin{array}{lclrrrrrrrrrlcl}
v_1 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & \mu, & \nu, & \lambda\mu, & \lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_2 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & -\lambda, & -\mu, & -\nu, & \lambda\mu, & \lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_3 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & -\lambda, & \mu, & \nu, & -\lambda\mu, & -\lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_4 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & -\mu, & -\nu, & -\lambda\mu, & -\lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_5 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & -\mu, & \nu, & -\lambda\mu, & \lambda\nu, & -\mu\nu & \rangle & \in & C,\\
v_6 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & \mu, & -\nu, & \lambda\mu, & -\lambda\nu, & -\mu\nu & \rangle & \in & C,\\
v_7 & := & \langle & \mu^2, & \nu^2, & \lambda^2, & \mu, & \nu, & \lambda, & \mu\nu, & \lambda\mu, & \lambda\nu & \rangle & \in & C,\\
v_8 & := & \langle & \nu^2, & \lambda^2, & \mu^2, & \nu, & \lambda, & \mu, & \lambda\nu, & \mu\nu, & \lambda\mu & \rangle & \in & C,\\
v_9 & := & \langle & \mu^2, & \lambda^2, & \nu^2, & \mu, & \lambda, & \nu, & \lambda\mu, & \mu\nu, & \lambda\nu & \rangle & \in & C.
\end{array}$$

For every $i \in 9$, 1_i denotes the i'th unit vector of the coordinate-system ${}^9F$. It is easy to check that $1_3 = \frac{1}{4\lambda}(v_1 - v_2 - v_3 + v_4)$. Hence $1_3 \in W$. Similarly $1_4, 1_5 \in W$. It is easy to check that $1_6 = \frac{1}{4\lambda\mu}(v_1 - v_4 - v_5 + v_6) + v$, for some v which is in the sub-space generated by $\{1_3, 1_4, 1_5\}$. Hence $1_6 \in W$. Similarly $1_7, 1_8 \in W$. It is easy to check that $1_0 + 1_1 + 1_2 = \frac{1}{\lambda^2+\mu^2+\nu^2}(v_1 + v_7 + v_8) + v$, for some v which is in the sub-space generated by $\{1_3, 1_4, 1_5, 1_6, 1_7, 1_8\}$. Thus $1_0 + 1_1 + 1_2 \in W$. It is easy to check that $1_0 - 1_2 = \frac{1}{\lambda^2-\mu^2}(v_1 - v_9) + v$, for some v which is in the sub-space generated by $\{1_3, 1_4, 1_5, 1_6, 1_7, 1_8\}$. Hence $1_0 - 1_2 \in W$. Similarly $1_0 - 1_3 \in W$. But $1_0 + 1_1 + 1_2, 1_0 - 1_1, 1_0 - 1_2 \in W$ imply $1_0 \in W$. Similarly $1_1, 1_2 \in W$. We proved $(\forall i \in 9)1_i \in W$, and this completes the proof of Claim 3.4.43 and Lemma 3.4.42. ∎

3.5 Simple models for Basax

For $n \leq 3$ we proved that **Basax**(n) is consistent in §2.4 ("Models for **Basax** in dimension 2") and in §3.2 ("Intuitive ... **Basax**(3)"). In this section we show that **Basax** is consistent for arbitrary $n \geq 2$ by defining a class of frame models, which we call the class of simple models, in symbols SM, and showing that $\mathsf{SM} \models$ **Basax**. The consistency proof for **Basax**(3) in §3.2 was an intuitive, visualizable, structuralist one. Although we do prefer such structuralist proofs to computational ones, the proof given below is a computational one.[233] We plan to replace the computational proof below with a structuralist one like the one in §3.2 for arbitrary n,[234] in a later version.

In §2.4, first we gave an intuitive idea for a consistency proof for **Basax**(2); and then in the second part of §2.4 we gave a concrete construction of models denoted there as $\mathfrak{M}_0^P$. The consistency proof below will be analogous with the construction of $\mathfrak{M}_0^P$ in the second part of §2.4.[235] This proof goes by first defining a class SM of n-dimensional frame models, for arbitrary n (cf. Def.3.5.5), and then proving $\mathsf{SM} \models$ **Basax** in Thm.3.5.6 (the proof of Thm.3.5.6 will be presented only for $n = 3$ and will be left to the reader to generalize it to $n \geq 3$).

For the definition of the class of simple models, first we single out some transformations of ${}^nF$. The intuitive meaning of these transformations will be discussed below the definition.

Definition 3.5.1 We define $\mathit{Triv}_0(n, \mathfrak{F}) = \mathit{Triv}_0$ and $\mathit{Triv}(n, \mathfrak{F}) = \mathit{Triv}$ as follows.

$$\begin{aligned} \mathit{Triv}_0 &\stackrel{\text{def}}{=} \{\mathsf{f} \in \mathit{Linb} : \mathsf{f}(1_t) = 1_t \ \wedge\ (\forall p \in {}^nF)\ \|\mathsf{f}(p)\| = \|p\|\}\ , \\ \mathit{Triv} &\stackrel{\text{def}}{=} \{\mathsf{f} \circ \tau : \mathsf{f} \in \mathit{Triv}_0 \ \wedge\ \tau \in \mathit{Tran}\}\ . \end{aligned}$$

$\lhd$

[233] The reason for this is that we do not have time to elaborate the 4-dimensional version of the structuralist proof given in §3.2, while the computational proof below is very easy to generalize to arbitrary $n \geq 3$. (To save space, we write up that proof for $n = 3$ only and leave the straightforward generalization to the reader.)

[234] which in our case means $n = 4$, because we made a convention for not caring about the $n > 4$ case

[235] Let $\mathsf{SM}(n)$ denote the n–dimensional version of the class SM of models to be defined in this section. The class $\mathsf{SM}(2)$ is almost the same as the class $\{\mathfrak{M}_0^P : P$ is an appropriate choice function (cf. §2.4, p.80)$\}$ defined in §2.4. The only difference is that $\mathsf{SM}(2)$ is slightly bigger in the following sense: In $\mathsf{SM}(2)$ we have an extra parameter N, which on the other hand cannot do too much for $n = 2$. Another difference is that in $\mathsf{SM}(2)$ $\mathfrak{F}$ is allowed to be an arbitrary Euclidean field, while in $\mathfrak{M}_0^P$ it was fixed to be $\mathfrak{R}$.

$Triv_0$ consists of linear transformations which are identity functions on the time-axis $\bar{t}$ and preserve the (squares of Euclidean) *lengths* (and consequently preserve the squares of Euclidean distances and orthogonality $\perp_e$). The intuitive importance of $Triv_0$ comes from the fact that the transformations in $Triv_0$ involve *no "relativistic effects"*, one could say that they are very non-relativistic or, so to speak, trivial. The abbreviation $Triv_0$ refers to this. Typical examples for such transformations, e.g. in the case $n = 3$ and $\mathfrak{F} = \mathfrak{R}$, are rotations around $\bar{t}$ axis and reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{x})$. Actually these two kinds generate all of $Triv_0(3, \mathfrak{R})$. For completeness, we note that an equivalent definition for $Triv_0$ would be the following.

$$\mathsf{f} \in Triv_0 \overset{\text{def}}{\iff} \Big((\forall p \in \bar{t}\,)\ \mathsf{f}(p) = p \ \wedge\ (\forall p \in S)\ (\mathsf{f}(p) \in S \ \wedge\ \|\mathsf{f}(p)\| = \|p\|) \Big),$$

where we recall that S denotes the space part $\{0\} \times {}^{n-1}F$ of our coordinate-system ${}^nF$.

$Triv$ consists of compositions of members of $Triv_0$ and translations. An equivalent definition for $Triv$ is represented in Figure 76. Since translations are also non-relativistic, the members of $Triv$ are non-relativistic, too.

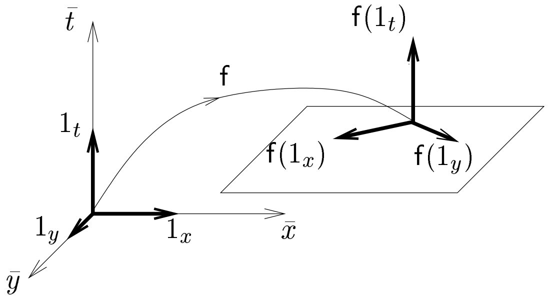


Figure 76: $\mathsf{f} \in Triv$ iff f is an affine transformation taking the unit vectors into pairwise orthogonal (in the Euclidean sense) vectors[237] of length 1, and leaving the direction of the time-unit vector unchanged.

The members of $Triv$ represent "changes" (e.g. actions, world-view transformations) which are sort of irrelevant from the point of view of relativity theory; therefore we will be a little bit casual (or "careless") in connection with situations when only members of $Triv$ make a difference.

[237]In this figure we use the word vector in an intuitive sense where a vector is represented by a pair of points (instead of a single point as is done in the rest of this work).

Remark 3.5.2 $\langle Triv_0, \circ, ^{-1}, \mathrm{Id}\rangle$ and $\langle Triv, \circ, ^{-1}, \mathrm{Id}\rangle$ are groups.

$\lhd$

Lemmas 3.5.3 and 3.5.4 below are needed for the definition of the class of simple models.

The following lemma says that any line ℓ can be mapped into $\mathsf{Plane}(\bar{t}, \bar{x})$ by a trivial transformation (taking any prescribed point on ℓ to $\bar{0}$). Somehow, this means that any "configuration" (involving two observers) can be transformed to a standard configuration by trivial transformations.

LEMMA 3.5.3 *Assume $\mathfrak{F}$ is Euclidean (i.e. "positive" square-roots exist*[238] *in $\mathfrak{F}$). Assume $\ell \in \mathsf{Eucl}$ and p is a point lying on ℓ. Then there is $N \in Triv$ such that $N[\ell] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N(p) = \bar{0}$.*

We will give the **proof** at the end of this section on pp.260-262.

LEMMA 3.5.4 *Assume $\ell \in \mathsf{Eucl}$ and $N \in Triv$. Then $ang^2(\ell) = ang^2(N[\ell])$.*

Proof: The proof is straightforward. We omit it. ∎

In Def.3.5.5 below the class SM of simple models will be defined in the following way. For each Euclidean ordered field $\mathfrak{F}$ and for each function P that to each $\ell \in \mathsf{SlowEucl}$ associates two distinct points o_ℓ and t_ℓ lying on ℓ and $N_\ell \in Triv$ with $N_\ell[\ell] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N_\ell(o_\ell) = \bar{0}$, a frame model $\mathfrak{M}$ will be defined. We suggest the reader to read the following definition only for $n = 2$ and $n = 3$, at the first reading.

Definition 3.5.5 **(Simple Models, SM)**
Let $\mathfrak{F}$ be a Euclidean ordered field (i.e. "positive" square roots exist in $\mathfrak{F}$). Let P be a "choice" function that to each $\ell \in \mathsf{SlowEucl}$ associates two distinct points o_ℓ and t_ℓ lying on ℓ and $N_\ell \in Triv$ with $N_\ell[\ell] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N_\ell(o_\ell) = \bar{0}$. By Lemma 3.5.3 above, such a P exists. We will denote $P(\ell)$ by $\langle o_\ell, t_\ell, N_\ell\rangle$. To each such $\mathfrak{F}$ and function P, we will define a frame model $\mathfrak{M}^P_\mathfrak{F}$.

We define $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_\mathfrak{F} \stackrel{\text{def}}{=} \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, G;\, \in, W\rangle$, where

$G \stackrel{\text{def}}{=} \mathsf{Eucl}$,

[238] Throughout by this expression we mean that square roots of positive elements exist in $\mathfrak{F}$.

$Obs \overset{\text{def}}{=} \mathsf{SlowEucl}$,

$Ph \overset{\text{def}}{=} \mathsf{PhtEucl}$,

$B \overset{\text{def}}{=} Ib \overset{\text{def}}{=} Obs \cup Ph = \{\, \ell \in \mathsf{Eucl} \ :\ ang^2(\ell) \leq 1 \,\} = \mathsf{SlowEucl} \cup \mathsf{PhtEucl}$.

By the above, **Ax1** and **Ax2** are true in $\mathfrak{M}$. It remains to define W. Let

$$m_0 \overset{\text{def}}{=} \bar{t}\,.$$

First we will define $w_{m_0} : {}^nF \longrightarrow \mathcal{P}(B)$ and $\mathsf{f}_{km_0} : {}^nF \longrightarrow {}^nF$ for all $k \in \mathsf{SlowEucl}$, $k \neq m_0$. To define w_{m_0}, let $p \in {}^nF$. Then

$$w_{m_0}(p) \overset{\text{def}}{=} \{\, \ell \in B \ :\ p \in \ell \,\}\ .$$

By this, we have that for all $\ell \in B$,

$$tr_{m_0}(\ell) = \ell\,,$$

in particular, $tr_{m_0}(m_0) = m_0 = \bar{t}$. Thus $\mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{AxE}$ are satisfied when m is replaced in them with m_0.

We turn now to defining f_{km_0}.

Let us recall that for every $k \in \mathsf{SlowEucl}$, the parameter P of the model $\mathfrak{M} \overset{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}}$ gives a triple $\langle o_k, t_k, N_k\rangle$ such that $\overline{o_k t_k} = k$ $(o_k \neq t_k)$ and $N_k \in \mathit{Triv}$ with $N_k[k] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N_k(o_k) = \bar{0}$.

First we will define f_{km_0} for the case $n = 2$, after that for the case $n = 3$, and finally for arbitrary $n \geq 2$.

Recall that Id is the identical transformation of ${}^nF$ taking p to p.

<u>*Definition of* f_{km_0} *for the case n=2:*</u>
Assume $n = 2$. Let $k \in \mathsf{SlowEucl}$, $k \neq m_0 \overset{\text{def}}{=} \bar{t}$ be arbitrary and fixed. First we will define f_{km_0} for the special case when $N_k = \mathrm{Id}$. This will be implemented in item (i) below. After that in item (ii) we will define f_{km_0} for $N_k \neq \mathrm{Id}$. Of (i) and (ii) only (i) i.e. $N_k = \mathrm{Id}$ case is “interesting” because the $N_k \neq \mathrm{Id}$ case has a kind of book-keeping character. (The same remark applies to the cases when $n \geq 3$.)

(i) We define f_{km_0} for the special case $N_k = \mathrm{Id}$ as described below. The following construction of f_{km_0} will be the same as the definition of a rhombus transformation was in Def.2.3.18 on p.72. The same remark applies to the cases of

$n > 2$ (when $N_k = \mathrm{Id}$) which we will discuss soon. In this connection we ask the reader to consult first Figures 15–18 (pp. 63–67) as well as the intuitive idea of "model-construction" on pp. 78–80.

$N_k = \mathrm{Id}$ implies that $o_k = \bar{0} \in k$.

Let x_k be the mirror image of t_k w.r.t. the line $\overline{\bar{0}\langle 1,1\rangle}$. In more detail: If $t_k = \langle t_0, t_1\rangle$ then we define $x_k \stackrel{\text{def}}{=} \langle t_1, t_0\rangle$.

Let f_{km_0} be the linear transformation which takes 1_t, 1_x to t_k, x_k, respectively. Clearly such an f_{km_0} exists and is unique. It is easy to check that this f_{km_0} is a bijective linear transformation.[239]

The reason why we chose x_k exactly the way we did can be explained by Prop.3.1.21 in §3.1.

(ii) We define f_{km_0} for the case $N_k \neq \mathrm{Id}$ as follows.

Let $t'_k \stackrel{\text{def}}{=} N_k[t_k]$. Then $t'_k \neq \bar{0}$ by $t_k \neq o_k$ and $\bar{0} = N_k(o_k)$.

Let x'_k be chosen for t'_k exactly as we chose x_k for t_k in item (i). Let $x_k \stackrel{\text{def}}{=} N_k^{-1}(x'_k)$.

We define f_{km_0} to be the affine transformation which takes $\bar{0}$, 1_t, 1_x to o_k, t_k, x_k, respectively. Clearly such an f_{km_0} exists[240] and is unique.

Definition of f_{km_0} *for the case n=3:*
Assume $n = 3$. Let $k \in \mathsf{SlowEucl}$, $k \neq m_0 \stackrel{\text{def}}{=} \bar{t}$ be arbitrary and fixed. Let us recall $P(k) = \langle o_k, t_k, N_k\rangle$ (see at the beginning of Def.3.5.5). First we will define f_{km_0} for the case when $N_k = \mathrm{Id}$. This will be implemented in item (i) below. After that in item (ii) we will define f_{km_0} for $N_k \neq \mathrm{Id}$.

(i) We define f_{km_0} for the case $N_k = \mathrm{Id}$ as described below. The construction below is essentially the same as the one in §3.2 ("Intuitive ... **Basax**(3)"). At this point, the reader is asked to compare Figure 77 below with the pictures on p.181 (i.e. in the middle of §3.2) and also to consult Figures 19, 20 (pp. 68–70). Throughout this definition the reader is asked to consult Figure 77.

$N_k = \mathrm{Id}$ implies that $o_k = \bar{0}$ and $\overline{o_k t_k} = k \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

[239] This is so because $t_k \neq o_k = \bar{0}$, and by $ang^2(k) \neq 1$ one can check that vectors t_k and x_k are linearly independent.

[240] f_{km_0} exists because vectors $t_k - o_k$, $x_k - o_k$ are linearly independent (this is so because vectors t'_k, x'_k are linearly independent and N_k^{-1} is the affine transformation taking $\bar{0}$, t'_k, k'_x to o_k, t_k, x_k, respectively).

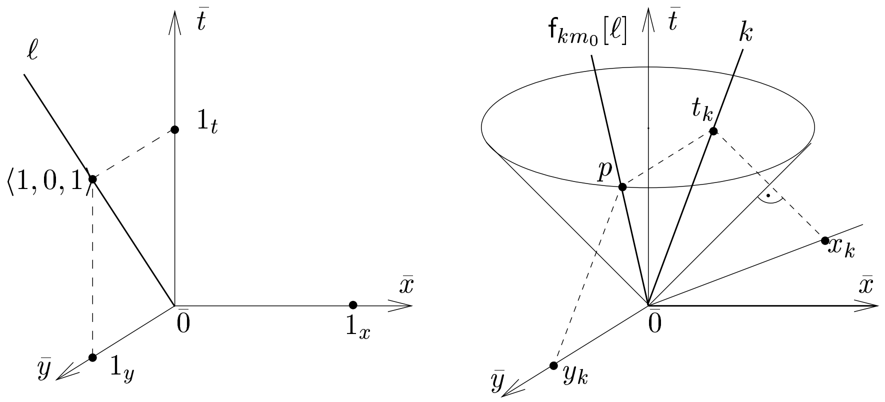


Figure 77: Illustration for item (i) of def. of f_{km_0} for the case n=3.

In what follows we will define points x_k and y_k, and f_{km_0} will be the linear transformation which will take 1_t, 1_x, 1_y to t_k, x_k, y_k, respectively. (We note that in §3.2, t_k, x_k, y_k were denoted by $1'_t$, $1'_x$, $1'_y$ respectively.) The reason why we will choose x_k and y_k exactly the way we will do, can be explained (and motivated) in a completely similar style as in the proof of Prop.3.1.21 in §3.1. Such an explanation will be included in the present work at a later stage of its development.

A more intuitive definition of the following choice of x_k and y_k was given around p.181.

First we define the point x_k as the mirror image of t_k w.r.t. the line $\overline{\bar{0}\langle 1,1,0\rangle}$. In more detail: Let $t_k = \langle t_0, t_1, 0\rangle$ then we define $x_k \stackrel{\text{def}}{=} \langle t_1, t_0, 0\rangle$.

Now, y_k is constructed from t_k as indicated in Figure 77. In more detail we project t_k to $\mathsf{LightCone}(\bar{0})$ by a line parallel with the axis $\bar{y}$, obtaining point p.[241] Then we project down point p to $\bar{y}$ along a line parallel with k. This way we obtain y_k. This completes the definition of f_{km_0}. In a more computational style y_k and f_{km_0} are defined as follows.

Let $\lambda \in {}^{+}F$. We will use λ as a parameter. We define $\mathsf{f}_\lambda : {}^3F \longrightarrow {}^3F$ to be the linear transformation for which $\mathsf{f}_\lambda(1_t) = t_k$, $\mathsf{f}_\lambda(1_x) = x_k$, and $\mathsf{f}_\lambda(1_y) = \lambda \cdot 1_y$. Clearly, such an f_λ exists and is unique. It is easy to check that this f_λ is a bijective linear transformation.[242]

[241] This p exists since $\mathfrak{F}$ is Euclidean.
[242] This is so because vectors t_k, x_k, $\lambda \cdot 1_y$ are linearly independent.

Let $\ell \stackrel{\text{def}}{=} \overline{\bar{0}\langle 1,0,1\rangle}$. Let us notice that $\ell \in \mathsf{PhtEucl}$. We claim that $ang^2(\mathsf{f}_\lambda[\ell])$ depends on the choice of λ. If λ is very big (e.g. $\lambda > 100 \cdot |t_k|$), then $ang^2(\mathsf{f}_\lambda[\ell]) > 1$, while for small λ (e.g. $\lambda < |t_k|/100$) $ang^2(\mathsf{f}_\lambda([\ell]) < 1$, the latter is so because $\overline{\bar{0}t_k} = k \in \mathsf{SlowEucl}$. Checking this claim is left to the reader.

Next we use our assumption that $\mathfrak{F}$ is Euclidean, i.e. that "positive" square-roots exist in $\mathfrak{F}$. Namely, we claim that between the two extremes (big and small choices of λ) there exists $\lambda \in {}^+F$ such that

$$ang^2(\mathsf{f}_\lambda[\ell]) = 1\,, \tag{71}$$

because $\lambda \mapsto ang^2(\mathsf{f}_\lambda[\ell])$ is a quadratic polynomial function (and "positive" square-roots exist).[243]

Let this λ be fixed. Now, we define

$$\begin{aligned} y_k &\stackrel{\text{def}}{=} \lambda \cdot 1_y\,, \\ \mathsf{f}_{km_0} &\stackrel{\text{def}}{=} \mathsf{f}_\lambda\,, \end{aligned}$$

for the above choice of λ. Let us notice that condition (71) above was needed because $\ell \in \mathsf{PhtEucl}$ and because of Prop.3.1.17 in §3.1.

By this, f_{km_0} is defined for the case $N_k = \mathrm{Id}$, i.e. f_{km_0} is the bijective linear transformation which takes 1_t, 1_x, 1_y to t_k, x_k, y_k, respectively.

We note that, as we already indicated on p.250, f_{km_0} turns out to be a rhombus transformation (cf. Def.2.3.18 on p.72). To see this it remains to prove that f_{km_0} takes photon-lines to photon-lines. An intuitive, structuralist proof of this was given in §3.2 (for the case when $\mathfrak{F} = \mathfrak{R}$, but the same proof goes through for arbitrary Euclidean $\mathfrak{F}$). A computational proof will be given as item (V) in the proof of Thm.3.5.6 on p.256.

(ii) We define f_{km_0} for $N_k \neq \mathrm{Id}$ as follows.[244]

Let $t'_k \stackrel{\text{def}}{=} N_k[t_k]$. Then $\overline{\bar{0}t'_k} \in \Big(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t},\bar{x})\Big)$ by $N_k(o_k) = \bar{0}$, by $\overline{o_k t_k} = k \in \mathsf{SlowEucl}$, by Lemma 3.5.4, and by $N_k[k] \subseteq \mathsf{Plane}(\bar{t},\bar{x})$.

Let x'_k and y'_k be chosen for t'_k exactly the way as we chose x_k and y_k for t_k in item (i). It can be done because $\overline{\bar{0}t'_k} \in \Big(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t},\bar{x})\Big)$.

[243] As a curiosity we mention that $\lambda = \sqrt{t_0^2 - t_1^2}$ is such, where $t_k = \langle t_0, t_1, 0\rangle$. We will see this in Claim 3.5.7, but it is irrelevant at the present point.

[244] The intuitive idea is the following: First we transform o_k, t_k by N_k into $\mathsf{Plane}(\bar{t},\bar{x})$ (o_k goes to $\bar{0}$). Then we define f_{km_0} to be the composition of a rhombus transformation taking 1_t to $N_k(t_k)$ with N_k^{-1}. This applies also to the cases when $n > 3$.

Let $x_k \stackrel{\text{def}}{=} N_k^{-1}(x'_k)$, and $y_k \stackrel{\text{def}}{=} N_k^{-1}(y'_k)$.

We define f_{km_0} to be the affine transformation which takes $\bar{0}$, 1_t, 1_x, 1_y to o_k, t_k, x_k, y_k, respectively. Clearly such an f_{km_0} exists[245] and is unique.

<u>*Definition of* f_{km_0} *for the case of arbitrary n:*</u>
The definition of f_{km_0} for arbitrary n is analogous to the definition of f_{km_0} for $n = 3$. (We recommend the reader to consult Figure 77 there. In the definition of f_{km_0} for $n = 3$ we gave a more intuitive version based on Figure 77, and a more computational version. Below we will generalize to arbitrary n the computational version. The interested reader is invited to generalize the intuitive picture-oriented version.) Let $n \geq 2$ be arbitrary. Let $k \in \mathsf{SlowEucl}$, $k \neq m_0$ be arbitrary and fixed. Recall again that $P(k) = \langle o_k, t_k, N_k \rangle$. First we will define f_{km_0} for the case when $N_k = \text{Id}$. This will be implemented in item (i) below. After that in item (ii) we will define f_{km_0} for $N_k \neq \text{Id}$.

as was done helyett is done!

(i) We define f_{km_0} for the case $N_k = \text{Id}$ as described below.

We note that f_{km_0} will turn out to be a rhombus transformation and this can be checked exactly as is done in the case of $n = 3$. (It might be a good idea for the reader to look up Def.2.3.18 [Rhombus transformations] before reading on).

$N_k = \text{Id}$ implies that $o_k = \bar{0}$ and $\overline{o_k t_k} = k \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

First we define the point x_k as the mirror image of t_k w.r.t. the line $\overline{\bar{0}\langle 1, 1, 0, \ldots, 0\rangle}$. In more detail: Let $t_k = \langle t_0, t_1, 0, \ldots, 0\rangle$ then we define $x_k \stackrel{\text{def}}{=} \langle t_1, t_0, 0, \ldots, 0\rangle$.

Let $\lambda \in {}^+F$. We will use λ as a parameter. We define $\mathsf{f}_\lambda : {}^nF \longrightarrow {}^nF$ to be the linear transformation for which $\mathsf{f}_\lambda(1_t) = t_k$, $\mathsf{f}_\lambda(1_x) = x_k$, and $\mathsf{f}_\lambda(1_i) = \lambda \cdot 1_i$, for all $i \in n \setminus 2$.[246] Clearly, such an f_λ exists and is unique. It is easy to check that f_λ is a bijective linear transformation.

Let $\ell_i \stackrel{\text{def}}{=} \overline{\bar{0}(1_t + 1_i)}$, for all $i \in n \setminus 2$. Let us notice that $\ell_i \in \mathsf{PhtEucl}$, for all $i \in n \setminus 2$.

It is easy to check that $ang^2(\mathsf{f}_\lambda[\ell_i]) = ang^2(\mathsf{f}_\lambda[\ell_j])$, for all $i, j \in n \setminus 2$.

[245] f_{km_0} exists because vectors $t_k - o_k$, $x_k - o_k$, $y_k - o_k$ are linearly independent (this is so because vectors t'_k, x'_k, y'_k are linearly independent and N_k^{-1} is the affine transformation taking $\bar{0}$, t'_k, x'_k, y'_k to o_k, t_k, x_k, y_k, respectively).

[246] Recall that $n \setminus 2 = \{2, \ldots, n-1\}$, cf. the notation list on p.26.

We claim that $ang^2(\mathsf{f}_\lambda[\ell_i])$ depends on the choice of λ. Now if λ is very big, then $ang^2(\mathsf{f}_\lambda[\ell_i]) > 1$, while for small λ, $ang^2(\mathsf{f}_\lambda[\ell_i]) < 1$, the latter is so because $\overline{\bar{0}t_k} = k \in \mathsf{SlowEucl}$. Checking this claim is left to the reader.

Next we use our assumption that $\mathfrak{F}$ is Euclidean, i.e. that "positive" square-roots exist in $\mathfrak{F}$. Namely, we claim that between the two extremes (big and small choices of λ) there exists $\lambda \in {}^+F$ such that

$$ang^2(\mathsf{f}_\lambda[\ell_i]) = 1\,, \text{ for all } i \in n \setminus 2\,,$$

because $\lambda \mapsto ang^2(\mathsf{f}_\lambda[\ell])$ is a quadratic polynomial function (and "positive" square-roots exist). Let this λ be fixed.

Now, we define

$$\begin{aligned} x_{k,i} &\stackrel{\text{def}}{=} \lambda \cdot 1_i\,, \text{ and}^{247} \\ \mathsf{f}_{km_0} &\stackrel{\text{def}}{=} \mathsf{f}_\lambda\,, \end{aligned}$$

for the above choice of λ, and for all $i \in n \setminus 2$.

By this, f_{km_0} is defined for the case $N_k = \mathrm{Id}$, i.e. f_{km_0} is the bijective linear transformation which takes $1_t,\ 1_x,\ 1_2, \ldots, 1_{n-1}$ to $t_k,\ x_k,\ x_{k,2}, \ldots, x_{k,n-1}$, respectively.

(ii) The definition of f_{km_0} for the case $N_k \neq \mathrm{Id}$ is obtained from item (i) in a completely analogous way as we did this for the case $n = 3$. In more detail:

Let $t'_k \stackrel{\text{def}}{=} N_k[t_k]$. Then $\overline{\bar{0}t'_k} \in \left(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x})\right)$ by $N_k(o_k) = \bar{0}$, by $\overline{o_k t_k} = k \in \mathsf{SlowEucl}$, by Lemma 3.5.4, and by $N_k[k] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

Let $x'_k,\ x'_{k,i}$ $(i \in n \setminus 2)$ be obtained from t'_k exactly as $x_k,\ x_{k,i}$ $(i \in n \setminus 2)$ were obtained from t_k in item (i). This can be done because $\overline{\bar{0}t'_k} \in \left(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x})\right)$. Then let $x_k \stackrel{\text{def}}{=} N_k^{-1}(x'_k)$ and $x_{k,i} \stackrel{\text{def}}{=} N_k^{-1}(x'_{k,i})$. Then f_{km_0} is defined to be the affine transformation which takes $\bar{0},\ 1_t,\ 1_x,\ 1_2, \ldots, 1_{n-1}$ to $o_k,\ t_k,\ x_k,\ x_{k,2}, \ldots, x_{k,n-1}$, respectively.

<u>*Definition of W:*</u>

By the above, f_{km_0} is defined for all $k \in \mathsf{SlowEucl}$, $k \neq m_0$. Recall that w_{m_0} was defined below the definition of m_0 at the beginning of Def.3.5.5. We now define

$w_k \stackrel{\text{def}}{=} \mathsf{f}_{km_0} \circ w_{m_0}$, for all $k \in Obs \setminus \{m_0\}$, and

[247] $x_{k,i}$ will be denoted as 1_i^k e.g. on p.325 (above Def.3.8.38).

$W \stackrel{\text{def}}{=} \{ \langle m,p,h\rangle \ : \ m \in Obs, h \in w_m(p) \}$.

By this, the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}} \stackrel{\text{def}}{=} \langle B, \dots, W\rangle$ has been defined.

For fixed $n \geq 2$, the class of the above defined models is called the class of *simple models*, and we denote this class by SM.
END OF DEF. OF SM.

$\lhd$

THEOREM 3.5.6 $\mathsf{SM} \models$ **Basax**.

Proof: We will give the proof for the case $n = 3$. The proof for arbitrary n is an easy generalization of the present one, and is left to the reader. (The proof for $n = 2$ is obtainable from that for $n = 3$ in the obvious way.) The organization of the proof will be analogous with that of a similar proof given for $n = 2$ for Thm.2.4.1 in the second part of §2.4 ("Models for **Basax** in dimension 2").

Let $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}}$ be such that $\mathfrak{F}$ is Euclidean. Recall that $P(k) = \langle o_k, t_k, N_k\rangle$ (cf. the beginning of Def.3.5.5). We have already observed that $\mathfrak{M} \models$ **Ax1**, **Ax2**, and that **Ax3**, **Ax4**, **Ax5**, **AxE** hold for the fixed observer $m_0 \in Obs$ (cf. the first two pages of Def.3.5.5, above "Definition of f_{km_0}"). Let $k \in Obs \setminus \{m_0\}$ be arbitrary but fixed.

We will prove that (I)-(V) hold for f_{km_0}:

(I) $\mathsf{f}_{km_0} : {}^3F \longrightarrow {}^3F$ is a bijection.

(II) $\mathsf{f}_{km_0}[\ell] \in \mathsf{Eucl}$, for all $\ell \in \mathsf{Eucl}$.

(III) $\mathsf{f}_{km_0}[\bar{t}\,] = k$.

(IV) $\mathsf{f}_{km_0}[\ell] \in Ph$ iff $\ell \in Ph$, for all $\ell \in \mathsf{Eucl}$.

(V) $\mathsf{f}_{km_0}[\ell] \in Obs$, for all $\ell \in Obs$.

Indeed, (I)-(II) hold because f_{km_0} is defined to be an affine transformation. (III) holds because of (II) and because we defined f_{km_0} to take $\bar{0}$, 1_t, respectively, to o_k, t_k, and $k = \overline{o_k t_k}$. We will prove (IV) and (V) at the end of the proof.

Now, in $\mathfrak{M}$ we have for all $\ell \in B$ that

$$\ell = tr_k(\mathsf{f}_{km_0}[\ell]) \, . \qquad (72)$$

(The proof of (72) is exactly like in §2.4, in the proof of Thm.2.4.1.) Since f_{km_0} is a bijection, by (II) we have that both f_{km_0} and $\mathsf{f}_{km_0}^{-1}$ preserve Eucl. Using this, together with (II)-(V), (72), and the fact that **Ax3**, **Ax4**, **Ax5**, **AxE** hold for $m_0 \stackrel{\text{def}}{=} \bar{t}$, we get that **Ax3**, **Ax4**, **Ax5**, **AxE** hold for k, too. From (I) and from the definition[248] of w_k we get that $Rng(w_k) = Rng(w_{m_0})$. Since k was arbitrary, this proves $\mathfrak{M} \models$ **Basax**.

Thm.3.5.6 is proved modulo (IV) and (V) above. Now we turn to prove these.

Proof of (IV): For $n = 3$ an intuitive, structuralist proof is in §3.2. The computational proof included below is easily generalized to arbitrary n.

To prove (IV), by $Ph = \mathsf{PhtEucl}$ it is sufficient to prove (73) below.

$$(73) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\ell \in \mathsf{PhtEucl} \;\Leftrightarrow\; \mathsf{f}_{km_0}[\ell] \in \mathsf{PhtEucl}\Big).$$

Next we turn to prove (73). The proof of (73) will consist of two cases: (i) $N_k = \mathrm{Id}$, (ii) $N_k \neq \mathrm{Id}$.

Proof of (73) for case $N_k = \mathrm{Id}$:
Let us recall that for this case f_{km_0} was defined in item (i) of the "definition of f_{km_0} for the case $n = 3$". Let us recall that in that definition we have that

$$(74) \qquad o_k = \bar{0} \quad \text{and} \quad \overline{o_k t_k} = k \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \,,$$

and f_{km_0} is the linear transformation which takes 1_t, 1_x, 1_y, respectively, to $t_k = \langle t_0, t_1, 0\rangle$, $x_k = \langle t_1, t_0, 0\rangle$, $y_k = \langle 0, 0, \lambda\rangle$, for fixed $\lambda \in {}^+F$, where λ was fixed in such a way that

$$ang^2(\mathsf{f}_{km_0}[\ell]) = 1, \text{ where } \ell \stackrel{\text{def}}{=} \overline{\bar{0}\langle 1, 0, 1\rangle} \,.$$

Claim 3.5.7 $t_0^2 - t_1^2 > 0$ and $\lambda = \sqrt{t_0^2 - t_1^2}$.

Proof of Claim 3.5.7: We have $t_0^2 - t_1^2 > 0$ because $t_k = \langle t_0, t_1, 0\rangle$, because $k \in Obs \stackrel{\text{def}}{=} \mathsf{SlowEucl}$ and because by (74), $\overline{\bar{0}t_k} = k$.

By f_{km_0} being a linear transformation taking 1_t, 1_x, 1_y to t_k, x_k, y_k, respectively, it is easy to see that $\mathsf{f}_{km_0}[\ell] = \overline{\bar{0}\langle t_0, t_1, \lambda\rangle}$. Now by this

$$ang^2(\mathsf{f}_{km_0}[\ell]) = 1 \quad \Leftrightarrow \quad \frac{t_1^2 + \lambda^2}{t_0^2} = 1 \,.$$

By this and by $\lambda \in {}^+F$, we have

$$ang^2(\mathsf{f}_{km_0}[\ell]) = 1 \quad \Leftrightarrow \quad \lambda = \sqrt{t_0^2 - t_1^2} \,;$$

[248] Recall that $w_k = \mathsf{f}_{km_0} \circ w_{m_0}$.

where $\sqrt{t_0^2-t_1^2}$ exists because $t_0^2-t_1^2>0$ and $\mathfrak{F}$ is Euclidean. This proves Claim 3.5.7.
QED (Claim 3.5.7)

We have that f_{km_0} takes parallel lines to parallel lines because f_{km_0} is a linear transformation. Hence to show (73) it is enough to show (75) below.

$$(75)\qquad (\forall \ell \in \mathsf{Eucl})\Big(\bar{0}\in\ell \ \Rightarrow\ (\ell\in\mathsf{PhtEucl}\ \Leftrightarrow\ \mathsf{f}_{km_0}[\ell]\in\mathsf{PhtEucl}\Big).$$

To show (75) let $\ell\in\mathsf{Eucl}$ with $\bar{0}\in\ell$. Without loss of generality we may assume that $\ell=\overline{\bar{0}\langle 1,a,d\rangle}$, for some $a,d\in F$. Let these a,d be fixed. Now by using Claim 3.5.7, we have that f_{km_0} is the linear transformation taking 1_t, 1_x, 1_y to $t_k=\langle t_0,t_1,0\rangle$, $x_k=\langle t_1,t_0,0\rangle$, $y_k=\langle 0,0,\sqrt{t_0^2-t_1^2}\rangle$, respectively ($t_0^2-t_1^2>0$). By this, it is easy to see that

$$\mathsf{f}_{km_0}(1,a,d)=\langle t_0+at_1,\ t_1+at_0,\ d\sqrt{t_0^2-t_1^2}\ \rangle,\ \text{hence}$$

$$(76)\qquad \mathsf{f}_{km_0}[\ell]=\overline{\bar{0}\ \langle t_0+at_1,\ t_1+at_0,\ d\sqrt{t_0^2-t_1^2}\ \rangle}\,.$$

Now

$$\begin{aligned}
ang^2(\mathsf{f}_{km_0}[\ell]) &= \frac{(t_1+at_0)^2+\left(d\sqrt{t_0^2-t_1^2}\right)^2}{(t_0+at_1)^2} && \text{(by (76))}\\
&= \frac{(t_0+at_1)^2+(a^2+d^2-1)(t_0^2-t_1^2)}{(t_0+at_1)^2} && \text{(by some computation).}
\end{aligned}$$

Now

$$\begin{aligned}
ang^2(\mathsf{f}_{km_0}[\ell]))=1 &\quad\Leftrightarrow\\
(a^2+d^2-1)(t_0^2-t_1^2)=0 &\quad\Leftrightarrow\quad \text{(by } t_0^2-t_1^2>0)\\
a^2+d^2=1 &\quad\Leftrightarrow\quad \text{(by def. of } a,d)\\
ang^2(\ell)=1.&
\end{aligned}$$

By the above computation, we have $\mathsf{f}_{km_0}[\ell]\in\mathsf{PhtEucl}$ iff $\ell\in\mathsf{PhtEucl}$. By this, (73) is proved for case $N_k=\mathrm{Id}$.

Proof of (73) for case $N_k\neq\mathrm{Id}$*:*
Let us recall that for this case f_{km_0} was defined in item (ii) of the "Definition of f_{km_0} for the case $n=3$". Let t'_k, x'_k, y'_k be exactly those points which were defined there and let f_{km_0} be the affine transformation which was defined there (in item (ii) of the "Definition of f_{km_0} for the case $n=3$"). Now, we define f'_{km_0} to be the linear transformation which takes 1_t, 1_x, 1_y to t'_k, x'_k, y'_k, respectively.

Now we can prove that f'_{km_0} satisfies condition (73) above in a completely analogous way as we did for case $N_k = \mathrm{Id}$ for f_{km_0} there (to see the analogy, consider the definition of x'_k, y'_k in line 4 of item (ii)).

It is easy to check that $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$. Now by f'_{km_0} satisfying condition (73) above, by $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$ and by Lemma 3.5.4, we have that f_{km_0} satisfies (73). We proved (73) for case $N_k \neq \mathrm{Id}$, too. This completes the proof of (IV).

Proof of (V): As we indicated at the beginning, we want the proof of Thm.3.5.6 to go through for arbitrary $2 < n \leq 4$. For $n = 3$, an intuitive, structuralist proof of (V) can be recovered from the proof in §3.2 ("Intuitive ... **Basax**(3)"). That intuitive, visualizable proof can be generalized for $n = 4$ too. As we indicated, in the present work we prefer intuitive, visualizable (structuralist) proofs to computational ones. Despite of this, for lack of time we include below a computational proof for (V) which is easily generalizable to arbitrary $n > 2$. In a later version we plan to replace it with a short, structuralist one like the one in §3.2.

To prove (V), by $Obs = \mathsf{SlowEucl}$ it is sufficient to prove (77) below.

$$(\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}\,. \tag{77}$$

Next we turn to prove (77). The proof of (77) will be similar to that of (73) in the proof of (IV). The proof again will consist of two cases.

Proof of (77) for case $N_k = \mathrm{Id}$:
As we have shown at the beginning of the proof of (IV) (cf. Claim 3.5.7), for this case we have that f_{km_0} is a linear transformation which takes 1_t, 1_x, 1_y to $t_k = \langle t_0, t_1, 0\rangle$, $x_k = \langle t_1, t_0, 0\rangle$, $y_k = \langle 0, 0, \sqrt{t_0^2 - t_1^2}\rangle$, respectively ($t_0^2 - t_1^2 > 0$).

By f_{km_0} being a linear transformation, we have that f_{km_0} takes parallel lines to parallel lines. Thus to prove (77) above it is enough to prove

$$(\forall \ell \in \mathsf{SlowEucl})\Big(\bar{0} \in \ell \ \Rightarrow\ \mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}\Big)\,.$$

To see this, let $\ell \in \mathsf{SlowEucl}$ with $\bar{0} \in \ell$. Then it is easy to see that $\ell = \overline{\bar{0}\langle 1, a, d\rangle}$, for some $a, d \in F$ with $a^2 + d^2 < 1$. Let these a, d be fixed. Now it is easy to see that

$$\mathsf{f}_{km_0}[\ell] = \overline{\bar{0}\ \langle t_0 + at_1,\ t_1 + at_0,\ d\sqrt{t_0^2 - t_1^2}\ \rangle}\,. \tag{78}$$

Now

$$
\begin{aligned}
ang^2(\mathsf{f}_{km_0}[\ell]) \quad &= \quad \frac{(t_1+at_0)^2+\left(d\sqrt{t_0^2-t_1^2}\right)^2}{(t_0+at_1)^2} && \text{(by(78))}\\
&= \quad \frac{(t_0+at_1)^2+(a^2+d^2-1)(t_0^2-t_1^2)}{(t_0+at_1)^2} && \text{(by some computation)}\\
&< \quad \frac{(t_0+at_1)^2}{(t_0+at_1)^2} && \text{(by } a^2+d^2<1 \text{ and } t_0^2-t_1^2>0)\\
&= \quad 1\,.
\end{aligned}
$$

If we summarize the above computation we get $ang^2(\mathsf{f}_{km_0}[\ell]) < 1$, hence $\mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}$. By this, (77) is proved for case $N_k = \mathrm{Id}$.

Proof of (77) for case $N_k \neq \mathrm{Id}$*:*
The proof of this will be analogous with that of (73) in the proof of (IV). Let us recall that for this case f_{km_0} was defined in item (ii) of the "Definition of f_{km_0} for the case $n = 3$". Let t'_k, x'_k, y'_k be exactly those points which were defined there and let f_{km_0} be the affine transformation which was defined there (in item (ii) of the "Definition of f_{km_0} for the case $n = 3$"). Now we define f'_{km_0} to be the linear transformation which takes 1_t, 1_x, 1_y to t'_k, x'_k, y'_k, respectively.

Now we can prove that f'_{km_0} satisfies condition (77) above in a completely analogous way as we did for case $N_k = \mathrm{Id}$ for f_{km_0} there (to see the analogy, consider the definition of x'_k, y'_k in line 4 of item (ii)).

It is easy to check that $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$. Now by f'_{km_0} satisfying condition (77) above, by $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$ and by Lemma 3.5.4 we have that f_{km_0} satisfies (77). We proved (77) for case $N_k \neq \mathrm{Id}$, too. This completes the proof of (V). By this, *Thm.3.5.6 is proved.* ∎

Proof of Lemma 3.5.3:
Definition: Assume $\mathfrak{F}$ is Euclidean. By a *congruence* transformation $\mathsf{h} : {}^nF \longrightarrow {}^nF$ we understand an affine transformation which preserves Euclidean distances, i.e. $(\forall p, q \in {}^nF)\ |\mathsf{h}(p) - \mathsf{h}(q)| = |p - q|$. We note that in the present proof we will use such transformations which preserve $\bar{0}$.

Let us turn to proving Lemma 3.5.3.

(1) Without loss of generality we may assume $p = \bar{0}$.

(2) The assumption that $\mathfrak{F}$ is Euclidean (i.e. "positive" square-roots exist) is essential in proving this lemma (it is not true without this assumption).

(3) We need a proof only for $n \leq 4$ because we mentioned earlier in this work that we treat $n > 4$ only if no extra effort is needed for that. (All the same, the present lemma is true for arbitrary n).

Sub-lemma: Let $q \in {}^jF$, $j \in \omega$ arbitrary, and let $\mathfrak{F}$ be Euclidean. Then there is a *congruence* transformation $\mathsf{f} : {}^jF \longrightarrow {}^jF$ with $\mathsf{f}(\bar{0}) = \bar{0}$ and $\mathsf{f}(q) \in \bar{x}$.

Proof of Sub-lemma:

We prove this only for $j \leq 3$ because that will be sufficient for $n \leq 4$, cf. item (3) above. Let $j = 3$. Throughout the proof of Sub-lemma the reader is asked to consult Figure 78.

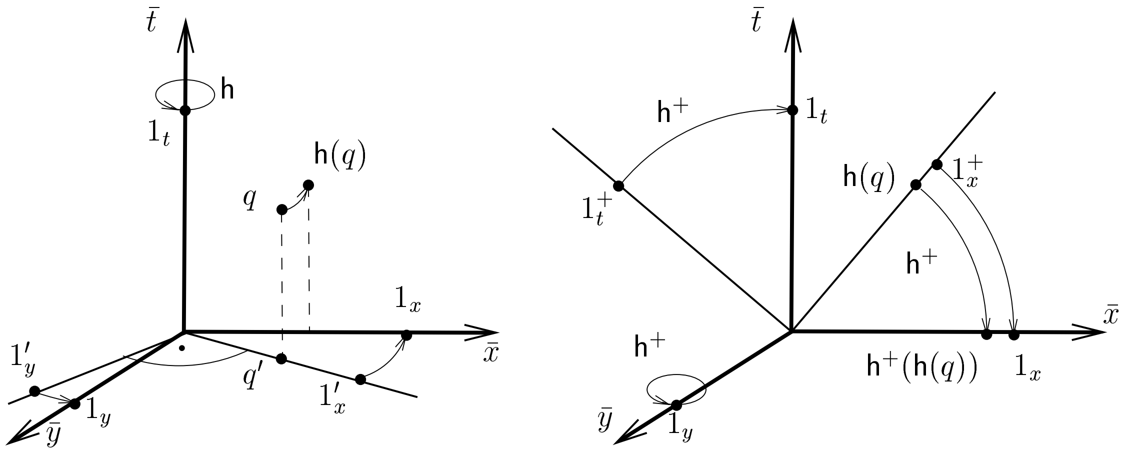


Figure 78: Illustration for the proof of Sub-lemma.

Let $q' \stackrel{\text{def}}{=} \langle 0, q_1, q_2 \rangle$ and $\lambda \in F$ be such that $1'_x \stackrel{\text{def}}{=} \lambda \cdot q'$ is of length 1, i.e. $|1'_x| = 1$. Such a λ exists because $\mathfrak{F}$ is Euclidean.

Let $1'_y = \langle 0, a, d \rangle$ be arbitrary but orthogonal (in the Euclidean sense) to $1'_x$ with $|1'_y| = 1$. This obviously exists.

Let h be the linear trasformation defined by $1_t, 1'_x, 1'_y \mapsto 1_t, 1_x, 1_y$. By the choice of $1'_x, 1'_y$, h exists and is bijective. Further, $\mathsf{h}(q) \in \mathsf{Plane}(\bar{t}, \bar{x})$.

By a completely similar argument, there is another bijective linear transformation h^+ with $\mathsf{h}^+(\mathsf{h}(q)) \in \bar{x}$. But then $\mathsf{f} = \mathsf{h} \circ \mathsf{h}^+$ has the desired properties. Further f is a *congruence* transformation because $1'_x, 1'_y$ had lenght 1, $1'_x \perp_e 1'_y$ etc.

The proof for $j < 3$ is obtained from the above one the obvious way.

END of Proof of Sub-lemma.

Let us turn to proving Lemma 3.5.3. Throughout the proof the reader is asked to consult Figure 79.

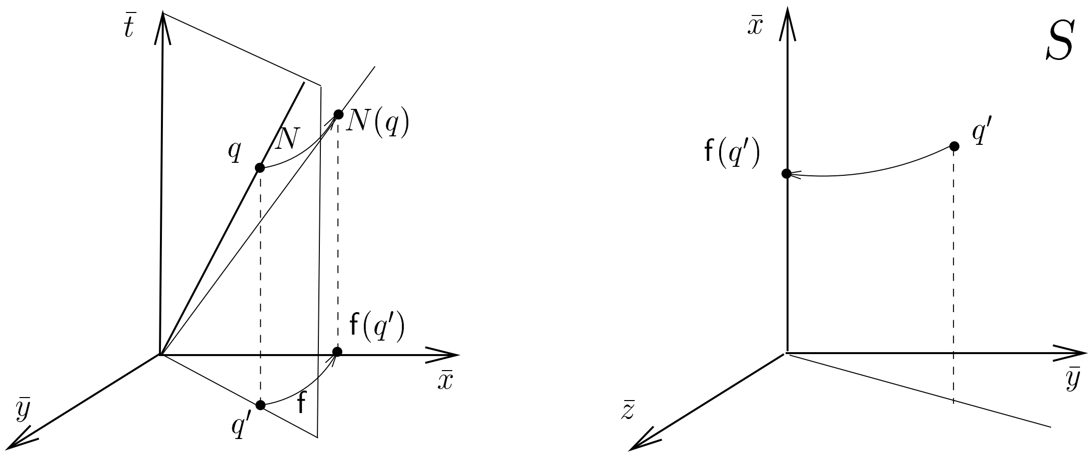


Figure 79: Illustration for the proof of Lemma 3.5.3.

In the present proof by the space part S of our coordinate system ${}^{n}F$ we understand ${}^{n-1}F$ (while in the rest of this work S denotes $\{0\} \times {}^{n-1}F$).

Let $\ell \in \mathsf{Eucl}$, $\bar{0} \in \ell$ and $\bar{0} \neq q \in \ell$. Let $q' = \langle q_1, \ldots, q_{n-1} \rangle \in S$.

Clearly, any congruence transformation $\mathsf{f} : S \longrightarrow S$ of S preserving $\bar{0}$ induces an $N \in \mathit{Triv}_0$ as follows.

$$N(p_0, \ldots, p_{n-1}) \stackrel{\text{def}}{=} \langle p_0, \mathsf{f}(p_1, \ldots, p_{n-1}) \rangle .$$

By Sub-lemma (applied to q', S in place of $q, {}^{j}F$), there is $\mathsf{f} : S \longrightarrow S$ with $\langle 0, \mathsf{f}(q_1, \ldots, q_{n-1}) \rangle \in \bar{x}$.

The "N" induced by this f has the desired properties.

This completes the proof of Lemma 3.5.3 for $n \leq 4$, because we proved Sub-lemma only for $j \leq 3$.

The generalization for $n > 4$ goes by proving Sub-lemma for arbitrary j. This can be done via a straightforward induction. We omit it for the already indicated reasons. ∎

3.6 Models of Basax

pezsgo után szétválasztani a gyökvonásos részt a modellektől, külön fejezetbe!

The purpose of the present section is to see[249] what the models of **Basax**(n) can look like.[250] We understand this goal in the spirit of the three drawings in Figure 29 (p.88) representing $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ (as typical models of **Basax**(2)).

In this section we will characterize the models of **Basax**(3) and **Basax**(n) + **Ax**$(\sqrt{\ })$, for $n \geq 3$. We will construct a class of frame models as concrete[251] structures which we call the class of general models, in symbols GM, and we will show that $\mathsf{GM} \models$ **Basax**. Let $\mathsf{GM}(n)$ denote the n–dimensional version of the class GM of models to be defined in this section. We will show that the class Mod(**Basax**(3)) coincides with $\mathsf{GM}(3)$; and more generally the class of models of **Basax**(n) + **Ax**$(\sqrt{\ })$ coincides with $\mathsf{GM}(n)$, for all $n > 2$. Mod(**Basax**(2)) was characterized in an earlier version [25] of this work, but cf. also §§ 2.3, 2.4 in particular Thm.2.3.12 of the present work. Concerning the $n = 2$ case, we note that $\mathsf{GM}(2) \subsetneq \mathsf{Mod}$(**Basax**$(2)$ + **Ax**$(\sqrt{\ })$) e.g. because of §2.7. On the other hand $\mathsf{GM}(2) = \mathsf{Mod}$(**Basax**$(2)$ + **Ax**$(\sqrt{\ })$ + **"$\nexists$ FTL observers"**).

We will also introduce an axiom **Ax7** which is more natural from the physical point of view than **Ax**$(\sqrt{\ })$ and which implies **Ax**$(\sqrt{\ })$, more precisely we will prove that **Basax**(n) + **Ax7** $\models$ **Ax**$(\sqrt{\ })$, for $n > 2$. We will also prove **Basax**(3) $\models$ **Ax**$(\sqrt{\ })$. The generalization of this to arbitrary $n > 3$ remains an

[249] I.e. to develop an understanding and ability to visualize and gain ***insights*** into the possible structures of the models in question.

[250] At the final reading, some parts of the present work seem to be ***unnecessarily computational*** for the authors. An example is the present section ("Models of **Basax**") whose purpose should be to help the reader imagine, visualize, and see in a simple and clear way all the various kinds of models compatible with **Basax**(n). Therefore we feel a strong temptation to rewrite the present section in the structuralist (or "visualist") style of the pictures on p.88 and of §3.2 ("Intuitive … **Basax**(3)"). We know that it ***is*** possible to write up a definition of $\mathsf{GM}(n)$ which is transparent, suggestive, intuitive and simple. However, we also know that if we keep on re-writing the least satisfactory part of this work then it will never be finished. Therefore we postpone writing up a more transparent and more visual definition of $\mathsf{GM}(n)$ to a future time. In the meantime, the interested reader is invited to write up his intuitive and more "pictures oriented" version.

[251] By "concrete" we mean that they are "set theoretically built up" i.e. "constructively defined" in the style of §2.4 pp. 80–84 or in the style of the definition of SM in §3.5. In passing, we note that an example of an abstract class of structures is the abstractly given class BA of Boolean algebras defined as complemented distributive lattices (i.e. they are defined by a set of axioms). The "matching" example for a concrete class of structures is the class Sba of Boolean set-algebras where the elements of an algebra are sets and the operations are the set theoretic intersection and complementation. Then the representation theorem of Boolean algebras says $\mathsf{BA} = \mathbf{I}\mathsf{Sba}$. We will return to representation theorems in §6 ("Observer independent geometry").

open question. The usefulness of these considerations about **Ax7** comes from the fact that we have several results using $\mathbf{Ax}(\sqrt{\ })$ and now in those one can replace $\mathbf{Ax}(\sqrt{\ })$ with a physically more natural **Ax7**.

The theorems saying, $\mathsf{Mod}(\mathbf{Basax}(3)) = \mathsf{GM}(3)$ etc. provide a kind of characterization of the world-view transformations, the f_{mk}'s. A different, more direct characterization of the f_{mk}'s will be given in a separate theorem (analogous with the one in §2.3).

The theorem saying, $\mathsf{GM}(n) \models \mathbf{Basax}(n)$ will imply that there are many different consistent extensions of **Basax**, which was one of our aims discussed in the Introduction (§1.1.(X)). The application of (our investigating) $\mathsf{GM}(n)$ to some of the "philosophical" goals formulated in the Introduction of the present work will be discussed in Remark 3.6.15 (pp. 271–273).

Indirectly we will also obtain a characterization of the models of **Newbasax**. Namely our characterization theorem (Thm.3.6.13) of the models of **Basax** will give a characterization of the models of **Newbasax** too via Thm.3.3.12 connecting models of **Newbasax** to models of **Basax**.

Below we formulate potential axiom **Ax7**.

Ax7 $(\forall m \in Obs)(\forall \ell \in \mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x}))(\exists k \in Obs)$
$$(tr_m(k) = \ell \ \wedge \ \mathsf{f}_{km}[\bar{y}] \parallel \bar{y})\,.$$

See Figure 80.

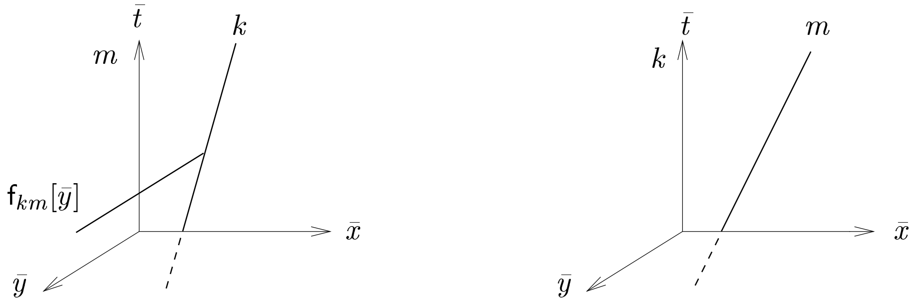


Figure 80: Illustration for **Ax7**.

Intuitively, **Ax7** says that there are observers moving in direction $\bar{x}$ whose $\bar{y}$ axis remains parallel with the "original" $\bar{y}$ axis. This assumption is almost always taken for granted in physics books (cf. e.g. Rindler [224]). Indeed, it sounds contrary to experience to assume that for some speed $v < 1$, if an observer k is moving with speed v in direction $\bar{x}$ then some "magical force" would force k to point his $\bar{y}$ axis in some direction different from the original $\bar{y}$ axis.

Summing up, we consider **Ax7** as a relatively weak and natural ("physically convincing") assumption. Actually we tend a feel that **Ax7** is more natural (in some sense) than e.g. $\mathbf{Ax}(\sqrt{\ })$.

Remark 3.6.1 We note that $\mathsf{SM} \not\models \mathbf{Ax7}$, where SM is defined in Def.3.5.5 in §3.5. However, we will see that the models of SM can be extended to richer models validating **Ax7** (cf. Prop.3.6.18).

$\lhd$

In what follows we will introduce the set PT of so called <u>*photon-preserving affine transformations*</u> which will be a subset of the set of all affine transformations. To motivate this definition we recall Thm.3.1.4 and Prop.3.1.17 from §3.1.

Thm.3.1.4 $\quad \mathbf{Basax} \models (\mathsf{f}_{mk} = \widetilde{\varphi} \circ \mathsf{f}, \quad \text{for some } \mathsf{f} \in Aftr \text{ and } \varphi \in Aut(\mathbf{F})).$

Prop.3.1.17 $\quad \mathbf{Basax} \models (\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \Leftrightarrow \mathsf{f}_{mk}[\ell] \in \mathsf{PhtEucl}).$

By these two, we also have that the "f" occurring in Thm.3.1.4 preserves $\mathsf{PhtEucl}$ in both directions. Next, we will collect these f's into something called PT.

Definition 3.6.2 The set of <u>*photon-preserving affine transformations*</u>, in symbols $PT = PT(n, \mathfrak{F})$, is defined as follows.

$$PT \stackrel{\text{def}}{=} \{\, \mathsf{f} \in Aftr \ : \ (\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \Leftrightarrow \mathsf{f}[\ell] \in \mathsf{PhtEucl})\,\} .^{252}$$

$\lhd$

[252]Perhaps it would be more didactic to define PT_0 to be the set of all photon-preserving transformations. Then we would define $PT = PT_0 \cap Aftr$. To keep the present work simple we do not do this. In §6.7 we will mention the Alexandrov-Zeeman theorem a kind of generalization of which are items 6.7.20–6.7.35 (cf. also Goldblatt [108, Appendix B]). By items 6.7.34 (p.1138) and 3.1.6 (p.163) if $\mathfrak{F}$ is Euclidean and $n > 2$ then PT_0 becomes superfluous in the following sense: $PT_0 = \{\, \mathsf{f} \circ \widetilde{\varphi} \ : \ \mathsf{f} \in PT \ \wedge \ \varphi \in Aut(\mathfrak{F})\}$.

Remark 3.6.3

(i) $\langle PT, \circ, ^{-1}, \mathrm{Id}\rangle$ is a group.

(ii) The distinguished classes $Rhomb, Lor, SLor, Poi, Exp, Tran, Triv_0, Triv$ of transformations introduced so far are all contained in PT. For this as well as for their relationships we refer to Lemma 3.7.1 (p.283). All of these classes are groups (w.r.t. $\circ$, $^{-1}$, Id).

$\triangleleft$

The set of rhombus transformations and the set of Poincaré transformations were defined in Def.2.3.18 (p.72) and in Def.2.9.1 (p.152), respectively. The just defined photon-preserving affine transformations are strongly related to rhombus and Poincaré transformations. In the next theorem we make this relationship explicit. Item (i) of the next theorem says that PT-transformations are basically the same as rhombus transformations modulo trivial transformations. More precisely PT transformations are rhombus ones composed with trivial ones.

THEOREM 3.6.4 *Assume $\mathfrak{F}$ is Euclidean. Then (i) and (ii) below hold.*

(i) $PT =$
$$\{\, triv_0 \circ rhomb \circ triv \ :\ triv_0 \in Triv_0 \ \wedge\ rhomb \in Rhomb \ \wedge\ triv \in Triv \,\}.$$

(ii) $PT = \{\, poi \circ \exp \ :\ poi \in Poi \ \wedge\ \exp \in Exp \,\}$.

The **proof** of item (i) will be given on p.277, and the proof of item (ii) will be given in §3.7 on p.283.

The next two propositions belong to the motivation of Def.3.6.2 above. Thm.3.6.9 below them describes how the elements of PT look like. Prop.3.6.5(ii) below also serves as a motivation for the definition of GM (Def.3.6.11 way below).

Intuitively, the following proposition says that f_{mk} can be written both in the form $\mathsf{f}\circ\widetilde{\varphi}$ and $\widetilde{\varphi}\circ\mathsf{f}$ for some $\mathsf{f} \in PT$ and $\varphi \in Aut(\mathbf{F})$. Actually the following is true.

<u>*Fact:*</u> Assume **Basax**. The f_{mk}'s can be written in the form $\mathsf{f}\circ\widetilde{\varphi}$ iff f_{mk}'s can be written in the form $\widetilde{\varphi}\circ\mathsf{f}$.

<u>*Proof:*</u> Assume the f_{mk}'s are of the form $\mathsf{f}\circ\widetilde{\varphi}$. Then $\mathsf{f}_{km} = \mathsf{f}_{mk}^{-1} = (\mathsf{f}\circ\widetilde{\varphi})^{-1} = \widetilde{\varphi}^{-1}\circ\mathsf{f}^{-1} = \widetilde{\varphi'}\circ\mathsf{f}'$ with $\varphi' \in Aut(\mathbf{F})$ and $\mathsf{f}' \in PT$. (Cf. also Remark 3.6.3.)

PROPOSITION 3.6.5

(i) **Basax** $\models (\forall m,k \in Obs)\Big(\mathsf{f}_{mk} = \widetilde{\varphi}\circ\mathsf{f},$ *for some* $\mathsf{f} \in PT$ *with* $\mathsf{f}[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathbf{F})\Big)$.
The converse statement

$$(\mathsf{f}_{mk} = \mathsf{f}\circ\widetilde{\varphi}, \quad \textit{for some } \mathsf{f} \in PT \textit{ and } \varphi \in Aut(\mathbf{F}))$$

is also true in all **Basax** *models.*

(ii) **Basax** + $\mathbf{Ax}(\sqrt{\ })$ $\models (\forall m,k \in Obs)\Big(\mathsf{f}_{mk} = \widetilde{\varphi}\circ\mathsf{f},$ *for some* $\mathsf{f} \in PT$ *with* $\mathsf{f}[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathfrak{F})\Big)$.
The converse statement

$$(\mathsf{f}_{mk} = \mathsf{f}\circ\widetilde{\varphi}, \quad \textit{for some } \mathsf{f} \in PT \textit{ and } \varphi \in Aut(\mathfrak{F}))$$

is also true in all **Basax** + $\mathbf{Ax}(\sqrt{\ })$ *models.*

Proof: Item (i) follows by Thm.3.1.4 (p.162), by **Ax4**, by Prop.2.3.3(vii),(ix), and by the following property (which follows by Lemma 3.6.22(iii) way below):

$$(\forall\varphi \in Aut(\mathbf{F}))(\forall\ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \Leftrightarrow \widetilde{\varphi}[\ell] \in \mathsf{PhtEucl})\,.$$

Item (ii) is a corollary of item (i) and Remark 3.6.7 below. ∎

The above proposition says that the f_{mk}'s can be obtained in a certain form, under some assumptions. Actually the other direction of this statement holds too. This way we will obtain a characterization theorem of the world-view transformations (i.e. of the f_{mk}'s), see Thm.3.6.16 on p.273.

The emphasis in Prop.3.6.5(ii) above and Prop.3.6.6 below is on φ being order preserving, i.e. on writing $\mathfrak{F}$ in place of $\mathbf{F}$.

PROPOSITION 3.6.6 *Let* $n \geq 3$. *Then (i), (ii) below hold.*

(i) **Basax**(n) + **Ax7** $\models (\forall m,k \in Obs)\Big(\mathsf{f}_{mk} = \widetilde{\varphi}\circ\mathsf{f},$ *for some* $\mathsf{f} \in PT$ *with* $\mathsf{f}[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathfrak{F})\Big)$.

(ii) **Basax**(3) $\models (\forall m,k \in Obs)\Big(\mathsf{f}_{mk} = \widetilde{\varphi}\circ\mathsf{f},$ *for some* $\mathsf{f} \in PT$ *with* $\mathsf{f}[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathfrak{F})\Big)$.

Proof: This proposition is a corollary of Prop.3.6.5(ii) above and Thm.3.6.17 way below which says that both **Basax**(3) and **Basax**(n) + **Ax7** for $n \geq 3$, implies **Ax**($\sqrt{\ }$). ∎

Remark 3.6.7 Assume $\mathfrak{F}$ is Euclidean, i.e. $\mathfrak{F} \models \mathbf{Ax}(\sqrt{\ })$. Then $Aut(\mathfrak{F}) = Aut(\mathbf{F})$ because of the following. $Aut(\mathfrak{F}) \subseteq Aut(\mathbf{F})$ is obvious. To prove the other inclusion let $\varphi \in Aut(\mathbf{F})$ and let $0 \leq x \in F$. By $\mathbf{Ax}(\sqrt{\ })$, we have $x = y^2$, for some $y \in F$. Let this y be fixed. Now $\varphi(x) = \varphi(y^2) = \varphi(y)^2 \geq 0$. Hence φ preserves the property of being positive, therefore φ is order preserving, i.e. $\varphi \in Aut(\mathfrak{F})$.

$\lhd$

The definition of the class GM of general models will consist of two steps. First we define certain distinguished transformations called the T_{pq}'s (Def.3.6.8), and then using this we will define GM if Def.3.6.11. More on the intuitive structure of the definition of general models can be found imediately above Def.3.6.11.

In Def.3.6.8 below, to each $p, q \in {}^nF$ we associate a PT-transformation T_{pq} such that it maps $\bar{0}$ and 1_t to p and q respectively. Prop.3.6.9 below says that two transformations with this property can differ from each other only to the extent of a $Triv_0$-transformation.

Definition 3.6.8 (T_{pq})
Assume $\mathfrak{F}$ is Euclidean.

Let $N : {}^nF \times {}^nF \longrightarrow Triv$ be a fixed function with the property ($\star$) below. $N_{pq} \stackrel{\text{def}}{=} N(p, q)$.

$$(\star) \qquad (\forall p, q \in {}^nF)\Big(p \neq q \;\Rightarrow\; (N_{pq}[\overline{pq}] \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \;\wedge\; N_{pq}(p) = \bar{0})\Big).$$

Such a function exists by Lemma 3.5.3 in §3.5. Throughout, this function N is fixed.[253]

For every distinct $p, q \in {}^nF$ with $\overline{pq} \in \mathsf{SlowEucl}$ we define T_{pq} exacly as f_{km_0} was defined in the definition of SM (Def.3.5.5 in §3.5) but with $\langle p, q, N_{pq}\rangle$ in place of $\langle o_k, t_k, N_k\rangle$.

$\lhd$

The following proposition describes how the elements of PT look like. (Cf. also Thm.3.6.4 on p.266).

[253]In some sense, N is also a "choice function" similarly to P in the definition of SM (Def.3.5.5 in §3.5).

PROPOSITION 3.6.9 *Assume $\mathfrak{F}$ is Euclidean. Assume $p, q \in {}^nF$ with $p \neq q$ and $\overline{pq} \in$ SlowEucl. For (i), (ii) below, we claim (i)$\Leftrightarrow$(ii).*

(i) $A \in PT$ *with* $A(\bar{0}) = p$, $A(1_t) = q$.

(ii) $A = M \circ T_{pq}$, *for some* $M \in Triv_0$.

We will give the **proof** later, after the proof of Lemma 3.6.21 on p.277.

Remark 3.6.10 (Connection between T_{pq} and $Rhomb$:) Assume $\mathfrak{F}$ is Euclidean. Let $p, q \in {}^nF$ and let T_{pq} as defined in Def.3.6.8 above. Then

$$T_{pq} = rhomb \circ triv, \quad \text{for some } rhomb \in Rhomb \text{ and } triv \in Triv\,.$$

More concretely $T_{pq} = rhomb \circ N_{pq}^{-1}$, where N_{pq} is as in Def.3.6.8 above, and $rhomb \in Rhomb$ taking 1_t to $N_{pq}(q)$.

$\triangleleft$

In the following definition we will define a class of frame models GM, so called *general models*. This definition is motivated by Prop.3.6.5(ii) and Prop.3.6.9.

a below itt egy kicsit hülyén hangzik!

On the intuitive structure of Def.3.6.11 (GM) below:

The definition of a general model below can be visualized from the point of view of a distinguished observer, call in m_0, (who is not explicitly indicated in the formal definition) as follows. Each general model will be determined by choosing two parameters α and β (they are functions). To each observer $k \in Obs$, α associates a tuple $\langle o_k, t_k, \varphi_k, M_k\rangle$ whose role is the following. o_k tells us where m_0 sees the origin of k's coordinate-system. t_k tells us where m_0 sees 1_t of k's coordinate-system. M_k tells us where the other unit vectors of k are seen by m_0. Finally φ_k tells us how f_{km_0} differs from the affine transformation determined by o_k and the images of the unit vectors of k.[254] The function β tells us where the distinguished observer m_0 sees the various bodies.

Definition 3.6.11 (General Models, GM)

Let $\mathfrak{F}$ be Euclidean. Let B, Obs, Ph, Ib be sets and let

$$\begin{array}{lcll} \alpha & : & Obs & \longrightarrow & {}^nF \times {}^nF \times Aut(\mathfrak{F}) \times Triv_0 \quad \text{and} \\ \beta & : & B & \longrightarrow & \mathcal{P}({}^nF) \end{array}$$

be functions with properties 1-6 below. For all $k \in Obs$ we denote $\alpha(k)$ by $\langle o_k, t_k, \varphi_k, M_k\rangle$.

[254]By the images of these unit vectors we mean the points (of ${}^nF$) where m_0 sees the unit vectors of k.

1. $Obs \cup Ph \subseteq Ib \subseteq B$.

2. $(\forall k \in Obs)\ o_k \neq t_k$.

3. $(\forall k \in Obs)\ \beta(k) = \overline{o_k t_k}$.

4. $\beta[Obs] = \mathsf{SlowEucl}$.

5. $\beta[Ph] = \mathsf{PhtEucl}$.

6. $\beta[Ib] \subseteq \mathsf{Eucl}$.

For every such $\mathfrak{F}$, and for every such sets B, Obs, Ph, Ib and functions α, β satisfying 1-6 we define a frame model $\mathfrak{M}$ as follows.
$\mathfrak{M} \stackrel{\text{def}}{=} \langle (B, Obs, Ph, Ib), \mathfrak{F}, G; \in, W \rangle$, where $G = \mathsf{Eucl}$.
It remains to define W. First we define a function $w_0 : {}^nF \longrightarrow \mathcal{P}(B)$ as follows. Let $p \in {}^nF$. Then

$$w_0(p) \stackrel{\text{def}}{=} \{\, b \in B \,:\, p \in \beta(b) \,\} \;.$$

Let $k \in Obs$ be arbitrary. We define

$$w_k \stackrel{\text{def}}{=} \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k} \circ w_0 \,;$$

where $T_{o_k t_k}$ was defined in Def.3.6.8. (Let us notice that $o_k \neq t_k$ (by 2) and $\overline{o_k t_k} \in \mathsf{SlowEucl}$ (by 3,4). Therefore $T_{o_k t_k}$ is defined.)

$$W \stackrel{\text{def}}{=} \{\, \langle m, p, h \rangle \,:\, m \in Obs, h \in w_m(p) \,\} \;.$$

By this $\mathfrak{M}$ is defined.
For a fixed $n \geq 2$, the class of the above defined models is called the class of *general models*, and we denote this class by $\mathsf{GM} = \mathsf{GM}(n)$.

$\triangleleft$

Proposition 3.6.12 $\mathsf{GM} \models$ **Basax**.

We will give the **proof** later, after the proof of Lemma 3.6.23 on p.278.

In the following theorem we give "structural" characterizations for the models of the theories **Basax**(3), **Basax**(n) + **Ax**($\sqrt{\ }$), and **Basax**(n) + **Ax7**, for $n \geq 3$.[255]

[255]Thm.3.6.13 is strongly related to what are called *representation theorems* in algebraic logic (and also in the Tarskian approach of first-order axiomatic geometry cf. Henkin-Monk-Tarski [129], Schwabhäuser-Szmielew-Tarski [237]).

THEOREM 3.6.13 *Let $n \geq 3$. Then (i)-(iii) below hold.*[256]

$$\begin{array}{rlcl} \textbf{(i)} & \mathsf{Mod}(\mathbf{Basax}(3)) & = & \mathsf{GM}(3). \\ \textbf{(ii)} & \mathsf{Mod}(\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ })) & = & \mathsf{GM}(n). \\ \textbf{(iii)} & \mathsf{Mod}(\mathbf{Basax}(n) + \mathbf{Ax7}) & = & \mathsf{GM}(n) \cap \mathsf{Mod}(\mathbf{Ax7}). \end{array}$$

On the **proof**: We will give the proof of item (ii) later, on p.280. Items (i) and (iii) directly follow by item (ii) and Thm.3.6.17 below.

QUESTION 3.6.14 *Is* $\mathsf{Mod}(\mathbf{Basax}(4)) = \mathsf{GM}(4)$ *true?*

$\lhd$

We note that there are very special, extremely symmetric elements of GM which have been thoroughly investigated in the literature for a long time. We call them Minkowski models. They will be defined and discussed in §§ 3.8.2, 3.8.3.

In connection with Items 3.6.11, 3.6.12 above we include the following.

Remark 3.6.15 (On our "philosophical" goals formulated in §1.1 (X) concerning proving many relativistic effects from few assumptions.)
In the introduction we indicated that we want to prove as many interesting predictions of relativity theory as possible from as few assumptions as possible, cf. item (X) of §1.1. I.e. we want to make our axiomatic relativity theory far from being complete (i.e. we want to make it logically weak or "flexible") but strong enough to get interesting theorems. In §2.5 we proved from **Basax** several of the typical predictions of relativity. (In other parts of §2 we proved some more such predictions). This can be interpreted to say that **Basax** is *strong enough* for proving most of the interesting results.[257]

Next, let us turn to seeing that **Basax** is not only "strong enough" but that it is also "flexible enough" (in the above outlined sense). Prop.3.6.12 above can be used to conclude that there are many *non-elementarily-equivalent* models of **Basax**. This will be actually proved in §3.8, but till then the reader is invited to convince himself by meditating over the definitions of GM and SM in the way outlined below. We know that $\mathsf{SM} \subseteq \mathsf{GM} \models \mathbf{Basax}$.

[256] According to our Convention 2.1.2, we *ignore* those of our frame models in which "E" is not the real set theoretic "$\in$". Therefore $\mathsf{Mod}(\mathbf{Basax}(n))$ denotes the class of only those models of $\mathbf{Basax}(n)$ in which "E" is the real $\in$. This is why our theorem is stated in the stronger form $\mathsf{Mod}(\ldots) = \mathsf{GM}(\ldots)$ as opposed to the weaker form $\mathsf{Mod}(\ldots) = \mathbf{I}\mathsf{GM}(\ldots)$.

[257] At the end of §2 we indicated that there are some exceptional cases where we do need to reinforce **Basax** (or **Newbasax**) with something like e.g. **Ax(symm)**.

First let us recall from the logic literature that $\mathfrak{M}$ is elementarily-equivalent with $\mathfrak{N}$, in symbols $\mathfrak{M} \equiv_{ee} \mathfrak{N}$ iff $\mathsf{Th}(\mathfrak{M}) = \mathsf{Th}(\mathfrak{N})$. Cf. also Def.3.8.17 on p.303.

Let $n > 1$ be arbitrary. In the definition of SM we had a "parameter" P which one could choose freely obtaining a model $\mathfrak{M}^P = \mathfrak{M}^P_{\mathfrak{F}}$ in SM. First we notice that any one of the (n-dimensional versions of the) models $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ in Figure 29 on p.88 can be obtained by choosing P appropriately. Clearly these models are distinguishable from each other by first-order formulas i.e. $\mathfrak{M}_1 \not\equiv_{ee} \mathfrak{M}_2$ etc. One easily sees that there is an infinity of non-elementarily-equivalent models which differ from each other the same way as the $\mathfrak{M}_i$'s do ($i < 3$) on p.88.

Further, the definition of GM gives us non-elementarily-equivalent models in other ways too. (E.g. if we assume **Ax(symm)** then the above kind of multiplicity of models gets ruled out, but there remain other respects in which we still can get many non-elementarily-equivalent models as follows.) For example if we play with the parameter β in the definition of GM, then we can have many members of $Ib \setminus (Obs \cup Ph)$ on the same straight line[258], or exactly one member of that set on each slow-line and none on fast-lines, or no $Ib \setminus (Obs \cup Ph)$ at all, or we can have them on some fast-lines (i.e. we can have tachyons) etc. This again, gives us a multiplicity of non-elementarily-equivalent models in a style (or regard) *different* from the above outlined $\mathfrak{M}_1$, $\mathfrak{M}_2$ etc. oriented multiplicity. Let us notice that the ordered field reduct $\mathfrak{F}^{\mathfrak{M}}$ is the same for all these different models (both of the ones obtained by "playing" with P and the ones obtained by playing with β). We leave exploration of further (kinds of) possibilities for constructing non-elementarily-equivalent models of **Basax**(n) to the reader, but the above hints already show that there must be many (even if we fix the ordered field reduct $\mathfrak{F}$ to be the same). More detailed investigation of this comes in Thm.3.8.18 (p.303) where we will see that there are indeed extremely many non-elementarily equivalent models of **Basax**(n) in many ways even if we add certain restrictions (like **Ax(symm)**) to **Basax**.[259]

The large number of non-elementarily-equivalent-models means that there is a large number of *different consistent* (deductively closed) *theories*[260] extending **Basax**. But this means that our goal formulated in item (X) of §1.1 has been achieved, because **Basax** is both flexible[261] (i.e. has many different extensions), *and* at the same time it proves the paradigmatic effects mentioned in §2.

At this point, we note that this

"flexibility + proving most of the paradigmatic effects"

[258] as their life-line

[259] We will also see that by adding further appropriate axioms one *can* cut down the number of these models basically to one *if* one *wants to* for some reason.

[260] actually a large number of maximal consistent theories, too

[261] or in other words assumes little

achievement can be improved a lot with the devices which we already have in the present section. Namely, **Basax** can be replaced with the weaker (i.e. more flexible) system **Newbasax**, and that in turn can be replaced with the even more flexible **Bax** and *we still can prove* all the paradigmatic effects mentioned in §2. Actually in §§3.4.2, 4.4, 4.5 we introduced even more flexible versions some of which are philosophically more significant than say **Bax**, and it would be interesting to discuss which one of these proves all or almost all the paradigmatic effects in §2. However, here we do not go into this, instead we stick with **Newbasax** and **Bax**.

As we said, the *paradigmatic effects* of relativity collected in §2.5 are all provable in **Newbasax** and even in **Bax**.[262]

We note that even (∗) below seems to be provable, but we did not check the details.

(∗) $$\mathbf{Bax} \models [\mathbf{Ax(symm)} \;\Rightarrow\; \text{“Twin paradox”}]\,.$$

Moreover even the following might be true

$$\mathbf{Bax} \models [\mathbf{Ax(syt)} \;\Rightarrow\; \text{“Twin paradox”}]\,.$$

This is made interesting by observing that[263]

$$\mathbf{Bax} \Dashv (\mathbf{Newbasax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ }))\not\models \mathbf{Basax}$$

i.e. **Newbasax** + **Ax(symm)** is a genuinely more flexible system than **Basax** + **Ax(symm)**.

◁

In Thm.3.6.16 below we generalize Thm.2.3.12 (p.65) to arbitrary dimensions. Let us recall that Thm.2.3.12 was a characterization of the f_{mk}'s in **Basax**(2) models. The reader is asked to have a quick glance at Thm.2.3.12 first.

THEOREM 3.6.16 (Characterization of the world-view transformations in Basax + Ax($\sqrt{\ }$)) *Assume $\mathfrak{F}$ is Euclidean and $\mathsf{f} : {}^nF \longrightarrow {}^nF$. Then (i)–(iv) below are* equivalent.

(i) f *is a world-view transformation in some* **Basax** *model whose ordered field reduct is $\mathfrak{F}$.*

[262] Some revision of how these effects are formalized in our frame language might be needed for this, but we do not go into discussing that in the present version of this work. We plan to do that in a future work.

[263] To make our formula more intuitive, we use "$\models$" in its reverse form where $Th_1 \Dashv Th$ means $Th \models Th_1$. Intuitively, the pattern of the above formula is the following
$\mathbf{Bax} \le (\mathbf{Newbasax} + \ldots) \not\ge \mathbf{Basax}$ implying automatically
$\mathbf{Bax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ }) \not\ge \mathbf{Basax}$.

(ii) f *is a photon-preserving bijective collineation of* ${}^n\mathbf{F}$.

(iii) $\mathsf{f} = \mathsf{g} \circ \widetilde{\varphi}$ *for some* $\mathsf{g} \in PT$ *and* $\varphi \in Aut(\mathfrak{F})$.

(iv) f *is a composition of a Poincaré transformation, an expansion and a map induced by an automorphism of* $\mathfrak{F}$. *That is,* $\mathsf{f} = poi \circ exp \circ \widetilde{\varphi}$, *for some* $poi \in Poi$, $exp \in Exp$ *and* $\varphi \in Aut(\mathfrak{F})$.

Proof: We have already seen that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), cf. Thm.3.1.1 (p.160), Prop.2.3.3(ix), (v) (p.58), Prop.3.1.17 (p.171), Prop.3.6.5(ii). (iii) $\Leftrightarrow$ (iv) follows by Thm.3.6.4(ii) on p.266. (iii) $\Rightarrow$ (i) follows from Thm.3.6.13 and Lemma 3.4.5 (p.205). ∎

Theorem 3.6.17 below says that if $n \geq 3$, then both $\mathbf{Basax}(3)$ and $\mathbf{Basax}(n) + \mathbf{Ax7}$ imply $\mathbf{Ax}(\sqrt{\ })$. We note that these implications do not hold backwards, i.e. $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) \not\models \mathbf{Ax7}$, for every $n \geq 3$. However, Proposition 3.6.18 on p.274 (below the proof of Thm.3.6.17) says that every model of $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ })$ can be extended to a model of $\mathbf{Basax}(n) + \mathbf{Ax7}$, for every $n \geq 3$. We also note that Thm.3.6.17(i) below does not generalize to $n = 2$, that is $\mathbf{Basax}(2) \not\models \mathbf{Ax}(\sqrt{\ })$.

THEOREM 3.6.17 *Assume* $n \geq 3$. *Then (i), (ii) below hold.*

(i) $\mathbf{Basax}(3) \models \mathbf{Ax}(\sqrt{\ })$.

(ii) $\mathbf{Basax}(n) + \mathbf{Ax7} \models \mathbf{Ax}(\sqrt{\ })$.

On the proof: In order to keep the complexity of the proof of Thm.3.6.13 relatively low we postpone the proof of Thm.3.6.17 to §3.7 (pp. 284–292).

The notation $\mathfrak{M} \subseteq \mathfrak{N}$ was introduced in Convention 3.1.2 (p.160).

PROPOSITION 3.6.18 *Assume* $n \geq 3$. *Assume* $\mathfrak{M}$ *is a frame model such that* $\mathfrak{M} \models \mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ })$. *Then there is a frame model* $\mathfrak{M}^+$ *such that* $\mathfrak{M} \subseteq \mathfrak{M}^+$ *and* $\mathfrak{M}^+ \models \mathbf{Basax}(n) + \mathbf{Ax7}$.

Proof: Let $n \geq 3$. Let
$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G; \in, W \rangle \models \mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ })$.
Let $\mathfrak{M}^+ \stackrel{\text{def}}{=} \langle (B^+, Obs^+, Ph^+, Ib^+), \mathfrak{F}, G; \in, W^+ \rangle$ be defined as follows.
$B^+ \stackrel{\text{def}}{=} B \times Triv_0$, $Obs^+ \stackrel{\text{def}}{=} Obs \times Triv_0$, $Ph^+ \stackrel{\text{def}}{=} Ph \times Triv_0$, $Ib^+ \stackrel{\text{def}}{=} Ib \times Triv_0$,
and for all $\langle m, N \rangle \in Obs^+$ and $p \in {}^nF$ $\quad w^+_{\langle m,N \rangle}(p) \stackrel{\text{def}}{=} (N \circ w_m(p)) \times Triv_0$. By this $\mathfrak{M}^+$ is defined. Now $\mathfrak{M} \subseteq \mathfrak{M}^+$ assuming that we treat $\langle b, \mathrm{Id} \rangle \in B^+$ as identical with $b \in B$, for all $b \in B$. We claim that $\mathfrak{M}^+ \models \mathbf{Basax}(n) + \mathbf{Ax7}$. Checking this claim is left to the reader. ∎

QUESTION 3.6.19 *Is* $\mathbf{Basax}(4) \models \mathbf{Ax}(\sqrt{\ })$ *true?*

$\lhd$

We note that Questions 3.6.14 and 3.6.19 are equivalent in the sense that the answer to Question 3.6.19 is "YES" iff the answer to Question 3.6.14 is "YES".

Now, we start preparations for proving Thm.3.6.13 (and some of the "lesser" theorems).

LEMMA 3.6.20 $\{ \mathsf{f} \in PT \,:\, \mathsf{f}(\bar{0}) = \bar{0} \,\wedge\, \mathsf{f}(1_t) = 1_t \} = Triv_0$

Proof: $\{ \mathsf{f} \in PT \,:\, \mathsf{f}(\bar{0}) = \bar{0} \,\wedge\, \mathsf{f}(1_t) = 1_t \} \supseteq Triv_0$ holds by $Triv_0 \subseteq PT$ (cf. Lemma 3.7.1 on p.283). To prove the other inclusion let $\mathsf{f} \in PT$ with $\mathsf{f}(\bar{0}) = \bar{0}$ and $\mathsf{f}(1_t) = 1_t$. First we prove that (80) holds.

$$(\forall 0 < i \in n)\, \mathsf{f}(1_i) \perp_e 1_t\,. \tag{80}$$

While proving (80) we will use some simple notions from elementary geometry without introducing them. Throughout the proof of (80) the reader is asked to consult Figure 81.

Assume that $0 < i \in n$. The f image $\langle 1_t, \mathsf{f}(1_i), -1_t, -\mathsf{f}(1_i)\rangle$ of the square $\langle 1_t, 1_i, -1_t, -1_i\rangle$ is a quadrangle whose sides are photon-lines since the sides of the square $\langle 1_t, 1_i, -1_t, -1_i\rangle$ are photon-lines and since f is an affine transformation taking photon-lines to photon-lines. Since in every plane containing the $\bar{t}$-axis two photon-lines are either parallel or orthogonal in the Euclidean sense, the statement that the sides of quadrangle $\langle 1_t, \mathsf{f}(1_i), -1_t, -\mathsf{f}(1_i)\rangle$ are photon-lines implies that $\langle 1_t, \mathsf{f}(1_i), -1_t, -\mathsf{f}(1_i)\rangle$ is a rectangle whose sides are photon lines and one diagonal lies on the $\bar{t}$-axis, which in turn implies that $\langle 1_t, \mathsf{f}(1_i), -1_t, -\mathsf{f}(1_i)\rangle$ is a square. Hence $\mathsf{f}(1_i) \perp_e 1_t$, and this completes the proof of (80).

Since f is a linear transformation statement (80) implies that $(\forall p \in S)\; \mathsf{f}(p) \in S$.

To prove that $\mathsf{f} \in Triv_0$, by having in mind that an equivalent definition for $Triv_0$ is

$$\mathsf{f} \in Triv_0 \;\stackrel{\text{def}}{\Longleftrightarrow}\; \Big((\forall p \in \bar{t}\,)\; \mathsf{f}(p) = p \;\wedge\; (\forall p \in S)\; (\mathsf{f}(p) \in S \;\wedge\; \|\mathsf{f}(p)\| = \|p\|) \Big)\,,$$

it remains to prove that $(\forall p \in S)\; \|p\| = \|\mathsf{f}(p)\|$. Let $p \in S$. Assume first that $\mathfrak{F}$ is Euclidean. Then there is $q \in \bar{t}$ such that $\overline{pq}$ is a photon-line, namely $q \in \bar{t}$ with $\|q\| = \|p\|$ is such. Let this q be fixed.

Then $\overline{\mathsf{f}(p)q}$ is a photon-line since $\overline{pq}$ is a photon-line and since f takes photon-lines to photon lines and leaves q fixed (by $q \in \bar{t}\,$). This and $\mathsf{f}(p) \in S$ implies that

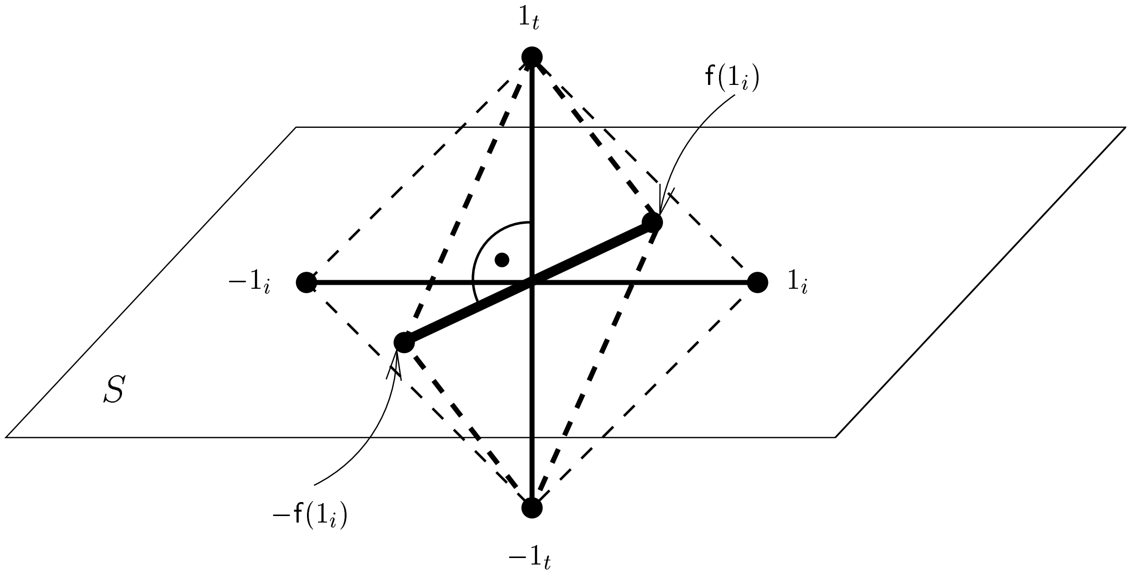


Figure 81: Illustration for the proof of Lemma 3.6.20.

$\|\mathsf{f}(p)\| = \|q\| = \|p\|$. Lemma 3.6.20 is proved for the case when $\mathfrak{F}$ is Euclidean. For non-Euclidean $\mathfrak{F}$ a proof can be obtained by applying Lemma 3.6.20 to the real closure[264] $\mathfrak{F}_*$ of $\mathfrak{F}$ and by using Lemma 3.4.7 on p.206 which says that for a transformation $\mathsf{f} \in PT(\mathfrak{F}, n)$ there is $\mathsf{f}_* \in PT(\mathfrak{F}_*, n)$ with $\mathsf{f} \subseteq \mathsf{f}_*$. Checking the details are left to the reader. ∎

LEMMA 3.6.21 *Assume $\mathfrak{F}$ is Euclidean. Let $p, q \in {}^nF$ with $p \neq q$ and $\overline{pq} \in \mathsf{SlowEucl}$. Then for T_{pq} defined in Def.3.6.8, (i)-(iii) below hold.*

(i) $T_{pq} \in PT$.

(ii) $T_{pq}(\bar{0}) = p$ *and* $T_{pq}(1_t) = q$.

(iii) $(\forall \ell \in \mathsf{SlowEucl})\ T_{pq}[\ell] \in \mathsf{SlowEucl}$.

Proof: The proof of the lemma follows by the definition of T_{pq} (Def.3.6.8). This definition says that T_{pq} is defined exactly as f_{km_0} was defined in the definition of SM (Def.3.5.5 in §3.5) but with $\langle p, q, N_{pq}\rangle$ in place of $\langle o_k, t_k, N_k\rangle$. Now in the definition

[264]The notion of the real closure of an ordered field can be found e.g. in [92].

of SM, f_{km_0} was defined to be an affine transformation which takes $\bar{0}$, 1_t to o_k, t_k, respectively. Further, in the proof of Thm.3.5.6 in §3.5, which says $\mathsf{SM} \models$ **Basax**, we have the following propositions. In the "Proof of (IV)" (73) says that

$$(\forall \ell \in \mathsf{Eucl})\Big(\ell \in \mathsf{PhtEucl} \ \Leftrightarrow\ \mathsf{f}_{km_0}[\ell] \in \mathsf{PhtEucl}\Big) .$$

And in the "Proof of (V)" (77) says that

$$(\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl} .$$

By the above, we conclude that (i)-(iii) hold for T_{pq}. ∎

Proof of Prop.3.6.9: Assume $\mathfrak{F}$ is Euclidean. Let $p, q \in {}^nF$ with $p \neq q$ and $\overline{pq} \in \mathsf{SlowEucl}$. By Lemma 3.6.21(i),(ii) we have that (81) and (82) below hold.

(81) $$T_{pq} \in PT .$$

(82) $$T_{pq}(\bar{0}) = p \quad \text{and} \quad T_{pq}(1_t) = q .$$

Proof of (i)⇒ (ii): Let $A \in PT$ with $A(\bar{0}) = p$ and $A(1_t) = q$. Then we have

(83) $$A \circ T_{pq}^{-1} \in PT \quad \& \quad A \circ T_{pq}^{-1}(\bar{0}) = \bar{0} \quad \& \quad A \circ T_{pq}^{-1}(1_t) = 1_t ,$$

by (81),(82), and by Remark 3.6.3. Now we conclude that $A \circ T_{pq}^{-1} \in \mathit{Triv}_0$ by Lemma 3.6.20 and by (83). Therefore $A = M \circ T_{pq}$, for some $M \in \mathit{Triv}_0$.

Proof of (ii)⇒ (i): Let $M \in \mathit{Triv}_0$ and $A = M \circ T_{pq}$. Then $A \in PT$ by $\mathit{Triv}_0 \subseteq PT$, by (81), and by Remark 3.6.3. By $M \in \mathit{Triv}_0$, we have $M(\bar{0}) = \bar{0}$ and $M(1_t) = 1_t$. Therefore $A(\bar{0}) = p$ and $A(1_t) = q$ by (82). ∎

Proof of Thm.3.6.4(i):
$PT \ \supseteq \ \{\, \mathit{triv}_0 \circ \mathit{rhomb} \circ \mathit{triv} \ :\ \mathit{triv}_0 \in \mathit{Triv}_0 \ \wedge\ \mathit{rhomb} \in \mathit{Rhomb} \ \wedge\ \mathit{triv} \in \mathit{Triv} \,\}$ holds, since $\mathit{Triv}_0, \mathit{Triv}, \mathit{Rhomb} \subseteq PT$ (cf. Lemma 3.7.1 on p.283) and since PT is closed under composition $\circ$. To prove the other inclusion let $A \in PT$. Then $A = \mathit{triv}_0 \circ T_{pq}$, for some $p, q \in F$ and $\mathit{triv}_0 \in \mathit{Triv}_0$ by Prop.3.6.9. Let this T_{pq} be fixed. By Remark 3.6.10, $T_{pq} = \mathit{rhomb} \circ \mathit{triv}$, for some $\mathit{rhomb} \in \mathit{Rhomb}$ and $\mathit{triv} \in \mathit{Triv}$. Let such $\mathit{rhomb}, \mathit{triv}$ be fixed. Now $A = \mathit{triv}_0 \circ \mathit{rhomb} \circ \mathit{triv}$, and this completes the proof of Thm.3.6.4(i). Item (ii) of Thm.3.6.4 will be proved in §3.7 on p.283. ∎

LEMMA 3.6.22 *Let $\varphi \in Aut(\mathbf{F})$. Then (i)-(iii) below hold.*

(i) $\widetilde{\varphi} : {}^nF \longrightarrow {}^nF$ *is a bijection.*

(ii) $(\forall \ell \in \mathsf{Eucl})\ \widetilde{\varphi}[\ell] \in \mathsf{Eucl}$.

(iii) $(\forall \ell \in \mathsf{Eucl})\ ang^2(\widetilde{\varphi}[\ell]) = \varphi(ang^2(\ell))$.

Proof: The proof is straightforward, we omit it. We note that (i) and (ii) follow by Lemma 3.1.6 in §3.1. ∎

LEMMA 3.6.23 *Let $\varphi \in Aut(\mathfrak{F})$. Then*

$$\ell \in \mathsf{SlowEucl} \iff \widetilde{\varphi}[\ell] \in \mathsf{SlowEucl}, \text{ for all } \ell.$$

Proof:

Proof of $\Rightarrow$: Let $\ell \in \mathsf{SlowEucl}$. Then $ang^2(\ell) < 1$. By Lemma 3.6.22(iii), we have $ang^2(\widetilde{\varphi}[\ell]) = \varphi(ang^2(\ell))$. Therefore we have $\varphi(ang^2(\ell)) < 1$ by $ang^2(\ell) < 1$ and by φ being order preserving. Hence $\widetilde{\varphi}[\ell] \in \mathsf{SlowEucl}$.

Proof of $\Leftarrow$: The proof is analogous with the proof of direction $\Rightarrow$, because $\varphi^{-1} \in Aut(\mathfrak{F})$. ∎

Proof of Prop.3.6.12: The proof will be analogous with the proofs of $\mathfrak{M}_0^P \models \mathbf{Basax}(2)$ in §2.4 (Thm.2.4.1) and of $\mathsf{SM} \models \mathbf{Basax}(n)$ in §3.5 (Thm.3.5.6). The essential novelty in the present proof is that we have to handle the field automorphisms φ_k and the new Newtonian transformations M_k too, for each observer k, (because now they too belong to an observer k).[265]

Let $\mathfrak{M} \in \mathsf{GM}$. We will show that $\mathfrak{M} \models \mathbf{Basax}$.

For every $k \in Obs$, by Lemma 3.6.21, we have that (84)-(86) below hold.

(84) $\quad T_{o_k t_k} \in PT$

(85) $\quad T_{o_k t_k}[\bar{t}\,] = \overline{o_k t_k} \quad$ (by $T_{o_k t_k}(\bar{0}) = o_k$ and $T_{o_k t_k}(1_t) = t_k$).

(86) $\quad (\forall \ell \in \mathsf{SlowEucl})\ T_{o_k t_k}[\ell] \in \mathsf{SlowEucl}$.

For every $k \in Obs$ let

$$\mathsf{f}_k \stackrel{\text{def}}{=} \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k} .$$

[265] In the definition of SM, an observer k (more precisely k's world-view) was determined by data $\langle o_k, t_k, N_k\rangle$, while in GM it is determined by more data like $\langle o_k, t_k, N_k, \varphi_k, M_k\rangle$.

We will prove that (87)-(91) below hold for f_k, for every $k \in Obs$.

(87) $\mathsf{f}_k : {}^nF \longrightarrow {}^nF$ is a bijection.

(88) $(\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_k[\ell],\ \mathsf{f}_k^{-1}[\ell] \in \mathsf{Eucl}\,.$

(89) $(\forall \ell \in \mathsf{PhtEucl})\ \mathsf{f}_k[\ell],\ \mathsf{f}_k^{-1}[\ell] \in \mathsf{PhtEucl}\,.$

(90) $(\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}_k[\ell] \in \mathsf{SlowEucl}\,.$

(91) $\mathsf{f}_k[\bar{t}\,] = \overline{o_k t_k}\,.$

(87) holds by $M_k \circ T_{o_k t_k} \in Aftr$ and by Lemma 3.6.22(i).
(88) holds by $M_k \circ T_{o_k t_k} \in Aftr$, by Lemma 3.6.22(ii), and by (87).
(89) holds by $M_k \in Triv_0 \subseteq PT$, by (84), and by Lemma 3.6.22(iii).
(90) holds by (86), by Lemma 3.5.4 in §3.5, and by Lemma 3.6.23.
(91) holds by $\widetilde{\varphi_k} \circ M_k[\bar{t}\,] = \bar{t}$ and by (85).

It is easy to see that for every $k \in Obs$ and $b \in B$

(92) $$w_k = \mathsf{f}_k \circ w_0\,,$$

(93) $$\beta(b) = \mathsf{f}_k[tr_k(b)]\,,$$

(94) $$tr_k(b) = \mathsf{f}_k^{-1}[\beta(b)]\,,$$

by the definitions of w_k and f_k.
Now

$\mathfrak{M} \models \mathbf{Ax1}, \mathbf{Ax2}$ by $\mathfrak{M} \in \mathsf{GM}$.

$\mathfrak{M} \models \mathbf{Ax3}$ because of the following. Let $h \in Ib$ and $m \in Obs$. By (94), we have $tr_m(h) = \mathsf{f}_k^{-1}[\beta(h)]$. By $h \in Ib$ and by item 6 in the definition of GM (Def.3.6.11), we have $\beta(h) \in \mathsf{Eucl}$. Therefore $tr_m(h) = \mathsf{f}_k^{-1}[\beta(h)]$ and $\beta(h) \in \mathsf{Eucl}$ imply $tr_m(h) \in \mathsf{Eucl}$ by (88).

$\mathfrak{M} \models \mathbf{Ax4}$ because of the following. Let $m \in Obs$. Then $\mathsf{f}_m[\bar{t}\,] = \overline{o_m t_m}$ by (91). We have $\overline{o_m t_m} = \beta(m)$ by item 3 in the definition of GM (Def.3.6.11). By these two, we have $\mathsf{f}_m[\bar{t}\,] = \beta(m)$. By (93), we have $\beta(m) = \mathsf{f}_m[tr_m(m)]$. By this, by $\mathsf{f}_m[\bar{t}\,] = \beta(m)$, and by (87), we have $tr_m(m) = \bar{t}$.

$\mathfrak{M} \models \mathbf{Ax5}$ because of the following. Let $m \in Obs$, $\ell_1 \in \mathsf{SlowEucl}$, $\ell_2 \in \mathsf{PhtEucl}$. By (90), we have $\mathsf{f}_m[\ell_1] \in \mathsf{SlowEucl}$, and by (89), we have $\mathsf{f}_m[\ell_2] \in \mathsf{PhtEucl}$. Therefore by items 4,5 in the definition of GM (Def.3.6.11), we have $\mathsf{f}_m[\ell_1] = \beta(k)$ and $\mathsf{f}_m[\ell_2] = \beta(ph)$, for some $k \in Obs$ and $ph \in Ph$. Let such k and ph be fixed. Now we conclude $tr_m(k) = \ell_1$ and $tr_m(ph) = \ell_2$ by $\mathsf{f}_m[\ell_1] = \beta(k)$, $\mathsf{f}_m[\ell_2] = \beta(ph)$, by (93), and by (87).

$\mathfrak{M} \models$ **Ax6** because (87) and (92) imply that $(\forall m \in Obs)$ $Rng(w_m) = Rng(w_0)$.

$\mathfrak{M} \models$ **AxE** because of the following. Let $m \in Obs$ and $ph \in Ph$. By (94), we have $tr_m(ph) = \mathsf{f}_m^{-1}[\beta(ph)]$. By item 5 in the definition of GM (Def.3.6.11), we have $\beta(ph) \in \mathsf{PhtEucl}$. By this, by $tr_m(ph) = \mathsf{f}_m^{-1}[\beta(ph)]$ and by (89), we have $tr_m(ph) \in \mathsf{PhtEucl}$, hence $v_m(ph) = 1$.

By the above, $\mathfrak{M} \models$ **Basax**. Therefore,
<u>*Prop.3.6.12 is proved.*</u> ∎

Proof of Thm.3.6.13(ii): Let $n \geq 3$. By Prop.3.6.12 and $\mathsf{GM} \models \mathbf{Ax}(\sqrt{\ })$, we have that $\mathsf{GM} \subseteq \mathsf{Mod}(\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }))$. To prove the other inclusion assume

$$\mathfrak{N} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G^{\mathfrak{N}}; \in, W^{\mathfrak{N}} \rangle \in \mathsf{Mod}(\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }))\,.$$

Now we will prove that $\mathfrak{N} \in \mathsf{GM}$. By $\mathfrak{N} \models \mathbf{Ax}(\sqrt{\ })$, we have that $\mathfrak{F}$ is Euclidean. Let $m_0 \in Obs$ be arbitrary and fixed. We will define functions

$$\begin{array}{llll} \alpha & : & Obs & \longrightarrow \ {}^nF \times {}^nF \times Aut(\mathfrak{F}) \times Triv_0 \quad \text{and} \\ \beta & : & B & \longrightarrow \ \mathcal{P}({}^nF). \end{array}$$

<u>*Definition of α:*</u>
Let $k \in Obs$. In what follows we will define $\alpha(k)$. By Prop.3.6.5(ii), we have

$$\mathsf{f}_{km_0} = \widetilde{\varphi_k} \circ A, \quad \text{for some } A \in PT \text{ with } A[\bar{t}\,] = tr_{m_0}(k) \text{ and } \varphi_k \in Aut(\mathfrak{F})\,.$$

Let these φ_k and A be fixed. Let

$$o_k \stackrel{\text{def}}{=} A(\bar{0}) \quad \text{ and } \quad t_k \stackrel{\text{def}}{=} A(1_t)\,.$$

By this and $A[\bar{t}\,] = tr_{m_0}(k)$, we have $o_k \neq t_k$ and $\overline{o_k t_k} = tr_{m_0}(k)$. There is no FTL observer in $\mathfrak{N}$ by Thm.3.4.1(i) in §3.4, hence $\overline{o_k t_k} = tr_{m_0}(k) \in \mathsf{SlowEucl}$. Now Prop.3.6.9 implies that

$$A = M_k \circ T_{o_k t_k}\,, \quad \text{ for some } M_k \in Triv_0\,,$$

because $o_k \neq t_k$, $\overline{o_k t_k} \in \mathsf{SlowEucl}$, $A \in PT$, $A(\bar{0}) = o_k$, and $A(1_t) = t_k$. Let this M_k be fixed. Now

$$\alpha(k) \stackrel{\text{def}}{=} \langle o_k, t_k, \varphi_k, M_k \rangle\,.$$

By this function α is defined. We note that

$$\mathsf{f}_{km_0} = \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k}\,, \tag{95}$$

by $\mathsf{f}_{km_0} = \widetilde{\varphi_k} \circ A$ and $A = M_k \circ T_{o_k t_k}$.

<u>*Definition of β:*</u>
Let $b \in B$. Then

$$\beta(b) \stackrel{\text{def}}{=} tr_{m_0}(b)\,.$$

By this function β is defined.

Now we will check that 1-6 in the definition of GM (Def.3.6.11) hold for B, Obs, Ph, Ib, α, and β.

1 holds by $\mathfrak{N} \in \mathsf{Mod}(\mathbf{Ax2})$.

In the definition of α we saw that 2 holds, i.e. for all $k \in Obs \quad o_k \neq t_k$.

3 holds because of the following. Let $k \in Obs$. In the definition of α we saw that $\overline{o_k t_k} = tr_{m_0}(k)$, and by def. of β we have $\beta(k) = tr_{m_0}(k)$. Hence $\beta(k) = \overline{o_k t_k}$.

4 holds because in $\mathfrak{N}$ there is no FTL observer by Thm.3.4.1(i), because of **Ax5**, and by the definition of β.

5 holds by def. of β and by **Ax5**,**AxE**.

6 holds by def. of β and by **Ax2**,**Ax3**.

It remains to show that W defined in the definition of GM coincides with $W^{\mathfrak{N}}$. To show this it is enough to prove $w_k^{\mathfrak{N}} = w_k$, for all $k \in Obs$. To see this let $k \in Obs$. By (95) above, we have

$$w_k^{\mathfrak{N}} \circ (w_{m_0}^{\mathfrak{N}})^{-1} = \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k}\,.$$

Therefore

(96) $$w_k^{\mathfrak{N}} = \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k} \circ w_{m_0}^{\mathfrak{N}}\,.$$

By the definition of β, it is easy to see that

$$w_{m_0}^{\mathfrak{N}} = w_0\,.$$

By this, by (96), and by the definition of w_k, we have

$$w_k^{\mathfrak{N}} = w_k\,.$$

This completes the proof of Thm.3.6.13(ii). Therefore,
<u>*Thm.3.6.13 is proved.*</u> ∎

On the case $n = 2$.

As we said in the introduction of this section $\mathsf{Mod}(\mathbf{Basax}(2))$ was characterized in an earlier version [25] of this work but cf. also Thm.2.3.12 in §2.3 herein where the f_{mk} transformations occurring in models of $\mathbf{Basax}(2)$ are completely characterized. (This does provide a *kind* of characterization of $\mathsf{Mod}(\mathbf{Basax}(2))$, from a certain point of view.) We think it would be useful to elaborate at this point the extension of the investigations in the present section to the case $n = 2$. Actually these investigations were presented on the series of seminars (1997 fall) on which the present material is based, but we did not have time to include them here. One example of those results is the following:

kell-e ide olyan, hogy proof?

THEOREM 3.6.24

$$\mathsf{GM}(2) = \mathsf{Mod}(\mathbf{Basax}(2) + \mathbf{Ax}(\sqrt{\ }) + \text{“}\nexists\ \textbf{FTL observers}\text{”}).$$

The **proof** will be filled in later. ∎

Actually, in many parts of the present section (§3), the case $n = 2$ seems to be somewhat neglected. If would be interesting to see *just a little bit* about how those parts extend to $n = 2$. As it turns out from the earlier version [25] of this work, in the case $n = 2$ the (non-order-preserving) automorphisms of the field $\mathbf{F}^{\mathfrak{M}}$ played a much more dominant role than in the cases $n > 2$.

3.7 Proofs of some earlier announced theorems

In this section we prove some earlier announced theorems.

Let us recall that Rhomb, Lor, SLor, Poi, PT, are the sets of rhombus, Lorentz, standard Lorentz, Poincaré, photon preserving (affine) transformations (over an ordered field $\mathfrak{F}$), respectively, defined in Definitions 2.3.18, 2.9.1 and 3.6.2. Further, Exp and Tran are the sets of expansions and translations, respectively, defined in Def.2.9.1. Triv_0 and Triv are the sets of so called trivial (non-relativistic) transformations defined in Def.3.5.1.

LEMMA 3.7.1 *Elements of Rhomb, Lor, SLor, Poi, Exp, Tran, Triv_0, Triv are all photon-preserving affine transformations (i.e. elements of PT), and are contained in each other in the following way:*

$$\begin{array}{ccccc}
 & & & & \mathit{Exp} \\
 & & & & \cap \\
\mathit{SLor} & & \subset & & \mathit{Rhomb} \\
\cap & & & & \cap \\
\mathit{Lor} & \subset & \mathit{Poi} & \subset & \boxed{PT} \\
\cup & & \cup & & \\
\mathit{Triv}_0 & \subset & \mathit{Triv}\,. & & \\
 & & \cup & & \\
 & & \mathit{Tran} & &
\end{array}$$

We omit the **proof**. ∎

Proof of Thm.3.6.4(ii): Assume $\mathfrak{F}$ is Euclidean. Then $PT \supseteq \{\, \mathit{poi} \circ \mathit{exp} \, : \, \mathit{poi} \in \mathit{Poi} \ \wedge \ \mathit{exp} \in \mathit{Exp} \,\}$ holds since $(\mathit{Poi} \cup \mathit{Exp}) \subseteq PT$ by Lemma 3.7.1 and since PT is closed under composition $\circ$. To prove the other inclusion let $\mathsf{f} \in PT$. Then, by Thm.3.6.4(i),

$$\mathsf{f} = \mathit{triv}_0 \circ \mathit{rhomb} \circ \mathit{triv}, \tag{97}$$

for some $\mathit{triv}_0 \in \mathit{Triv}_0$, $\mathit{rhomb} \in \mathit{Rhomb}$ and $\mathit{triv} \in \mathit{Triv}$. Let such triv_0, rhomb, triv be fixed. Now

$$\mathit{rhomb} = \mathit{slor} \circ \mathit{exp}, \tag{98}$$

for some $\mathit{slor} \in \mathit{SLor}$ and $\mathrm{exp} \in \mathit{Exp}$ by Thm.2.9.6 on p.156. Let such slor, exp be fixed. It is easy to see that $\mathit{exp} \circ \mathit{triv} = \mathit{triv} \circ \mathit{exp}$, for any $\mathit{triv} \in \mathit{Triv}$ and

$exp \in Exp$. Now

$$\begin{array}{rcll} \mathsf{f} & = & triv_0 \circ rhomb \circ triv & \text{by (97)} \\ & = & triv_0 \circ slor \circ exp \circ triv & \text{by (98)} \\ & = & triv_0 \circ slor \circ triv \circ exp & \text{by } exp \circ triv = triv \circ exp \\ & = & poi \circ exp & \text{for some } poi \in Poi \text{ because of the following:} \end{array}$$

$triv_0, slor, triv \in Poi$ by Lemma 3.7.1, and it is easy to check (by the definition of Poi) that Poi is closed under composition $\circ$. By the above computation we have $\mathsf{f} = poi \circ exp$, for some $poi \in Poi$ and this completes the proof of Thm.3.6.4(ii). ∎

Next, we turn to the proof of Thm.3.6.17. Let us recall that Thm.3.6.17 says that

(i)	**Basax**(3)	$\models$	**Ax**($\sqrt{\ }$), and
(ii)	**Basax**(n) + **Ax7**	$\models$	**Ax**($\sqrt{\ }$), for $n \geq 3$.

Proof of Thm.3.6.17: First we give a proof for (ii), and then a proof for (i).

Proof of (ii):

The idea of the proof is in Figure 82. Let $n \geq 3$ and let $\mathfrak{M} \models$ **Basax**(n) + **Ax7**. First we prove that (99) below holds.

$$(99) \quad (\forall a \in F) \Big(1 - a^2 > 0 \;\Rightarrow\; (\sqrt{1 - a^2} \in F, \text{ i.e. } (\exists d \in F)\, d^2 = 1 - a^2)\Big).$$

Throughout the proof of (99) the reader is asked to consult Figure 82. To prove (99) let $a \in F$ such that $1 - a^2 > 0$. Let $q \stackrel{\text{def}}{=} \langle 1, a, 0, \ldots, 0 \rangle$. Let P be the plane parallel with $\mathsf{Plane}(\bar{x}, \bar{y})$ and with height 1, i.e.

$$P \stackrel{\text{def}}{=} \{ \langle 1, x, y, 0, \ldots, 0 \rangle \;:\; x, y \in F \}.$$

Let $C \stackrel{\text{def}}{=} \mathsf{LightCone}(\bar{0}) \cap P$. Then C is a circle with radius 1 and center $o \stackrel{\text{def}}{=} 1_t$. Let $\ell \in \mathsf{Eucl}$ such that $q \in \ell$ and $\ell \parallel \bar{y}$. Notice that $\ell \subseteq P$.

Claim **3.7.2** $\quad \ell \cap C \neq \emptyset$.

Proof of the claim: Let $m \in Obs$. Let $k \in Obs$ such that $tr_m(k) = \overline{\bar{0}q}$ and $\mathsf{f}_{km}[\bar{y}] \parallel \bar{y}$. By **Ax7**, such an observer k exists[266]. Figure 82 illustrates the world-view of m. The intuitive idea of the proof of the claim is that we switch over to the world-view of k; then in the world-view of k we prove what we want, and then we transform back the result to the world-view of m.

[266]since $\overline{\bar{0}q} \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $ang^2(\overline{\bar{0}q}) = a^2 < 1$

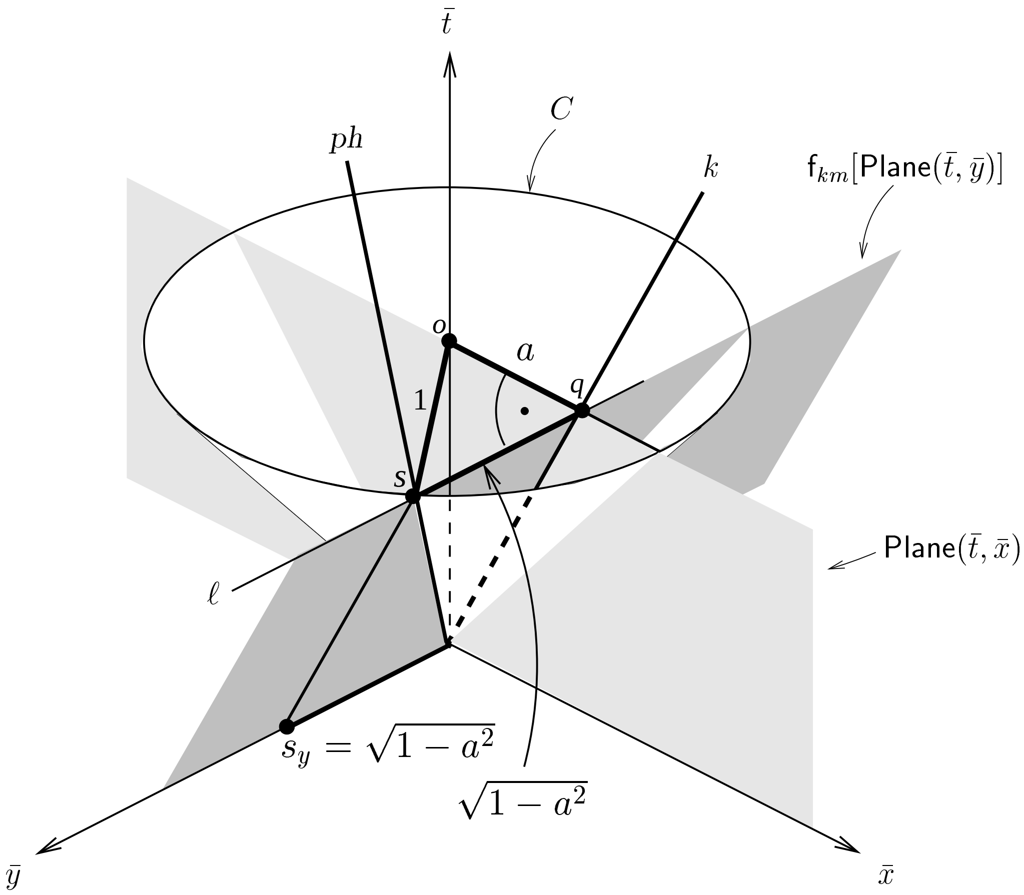


Figure 82: Illustration for the proof of Thm.3.6.17(ii).

Let us turn to the details. Let $ph \in Ph$ such that

$$\mathsf{f}_{mk}(\bar{0}) \in tr_k(ph) \subseteq \mathsf{Plane}(\bar{t}, \bar{y}).$$

Such a ph exists by **Ax5**.[267] Then

$$\bar{0} \in tr_m(ph) \subseteq \mathsf{f}_{km}[\mathsf{Plane}(\bar{t}, \bar{y})],$$

$\mathsf{f}_{km}[\mathsf{Plane}(\bar{t}, \bar{y})]$ is a plane (cf. 3.1.16). Then

$$\ell \subseteq \mathsf{f}_{km}[\mathsf{Plane}(\bar{t}, \bar{y})]$$

since $tr_m(k) = \mathsf{f}_{km}[\bar{t}]$, $\ell \parallel \bar{y} \parallel \mathsf{f}_{km}[\bar{y}]$, and $\ell \cap tr_m(k) \neq \emptyset$. But then $\ell \cap tr_m(ph) \neq \emptyset$. Let $s \in \ell \cap tr_m(ph)$. Then $s \in C$ (because $s \in \ell \subseteq P$, $s \in tr_m(ph) \subseteq \mathsf{LightCone}(\bar{0})$, and $C = P \cap \mathsf{LightCone}(\bar{0})$). Hence $s \in \ell \cap C$, and this completes the proof of the claim. QED (Claim 3.7.2)

Let $s \in \ell \cap C$. Such an s exists by Claim 3.7.2. Let us consider the triangle oqs. This triangle is rectangular, i.e. $\overline{oq} \perp_e \overline{qs}$. The length of hypotenuse os (of triangle oqs) is 1 since os is a radius of the circle C. The length of side oq (of triangle oqs) is a. Now, by Pythagoras' theorem, the square of length of side qs, which is the same as s_y^2, is $1 - a^2$. So, for the choice $d = s_y$ we have $d^2 = 1 - a^2$, and by this (99) above is proved.

Now, from (99) we prove $\mathbf{Ax}(\sqrt{\ })$, that is, $(\forall 0 < x \in F)(\exists y \in F)\, y^2 = x$.[268] This can be done by noticing that for every $0 < x \in F$

$$x = \left(\tfrac{x+1}{2}\right)^2 \left(1 - \left(\tfrac{x-1}{x+1}\right)^2\right),$$

$$1 - \left(\tfrac{x-1}{x+1}\right)^2 > 0$$

and by applying (99) for $a := \frac{x-1}{x+1}$. This completes the proof of item (ii) of Thm.3.6.17.

Proof of (i): The proof will be based on that of (ii), that is, we will eliminate the usage of **Ax7** from the proof of (ii). Let us notice that (in the proof of (ii)) **Ax7** was used only in the proof of Claim 3.7.2. Thus to prove (i) it is sufficient to prove Claim 3.7.2 using **Basax**(3) only. Let us recall from the proof of (ii) that $a \in F$ is fixed such that $1 - a^2 > 0$, $q = \langle 1, a, 0 \rangle$, P is a plane parallel with $\mathsf{Plane}(\bar{x}, \bar{y})$ and height 1 (i.e. $P = \{ \langle 1, x, y \rangle : x, y \in F \}$), $C = P \cap \mathsf{LightCone}(\bar{0})$, and $\ell \in \mathsf{Eucl}$ such that $q \in \ell \parallel \bar{y}$. Now, we recall Claim 3.7.2.

[267] This is so because e.g. $\{ \mathsf{f}_{mk}(\bar{0}) + \lambda \cdot (1_t + 1_y) : \lambda \in F \}$ is a photon-line in $\mathsf{Plane}(\bar{t}, \bar{y})$ passing through $\mathsf{f}_{mk}(\bar{0})$.

[268] We note that "(99) $\Rightarrow \mathbf{Ax}(\sqrt{\ })$" is a property of ordered fields in general.

Claim 3.7.2 $\quad \ell \cap C \neq \emptyset$.

Let us turn to proving Claim 3.7.2 from **Basax**(3). Assume $\mathfrak{M} \models$ **Basax**(3). Let $m \in Obs$. Let $k \in Obs$ such that $tr_m(k) = \overline{\bar{0}q}$. Such a k exists by **Ax5**. Figures 83–87 represent the idea of the proof. The intuitive idea of the proof (as the idea of that of (ii)) is that we switch over from the world-view of m to the world-view of k; then in the world-view of k we prove what we want, and then we transform back the result to the world-view of m.

Let us turn to the details. Let $S_k \subseteq {}^3F$ be a simultaneity of k (in the world-view of m) such that $q \in S_k$, that is, let $S_k \subseteq {}^3F$ be such that

$$\mathsf{f}_{mk}[S_k] = \{\, p + \mathsf{f}_{mk}(q) \ : \ p \in \mathsf{Plane}(\bar{x}, \bar{y}) \,\} \ ,$$

see Figure 83.

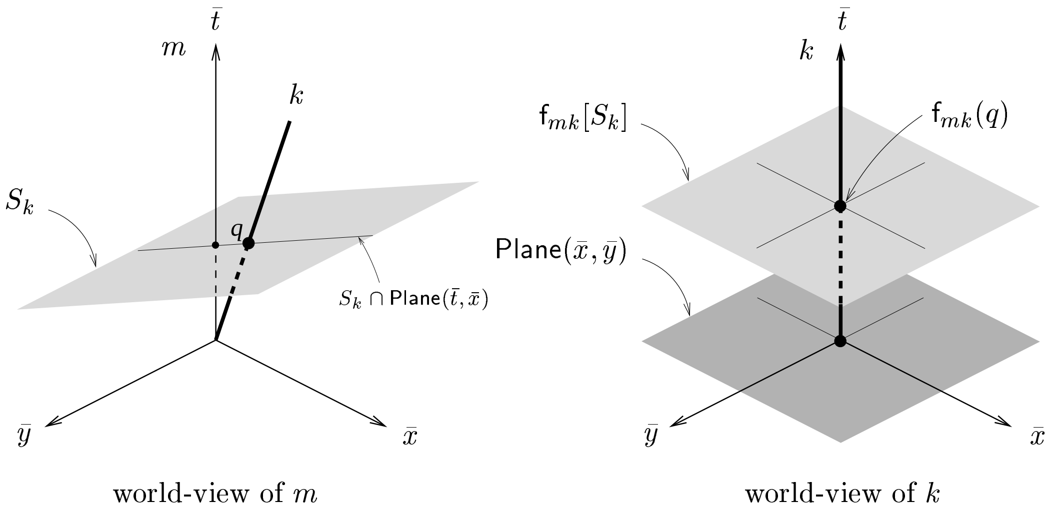


Figure 83: Let S_k be a simultaneity of k containing q.

S_k is a plane, cf. 3.1.16.

$$\ell \subseteq S_k$$

by Thm.2.5.6 saying that clocks do not get out of synchronism in direction orthogonal to movement.

$$\bar{0} \notin S_k$$

because $q \in S_k$, $\overline{\bar{0}q} = tr_m(k)$ and $tr_m(k) \not\subseteq S_k$ (since S_k is a simultaneity of k). Let

$$C_k \stackrel{\text{def}}{=} S_k \cap \mathsf{LightCone}(\bar{0}).$$

Then

$$\ell \cap C = \ell \cap C_k,^{269}$$

see Figure 84. Thus, to complete the proof it is enough to prove that $\ell \cap C_k \neq \emptyset$. Now, we turn to prove this. Throughout the proof we will use some basic theorems from elementary geometry (without recalling them) which are true in all 3-dimensional geometries over arbitrary ordered fields. We will use the notion of parallelism and the symbol $\|$ not only for lines, but also for planes in the usual way.

Let $\ell_1, \ell_2, \ell_3 \in \mathsf{PhtEucl}$ such that $\ell_1 \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$, $\ell_2, \ell_3 \subseteq \mathsf{Plane}(\bar{t}, \bar{y})$, $\ell_2 \neq \ell_3$, and $\bar{0} \in \ell_1 \cap \ell_2 \cap \ell_3$. Such ℓ_1, ℓ_2, ℓ_3 exist (e.g. $\ell_1 = \overline{\bar{0}\langle 1, -1, 0\rangle}$, $\ell_2 = \overline{\bar{0}\langle 1, 0, 1\rangle}$ and $\ell_3 = \overline{\bar{0}\langle 1, 0, -1\rangle}$ are such). See Figure 85. Let $r \in \ell_1 \cap S_k$, $u \in \ell_2 \cap S_k$ and $v \in \ell_3 \cap S_k$ (cf. Figure 85). Such r, u, v exist because we are in 3 dimensions and because there is no photon-line which is parallel with S_k since S_k is a simultaneity of observer k.[270] r, u, v, q are distinct points by the choice of r, u, v, q since $\bar{0} \notin S_k$ and $\overline{\bar{0}q} \notin \mathsf{PhtEucl}$. Let us notice that

$$r, u, v \in C_k,^{271}$$

$\overline{rq} \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \cap S_k$ and that $\overline{uv} \subseteq \mathsf{Plane}(\bar{t}, \bar{y}) \cap S_k$. Moreover $\overline{rq} = \mathsf{Plane}(\bar{t}, \bar{x}) \cap S_k$ and $\overline{uv} = \mathsf{Plane}(\bar{t}, \bar{y}) \cap S_k$ since $\mathsf{Plane}(\bar{t}, \bar{x}) \neq S_k \neq \mathsf{Plane}(\bar{t}, \bar{y})$ by $\bar{0} \notin S_k$. So

$$\bar{t} \cap \overline{rq} \cap \overline{uv} = \mathsf{Plane}(\bar{t}, \bar{x}) \cap \mathsf{Plane}(\bar{t}, \bar{y}) \cap S_k\,. \tag{100}$$

Further

$$\mathsf{Plane}(\bar{t}, \bar{x}) \cap \mathsf{Plane}(\bar{t}, \bar{y}) \cap S_k \neq \emptyset \tag{101}$$

because of the following. Assume that (101) above does not hold. Then $\bar{t} \cap S_k = \emptyset$, so $\bar{t} \parallel S_k$,[272] which together with $\bar{y} \parallel \ell \subseteq S_k$ imply that $\mathsf{Plane}(\bar{t}, \bar{y}) \parallel S_k$. $\mathsf{Plane}(\bar{t}, \bar{y}) \parallel S_k$ contradicts $\overline{uv} = \mathsf{Plane}(\bar{t}, \bar{y}) \cap S_k$. Let

$$w \in \bar{t} \cap \overline{rq} \cap \overline{uv}\,.$$

Such a w exists by (100) and (101). Since $\ell \parallel \mathsf{Plane}(\bar{t}, \bar{y})$, $\ell \subseteq S_k$ and $\overline{uv} = \mathsf{Plane}(\bar{t}, \bar{y}) \cap S_k$ we have that

$$\ell \parallel \overline{uv}\,.^{273} \tag{102}$$

[269] This is so because $\ell \subseteq S_k \cap P$, $C_k = S_k \cap \mathsf{LightCone}(\bar{0})$ and $C = P \cap \mathsf{LightCone}(\bar{0})$.

[270] I.e. if a photon-line parallel with S_k existed then, by **Ax5**, $ph \in Ph$ with $tr_m(ph) \parallel S_k$ would exist. Then (for this ph) $v_k(ph) = \infty$ would hold and this would contradict **AxE**.

[271] This is so by $\ell_1, \ell_2, \ell_3 \subseteq \mathsf{LightCone}(\bar{0})$, by $C_k = S_k \cap \mathsf{LightCone}(\bar{0})$ and by the choice of r, u, v.

[272] Since we are in 3 dimensions if a line is disjoint from a plain then they are parallel.

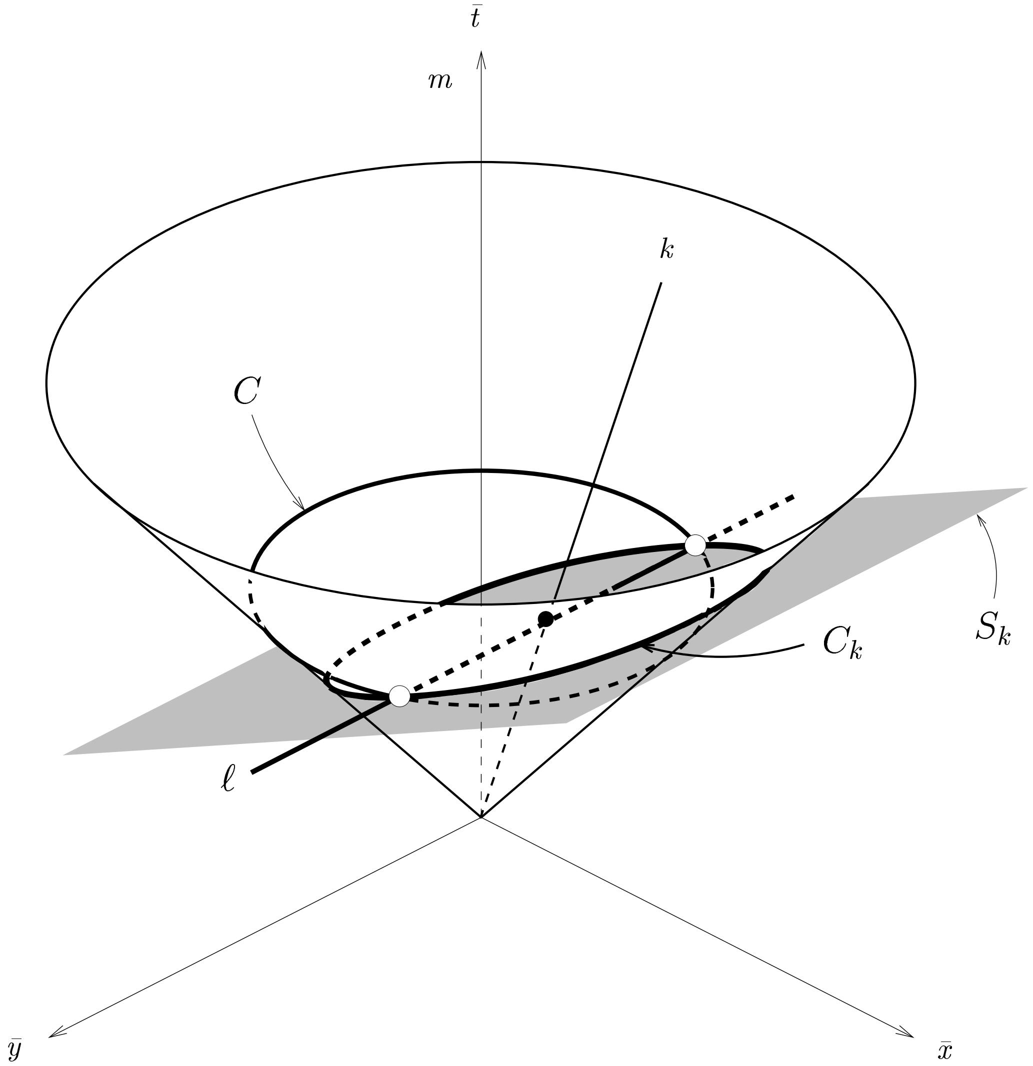


Figure 84: $\ell \cap C = \ell \cap C_k$. We will prove that $\ell \cap C_k \neq \emptyset$.

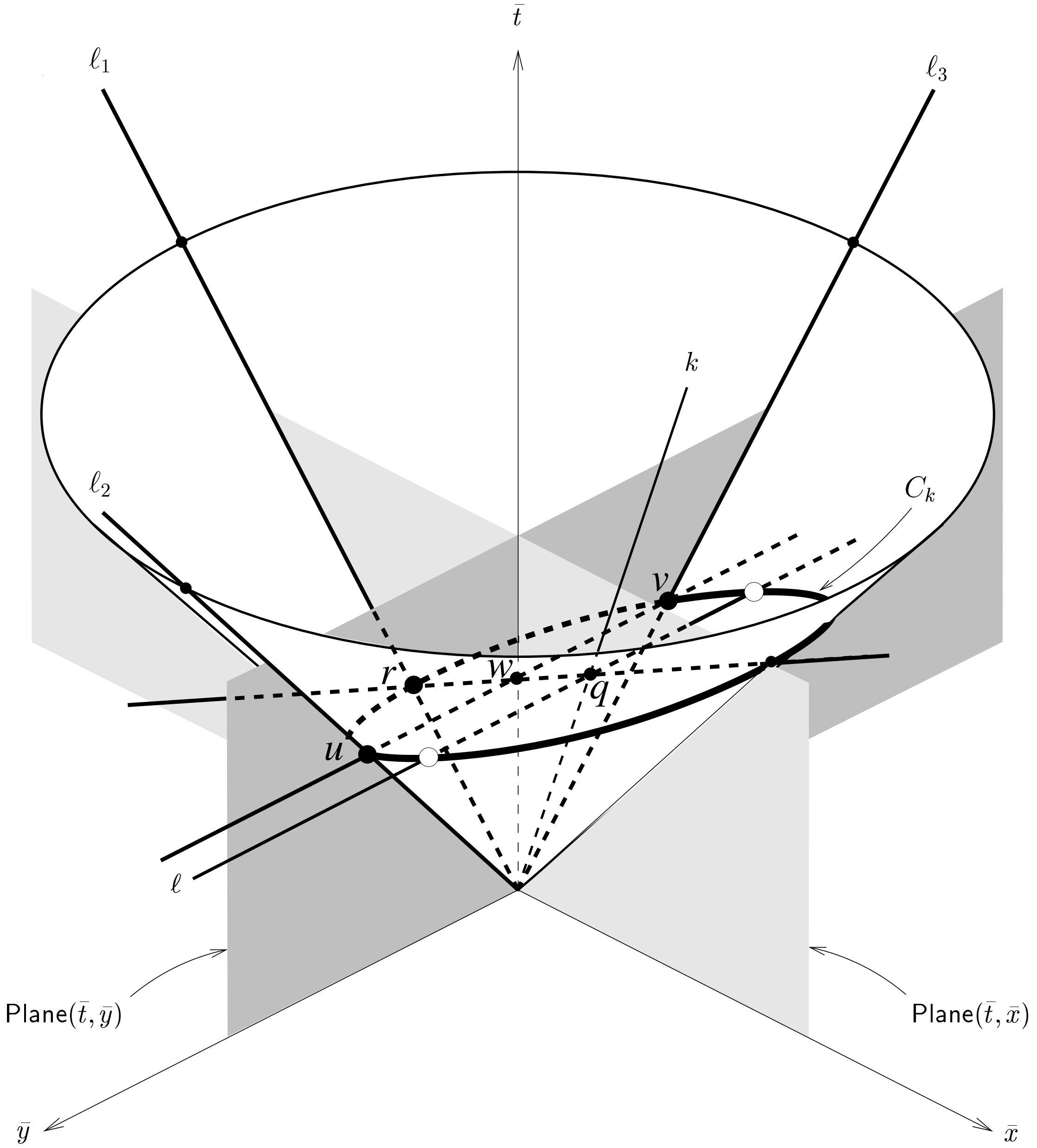


Figure 85: Let r, u, v, w be as above. Then w is a midpoint of segment uv.

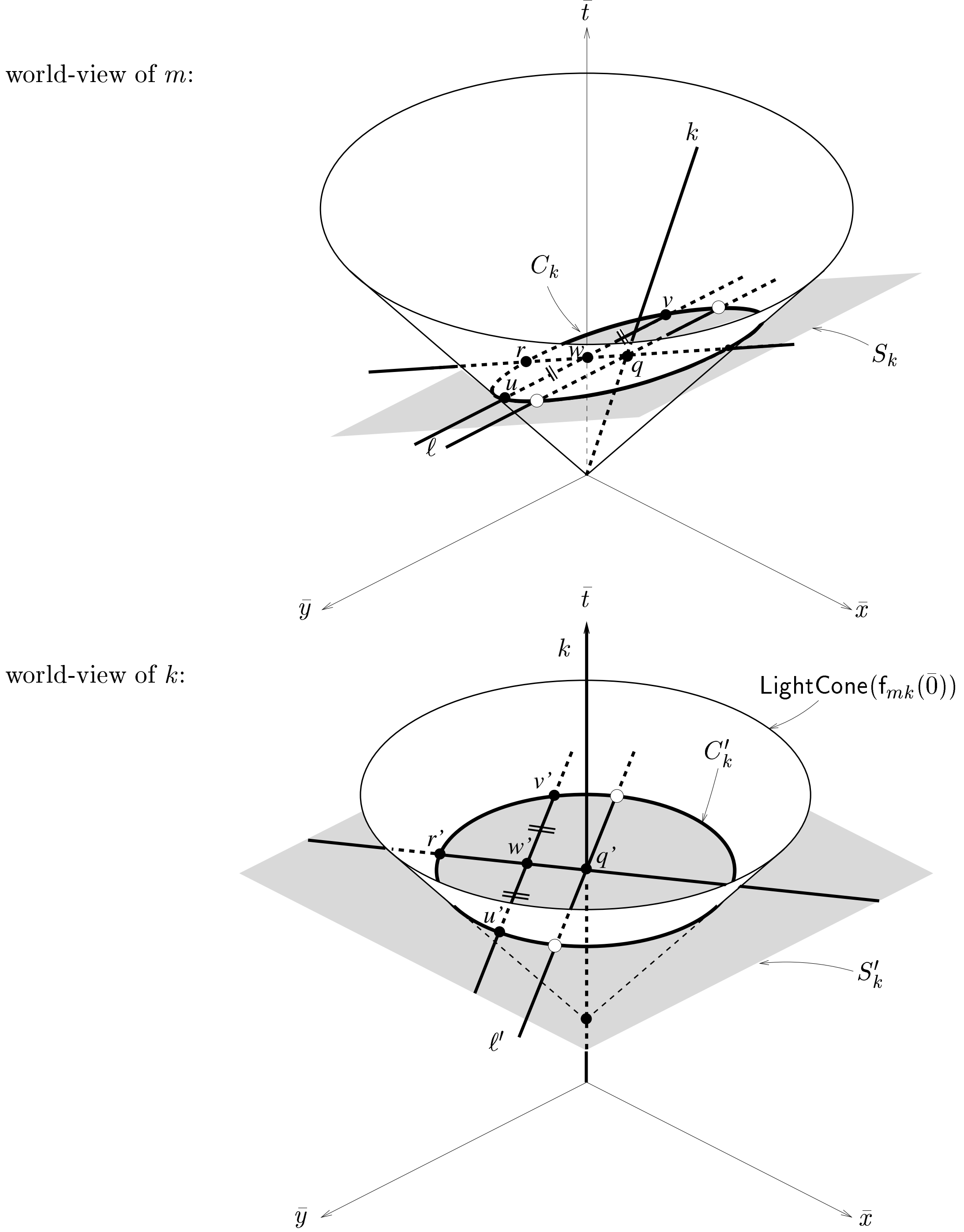


Figure 86: Let us switch over from the world-view of m to the world-view of k.

Now, $\bar{t} \perp_e \overline{uv}$ by $\bar{t} \perp_e \bar{y} \parallel \ell \parallel \overline{uv}$. Therefore, by $u, v \in \mathsf{LightCone}(\bar{0})$ and $w \in \bar{t} \cap \overline{uv}$, we have that

(103) $$w \text{ is the midpoint of segment } uv,$$

(i.e. $w = \frac{u+v}{2}$) since $\mathsf{LightCone}(\bar{0})$ is symmetric w.r.t. the time axis $\bar{t}$, cf. Figure 85.

Let us switch over to the world-view of k, cf. Figiure 86. Throughout we will use that $\mathsf{f}_{mk} : {}^3F \longrightarrow {}^3F$ is a bijection without mentioning this. Let $S'_k, C'_k, \ell', q', r', u', v', w'$ be the f_{mk}-images of $S_k, C_k, \ell, q, r, u, v, w$ respectively. See Figure 87.

Then
$$q' \in \bar{t}$$
by **Ax4** since $q \in tr_m(k)$.

C'_k is a circle with with center q' (lying in the plane S'_k which is parallel with $\mathsf{Plane}(\bar{x}, \bar{y})$) because of the following. Since $\bar{0} \in tr_m(k)$ we have that $\mathsf{f}_{mk}(\bar{0}) \in \bar{t}$ by **Ax4**. Further $\mathsf{f}_{mk}[\mathsf{LightCone}(\bar{0})] = \mathsf{LightCone}(\mathsf{f}_{mk}(\bar{0}))$ by f_{mk} being a bijection with the property ($\ell \in \mathsf{PhtEucl} \Leftrightarrow \mathsf{f}_{mk}[\ell] \in \mathsf{PhtEucl}$), cf. 3.1.17. But then
$C'_k = \mathsf{LightCone}(\mathsf{f}_{mk}(\bar{0})) \cap S'_k$, since $C_k = \mathsf{LightCone}(\bar{0}) \cap S_k$. So C'_k is a circle with center q'.

Since f_{mk} preserves Eucl (cf. 3.1.1) and w is the midpoint of segment uv, cf. (103), we have that
$$w' \text{ is the midpoint of segment } u'v'.\text{[274]}$$

(Let us notice that triangle $u'q'v'$ is isosceles triangle since segments $q'u'$ and $q'v'$ are both radiuses of circle C'_k.) But then $q'w'$ is the median line of isosceles triangle $u'q'v'$ (cf. Figure 87). This implies that $\overline{u'v'} \perp_e \overline{r'q'}$. Therefore, since $\ell' \parallel \overline{u'v'}$ (by (102) and f_{mk} being line preserving) we have that
$$\ell' \perp_e \overline{r'q'}\,.$$

But then
$$\langle r'_t, r'_y, -r'_x \rangle \in \ell' \cap C'_k$$
since $r' = \langle r'_t, r'_x, r'_y \rangle \in C'_k$, $q' \in \ell' \cap \bar{t}$, C'_k is a circle with center q' lying in the plane S'_k, $\ell' \subseteq S'_k$ and $S'_k \parallel \mathsf{Plane}(\bar{x}, \bar{y})$. Hence $\ell' \cap C'_k \neq \emptyset$. So $\ell \cap C_k \neq \emptyset$, and this completes the proof. Therefore,
<u>*Thm.3.6.17 is proved.*</u> ∎

[273]Here we used that if there are two intersecting planes such that there is a line in one plane parallel with the other plane then this line is parallel with the intersection of the two planes.

[274]Any $f : {}^nF \longrightarrow {}^nF$ preserving Eucl has the property that for any $p, q \in {}^nF$, f takes the midpoint of segment pq to the midpoint of segment $f(p)f(q)$, cf. the proof of Lemma 3.1.10.

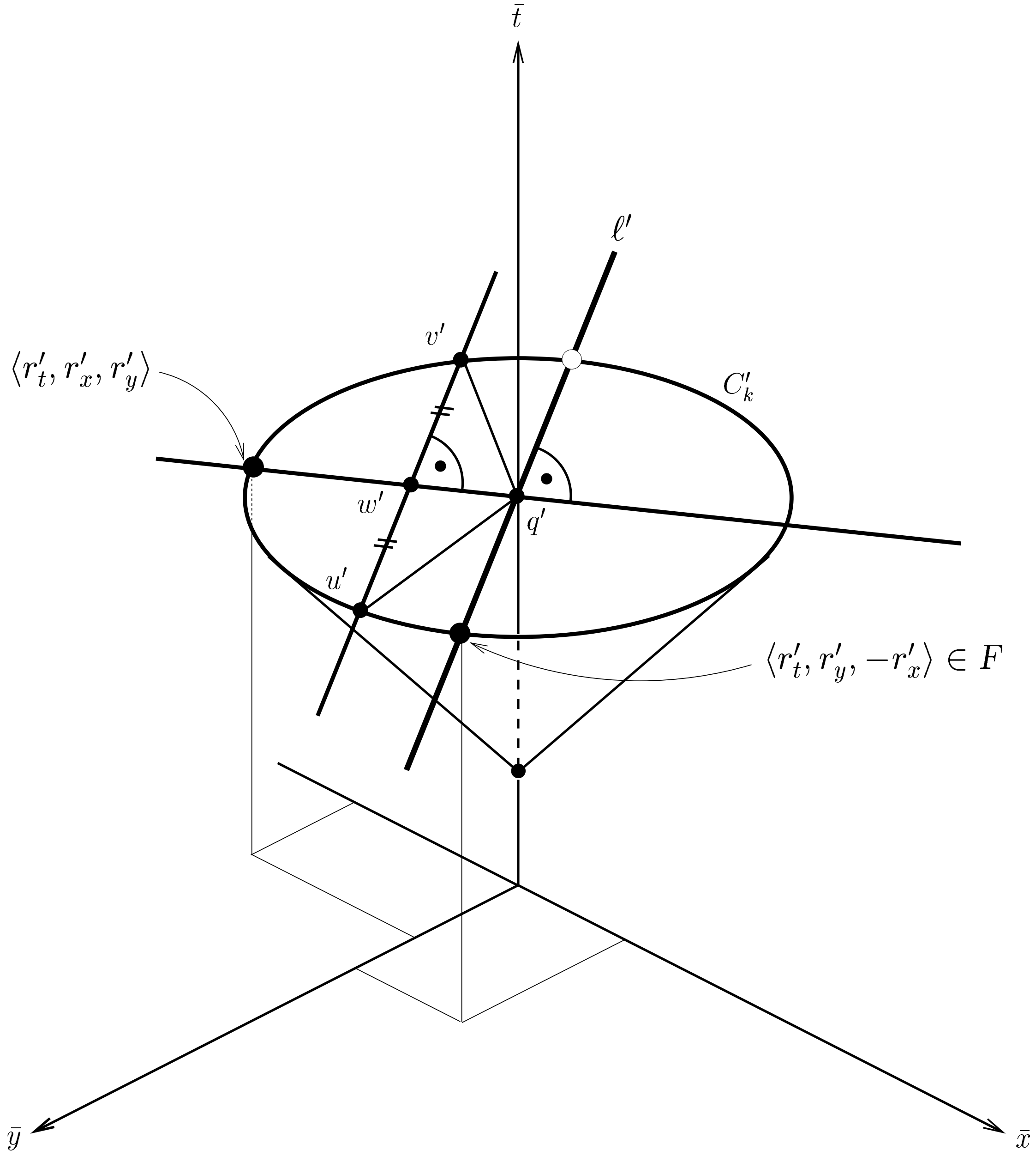


Figure 87: $\ell' \cap C'_k \neq \emptyset$, and this completes the proof of Thm. 3.6.17(i).

3.8 Making Basax complete and "Einsteinian" (BaCo)

Very roughly speaking, this section is about the following. We will see that our symmetry principle **Ax(symm)** makes **Basax** almost (but not perfectly) complete (cf. also Figure 29 and Thm.2.9.5) and we will also show how to make it perfectly complete (cf. Thm's 3.8.7, 3.8.14).[275] The main price of this "perfect" completion is that for this, we will have to assume an axiom, called **Ax♡**, which (among others) excludes non-inertial bodies from our domain of discourse. Therefore in this "perfectly" complete theory we will *not* be allowed to talk about *accelerated* bodies (like e.g. the Earth circling the Sun). As we will see in the "Gödel incompleteness" chapter, **Ax♡** is a very restrictive axiom, which is extremely useful if we want to do certain things, but at the same time it is important to be aware (of the future theorem saying) that we have to pay a considerably big price for assuming **Ax♡**. In some sense to be explained elsewhere, **Ax♡** amounts to deciding that we want to formalize *only the "heart"* of our theory "of motion etc." as opposed to formalizing the whole of the theory, cf. §1.1.IV (item IV of "Broad introduction"). More explanation of these thoughts will be included in a later version.

More precisely: In this section we will extend $\{\mathbf{Ax(symm)}, \mathbf{Ax}\heartsuit\}$ to a finite axiom system **Compl** that will make **Basax** categorical over any given Euclidean $\mathfrak{F}$. That is, we will show that, for any Euclidean $\mathfrak{F}$, $\mathbf{BaCo} := \mathbf{Basax} + \mathbf{Compl}$ admits exactly one model, up to isomorphisms, whose ordered field reduct is $\mathfrak{F}$. This will imply that $\mathbf{BaCo} + \mathsf{Th}(\mathfrak{F})$ is a *complete theory*,[276] for any Euclidean $\mathfrak{F}$, where $\mathsf{Th}(\mathfrak{F})$ is the first-order theory of $\mathfrak{F}$. E.g. $\mathbf{BaCo} + \mathsf{Th}(\mathfrak{R})$ is complete, where $\mathsf{Th}(\mathfrak{R})$ is the first-order theory of the reals.[277]

As a contrast to the above discussed completeness (of $\mathbf{BaCo} + \mathsf{Th}(\mathfrak{R})$) all the axioms in **Compl** will turn out to have an interesting property except for the axiom named $\mathbf{Ax(\uparrow)}$. This property is that if we omit the axiom in question from $\mathbf{BaCo} + \mathsf{Th}(\mathfrak{R})$ then all of a sudden we will have the opposite of completeness. In more detail: We will see that for any $\mathbf{Ax} \in \mathbf{Compl} \setminus \{\mathbf{Ax(\uparrow)}\}$,

[275] If we are interested only in the world-view transformations between observers then $\mathbf{Basax} + \mathbf{Ax(symm)}$ almost completely describes the situation. Namely, if we add some very simple auxiliary axioms together with $\mathsf{Th}(\mathfrak{R}) + \mathbf{Ax(\uparrow)}$ to be introduced soon, we get a complete description of the world-view transformations, clocks, meter rods oriented aspects of the "world".

[276] For the definition of complete theory cf. Def.3.8.13 on p.301.

[277] It happens to be the case that $\mathsf{Th}(\mathfrak{R})$ is the same as the usual theory of real-closed fields, cf. Def.3.8.12 on p.301, Fact 3.8.15 on p.302 and e.g. Chang-Keisler [59] or Goldblatt [108]. An important aspect of $\mathsf{Th}(\mathfrak{R})$ is that it is axiomatizable by a finite schema of first-order formulas (cf. Def.3.8.12).

beleírni, hogy annak örülünk ha sok modell van!

$\mathbf{BaCo} \setminus \{\mathbf{Ax}\} + \mathsf{Th}(\mathfrak{R})$ has 2^ω (i.e. continuum) many non-elementarily-equivalent models.[278] Equivalently, there are 2^ω many[279] different complete deductively closed[280] theories extending $\mathbf{BaCo} \setminus \{\mathbf{Ax}\} + \mathsf{Th}(\mathfrak{R})$.

Further, we will see, that the effects of $\mathbf{Ax(\uparrow)}$ are different from those of the remaining elements of **Compl**. Namely, for every Euclidean $\mathfrak{F}$, $\mathbf{BaCo} \setminus \{\mathbf{Ax(\uparrow)}\}$ has only a finite number of models, up to isomorphisms, whose ordered field reduct is $\mathfrak{F}$ (cf. items 3.8.8, 3.8.9).

The theory $\mathbf{BaCo} \setminus \{\mathbf{Ax}\heartsuit\}$ agrees with the "Einsteinian", standard version of special relativity, while **BaCo** is strongly related to Minkowskian geometry, cf. §§ 6.2, 6.6 in the geometry chapter.[281]

Our set of extra axioms **Compl** will consist of two groups: the essential ones and the auxiliary ones. The essential ones will be $\mathbf{Ax(symm_0)}$, $\mathbf{Ax}\heartsuit$ and $\mathbf{Ax(\uparrow)}$ (where $\mathbf{Ax(symm_0)}$ was introduced in §2.8, while $\mathbf{Ax}\heartsuit$ and $\mathbf{Ax(\uparrow)}$ will be introduced below). The auxiliaries are the rest. We call $\mathbf{Ax(symm_0)}$, $\mathbf{Ax}\heartsuit$ and $\mathbf{Ax(\uparrow)}$ essential because it is worth to discuss versions of relativity theory both with them and without them. For example consider $\mathbf{Ax}\heartsuit$. As we mentioned above, the choice between adding or not adding $\mathbf{Ax}\heartsuit$ to our theory might be interpreted as deciding whether we want to develop an "only the heart approach" or a "not only the heart approach". And of course both of these choices make sense (i.e. they both are worth of discussing). Therefore we call $\mathbf{Ax}\heartsuit$ one of the "essential" axioms. As a contrast let us look at one of the auxiliaries, e.g. consider $\mathbf{Ax}(\mathit{Triv}_t)$ to be introduced below. This axiom is a very natural one, namely it says that every observer can "re-coordinatize" his world-view by any trivial transformation.[282] The reason why we did not include $\mathbf{Ax}(\mathit{Triv}_t)$ into **Basax** is that we could derive our main theorems (e.g. no FTL observers, clocks getting out of synchronism) without $\mathbf{Ax}(\mathit{Triv}_t)$. Nobody is extremely curious about the question what is the difference between relativity theory without $\mathbf{Ax}(\mathit{Triv}_t)$ or with $\mathbf{Ax}(\mathit{Triv}_t)$. (One could say, that whenever one would need an axiom like $\mathbf{Ax}(\mathit{Triv}_t)$ for something, one will assume it without a second thought.)

Let us turn to discussing the axioms in **Compl**. As we said these axioms will be divided into two groups: the essentials and the auxiliaries.

[278] For the definition of elementarily-equivalence cf. Def.3.8.17 on p.303.

[279] This is as much as possible for administrative reasons because of the following. In our frame language there are ω (i.e. countable) many formulas, hence there are 2^ω many sets of formulas, therefore there are at most 2^ω many different complete closed theories.

[280] A theory Th is called ***deductively closed*** iff for any formula ψ in the language of Th ($Th \models \psi \ \Rightarrow \ \psi \in Th$). As usual, by a ***theory*** we understand a set of (first-order) formulas.

[281] The duality theory in §6.6 makes **BaCo** match nicely with Minkowskian geometry.

[282] And we guess, nobody would seriously doubt that it is true in the real world.

Essential axioms

First, we recall the symmetry principle $\mathbf{Ax(symm_0)}$ from §2.8. Intuitively, $\mathbf{Ax(symm_0)}$ says that as m sees k so does some sister k' of k see some brother m' of m. (Two observers say m and m' are called brothers if they have the same life-line.) For more on the intuitive meaning of $\mathbf{Ax(symm_0)}$ the reader is referred to §2.8.

$\mathbf{Ax(symm_0)}$ $(\forall m, k \in Obs)(\exists m', k' \in Obs)$
$$\Big(tr_m(m') = tr_k(k') = \bar{t} \ \wedge \ \mathsf{f}_{mk} = \mathsf{f}_{k'm'}\Big).$$

We also recall that the symmetry principle $\mathbf{Ax(symm)}$ was defined to be $\mathbf{Ax(symm_0)}$+$\mathbf{Ax(eqtime)}$, where we will recall $\mathbf{Ax(eqtime)}$ in the list of the auxiliary axioms below.

Ax$\heartsuit$ below says that every body is an observer or a photon. (The main idea behind **Ax**$\heartsuit$ is to exclude non-inertial bodies. As it happens, **Ax**$\heartsuit$ says slightly more than this.)

Ax$\heartsuit$ $\quad B = Obs \cup Ph.$

Our next axiom deals with the direction of flow of time. To prepare the formulation of this axiom, we introduce the following definition.

Definition 3.8.1 We define binary relations $\uparrow, \downarrow \ \subseteq \ Obs \times Obs$ as follows. Let $m, k \in Obs$.

$$m \uparrow k \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t > 0, \quad \text{and}$$
$$m \downarrow k \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t < 0\,.$$

Intuitively, $m \uparrow k$ means that m thinks that k's clock runs forwards, while $m \downarrow k$ means that m thinks that k's clock runs backwards, see Figure 88.

$\lhd$

Intuitively, $\mathbf{Ax(\uparrow)}$ below says that for "observer brothers" time flows in the same direction.

$\mathbf{Ax(\uparrow)}$ $\quad (\forall m, m' \in Obs)\ \Big(tr_m(m') = \bar{t} \quad \Rightarrow \quad m \uparrow m'\Big).$

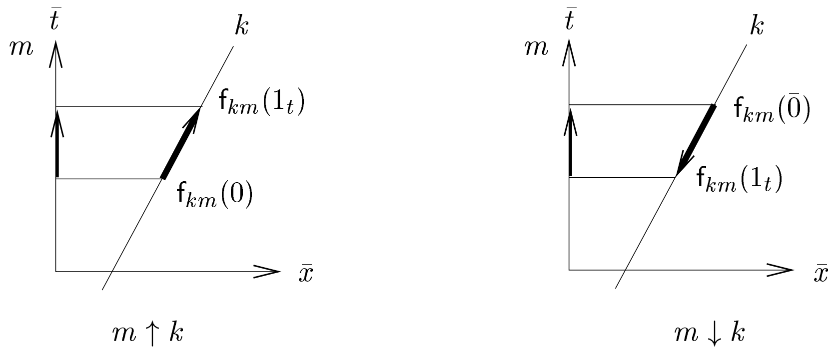


Figure 88: $m \uparrow k$ means that m sees that k's clock runs forwards.

Now, we turn to the auxiliary axioms.

Auxiliary axioms

In the axioms below we use the standard custom that free variables should be understood as universally quantified; e.g. if $\psi(x)$ is postulated as an axiom then it means $(\forall x)\ \psi(x)$.

Ax5$^+$ below is a stronger version of the "observer part" of **Ax5**, it says that any speed smaller than the speed of light is realized by an observer whose clock runs forwards (that is, every appropriate line is the life-line of an observer whose clock runs forwards.)

Ax5$^+$ $\quad \ell \in \mathsf{SlowEucl} \quad \Rightarrow \quad (\exists k \in Obs)\ (\ell = tr_m(k) \quad \wedge \quad m \uparrow k).$

The next axiom says that every observer can "re-coordinatize" his world-view by any trivial transformation.

Ax(*Triv*) $\quad (\forall f \in Triv)(\exists k \in Obs)\ \mathsf{f}_{mk} = f.$

To "make **Basax** complete" it will be enough to use the following weaker form **Ax(*Triv$_t$*)** of **Ax(*Triv*)** (instead of **Ax(*Triv*)**).

Ax(*Triv$_t$*) $\quad (\forall f \in Triv)\ \Big(f[\bar{t}] = \bar{t} \quad \Rightarrow \quad (\exists k \in Obs)\ \mathsf{f}_{mk} = f\Big).$

The next axioms say, intuitively, that of each kind of observers and bodies we have only one copy (or in other words, according to Leibniz's principle, if we cannot distinguish two observers or two bodies with some observable properties, then we treat them as equal). We also refer to this axiom as an axiom of extensionality.

Ax(ext) $(\forall m,k \in Obs)\ [w_m = w_k \ \Rightarrow\ m = k] \quad \wedge$
$$(\forall b, b_1 \in B \setminus Obs)(\forall m \in Obs)\ [tr_m(b) = tr_m(b_1) \ \Rightarrow\ b = b_1].$$

Now, we recall **Ax(eqtime)** from §2.8. Intuitively, it says that time passes with the same rate for "observer brothers" m and m'.

Ax(eqtime) $(\forall m, m' \in Obs)$
$$\Big(tr_m(m') = \bar{t} \quad \Rightarrow \quad (\forall p, q \in \bar{t}\,)\ |p - q| = |\mathsf{f}_{mm'}(p) - \mathsf{f}_{mm'}(q)|\Big).$$

Definition 3.8.2

$$\mathbf{Compl} \stackrel{\text{def}}{=} \{\mathbf{Ax(symm)},\ \mathbf{Ax}\heartsuit,\ \mathbf{Ax(\uparrow)},\ \mathbf{Ax5^+},\ \mathbf{Ax(ext)},\ \mathbf{Ax(}\mathit{Triv_t}\mathbf{)}\}\,.$$
$$\mathbf{BaCo} \stackrel{\text{def}}{=} \mathbf{Basax} + \mathbf{Compl}\,.$$

◁

CONVENTION 3.8.3 By $\mathbf{Compl} \setminus \{\mathbf{Ax(symm_0)}\}$ we understand $\{\mathbf{Ax(eqtime)},\ \mathbf{Ax}\heartsuit,\ \mathbf{Ax(\uparrow)},\ \mathbf{Ax5^+},\ \mathbf{Ax(ext)},\ \mathbf{Ax(}\mathit{Triv_t}\mathbf{)}\}$ and similarly for $\mathbf{BaCo} \setminus \{\mathbf{Ax(symm_0)}\}$.

◁

Let us start working in the direction of studying categoricity, completeness, and similar properties of our theories. From this point on, in the rest of the present work we will use the following two conventions.

CONVENTION 3.8.4 In the spirit of Convention 3.1.2 (p.160) we use the notions of <u>*homomorphism*</u>, <u>*isomorphism*</u>, and <u>*automorphism*</u> of both one-sorted and many-sorted structures in the usual sense. E.g. if $\mathfrak{N}$ and $\mathfrak{M}$ are similar many-sorted structures then by a homomorphism $h : \mathfrak{M} \longrightarrow \mathfrak{N}$ between them we mean the usual, structure-preserving map as defined in textbooks on many-sorted universal algebra or model theory, cf. e.g. Ehrig-Mahr [79] or Burmeister [53] or Lugowski [167] or Barwise-Feferman [43]. Cf. also footnote 310 on p.342.

◁

CONVENTION 3.8.5 Throughout this work by a <u>*theory*</u> we will understand an arbitrary set of first-order formulas in some fixed language of first-order logic (i.e. we will not assume that it is closed under semantical consequence). E.g. **Basax**, **BaCo** are theories.

$\triangleleft$

Definition 3.8.6 Let **Th** be a theory in our frame language and $\mathfrak{F}$ be an ordered field. Then **Th** is said to be <u>*$\mathfrak{F}$-categorical*</u> iff **Th** has exactly one model with ordered field reduct $\mathfrak{F}$, up to isomorphisms (i.e. iff any two models of **Th** with ordered field reduct $\mathfrak{F}$ are isomorphic).

$\triangleleft$

THEOREM 3.8.7 *Let $\mathfrak{F}$ be Euclidean (and let $n \geq 2$ be arbitrary). Then there is a unique model of* **BaCo** *with ordered field reduct $\mathfrak{F}$, up to isomorphisms. That is,* **BaCo** *is $\mathfrak{F}$-categorical.*

Proof: In §3.8.2 (p.322) for each Euclidean $\mathfrak{F}$ we will construct a model $\mathfrak{M}^M_{\mathfrak{F}}$, validating **BaCo**, which we will call the Minkowski model over $\mathfrak{F}$. In §3.8.3 (p.341) we will show that for each Euclidean $\mathfrak{F}$ every model of **BaCo** "over $\mathfrak{F}$" is isomorphic with the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ to be constructed in §3.8.2. Summing up, §3.8.2 is the proof of the existence part, while §3.8.3 is the proof of the uniqueness part of the theorem. ∎

In connection with the above proof, we note that §3.8.1, on median observers, is included in the present section to motivate the model construction in §3.8.2.

THEOREM 3.8.8 *Let $n > 2$ and let $\mathfrak{F}$ be Euclidean. Then there are exactly 2 models of* $\mathbf{BaCo} \setminus \{\mathbf{Ax(\uparrow)}\}$ *with ordered field reduct $\mathfrak{F}$, up to isomorphisms.*

The **proof** will be filled in later. ∎

Let us see how the conclusion of the above theorem changes for $n = 2$.

Conjecture 3.8.9 *Let $n = 2$ and let $\mathfrak{F}$ be Euclidean. Then there are exactly 4 models of* $\mathbf{BaCo} \setminus \{\mathbf{Ax(\uparrow)}\}$ *with ordered field reduct $\mathfrak{F}$, up to isomorphisms.*

$\triangleleft$

Remark 3.8.10 The reason for having more models for $\mathbf{BaCo} \setminus \{\mathbf{Ax}(\uparrow)\}$ in 2 dimensions (cf. 3.8.9) than in $n > 2$ dimensions (cf. 3.8.8) is the following. For $n = 2$, $\mathbf{BaCo} \setminus \{\mathbf{Ax}(\uparrow)\}$ allow FTL observers (cf. the proof of Thm.2.8.2), while for $n > 2$ already **Basax** excludes FTL observers (cf. 3.4.1).

$\lhd$

The above remark is in contrast with Thm.3.8.7 above. In view of the above remark one may ask, why we have then the same number of models for **BaCo** for all n, cf. Thm.3.8.7 above. In connection with this we state the following.

THEOREM 3.8.11
$\mathbf{Basax}(2) + \mathbf{Ax(symm_0)} + \mathbf{Ax}(\uparrow) + \mathbf{Ax}(\sqrt{\ }) \models$ **"$\nexists$ FTL observers"**.

Outline of proof: The proof goes by contradiction. Assume m sees k moving FTL. By $\mathbf{Ax(symm_0)}$, as m sees k so does some sister k' of k see some brother m' of m. (Let such k', m' be fixed.) This implies that if m sees that the clock of k is running forwards/backwards then k' sees that the clock of m' is doing the same, formally:

$$(m \uparrow k \;\Rightarrow\; k' \uparrow m') \quad \text{and} \quad (m \downarrow k \;\Rightarrow\; k' \downarrow m')\,. \tag{104}$$

By $\mathbf{Ax}(\uparrow)$ we have that for observers brothers/sisters times flow in the same direction, i.e.

$$m \uparrow m' \quad \text{and} \quad k \uparrow k'\,.$$

Now, from this, from (104), and from the facts that the world-view transformations (the f_{mk}'s) are both photon-line preserving and betweenness preserving (by $\mathbf{Ax}(\sqrt{\ })$ and Prop.3.6.5(ii)), one can prove the following.[283] If m sees that clock of k is running forwards/backwards then k sees that clock of m is doing the same, formally:

$$(m \uparrow k \;\Rightarrow\; k \uparrow m) \quad \text{and} \quad (m \downarrow k \;\Rightarrow\; k \downarrow m)\,.$$

This contradicts Thm.2.7.4 which says that if m sees k moving FTL then m and k see each others clocks differently. ∎

Now, we turn to formulating completeness type theorems. As usual, first we need some definitions.

[283] We omit the proof of this step.

Definition 3.8.12

(i) An ordered field $\mathfrak{F}$ is called *real-closed* if it is Euclidean (i.e. every positive element has a square root), and if every polynomial of odd degree has a zero. The latter requirement can be expressed with the set $\{ \phi_{2n+1} : n \in \omega \}$ of first-order formulas, where for every $n \in \omega$, ϕ_n denotes the following formula

$$(\phi_n) \qquad \forall x_0 \ldots \forall x_n \exists y \, [x_n \neq 0 \;\to\; (x_0 + x_1 \cdot y + \ldots + x_n \cdot y^n = 0)].$$

(ii) The *axiom system* $\mathbf{Ax(rc)}$ of *real-closed fields* is defined as follows.

$$\mathbf{Ax(rc)} \stackrel{\text{def}}{=} \mathbf{Ax}(\sqrt{\ }\,) + \{ \phi_{2n+1} \; : \; n \in \omega \} ,$$

where ϕ_n is as given above. Let us notice that in some sense $\mathbf{Ax(rc)}$ is defined by a *finite schema* of axioms. Therefore $\mathbf{BaCo} + \mathbf{Ax(rc)}$ remains in some sense finitistic. (Therefore the deductively closed theory generated by $\mathbf{BaCo} + \mathbf{Ax(rc)}$ remains axiomatizable by a finite schema.)

$\lhd$

Definition 3.8.13 A theory $\mathbf{Th}$ is called *complete* iff it is consistent and it implies either ψ or $\neg\psi$, for each closed (first-order) formula ψ (of its language), that is, for each closed ψ, either $\mathbf{Th} \models \psi$ or $\mathbf{Th} \models \neg\psi$ holds.[284]

$\lhd$

THEOREM 3.8.14

(i) $\mathbf{BaCo} + \mathbf{Ax(rc)}$ *is a complete theory.*

(ii) *Assume $\mathfrak{F}$ is Euclidean. Then* $\mathbf{BaCo} + \mathsf{Th}(\mathfrak{F})$ *is a complete theory.*[285]

In the proof we will use the following known fact, cf. e.g. Chang-Keisler [59].

[284] A theory is *consistent* iff it has a model; a formula is *closed* iff it has no free variables (i.e. every variable in the formula occurs under the scope of some quantifier.)

[285] Recall that $\mathsf{Th}(\mathfrak{F}) := \{ \psi \; : \; \mathfrak{F} \models \psi \}$.

FACT 3.8.15

(a) *The theory* $\mathbf{Ax(rc)}$ *of real-closed fields is complete.*

(b) $\mathfrak{R} \models \mathbf{Ax(rc)}$, *hence the closed theory generated*[286] *by* $\mathbf{Ax(rc)}$ *is* $\mathsf{Th}(\mathfrak{R})$.

$\triangleleft$

Proof of Thm.3.8.14: The proof is based on Thm.3.8.7. Let us notice that by Fact 3.8.15 it is enough to prove (ii) of the theorem. Let $\mathfrak{F}$ be Euclidean. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of $\mathbf{BaCo} + \mathsf{Th}(\mathfrak{F})$. We cannot apply Thm.3.8.7 yet, because the ordered field reducts $\mathfrak{F}_0$ and $\mathfrak{F}'_0$ of $\mathfrak{M}$ and $\mathfrak{M}'$ respectively may not be the same. But they are elementarily-equivalent[287], since $\mathsf{Th}(\mathfrak{F})$ is complete, so by the Keisler-Shelah isomorphic ultrapowers theorem (cf. e.g. Chang-Keisler [59]) they have isomorphic ultrapowers, say $\mathfrak{F}_1$ and $\mathfrak{F}'_1$ taken by an ultrafilter, say, U. Let $\mathfrak{M}_1$ and $\mathfrak{M}'_1$ be the ultrapowers of $\mathfrak{M}$ and $\mathfrak{M}'$ respectively, taken by the ultrafilter U used before. Then the field-reducts of these are $\mathfrak{F}_1$ and $\mathfrak{F}'_1$ respectively. Now we can apply Thm.3.8.7 to $\mathfrak{M}_1$ and $\mathfrak{M}'_1$ because $\mathfrak{F}_1$ and $\mathfrak{F}'_1$ are isomorphic, getting that $\mathfrak{M}_1$ and $\mathfrak{M}'_1$ are isomorphic, so elementarily-equivalent. But then $\mathfrak{M}$ and $\mathfrak{M}'$ are elementarily-equivalent, too, since the former two models are ultrapowers of these. This completes the proof since if all models of a theory $\mathbf{Th}$ are elementarily-equivalent then $\mathbf{Th}$ is complete. ∎

THEOREM 3.8.16 $\mathbf{BaCo} + \mathbf{Ax(rc)}$ *is a* <u>decidable</u> *theory.*[288]

Proof: The theorem is a corollary of Thm.3.8.14(i) because of the following. The theory $\mathbf{BaCo} + \mathbf{Ax(rc)}$ is recursively enumerable[289] since it is axiomatized by a finite schema. This, by the fact (known from basic logic) that every recursively enumerable complete theory is decidable, completes the proof. ∎

Results related to the above theorem will be discussed in §7 the subject of which is decidability and Gödel incompleteness.

nem kéne-e utalni az $\mathbf{Ax}(\uparrow\ )$-as tételre? $\mathbf{Ax(rc)}$n jó-e itt?

[286]By the *closed theory generated* by say $\mathbf{Ax(rc)}$, we understand the smallest deductively closed theory containing $\mathbf{Ax(rc)}$ (cf. footnote 280 on p.295).

[287]Cf. Def.3.8.17 below for elementary equivalence of models.

[288]A theory $\mathbf{Th}$ is *decidable* iff there is an algorithm which decides the set $\{\psi : \mathbf{Th} \models \psi\}$ of its consequences.

[289]A set T (of formulas or numbers) is called *recursively enumerable* iff there is an algorithm (or Turing machine) which lists (i.e. enumerates) all the elements of T, cf. any logic book for details. We note that any decidable T is recursively enumerable, but there are recursively enumerable sets which are not decidable.

As we already said, all the axioms in **Compl** turn out to have an interesting property except for **Ax(↑)**. This property is that if we omit the axiom in question from **BaCo+Ax(rc)** then all of a sudden we will have the opposite of completeness. This is the intuitive content of our next theorem (Thm.3.8.18).

Ax(rc) biztos-e, hogy megfelelő!

Definition 3.8.17 Two models $\mathfrak{M}$ and $\mathfrak{N}$ (of the same fist-order language) are called <u>*elementarily-equivalent*</u> iff there is no formula (on their language) which would distinguish them, that is, $\mathsf{Th}(\mathfrak{M}) = \mathsf{Th}(\mathfrak{N})$.

$\triangleleft$

THEOREM 3.8.18 *Let $\mathfrak{F}$ be Euclidean and* $\mathbf{Ax} \in \mathbf{Compl} \setminus \{\mathbf{Ax(\uparrow)}\}$. *Then (i) and (ii) below hold.*

- **(i)** *The theory* $\mathbf{BaCo} \setminus \{\mathbf{Ax}\} + \mathsf{Th}(\mathfrak{F})$ *has* 2^ω *(i.e. continuum) many non-elementarily-equivalent models.*
- **(ii)** *There are* 2^ω *many different complete deductively closed* [290] *theories extending* $\mathbf{BaCo} \setminus \{\mathbf{Ax}\} + \mathsf{Th}(\mathfrak{F})$.

as examples

On the proof: We note that item (ii) is an equivalent reformulation of item (i). A proof can be obtained as follows. Assume $\mathfrak{F}$ is Euclidean. Assume $\mathbf{Ax} \in \mathbf{Compl} \setminus \{\mathbf{Ax(\uparrow)}\}$. Assume that a set $\{\, \psi_j \,:\, j \in \omega \,\}$ of formulas satisfying $(\star)$ below has already been given.

$$(\star) \qquad \begin{array}{l} \text{For every } I \subseteq \omega \text{ there is a model } \mathfrak{M}_I \text{ of } \quad \mathbf{BaCo}\setminus\{\mathbf{Ax}\}+\mathsf{Th}(\mathfrak{F}) \\ \text{such that} \quad \mathfrak{M}_I \models \{\, \psi_j \,:\, j \in I \,\} \quad \text{and} \quad (\forall j \in \omega \setminus I)\ \mathfrak{M}_I \not\models \psi_j. \end{array}$$

Let such $\mathfrak{M}_I$'s be fixed. Then, clearly, if $I, J \subseteq \omega$ with $I \neq J$ then $\mathfrak{M}_I$ and $\mathfrak{M}_J$ will not be elementarily-equivalent. So, the set $\{\, \mathfrak{M}_I \,:\, I \subseteq \omega \,\}$ will consist of 2^ω many non-elementarily-equivalent $\mathbf{BaCo} \setminus \{\mathbf{Ax}\} + \mathsf{Th}(\mathfrak{F})$ models. As examples, below we give sets $\{\, \psi_j \,:\, j \in \omega \,\}$ of formulas satisfying $(\star)$ for the cases $\mathbf{Ax} = \mathbf{Ax(ext)}$ and $\mathbf{Ax} = \mathbf{Ax}\heartsuit$.

<u>*Case* **Ax = Ax(ext)**:</u> Let $j \in \omega$. The formula ψ_j below says that there cannot be exactly j many different observers having the same world-view function (i.e. if there are j many different observers having the same world-view function then there is a $(j+1)$'th observer with the same world-view function).

$$(\psi_j) \qquad \begin{array}{c} (\forall m_0, m_1, \ldots, m_{j-1} \in Obs) \Big((\forall i, i' \in j)(m_i \neq m_{i'} \ \wedge \ w_{m_i} = w_{m_{i'}}) \ \Rightarrow \\ (\exists m \in Obs)(\forall i \in j)(m \neq m_i \ \wedge \ w_m = w_{m_i}) \Big). \end{array}$$

monat?

Checking that $\{ \psi_j : j \in \omega \}$ has the property $(\star)$ above is left to the reader.

<u>*Case* $\mathbf{Ax} = \mathbf{Ax}\heartsuit$*:*</u> Let $j \in \omega$. The formula ψ_j below says that there are no two bodies meeting each other exactly j times (i.e. if there are two bodies meeting each other j times then they will meet each other $j+1$ times too).

$$
(\psi_j) \quad \begin{array}{l} (\forall b, b' \in B)(\forall p^0, p^1, \ldots, p^{j-1} \in {}^nF) \\ \Big([\, (\forall i, i' \in j) p^i \neq p^{i'} \;\wedge\; p^0, p^1, \ldots, p^{j-1} \in tr_m(b) \cap tr_m(b') \,] \;\Rightarrow \\ \qquad (\exists p \in {}^nF)\, [\, (\forall i \in j)\; p \neq p^i \;\wedge\; p \in tr_m(b) \cap tr_m(b_1) \,] \Big). \end{array}
$$

Checking that for $\{ \psi_j : j \in \omega \}$, $(\star)$ above holds is left to the reader.

The rest of the proof will be filled in at a later stage of development. ∎

We will see in §7 (on "decidability and Gödel incompleteness") that to any one of the theories $\mathbf{BaCo} \setminus \{\mathbf{Ax}\} + \mathsf{Th}(\mathfrak{F})$, discussed in Thm.3.8.18 above, the conclusion of Gödel's incompleteness theorem also applies.

Remark 3.8.19 The conclusions of the above theorem (Thm.3.8.18) apply to the finite schema axiomatizable theory $\mathbf{BaCo} + \mathbf{Ax(rc)}$, for the obvious reason that the closed theory generated by $\mathbf{Ax(rc)}$ is $\mathsf{Th}(\mathfrak{R})$ (cf. 3.8.15).

$\lhd$

meg kell-e említeni, hogy $\mathbf{Ax(symm_0)} \notin$ *Compl*?

Conjecture 3.8.20 *We conjecture that* $\mathbf{BaCo} \setminus \{\mathbf{Ax(symm_0)}\} + \mathbf{Ax(rc)}$ *will turn out to have* 2^ω *many non-elementarily-equivalent models.*

$\lhd$

The following is a corollary of the proof (to be given in §3.8.3) for the uniqueness part of the categoricity theorem in this section (Thm.3.8.7). A further reason for stating this corollary here is that it is strongly connected to Einstein's Special Principle of Relativity, and the same applies to one of the "main characters" of this section, namely to $\mathbf{Ax(symm)}$. The connection between $\mathbf{Ax(symm)}$ and Einstein's Special Principle of Relativity was discussed in §2.8. More discussion of Einstein's Special Principle of Relativity (in terms of logic, of course) comes in §3.9.

automorfizmus def.?

COROLLARY 3.8.21 Assume $\mathfrak{M} \models \mathbf{BaCo}$. Then

$$
(\forall m, k \in Obs) \Big(\text{there is an automorphism } \alpha \text{ of } \mathfrak{M} \text{ with } \alpha(m) = k\Big). \qquad \blacksquare
$$

[290] As we already said, a theory $\mathbf{Th}$ is called deductively closed iff for any formula ψ in the language of $\mathbf{Th}$ ($\mathbf{Th} \models \psi \;\Rightarrow\; \psi \in \mathbf{Th}$).

As we already said, the above corollary is strongly related to Einstein's Special Principle of Relativity, cf. e.g. Friedman [90, §IV.5].[291] Clearly, the corollary above says that all the observers are equivalent in a rather strong sense, under assuming **BaCo** of course. (Namely, being connected by an automorphism is an extremely strong form of equivalence.) In a later version we will discuss, under what assumptions (in place of **BaCo**) is the above corollary still true. It is not impossible that the answer will turn out to be something like either **Newbasax** + **Compl** or **BaCo** $\setminus$ {**Ax(↑)**} + **(no FTL obs.)**.

Since in this section we are discussing theories containing **Basax** + **Ax(symm)**, we include the following conjecture (though it does not strictly belong to the subject matter of this section).

Conjecture 3.8.22 **(i)** **Basax** + **Ax(symm)** $\models$ **Ax**($\sqrt{\ }$).

(ii) **Basax** + **Ax(symm)** $\models (\forall m, k \in Obs)\ \mathsf{f}_{mk} \in Poi$.

$\lhd$

In connection with the above conjecture, we note that
Basax + **Ax(symm)** + **Ax**($\sqrt{\ }$) $\models (\forall m, k \in Obs)\ \mathsf{f}_{mk} \in Poi$ (cf. Thm.2.9.5 on p.155), and that **Basax**(3) $\models$ **Ax**($\sqrt{\ }$) (cf. Thm.3.6.17 on p.274).

We conjecture that **Newbasax** + **Compl** + **Ax(rc)** has 2^ω many non-elementarily-equivalent models, but we did not have time to check this.

[291] Cf. also the discussion of Friedman's conceptual analysis of the various principles occurring in relativity theory in our §§3.4.2, 4.4.

3.8.1 Median observer

This sub-section is about the following idea. Whenever we have two observers m and k then to this two ones there is a third observer h who sees m and k exactly the same way. More precisely h will think that m and k are mirror images of each other w.r.t. the time-axis $\bar{t}$. (We mean this under some axioms, of course.) This third observer h will be called a median observer for observers m and k. These median observers will be used in several proofs, and the will also help us to visualize (in a simple way) certain situations.

$\sigma_{\bar{t}}$ defined below is a special case of σ_ℓ which was defined in Def.3.1.20(i) on p.173.

Definition 3.8.23

(i) We let $\sigma_{\bar{t}} : {}^nF \longrightarrow {}^nF$ denote the reflection w.r.t. the time-axis $\bar{t}$, more concretely:

$$\sigma_{\bar{t}} \in Linb \qquad \text{with} \qquad \Big(\sigma_{\bar{t}}(1_t) = 1_t \quad \text{and} \quad (\forall 0 < i \in n)\ \sigma_{\bar{t}}(1_i) = -1_i\Big).$$

(ii) Let $p, q \in {}^nF$. We say that p and q are <u>$\bar{t}$-symmetric</u> iff $\sigma_{\bar{t}}(p) = q$. Let $P, Q \subseteq {}^nF$. We say that P and Q are <u>$\bar{t}$-symmetric</u> iff $\sigma_{\bar{t}}[P] = Q$.

$\lhd$

Definition 3.8.24 Let $m, k \in Obs$. Then $h \in Obs$ is called a <u>median</u> observer for observers m and k iff $tr_h(m)$ and $tr_h(k)$ are $\bar{t}$-symmetric.

$\lhd$

THEOREM 3.8.25 *Assume* **Basax** + **Ax**$(\sqrt{\ })$.

(i) *Then for every $m, k \in Obs$ with $v_m(k) < 1$ there is a median observer h for observers m and k.*

(ii) *A median observer h for observers m and k, with $v_m(k) < 1$, can be constructed as illustrated in Figures 90, 91, 93.*

The idea of the proof of the above theorem is in Figures 90–94. For this proof we will need Lemma 3.8.28 and its corollary Cor.3.8.30 below, therefore the proof of the theorem comes below Cor.3.8.30 on p.309.

Remark 3.8.26 Assume **Basax** + $\mathbf{Ax}(\sqrt{\ })$. Assume $m, k \in Obs$ such that k moves FTL relative to m (i.e. $v_m(k) > 1$). Then, by Thm.2.7.2,[292] there is no median observer for observers m and k.

$\triangleleft$

ez már jó lenne a **Basax** $\models$ $\mathbf{Ax}(\sqrt{\ })$ bizjához is!

CONVENTION 3.8.27 Throughout, we will use the fact that all axioms of Euclidian geometry, with the exception of the Axiom of Continuity are true in geometries over arbitrary Eucledian fields. Therefore we will use those elementary theorems of geometry which do not need the continuity axiom.

$\triangleleft$

$v_m(k) = v_k(m)$-es tételnél bizonyításánál a §2.8 végén azt mondani, hogy tetszőleges n-re a bizonyítás átmegy, és ez a következő lemma miatt. Be kell vezetni, hogy egy sík mikor párhuzamos egy egyenessel! mirror image w.r.t. a line-t bevezetni!

LEMMA 3.8.28 *Assume* **Basax**. *Assume $m, k \in Obs$. Assume P is a plane parallel with $\bar{t}$ such that $tr_m(k) \subseteq P$. Assume $\ell \in$ Eucl such that ℓ is the mirror image of $tr_m(k)$ w.r.t. a photon-line ℓ' with $\ell' \subseteq P$. (Let us notice that then $\ell \subseteq P$.) Then events happening on ℓ in the world-view of m are all simultaneous for k. That is,* $(\forall p, q \in \ell)\ \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t$.

Proof: The proof will be similar to that of Prop.3.1.21. The idea of the proof is illustrated in Figure 89. Throughout the proof we will need the following claim.

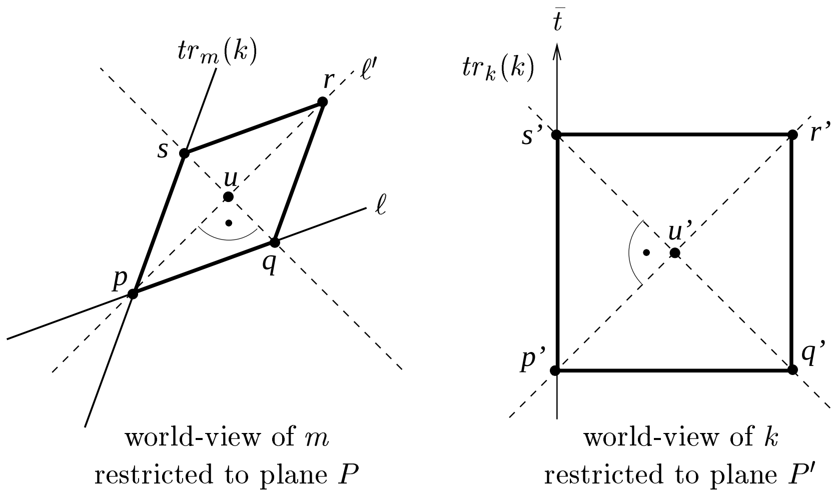


Figure 89: Illustration for the proof of Lemma 3.8.28.

[292]and by the no FTL thm. for $n > 2$, cf. Thm.3.4.1

Claim **3.8.29**

(i) Assume P is a plane parallel with $\bar{t}$. Assume $\ell_1 \in \mathsf{Eucl}$ and $\ell_2 \in \mathsf{PhtEucl}$ such that $\ell_1, \ell_2 \subseteq P$ and $\ell_1 \perp_e \ell_2$. Then $\ell_2 \in \mathsf{PhtEucl}$.

(ii) Assume $v, w, z \in {}^nF$ are distinct points such that $\overline{vw} = \bar{t}$ and $\overline{vz}, \overline{wz} \in \mathsf{PhtEucl}$. Then $\overline{vz} \perp_e \overline{wz}$ and segments vz and wz are of equal length (i.e. $\|v - z\| = \|w - z\|$).

The proof of this claim is straightforward, therefore we omit it.

Let us turn to proving Lemma 3.8.28. Throughout the proof the reader is asked to consult Figure 89. Assume **Basax**. Assume that $m, k \in Obs$, that P is a plane parallel with $\bar{t}$ such that $tr_m(k) \subseteq P$, and that $\ell' \in \mathsf{PhtEucl}$ such that $\ell' \subseteq P$. Let ℓ be the mirror image of $tr_m(k)$ w.r.t. ℓ'. (Then $\ell \subseteq P$.) Let $p \in tr_m(k) \cap \ell \cap \ell'$. Such p exists and is unique. Let $q \in \ell$ with $q \neq p$ be arbitrary. To prove the lemma, we will prove that events $w_m(p)$ and $w_m(q)$ are simultaneous for observer k. Let s be the mirror image of q w.r.t. ℓ', let u be the midpoint of segment qs, and let $r \in \ell'$ be such that u will be the midpoint of segment pr. Then, obviously, $s \in tr_m(k)$. Further, $\overline{qs} \in \mathsf{PhtEucl}$ by Claim 3.8.29(i). So,

Hogyan kell a bizonyítást elkezdeni úgy, hogy ne kelljen a feltételeket megismételni?

(109) $$\overline{qs} \in \mathsf{PhtEucl} \quad \text{and} \quad \overline{pr} = \ell' \in \mathsf{PhtEucl}\,.$$

Let us consider the quadrangle $\langle p, q, r, s\rangle$. Its diagonals bisect each other, so we have that

(110) $$\langle p, q, r, s\rangle \text{ is a parallelogram.}$$

Let p', q', r', s', u', P' denote, respectively, the f_{mk} images of p, q, r, s, u, P. (Let us notice that P' is a plane since P is a plane, cf. 3.1.16.) Then by (109) and (110) (and by f_{mk} being a bijection preserving Eucl and $\mathsf{PhtEucl}$) we have that

(111) $$\langle p', q', r', s'\rangle \text{ is a parallelogram} \quad \text{and} \quad \overline{q's'},\ \overline{p'r'} \in \mathsf{PhtEucl}\,.$$

Since $p, s \in tr_m(k)$, we have that
$$\overline{p's'} = \bar{t}$$
by **Ax4**. Now, by (111) and by applying Claim 3.8.29(ii) for $v := p'$, $z := u'$, $w := s'$ one concludes that $\langle p', q', r', s'\rangle$ is a square whose side $p's'$ is lying on the $\bar{t}$ axis. This implies that $p'_t = q'_t$. Hence, events $w_m(p)$ and $w_m(q)$ are simultaneous for k. ∎

square-t definiélni!

COROLLARY 3.8.30 Assume **Basax**. Let $m, k \in Obs$ such that m and k do not move relative to each other, that is, $tr_m(k) \parallel \bar{t}$. Then events which are simultaneous for m remain simultaneous for k.

Proof: Assume **Basax**. Let $m, k \in Obs$ with $tr_m(k) \parallel \bar{t}$. Let $p \in tr_m(k)$ be fixed. By Lemma 3.8.28, we have that

$$(112) \quad (\forall 0 < i \in n)\Big(\text{events } w_m(p) \text{ and } w_m(p+1_i) \text{ are simultaneous for } k\Big),$$

since $(\forall 0 < i \in n)$ $\Big(\overline{p(p+1_i)}$ is the mirror image of $tr_m(k)$ w.r.t. the photon-line $\{\, p + \lambda \cdot (1_t + 1_i) \,:\, \lambda \in F \,\}\Big)$. By f_{mk} being a bijection taking straight lines to straight lines (and therefore taking planes to planes, parallel lines to parallel ones etc.), (112) implies that any two events which are simultaneous for m remain simultaneous for k. ∎

Proof of Thm. 3.8.25: Assume **Basax** + **Ax**($\sqrt{\ }$). Let $m, k \in Obs$ with $v_m(k) < 1$. We distinguish two cases:
Case (I): m and k meet each other, or their life-lines are parallel (i.e. $tr_m(k) \cap \bar{t} \neq \emptyset$ or $tr_m(k) \parallel \bar{t}$).
Case (II): The life-lines of m and k are skew lines (i.e. there is no plane containing both $\bar{t}$ and $tr_m(k)$).

perpendicular bisector in a plane-t kéne definiálni!

Proof for Case (I): Assume first, that the life-lines of m and k are parallel, i.e. $tr_m(k) \parallel \bar{t}$. Let $\mathrm{A} \in \bar{t}$ and $\mathrm{B} \in tr_m(k)$ such that $\mathrm{A}_t = \mathrm{B}_t$. Let $\ell \in \mathsf{Eucl}$ be the perpendicular bisector of segment AB in $\mathsf{Plane}(\bar{t}, tr_m(k))$, see Figure 90. Let h be an observer whose life-line is ℓ, i.e. let $h \in Obs$ such that $tr_m(h) = \ell$. (Such an h exists by **Ax5**.) This h is a median observer for observers for m and k. Checking the details is left to the reader.

Now, assume that m and k meet each other. See Figure 91. We note that the proof might be easier to understand if the reader assumes (at lest at the first reading) that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$, $tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and that the whole construction is happening in $\mathsf{Plane}(\bar{t}, \bar{x})$.[293]

Let $\mathrm{O} \in \bar{t} \cap tr_m(k)$, i.e. m and k meet each other at point O in the world-view of m. Let $\mathrm{A} \in \bar{t}$ with $\mathrm{A} \neq \mathrm{O}$ and let $\mathrm{B} \in \ell$ such that $\mathrm{B}_t = \mathrm{A}_t$. Let $\ell' \in \mathsf{Eucl}$ be the perpendicular bisector of segment AB in $\mathsf{Plane}(\bar{t}, tr_m(k))$. (Let us notice that $\ell' \parallel \bar{t}$ by $\mathrm{A}_t = \mathrm{B}_t$.) Let $\mathcal{C}$ be the circle with diameter OA in $\mathsf{Plane}(\bar{t}, tr_m(k))$. By $\mathfrak{F}$ being Euclidean $\mathcal{C} \cap \ell' \neq \emptyset$.[294] Let $\mathrm{C} \in \mathcal{C} \cap \ell'$ such that $\overline{\mathrm{OC}} \in \mathsf{SlowEucl}$. Such a C exists, and is unique. Let $h \in Obs$ such that $tr_m(h) = \overline{\mathrm{OC}}$. (Such an h exists by **Ax5**.) We will prove that this h is a median observer for m and k. The proof of this is

circle definilálni!

[293] By this we mean to say that all the basic ideas come up in this special case, therefore it might be a practical idea to concentrate on this case only.
[294] This is so because the length of segment OA is greater than that of AB (by $v_m(k) < 1$).

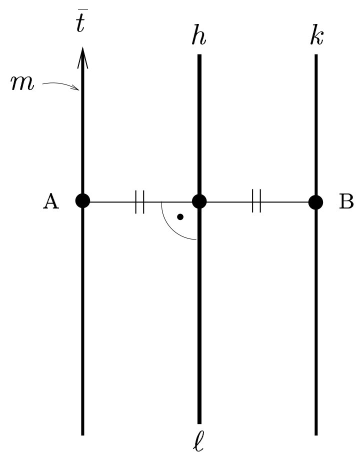


Figure 90: h is a median observer for m and k (for the case when $tr_m(k) \parallel \bar{t}$).

illustrated in Figure 92 and goes as follows. Let $\mathrm{D} \in \bar{t} \cap \overline{\mathrm{BC}}$. (Such a D exists and is unique.) First, we prove that events $w_m(\mathrm{D})$, $w_m(\mathrm{C})$ and $w_m(\mathrm{B})$ are simultaneous for observer h. $\overline{\mathrm{OC}} \perp_e \overline{\mathrm{AC}}$ since C is on the circle $\mathcal{C}$ whose diameter is segment OA. Let us recall that ℓ' is the perpendicular bisector of segment AB (in $\mathsf{Plane}(\bar{t}, tr_m(k))$), and that $\ell' \parallel \bar{t}$. Let $\mathcal{T}$ be the composition of the following two transformations of[295] $\mathsf{Plane}(\bar{t}, tr_m(k))$: "the rotation by 90^o about C" and "the reflection w.r.t. line ℓ' ". Transformation $\mathcal{T}$ takes $tr_m(h) = \overline{\mathrm{OC}}$ to $\overline{\mathrm{CB}}$. On the other hand, it is not hard to prove that $\mathcal{T}$ is a reflection [of $\mathsf{Plane}(\bar{t}, tr_m(k))$] w.r.t. a photon-line ℓ_{ph} with $\mathrm{C} \in \ell_{ph} \subseteq \mathsf{Plane}(\bar{t}, tr_m(k))$. Therefore $\overline{\mathrm{CB}}$ is the mirror image of $tr_m(h)$ w.r.t. a photon-line $\ell_{ph} \subseteq \mathsf{Plane}(\bar{t}, tr_m(k))$. But this, by Lemma 3.8.28, implies that

(113) events $w_m(\mathrm{B})$, $w_m(\mathrm{C})$ and $w_m(\mathrm{D})$ are simultaneous for h.

Let us notice that by our construction

(114) C is the midpoint of segment BD. [296]

Let us switch over (from the world-view of m) to the world-view of h. Let $\mathrm{B}', \mathrm{C}', \mathrm{D}'$ denote the f_{mh} images of B, C, D, respectively. Then by (113) and (114), respectively, we have that (115) and (116) below hold.

(115) $$\mathrm{B}'_t = \mathrm{C}'_t = \mathrm{D}'_t .$$

[295] I.e. the transformations take the plane into itself.

[296] This is so because ℓ' is the midline of triangle ABD by $\ell' \parallel \bar{t}$ and by ℓ' being a bisector of segment AB.

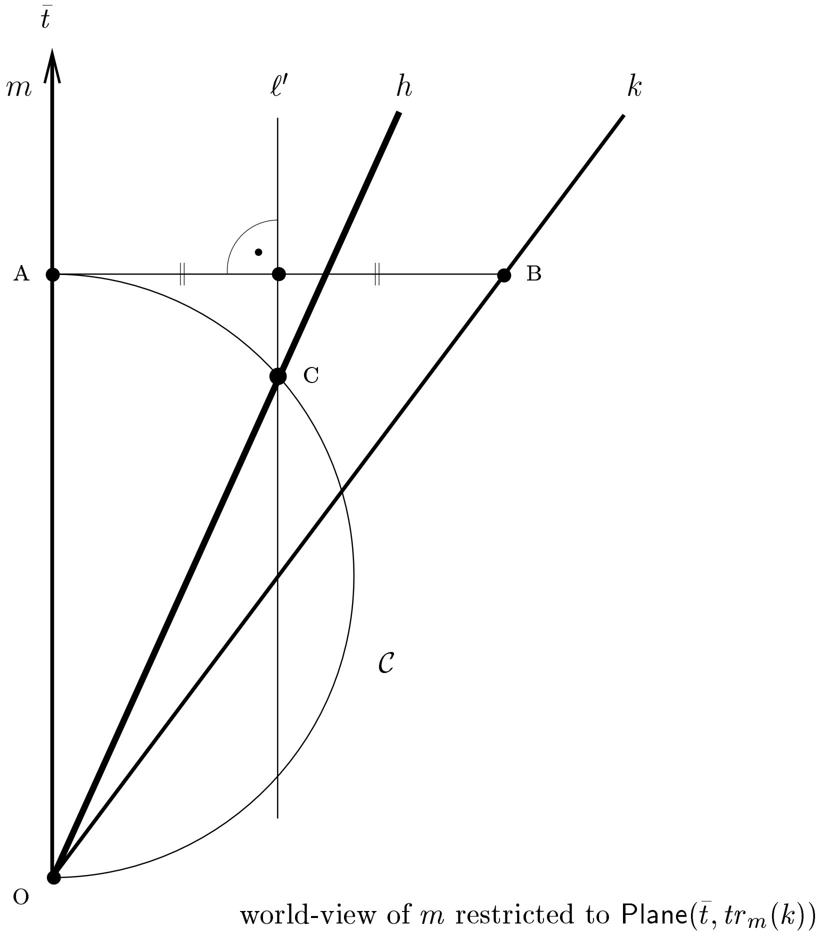
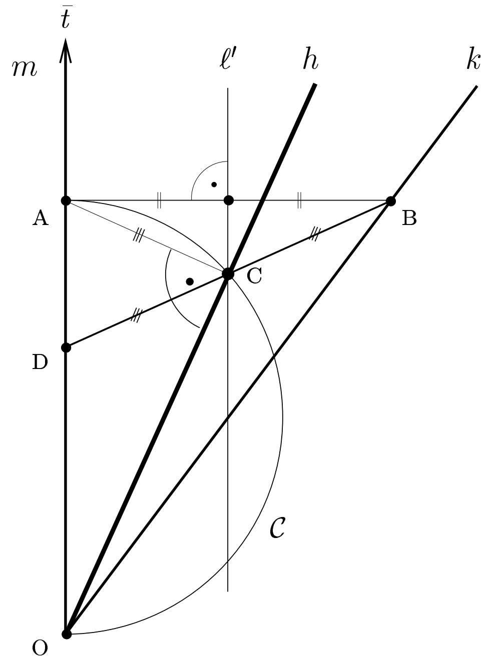


Figure 91: h is a median observer for m and k. (Construction for the case when m and k meet each other).

world-view of m restricted
to $\mathsf{Plane}(\bar{t}, tr_m(k))$:



world-view of h restricted
to $\mathsf{Plane}(\bar{t}, tr_h(k))$:

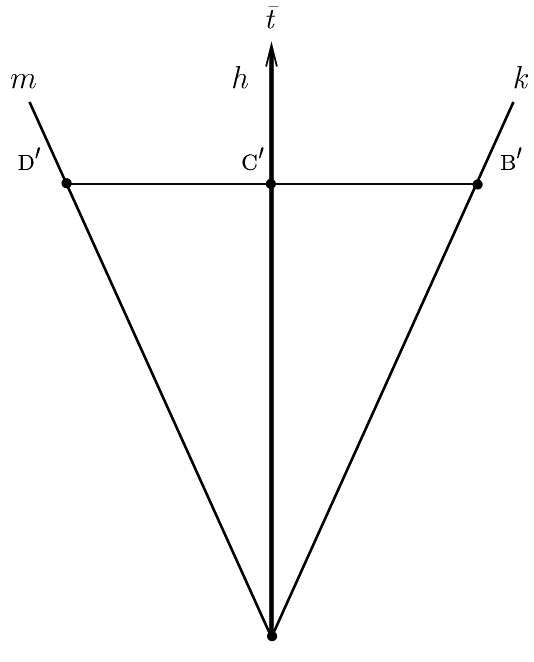


Figure 92: This is the main idea of the proof that h constructed as in Figure 91 is a median observer for observers m and k.

(116) $\quad$ C' is the midpoint of segment $B'D'$.

Let us notice that $C' \in \bar{t}$, [297] and that $B' \in tr_h(k)$, $D' \in tr_h(m)$. But then, by (115) and (116), we conclude that $tr_h(m)$ and $tr_h(k)$ are $\bar{t}$-symmetric, so h is a median observer for observers m and k. By this the theorem is proved for Case (I).

<u>*Proof for Case (II):*</u> Let us recall that in this case the life-lines of m and k are skew lines. The proof is illustrated in Figures 93, 94.

Let $k' \in Obs$ such that $tr_m(k') \parallel tr_m(k)$ and $tr_m(k') \cap \bar{t} \neq \emptyset$. Such a k' exists. Let h' be a median observer for observers m and k'. Such an h' exists by Case (I). We will construct a median observer h for m and k in the world-view of h'. Let us notice that $tr_{h'}(k') \parallel tr_{h'}(k)$. See Figure 94. Let $A \in \bar{t} \cap tr_{h'}(k') \cap tr_{h'}(m)$. Such an A exists and is unique. Let $B \in tr_{h'}(k)$ such that

(117) $$B_t = A_t\,.$$

Such a B exists. Let C be such that

(118) $\quad$ C is the midpoint of segment AB.

Let $\ell \in \mathsf{Eucl}$ such that $C \in \ell \parallel \bar{t}$. Let $h \in Obs$ such that $tr_{h'}(h) = \ell$. We will prove that h is a median observer for observers m and k. Let $M \in tr_{h'}(m)$, such that $M \neq A$, further let $H \in tr_{h'}(h)$ and $K \in tr_{h'}(k)$ such that

(119) $$M_t = H_t = K_t\,.$$

We will prove that

(120) $\quad$ H is the midpoint of segment MK.

To prove (120) let K' and H' be, respectively, points on $tr_{h'}(k')$ and $tr_{h'}(h') = \bar{t}$ such that ${K'}_t = {H'}_t = M_t$. So, events happening at points M, H, H', K, K' are simultaneous for h' by the choice of these points. It is easy to check that (121) and (122) below hold.

(121) $\quad$ $\overline{H'H} \parallel \overline{AC}$ and (segments $H'H$ and AC are of equal length).

(122) $\quad$ $\overline{K'K} \parallel \overline{AB}$ and (segments $K'K$ and AB are of equal length).

Further, by h' being a median observer for observers m and k', we have that

(123) $\quad$ H' is the midpoint of segment MK'.

[297]This is so by $C \in tr_m(h)$ and **Ax4**.

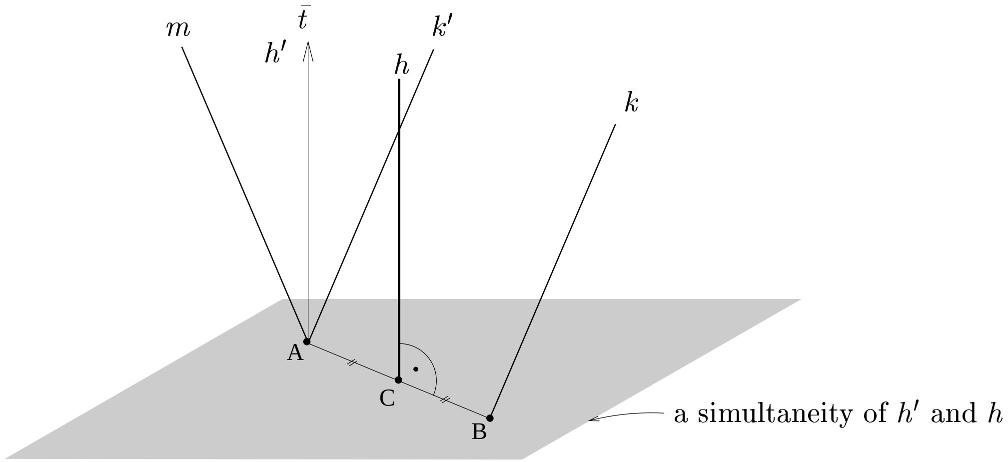


Figure 93: h is a median observer for observers m and k. (Construction for the case when the traces of m and k are skew lines.)

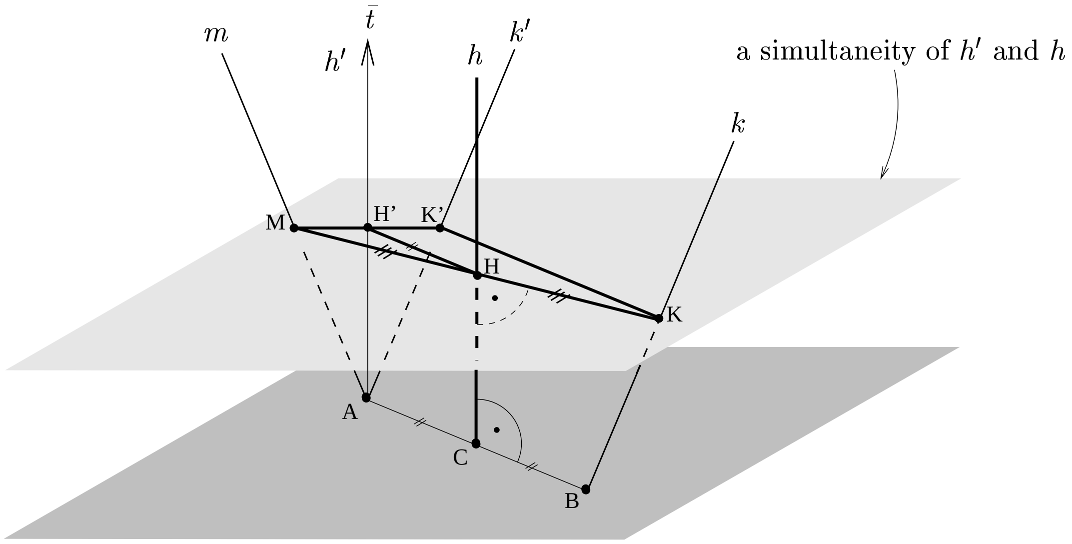


Figure 94: This illustrates why the construction in Figure 93 works.

But then, by (121)–(123) and by C being the midpoint of segment AB (cf. 118), we conclude that $\text{H}'\text{H}$ is the midline of triangle $\text{MK}'\text{K}$. Therefore, (120) above holds.

By 3.8.30 we have that events which ate simultaneous for h' remain simultaneous for h. Let us switch over from the world-view of h' to the world-view of h. Let $\text{A}^*, \text{B}^*, \text{C}^*, \text{M}^*, \text{K}^*, \text{H}^*$ denote, respectively, the $\mathsf{f}_{h'h}$ images of $\text{A}, \text{B}, \text{C}, \text{M}, \text{K}, \text{H}$. Let us notice that $\text{C}^*, \text{H}^* \in \bar{t}$ (by $\text{C}, \text{H} \in tr_{h'}(h)$ and by **Ax4**). Now, by (117, 118) (and by 3.8.30), we have that

$$\text{A}^*{}_t = \text{B}^*{}_t = \text{C}^*{}_t \quad \text{and} \quad \text{C}^* \text{ is the midpoint of segment } \text{A}^*\text{B}^*.$$

Therefore, since $\text{C}^* \in \bar{t}$, we conclude that

(124) $$\text{A}^* \text{ and } \text{B}^* \text{ are } \bar{t}\text{-symmetric.}$$

We conclude (125) to be formalized below from (119, 120) completely analogously to the way we proved (124) from (117, 118).

(125) $$\text{M}^* \text{ and } \text{K}^* \text{ are } \bar{t}\text{-symmetric.}$$

Let us notice that $\overline{\text{A}^*\text{M}^*} = tr_h(m)$ and that $\overline{\text{B}^*\text{K}^*} = tr_h(k)$. Therefore, by (124, 125) we have that $tr_h(m)$ and $tr_h(k)$ are $\bar{t}$-symmetric, so h is a median observer for m and k. ∎

The following six propositions are included to motivate the model construction in §3.8.2 below. The most important two of these are Propositions 3.8.31 and 3.8.32 below. The intuitive content of these two is the following: The symmetry principle **Ax(symm)** implies that a median observer h (for observers m and k) sees the clocks of m and k slowing down with the same rate.

PROPOSITION 3.8.31 *Assume*
Basax + **Ax**($\sqrt{\ }$) + **Ax(symm)**. *Let*
$m, k \in Obs$. Assume that h is a median observer for observers m and k. Then h sees the clocks of m and k slowing down with the same rate; formally:

$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t| \,.$$

Proof: The proof will be filled in later. ∎

The following proposition is a variant of Prop.3.8.31 above. For the intuitive meaning of the following proposition the reader is referred to Figure 95 on p.319.

PROPOSITION 3.8.32
Assume **Basax** + **Ax**($\sqrt{\ }$) + **Ax(symm)** + **Ax5**$^+$ + **Ax(↑)**. *Assume* $m, k \in Obs$ *such that* $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. *Assume* h *is a median observer for observers* m *and* k. *Then in the world-view of* h *the time-unit vectors of* m *and* k *are* $\bar{t}$*-symmetric, that is,*

$$\mathsf{f}_{mh}(1_t) \textit{ and } \mathsf{f}_{kh}(1_t) \textit{ are } \bar{t}\textit{-symmetric.}$$

For the proof of the above proposition we need items 3.8.33–3.8.37 below, therefore the proof (of Prop.3.8.32) comes below these items (on p.318). We note that the idea of the proof is in Figure 95.

Intuitively, Prop.3.8.33 below says, that under assuming **Ax(eqtime)** the world-view transformation between "observer brothers" m and m' is a trivial one.

Proposition 3.8.33 *Assume* **Basax** + **Ax(eqtime)**. *Assume* $m, m' \in Obs$ *are brothers, formally:* $tr_m(m') = \bar{t}$. *Then (i) and (ii) below hold.*

(i) $\mathsf{f}_{mm'} \in Aftr$.

(ii) Assume in addition **Ax(↑)**. *Then*

$$\mathsf{f}_{mm'} \in Triv\,.$$

Proof: Assume **Basax** + **Ax(eqtime)**. Assume $m, m' \in Obs$ such that $tr_m(m') = \bar{t}$. Then, by Prop.3.6.5,

(126) $\mathsf{f}_{mm'} = \widetilde{\varphi} \circ f$, for some $f \in PT$ with $f[\bar{t}] = \bar{t}$ and for some $\varphi \in Aut(\mathbf{F})$.

Let such f and φ be fixed. Let $f_0 : F \longrightarrow F$ be the restriction of f to $\bar{t}$, that is, $(\forall t \in F)\ f_0(t) \stackrel{\text{def}}{=} f(\langle t, 0, \ldots, 0\rangle)_0$. By $f \in PT \subseteq Aftr$ we have that f_0 is a linear function.[298] Now, by **Ax(eqtime)**, we have that $(\forall t \in F)\ |f_0(\varphi(t)) - f_0(0)| = |t|$ (by $\varphi(0) = 0$), in particular $|f_0(1) - f_0(0)| = 1$ (by $\varphi(1) = 1$). From these two, by f_0 being a linear function, one concludes that for the automorphism φ we have $(\forall t \in F)\ |\varphi(t)| = |t|$, therefore φ is the trivial automorphism (i.e. the identity function). Hence

(127) $$\mathsf{f}_{mm'} \in PT$$

by (126), and this completes the proof of (i) since $PT \subseteq Aftr$. Let us notice that $\mathsf{f}_{mm'}[\bar{t}] = \bar{t}$. To prove (ii), assume in addition **Ax(↑)**. Now, $\mathsf{f}_{mm'}(1_t) - \mathsf{f}_{mm'}(\bar{0}) = 1_t$ by **Ax(eqtime)** and **Ax(↑)**. Therefore $\mathsf{f}_{mm'} = g \circ \tau$ for some $g \in PT$ with $g(\bar{0}) = \bar{0}$

[298] A function $h : F \longrightarrow F$ is called a linear function iff $(\exists a, d \in F)(\forall x \in F)\ h(x) = ax + d$.

and $g(1_t) = 1_t$ and for some translation τ (by 127). Let such g and τ be fixed. Applying Lemma 3.6.20 we get that $g \in \mathit{Triv}_0$, hence $\mathsf{f}_{mm'} \in \mathit{Triv}$ by the definition of Triv, and this completes the proof of (ii). ∎

The following proposition is a variant of Thm.2.8.9. Intuitively, it says that under assuming **Ax(symm)**, any two observers see each other clocks slowing down with the same rate.

Proposition 3.8.34 *Assume* **Basax + Ax(symm) + Ax(↑)**. *Let $m, k \in Obs$. Then*

$$\mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t = \mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t\,.$$

Outline of proof: Assume **Basax + Ax(symm) + Ax(↑)**. Let $m, k \in Obs$. Let $k', m' \in Obs$ such that $\mathsf{f}_{km} = \mathsf{f}_{m'k'}$ and that $tr_k(k') = tr_m(m') = \bar{t}$. Such k', m' exist by **Ax(symm)**. By noticing that

$$\mathsf{f}_{km} = (\mathsf{f}_{m'k'} =)\ \mathsf{f}_{m'm} \circ \mathsf{f}_{mk} \circ \mathsf{f}_{kk'}$$

holds and by (128)–(130) below, one can complete the proof; where (128) holds by **Ax(eqtime)**, **Ax(↑)** and $tr_m(m') = \bar{t}$; (129) can be checked by Thm.3.1.4; and (130) is true since $\mathsf{f}_{kk'} \in \mathit{Triv}$ by Prop.3.8.33.

(128) $\qquad \mathsf{f}_{m'm}(1_t) - \mathsf{f}_{m'm}(\bar{0}) = 1_t \quad \text{and} \quad \mathsf{f}_{m'm}(1_t), \mathsf{f}_{m'm}(\bar{0}) \in \bar{t}\,.$

(129) $\qquad (\forall p, q \in \bar{t}\,) \left(p - q = 1_t \ \Rightarrow\ \mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q) = \mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0})\right).$

(130) $\qquad (\forall p, q \in {}^nF)\ \mathsf{f}_{kk'}(p)_t - \mathsf{f}_{kk'}(q)_t = p_t - q_t\,.$ ∎

The following is a variant of Thm.2.8.3. Cf. also Thm.2.9.5.

Proposition 3.8.35 *Assume* **Basax + Ax($\sqrt{\ }$) + Ax(symm)**. *Let $m, k \in Obs$. Then*

(i) $\mathsf{f}_{mk} \in \mathit{Aftr}$, *moreover:*

(ii) $\mathsf{f}_{mk} \in PT$.

For the proof of Prop.3.8.35 we will need Lemma 3.8.36 below, so we will give the proof of Prop.3.8.35 below the lemma.

LEMMA 3.8.36

(i) Assume $g \in \mathit{Aftr}(n, \mathbf{F})$ and $\varphi \in \mathit{Aut}(\mathbf{F})$. Then $g \circ \widetilde{\varphi} = \widetilde{\varphi} \circ g'$, for some $g' \in \mathit{Aftr}(n, \mathbf{F})$.

(ii) Assume $g, g' \in Aftr(n, \mathbf{F})$ and $\varphi, \varphi' \in Aut(\mathbf{F})$ such that $\widetilde{\varphi} \circ g = \widetilde{\varphi'} \circ g'$. Then $\varphi = \varphi'$.

The **proof** of this lemma is straightforward, we omit it. ∎

Proof of Prop.3.8.35: Assume $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(symm)}$. Let $m, k \in Obs$. Then, by Prop.3.6.5,

$$\mathsf{f}_{mk} = f \circ \widetilde{\varphi}, \qquad \text{for some } \varphi \in Aut(\mathfrak{F}) \text{ and } f \in PT. \tag{131}$$

Let such φ and f be fixed. Let $m', k' \in Obs$ such that $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$ and $tr_m(m') = tr_k(k') = \bar{t}$. Such m' and k' exist by $\mathbf{Ax(symm)}$. Then, by $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$, we have

$$\mathsf{f}_{mk} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km} \circ \mathsf{f}_{mm'}\,. \tag{132}$$

(131) and (132) imply that

$$\widetilde{\varphi} \circ f = \mathsf{f}_{k'k} \circ \widetilde{\varphi^{-1}} \circ f^{-1} \circ \mathsf{f}_{mm'}\,. \tag{133}$$

By Prop.3.8.33(i), we have that $\mathsf{f}_{k'k}, \mathsf{f}_{mm'} \in Aftr$. Applying Lemma 3.8.36, (133) implies[299] that $\varphi = \varphi^{-1}$, which means that $\varphi^2 = \mathrm{Id}$. Since an <u>*ordered*</u> field cannot have a non-trivial automorphism whose order is finite, we conclude that $\varphi = \mathrm{Id}$, which by (131) completes the proof. ∎

Intuitively, Prop.3.8.37 below says that, under assuming $\mathbf{Ax5^+}$+$\mathbf{Ax(\uparrow)}$, clocks are ticking forwards only.

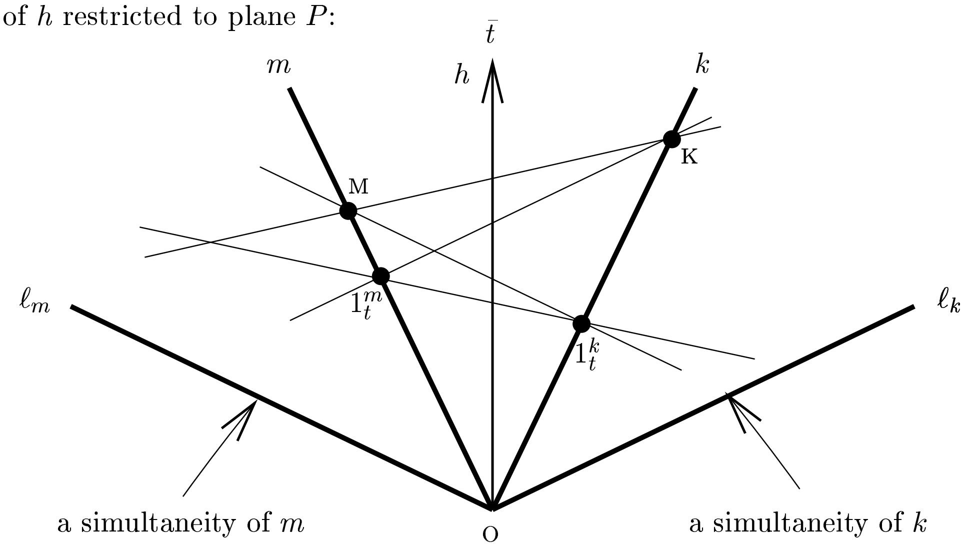
Proposition 3.8.37 *Assume $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax5^+} + \mathbf{Ax(\uparrow)}$. Assume $m, k \in Obs$. Assume any one of the following three assumptions: (i) $v_m(k) < 1$ or (ii) $\mathbf{Ax(symm)}$ or (iii) $n > 2$.*

Then m sees the clock of k ticking forwards, that is, $m \uparrow k$.

On the proof: Assume $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax5^+} + \mathbf{Ax(\uparrow)}$. Assume $m, k \in Obs$. Let us notice that both (ii) $\mathbf{Ax(symm)}$ and (iii) $n > 2$ imply (i) $v_m(k) < 1$, by Thm.3.8.11 and Thm.3.4.1. Assume $v_m(k) < 1$. Then by $\mathbf{Ax5^+}$, there is $h \in Obs$ such that $tr_m(h) = tr_m(k)$ and $m \uparrow h$. Then, by $\mathbf{Ax(\uparrow)}$, $h \uparrow k$. But then $m \uparrow k$ must hold since the world-view transformations are "betweenness preserving" by Prop.3.6.5(ii) (let us notice that we assumed $\mathbf{Ax}(\sqrt{\ })$). ∎

betweenness preserving, nincs még definiálva!

Proof of Prop.3.8.32: We note that a more detailed and careful proof will be included at a later stage of development of the present work. The idea of the proof world-view of h restricted to plane P:



If 1_t of m and k were not $\bar{t}$-symmetric as seen by h then m and k would see each other clocks differently, i.e. slowing down with different rate.

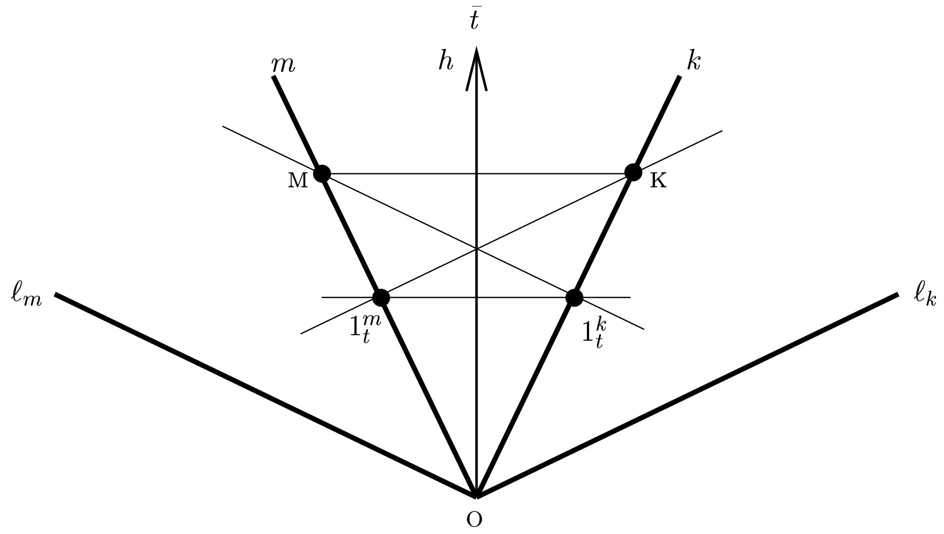


Figure 95: Illustration for the proof of Prop.3.8.32.

is illustrated in Figure 95. Assume **Basax** + $\mathbf{Ax}(\sqrt{\ })$ + **Ax(symm)** + $\mathbf{Ax5}^+$ + **Ax(↑)**. Assume $m, k \in Obs$ such that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ and assume that h is a median observer for observers m and k. (Let us notice that then $\mathsf{f}_{mh}(\bar{0}) = \mathsf{f}_{kh}(\bar{0})$ and that $\mathsf{Plane}(\bar{t}, tr_h(m)) = \mathsf{Plane}(\bar{t}, tr_h(k))$.) Let

$$\begin{aligned} \textsc{o} &\stackrel{\text{def}}{=} \mathsf{f}_{mh}(\bar{0}) = \mathsf{f}_{kh}(\bar{0}), \quad \text{and} \\ P &\stackrel{\text{def}}{=} \mathsf{Plane}(\bar{t}, tr_h(m)) = \mathsf{Plane}(\bar{t}, tr_h(k)) \,. \end{aligned}$$

Let 1_t^m and 1_t^k denote, respectively, the time unit vectors of m and k as seen by h, i.e. $1_t^m \stackrel{\text{def}}{=} \mathsf{f}_{mh}(1_t)$ and $1_t^k = \mathsf{f}_{kh}(1_t)$. (This notation corresponds to our intuition in the case when $\textsc{o} = \bar{0}$. Out of "lazyness" we use the same notation in the general case too.) Obviously,

(134) $$1_t^m \in tr_h(m) \quad \text{and} \quad 1_t^k \in tr_h(k)\,,$$

see Figure 95. Further, by Prop.3.8.37,

(135) $$(1_t^m)_t - \textsc{o}_t > 0 \quad \text{and} \quad (1_t^k)_t - \textsc{o}_t > 0\,.$$

By h being a median observer for m and k

(136) $$tr_h(m) \quad \text{and} \quad tr_h(k) \quad \text{are } \bar{t}\text{-symmetric.}$$

We shall prove that 1_t^m and 1_t^k are $\bar{t}$-symmetric, too (this is what the proposition states). Let ℓ_m and ℓ_k be, respectively, the simultaneities of m and k passing through $\textsc{o}$ in the world-view of h restricted to plane P, formally

$$\ell_m \stackrel{\text{def}}{=} \mathsf{f}_{mh}[S] \cap P \quad \text{and} \quad \ell_k \stackrel{\text{def}}{=} \mathsf{f}_{kh}[S] \cap P\,.$$

See Figure 95. Then $\ell_m, \ell_k \in \mathsf{Eucl}$ and

(137) $$\ell_m \quad \text{and} \quad \ell_k \quad \text{are } \bar{t}\text{-symmetric,}$$

where (137) can be proved e.g. by using Lemma 3.8.28. Let $\textsc{m} \in tr_h(m)$ such that events $w_h(\textsc{m})$ and $w_h(1_t^k)$ will be simultaneous for m. (Such an $\textsc{m}$ exists and is unique.) Then

(138) $$\overline{\textsc{m}1_t^k} \parallel \ell_m\,.$$

Let $\textsc{k} \in tr_h(k)$ such that events $w_h(\textsc{k})$ and $w_h(1_t^m)$ will be simultaneous for k. (Such an $\textsc{k}$ exists and is unique.) Then

(139) $$\overline{\textsc{k}1_t^m} \parallel \ell_k\,.$$

Further, by Lemma 3.8.28, by (135, 136), by $v_h(m) < 1$ by $v_h(k) < 1$, [300]

(140) $$\textsc{m}_t - \textsc{o}_t > 0 \quad \text{and} \quad \textsc{k}_t - \textsc{o}_t > 0\,.$$

[299]First, one applies item (i) of Lemma 3.8.36 for $\mathsf{f}_{k'k}$ and φ^{-1} (in place of g and φ), after that one applies item (ii) of Lemma 3.8.36.

[300]That $v_h(m) < 1$ and $v_h(k) < 1$ are true by Thm's 3.8.11 and 3.4.1.

By Prop.3.8.34, m and k see each other clocks slowing down with the same rate, i.e. $\mathsf{f}_{mk}(1_t)_t = \mathsf{f}_{km}(1_t)_t$. This phenomenon appears in the world-view of h as (141) below because of the following. By Proposition 3.8.33(i), we have that f_{mh} and f_{kh} are affine transformations. By this and by Prop.3.8.37, it is easy to prove that $\mathsf{f}_{km}(1_t)_t = \frac{|\mathrm{M}-\mathrm{O}|}{|1_t^m-\mathrm{O}|}$ and $\mathsf{f}_{mk}(1_t)_t = \frac{|\mathrm{K}-\mathrm{O}|}{|1_t^k-\mathrm{O}|}$.

$$\frac{|\mathrm{M}-\mathrm{O}|}{|1_t^m-\mathrm{O}|} = \frac{|\mathrm{K}-\mathrm{O}|}{|1_t^k-\mathrm{O}|}\,. \tag{141}$$

Let us notice that by (135, 140), (141) is equivalent with

$$\overline{\mathrm{MK}} \parallel \overline{1_t^m 1_t^k}\,.$$

Now (134–141) imply that

$$1_t^m \quad \textit{and} \quad 1_t^k \quad \text{are } \bar{t}\text{-symmetric,}$$

see Figure 95. This is what we wanted to prove. ∎

3.8.2 Model construction for BaCo

This sub-section is devoted to the proof of the existence part of Thm.3.8.7 which says that **BaCo** is $\mathfrak{F}$-categorical (for any Euclidean $\mathfrak{F}$); i.e. we are concerned with the <u>"consistency"</u> part now (while the "completeness" part was in Thm.3.8.14 on p.301, and the "uniqueness" part comes in §3.8.3).

On the <u>*intuitive idea*</u> of our model construction for **BaCo**:

We could base our present model construction on the intuitive model construction for **Basax**(3) given in §3.2; and probably this would yield the most satisfying version, from the point of view of visualizability and easy understandability. However, for historical (and related) reasons the construction in the present sub-section will be closer to the somewhat more computational §3.6 ("Models of **Basax**"). But before going into that, let us have a quick glance at how the construction would go if we based it on the more intuitive §3.2 ("Intuitive ... **Basax**(3)"). So, let us imagine that we start out from §3.2 and want to modify it to obtain a construction of a model for **BaCo**. The model constructed in §3.2 satisfies all of **BaCo** except for $\mathbf{Ax(symm_0)}$, $\mathbf{Ax5^+}$ and $\mathbf{Ax}(\mathit{Triv}_t)$; more precisely we can throw away superfluous bodies such that $\mathbf{Ax}\heartsuit$ and **Ax(ext)** will also be satisfied by the so trimmed version of the model constructed in §3.2. We note that the construction in §3.2 goes through for any Euclidean field $\mathfrak{F}$ in place of $\mathfrak{R}$. The reader is invited to check this. So we have to modify that construction such that the missing three axioms ($\mathbf{Ax5^+}$, $\mathbf{Ax(symm_0)}$, $\mathbf{Ax}(\mathit{Triv}_t)$) become satisfied too. As a first step the reader is asked to look up the picture on p.181 representing the choice of the unit vectors $1'_t$, $1'_x$, $1'_y$. Now, we choose the direction of $1'_t$ to be positive such that $\mathbf{Ax5^+}$ becomes true (for m). Then we add to the model enough "brothers" for each observer such as to make $\mathbf{Ax}(\mathit{Triv}_t)$ true (but do not destroy any of the other axioms). The construction in §3.2 is flexible enough to accommodate all these changes. Therefore (this way) one can obtain a relatively simple and easily visualizable model all of **BaCo**(3) except for $\mathbf{Ax(symm_0)}$. The interested reader is invited to fill in the details and to try to visualize the so obtained model. Summing up, so far we have a model $\mathfrak{M}$ satisfying $\mathbf{BaCo} \setminus \{\mathbf{Ax(symm_0)}\}$ for $n = 3$. The remaining part (below) of the present intuitive text shows how to modify this $\mathfrak{M}$ to obtain a model for **BaCo**(3). However, we would like to obtain a model of **BaCo**(n) with n arbitrary. This could be done by generalizing the intuitive model construction in §3.2.

In order to not to loose time with generalizing the intuitive proof in §3.2 to arbitrary n, instead of the above constructed model we will use the following. Let n be arbitrary. Using Def.3.6.11 ("General Models") and Thm.3.6.12 on p.270 it is easy to construct a model $\mathfrak{M}$ satisfying **BaCo** except for $\mathbf{Ax(symm_0)}$ such that all world-view transformations in $\mathfrak{M}$ are affine and there are no FTL observers in it. Further the $\mathfrak{F}$-reduct of $\mathfrak{M}$ is Euclidean.[301] (The above outlined construction using §3.2 satisfied all the just outlined criteria.) Let us note that in $\mathfrak{M}$ all clocks are ticking forwards, by Prop.3.8.37 (i.e. each observer sees the clock of an other observer ticking forwards).

Next we modify this model $\mathfrak{M}$ such that $\mathbf{Ax(symm_0)}$ becomes true in it (and of course all the other axioms remain true). We do this the following way: First we choose an observer m_0.

Instead of the symbols $1'_t$ and m (used in §3.2) in the present sub-section we use 1^k_t and m_0.

All what we do in the next 12 lines is understood in the world-view of m_0.

For each one k of the remaining observers we change the length of time-unit vector 1^k_t of k such that a median observer h (for observers m_0 and k) thinks that clocks of m_0 and k slow down with the same rate and clock of k remains ticking forwards, see Figures 96, 97. (This step is motivated by Propositions 3.8.31, 3.8.32.) We think of 1^k_t as an ordered pair $\langle \mathsf{f}_{km_0}(\bar{0}), \mathsf{f}_{km_0}(1_t)\rangle$. When we change the length of a vector represented as such a pair then we leave $\mathsf{f}_{km_0}(\bar{0})$ fixed and we change only $\mathsf{f}_{km_0}(1_t)$, i.e. we change only the tip of the arrow. See Figure 97.

After these we have to adjust the lengths of the rest (like e.g. 1^k_x) of the unit vectors of k such that they match nicely the new time unit-vector of k. Formally all this can by done by composing w_k with an expansion exp from the left (then the new version of w_k will be $exp \circ w_k$).

It is not hard to prove that if this construction is carried through then the new model, call it $\mathfrak{M}_1$, validates $\mathbf{Ax(eqspace)}$[302] for the particular observer m_0 in place of m (k remains universally quantified). This can be seen by studying the world-view of the median observer h for m_0 and k. We note that the median observers remain the same in the new model $\mathfrak{M}_1$, as they were in $\mathfrak{M}$. It remains to prove that $\mathfrak{M} \models \mathbf{Ax(symm_0)}$. This goes as follows. From $\mathbf{Ax(eqspace)}$ (for m_0 and k) one can infer that f_{m_0k} preserves Minkowski-distance. This implies that

$(*)$ for every m and k, f_{mk} is Minkowski-distance preserving.

Minkowski distance vagy distances?

Let m, k be arbitrary, and let h be their median observer. From the world-view of h we see the following: Since f_{mh} and f_{kh} are Minkowski-distance preserving by $(*)$, h thinks that the time-unit vectors 1^m_t and 1^k_t of m and k have the same Minkowski-length (namely 1). But then, since m and k have the same speed, the Euclidean length of 1^m_t and 1^k_t coincide. For simplicity we assume that $n = 3$. Let $1^{m'}_t, 1^{m'}_x, 1^{m'}_y$

[301]We note that such a model will be constructed in Def.3.8.38, cf. also Prop.3.8.40.
[302]For $\mathbf{Ax(eqspace)}$ cf. §2.8, p.136.

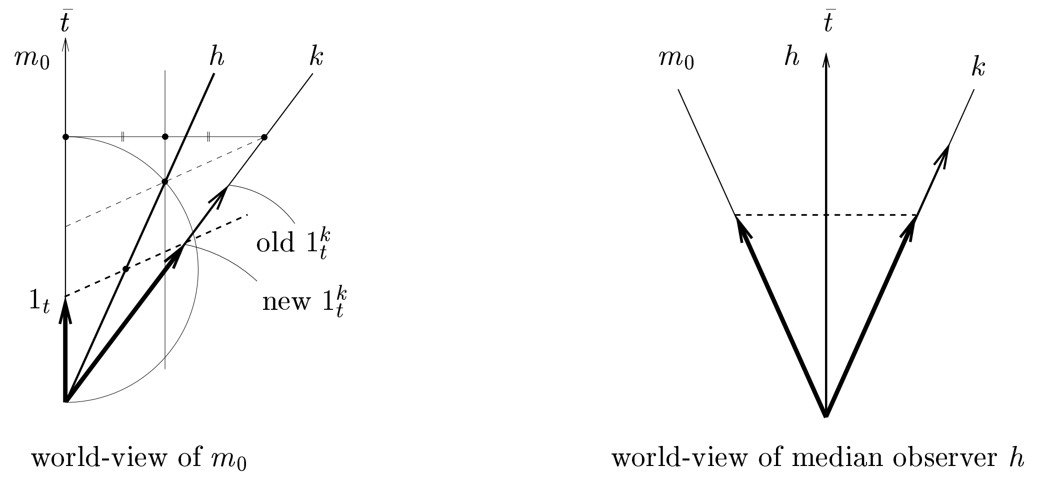


Figure 96: We change the length of time-unit vector 1_t^k of k such that a median observer h (for observers m_0 and k) thinks that clocks of m_0 and k slow down with the same rate, cf. also Figure 97.

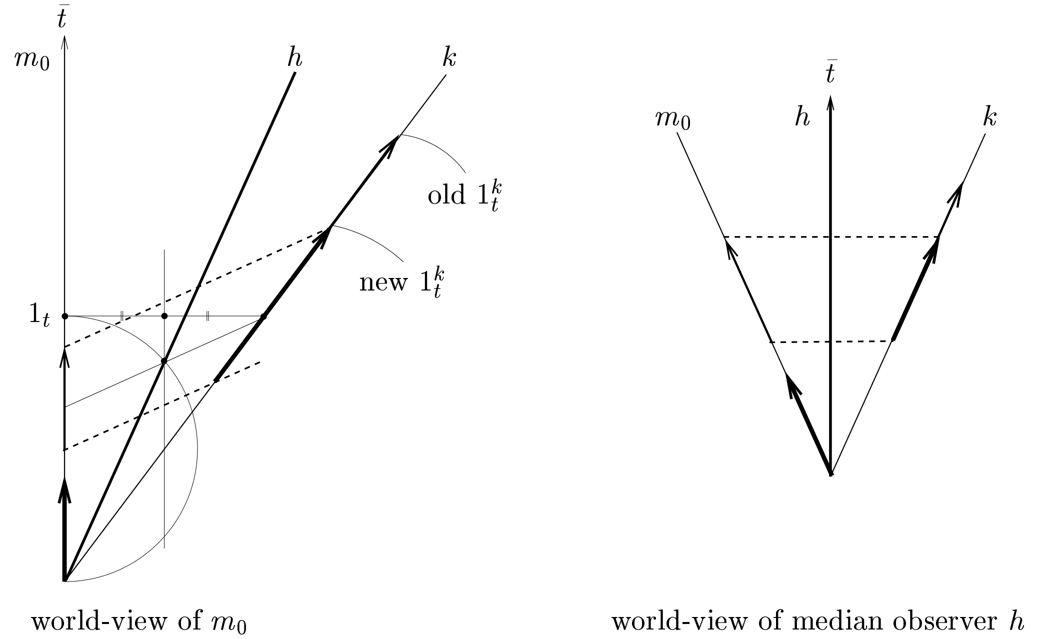


Figure 97: We change only the tip of the arrow.

be, respectively, $\bar{t}$-symmetric to $1_t^k, 1_x^k, 1_y^k$, see Figure 98.[303] Similarly let $1_t^{k'}, 1_x^{k'}, 1_y^{k'}$ be, respectively, $\bar{t}$-symmetric to $1_t^m, 1_x^m, 1_y^m$. Then since 1_t^m and 1_t^k have the same Euclidean length (and since clocks are ticking only forwards), by $\mathbf{Ax}(\mathit{Triv}_t)$, one can prove that there are observers m' and k' whose unit vectors are $\langle 1_t^{m'}, 1_x^{m'}, 1_y^{m'}\rangle$ and $\langle 1_t^{k'}, 1_x^{k'}, 1_y^{k'}\rangle$, respectively. Then viewing the situation from the world-view of h (see Figure 98), we conclude that, $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$ (and $tr_m(m') = tr_k(k') = \bar{t}$). This completes the intuitive proof of validity of $\mathbf{Ax(symm_0)}$ in our model (i.e. the proof of consistency of **BaCo**); the formal proof will be slightly different and it will be given in the form of the proofs of Propositions 3.8.40 and 3.8.44. The summary of this formal proof is the following:

For every Euclidean $\mathfrak{F}$, first we will define a class of models

$$\{\mathfrak{M}_{\mathfrak{F}}^Q : Q \text{ is an appropriate choice function}\},$$

in a similar style as models were defined in §2.4 ("Models for **Basax** in dimension 2"), Def.3.5.5 ("Simple Models") or in Def.3.6.11 ("General Models").[304] We will show that this class validates all of **BaCo** except for $\mathbf{Ax(symm_0)}$. Intuitively, Q will determine the lengths of the time-unit vectors of observers as seen by a particular observer m_0. Then we will choose Q, such that $\mathfrak{M}_{\mathfrak{F}}^Q \models \mathbf{Ax(symm_0)}$ will become true. This special choice of Q will be denoted by M, and the model corresponding to M will be denoted by $\mathfrak{M}_{\mathfrak{F}}^M$. In the symbol $\mathfrak{M}_{\mathfrak{F}}^M$ the letter M intends to remind us that model $\mathfrak{M}_{\mathfrak{F}}^M$ is the standard Minkowskian one (cf. Def.6.2.58) over $\mathfrak{F}$.

Definition 3.8.38 ($\mathfrak{M}_{\mathfrak{F}}^Q$)
Let $\mathfrak{F}$ be Euclidean (and $n \geq 2$). Let

$$\mathsf{Speeds} \overset{\text{def}}{=} \{ x \in F : 0 \leq x < 1 \}.$$

Let

$$Q : \mathsf{Speeds} \longrightarrow {}^+F$$

be a function such that $Q(0) = 1$.[305] Intuitively, $Q(v)$ (for $v \in \mathsf{Speeds}$) will be the length of the time-unit vectors of those observers k which are moving with

[303] $1_t^k \ldots 1_y^k$ were denoted as $x_{k,0} \ldots x_{k,2}$ on p.255 (around the end of the definition of SM).

[304] In the present context Q denotes a function while in the definition of frame models Q was a sort. These two Q's are of course completely different things. We hope that this coincidence of notation will cause no confusion. Anyway, we emphasize that in the present sub-section (§3.8) our Q is <u>*not*</u> a sort of our language, but a so called choice function.

[305] The assumption $Q(0) = 1$ is not important in this definition, we made it only for convenience.

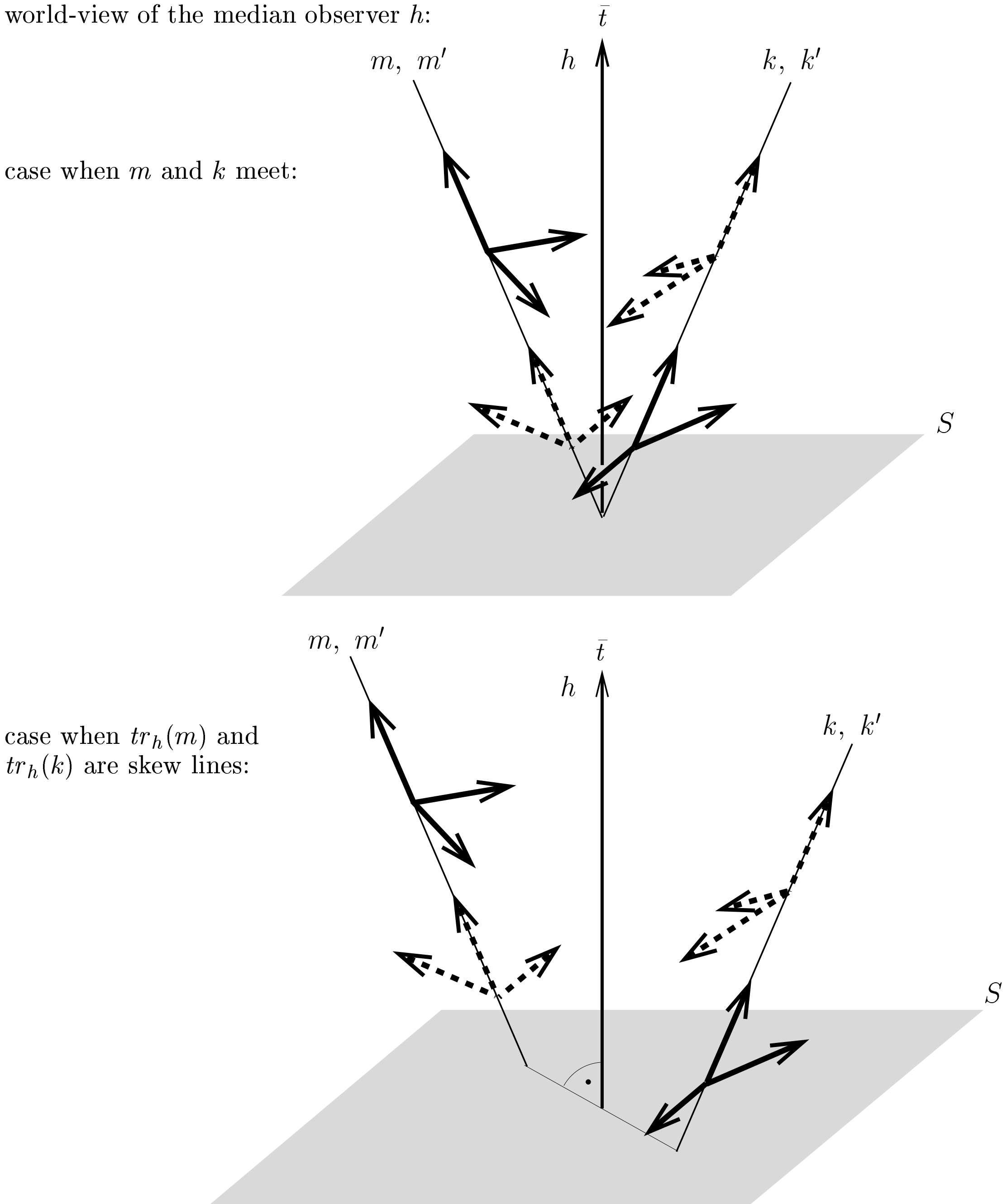


Figure 98: The "solid" vectors are the unit vectors of m and k, while the "dashed" vectors are the unit vectors of m' and k'. Viewing the situation from the world-view of h, we conclude that $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$.

speed v relative to a particular observer m_0. Soon, we will define the model $\mathfrak{M}^Q_{\mathfrak{F}}$ corresponding to $\mathfrak{F}$ and Q (and n).[306]

Let $Rhomb^Q = Rhomb^Q(n, \mathfrak{F})$ and $PT^Q = PT^Q(n, \mathfrak{F})$, corresponding to Q, be the sets of transformations defined as follows.

$$\begin{aligned}
Rhomb^Q &\stackrel{\text{def}}{=} \{\, f \in Rhomb(n, \mathfrak{F}) \,:\, (*) \text{ below holds for } f \,\} \,. \\
(*) \qquad & f(1_t)_t > 0 \;\wedge\; f[\bar{t}\,] \in \mathsf{SlowEucl} \;\;\wedge\;\; |f(1_t)| = Q\Big(ang^2(f[\bar{t}\,])\Big) \,. \\
PT^Q &\stackrel{\text{def}}{=} \Big\{\, g \circ f \circ g_0 \,:\, f \in Rhomb^Q,\; g \in Triv,\; g_0 \in Triv_0 \,\Big\} \,.
\end{aligned}$$

We define

$$\begin{aligned}
\mathfrak{M}^Q_{\mathfrak{F}} &\stackrel{\text{def}}{=} \langle (B;\, Obs, Ph, Ib), \mathfrak{F}, \mathsf{Eucl}(n, \mathfrak{F});\, \in, W \rangle \,, \text{ where} \\
Obs &\stackrel{\text{def}}{=} PT^Q \\
Ph &\stackrel{\text{def}}{=} \mathsf{PhtEucl} \\
B &\stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} Obs \cup Ph \,.
\end{aligned}$$

It remains to define W. (By $Q(0) = 1$ we have that $\mathrm{Id} \in Obs$. Now, ituitively, $k \in Obs$ $(= PT^Q)$, as a "mathematical entity" will happen to be the world-view transformation between observers k and $m_0 := \mathrm{Id}$.) First, we define a function $w_0 : {}^nF \longrightarrow \mathcal{P}(B)$ as follows. For every $p \in {}^nF$, let

$$w_0(p) \stackrel{\text{def}}{=} \{\, k \in Obs \,:\, p \in k[\bar{t}\,] \,\} \cup \{\, ph \in Ph \,:\, p \in ph \,\} \,.$$

For every $k \in Obs$ let

$$w_k \stackrel{\text{def}}{=} k \circ w_0 \,.$$

Let

$$W \stackrel{\text{def}}{=} \{\, \langle k, p, b \rangle \,:\, k \in Obs,\; b \in w_m(p) \,\} \,.$$

By this, the model $\mathfrak{M}^Q_{\mathfrak{F}}$ has been defined.

$\lhd$

Remark 3.8.39 (On the definition of $\mathfrak{M}^Q_{\mathfrak{F}}$) As a heart of the definition of $\mathfrak{M}^Q_{\mathfrak{F}}$ we used the class PT^Q. But the members of PT^Q are put together from members of *Triv* and members of $Rhomb^Q$. Below the definition of *Triv*, we emphasized that the members of *Triv* are irrelevant from the point of view of relativity theory and therefore we will downplay their role in this material. Accordingly, we say that the heart $\mathfrak{M}^Q_{\mathfrak{F}}$ is the class $Rhomb^Q$ of rhombus transformations (and *Triv* is added only

[306]Therefore the model $\mathfrak{M}^Q_{\mathfrak{F}}$ will have three parameters Q, $\mathfrak{F}$ and n. Since n and $\mathfrak{F}$ are usually understood from context most of the time explicitly we will mention Q as a *parameter* of our model $\mathfrak{M}^Q_{\mathfrak{F}}$.

for "book-keeping" purposes; and this is how we arrive at PT^Q whose heart remains $Rhomb^Q$).

$\triangleleft$

PROPOSITION 3.8.40 *For any Euclidean $\mathfrak{F}$ (for every $n \geq 2$) and for every function Q as in Def.3.8.38 above,*

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \mathbf{BaCo} \setminus \{\mathbf{Ax(symm_0)}\},$$

where the model $\mathfrak{M}^Q_{\mathfrak{F}}$ was defined in Def.3.8.38 above.

Before turning to the proof of Prop.3.8.40 we list some simple properties of $Rhomb^Q$ and PT^Q introduced in Def.3.8.38.

Lemma 3.8.41 Assume $\mathfrak{F}$ is Euclidean. Assume Q, $Rhomb^Q$ and PT^Q are as in Def.3.8.38. Then (i)–(iv) below hold.

(i) $(\forall f \in PT^Q)\ f[\bar{t}] \in \mathsf{SlowEucl}$.

(ii) $(\forall \ell \in \mathsf{SlowEucl}) \Big(\bar{0} \in \ell \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \;\Rightarrow\; (\exists f \in Rhomb^Q)\ f[\bar{t}] = \ell\Big)$.

(iii) $(\forall \ell \in \mathsf{SlowEucl})\ (\exists f \in PT^Q)\ f[\bar{t}] = \ell$.

(iv) $Triv \subseteq PT^Q$ and $(\forall g \in Triv)(\forall f \in PT^Q)\ g \circ f \in PT^Q$.

Proof: Item (i) holds by the definition of PT^Q. Item (ii) can be proved by Lemma 3.8.46 below. Item (iii) follows from (ii), by Lemma 3.5.3 and by the definition of PT^Q. Item (iv) follows by the definition of PT^Q, by noticing that $\mathrm{Id} \in PT^Q$,[307] and by the fact that $Triv$ is closed under composition. ∎

Proof of Prop.3.8.40: Let $\mathfrak{F}$, n, Q, PT^Q and $\mathfrak{M}^Q_{\mathfrak{F}}$ be as in Def.3.8.38. Further let

$$w_0 : {}^nF \longrightarrow {}^nF$$

be as in Def.3.8.38 on p.327. We will check that the axioms in $\mathbf{BaCo} \setminus \{\mathbf{Ax(symm_0)}\}$ are valid in $\mathfrak{M}^Q_{\mathfrak{F}}$.

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax\heartsuit}\}$$

[307] $\mathrm{Id} \in PT^Q$ by $Q(0) = 1$.

by the definition of $\mathfrak{M}^Q_{\mathfrak{F}}$. Further

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \mathbf{Ax6}$$

since $(\forall k \in Obs) Rng(w_k) = Rng(w_0)$. Let

$$m_0 \stackrel{\text{def}}{=} \mathrm{Id} \in Obs \ (= PT^Q)\,.$$

Then, obviously,

$$w_{m_0} = w_0\,.$$

It is easy to check that for every $k \in Obs$ and for every $ph \in Ph$ ($=$ $\mathsf{PhtEucl}$) (143) and (144) below hold.

$$(143) \qquad tr_{m_0}(k) \ = \ k[\bar{t}\,]\,.$$

$$(144) \qquad tr_{m_0}(ph) \ = \ ph\,.$$

By Lemma 3.8.41(iii) we have that (145) below holds.

$$(145) \qquad (\forall \ell \in \mathsf{SlowEucl})(\exists k \in Obs)\, k[\bar{t}\,] = \ell\,.$$

By (143)–(145), we have that **Ax3**, **Ax4**, **Ax5**, **AxE** are satisfied when m is replaced in them with m_0. Let $k \in Obs$ be arbitrary, but fixed. We will prove that **Ax3**–**Ax5**, **AxE** hold for k, too. By the definition of $\mathfrak{M}^Q_{\mathfrak{F}}$, it is easy to check that (146) and (147) below hold.

$$(146) \qquad \mathsf{f}_{km_0} \ = \ k\,.$$

$$(147) \qquad k[tr_k(b)] \ = \ tr_{m_0}(b), \quad \text{for all } b \in B.$$

By the definition of PT^Q, we have that $k[\bar{t}\,] \in \mathsf{SlowEucl}$. Let us notice that $PT^Q \subseteq PT$ (by Remarks 3.5.2, 3.6.3 and Lemma 3.7.1). Hence $k \in PT$. By (146) and (147), and by the fact that **Ax3**–**Ax5**, **AxE** hold for m_0, and by some properties of PT's, we get that **Ax3**–**Ax5**, **AxE** hold for k, since $k \in PT$ and $k[\bar{t}\,] \in \mathsf{SlowEucl}$. Hence,

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \{\mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{AxE}\}\,.$$

So far, we have proved

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \mathbf{Basax} + \mathbf{Ax}\heartsuit\,.$$

By (146), we have that

$$(148) \qquad (\forall m, k \in Obs)\ \mathsf{f}_{mk} = m \circ k^{-1}\,,$$

since $\mathsf{f}_{mk} = \mathsf{f}_{mm_0} \circ \mathsf{f}_{m_0k} = \mathsf{f}_{mm_0} \circ \mathsf{f}^{-1}_{km_0}$. For every $m \in Obs$ $(= PT^Q)$, by the def. of PT^Q, we have

$$(149) \qquad m \in PT \quad \wedge \quad m[\bar{t}] \in \mathsf{SlowEucl} \quad \wedge \quad m(1_t)_t - m(\bar{0})_t > 0\,.$$

Since the property (149) is preserved under composition and taking inverse, by (148), we get that

$$(150) \qquad (\forall m, k \in Obs)\ \mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t > 0\,.$$

By (150) and **Ax5**, we have

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \{\mathbf{Ax5^+}, \mathbf{Ax(\uparrow)}\}\,.$$

By the definition of $PT^Q = Obs$, one can check that for every $m, k \in Obs$ (151) below holds. Further, since we have already proved $\mathfrak{M}^Q_{\mathfrak{F}} \models$ **Basax**, we have that (152) below holds by (146).

$$(151) \qquad m[\bar{t}] = k[\bar{t}] \quad \Rightarrow \quad (\forall p, q \in \bar{t})\ |m(p)_t - m(q)_t| = |k(p)_t - k(q)_t|\,.$$
$$(152) \qquad tr_m(k) = \bar{t} \quad \Rightarrow \quad m[\bar{t}] = k[\bar{t}]\,.$$

By (148), (151) and (152), we conclude that

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \mathbf{Ax(eqtime)}\,.$$

Now, we turn to proving that $\mathbf{Ax(}Triv_t\mathbf{)}$ holds in our model. We will prove the stronger form $\mathbf{Ax(}Triv\mathbf{)}$ of $\mathbf{Ax(}Triv_t\mathbf{)}$. Let $m \in Obs$ $(= PT^Q)$ and $g \in Triv$. We will prove that there is $k \in Obs$ such that $\mathsf{f}_{mk} = g$. By $g \in Triv$, we have that $g^{-1} \in Triv$ (cf. Remark 3.5.2.). Applying Claim 3.8.41(iv), we get $g^{-1} \circ m \in PT^Q$ $(= Obs)$. Hence $g^{-1} \circ m = k$, for some $k \in Obs$. Let this k be fixed. Now, $m \circ k^{-1} = g$ holds (by $g^{-1} \circ m = k$). This, by (148), implies that $\mathsf{f}_{mk} = g$. Hence,

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \mathbf{Ax(}Triv\mathbf{)}\,.$$

Finally, we turn to prove **Ax(ext)**. It is easy to see that the function $w_0 : {}^nF \longrightarrow \mathcal{P}(B)$ (defined in Def.3.8.38) is injective. Hence for $m, k \in Obs$ with $m \neq k$ we have $w_m := m \circ w_0 \neq k \circ w_0 =: w_k$. So, the "observer" part of **Ax(ext)** has been proved, and it is easy to prove the remaining part of **Ax(ext)**. We leave it to the reader. So,

$$\mathfrak{M}^Q_{\mathfrak{F}} \models \mathbf{Ax(ext)}\,,$$

and this completes the proof of Prop.3.8.40. ∎

Next, we will define a special model $\mathfrak{M}^M_{\mathfrak{F}}$ which we will call the *Minkowski model* (because of its connections with Minkowskian geometry defined in Def.6.2.58). We will define this model by choosing the parameter Q of the model $\mathfrak{M}^Q_{\mathfrak{F}}$ (defined in Def.3.8.38) in a special way. This special choice of Q will be denoted by M. Our purpose with this special choice M of Q is to ensure that $\mathfrak{M}^M_{\mathfrak{F}} \models$ **BaCo**. Of course, by Prop.3.8.40, the only axiom we will have to worry about is $\mathbf{Ax(symm_0)}$.

Let us turn to discussing how to choose M. In this Propositions 3.8.31, 3.8.32 and the intuitive discussion (on pp. 322–325) at the beginning of the present sub-section will help us. We recall that, intuitively, Propositions 3.8.31 and 3.8.32, say that the median observer for observers m and k thinks that the clocks of m and k slow down with the same rate, under assuming the symmetry principle **Ax(symm)**. In choosing the function M we will also use the construction we gave in Thm.3.8.25(ii) above (cf. Figure 91 on p.311) for constructing the median observer. The idea is the following. Assume we are given a speed $v \in \mathsf{Speeds}$. Consider $\ell \in \mathsf{Eucl}$ with $\bar{0} \in \ell$ and $ang^2(\ell) = v$. Let us think of ℓ and $\bar{t}$ as two observers. Construct h as the median one of ℓ and $\bar{t}$ (cf. Figures 91, 99). Consider the simultaneity of h containing 1_t (in the world-view of observer $\bar{t}$). The intersection of this simultaneity with ℓ gives us the time-unit vector of observer ℓ. Then, we will define $M(v)$ to be the length of this vector.

Definition 3.8.42 (The Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$)

Let $\mathfrak{F}$ be Euclidean, $(n \geq 2)$, and $\mathsf{Speeds} \stackrel{\text{def}}{=} \{ x \in F : 0 \leq x < 1 \}$ as in Def.3.8.38. We are going to define a function $M : \mathsf{Speeds} \longrightarrow F^+$. To define this function, let $0 \neq v \in \mathsf{Speeds}$. Throughout the following construction the reader is asked to consult Figure 99.

Let $\ell \in \mathsf{Eucl}(2, \mathfrak{F})$ such that $\bar{0} \in \ell$ and $ang^2(\ell) = v$. By $\mathfrak{F}$ being Euclidean such an ℓ exists. Let $\mathrm{A} \in \bar{t}$ such that $\mathrm{A}_t \neq 0$. Let $\mathrm{B} \in \ell$ such that $\mathrm{B}_t = \mathrm{A}_t$. (Such a B exists and is unique.) Let $\ell' \in \mathsf{Eucl}$ be the perpendicular bisector of segment AB. Let $\mathcal{C}$ be the circle with diameter $\bar{0}\mathrm{A}$. By $\mathfrak{F}$ being Euclidean and $ang^2(\ell) = v < 1$, we have that $\mathcal{C} \cap \ell' \neq \emptyset$. Let $\mathrm{C} \in \mathcal{C} \cap \ell'$ be such that $\overline{\bar{0}\mathrm{C}} \in \mathsf{SlowEucl}$. Such a C exists and is unique. Let $\ell'' \in \mathsf{Eucl}$ such that $1_t \in \ell''$ and $\ell'' \parallel \overline{\mathrm{CB}}$. Let $\mathrm{E} \in \ell'' \cap \ell$. Such an E exists and is unique. Now, we let

$$M(v) \stackrel{\text{def}}{=} |\mathrm{E}| .$$

Further, we let

$$M(0) \stackrel{\text{def}}{=} 1 .$$

By all these, the function $M : \mathsf{Speeds} \longrightarrow {}^+F$ has been defined. We claim that the definition of M is unambiguous in the sense that the value $M(v)$ is independent

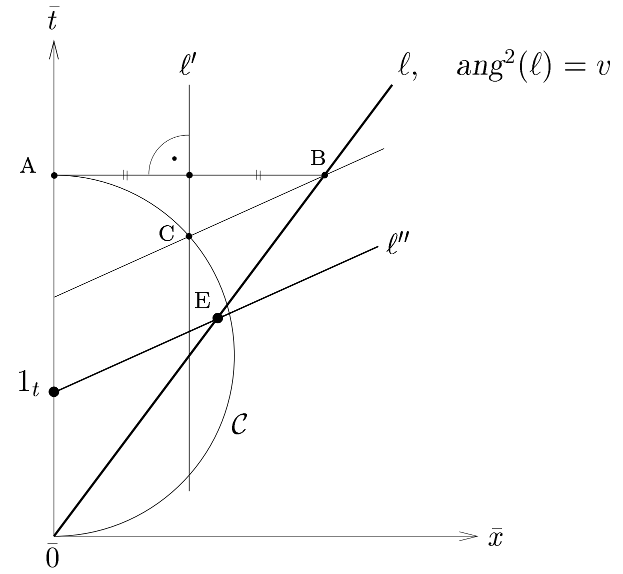


Figure 99: Illustration for the definition of M. $M(v) \stackrel{\text{def}}{=} |\mathrm{E}|$. (Cf. Figures 91, 92.)

form the choice of parameters (e.g. ℓ, A) which we used in defining M. (Checking this claim is easy, and is left to the reader.)

Recall that in Def.3.8.38 the model $\mathfrak{M}^Q_{\mathfrak{F}}$ was defined. Now, the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ is defined by substituting our special M in place of Q (in $\mathfrak{M}^Q_{\mathfrak{F}}$).

Recall also that in Def.3.8.38 the sets of transformations $Rhomb^Q$ and PT^Q were defined. Now, $Rhomb^M = Rhomb^M(n, \mathfrak{F})$ and $PT^M = PT^M(n, \mathfrak{F})$ are defined by substituting our special M in place of Q (in $Rhomb^Q$ and PT^Q, respectively).

$\triangleleft$

PROPOSITION 3.8.43 *Assume $\mathfrak{F}$ is Euclidean. Then*

$$f \in Rhomb^M$$
$$\Longleftrightarrow$$
$$\Big(f \in Rhomb, \quad f[\bar{t}] \in \mathsf{SlowEucl}, \quad \textit{and either } (\star) \textit{ below holds or } f = \mathrm{Id}\Big).$$

($\star$) *$f(1_t)$ is constructed from $f[\bar{t}]$ (and from $\bar{t}$, 1_t) as in Figure 100; completely analogously*[308] *as* E *was constructed from ℓ (and from $\bar{t}$, 1_t) in Def.3.8.42 (cf. Figure 99.)*

Proof: The proof is straightforward by the definition of M and $Rhomb^M$. ∎

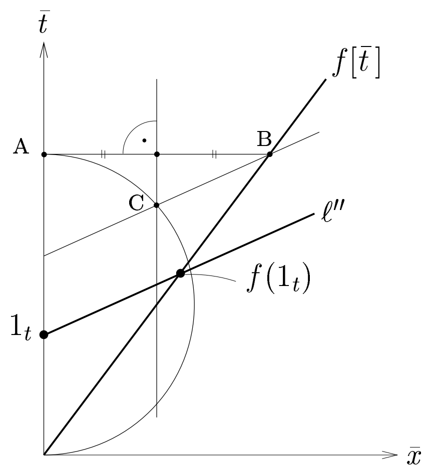


Figure 100: $f \in Rhomb^M \;\Leftrightarrow\; (f \in Rhomb \;\wedge\; f(1_t)$ is as above). (Illustration for Prop.3.8.43.)

PROPOSITION 3.8.44 *Assume $\mathfrak{F}$ is Euclidean. Then*

$$\mathfrak{M}^M_{\mathfrak{F}} \models \mathbf{BaCo}\,.$$

In the proof of Prop.3.8.44 we will need six lemmas. The first four of these are about some simple properties of the set $Rhomb = Rhomb(n, \mathfrak{F})$ of rhombus transformations (cf. 3.8.46–3.8.49). The fifth lemma is about the set PT of photon preserving (affine) transformations (cf. 3.8.50). The sixth lemma is about Minkowski-distance (cf. 3.8.51). The proof of Prop.3.8.44 comes below these lemmas on p.336.

Definition 3.8.45 We let $\iota_x : {}^nF \longrightarrow {}^nF$ be the linear transformation which inverts the x unit vector and leaves all the other unit vectors fixed, that is,

$$\iota_x \in Linb \qquad \text{with} \qquad \Big(\iota_x(1_x) = -1_x \quad \text{and} \quad (\forall 1 \neq i \in n)\ \iota_x(1_i) = 1_i\Big)\,.$$

We note, that for $n = 3$, ι_x is the reflection w.r.t. $\mathsf{Plane}(\bar{t}, \bar{y})$. The symbol ι_x intends to remind the reader that the x component is being inverted. ◁

[308] The just quoted part of Def.3.8.42 (i.e. the construction of E) was given in two dimensions only, and now we are in n dimensions. All the same this causes no problem, because the construction in which we want to imitate the corresponding part of Def.3.8.42 happens in $\mathsf{Plane}(\bar{t}, \bar{x})$.

LEMMA 3.8.46 *Assume $\mathfrak{F}$ is Euclidean. Let $p \in \mathsf{Plane}(\bar{t},\bar{x})$ such that $p \neq \bar{0}$ and $\overline{\bar{0}p} \in \mathsf{SlowEucl}$. Then there is $f \in Rhomb$ with $f(1_t) = p$.*

The **proof** will be filled in later. ∎

LEMMA 3.8.47 $\langle Rhomb, \circ, {}^{-1}, \mathrm{Id}\rangle$ *forms a group.*

The **proof** will be filled in later. ∎

Roughly, item (ii) of the following lemma says that there are exactly two rhombus transformations which agree on 1_t. Cf. Figure 101.

LEMMA 3.8.48 *Assume $n \geq 3$. Let $f, g \in Rhomb$ such that $f(1_t) = g(1_t)$. Then (i) and (ii) below hold.*

(i) *Assume $n \geq 3$. Then $f(1_y) = g(1_y)$.*

(ii) *Either $f = g$ or $f = \iota_x \circ g$. See Figure 101.*

The **proof** will be filled in later. ∎

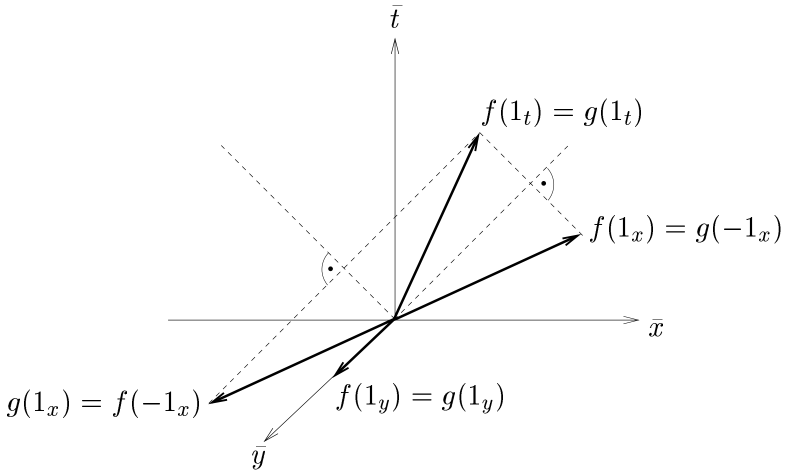


Figure 101: Illustration for Lemma 3.8.48: Assume $f, g \in Rhomb$ such that $f(1_t) = g(1_t)$ and $f \neq g$. Then $f = \iota_x \circ g$.

esetleg squares of M-distances?

LEMMA 3.8.49 *Assume $n \geq 3$. Assume $f \in Rhomb$ such that $f(1_y) = 1_y$. Then f preserves the square of Minkowski-distance, that is,*

$$(\forall p, q \in {}^nF)\ g_\mu^2(p,q) = g_\mu^2(f(p), f(q))\,.$$

On the proof: Assume $n \geq 3$ and assume $f \in Rhomb$ such that $f(1_y) = 1_y$. By $f(1_y) = 1_y$ and by the definition of *Rhomb*, we have that (i)–(iii) below hold.

(i) $(\forall 1 < i \in n)\ f(1_i) = 1_i$.

(ii) $f(1_t), f(1_x) \in \mathsf{Plane}(\bar{t}, \bar{x})$.

(iii) $f(1_t)$ and $f(1_y)$ are mirror images of each other w.r.t. a photon-line ℓ with $\bar{0} \in \ell \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

Also, by the definition of *Rhomb*, f takes photon-lines to photon-lines. Hence,

the f image $\overline{\bar{0}(f(1_t) + 1_y)}$ of the photon-line $\overline{\bar{0}(1_t + 1_y)}$ is a photon line too.

By this and by (ii) above, one can "compute" that

$$g^2_\mu(\bar{0}, f(1_t)) = 1\,. \tag{154}$$

By (ii), (iii) and (154), we conclude (155) below.

$$g^2_\mu(\bar{0}, f(1_x)) = 1\,. \tag{155}$$

Now we claim that (i)–(iii) and (154, 155) imply that f preserves the square of Minkowski-distance. Checking this claim is left to the reader. (The proof of this is a straightforward computation and we guess it should be known from the "standard part" of the literature. We note that f is a standard Lorentz transformation.)

∎

LEMMA 3.8.50 *Assume $f, f' \in PT$ such that $f[\bar{t}] = f'[\bar{t}]$ and $f(1_t)_t - f(\bar{0})_t = f'(1_t)_t - f'(\bar{0})_t$. Then*

$$f = g \circ f' \text{ for some } g \in \mathit{Triv} \text{ with } g[\bar{t}] = \bar{t}.$$

Proof: The proof will be filled in later. We note that a proof can be obtained using Lemma 3.6.20. ∎

LEMMA 3.8.51 *Assume $\ell, \ell' \in \mathsf{Eucl}$ such that $\mathrm{ang}^2(\ell) = \mathrm{ang}^2(\ell')$. Assume $p, q \in \ell$ and $p', q' \in \ell'$ such that $g^2_\mu(p, q) = g^2_\mu(p', q')$. Then,*

$$|p_t - q_t| = |p'_t - q'_t|\,.$$

Proof: The proof is straightforward. We omit it. ∎

Proof of Prop.3.8.44: Assume $\mathfrak{F}$ is Euclidean. To prove $\mathfrak{M}_{\mathfrak{F}}^{M} \models$ **BaCo**, by Prop.3.8.40, it remains to prove only that $\mathfrak{M}_{\mathfrak{F}}^{M} \models \mathbf{Ax(symm_0)}$. Let us recall that $Obs = PT^M$, $Rhomb^M \subseteq PT^M$. $\mathrm{Id} \in Rhomb^M$ by $M(0) = 1$. Let

$$m_0 \stackrel{\text{def}}{=} \mathrm{Id} \in Obs\,.$$

By the definition of $\mathfrak{M}_{\mathfrak{F}}^{M}$, it is easy to see that

$$(156) \qquad (\forall k \in Obs)\, \mathsf{f}_{km_0} = k \quad \wedge \quad (\forall m, k \in Obs)\, \mathsf{f}_{mk} = m \circ k^{-1}\,,$$

cf. (146) and (148) on p.329 in the proof of Prop.3.8.40.

Claim **3.8.52** Assume $n \geq 3$. Then $(\forall k \in Rhomb^M)\; k(1_y) = 1_y$.

Proof: Assume $k \in Rhomb^M$ ($\subseteq Obs$). Let h be a median observer for observers m_0 and k as constructed in Figure 91 on p.311, cf. Thm.3.8.25(ii). (Of course, we have to replace m in that picture with m_0.) Let $\mathrm{A}, \mathrm{B}, \mathrm{C}$ be points as in that construction. By Lemma 3.8.41(ii) (and Thm.3.8.25(ii)) we may assume that

$$h \in Rhomb^M\,.$$

Throughout the proof of this claim (i.e. 3.8.52) the reader is asked to consult Figure 102. Let us notice that $k(1_t) \in tr_{m_0}(k)$ since $k = \mathsf{f}_{km_0}$ (cf. 156). Now, by Prop.3.8.43,

$$(157) \qquad \overline{1_t k(1_t)} \parallel \overline{\mathrm{CB}}\,,$$

since $k \in Rhomb^M$. By the proof of Thm.3.8.25 (cf. (113) on p.310),

events $w_{m_0}(\mathrm{B})$ and $w_{m_0}(\mathrm{C})$ are simultaneous for h.

This and (157) imply that

(158) events $w_{m_0}(1_t)$ and $w_{m_0}(k(1_t))$ are simultaneous for h.

Let us notice that $\mathsf{f}_{m_0h}(k(1_t)) = \mathsf{f}_{kh}(1_t)$.[309] By this and by (158), we have

$$(159) \qquad \mathsf{f}_{m_0h}(1_t)_t \quad = \quad \mathsf{f}_{kh}(1_t)_t\,.$$

Obviously (by $\mathfrak{M}_{\mathfrak{F}}^{M} \models$ **Basax**),

[309] This is so because $k = \mathsf{f}_{km_0}$ (cf. (156)) and because $\mathsf{f}_{kh} = \mathsf{f}_{km_0} \circ \mathsf{f}_{m_0h}$.

world-view of m_0:

world-view of h:

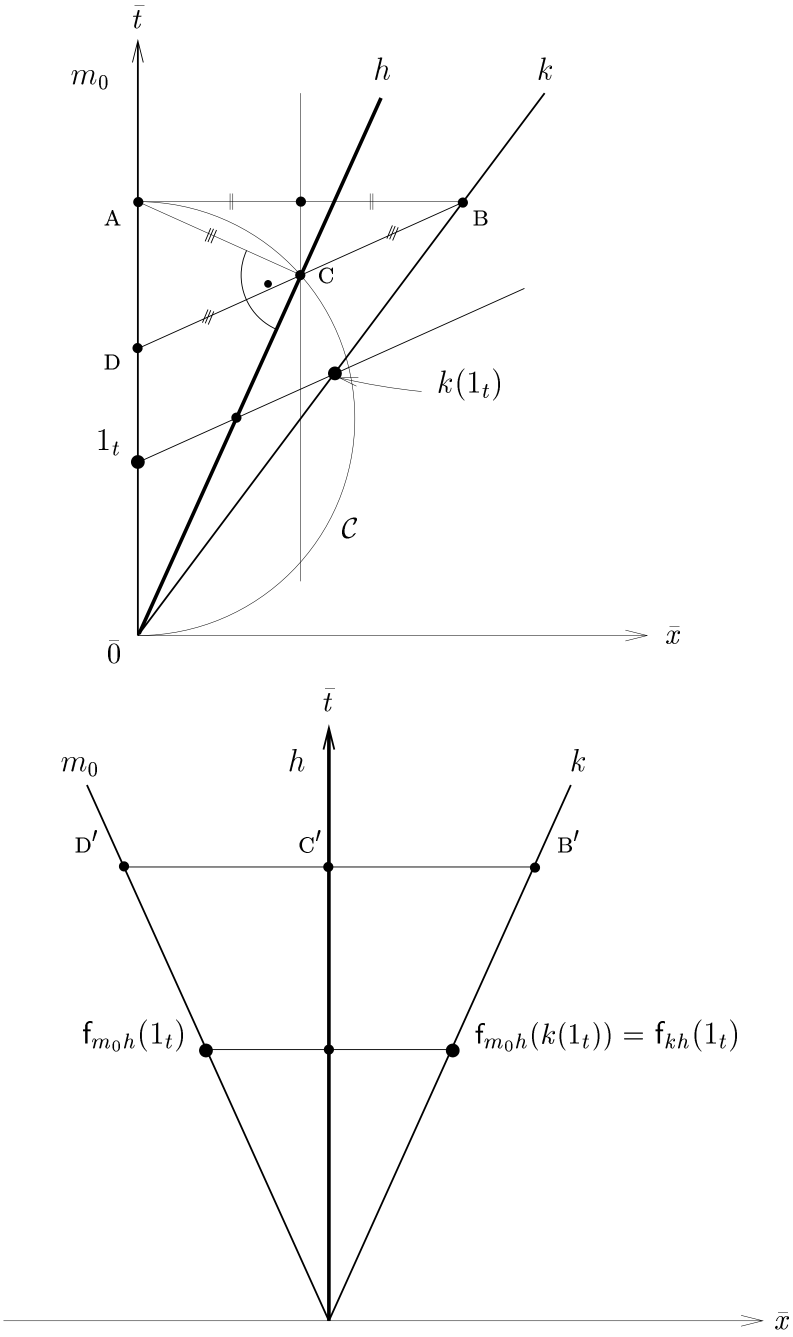


Figure 102: Illustration for the proof of Claim 3.8.52.

(160) $$\mathsf{f}_{m_0h}(1_t) \in tr_h(m_0) \quad \text{and} \quad \mathsf{f}_{kh}(1_t) \in tr_h(k)\,.$$

Now, by h being a median observer (for m_0 and k), we have that

(161) $$tr_h(m_0) \quad \text{and} \quad tr_h(k) \quad \text{are } \bar{t}\text{-symmetric.}$$

By (159)–(161), we conclude that

(162) $$\mathsf{f}_{m_0h}(1_t) \quad \text{and} \quad \mathsf{f}_{kh}(1_t) \quad \text{are } \bar{t}\text{-symmetric.}$$

Since $k, h \in \mathit{Rhomb}$, we have that

(163) $$\mathsf{f}_{m_0h} \in \mathit{Rhomb} \quad \text{and} \quad \mathsf{f}_{kh} \in \mathit{Rhomb}$$

by (156) (and Lemma 3.8.47). By (163), we have that

(164) $$\mathsf{f}_{m_0h}(1_t) \in \mathsf{Plane}(\bar{t}, \bar{x}) \quad \text{and} \quad \mathsf{f}_{kh}(1_t) \in \mathsf{Plane}(\bar{t}, \bar{x})\,.$$

Consider the transformation $\mathsf{f}_{m_0h} \circ \iota_x$. See Figure 103.

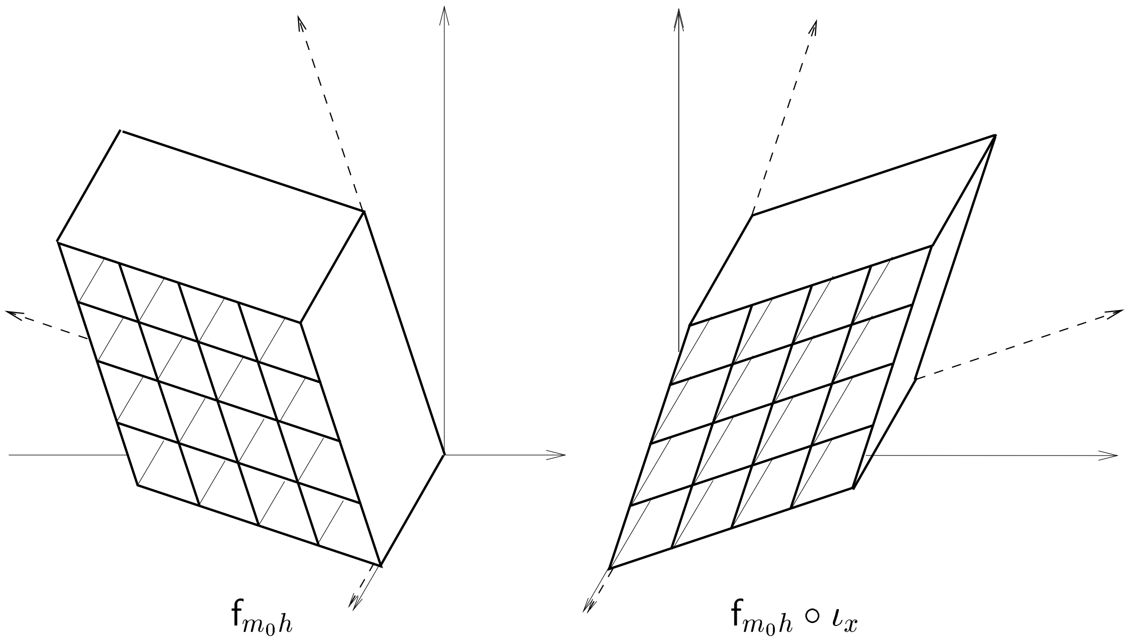


Figure 103: For $n = 3$, composing with ι_x from the right amounts to taking the mirror image w.r.t. $\mathsf{Plane}(\bar{t}, \bar{y})$ of the unit vectors determining the transformation f_{m_0h}.

(165) $$\mathsf{f}_{m_0h} \circ \iota_x \in \mathit{Rhomb}$$

by $\mathsf{f}_{m_0h},\ \iota_x \in \mathit{Rhomb}$ and by Lemma 3.8.47. Further, by (162, 164), we have

(166) $$(\mathsf{f}_{m_0h} \circ \iota_x)(1_t) = \mathsf{f}_{kh}(1_t)\,.$$

So, $\mathsf{f}_{m_0h} \circ \iota_x$ and f_{kh} are both rhombus transformations (cf. 163, 165) and they take 1_t to the same place (cf. 166). Hence, by Lemma 3.8.48(i), we conclude that $(\mathsf{f}_{m_0h} \circ \iota_x)(1_y) = 1_y$. By this, since ι_x leaves $\bar{y}$ point-wise fixed (and since $\mathsf{f}_{m_0h} \in$ *Rhomb*) we have that $\mathsf{f}_{m_0h}(1_y) = \mathsf{f}_{kh}(1_y)$. Therefore, by $\mathsf{f}_{km_0} = \mathsf{f}_{kh} \circ \mathsf{f}_{m_0h}^{-1}$, we conclude that $\mathsf{f}_{km_0}(1_y) = 1_y$. Hence, by $k = \mathsf{f}_{km_0}$, we have $k(1_y) = 1_y$, and this completes the proof of Claim 3.8.52.

QED (Claim 3.8.52)

Claim **3.8.53** $(\forall m, k \in Obs)$ [f_{km} preserves (the square of) Minkowski-distance].

Proof: If $n \geq 3$ then Claim 3.8.52 and Lemma 3.8.49 imply that every $k \in Rhomb^M$ preserves (the square of) Minkowski-distance. The same holds for $n = 2$ because of the following. Assume that $f \in Rhomb^M(2, \mathfrak{F})$. Then by Lemma 3.8.46 (and by Lemma 3.8.41) f can be extended in a natural way to an $f^* \in Rhomb(3, \mathfrak{F})$ (i.e. there is $f^* \in Rhomb(3, \mathfrak{F})$ such that f is the natural restriction of f^* to $\mathsf{Plane}(\bar{t}, \bar{x})$). But then, by Prop.3.8.43, we conclude that $f^* \in Rhomb^M(3, \mathfrak{F})$. So, since, as we said, the members $Rhomb^M(3, \mathfrak{F})$ preserve Minkowski-distance, f^* preserves Minkowski-distance. Therefore, f preserves Minkowski-distance.

The members of *Triv* and $Triv_0$ preserve Minkowski-distance. Further, the property of "preserving Minkowski-distance" is preserved under composition. But then, since $PT^M = \{ g \circ f \circ g_0 : f \in Rhomb,\ g \in Triv,\ g_0 \in Triv_0 \}$ by definition, we have that every $k \in PT^M$ $(= Obs)$ preserves Minkowski-distance. Therefore, $(\forall m, k \in Obs)$ $\mathsf{f}_{mk} = m \circ k^{-1}$ (cf. 156) completes the proof of the claim.

QED (Claim 3.8.53)

Now, we turn to proving $\mathfrak{M}^M_{\mathfrak{F}} \models \mathbf{Ax(symm_0)}$. The idea of this proof is illustrated in Figure 98 on p.326 and is explained in the intuitive text on p.323. Now, we include below a "computational" proof for this. To prove $\mathbf{Ax(symm_0)}$ let $m, k \in Obs$. Let h be a median observer for m and k. By Claim 3.8.53, we have $g^2_\mu(1_t, 0) = g^2_\mu(\mathsf{f}_{mh}(1_t), \mathsf{f}_{mh}(\bar{0}))$ and $g^2_\mu(1_t, 0) = g^2_\mu(\mathsf{f}_{kh}(1_t), \mathsf{f}_{kh}(\bar{0}))$. Hence,

$$g^2_\mu(\mathsf{f}_{mh}(1_t), \mathsf{f}_{mh}(\bar{0})) = g^2_\mu(\mathsf{f}_{kh}(1_t), \mathsf{f}_{kh}(\bar{0}))\,. \tag{167}$$

By (150) on p.330 in the proof of Prop.3.8.40, we have

$$\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t > 0 \qquad \text{and} \qquad \mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t > 0\,. \tag{168}$$

From (167, 168) and Lemma 3.8.51, we get

$$\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t = \mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t\,, \tag{169}$$

since by h being "the median", $ang^2(tr_h(m)) = ang^2(tr_h(k))$ holds, and since $\mathsf{f}_{mh}(1_t)$, $\mathsf{f}_{mh}(\bar{0}) \in tr_h(m)$ and $\mathsf{f}_{kh}(1_t)$, $\mathsf{f}_{kh}(\bar{0}) \in tr_h(k)$. It is easy to check that

(170)–(172) below hold; E.g. (171) holds by h being "the median" (and by **Ax4**); (172) holds by (169).

$$\mathsf{f}_{mh},\ \mathsf{f}_{kh},\ \mathsf{f}_{mh}\circ\sigma_{\bar{t}},\ \mathsf{f}_{kh}\circ\sigma_{\bar{t}}\in PT\,. \tag{170}$$

$$\mathsf{f}_{mh}[\bar{t}\,]=(\mathsf{f}_{kh}\circ\sigma_{\bar{t}})[\bar{t}\,]\quad\text{ and }\quad\mathsf{f}_{kh}[\bar{t}\,]=(\mathsf{f}_{mh}\circ\sigma_{\bar{t}})[\bar{t}\,]\,. \tag{171}$$

$$\begin{array}{rcll}\mathsf{f}_{mh}(1_t)_t-\mathsf{f}_{mh}(\bar{0})_t&=&(\mathsf{f}_{kh}\circ\sigma_{\bar{t}})(1_t)_t-(\mathsf{f}_{kh}\circ\sigma_{\bar{t}})(\bar{0})_t&\text{and}\\ \mathsf{f}_{kh}(1_t)_t-\mathsf{f}_{kh}(\bar{0})_t&=&(\mathsf{f}_{mh}\circ\sigma_{\bar{t}})(1_t)_t-(\mathsf{f}_{mh}\circ\sigma_{\bar{t}})(\bar{0})_t\,.&\end{array} \tag{172}$$

By (170)–(172) and Lemma 3.8.50, there are $f,g\in \mathit{Triv}$ such that $f[\bar{t}\,]=g[\bar{t}\,]=\bar{t}$ and

$$\mathsf{f}_{mh}=f\circ\mathsf{f}_{kh}\circ\sigma_{\bar{t}}\quad\text{ and }\quad\mathsf{f}_{kh}=g\circ\mathsf{f}_{mh}\circ\sigma_{\bar{t}}\,. \tag{173}$$

Let such f,g be fixed. Now, by $\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}$ (and by $f^{-1},g^{-1}\in \mathit{Triv}$ with $f^{-1}[\bar{t}\,]=g^{-1}[\bar{t}\,]=\bar{t}\,$) there are $k',m'\in \mathit{Obs}$ such that

$$tr_k(k')=tr_m(m')=\bar{t}\,,\quad\text{ and } \tag{174}$$

$$\mathsf{f}_{k'k}=f\qquad\text{ and }\qquad\mathsf{f}_{m'm}=g\,. \tag{175}$$

By (173) and (175), we conclude

$$\mathsf{f}_{mh}=\mathsf{f}_{k'h}\circ\sigma_{\bar{t}}\qquad\text{ and }\qquad\mathsf{f}_{kh}=\mathsf{f}_{m'h}\circ\sigma_{\bar{t}}\,. \tag{176}$$

Now,

$$\begin{array}{rcll}\mathsf{f}_{mk}&=&\mathsf{f}_{mh}\circ\mathsf{f}_{hk}&\\ &=&(\mathsf{f}_{k'h}\circ\sigma_{\bar{t}})\circ(\mathsf{f}_{m'h}\circ\sigma_{\bar{t}})^{-1}&\text{by (176)}\\ &=&\mathsf{f}_{k'h}\circ\sigma_{\bar{t}}\circ\sigma_{\bar{t}}\circ\mathsf{f}_{hm'}&\text{by } \sigma_{\bar{t}}^{-1}=\sigma_{\bar{t}}\text{ and } \mathsf{f}_{m'h}^{-1}=\mathsf{f}_{hm'}\\ &=&\mathsf{f}_{k'h}\circ\mathsf{f}_{hm'}&\text{by } \sigma_{\bar{t}}^{2}=\mathrm{Id}\\ &=&\mathsf{f}_{k'm'}\,.&\end{array}$$

So $\mathsf{f}_{mk}=\mathsf{f}_{k'm'}$, and this by (174) completes the proof of Prop.3.8.44. ∎

3.8.3 $\mathfrak{F}$-categoricity, completeness etc. of BaCo

intuitív szöveg kéne!

This sub-section is devoted to proving the uniqueness part of Thm.3.8.7 saying that **BaCo** is $\mathfrak{F}$-categorical for any Euclidean $\mathfrak{F}$; that is, to every Euclidean $\mathfrak{F}$ there is exactly one model up to isomorphisms of **BaCo** extending $\mathfrak{F}$.

Let us recall that in §3.8.2 (Def.3.8.42) for every Euclidean $\mathfrak{F}$ (and $n \geq 2$) the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ was defined. In the present sub-section we will prove that every model $\mathfrak{N}$ of **BaCo**, with ordered field reduct $\mathfrak{F}$, is isomorphic with the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$. Therefore any two models of **BaCo** with the same ordered field reduct are isomorphic.

First, we state a lemma.

LEMMA 3.8.54 **Basax+Ax(symm)+Ax($\uparrow$)+Ax5$^+$+Ax(**$Triv_t$**)** $\models$ **Ax(**$Triv$**)**.

The **proof** of this lemma will be included at a later stage of development. ∎

PROPOSITION 3.8.55 *Assume $\mathfrak{N}$ is a model of* **BaCo** *(and $n \geq 2$). Assume $\mathfrak{F}$ is the ordered field reduct of $\mathfrak{N}$. Then $\mathfrak{N}$ is isomorphic with the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ (defined in Def.3.8.42).*

Proof: Assume $\mathfrak{N}$ is a model of **BaCo**, and assume that the ordered field reduct is $\mathfrak{F}$. Then $\mathfrak{F}$ is Euclidean (by $\mathbf{Ax}(\sqrt{\ }) \in$ **BaCo**). Recall that the sets $Rhomb^M$ and PT^M of transformations were defined in Def.3.8.42. Let $m_0 \in Obs^{\mathfrak{N}}$ be *arbitrary*, but fixed.

Claim 3.8.56 $\left\{ \mathsf{f}_{km_0} \ : \ k \in Obs^{\mathfrak{N}} \right\} = PT^M$.

We will give the *proof* of this claim at the end of the proof of Prop.3.8.55.

Let $\mathfrak{M} := \mathfrak{M}^M_{\mathfrak{F}}$ be the Minkowski model over $\mathfrak{F}$. Recall that by the definition of $\mathfrak{M}^M_{\mathfrak{F}}$, we have that

$$Obs^{\mathfrak{M}} = PT^M \qquad \text{and} \qquad Ph^{\mathfrak{M}} = \mathsf{PhtEucl}\,,$$

cf. Def.3.8.38. Let

$$\alpha_0 \stackrel{\text{def}}{=} \left\{ \langle k, \mathsf{f}_{km_0} \rangle \ : \ k \in Obs^{\mathfrak{N}} \right\} ,$$
$$\alpha_1 \stackrel{\text{def}}{=} \left\{ \langle ph, tr_{m_0}(ph) \rangle \ : \ ph \in Ph^{\mathfrak{N}} \right\} .$$

Then by Claim 3.8.56 (and by $Obs^{\mathfrak{M}} = PT^M$) $\alpha_0 : Obs^{\mathfrak{N}} \longrightarrow Obs^{\mathfrak{M}}$ is a surjective function. By $\mathfrak{N} \models \mathbf{Basax} + \mathbf{Ax(ext)}$, we have that $(k \neq m \Rightarrow \mathsf{f}_{km_0} \neq \mathsf{f}_{mm_0})$. Hence α_0 is a bijection. Similarly, $\alpha_1 : Ph^{\mathfrak{N}} \longrightarrow Ph^{\mathfrak{M}}$ is a bijection too by $Ph^{\mathfrak{M}} = \mathsf{PhtEucl}$ and by $\mathfrak{N} \models \mathbf{Basax} + \mathbf{Ax(ext)}$.

$$(\alpha_0 \cup \alpha_1) : B^{\mathfrak{N}} \longrightarrow B^{\mathfrak{M}} \quad \text{is a bijection,}$$

since $B = Obs \cup Ph$ (by $\mathbf{Ax}\heartsuit$) and $Obs \cap Ph = \emptyset$ by Prop.2.3.3(i). This bijection extends in a natural way to a "potential isomorphism"[310] α between models $\mathfrak{N}$ and $\mathfrak{M}$; namely α is the identity function on the sorts F and G (and of course is $\alpha_0 \cup \alpha_1$ on sort B). Next, we prove that the so defined extended function α is indeed a homomorphism, namely that it commutes with the operations and relations. Moreover, we will prove that it is also an isomorphism. Consider the relation $W \subseteq Obs \times {}^nF \times B$. We will prove that

$$W^{\mathfrak{N}}(k, p, b) \quad \Leftrightarrow \quad W^{\mathfrak{M}}(\alpha(k), p, \alpha(b))\,.$$

Claim **3.8.57** $(\forall k \in Obs^{\mathfrak{N}})(\forall b \in B^{\mathfrak{N}})\ tr^{\mathfrak{N}}_k(b) = tr^{\mathfrak{M}}_{\alpha(k)}(\alpha(b))$.

Proof: Let us notice that $\alpha(m_0) = \mathrm{Id}$ by the definition of α. Now, by the definition of $\mathfrak{M} = \mathfrak{M}^M_{\mathfrak{F}}$ is easy to check that for every $k \in Obs^{\mathfrak{N}}$, $ph \in Ph^{\mathfrak{N}}$ and $b \in Obs^{\mathfrak{N}}$ (177)–(179) below hold (cf. (143), (144) and (147) around p.329 in the Proof of Prop.3.8.40).

$$tr_{\alpha(m_0)}(\alpha(k)) = \alpha(k)[\bar{t}\,]\,. \tag{177}$$

$$tr_{\alpha(m_0)}(\alpha(ph)) = \alpha(ph)\,. \tag{178}$$

$$\alpha(k)\left[tr_{\alpha(k)}(\alpha(b))\right] = tr_{\alpha(m_0)}(\alpha(b))\,. \tag{179}$$

Next, we prove that (180) below holds.

$$(\forall b \in B^{\mathfrak{N}})\ (tr^{\mathfrak{N}}_{m_0}(b) = tr^{\mathfrak{M}}_{\alpha(m_0)}(\alpha(b)) \tag{180}$$

To prove (180) let $b \in Obs^{\mathfrak{N}}$. By $\mathbf{Ax}\heartsuit$, $B^{\mathfrak{N}} = Obs^{\mathfrak{N}} \cup Ph^{\mathfrak{N}}$. Assume first that $b = k$, for some $k \in Obs^{\mathfrak{N}}$. Let this k be fixed. Then, $tr_{m_0}(k) = \mathsf{f}_{m_0k}[\bar{t}\,] = \alpha(k)[\bar{t}\,] = tr_{\alpha(m_0)}(\alpha(k))$ by (177) above and by the definition of α. Hence, (180) holds when $b \in Obs^{\mathfrak{N}}$. Assume now that $b = ph$, for some $ph \in Ph^{\mathfrak{N}}$. Let this ph be fixed. Then, $tr_{m_0}(ph) = \alpha(ph) = tr_{\alpha(m_0)}(\alpha(ph))$ by (178) and by the definition of α. So, (180) has been proved.

[310] We recall from the literature that a homomorphism between our kind of 3-sorted models $\mathfrak{N}$ and $\mathfrak{M}$ consists of 3 functions h_B, h_F, h_G such that $h_B : B^{\mathfrak{N}} \longrightarrow B^{\mathfrak{M}}$, $h_F : F^{\mathfrak{N}} \longrightarrow F^{\mathfrak{M}}$, $h_G : G^{\mathfrak{N}} \longrightarrow G^{\mathfrak{M}}$ and the usual commutativity conditions hold. By a potential homomorphism we understand such a triple of functions without requiring the commutativity conditions.

Now, we turn to prove Claim 3.8.57. Let $k \in Obs^{\mathfrak{N}}$ and $b \in B^{\mathfrak{N}}$. Then,

$$\begin{array}{rcll} tr_k(b) & = & \mathsf{f}_{km_0}^{-1}[tr_{m_0}(b)] & \text{by Prop.2.3.3} \\ & = & \alpha(k)^{-1}[tr_{\alpha(m_0)}(\alpha(b))] & \text{by the def. } \alpha \text{ and by (180)} \\ & = & tr_{\alpha(k)}(\alpha(b)) & \text{by (179)}\,. \end{array}$$

This completes the proof of Claim 3.8.57.

QED (Claim 3.8.57)

Now, by Claim 3.8.57 and by the fact that in every frame model

$$W(k,p,b) \quad \Leftrightarrow \quad p \in tr_k(b)$$

holds, we conclude that, for every $k \in Obs^{\mathfrak{N}}$, $p \in {}^nF$ and $b \in B^{\mathfrak{N}}$,

$$W^{\mathfrak{N}}(k,p,b) \quad \Longleftrightarrow \quad W^{\mathfrak{M}}(\alpha(k),p,\alpha(b))\,.$$

With this we proved that α is an isomorphism between $\mathfrak{N}$ and $\mathfrak{M}$, and this is what we wanted. So, Prop.3.8.55 is proved modulo Claim 3.8.56.

Let us turn proving Claim 3.8.56.

Proof of Claim 3.8.56: Let us recall that the claim states that
$\left\{\, \mathsf{f}_{km_0} \,:\, k \in Obs^{\mathfrak{N}} \,\right\} = PT^M$.

Claim **3.8.58** Assume $m \in Obs$. Assume $\ell \in \mathsf{SlowEucl}$ such that $\bar{0} \in \ell \subseteq \mathsf{Plane}(\bar{t},\bar{x})$. Then there is $k \in Obs$ such that $\mathsf{f}_{km} \in Rhomb$ and $\mathsf{f}_{km}[\bar{t}\,] = \ell$.

Proof: Let $m \in Obs$. Let $\ell \in \mathsf{SlowEucl}$ such that $\bar{0} \in \ell \subseteq \mathsf{Plane}(\bar{t},\bar{x})$. Let $k' \in Obs$ such that

$$tr_m(k') = \ell$$

and $\mathsf{f}_{k'm}(\bar{0}) = \bar{0}$. Such a k' exist by $\mathbf{Ax}(Triv_t)$ and $\mathbf{Ax5}$. Let $f \in Rhomb$ such that $f(1_t) = \mathsf{f}_{k'm}(1_t)$. Such an f exists by Lemma 3.8.46. Let us notice that

(181) $$f[\bar{t}\,] = tr_m(k') = \ell\,.$$

Now, it is easy to check that

(182) $$(\mathsf{f}_{k'm} \circ f^{-1})(1_t) = 1_t \quad \text{and} \quad (\mathsf{f}_{k'm} \circ f^{-1})(\bar{0}) = \bar{0}\,.$$

Further, by Prop.3.8.35(ii), we have that

(183) $$\mathsf{f}_{k'm} \circ f^{-1} \in PT\,.$$

By (182, 183) and Lemma 3.6.20, we conclude that

$$\mathsf{f}_{k'm} \circ f^{-1} \in Triv_0\,.$$

By this, and by $\mathbf{Ax}(\mathit{Triv}_t)$, there is $k \in \mathit{Obs}$ with $tr_{k'}(k) = \bar{t}$ and $\mathsf{f}_{k'k} = \mathsf{f}_{k'm} \circ f^{-1}$. Let such a k be fixed. But, then $\mathsf{f}_{km} = f$ and $\mathsf{f}_{km}[\bar{t}\,] = tr_m(k) = tr_m(k') = \ell$. By $f \in \mathit{Rhomb}$, this completes the proof of Claim 3.8.58.

QED (Claim 3.8.58)

Throughout this proof $\mathit{Obs} := \mathit{Obs}^{\mathfrak{N}}$.

Claim **3.8.59** $(\forall m, k \in \mathit{Obs})\ (\mathsf{f}_{km} \in \mathit{Rhomb} \ \Rightarrow\ \mathsf{f}_{km} \in \mathit{Rhomb}^M)$.

Proof: Let $m, k \in \mathit{Obs}$ such that $\mathsf{f}_{km} \in \mathit{Rhomb}$. Let h be the median observer for observers m and k as constructed in Figure 91, 92 (cf. Thm.3.8.25). By Prop.3.8.32, in the world-view of h the time unit vectors of m and k are $\bar{t}$-symmetric. Hence, events $w_m(1_t)$ and $w_m(\mathsf{f}_{km}(1_t))$ are simultaneous for h. Therefore, by Prop.3.8.43 (see Figure 100 on p.333), by Thm.3.8.25(ii) (see Figures 91, 92 on pp.311–312) and by the proof of Thm.3.8.25, we conclude that $\mathsf{f}_{km} \in \mathit{Rhomb}^M$.

QED (Claim 3.8.59)

Claim **3.8.60** Assume $m \in \mathit{Obs}$. Then $(\forall f \in \mathit{Rhomb}^M)(\exists k \in \mathit{Obs})\ \mathsf{f}_{km} = f$.

Proof: Let $m \in \mathit{Obs}$. Let $f \in \mathit{Rhomb}^M$. Then by Claim 3.8.58 (and by Lemma 3.8.41(i)), there is $k \in \mathit{Obs}$ such that

$$\mathsf{f}_{km} \in \mathit{Rhomb} \qquad \text{and} \qquad \mathsf{f}_{km}[\bar{t}\,] = f[\bar{t}\,]\,.$$

But then, by Claim 3.8.59, we have

$$\mathsf{f}_{km} \in \mathit{Rhomb}^M$$

(since $\mathsf{f}_{km} \in \mathit{Rhomb}$). Further, by f, $\mathsf{f}_{km} \in \mathit{Rhomb}^M$, by $\mathsf{f}_{km}[\bar{t}\,] = f[\bar{t}\,]$ and by the definition of Rhomb^M, we have

$$\mathsf{f}_{km}(1_t) = f(1_t)\,.$$

Then by Lemma 3.8.48(ii) either $\mathsf{f}_{km} = f$ or $\mathsf{f}_{km} = \iota_x \circ f$. If $\mathsf{f}_{km} = f$ is the case then we are done. So, assume $\mathsf{f}_{km} = \iota_x \circ f$. Then by $\mathbf{Ax}(\mathit{Triv}_t)$ (and by $\iota_x \in \mathit{Triv}$ with $\iota_x[\bar{t}\,] = \bar{t}\,$), there is $k' \in \mathit{Obs}$, such that $\mathsf{f}_{kk'} = \iota_x$. For this k', we have $\mathsf{f}_{k'm} = f$ (by $\mathsf{f}_{km} = \iota_x \circ f$ and $\mathsf{f}_{kk'} = \iota_x$). This completes the proof of Claim 3.8.60.

QED (Claim 3.8.60)

Claim **3.8.61** Assume $m \in \mathit{Obs}$. Then $(\forall f \in PT^M)(\exists k \in \mathit{Obs})\ \mathsf{f}_{km} = f$.

Proof: Let $m \in Obs$. Let $f \in PT^M$. We will prove that there is $k \in Obs$ such that $\mathsf{f}_{km} = f$. By the definition of PT^M

$$f = g \circ f' \circ g_0, \tag{184}$$

for some $f' \in Rhomb^M$, $g \in Triv$, and $g_0 \in Triv_0$. Let such f', g, g_0 be fixed. By Lemma 3.8.54, we have that $\mathfrak{N} \models \mathbf{Ax}(Triv)$. Now, let $m' \in Obs$ such that

$$\mathsf{f}_{m'm} \;=\; g_0\,. \tag{185}$$

Such an m' exists by $\mathbf{Ax}(Triv_t)$. Let $k' \in Obs$ such that

$$\mathsf{f}_{k'm'} \;=\; f'\,. \tag{186}$$

Such a k' exists by Claim 3.8.60. Let $k \in Obs$ such that

$$\mathsf{f}_{kk'} \;=\; g\,. \tag{187}$$

Such a k exists by $\mathbf{Ax}(Triv)$.

Now, by (184)–(187), we have $\mathsf{f}_{km} = \mathsf{f}_{kk'} \circ \mathsf{f}_{k'm'} \circ \mathsf{f}_{m'm} = g \circ f' \circ g_0 = f$. This completes the proof of Claim 3.8.61.

QED (Claim 3.8.61)

Claim **3.8.62** Assume $m \in Obs$. Then $(\forall k \in Obs)\ \mathsf{f}_{km} \in PT^M$.

Proof: Let $m, k \in Obs$. We will prove that $\mathsf{f}_{km} \in PT^M$. By Claim 3.8.41(iii) we have that (188) below holds.

$$(\forall \ell \in \mathsf{SlowEucl})(\exists f \in PT^M)\ f[\bar{t}] = \ell\,. \tag{188}$$

$tr_m(k) \in \mathsf{SlowEucl}$ since there are no FTL observers in $\mathfrak{N}$ by Thm's 3.8.11, 3.4.1. Let $f \in PT^M$ such that $f[\bar{t}] = tr_m(k)$. Such an f exists by (188) above. Let $k' \in Obs$ such that

$$\mathsf{f}_{k'm} = f\,.$$

Such a k' exist by Claim 3.8.60. By $\mathsf{f}_{k'm} = f$ and $f[\bar{t}] = tr_m(k)$ we have that k and k' are "brothers", i.e. $tr_k(k') = \bar{t}$. But then $\mathsf{f}_{kk'} \in Triv$ by Prop.3.8.33(ii). Since $f \in PT^M$ and $\mathsf{f}_{kk'} \in Triv$, by Lemma 3.8.41(iv) (or by the definition of PT^M) we have that $\mathsf{f}_{kk'} \circ f \in PT^M$. Now, $\mathsf{f}_{kk'} \circ f \in PT^M$ and $f = \mathsf{f}_{k'm}$ imply that $\mathsf{f}_{km} = \mathsf{f}_{kk'} \circ \mathsf{f}_{k'm} \in PT^M$. This completes the proof of Claim 3.8.62.

QED (Claim 3.8.62)

Now, by Claims 3.8.61 and 3.8.62, we have that

$$(\forall m \in Obs)\ \{\,\mathsf{f}_{km}\ :\ k \in Obs\,\} = PT^M.$$

This completes the proof of Claim 3.8.56 and the proof of Prop.3.8.55. ∎

At this point all parts of the proof of Thm.3.8.7 has been taken care of.

Next, we formulate characterizations of the rhombus transformations and the photon preserving transformations in Minkowski models (or equivalently in **BaCo** models).

PROPOSITION 3.8.63

$$\begin{array}{lrcll} \textbf{(i)} & \mathit{Rhomb}^M & = & \{ f \in \mathit{SLor} & : \quad f(1_t)_t > 0, \quad f[\bar{t}\,] \in \mathsf{SlowEucl}\,\}. \\ \textbf{(ii)} & \mathit{PT}^M & = & \{ f \in \mathit{Poi} & : \quad f(1_t)_t > 0, \quad f[\bar{t}\,] \in \mathsf{SlowEucl}\,\}. \end{array}$$

The **proof** will be filled in later. ∎

PROPOSITION 3.8.64 *Assume $n \geq 3$. Then (i) and (ii) below hold.*

$$\begin{array}{lrcll} \textbf{(i)} & \mathit{Rhomb}^M & = & \{ f \in \mathit{SLor} & : \quad f(1_t)_t > 0\,\}. \\ \textbf{(ii)} & \mathit{PT}^M & = & \{ f \in \mathit{Poi} & : \quad f(1_t)_t > 0\,\}. \end{array}$$

Proof: The proposition follows from Prop.3.8.63 above and Lemma 3.4.5 on p.205. ∎

It might be of interest to notice that for any $m \in \mathit{Obs}$ the set of world-view transformations involving m is the whole of PT^M, in Minkowski models by Claim 3.8.56.

3.9 Symmetry axioms

In §2.8 we began with the formalization of symmetry assumptions used in relativistic arguments, which are understood as part of Einstein's Special Principle of Relativity (SPR). We introduced our distinguished symmetry axiom **Ax(symm)** as a possible candidate for the role of symmetry axiom. In this section we intend to push this analysis further.

First, we shall introduce several (rather strong) symmetry axioms that formalize and logically analyze the "remaining part" of Einstein's SPR (in a sense we are going to discuss soon). We shall introduce $\mathbf{Ax}(\omega)$ for their conjuction to abstract from the subtle differences among them. Then we shall recall and discuss some weaker axioms occurring in this study, which are also partial formalizations of SPR expressing symmetry.

Axiom $\mathbf{Ax}(\omega)$ is used for the same purpose as **Ax(symm)** and **Ax(syt)**[311]. Namely, whenever we have a "core theory" like **Basax**, **Flxbasax** or **Bax**, we shall examine the way the introduction of a symmetry axiom as a *methodological principle* simplifies our description of the world. By considering symmetry as a matter of methodology (or aesthetics) we entertain the following idea. While axioms of our core theories from **Basax** to $\mathbf{Bax}^-$ and further can be defended to some extent by referring to experiments, perhaps combined with, say, "inductive logic", accepting symmetry is some sort of "aesthetic" decision: we want to describe the world as simply as possible, excluding e.g. the possibility that different observers use different measurement units thereby disguising physical phenomena. This kind of methodological decision is related to ideas/principles known as "Occam's razor" or "Mach's principle". In this connection it is very important to choose for each theory an *adequate* symmetry principle, in the sense that our methodological decision should not falsify the essential physical assumptions of the theory. For example, both $\mathbf{Ax}(\omega)$ and **Ax(symm)** are adequate for **Basax**(n) for $n \geq 3$; if $n = 2$ then only **Ax(symm)** is adequate; for **Bax** only **Ax(syt)** seems to be appropriate. At first sight we can use the "rule of thumb" that $\mathbf{Ax}(\omega) > \mathbf{Ax(symm)} > \mathbf{Ax(syt)}$.

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We note that the first two pages of §3.9.1 below contain important information, relevant to the present introduction. Similarly, Remark 3.9.10 (p.355) is strongly relevant to this introduction. Further Corollary 3.8.21 on p.304 is highly relevant. Actually, that corollary implies the following model theoretical characterization of symmetry principles (in the spirit of Einstein's formulation of SPR).

Definition 3.9.1 Let

$$\mathbf{BaCo}^- \stackrel{\text{def}}{:=} \mathbf{BaCo} \setminus \{\mathbf{Ax(symm)}\} + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\sqrt{\ }), \quad \text{i.e.}$$

[311] Cf. §§2.8, 4.2.

$$\mathbf{BaCo}^- \quad = \quad \mathbf{Basax} + (\mathbf{Compl} \setminus \{\mathbf{Ax(symm)}\}) + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\sqrt{\ }),$$

cf. p.298 (Def.3.8.2).

$\lhd$

THEOREM 3.9.2 *Assume $\mathfrak{M} \models \mathbf{BaCo}^-$. Then (i) and (ii) below are equivalent.*

(i) $(\forall m, k \in Obs^{\mathfrak{M}})(\exists \alpha \in Aut(\mathfrak{M}))\, \alpha(m) = k$.

I.e. any two observers are connected by an automorphism of $\mathfrak{M}$ (i.e. all observers are alike).

(ii) $\mathfrak{M} \models \mathbf{Ax(symm)}$. ∎

The above theorem says the following.

$$\mathbf{Ax(symm)} \quad \Longleftrightarrow \quad (\forall m, k \in Obs)(\exists \alpha \in Aut(\mathfrak{M}))\, \alpha(m) = k,$$

assuming $\mathbf{BaCo}^-$.

We should emphasize two things here: (1) This is a model theoretic characterization of **Ax(symm)** in Einstein's spirit. (2) We had to pay a very high price for this characterization in the sense that $\mathbf{BaCo}^-$ is a very strong assumption. We will return to giving satisfactory model theoretical characterizations under weaker assumptions in chapter 6 (cf. §6.2.8).

3.9.1 Alternative symmetry axioms

As we promised, we are introducing a couple of axioms which are candidates for the role of symmerty axiom. Each can be considered as a partial formalization of Einstein's Special Principle of Relativity. They assert symmetry (the claim that the way two observers see each other is "the same") in special characteristic situations. As we shall see later, they are not necessarily equivalent assuming **Basax** only.[312]

Let us discuss first in which sense we speak about formalizing parts of SPR. Roughly speaking, SPR states that from a certain point of view all inertial observers are alike. Then we only have to specify the relevant aspect from which inertial observers are all the same. The usual stipulation is that no law of nature distinguishes any observer from the others. More formally,

($\star$) If a formula φ in the frame language of relativity qualifies as a law of nature, $\mathfrak{M}$ is a frame model and $m, k \in Obs^{\mathfrak{M}}$, then $(\mathfrak{M} \models \varphi(m) \iff \mathfrak{M} \models \varphi(k))$.

[312] We note, however, that when analyzing symmetrical versions of theories we shall only use the concatenation (conjuction) of the axioms to be introduced (see $\mathbf{Ax}(\omega)$ below).

Which formulae are then potential laws of nature? This is not easy to tell precisely. We obviously do not want φ to talk about something accidental (e.g. the cardinality $\mathrm{card}(w_m^{\mathfrak{M}}(\bar{0}))$ of bodies at the origin of m's coordinate system). Clearly, our earlier axiom systems like **Basax**, **Newbasax**, **Bax** all can be regarded as capturing parts of SPR in the sense that they do not distinguish any subset of inertial observers. (E.g. by **Basax** all observers see photons the same way. This can be understood as part of SPR. Also the very philosophy of **Basax** that does not distinguish a particular observer as the main observer is part of SPR.)

Since we cannot tell which formulae are laws of nature and which are not,[313] we must formalize parts (or instances) of SPR in a form different from the above schema. To be sure, we do not want to use the whole of SPR. The reason is that it would be too strong for helping the logical analysis. Axioms **Ax□1**, **Ax△1**, **Ax□2**, **Ax△2**, to be introduced below, as well as **Ax(symm)** and **Ax(syt)**, are formalized instances of SPR in the sense that they assert symmetry (the claim that the way two observers see each other is "the same") in special characteristic situations.[314] As a "limiting case", Corollary 3.8.21 in the previous section shows that in our special axiom system **BaCo** we have collected enough instances of SPR to recover the whole principle. But we emphasize again that our main concern is not to obtain stronger and stronger axiom systems.[315] Instead, we thrive for understanding, insight, simplicity, decomposition, analysis, as explained in the introduction. In case of SPR, for instance, different parts may be acceptable for the Reichenbachian theories and for the conventional Einsteinian theory.

Let us turn to introducing some of the promised symmetry axioms. First of all, we introduce below the notion of an isometry (over an arbitrary field) and include a technical result, because they will help us in handling the sort of transformations that comes up naturally when formulating some of our symmetry axioms.

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[313]No doubt the concept of a law of nature has a more or less clear intuitive meaning. We only assert that this concept has no precise formal definition which could be translated to our plain language.

The issue of the definiability of laws of nature is not at all an idle question. For example, the literature writes about a conflict between Gödel's incompleteness theorem and Hawking's program of searching for a TOE (also called the Final Laws of Nature). The logical part of this debate could be resolved if one could answer the above question, i.e. if one could tell exactly which formulae are potential laws of nature. For the discussion of this problem (about laws of nature) with an emphasis different from ours cf. Friedman [90], pp. 150-151.

[314]At first sight principle $(\star)$ sounds different from saying that "the way I see you is the way you see me". But a little reflection on the ideas involved reveals that the "as I see you so do you see me" statement is a special case of $(\star)$.

[315]The stronger your axiom system gets the weaker (and less general) your theorems get.

Definition 3.9.3 Let $\mathbf{F}$ be a field. (Note that $\mathbf{F}$ is not necessarily Euclidean, i.e. square root may not exist.) Let $h : {}^nF \to {}^nF$ be an arbitrary function. We call h an isometry if h preserves the square of Euclidean distances. That is, $\|p - q\| = \|h(p) - h(q)\|$ for any $p, q \in {}^nF$.

We shall use the expressions "congruence transformation" or "distance preserving transformation" as synonymes for "isometry".

Remark 3.9.4

(i) Isomeries on ${}^n\mathbf{F}$ form a group, for any field $\mathbf{F}$.

(ii) The connection between our class of transformations *Triv* (introduced in Def. 3.5.1) and the class of isometries is as follows. *Triv* is the set of those isometries f for which $f[\bar{t}] \parallel \bar{t}$ and $f(1_t)_t > 0$.

LEMMA 3.9.5

Assume $\mathfrak{F}$ is an ordered field. If $h : {}^nF \to {}^nF$ is an isometry, then h is an affine transformation. Briefly:

$$\textit{Isometry} \Rightarrow \textit{affine.}$$

We postpone the proof Lemma 3.9.5. It comes after Thm. 3.9.31.

Ax□1 $(\forall m, k, m' \in Obs)(\exists k' \in Obs)\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$.

That is, any two observers m and m' are equivalent in the sense that if m sees some k a certain way, then m' too sees some observer, call it k', exactly the same way as m sees k. (This intuitive formulation clearly shows that **Ax□1** is a special case of the $(\star)$ form of SPR.)

Ax□2 $(\forall m, k, m', k' \in Obs)(tr_m(k) = tr_{m'}(k') \to$
there is an (affine) isometry N of ${}^n\mathbf{F}$ such that $N[\bar{t}] \parallel \bar{t}$ and $\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ N)$.[316]

[316]Lemma 3.9.5 ensures that we "lose" no isometry by quantifying over *affine transformations*. This is necessary to keep **Ax□2** in first order logic.

Alternatively, in the conclusion one could use the composition $g = \mathsf{f}_{mk} \circ \mathsf{f}_{k'm'}$ and state that this function is an isometry preserving the $\bar{t}$ axis. Doing so one relies on Prop. 2.3.3(x).

In most cases it would not matter if we required $N[\bar{t}] = \bar{t}$ instead of $N[\bar{t}] \parallel \bar{t}$ in this axiom (e.g. assuming **Bax**).

Intuitively, the way a particular observer m sees another observer k move determines how their world-views are related, up to a trivial recoordinatization of space-time and possibly a reversal of the arrow of time. The isometry (distance-preserving transformation) N permits that, letting observers m, k, m' be fixed, k' is still "free" to choose the orientation of its coordinate axes and the origin of its frame of reference (although the latter freedom is restricted, of course, by **Ax4**).

Ax△1 $(\forall m, k \in Obs)(\exists k' \in Obs)(tr_m(k) = tr_m(k') \ \wedge \ \mathsf{f}_{mk'} = \mathsf{f}_{k'm})$.

This means that although we cannot require from a pair of observers, say m and k, that one of them sees the other the same way as the other sees him (because he may e.g. "turn his head in the wrong direction"), we can require that it should be possible to find a brother k' of k who can see m just like m sees k'.

Ax△2 $(\forall m, k \in Obs)$ (there is an (affine) isometry N of ${}^n\mathbf{F}$ such that $N[\bar{t}] \parallel \bar{t}$ and $\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N$).[317]

That is, the way two observers see each other cannot be very different; a trivial recoordinatization of space-time by one of them[318] is enough to make them see each other the same way. Note also that we allow the direction of time to be different for the two observers.

As the reader may have noticed, **Ax□1** and **Ax□2** assert the equivalence of two observers, m and m' in connection with other observers (k and k'), while **Ax△1** and **Ax△2** characterize the way two observers, m and k, see each other. While **Ax△1** and **Ax□1** solve the problem of the possibly wrong orientation of spatial coordinate axes by taking a new observer with the appropriate trace (and orientation), **Ax△2** and **Ax□2** admit a distance preserving transformation in the relationship of world-view transformations.

We think that from the point of view of logical analysis pursued in this study the subtle differences between the above introduced symmetry axioms (**Ax△1**, **Ax□1**, **Ax△2** and **Ax□2**) are often irrelevant. We shall see soon that if $n \geq 3$, then they can be shown to be equivalent assuming **Basax** and a couple of auxiliary axioms. For these reasons we shall usually refer to their conjunction $\mathbf{Ax}(\omega)$ or their disjunction $\mathbf{Ax}(\omega^-)$ introduced below.

Definition 3.9.6

$$\mathbf{Ax}(\omega) \stackrel{\text{def}}{=} \mathbf{Ax\square 1} \wedge \mathbf{Ax\square 2} \wedge \mathbf{Ax\triangle 1} \wedge \mathbf{Ax\triangle 2}.$$
$$\mathbf{Ax}(\omega^-) \stackrel{\text{def}}{=} \mathbf{Ax\square 1} \vee \mathbf{Ax\square 2} \vee \mathbf{Ax\triangle 1} \vee \mathbf{Ax\triangle 2}.$$

[317]Again, because of Lemma 3.9.5, one can imagine quantification over isometries in general in place of quantification over affine isometries.

[318]Rearranging the equation in this axiom yields $N^{-1} \circ \mathsf{f}_{mk} = \mathsf{f}_{km} \circ N$.

Thus, whenever we intend to extend a theory like **Basax**, **Flxbasax**, **Bax** etc. with a natural symmetry principle, we can consider **Basax** + $\mathbf{Ax}(\omega)$, **Flxbasax** + $\mathbf{Ax}(\omega)$, **Bax** + $\mathbf{Ax}(\omega)$ etc. *provided that* we are at a point that we need not worry about the fine distinctions between $\mathbf{Ax}\triangle\mathbf{1}$ and its variants. In this connection we note that $\mathbf{Ax}(\omega)$ is definitely stronger than **Ax(symm)**. We shall argue this claim soon.[319]

PROPOSITION 3.9.7 $\mathbf{Ax}(\omega)$ *is consistent with* **Basax***; moreover* $\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}(\omega)$ *(and we have already seen that* $\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ })$ *is consistent).*

Proof: If $n \geq 3$, then Prop. 3.9.7 follows by Thm. 3.8.7, saying that $\mathbf{BaCo}+\mathbf{Ax}(\sqrt{\ })$ is consistent, and the following items:

- $\mathbf{BaCo}(n) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}\triangle\mathbf{1}$ by Thm. 3.9.26(ii),
- $\mathbf{BaCo}(n) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}\triangle\mathbf{2}$ by Thm. 3.9.27(ii),
- $\mathbf{BaCo}(n) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}\square\mathbf{1}$ by Thm. 3.9.31(ii),
- $\mathbf{BaCo}(n) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}\square\mathbf{2}$ by Thm. 3.9.29(ii).

For the case $n = 2$ one has to use the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$, and derive the items of $\mathbf{Ax}(\omega)$ one by one. We omit this part of the proof. ∎

First, to gain insight into the strength of these axioms we shall compare their implications to those of **Ax(symm)**. Recall that Theorem 2.8.2 said that the existence of FTL observers is consistent with $\mathbf{Basax}(2) + \mathbf{Ax(symm)}$. Moreover, in Thm. 3.8.11 we proved that adding $\mathbf{Ax}(\uparrow)$ (plus the auxiliary axiom $\mathbf{Ax}(\sqrt{\ })$) to $\mathbf{Basax}(2) + \mathbf{Ax(symm)}$ is enough to exclude faster than light observers.

The following theorem says that assuming $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$, both $\mathbf{Ax}\triangle\mathbf{1}$ and $\mathbf{Ax}\triangle\mathbf{2}$ exclude faster than light observers even in 2 dimensions, while $\mathbf{Ax(syt_0)}$, **Ax(symm)**, $\mathbf{Ax}\square\mathbf{1}$ and $\mathbf{Ax}\square\mathbf{2}$ do not.

THEOREM 3.9.8 *The following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}\triangle\mathbf{1} \models$ *"there are no FTL observers",*

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}\triangle\mathbf{2} \models$ *"there are no FTL observers",*

(iii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax(syt_0)} + \mathbf{Ax(symm)} + \mathbf{Ax}\square\mathbf{1} + \mathbf{Ax}\square\mathbf{2} \not\models$ *"there are no FTL observers".*

[319]Cf. Thm. 3.9.8.

To prove Thm. 3.9.8 we shall need the following lemma.

LEMMA 3.9.9 $\mathbf{Bax}^- + \mathbf{Ax}\triangle\mathbf{2} + \mathbf{Ax}(\sqrt{\ }) \models \mathsf{f}_{mk} \in Aftr$.

Proof: Assume $\mathbf{Bax}^- + \mathbf{Ax}\triangle\mathbf{2} + \mathbf{Ax}(\sqrt{\ })$. Let $m, k \in Obs$ be arbitrary. We have $\mathsf{f}_{mk} = \widetilde{\varphi} \circ g$ for some $g \in Aftr$ and $\varphi \in Aut(\mathbf{F})$ by Thm. 4.3.11 and Lemma 3.1.6. But $\varphi \in Aut(\mathfrak{F})$ by Lemma 6.6.6 (i.e. φ preserves order).

Now, from $\mathbf{Ax}\triangle\mathbf{2}$ we get the following statement:

$$\widetilde{\varphi} \circ g = N \circ g^{-1} \circ \widetilde{\varphi}^{-1} \circ N, \tag{189}$$

for some isometry N. We have $N \in Aftr$ by Lemma 3.9.5.

Using (189), Lemma 3.8.36 and the fact that N and g are affine transformations, one obtains $\varphi^2 = \mathrm{Id}$. Since φ preserves order, we have $\varphi = \mathrm{Id}$. ∎

Proof of Thm. 3.9.8(i): This item is only a restatement of Cor. 2.7.6. Moreover, we conjecture that the proof might be able to be generalized so that it omits $\mathbf{Ax}(\sqrt{\ })$.

Proof of Thm. 3.9.8(ii): Informally, the proof is based on the observation, proven in §2.7, that a pair of observers moving faster than light relative to one another must see each other's clocks run differently. If m sees k's clock run forwards, then k sees m's clock run backwards; or, if m sees k's clock run backwards, then k sees m's clock run forwards. This fact conflicts with axiom $\mathbf{Ax}\triangle\mathbf{2}$, which implies that m sees k's clock run forwards if and only if k sees m's clock run forwards.

Let us work out this idea formally. Assume $\mathbf{Basax} + \mathbf{Ax}\triangle\mathbf{2} + \mathbf{Ax}(\sqrt{\ })$. Suppose for contradiction that there are $m, k \in Obs$ such that $v_m(k) > 1$. (Then $v_k(m) > 1$ by Thm. 2.7.1.) By $\mathbf{Ax}\triangle\mathbf{2}$ we have

$$\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N, \tag{190}$$

for some isometry N such that $N[\bar{t}] \parallel \bar{t}$. On the other hand, by Thm. 2.7.4 we have

$$\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t > 0 \quad \Longleftrightarrow \quad \mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t < 0. \tag{191}$$

Case 1. Assume $N(1_t)_t - N(\bar{0})_t > 0$. In this case N does not turn back time sequences. That is, one can check that

$$(\forall p, q \in {}^nF)(p_t > q_t \iff N(p)_t > N(q)_t). \tag{192}$$

Using that $\mathsf{f}_{mk} \in Aftr$ by Lemma 3.9.9,[320]

$$(N \circ \mathsf{f}_{km} \circ N)(1_t) = N(\mathsf{f}_{km}(1_t)) - N(\mathsf{f}_{km}(\bar{0})) + N(\mathsf{f}_{km}(N(\bar{0}))),$$

[320] We omit some simple computational steps.

thus by (190) one gets $\mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0}) = N(\mathsf{f}_{km}(1_t)) - N(\mathsf{f}_{km}(\bar{0}))$. Hence

$$\begin{aligned} \mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t > 0 \quad &\iff \quad N(\mathsf{f}_{km}(1_t))_t - N(\mathsf{f}_{km}(\bar{0}))_t > 0 \\ &\iff \quad \mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t > 0, \end{aligned}$$

using (192). This obviously contradicts (191).

Case 2: $N(1_t)_t - N(\bar{0})_t < 0$. This case can be handled analogously. We omit the details.

Idea of the proof of Thm. 3.9.8(iii): We need to build a model of **Basax**(2) + **Ax**$(\sqrt{\ })$ + **Ax**$($*Triv*$)$ + **Ax**$(\|)$ + **Ax(syt$_0$)** + **Ax(symm)** + **Ax□1** + **Ax□2** that contains faster than light observers.

Consider the standard Minkowski-plane on ${}^2R$. Let the set of our observers include *all* the possible reference frames whose "unit vectors" have Minkowski-length 1. One has to apply the model building algorithm of Def. 3.6.11.

Let $\mathfrak{M}$ denote this model. Then $\mathfrak{M} \models$ **Basax**(2)+**Ax**$(\sqrt{\ })$+**Ax**$($*Triv*$)$+**Ax**$(\|)$. We suppose that the symmetry principles **Ax(syt$_0$)**, **Ax(symm)**, **Ax□1** and **Ax□2** hold as well. We admit that we have not checked this claim carefully. ∎

For completeness we shall show that **Basax**(2) + **Ax□1'** admits FTL observers, where the "simple axiom" **Ax□1'** is a straightforward consequence[321] of **Ax□1**.

Ax□1' $(\forall m, k \in Obs)(\exists m' \in Obs)(tr_k(m') = tr_m(k) \quad \wedge \quad \mathsf{f}_{mk} = \mathsf{f}_{km'})$.

Compare **Ax□1'** with **Ax△1**. Both axioms aim at saying that the way m sees k is the same as the way k sees m, but the possibility must be allowed for that their coordinate axes are not suitably oriented to validate this claim. Now, in case of **Ax△1** we took an appropriately oriented brother of k instead of k itself; while in **Ax□1'** we took third observer m' (instead of m) whose orbit in the world view of k is the same as k's trace for m, thereby allowing for a similar relationship of world-views (remember that m' is *not* a brother of m in **Ax□1'**).[322]

The basic idea is depicted on Figure 104. Somewhat more formally, let $\mathfrak{M}$ be a model of **Basax**(2) + **Ax□1'**. We shall construct another model $\mathfrak{M}_1$ such that $\mathfrak{M}_1 \models$ **Basax**(2) + **Ax□1'** + $(\exists m, k \in Obs) v_m(k) > 1$. The construction consists of two steps:

1. We extended $\mathfrak{M}$ so that the "mirror image" of any observer will be present in the new model $\mathfrak{M}_0$. By this we mean that for any $m \in Obs$ we have an $m' \in Obs$ such that $(\forall p \in {}^nF)\mathsf{f}_{mm'}(p) = -p$.

[321] Statement **Ax□1'** follows by applying **Ax□1** for $m, k, m' = k$, and checking $tr_k(m') = tr_m(k)$. For the latter step we use **Ax4** and Prop. 2.3.3(vii), which in turn requires **Basax**.

[322] Further we note that **Ax□1** in its original form asserts the equivalence of m and m', while **Ax△1** deals with the equivalence of m and k.

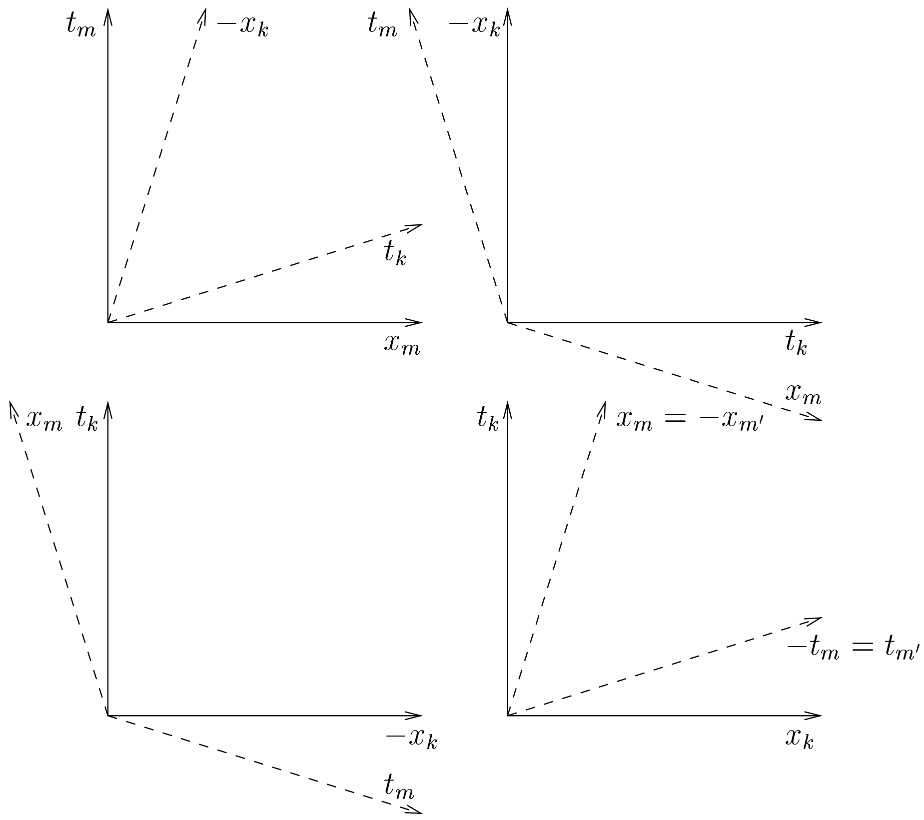


Figure 104: Intuitive idea of the proof of statement $(\exists m' \in Obs)\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$.

2. We add an observer to $\mathfrak{M}_0$ that travels faster than light to some other observer. Thus we arrive at model $\mathfrak{M}_1$

The first step is needed to ensure that adding an FTL observer does not invalidate **Ax□1'**. Figure 104 shows how we can find for $m, k \in Obs$ the appropriate $m' \in Obs$ (referred to in **Ax□1'**) in the case $v_m(k) > 1$. The picture actually shows how $\mathsf{f}_{km'}$ is built. Filling in the details of the model construction is straightforward. The reader is challenged to do that. ∎

Remark 3.9.10 [On the intuition behind Ax△1 etc.] Consider the following equivalence principle (which is called homogenity in model theory): All observers are alike in the sense that

(⋆) $(\forall m, k \in Obs)(\exists h \in Aut(\mathfrak{M}))h(m) = k.$

Clearly, ($\star$) states a very strong form of equivalence of inertial observers (although not in first order logic). It would be interesting to know how much of $\mathbf{Ax\triangle 1}$ and the other recently introduced symmetry axioms is derivable from ($\star$) (perhaps using other items of SPR like $\mathbf{Ax(\|)}$, $\mathbf{Ax(}\mathit{Triv}\mathbf{)}$, $\mathbf{Ax(ext)}$ or $\mathbf{Ax\heartsuit}$).

Of course, we cannot replace the conclusion of ($\star$) with the stronger $h(m) = k \wedge h(k) = m$, since m and k might look in the wrong direction spatially. However, the following version seems to be correct.

($\star\star$) $(\forall m, k \in Obs)(\exists k' \in Obs)[tr_m(k) = tr_m(k') \quad \wedge$
$(\exists h \in Aut(\mathfrak{M}))(h(m) = k' \quad \wedge \quad h(k') = m)].$

In ($\star\star$) we have followed the usual strategy of quantifing over observers in order to abstract from the accidental orientation of k's spatial coordinate axes. That is, ($\star\star$) requires that at least a brother of k should be equivalent with a given m up to an automorphism (of the entire model). Now we may ask the same question about ($\star\star$) as about ($\star$).

A different "derivation" (or justification) of $\mathbf{Ax\triangle 1}$ would be using the following potential axiom.

($\star\star\star$) f_{mk} depends only on the velocity vector $\vec{v}_m(k)$ together with the orientation of k's space coordinates (and, of course, on k's origin as seen by m, $\mathsf{f}_{km}(\bar{0})$).

For instance, the orientation of k's $\bar{x}$ axis is the direction in which the space component of the vector $\mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0})$ points. In other words, ($\star\star\star$) says that f_{mk} is a (partial) function of three variables, a vector $\vec{v} \in {}^{n-1}F$, a point (k's origin), and an "orientation", where the latter is a triple of directions.[323] Then we conjecture that ($\star\star\star$) plus some auxiliary axioms would imply $\mathbf{Ax\triangle 1}$. But we observe that ($\star\star\star$) would be a very strong axiom.

It can be checked that items ($\star$) to ($\star\star\star$) are true in models of **BaCo**. For item ($\star$) cf. Cor. 3.8.21 on p. 304. The reader is invited to check the validity of the other items in models of **BaCo**.

The following theorem says that all the symmetry axioms studied in the present sub-section are equivalent, assuming **Basax**, some auxiliary axioms and $n \geq 3$. Note that the assertions of this theorem can be made sharper is some cases, as we shall see below in this sub-section when proving the equivalence statements in detail.

[323] I.e. of elements of directions $\subseteq {}^{n-1}F$.

THEOREM 3.9.11 *Assume $n \geq 3$. Let*

$$H \stackrel{\text{def}}{=} \{\mathbf{Ax(syt_0)}, \mathbf{Ax(eqspace)}, \mathbf{Ax\triangle 2}\},$$
$$H' \stackrel{\text{def}}{=} \{\mathbf{Ax(symm)}, \mathbf{Ax\Box 2}, \mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)}\},$$
$$H'' \stackrel{\text{def}}{=} \{\mathbf{Ax\Box 1} + \mathbf{Ax(eqtime)}\}.$$

Then items (i) to (iii) below hold.

(i) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) \models$ *all the axioms in H are equivalent with one another,*

(ii) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models$ *all the axioms in $H \cup H'$ are equivalent with one another,*

(iii) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax5}^+ \models$ *all the axioms in $H \cup H' \cup H''$ are equivalent with one another.*

Proof: This theorem is based on Prop. 3.9.12 below. One can proceed as follows:

- Prop. 3.9.12(v)-(vi) assert that $\mathbf{Ax(syt_0)}$ is implied by $\mathbf{Ax\triangle 2}$ or $\mathbf{Ax(eqspace)}$ if $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ })$ is assumed. Axiom $\mathbf{Ax(syt_0)}$ implies both $\mathbf{Ax\triangle 2}$ and $\mathbf{Ax(eqspace)}$ by items (ix) and (vi).

- Items (ii) and (iv) of Prop. 3.9.12 assert that $\mathbf{Ax\triangle 2} \in H$ is implied by $\mathbf{Ax\Box 2}$ or $\mathbf{Ax\triangle 1} \wedge \mathbf{Ax(eqtime)}$ if $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)$ is assumed. Axiom $\mathbf{Ax\triangle 2}$ implies $\mathbf{Ax\Box 2}$ by item (xi), and $\mathbf{Ax(syt_0)}$ implies $\mathbf{Ax\triangle 1}$+$\mathbf{Ax(eqtime)}$ by item (vii). On the other hand, items (iii) and (viii) establish the equivalence of $\mathbf{Ax(syt_0)}$ and $\mathbf{Ax(symm)}$ assuming $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)$.

- Items (i) and (x) assert the equivalence of $\mathbf{Ax\Box 1} \wedge \mathbf{Ax(eqtime)}$ to $\mathbf{Ax\Box 2}$ assuming $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax5}^+$ for $n \geq 3$. Thereby the pairwise equivalence of members of $H \cup H' \cup H''$ is implied.

∎

PROPOSITION 3.9.12 *Assume* **Basax**. *Then items (i) to (xi) below hold.*

(i) $(\mathbf{Ax\Box 1} + \mathbf{Ax(eqtime)}) \to \mathbf{Ax\Box 2}$.

(ii) $(\mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)}) \to \mathbf{Ax\triangle 2}$.

(iii) $\mathbf{Ax(symm)} \to \mathbf{Ax(syt_0)}$.

In the following items always assume $\mathbf{Ax}(\sqrt{\ })$.

(iv) *Assume* $n \geq 3$ *and* $\mathbf{Ax}(\mathit{Triv}_t)$. *Then* $\mathbf{Ax\Box 2} \to \mathbf{Ax\triangle 2}$.

(v) $\mathbf{Ax\triangle 2} \to \mathbf{Ax(syt_0)}$.

(vi) $\mathbf{Ax(eqspace)} \leftrightarrow \mathbf{Ax(syt_0)}$.

(vii) *Assume* $n \geq 3$ *and* $\mathbf{Ax}(\mathit{Triv}_t)$. *Then* $\mathbf{Ax(syt_0)} \to (\mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)})$.

(viii) *Assume* $n \geq 3$ *and* $\mathbf{Ax}(\mathit{Triv}_t)$. *Then* $\mathbf{Ax(syt_0)} \to \mathbf{Ax(symm)}$.

(ix) *Assume* $n \geq 3$. *Then* $\mathbf{Ax(syt_0)} \to \mathbf{Ax\triangle 2}$.

(x) *Assume* $n \geq 3$ *and* $\mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax5^+}$. *Then* $\mathbf{Ax\Box 2} \to (\mathbf{Ax\Box 1} + \mathbf{Ax(eqtime)})$.

(xi) $\mathbf{Ax\triangle 2} \to \mathbf{Ax\Box 2}$.

Proof: This proposition follows from several theorems, propositions and corollaries demonstrated in course of this sub-section. We shall give the references to them.

- Item (i) follows from Thm. 3.9.20(i),
- item (ii) follows from Thm. 3.9.23(i),
- item (iii) follows from Prop. 3.9.47(i),
- item (iv) follows from Thm. 3.9.19(ii),
- item (v) follows from Prop. 3.9.51(i),
- item (vi) follows from Prop 3.9.74(i),
- item (vii) follows from Cor. 3.9.53(iii),
- item (viii) follows from Prop. 3.9.47(ii),
- item (ix) follows from Prop. 3.9.51(ii),
- item (x) follows from Thm. 3.9.20(ii),
- item (xi) follows from Thm. 3.9.19(i).

$\blacksquare$

The following simple proposition says, roughly, that we could have used *Triv* instead of isometries when formulating **Ax△2**. Apart from its role as a lemma, it illustrates a very useful proof procedure for symmetry statements. The reader will find many similar arguments throughout this sub-section.

PROPOSITION 3.9.13 **Basax + Ax($\sqrt{\ }$)** $\models$ **Ax△2** $\leftrightarrow$ **Ax△2***, *where* **Ax△2*** *is defined as follows:*

Ax△2* $(\forall m, k \in Obs)(\text{there is } N \in Triv \text{ such that } N[\bar{t}] = \bar{t} \quad \wedge \quad \mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N)$.

That is, the way a certain observer sees another differs only trivially from the way the other sees him. To show Prop. 3.9.13 as well as some other propositions we shall need the following lemmas.

Definition 3.9.14 Let σ_S denote the reflection around the plane S (where S denotes the spatial subspace $S = \{p \in {}^nF : p_t = 0\}$). That is,

$$\sigma_S(p)_t = -p_t \quad \text{and} \quad (\forall 0 < i < n)\sigma_S(p)_i = p_i.$$

LEMMA 3.9.15 *The following items hold.*

(i) **Basax + Ax($\sqrt{\ }$)** $\models$ **Ax△2** $\to$ **Ax(eqtime)**.

(ii) **Basax + Ax($\sqrt{\ }$)** $\models$ **Ax△2** $\to$ **Ax($\|$)**.

Proof: Item (i) is a corollary of item (ii), hence we shall prove item (ii) only. Assume **Basax + Ax△2 + Ax($\sqrt{\ }$)**. Let $p, q \in \bar{t}$. Let $m, k \in Obs$ be such that $tr_m(k) \parallel \bar{t}$. We have $\mathsf{f}_{mk} \in Aftr$ by Lemma 3.9.9. By **Ax△2** there is an isometry N such that $N[\bar{t}] \parallel \bar{t}$ and

(193) $$\mathsf{f}_{km} = N \circ \mathsf{f}_{mk} \circ N.$$

Then

$$\begin{aligned} |p - q| &= |(\mathsf{f}_{mk} \circ \mathsf{f}_{km})(p) - (\mathsf{f}_{mk} \circ \mathsf{f}_{km})(q)| \\ &= |(\mathsf{f}_{mk} \circ N \circ \mathsf{f}_{mk} \circ N)(p) - (\mathsf{f}_{mk} \circ N \circ \mathsf{f}_{mk} \circ N)(q)|, \end{aligned}$$

by $\mathsf{f}_{mk} \circ \mathsf{f}_{km} = \mathrm{Id}$ and (193). Since $N, \mathsf{f}_{mk}, \mathsf{f}_{km} \in Aftr$, letting $\hat{g}$ denote the linear part of any $g \in Aftr$,

(194) $$|p - q| = |(\hat{\mathsf{f}}_{mk} \circ \hat{N} \circ \hat{\mathsf{f}}_{mk} \circ \hat{N})(p) - (\hat{\mathsf{f}}_{mk} \circ \hat{N} \circ \hat{\mathsf{f}}_{mk} \circ \hat{N})(q)|.$$

Now, by $N[\bar{t}] \parallel \bar{t} \parallel \mathsf{f}_{mk}[\bar{t}]$ we have

$$(195) \qquad (\forall x \in \bar{t})\hat{N}(x) = \pm x, \text{ and}$$

$$(196) \qquad (\forall x \in \bar{t})\hat{\mathsf{f}_{mk}}(x) \in \bar{t}.$$

Applying (195) and (196) for (194) yields

$$|p - q| = |p - q||\mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0})|^2,$$

Now, (193) implies $|\mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0})| = 1$. Thus $|p - q| = |\mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q)|$. (This would be enough to derive **Ax(eqtime)**.)

Now let $g = \mathsf{f}_{mk} \circ \tau_{-\mathsf{f}_{mk}(\bar{0})}$, if $\mathsf{f}_{mk}(1_t) > \mathsf{f}_{mk}(\bar{0})$, else let $g = \mathsf{f}_{mk} \circ \sigma_S \circ \tau_{\mathsf{f}_{mk}(\bar{0})}$. We have $g \in PT$ by $\mathsf{f}_{mk} \in \mathit{Aftr}$ and **Basax**. Further, from the above definition and the fact that f_{mk} preserves distance between points in $\bar{t}$, $g(1_t) = 1_t$ and $g(\bar{0}) = \bar{0}$ follow. Then, using Lemma 3.6.20 one obtains $g \in \mathit{Triv}_0$. Hence f_{mk} is an isometry. ∎

The following lemma says that, assuming **Basax** + **Ax△2** + **Ax**($\sqrt{\ }$), the median observer for a pair of observers m and k can see the clocks of m and k slow down with the same rate.

LEMMA 3.9.16

$$\mathbf{Basax} + \mathbf{Ax\triangle 2} + \mathbf{Ax}(\sqrt{\ }) \models (h \textit{ is a median observer for } m, k) \rightarrow$$
$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|.$$

Proof: The proof is analogous to that of Propositions 3.8.31 and 3.8.32. Indeed, we shall present the same argument while referring to different lemmas. The reader is asked to consult Figure 105 when following the argument.

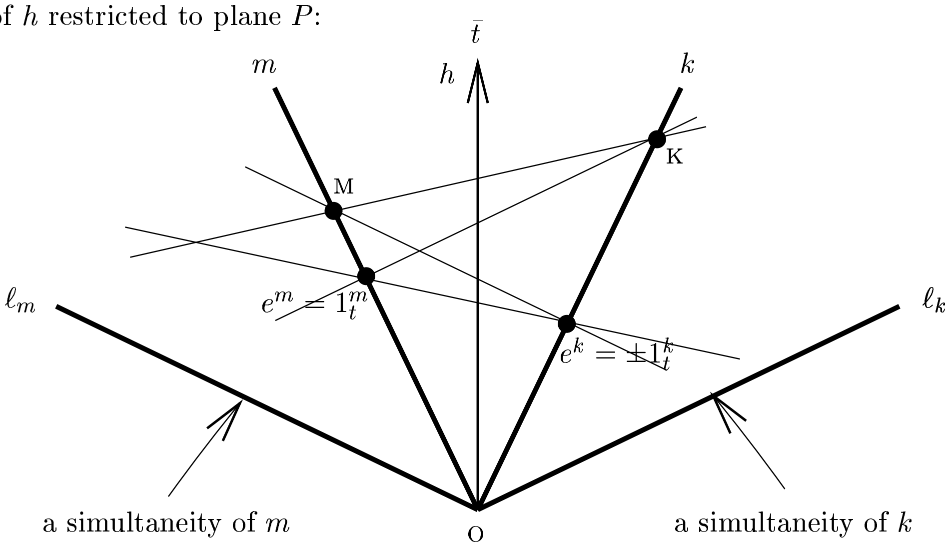
Assume **Basax**+**Ax△2**+**Ax**($\sqrt{\ }$). Let $m, k, h \in \mathit{Obs}$ be such that h is a median observer for m and k. (Actually, m and k can be chosen freely, because $v_m(k) < 1$ by Thm. 3.9.8(ii), and hence the existence of a median observer is guaranteed by Thm. 3.8.25.) We shall approach the general case in two steps.

Special case 1: $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. We shall use the notation of the proof of Prop. 3.8.32 as closely as possible. Let

$$\begin{aligned} \mathrm{o} &\stackrel{\text{def}}{=} \mathsf{f}_{mh}(\bar{0}) = \mathsf{f}_{kh}(\bar{0}), \\ P &\stackrel{\text{def}}{=} \mathsf{Plane}(\bar{t}, tr_h(m)) = \mathsf{Plane}(\bar{t}, tr_h(k)), \\ \ell_m &\stackrel{\text{def}}{=} \mathsf{f}_{mh}[S] \cap P \quad \text{and} \quad \ell_k \stackrel{\text{def}}{=} \mathsf{f}_{kh}[S] \cap P. \end{aligned}$$

Lines ℓ_m and ℓ_k contain all events in the plane P simultaneous with $w_m(\bar{0}) = w_k(\bar{0}) = w_h(\mathrm{o})$ for m and k, respectively. See Figure 105. Recall that $v_m(k) < 1$ by Thm. 3.9.8(ii).

world-view of h restricted to plane P:



If e^m and e^k were not $\bar{t}$-symmetric as seen by h then m and k would see each other clocks differently, i.e. slowing down with different rate.

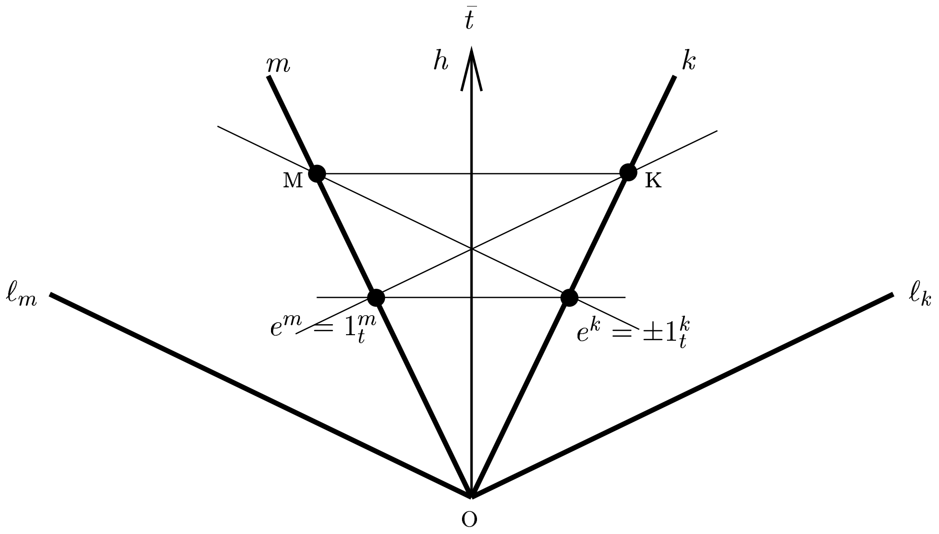


Figure 105: Illustration for the proof of Lemma 3.9.16.

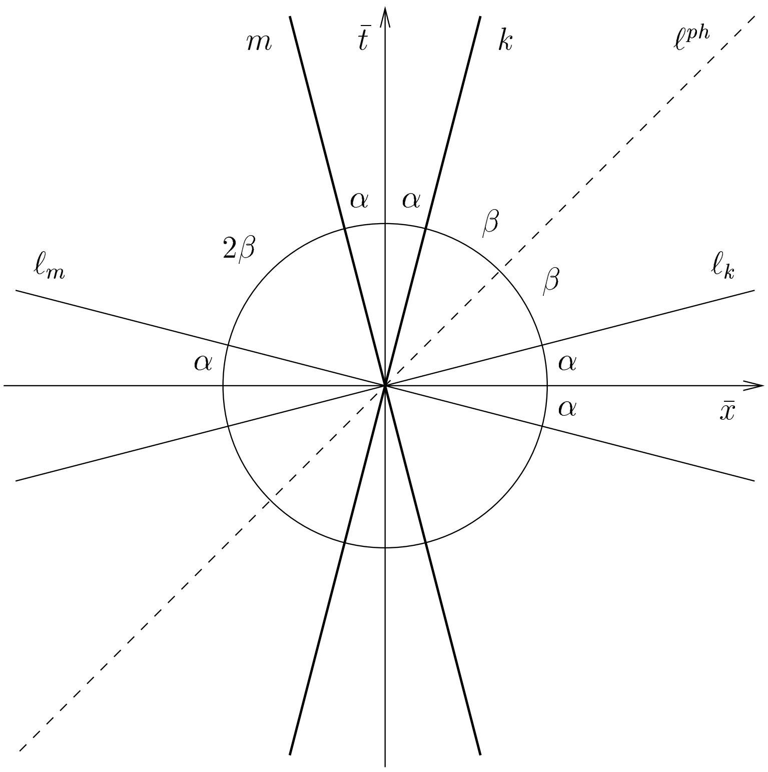


Figure 106: Illustration for the proof of Claim 3.9.17.

Claim 3.9.17 $\ell_m = \sigma_{\bar{t}}[\ell_k]$.

Let $\ell^{ph} \in \mathsf{PhtEucl}$ be such that $\mathrm{o} \in \ell^{ph} \subseteq P$. (Actually, there are two such lines.) Let

$$\ell'_m = \sigma_{\ell^{ph}}[tr_h(m)] \quad \text{and} \quad \ell'_k = \sigma_{\ell^{ph}}[tr_h(k)].$$

It is easy to check that $\ell'_m = \sigma_{\bar{t}}[\ell'_k]$. See Figure 106. Now, by Lemma 3.8.28 events on ℓ'_m are simultaneous for m, and, similarly, events on ℓ'_k are simultaneous for k. Then $\ell'_m \subseteq \mathsf{f}_{mh}[S]$ and $\ell'_k \subseteq \mathsf{f}_{kh}[S]$. On the other hand, we have $\ell'_m, \ell'_k \subseteq P$ by construction. Therefore $\ell_m = \ell'_m$ and $\ell_k = \ell'_k$. (Claim 3.9.17) ∎

Let 1^m_t and 1^k_t denote the time unit vectors of observers m and k, respectively. Formally,

$$1^m_t \stackrel{\text{def}}{=} \mathsf{f}_{mh}(1_t) \in tr_h(m), \quad \text{and} \quad 1^k_t \stackrel{\text{def}}{=} \mathsf{f}_{kh}(1_t) \in tr_h(k).$$

Let $e^m = 1_t^m$. If $m \uparrow k$, let $e^k = 1_t^k$; otherwise let $e^k = -1_t^k$. Hence e^m and e^k are on "the same side" of the horizontal line going through O in P. Formally,

$$(e^m)_t > \mathrm{O}_t \iff (e^k)_t > \mathrm{O}_t.$$

Let $\mathrm{M} \in tr_h(m)$ be such that events $w_h(\mathrm{M})$ and $w_h(e^k)$ are simultaneous for m. Formally, $\mathsf{f}_{hm}(\mathrm{M})_t = \mathsf{f}_{hm}(e^k)_t$. Analogously, let $\mathrm{K} \in tr_h(k)$ be such that events $w_h(\mathrm{K})$ and $w_h(e^m)$ are simultaneous for k (i.e. $\mathsf{f}_{hk}(\mathrm{K})_t = \mathsf{f}_{hk}(e^m)_t$). It is easy to check that M and K are uniquely defined. By the fact that world-view transformations are bijective collineations (cf. Prop. 2.3.3(v) and Thm. 3.1.1) we have

$$\overline{\mathrm{M}e^k} \parallel \ell_m \quad \text{and} \quad \overline{\mathrm{K}e^m} \parallel \ell_k. \tag{197}$$

Claim 3.9.18 $\frac{|\mathrm{M}-\mathrm{O}|}{|e^m-\mathrm{O}|} = \frac{|\mathrm{K}-\mathrm{O}|}{|e^k-\mathrm{O}|}$.

By (197) we have

$$|\mathsf{f}_{km}(1_t)_t| = \left|\frac{\mathsf{f}_{hm}(\mathrm{M})_t}{\mathsf{f}_{hm}(e^m)_t}\right| = \frac{|\mathsf{f}_{hm}(\mathrm{M})_t|}{|1_t|}. \tag{198}$$

We have $\mathsf{f}_{hm} \in Aftr$ by Lemma 3.9.9. Then f_{hm} preserves proportion of distances between collinear points. Since $\mathrm{O}, e^m, \mathrm{M} \in tr_h(m)$, their f_{hm}-images, $\bar{0}$, 1_t $\mathsf{f}_{hm}(\mathrm{M})$ are collinear (indeed, they fall on $\bar{t}$). Thus, calculating (198) in the world-view of h, one gets

$$|\mathsf{f}_{km}(1_t)_t| = \frac{|\mathrm{M}-\mathrm{O}|}{|e^m-\mathrm{O}|}. \tag{199}$$

By a similar argument one obtains

$$|\mathsf{f}_{mk}(1_t)_t| = \frac{|\mathrm{K}-\mathrm{O}|}{|e^k-\mathrm{O}|}. \tag{200}$$

By Prop. 3.9.51(i) $\mathbf{Ax(syt_0)}$ holds. Then

$$|\mathsf{f}_{km}(1_t)_t| = |\mathsf{f}_{mk}(1_t)_t|. \tag{201}$$

Now, (201) together with (199) and (200) imply Claim 3.9.18. (Claim 3.9.18) ∎

We have by construction that e^m and K, and e^k and M, respectively, are on the same side. That is,

$$\mathrm{K}_t > \mathrm{O}_t \iff (e^m)_t > \mathrm{O}_t \iff (e^k)_t > \mathrm{O}_t \iff \mathrm{M}_t > \mathrm{O}_t. \tag{202}$$

The triplets $\textsc{o}$, e^m, $\textsc{m}$ and $\textsc{o}$, e^k, $\textsc{k}$ are collinear by construction. Moreover, by (202) $\textsc{o}$ is neither between e^m and $\textsc{m}$, nor between e^k and $\textsc{k}$. It is known from elementary geometry that Claim 3.9.18 plus the above facts imply

$$\overline{\textsc{mk}} \parallel \overline{e^m e^k}.$$

Using the definition of $\textsc{m}$, $\textsc{k}$, $\textsc{o}$, e^m and e^k one gets

$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|,$$

as required.

Special case 2: $tr_m(k) \cap \bar{t} \neq \emptyset$. Clearly, this case is slightly less special than Case 1. We assume that m and k meet, but we do not expect that their meeting point is the origin of either. We can reduce Case 2 to Case 1 by observing that only the length of the unit vectors, and not their actual location, played a role in the proof of Case 1.

Let us formalize this idea. By assumption, there are $p, q \in \bar{t}$ such that $\mathsf{f}_{mk}(p) = q$. We can use the same argument as for Case 1, but for the following entities:

$$\begin{aligned}
\textsc{o} &\stackrel{\text{def}}{=} \mathsf{f}_{mh}(p) = \mathsf{f}_{kh}(q),\\
\ell_m &\stackrel{\text{def}}{=} \mathsf{f}_{mh}[S+p] \cap P \quad \text{and} \quad \ell_k \stackrel{\text{def}}{=} \mathsf{f}_{kh}[S+q] \cap P\\
e^m &\stackrel{\text{def}}{=} \mathsf{f}_{mh}(p+1_t),\\
e^k &\stackrel{\text{def}}{=} \mathsf{f}_{kh}(q+1_t),\ \text{if } m \uparrow k \text{ and } \mathsf{f}_{kh}(q-1_t) \text{ otherwise.}
\end{aligned}$$

Plane P is defined as previously. The interested reader might fill in the details.

General case: Let $m, k \in Obs$ be arbitrary. By **Ax5** there is an observer k' such that $tr_k(k') \parallel \bar{t}$ and m and k' meet ($tr_m(k') \cap \bar{t} \neq \emptyset$). (Recall that the traces of k and k' are parallel in any observer's world-view because world-view transformations are bijective collineations.) Let h' be a median observer for m and k'. Such h' exists by Thm's 3.9.8(ii) and 3.8.25. Since m, k', h' fulfill the conditions of Case 2, we have

$$|\mathsf{f}_{mh'}(1_t)_t - \mathsf{f}_{mh'}(\bar{0})_t| = |\mathsf{f}_{k'h'}(1_t)_t - \mathsf{f}_{k'h'}(\bar{0})_t|. \tag{203}$$

Now, axiom $\mathbf{Ax(\|)}$ holds by Lemma 3.9.15(ii). Then the following items hold.

1. Observer h' can see the time unit vectors of k and k' of exactly the same length. Formally,

$$|\mathsf{f}_{kh'}(1_t)_t - \mathsf{f}_{kh'}(\bar{0})_t| = |\mathsf{f}_{k'h'}(1_t)_t - \mathsf{f}_{k'h'}(\bar{0})_t|. \tag{204}$$

2. Observers h and h' agree on the length of unit vectors of m and k. That is,

$$|\mathsf{f}_{kh'}(1_t)_t - \mathsf{f}_{kh'}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|, \tag{205}$$

$$|\mathsf{f}_{mh'}(1_t)_t - \mathsf{f}_{mh'}(\bar{0})_t| = |\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t|. \tag{206}$$

Now, Lemma 3.9.16 follows in the general case from items (203) to (206). We omit the details, but see Figure 107. ∎

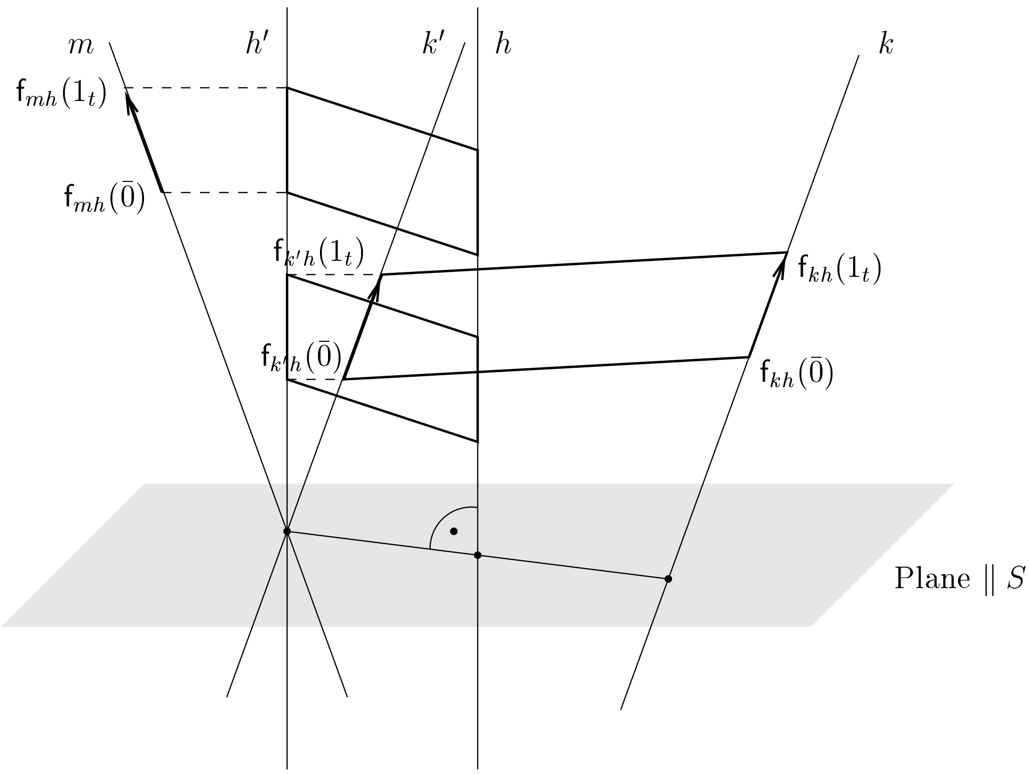


Figure 107: Idea of the proof of Lemma 3.9.16, General case. The picture show the world-view of observer h. The thickened polygons can be demonstrated to be parallelograms. Although the picture assumes $m \uparrow k \uparrow k'$, all other cases can be handled similarly.

Proof of Prop. 3.9.13: This proof is the first of a whole series of structurally similar proofs. All these aim at deriving **Ax△1**, **Ax△2** or **Ax(symm$_0$)** from an appropriate set of axioms. Hence we are going to present it in much detail, while we shall try to give the forthcoming similar proofs as briefly as possible.

Assume $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}\triangle\mathbf{2}$. We have to prove $\mathbf{Ax}\triangle\mathbf{2}^*$.

Let $m, k \in Obs$ be arbitrary. Faster than light observers are excluded by Thm. 3.9.8(ii). Then by Thm. 3.8.25 there is a median observer h for m and k. Thus we have

$$tr_h(k) = \sigma_{\bar{t}}[tr_h(m)]. \tag{207}$$

We have $\mathsf{f}_{mh}, \mathsf{f}_{kh}, \mathsf{f}_{mk} \in Aftr$ by Lemma 3.9.9. On the other hand, by Lemma 3.9.16 we have that h can see the clocks of m and k tick with the same rate:

$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|. \tag{208}$$

Case 1: $m \uparrow k$. The reader is asked to consult Figure 108. Consider transforma-

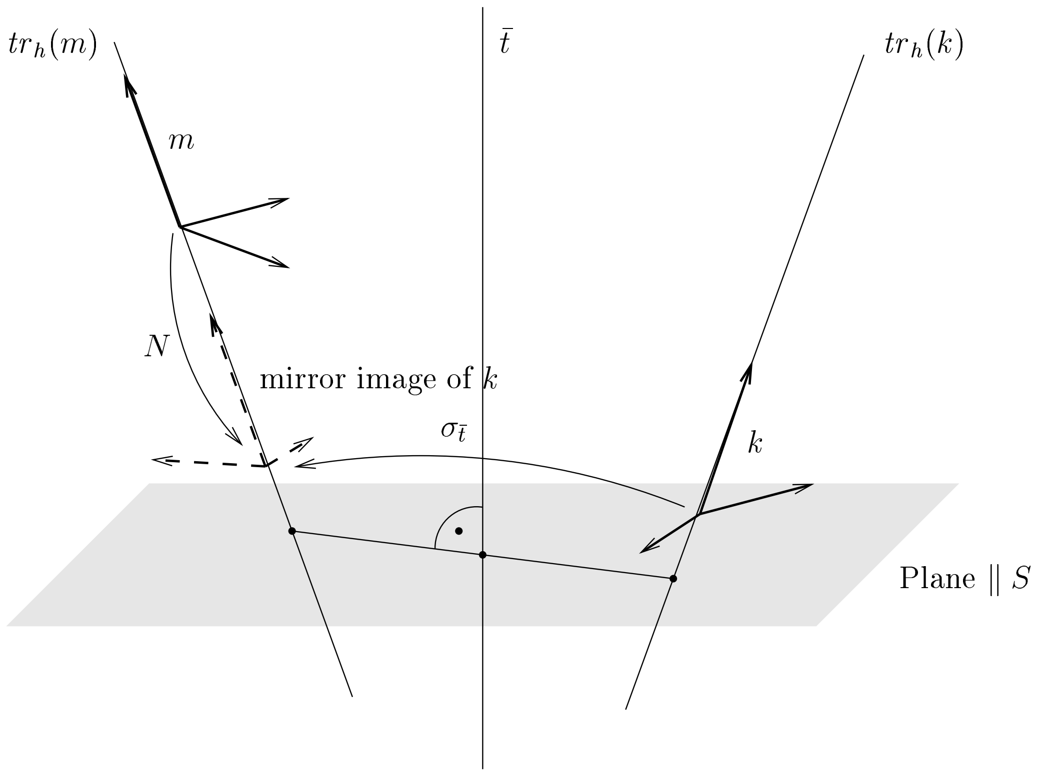


Figure 108: Idea of the proof of Prop. 3.9.13, Case 1. The picture shows the world-view of observer h.

tions f_{mh} and $\mathsf{f}_{kh} \circ \sigma_{\bar{t}}$. Both map $\bar{t}$ axis to the same line, $tr_h(m)$ by Prop. 2.3.3 and (207). Since $m \uparrow k$ and $\sigma_{\bar{t}} \restriction \bar{t} = \mathrm{Id}$, (208) implies

$$\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t = (\mathsf{f}_{kh} \circ \sigma_{\bar{t}})(1_t)_t - (\mathsf{f}_{kh} \circ \sigma_{\bar{t}})(\bar{0})_t. \tag{209}$$

Then by Lemma 3.8.50 there is a $N \in \mathit{Triv}$ such that $N[\bar{t}] = \bar{t}$ and

$$N \circ \mathsf{f}_{mh} = \mathsf{f}_{kh} \circ \sigma_{\bar{t}}. \tag{210}$$

Inverting both sides of (210), and using Prop. 2.3.3(x) one obtains

$$\mathsf{f}_{hm} \circ N^{-1} = \sigma_{\bar{t}} \circ \mathsf{f}_{hk}.$$

Multiplying with $\sigma_{\bar{t}}$ from the left and with N from the right yields

$$\sigma_{\bar{t}} \circ \mathsf{f}_{hm} = \mathsf{f}_{hk} \circ N. \tag{211}$$

Composing the appropriate sides of (210) and (211) yields $N \circ \mathsf{f}_{mk} \circ N = \mathsf{f}_{km}$, as required.

Case 2: $m \downarrow k$. This case is similar to Case 1, but we should consider $\mathsf{f}_{kh} \circ \sigma_\ell$ instead of $\mathsf{f}_{kh} \circ \sigma_{\bar{t}}$, where ℓ is the line defined by the following conditions:

- ℓ is perpendicular (in the Euclidean sense) both to $\bar{t}$ and the common perpendicular ℓ_p of lines $tr_h(m)$ and $tr_h(k)$,
- ℓ is parallel to the pair of parallel planes containing $tr_h(m)$ and $tr_h(k)$,
- $\ell \cap \bar{t} \neq \emptyset$, and
- $\ell \cap \ell_p \neq \emptyset$.

See Figure 109. Transformation σ_ℓ reverses the arrow of time, thereby ensuring that a relation analogous to (210) holds. Cf. (212) below.

To elaborate this idea formally, consider transformations f_{mh} and $\mathsf{f}_{kh} \circ \sigma_\ell$. We have $\mathsf{f}_{mh}[\bar{t}] = tr_h(m) = (\mathsf{f}_{kh} \circ \sigma_\ell)[\bar{t}]$ by Prop. 2.3.3 and (207). Since $m \downarrow k$ and $\sigma_\ell \restriction \bar{t} = -\mathrm{Id}$, (208) implies

$$\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t = (\mathsf{f}_{kh} \circ \sigma_\ell)(1_t)_t - (\mathsf{f}_{kh} \circ \sigma_\ell)(\bar{0})_t.$$

Then by Lemma 3.8.50 there is a $N \in \mathit{Triv}$ such that $N[\bar{t}] = \bar{t}$ and

$$N \circ \mathsf{f}_{mh} = \mathsf{f}_{kh} \circ \sigma_\ell. \tag{212}$$

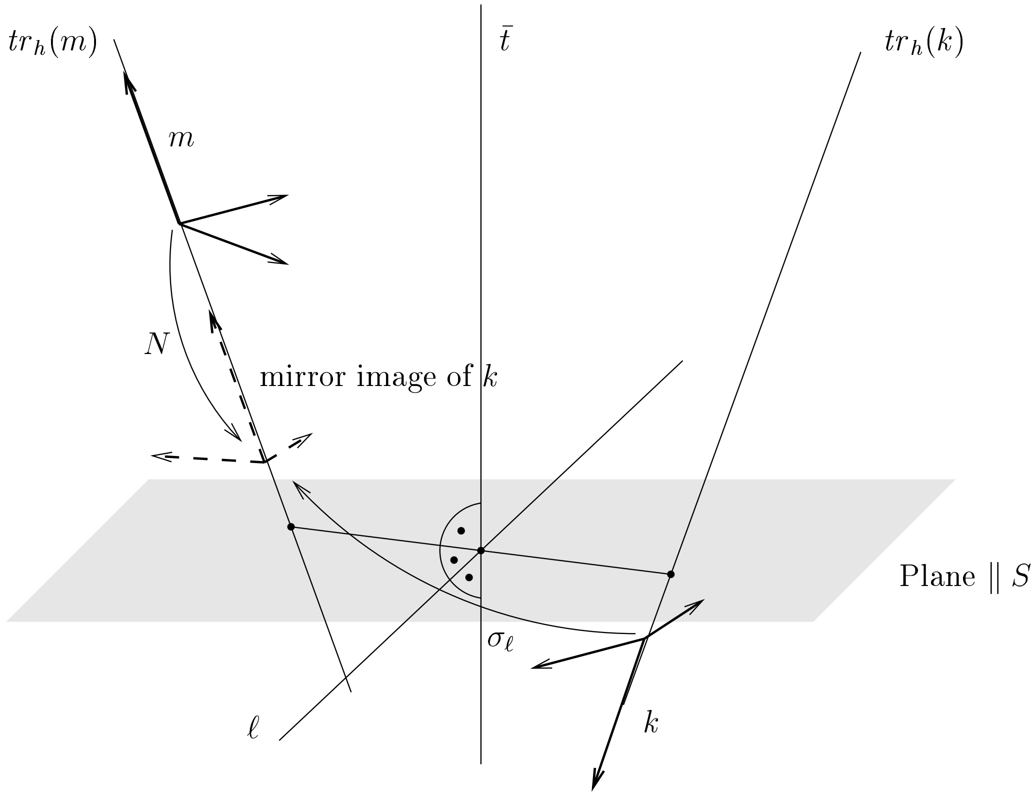


Figure 109: Idea of the proof of Prop. 3.9.13, Case 2. Again, we can see the world-view of observer h.

Form this point the argument continues like in Case 1, using (212) instead of (210). We omit the details. ∎

Now let us turn to the issue of the interconnections of **Ax□1**, **Ax△1**, **Ax□2** and **Ax△2**. In what follows we shall need some auxiliary axioms like **Ax(eqtime)**, **Ax(**$Triv_t$**)**. We consider these axioms rather harmless and natural. They are not part of **Basax** only because they were unnecessary to prove our basic theorems.

THEOREM 3.9.19 *The following items hold:*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}\triangle\mathbf{2} \rightarrow \mathbf{Ax}\square\mathbf{2}$,

(ii) If $n \geq 3$, then[324]

$$\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax\triangle 2} \leftrightarrow \mathbf{Ax\square 2}.$$

Proof of item (i): The argument is illustrated in Figure 110. Assume **Basax** +

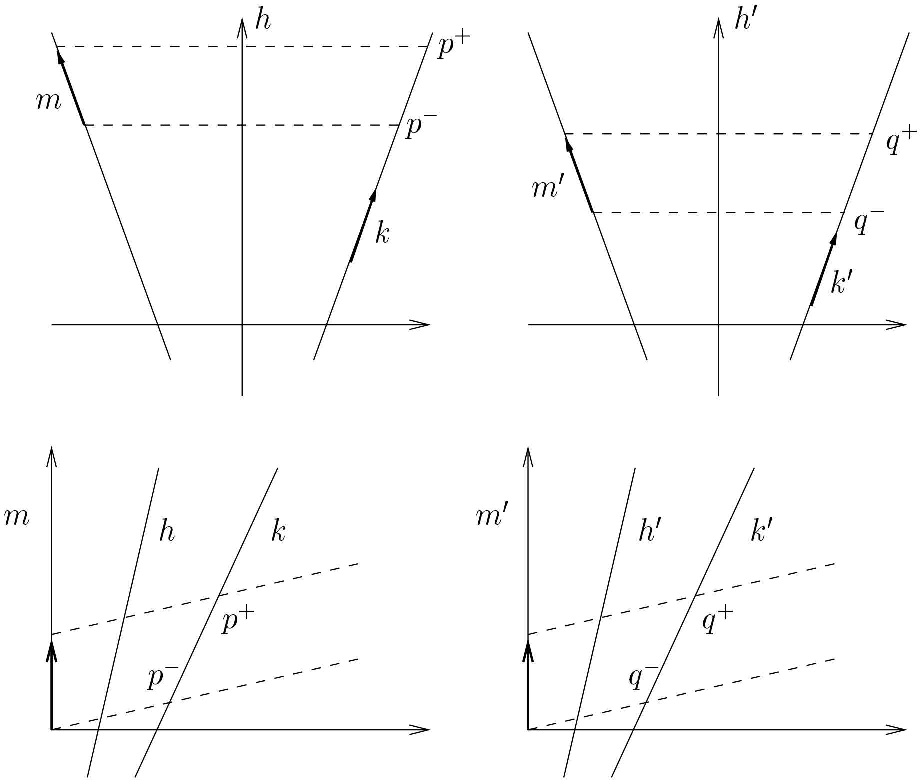


Figure 110: Illustration for the proof of Thm. 3.9.19(i).

$\mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax\triangle 2}$. Let $m, k, m', k' \in \mathit{Obs}$ be such that $tr_m(k) = tr_{m'}(k')$. By Thm. 3.9.8(ii) $v_m(k) = v_{m'}(k') < 1$. Then by Thm. 3.8.25 there are $h, h' \in \mathit{Obs}$ such that h is a median observer for m and k, and h' is a median observer for m' and k'. By the construction of Thm. 3.8.25(ii), the trace of the median observer for a pair $m_0, m_1 \in \mathit{Obs}$ is uniquely determined by $tr_{m_0}(m_1)$. Hence

(213) $$tr_{m'}(h') = tr_m(h).$$

[324]The condition $n \geq 3$ is needed because $\mathbf{Ax\square 2}$ does not exclude FTL observers, while $\mathbf{Ax\triangle 2}$ does (assuming $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$).

On the other hand, by Lemma 3.9.16 we have

$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|, \tag{214}$$

$$|\mathsf{f}_{m'h'}(1_t)_t - \mathsf{f}_{m'h'}(\bar{0})_t| = |\mathsf{f}_{k'h'}(1_t)_t - \mathsf{f}_{k'h'}(\bar{0})_t|. \tag{215}$$

Let p^+ be the point on $tr_h(k)$ that is simultaneous with $\mathsf{f}_{mh}(1_t)$ for h, and let $p^- \in tr_h(k)$ be simultaneous with $\mathsf{f}_{mh}(\bar{0})$ for h. Similarly, let $q^+, q^- \in tr_{h'}(k')$ be such that for h' q^+ and $\mathsf{f}_{m'h'}(1_t)$, and q^- and $\mathsf{f}_{m'h'}(\bar{0})$, respectively, are simultaneous. Then from (214) and (215) we obtain

$$|p^+_t - p^-_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|, \tag{216}$$

$$|q^+_t - q^-_t| = |\mathsf{f}_{k'h'}(1_t)_t - \mathsf{f}_{k'h'}(\bar{0})_t|. \tag{217}$$

Turning to the world-view of m and m', it is easy to check that f_{hm} maps p^+ and p^- to the points where $\mathsf{f}_{h'm'}$ maps q^+ and q^-. Indeed, $\mathsf{f}_{hm}(p^-)$ is the intersection of $tr_m(k)$ and the plane P that is parallel to $\mathsf{f}_{hm}[S]$ and goes though $\bar{0}$ (since f_{hm} is a bijective collineation, it takes parallel planes to parallel planes). Similarly, $\mathsf{f}_{hm}(p^+) = tr_m(k) \cap (P + 1_t)$. Regarding m', q^+, q^-, it is easy to see that the corresponding plane, i.e. that is parallel to $\mathsf{f}_{h'm'}[S]$ and goes though $\bar{0}$, is P. This follows by $tr_m(h) = tr_{m'}(h')$ and Lemma 3.8.28. To sum up, we have

$$\begin{aligned} \mathsf{f}_{hm}(p^+) &= \mathsf{f}_{h'm'}(q^+), \\ \mathsf{f}_{hm}(p^-) &= \mathsf{f}_{h'm'}(q^-). \end{aligned}$$

Since $\mathsf{f}_{hm}, \mathsf{f}_{h'm'} \in \mathit{Aftr}$ by Lemma 3.9.9, both functions preserve proportions along the same line. Hence (216) and (217) imply

$$|\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| = |\mathsf{f}_{k'm'}(1_t)_t - \mathsf{f}_{k'm'}(\bar{0})_t|.$$

Then, comparing f_{km} and $\mathsf{f}_{k'm'}$, by Lemma 3.8.50 there is $N \in \mathit{Triv}$ such that $N[\bar{t}] = \bar{t}$ and

$$N \circ \mathsf{f}_{km} = \mathsf{f}_{k'm'}, \quad \text{or} \quad N \circ \mathsf{f}_{km} = \sigma_S \circ \mathsf{f}_{k'm'}.$$

(We have the first case if ($m \uparrow k$ and $m' \uparrow k'$) or ($m \downarrow k$ and $m' \downarrow k'$), the second case applies if ($m \downarrow k$ and $m' \uparrow k'$) or ($m \uparrow k$ and $m' \downarrow k'$).) Both cases imply $\mathbf{Ax\Box 2}$.

Proof of item (ii): Direction "$\rightarrow$" only recapitulates item (i). Hence we shall prove the other direction only. Assume $\mathbf{Basax}(n) + \mathbf{Ax\Box 2} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ })$ for some $n \geq 3$. We have to derive $\mathbf{Ax\triangle 2}$. Figure 111 shows the elements of the argument.

Let $m, k \in \mathit{Obs}$ be arbitrary. We have $v_m(k) < 1$ by Thm. 3.4.1. Then by Thm. 3.8.25 there is a median observer h for m and k. Thus $tr_h(k) = \sigma_{\bar{t}}[tr_h(m)]$.

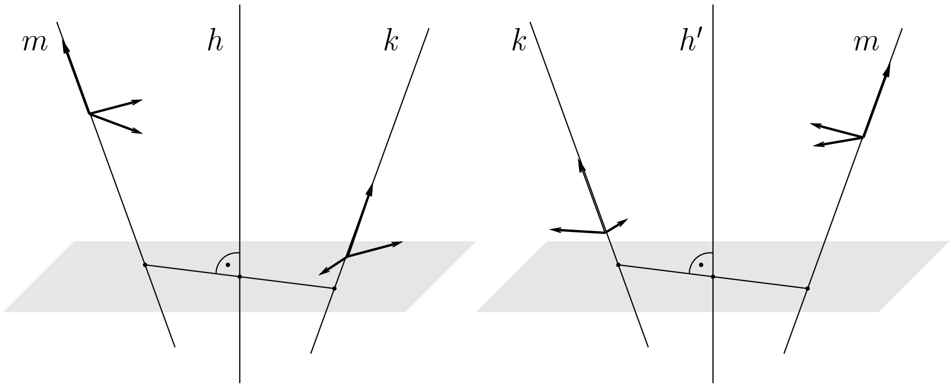


Figure 111: Illustration for the proof of Thm. 3.9.19(ii).

On the other hand, by $\mathbf{Ax(}Triv_t\mathbf{)}$ we have an $h' \in Obs$ such that $\mathsf{f}_{hh'} = \sigma_{\bar{t}}$. Then $tr_{h'}(k) = tr_h(m)$ by $tr_h(k) = \sigma_{\bar{t}}[tr_h(m)]$. Applying $\mathbf{Ax\square 2}$ one gets

$$\mathsf{f}_{hm} = \mathsf{f}_{h'k} \circ N = \sigma_{\bar{t}} \circ \mathsf{f}_{hk} \circ N,$$

for some isometry N such that $N[\bar{t}] \parallel \bar{t}$. Then by Prop. 2.3.3(x)

$$\begin{aligned} \mathsf{f}_{mh} &= N^{-1} \circ \mathsf{f}_{kh} \circ \sigma_{\bar{t}}, \\ N \circ \mathsf{f}_{mh} &= \mathsf{f}_{kh} \circ \sigma_{\bar{t}}. \end{aligned}$$

From this point the proof follows the same steps as that of Prop. 3.9.13, Case 1, from (210). ∎

THEOREM 3.9.20 *The following items hold:*

(i) $\mathbf{Basax} + \mathbf{Ax(eqtime)} \models \mathbf{Ax\square 1} \to \mathbf{Ax\square 2}$,

(ii) If $n \geq 3$, *then* $\mathbf{Basax}(n) + \mathbf{Ax(}Triv_t\mathbf{)} + \mathbf{Ax(eqtime)} + \mathbf{Ax5^+} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax\square 1} \leftrightarrow \mathbf{Ax\square 2}$.

To prove Thm. 3.9.20 we shall use the following lemma, which in turn is a variant of Lemma 3.8.33.

LEMMA 3.9.21

$$\mathbf{Basax} + \mathbf{Ax(eqtime)} \models tr_m(m') = \bar{t} \to (\mathsf{f}_{mm'} \textit{ is an isometry and } \mathsf{f}_{mm'}[\bar{t}] = \bar{t}).$$

Proof: Assume **Basax** + **Ax(eqtime)**. Let $m, m' \in Obs$ be such that $tr_m(m') = \bar{t}$. Now, $\mathsf{f}_{mm'}[\bar{t}] = \bar{t}$ follows by Prop. 2.3.3(vii).

By Prop. 3.8.33(i) we have $\mathsf{f}_{mm'} \in Aftr$; but the proof of Prop. 3.8.33(i) actually establishes $\mathsf{f}_{mm'} \in PT$ (see (127) in that proof).

Case 1: $m \uparrow m'$. In this case the proof of Prop. 3.8.33(ii) goes through; $\mathsf{f}_{mm'} \in Triv$ and thus $\mathsf{f}_{mm'}$ is an isometry.

Case 2: $m \downarrow m'$. Let σ_S denote the reflection around S. Consider $f = \mathsf{f}_{mm'} \circ \sigma_S$. Clearly, $f \in PT$ because $\mathsf{f}_{mm'}, \sigma_S \in PT$ and PT is a group. We have $f[\bar{t}] = \bar{t}$ by $\mathsf{f}_{mm'}[\bar{t}] = \bar{t}$ and $\sigma_S[\bar{t}] = \bar{t}$. On the other hand, $\mathsf{f}_{mm'}(1_t) - \mathsf{f}_{mm'}(\bar{0}) = -1_t$ by **Ax(eqtime)** and $m \downarrow m'$; thus $f(1_t) - f(\bar{0}) = 1_t$. Then $f = g \circ \tau$ for some $g \in PT$, $g(\bar{0}) = \bar{0}$, $g(1_t) = 1_t$, and a translation τ. Now, $g \in Triv_0$ by Lemma 3.6.20, and hence $f \in Triv$ by the definition of $Triv$.

Clearly, f is an isometry by $f \in Triv$. Then $\mathsf{f}_{mm'} = f \circ \sigma_S$ is an isometry, too. ∎

LEMMA 3.9.22 *If $n \geq 3$, then*

$$\mathbf{Basax}(n) + \mathbf{Ax}(\surd\,) \models (\text{“}\uparrow\text{” } \textit{is an equivalence relation}).$$

Proof: We omit the proof. ∎

Proof of Thm. 3.9.20(i): Assume **Basax** + **Ax(eqtime)** + **Ax□1**. Let $m, k, m', k' \in Obs$ be such that $tr_m(k) = tr_{m'}(k')$, as assumed by **Ax□2**.

By **Ax□1** there is a $k_0 \in Obs$ such that $\mathsf{f}_{mk} = \mathsf{f}_{m'k_0}$. Thus by Prop. 2.3.3(x)

$$\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ \mathsf{f}_{k'k_0}. \tag{218}$$

Applying Prop. 2.3.3(vii) we get $tr_{m'}(k_0) = tr_m(k)$. Then $tr_{m'}(k_0) = tr_{m'}(k')$ by $tr_m(k) = tr_{m'}(k')$, and $tr_{k'}(k_0) = \bar{t}$ because $\mathsf{f}_{m'k'}$ is a bijection and by **Ax4**.

Now, applying Lemma 3.9.21 we have that $\mathsf{f}_{k'k_0}$ is an isometry with $\mathsf{f}_{k'k_0}[\bar{t}] = \bar{t}$. Therefore (218) is exactly the conclusion of **Ax□2** (letting $N = \mathsf{f}_{k'k_0}$).

Proof of item (ii): Direction “$\to$” is a corollary of item (i). To prove the other direction, assume $\mathbf{Basax}(n) + \mathbf{Ax}(Triv_t) + \mathbf{Ax5}^+ + \mathbf{Ax}(\surd\,) + \mathbf{Ax□2}$ for $n \geq 3$. (We do not need **Ax(eqtime)** here.) Let $m, k, m' \in Obs$ be arbitrary.

We have $v_m(k) < 1$ by Thm. 3.4.1. We have that “$\uparrow$” is an equivalence relation by Lemma 3.9.22. We have three cases according to the direction of time for m, k, m'. Let $\ell = tr_m(k)$.

Case 1: $m \uparrow k$. See Figure 112. By $\mathbf{Ax5}^+$ we have an observer k_0 such that $tr_{m'}(k_0) = \ell$ and $m' \uparrow k_0$. Then, by **Ax□2**,

$$\mathsf{f}_{mk} = \mathsf{f}_{m'k_0} \circ N, \tag{219}$$

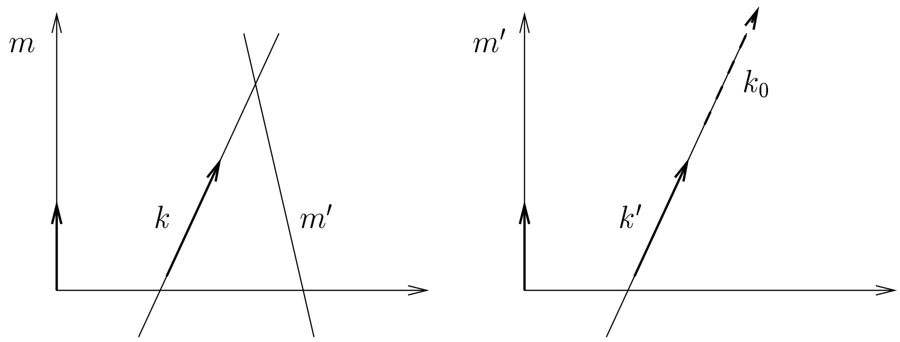


Figure 112: Illustration for the proof of Thm. 3.9.20(ii), Case 1.

for some isometry N such that $N[\bar{t}] \parallel \bar{t}$. It is easy to see that $N[\bar{t}] = \bar{t}$ must hold. Now, because neither f_{mk} nor $\mathsf{f}_{m'k_0}$ reverses the arrow of time (formally, $\mathsf{f}_{mk}(1_t) > \mathsf{f}_{mk}(\bar{0})$ and $\mathsf{f}_{m'k_0}(1_t) > \mathsf{f}_{m'k_0}(\bar{0})$), N must not do so either. Hence $N \in \mathit{Triv}$. Then, by **Ax(**Triv_t**)** there is another observer k' such that $\mathsf{f}_{k_0k'} = N$. Thus (219) becomes $\mathsf{f}_{mk} = \mathsf{f}_{m'k_0} \circ \mathsf{f}_{k_0k'} = \mathsf{f}_{m'k'}$, as required.

Case 2: $m \downarrow k$, $m \uparrow m'$. See Figure 113. In this case must we apply $\mathbf{Ax5^+}$ differently. We apply $\mathbf{Ax5^+}$ in the world-view of k. We get that there is $k_0 \in \mathit{Obs}$ such that $tr_k(k_0) = \mathsf{f}_{m'k}[\ell]$ and $k \uparrow k_0$. We have $m' \downarrow k_0$ by $m \downarrow k$, $m \uparrow m'$, $k \uparrow k_0$, and because "$\uparrow$" is an equivalence relation. It is straightforward to check that $tr_{m'}(k_0) = \ell$. By $\mathbf{Ax\Box 2}$ get that $\mathsf{f}_{mk} = \mathsf{f}_{m'k_0} \circ N$ for an isometry N such that $N[\bar{t}] = \bar{t}$. Because both f_{mk} and $\mathsf{f}_{m'k_0}$ turn back the arrow of time, N must not do so. The rest of the proof is the same as for Case 1.

Case 3: $m \downarrow k$, $m \downarrow m'$. See Figure 114. We apply $\mathbf{Ax5^+}$ in the world-view of m. We get that there is $k_0 \in \mathit{Obs}$ such that $tr_m(k_0) = \mathsf{f}_{m'm}[\ell]$ and $m \uparrow k_0$. We have $m' \downarrow k_0$ by $m \downarrow m'$, $m \uparrow k_0$, and because "$\uparrow$" is an equivalence relation. Again, it can be checked that $tr_{m'}(k_0) = \ell$. By $\mathbf{Ax\Box 2}$ get that $\mathsf{f}_{mk} = \mathsf{f}_{m'k_0} \circ N$ for an isometry N such that $N[\bar{t}] = \bar{t}$. Because both f_{mk} and $\mathsf{f}_{m'k_0}$ turn back the arrow of time, N must not do so. The rest of the proof is the same as for Cases 1 and 2. ∎

THEOREM 3.9.23 *The following items hold:*

(i) **Basax** + **Ax(eqtime)** $\models$ $\mathbf{Ax\triangle 1} \to \mathbf{Ax\triangle 2}$,

(ii) **Basax** + **Ax(**Triv_t**)** + **Ax(**$\sqrt{\ }$**)** $\models$ $\mathbf{Ax\triangle 2} \to \mathbf{Ax\triangle 1}$,

(iii) **Basax** + **Ax(**Triv_t**)** + **Ax(eqtime)** + **Ax(**$\sqrt{\ }$**)** $\models$ $\mathbf{Ax\triangle 1} \leftrightarrow \mathbf{Ax\triangle 2}$.

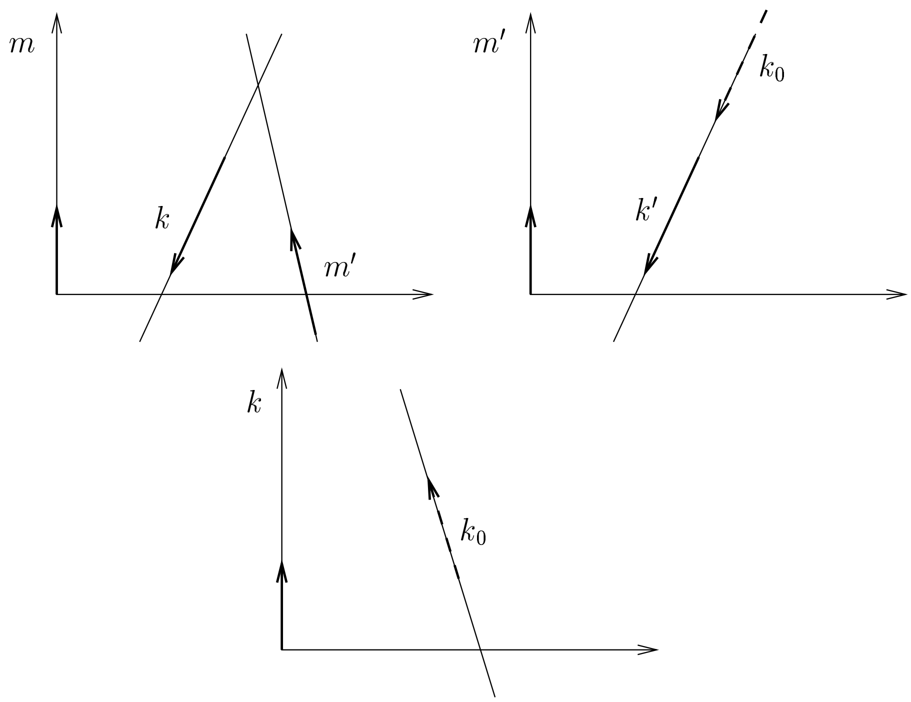


Figure 113: Illustration for the proof of Thm. 3.9.20(ii), Case 2.

Proof of item(i): Assume **Basax** + **Ax(eqtime)** + **Ax△1**. Let $m, k \in Obs$ be arbitrary. By **Ax△1** there is a $k' \in Obs$ such that $tr_m(k) = tr_m(k')$ and $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$. Thus by Prop. 2.3.3(x)

(220) $$\mathsf{f}_{mk} \circ \mathsf{f}_{kk'} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km}.$$

Now, $tr_k(k') = \bar{t}$ by Prop. 2.3.3(vii) and $tr_k(k) = \bar{t}$. Then by Lemma 3.9.21 we have that $\mathsf{f}_{k'k}$ is an isometry with $\mathsf{f}_{k'k}[\bar{t}] = \bar{t}$. Letting $N = \mathsf{f}_{k'k}$, from (220) we get

$$\begin{aligned} \mathsf{f}_{mk} \circ N^{-1} &= N \circ \mathsf{f}_{km}; \\ \mathsf{f}_{mk} &= N \circ \mathsf{f}_{km} \circ N, \end{aligned}$$

as required.

Proof of item (ii): Assume **Basax**+**Ax(**$Triv_t$**)**+**Ax(**$\sqrt{\ }$)+**Ax△2**. Let $m, k \in Obs$ be arbitrary. By Prop. 3.9.13 we have that there is $N \in Triv$ such that

$$N[\bar{t}] = \bar{t} \quad \wedge \quad \mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N.$$

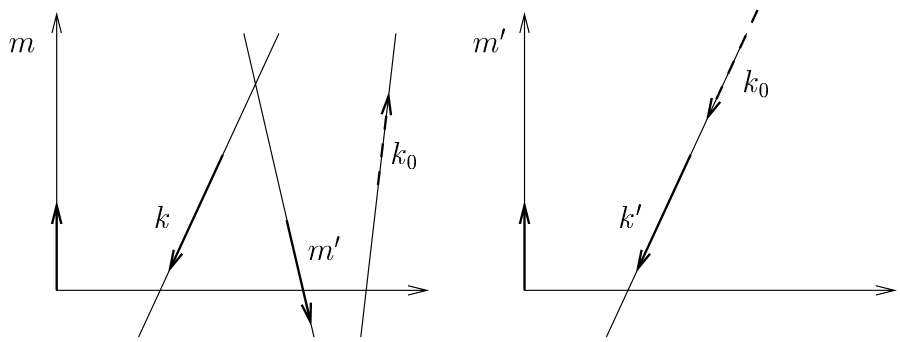


Figure 114: Illustration for the proof of Thm. 3.9.20(ii), Case 3.

Consequently (using Prop. 2.3.3(x)),

$$\mathsf{f}_{mk} \circ N^{-1} = N \circ \mathsf{f}_{km}.$$

By $\mathbf{Ax}(\mathit{Triv}_t)$ we have an $k' \in \mathit{Obs}$ such that $\mathsf{f}_{k'k} = N$. Then

$$\mathsf{f}_{mk'} = \mathsf{f}_{k'm}.$$

On the other hand, $tr_m(k') = (\mathsf{f}_{k'k} \circ \mathsf{f}_{km})[\bar{t}] = \mathsf{f}_{km}[\bar{t}] = tr_m(k)$, as required.

Proof of item (iii): This item is a corollary of items (i) and (ii). ∎

COROLLARY 3.9.24 $\mathbf{Basax} + \mathbf{Ax}\triangle\mathbf{1} + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\sqrt{\ }) \models \mathsf{f}_{mk} \in \mathit{Aftr}$.

Proof: This corollary follows by Thm. 3.9.23(i) and Lemma 3.9.9. ∎

PROPOSITION 3.9.25 *Let $n \geq 3$. Then*

(i) $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}\square\mathbf{1} \to \mathbf{Ax}\triangle\mathbf{1}$,

(ii) $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)} + \mathbf{Ax5}^+ \models \mathbf{Ax}\square\mathbf{1} \leftrightarrow \mathbf{Ax}\triangle\mathbf{1}$.

Proof: Item (ii) follows by Thm's 3.9.19(i), 3.9.20(ii) and 3.9.23(i). We shall prove item (i) directly.

Assume $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}\square\mathbf{1}$ for $n \geq 3$. Let $m, k \in \mathit{Obs}$ be arbitrary. Let $h \in \mathit{Obs}$ be a median observer for m and k (such an h exists by Thm's 3.4.1 and 3.8.25).

By $\mathbf{Ax}(\mathit{Triv}_t)$ there is an $h' \in \mathit{Obs}$ such that $\mathsf{f}_{hh'} = \sigma_{\bar{t}}$. Consider observers h, m, h'. Axiom $\mathbf{Ax}\square\mathbf{1}$ implies that there is a $k' \in \mathit{Obs}$ such that

$$\mathsf{f}_{hm} = \mathsf{f}_{h'k'} = \mathsf{f}_{h'h} \circ \mathsf{f}_{hk'} = \sigma_{\bar{t}} \circ \mathsf{f}_{hk'}. \tag{221}$$

Then $tr_h(k) = tr_h(k')$. Since the world-view transformations are bijective, we have $tr_k(k') = \bar{t}$. Inverting (221), multiplying by $\sigma_{\bar{t}}$ from the right, and composing the appropriate sides yields

$$\mathsf{f}_{mk'} = \mathsf{f}_{k'h} \circ \sigma_{\bar{t}} \circ \sigma_{\bar{t}} \circ \mathsf{f}_{hm} = \mathsf{f}_{k'm},$$

as required. ∎

We shall now turn to this issue of the relationship of our preferred symmetry axiom, $\mathbf{Ax(symm)}$, to the recently introduced axioms. We shall prove a sequence of theorems and corollaries (items 3.9.26 to 3.9.31) which state that under certain assumptions $\mathbf{Ax(symm)}$ follows from any of $\{\mathbf{Ax}\triangle\mathbf{1}, \mathbf{Ax}\triangle\mathbf{2}, \mathbf{Ax}\square\mathbf{1}, \mathbf{Ax}\square\mathbf{2}\}$ or conversely.

THEOREM 3.9.26 *The following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)} \models \mathbf{Ax}\triangle\mathbf{1} \rightarrow \mathbf{Ax(symm)}$.

(ii) If $n \geq 3$, *then*[325]

$$\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models \mathbf{Ax(symm)} \rightarrow \mathbf{Ax}\triangle\mathbf{1}.$$

Proof of item (i): Assume $\mathbf{Basax} + \mathbf{Ax}\triangle\mathbf{1} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)}$. We only need to derive $\mathbf{Ax(symm_0)}$, because $\mathbf{Ax(eqtime)}$ is assumed.

Let $m, k \in \mathit{Obs}$ be arbitrary. Faster than light observers are excluded by Cor. 2.7.6. Then by Thm. 3.8.25 there is a median observer h for m and k. Thus we have

$$tr_h(k) = \sigma_{\bar{t}}[tr_h(m)]. \tag{222}$$

On the other hand, by Thm. 3.9.23(i) $\mathbf{Ax}\triangle\mathbf{2}$ holds. Then by Lemma 3.9.16 we have that h sees the clocks of m and k tick with the same rate:

$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|. \tag{223}$$

We have $\mathsf{f}_{mh}, \mathsf{f}_{kh}, \mathsf{f}_{mk} \in PT$ by Cor. 3.9.24.

[325] Note that $\mathbf{Ax}\triangle\mathbf{1}$ excludes FTL observers together with the rest of the premises, unlike $\mathbf{Ax(symm)}$.

Case 1: $m \uparrow k$. The world-view of h for Case 1 is shown in Figure 115. The basic idea is to reflect the origin and the unit vectors of observers m and k to the $\bar{t}$ axis. We shall show that this operation yields the origin and unit vectors of two other observers m' and k' that must be present in the model because of $\mathbf{Ax}(\mathit{Triv}_t)$. It will be easy to see that $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$ by this construction, and that m and m', and k and k', respectively, are brothers.

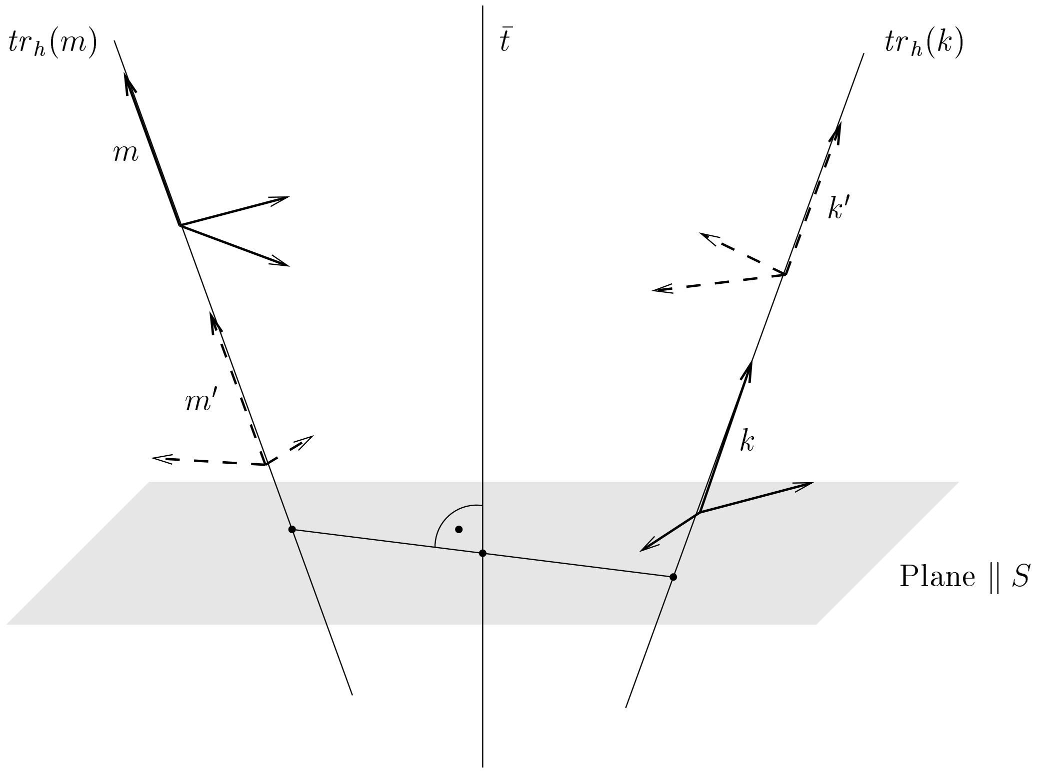


Figure 115: Illustration for the proof of Prop. 3.9.26, Case 1.

Consider transformations f_{mh} and $\mathsf{f}_{kh} \circ \sigma_{\bar{t}}$. Both map $\bar{t}$ axis to the same line, $tr_h(m)$ by Prop. 2.3.3 and (222). Since $m \uparrow k$ and $\sigma_{\bar{t}} \restriction \bar{t} = \mathrm{Id}$, (223) implies

$$\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t = (\mathsf{f}_{kh} \circ \sigma_{\bar{t}})(1_t)_t - (\mathsf{f}_{kh} \circ \sigma_{\bar{t}})(\bar{0})_t. \tag{224}$$

Hence by Lemma 3.8.50 there is a $N \in \mathit{Triv}$ such that $N[\bar{t}] = \bar{t}$ and

$$N \circ \mathsf{f}_{mh} = \mathsf{f}_{kh} \circ \sigma_{\bar{t}}. \tag{225}$$

Now by $\mathbf{Ax(}Triv_t\mathbf{)}$ there is an observer m' such that $\mathsf{f}_{m'm} = N$. On the other hand, (225) also implies by a simple calculation $N^{-1} \circ \mathsf{f}_{kh} = \mathsf{f}_{mh} \circ \sigma_{\bar{t}}$. Again by $\mathbf{Ax(}Triv_t\mathbf{)}$ there is $k' \in Obs$ such that $\mathsf{f}_{k'k} = N^{-1}$. Thus we arrive at

$$\mathsf{f}_{m'h} = \mathsf{f}_{kh} \circ \sigma_{\bar{t}} \quad \text{and} \quad \mathsf{f}_{k'h} = \mathsf{f}_{mh} \circ \sigma_{\bar{t}}.$$

By inverting one of these equations and combining with the other we get $\mathsf{f}_{m'k'} = \mathsf{f}_{km}$, as required.

Case 2: $m \downarrow k$. This case is similar to Case 1, but we should consider $\mathsf{f}_{kh} \circ \sigma_\ell$ instead of $\mathsf{f}_{kh} \circ \sigma_{\bar{t}}$, where ℓ is the line used in the proof of Prop. 3.9.13 Case 2, defined on p. 367. (Transformation σ_ℓ reverses the arrow of time, thereby ensuring that a relation analogous to (224) holds.) The reader is invited to fill in the details.

Proof of item (ii): Let $n \geq 3$. Assume $\mathbf{Basax}(n) + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(}Triv_t\mathbf{)}$. Let $m, k \in Obs$ be arbitrary. We have to show the existence of an observer k' such that $tr_m(k) = tr_m(k')$ and $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$.

By Thm. 3.8.25 and Thm. 3.4.1 there is a median observer h for m and k. Thus we have

(226) $$tr_h(k) = \sigma_{\bar{t}}[tr_h(m)].$$

On the other hand, by Prop. 3.8.31 we have that h sees the clocks of m and k tick with the same rate:

(227) $$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|.$$

By Prop. 3.8.34(ii) $\mathsf{f}_{mh}, \mathsf{f}_{kh}, \mathsf{f}_{mk} \in PT$.

Case 1: $m \uparrow k$. Because of (226) and (227) the world-view of h looks as shown on Figure 108. Consider transformations f_{kh} and $\mathsf{f}_{mh} \circ \sigma_{\bar{t}}$. Both map $\bar{t}$ axis to the same line, $tr_h(k)$ by Prop. 2.3.3 and (226). Since $m \uparrow k$ and $\sigma_{\bar{t}} \restriction \bar{t} = \mathrm{Id}$, (227) implies

(228) $$\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t = (\mathsf{f}_{mh} \circ \sigma_{\bar{t}})(1_t)_t - (\mathsf{f}_{mh} \circ \sigma_{\bar{t}})(\bar{0})_t.$$

Hence by Lemma 3.8.50 there is a $N \in Triv$ such that $N[\bar{t}] = \bar{t}$

$$N \circ \mathsf{f}_{kh} = \mathsf{f}_{mh} \circ \sigma_{\bar{t}}.$$

It is easy to check that $tr_m(k) = tr_m(k')$.

Now by $\mathbf{Ax(}Triv_t\mathbf{)}$ there is an observer k' such that $\mathsf{f}_{k'k} = N$.

(229) $$\mathsf{f}_{k'h} = \mathsf{f}_{mh} \circ \sigma_{\bar{t}}.$$

Using Figure 108, (229) means that the unit vector shown with dashed lines are actually the unit vectors of a "real" observer (k').

Inverting both sides of (229) and multiplying with $\sigma_{\bar{t}}$ we get $\sigma_{\bar{t}} \circ \mathsf{f}_{hk'} = \mathsf{f}_{hm}$. Composing this with equality the corresponding sides of (229) we arrive at $\mathsf{f}_{k'm} = \mathsf{f}_{mk'}$.

Case 2: $m \downarrow k$. This case is similar to Case 1, but we should consider $\mathsf{f}_{mh} \circ \sigma_\ell$ instead of $\mathsf{f}_{mh} \circ \sigma_{\bar{t}}$, where ℓ is as defined in the proof of Prop. 3.9.13, Case 2. The reader is invited to fill in the details, based on the analogy with the two cases of Prop. 3.9.13. ∎

THEOREM 3.9.27 *The following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models \mathbf{Ax\triangle 2} \to \mathbf{Ax(symm)}$.

(ii) If $n \geq 3$, then $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax(symm)} \to \mathbf{Ax\triangle 2}$.

Proof of item (i): Assume $\mathbf{Basax}+\mathbf{Ax\triangle 2}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax}(\mathit{Triv}_t)$. Let $m, k \in \mathit{Obs}$ be arbitrary. By Prop. 3.9.13 $\mathbf{Ax\triangle 2}^*$ holds. By $\mathbf{Ax\triangle 2}^*$ we have an $N \in \mathit{Triv}$ such that

$$N[\bar{t}] = \bar{t} \quad \text{and} \quad \mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N.$$

Now, applying $\mathbf{Ax}(\mathit{Triv}_t)$ twice we get that there are $m', k' \in \mathit{Obs}$ such that $\mathsf{f}_{k'k} = N = \mathsf{f}_{mm'}$. Then, $\mathsf{f}_{mk} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km} \circ \mathsf{f}_{mm'} = \mathsf{f}_{k'm'}$, as required. Noting that $tr_m(m') = \mathsf{f}_{m'm}[\bar{t}] = N^{-1}[\bar{t}] = \bar{t}$ and similarly $tr_k(k') = \bar{t}$ completes the proof of $\mathbf{Ax(symm_0)}$. Axiom $\mathbf{Ax(eqtime)}$ follows by Lemma 3.9.15(i).

Proof of item (ii): Assume $\mathbf{Basax}(n)+\mathbf{Ax(symm)}+\mathbf{Ax}(\sqrt{\ })$. We have to prove $\mathbf{Ax\triangle 2}$. Let $m, k \in \mathit{Obs}$ be arbitrary. By Prop. 3.8.34(ii) $\mathsf{f}_{mh}, \mathsf{f}_{kh}, \mathsf{f}_{mk} \in PT$. Faster than light observers are excluded by $n \geq 3$ and Thm. 3.4.1. Then by Thm. 3.8.25 there is a median observer h for m and k. Thus we have

$$tr_h(k) = \sigma_{\bar{t}}[tr_h(m)]. \tag{230}$$

On the other hand, by Prop. 3.8.31 we have that h sees the clocks of m and k tick with the same rate:

$$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|. \tag{231}$$

Having (230) and (231), one can continue like in the proof of Prop. 3.9.13. Again, there are two cases whether $m \uparrow k$ or $m \downarrow k$. Indeed, (230) and (231) are equivalent to (207) and (208). The reader is invited to fill in the details. ∎

COROLLARY 3.9.28 $\mathbf{Basax} + \mathbf{Ax\triangle 2} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models \mathsf{f}_{mk} \in \mathit{Poi}$.

Proof: This corollary follows by Thm's 3.9.27(i) and 2.9.5(i). ∎

THEOREM 3.9.29 *Assume $n \geq 3$. Then the following items hold.*

(i) $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax\Box 2} \rightarrow \mathbf{Ax(symm)}$,

(ii) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax(symm)} \rightarrow \mathbf{Ax\Box 2}$.

Proof of item (i): Let $n \geq 3$. Assume $\mathbf{Basax}(n) + \mathbf{Ax\Box 2} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ })$. By Thm. 3.9.19(ii) axiom $\mathbf{Ax\triangle 2}$ holds. Then by Thm. 3.9.27(i) we get that $\mathbf{Ax(symm)}$ holds as well.

Proof of item (ii): Let $n \geq 3$. Assume $\mathbf{Basax}(n) + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ })$. Thm. 3.9.27(ii) ensures that $\mathbf{Ax\triangle 2}$ holds. By Thm. 3.9.19(i) $\mathbf{Ax\Box 2}$ is implied.[326] ∎

COROLLARY 3.9.30 Let $n \geq 3$. Then

$$\mathbf{Basax}(n) + \mathbf{Ax\Box 2} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models \mathsf{f}_{mk} \in \mathit{Poi}.$$

Proof: This corollary follows by Thm's 3.9.29(i) and 2.9.5(i). ∎

THEOREM 3.9.31 *Assume $n \geq 3$. Then the following items hold.*

(i) $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax\Box 1} \rightarrow \mathbf{Ax(symm_0)}$,

(ii) $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax5^+} \models \mathbf{Ax(symm)} \rightarrow \mathbf{Ax\Box 1}$.

Proof of item(i): Assume $\mathbf{Basax}(n) + \mathbf{Ax\Box 1} + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax}(\sqrt{\ })$ for $n \geq 3$. We have to prove $\mathbf{Ax(symm_0)}$. Throughout the proof the reader is asked to consult Figure 116.

Let $m, k \in \mathit{Obs}$ be arbitrary. Faster than light observers are excluded by $n \geq 3$ and Thm. 3.4.1. Then by Thm. 3.8.25 there is a median observer h for m and k. Thus we have

(232) $$tr_h(k) = \sigma_{\bar{t}}[tr_h(m)].$$

By $\mathbf{Ax}(\mathit{Triv}_t)$ there is another observer h' such that $\mathsf{f}_{hh'} = \sigma_{\bar{t}}$. Of course, h' is also a median observer for m and k. Now, by $\mathbf{Ax\Box 1}$ there are observers m', k' such that the following statements hold:

(233) $$\mathsf{f}_{k'h'} = \mathsf{f}_{mh},$$

(234) $$\mathsf{f}_{m'h'} = \mathsf{f}_{kh}.$$

[326] Alternatively, one could prove item (i) and (ii) by exploiting the same method that we used to show Theorems 3.9.26 and 3.9.27. The reader is invited to fill in the details.

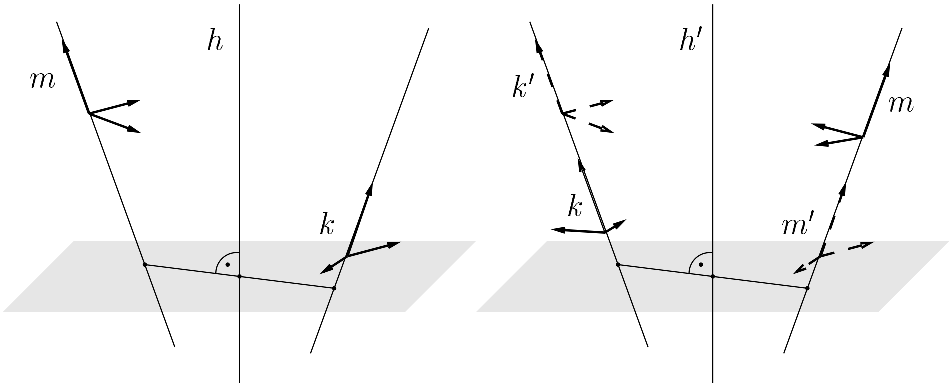


Figure 116: Illustration for the proof of Thm. 3.9.31(i).

Then, from (233) and (234) one can derive $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$.

We still have to show $tr_m(m') = tr_k(k') = \bar{t}$. Now, using (234), (232), $\mathsf{f}_{hh'} = \sigma_{\bar{t}}$, and Prop. 2.3.3(vii) in turn,

$$\begin{aligned} tr_{h'}(m') &= tr_h(k) \\ &= \sigma_{\bar{t}}[tr_h(m)] \\ &= \mathsf{f}_{hh'}[tr_h(m)] \\ &= tr_{h'}(m). \end{aligned}$$

Then, Prop. 2.3.3(vii) and $tr_m(m) = \bar{t}$ imply $tr_m(m') = \bar{t}$. Showing $tr_k(k') = \bar{t}$ is analogous.

Proof of item (ii): This is a corollary of Thm. 3.9.29(ii) and Thm. 3.9.20(ii). ∎

Proof of Lemma 3.9.5: We shall give the proof for an Euclidean ordered field $\mathfrak{F}$. A more general proof will be filled in later. For the time being we note that the only step that requires square roots (i.e. an Euclidean field) is the proof of Claim 3.9.32.

Assume $h : {}^nF \to {}^nF$ is an isometry. Formally, the following formula holds:

$$(\forall p, q \in {}^nF)\|p - q\| = \|h(p) - h(q)\|. \tag{235}$$

We have to show that $h \in Aftr$.

Claim 3.9.32 Function h is a collineation. Formally, $(\forall \ell \in \mathsf{Eucl})h[\ell] \in \mathsf{Eucl}$.

To show Claim 3.9.32 we shall use the notion of Euclidean distance. The distance of points $p, q \in {}^nF$ is $|p-q| = \sqrt{\|p-q\|}$. Of course, the field $\mathfrak{F}$ must be Euclidean. It is trivial that

$$(236)\qquad (\forall p, q \in {}^nF)\Big(|h(p)-h(q)| = |p-q| \leftrightarrow \|h(p)-h(q)\| = \|p-q\|\Big).$$

Let $p, q, r \in {}^nF$ be collinear. They can be chosen so that q is *between* p and r. Then, by the definition of distance, we have

$$(237)\qquad |p-q| + |q-r| = |p-r|.$$

By (236), (237) and h being an isometry we have

$$(238)\qquad |h(p)-h(q)| + |h(q)-h(r)| = |h(p)-h(r)|.$$

It is well known from Euclidean geometry that (238) implies that $h(p)$, $h(q)$, and $h(r)$ are collinear. Cf. e.g. [121, p. 59]. (Claim 3.9.32) ∎

Returning to the mainstream of the proof, let us notice the following. It is enough to deal with the case $h(\bar{0}) = \bar{0}$, because in the general case we have $h = h' \circ \tau$ for some isometry h' and translation τ, and if h' is an isometry, then so is h. Therefore we assume $h(\bar{0}) = \bar{0}$ in the rest of the proof. Lemma 3.9.5 follows by Claims 3.9.33 and 3.9.34

Claim 3.9.33 $(\forall u \in {}^nF)(\forall \lambda \in F)h(\lambda u) = \lambda h(u)$.

Assume $u \neq \bar{0}$, $\lambda \neq 0$ (otherwise Claim 3.9.33 becomes trivial). Since $\bar{0}$, u, λu are collinear, by Claim 3.9.32 $h(\bar{0}) = \bar{0}$, $h(u)$, $h(\lambda u)$ are collinear, too. Because h is an isometry, $\|h(\lambda u)\| = \|\lambda u\| = \lambda^2\|u\|$. Since $h(\lambda u) \in \overline{\bar{0}u}$, $h(\lambda u)$ can be either λu or $-\lambda u$.

Now, suppose $h(\lambda u) = -\lambda u$. Then

$$|\lambda - 1|^2\|u\| = \|\lambda u - u\| = \|h(\lambda u) - h(u)\| = \| - \lambda u - h(u)\| = |\lambda + 1|^2\|u\|.$$

Then $|\lambda - 1|^2 = |\lambda + 1|^2$, and hence $\lambda = 0$, contrary to our assumption $\lambda \neq 0$.

Claim 3.9.34 $(\forall u, v \in {}^nF)h(u+v) = h(u) + h(v)$.

Assume $u \neq \bar{0} \neq v$ and $u \nparallel v$ (if $u = \bar{0} \lor v = \bar{0}$, Claim 3.9.34 is trivial; if $u \parallel v$, then $u = \lambda v$ and the previous argument applies). By the definition of addition, $\bar{0}uv(u+v)$ is a parallelogram. That is,

$$\overline{\bar{0}u} \parallel \overline{v(u+v)} \quad \text{and} \quad \overline{\bar{0}v} \parallel \overline{u(u+v)}.$$

Since h is a *bijective* collineation, it takes parallel lines to parallel lines, and we have

$$\begin{aligned}\overline{h(\bar{0})h(u)} &\parallel \overline{h(v)h(u+v)},\\ \overline{h(\bar{0})h(v)} &\parallel \overline{h(u)h(u+v)},\\ h(u) &\not\parallel h(v).\end{aligned}$$

Now, since $h(u) \not\parallel h(v)$, $\overline{h(v)h(u+v)}$ and $\overline{h(u)h(u+v)}$ intersect in a single point, $h(u+v)$. But this point must be $h(u)+h(v)$, by the definition of addition. Thus Claim 3.9.34 holds. ∎

We are finishing this paragraph by dealing with the problem when two observers see each other *exactly* the same way (i.e. $\mathsf{f}_{mk} = \mathsf{f}_{km}$ for a pair $m, k \in Obs$).

Proposition 3.9.35 *Let* $n \geq 3$. *Assume* $\mathbf{Basax}(n) + \mathbf{Ax(symm)} + \mathbf{Ax}(Triv_t) + \mathbf{Ax}(\sqrt{\ })$. *Let* $m, k \in Obs$. *The following items hold.*

(i) There are $m', k' \in Obs$ *such that*

$$(\star) \qquad tr_m(m') = tr_k(k') = \bar{t} \quad \wedge \quad \mathsf{f}_{m'k'} = \mathsf{f}_{k'm'}.$$

(ii) Assume $tr_m(k) \cap \bar{t} \neq \emptyset$. *Then there are* $m', k' \in Obs$ *in* standard configuration *such that* $(\star)$ *above holds for* m' *and* k', *and* $\mathsf{f}_{m'k'} \in Rhomb$.[327]

Proof of item (i): This is a corollary of Thm. 3.9.26(ii).

Proof of item (ii): Assume $\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(Triv_t) + \mathbf{Ax}(\sqrt{\ })$ and $n \geq 3$. Let $m, k \in Obs$ be such that $tr_m(k) \cap \bar{t} \neq \emptyset$. Let h be a median observer for observers m and k. Such an h exists by Theorems 3.4.1 and 3.8.25. By $\mathbf{Ax}(Triv_t)$, h can be chosen such that $\bar{0} \in tr_h(m) \cap tr_h(k)$ and $tr_h(m) \cup tr_h(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. Let h be such. We will choose $m', k' \in Obs$ so that $\mathsf{f}_{hm'}, \mathsf{f}_{hk'} \in Rhomb$ and $tr_m(m') = tr_k(k') = \bar{t}$. The existence of such m', k' can be proved by $\mathbf{Ax}(Triv_t)$ as follows. Let $p \in tr_h(m)$ and $q \in tr_h(k)$ such that

$$p_t = \mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t, \quad \text{and} \tag{239}$$

$$q_t = \mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t. \tag{240}$$

Let $f, g \in Rhomb$ such that $f(1_t) = p$ and $g(1_t) = q$. Such f, g exist by Lemma 3.8.46. Clearly $f[\bar{t}] = \mathsf{f}_{mh}[\bar{t}]$ and $g[\bar{t}] = \mathsf{f}_{kh}[\bar{t}]$. By this, $Rhomb \subseteq PT$, (239), (240) and Lemma 3.8.50, we have that $f = N \circ \mathsf{f}_{mh}$ and $g = M \circ \mathsf{f}_{kh}$,

[327] Let us recall that if for observers m' and k' we have $\mathsf{f}_{m'k'} \in Rhomb$ then m' and k' are in standard configuration.

for some $N, M \in \mathit{Triv}$ with $N[\bar{t}] = M[\bar{t}] = \bar{t}$. Let such N, M be fixed. By $\mathbf{Ax}(\mathit{Triv}_t)$ there are $m', k' \in \mathit{Obs}$ such that $N = \mathsf{f}_{m'm}$ and $M = \mathsf{f}_{k'k}$. Let such m', k' be fixed. Thus, $f = \mathsf{f}_{m'h}$ and $g = \mathsf{f}_{k'h}$. So $tr_m(m') = tr_k(k') = \bar{t}$ and $\mathsf{f}_{m'k'} = \mathsf{f}_{m'h} \circ \mathsf{f}_{hk'} = f \circ g^{-1} \in \mathit{Rhomb}$. By $\mathsf{f}_{m'k'} \in \mathit{Rhomb}$, we have that m' and k' are in standard configuration. ∎

The next proposition answers the following natural question. Given observers m and k, how many brothers k' of k exist such that m and k' see each other symmetrically (i.e. $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$). Recall that $\mathbf{Ax}\triangle\mathbf{1}$ is of the following pattern:

$$(\forall m, k)(\exists k')(tr_k(k') = \bar{t} \quad \wedge \quad \mathsf{f}_{mk'} = \mathsf{f}_{k'm}).$$

In other words, in the proposition below we will answer the following question about $\mathbf{Ax}\triangle\mathbf{1}$: "How many different choices of k' satisfy the conclusion of $\mathbf{Ax}\triangle\mathbf{1}$ (assuming $\mathbf{BaCo}+\mathbf{Ax}(\sqrt{\ })$)?"

PROPOSITION 3.9.36

(i) *Assume* $\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ })$. *Let $m, k \in \mathit{Obs}$. Let h be a median observer for observers m, k (h exists by Thm's 3.8.11, 3.4.1, 3.8.25). Then the number of brothers of k satisfying the conclusion of* $\mathbf{Ax}\triangle\mathbf{1}$ *is the same as the cardinality of T below. In more detail: Consider the following sets:*

$$O \stackrel{\text{def}}{=} \{ k' \in \mathit{Obs} : tr_k(k') = \bar{t},\ \mathsf{f}_{mk'} = \mathsf{f}_{k'm} \}.$$
$$T \stackrel{\text{def}}{=} \left\{ \sigma \in \mathit{Triv} : \sigma[\bar{t}] = \bar{t},\ \sigma^2 = \mathrm{Id},\ \sigma[tr_h(m)] = tr_h(k) \right\}.$$

Then $|O| = |T|$. Moreover:

(ii) *There is a* natural bijection *β between O and T as follows.*

$$\beta \stackrel{\text{def}}{=} \{ \langle k', \sigma \rangle \in O \times \mathit{Aftr} : \mathsf{f}_{k'h} = \mathsf{f}_{mh} \circ \sigma \},$$

where the intuitive meaning of $\mathsf{f}_{k'h} = \mathsf{f}_{mh} \circ \sigma$ is the following. In the world-view of h the world-view of k' is the σ-image of that of m, in particular the unit vectors of k' are the σ-images of that of m (i.e. for every $i \in n$ letting $1_i^m \stackrel{\text{def}}{=} \mathsf{f}_{mh}(1_i)$ and $1_i^{k'} \stackrel{\text{def}}{=} \mathsf{f}_{k'h}(1_i)$ we have $1_i^{k'} = \sigma(1_i^m)$).

Proof: Assume **BaCo**. Let $m, k \in \mathit{Obs}$. Let h be a median observer for observers m, k. Let O, T be the sets as in the formulation of the proposition. Let β be as in

(ii) above. We will prove that β is a bijection between O and T. By the definition of β and by Prop.3.8.35, we have that $Dom(\beta) = O$.

By **Ax(ext)** it is easy to check that β is an injection. Thus to prove that β is a bijection between O and T it remains to prove that $Rng(\beta) = T$.

To prove $Rng(\beta) \subseteq T$, let $\sigma \in Rng(\beta)$. Then there is $k' \in O$ such that $\mathsf{f}_{k'h} = \mathsf{f}_{mh} \circ \sigma$. Let this k' be fixed. Then $\sigma = \mathsf{f}_{hm} \circ \mathsf{f}_{k'h}$. By $k' \in O$, we have $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$ and $tr_k(k') = \bar{t}$. So $\sigma^2 = \mathsf{f}_{hm} \circ \mathsf{f}_{k'h} \circ \mathsf{f}_{hm} \circ \mathsf{f}_{k'h} = \mathsf{f}_{hm} \circ \mathsf{f}_{k'm} \circ \mathsf{f}_{k'h} = \mathsf{f}_{hm} \circ \mathsf{f}_{mk'} \circ \mathsf{f}_{k'h} = \mathsf{f}_{hh} = \mathrm{Id}$. By the above computation

(241) $$\sigma^2 = \mathrm{Id}.$$

By $\mathsf{f}_{k'h} = \mathsf{f}_{mh} \circ \sigma$ and by $tr_k(k') = \bar{t}$, we have that

(242) $$\sigma[tr_h(m)] = tr_h(k).$$

$tr_m(h) = tr_{k'}(h)$ holds because of the following. By $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$, we have $tr_m(k') = tr_{k'}(m)$. Since $tr_k(k') = \bar{t}$, we have that h is a median observer for observers m and k'. By this, by $tr_m(k') = tr_{k'}(m)$ and by the proof of Thm.3.8.25 we have $tr_m(h) = tr_{k'}(h)$.

Therefore $\sigma[\bar{t}\,] = (\mathsf{f}_{hm} \circ \mathsf{f}_{k'h})[\bar{t}\,] = \mathsf{f}_{k'h}[tr_m(h)] = \mathsf{f}_{k'h}[tr_{k'}(h)] = \bar{t}$. Thus

(243) $$\sigma[\bar{t}\,] = \bar{t}.$$

$\sigma \in PT$ by Prop.3.8.35. By this, by (241) and (243), it can be checked that σ is an isometry. Since we assumed **BaCo**, we have

(244) $$\sigma \in Triv.$$

By (241)–(244), we have that $\sigma \in T$. Thus $Rng(\beta) \subseteq T$.

To prove $T \subseteq Rng(\beta)$, let $\sigma \in T$. Then, by the definition of T, $\sigma \in Triv$, $\sigma[\bar{t}\,] = \bar{t}$, $\sigma^2 = \mathrm{Id}$ and $\sigma[tr_h(m)] = tr_h(k)$. Consider the transformations $\mathsf{f}_{mh} \circ \sigma$ and f_{kh}. By $\sigma[tr_h(m)] = tr_h(k)$, we have

(245) $$(\mathsf{f}_{mh} \circ \sigma)[\bar{t}\,] = tr_h(k) = \mathsf{f}_{kh}[\bar{t}\,].$$

By Prop.3.8.32 and $\sigma \in T$, we have

(246) $$(\mathsf{f}_{mh} \circ \sigma)(1_t)_t - (\mathsf{f}_{mh} \circ \sigma)(\bar{0})_t = \mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t.$$

By (245), (246) and Lemma 3.8.50, we have that

(247) $$\mathsf{f}_{mh} \circ \sigma = N \circ \mathsf{f}_{kh},$$

for some $N \in Triv$ with $N[\bar{t}] = \bar{t}$. Let this N be fixed. By **Ax(**$Triv_t$**)** there is $k' \in Obs$ such that $\mathsf{f}_{k'k} = N$. Let this k' be fixed. Now, by $\mathsf{f}_{k'k} = N$ and (247) we have

(248) $$\mathsf{f}_{mh} \circ \sigma = \mathsf{f}_{k'h}$$

By (248),

(249) $$\sigma \circ \mathsf{f}_{hk'} = \mathsf{f}_{hm}$$

By (248), (249) and $\sigma^2 = \mathrm{Id}$, we have $\mathsf{f}_{mk'} = \mathsf{f}_{mh} \circ \mathsf{f}_{hk'} = \mathsf{f}_{mh} \circ \sigma \circ \sigma \circ \mathsf{f}_{hk'} = \mathsf{f}_{k'h} \circ \mathsf{f}_{hm} = \mathsf{f}_{k'm}$. So $\mathsf{f}_{mk'} = \mathsf{f}_{k'm}$. By $\mathsf{f}_{k'k} = N$ and $N[\bar{t}] = \bar{t}$, $tr_k(k') = \bar{t}$. Therefore $k' \in O$. Thus, by (248), $\sigma \in Rng(\beta)$. So $T \subseteq Rng(\beta)$. This completes the proof. ∎

3.9.2 A weaker symmetry axiom: Ax(syt)

Recall that we had to introduce **Ax(syt)** for several reasons.[328] First, it was necessary for the careful analysis of our sequence of weak systems of relativity introduced in §4. Second, it will be useful in Chapter 6 (Observer Independent Geometry). The main point in introducing **Ax(syt)** is that it may be a more *adequate* symmetry principle for certain axiom systems of relativity than the rather strong **Ax(**ω**)** or even **Ax(symm)**. It can be considered as a weakening of **Ax(symm)**. We can also conceive of **Ax(syt)** as a stronger version of **Ax(eqtime)**, which in turn is part of **Ax(symm)**.

First we are going to show that **Ax(syt)** is necessary in the sense that using **Ax(symm)** would blur the essential distinction between **Bax** and **Newbasax**. Recall that Thm. 3.4.34 states that $\stackrel{\odot}{\to}$ is an equivalence relation on Obs in models of **Bax**. Thus models of **Bax** fall apart to worlds whose observers see only observers of the same world. The following proposition states that the speed of light is the same for all those observers who see each other, i.e. that belong to the same world.

In a sense, we can interpret Prop. 3.9.37 below as saying that **Ax(symm)** is a *too strong* symmetry principle for someone who "seriously" wants to study **Bax** and its symmetric version, because it blurs the distinction between **Bax** and **Flxbasax** (modulo the assumption $(\forall m, k \in Obs) m \stackrel{\odot}{\to} k$ and the natural auxiliary axiom **Ax(||)**).

PROPOSITION 3.9.37 *If* $n \geq 3$, *then*

$$\mathbf{Bax}(n) + \mathbf{Ax(symm)} + \mathbf{Ax(\|)} \models (\forall m, k \in Obs)(m \stackrel{\odot}{\to} k \to c_m = c_k).$$

[328] Cf. §4.2.

Proof: The basic idea of the proof is as follows. If two observers, say m and k, see each other, but disagree on the speed of light, then there is an observer whose speed for m is the same as the speed of light for k. But symmetry requires that "the way I see you is the way you see me", and using this principle we can derive that k must think that the speed of this observer is the speed of light in m's opinion. But since the speed of light is different for m and k, we get that a faster than light observer exists for at least one of them, which is excluded by $n \geq 3$.

Now let us work out this idea formally. Let $\mathfrak{M}$ be a model of $\mathbf{Bax}(n) + \mathbf{Ax(symm)} + \mathbf{Ax}(\|)$ for some $n \geq 3$. Suppose that there are $m, k \in Obs$ such that $m \xrightarrow{\odot} k$ and $c_k < c_m$. By $\mathbf{Ax(symm_0)}$ there are $m', k' \in Obs$ such that $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$ and $tr_m(m') = tr_k(k') = \bar{t}$.

Claim 3.9.38 $c_m = c_{m'}$ and $c_k = c_{k'}$

We have that $\mathsf{f}_{mm'}$ and $\mathsf{f}_{kk'}$ are isometries by $\mathbf{Ax}(\|)$. The claim follows by the fact that isometries preserve angles. Cf. e.g. [121, p. 64].[329]

Claim 3.9.39 $(\forall \ell \in \mathsf{Eucl})\,[ang^2(\ell) = c_k \rightarrow ang^2(\mathsf{f}_{km}[\ell]) = c_m]$.

We postpone the proof of this claim to the end of this argument.

Let $k^\star$ be such an observer that $m' \xrightarrow{\odot} k^\star$ and $v_{m'}(k^\star) = c_k$ holds. Such an observer exists by $\mathbf{Ax5^{Obs}}$ and $c_k < c_m = c_{m'}$. We get $k' \xrightarrow{\odot} k^\star$ by $m' \xrightarrow{\odot} k^\star, k'$ and $\xrightarrow{\odot}$ being an equivalence relation by Thm. 3.4.34. Let us calculate $v_{k'}(k^\star)$.

$$\begin{aligned}
v_{k'}(k^\star) &= ang^2(tr_{k'}(k^\star)) && \text{by def. of } v_{k'}, \\
&= ang^2(\mathsf{f}_{m'k'}[tr_{m'}(k^\star)]) && \\
&= ang^2(\mathsf{f}_{km}[tr_{m'}(k^\star)]) && \text{by } \mathsf{f}_{mk} = \mathsf{f}_{k'm'} \text{ and Prop. 3.4.35,} \\
&= c_m && \text{by } v_{m'}(k^\star) = c_k \text{ and Claim 3.9.39.}
\end{aligned}$$

Now $v_{k'}(k^\star) = c_m > c_k = c'_k$ by Claim 3.9.38. This contradicts Thm. 3.4.19, which states that there are no FTL observers in models of $\mathbf{Bax}(\mathrm{n})$ if $n \geq 3$.

Proof of Claim 3.9.39: Let $\ell \in \mathsf{Eucl}$ be such that $ang^2(\ell) = c_k$. By $\mathbf{Ax5^{Ph}}$ there is $ph \in Ph$ for which $tr_k(ph) = \ell$. We get $\mathsf{f}_{km}[tr_k(ph)] = tr_m(ph)$. Then $ang^2(\mathsf{f}_{km}[tr_k(ph)]) = ang^2(tr_m(ph)) = c_m$, as required. ∎

COROLLARY 3.9.40 If $n \geq 3$, then

$$\mathbf{Bax}(n) + \mathbf{Ax(symm)} + \mathbf{Ax}(\|) + (\forall m, k \in Obs) m \xrightarrow{\odot} k \models \mathbf{Flxbasax}.$$

[329] Although this fact comes from standard Euclidean geometry, where $\mathbf{Ax}(\sqrt{\ })$ is always assumed, square roots are not essential to prove it.

Corollary 3.9.40 implies that **Bax** + **Ax(symm)** + **Ax(||)** is very close to **Newbasax**, since **Flxbasax** (introduced in §4.1) differs from **Newbasax** only by saying that the speed of light is constant without specifying what the exact value of this constant is. From the theoretical point of view this difference is negligible.

Proposition 3.9.37 also implies that **Bax** + **Ax(symm)** is extremely close to **Flxbasax**, because **Ax(||)** is a quite natural auxiliary axiom, and in **Bax** + **Ax(symm)** + **Ax(||)** all observers inside any fixed observer's world-view think that the speed of light is the same for everyone; therefore, roughly speaking, inside the world-view of any observer **Newbasax** is almost true (except for the precise value of c). Thus we feel that adding **Ax(symm)** to **Bax** removes the *essential* difference between **Bax** and **Newbasax** (or, more precisely, **Flxbasax**). We express this by saying that **Ax(symm)** "blurs" the difference between **Bax** and **Flxbasax**; hence **Ax(symm)** is *not* an *adequate* symmetry principle for **Bax**. In other words, if we want to study the "symmetric version" of **Bax**, and it is important that this should be distinct from **Newbasax** or **Flxbasax**, then we need a more refined, more subtle symmetry principle than **Ax(symm)**.

Remark 3.9.41 Recall that Theorem 3.3.12 characterized models of **Newbasax** in terms of models of **Basax** in the following manner. Any model $\mathfrak{M}$ of **Newbasax** was a "union" of models of **Basax** in the sense that a class $\mathcal{C}$ of **Basax**-models existed such that the observers, photons, inertial bodies, bodies and world-views in $\mathfrak{M}$ were unions of the corresponding entities in models belonging to $\mathcal{C}$; and observers were *not shared* in models in $\mathcal{C}$. Now, we feel that an analogous construction might be found for **Bax** + **Ax(symm)** + **Ax(||)**. That is, models of **Bax**(n) + **Ax(symm)** + **Ax(||)** could be made up as similar "disjoint unions" of models of **Flxbasax** (at least for $n \geq 3$).

Conjecture 3.9.42 **Bax**(n) + **Ax(syt)** $\not\models$ $(m \overset{\odot}{\to} k \to c_m = c_k)$, *for any* $n \geq 2$.

Idea of a possible proof: Let us take a model $\mathfrak{M}$ of **Basax**. Consider the following relation on the sort of observers:

$$m \parallel k \overset{\text{def}}{\Longleftrightarrow} (\exists h \in Obs) tr_h(m) \parallel tr_h(k).$$

We guess that "$\parallel$" is well defined and it is an equivalence relation. Let $f : Obs \to F^+$ be a function such that $m \parallel k \Rightarrow f(m) = f(k)$. Now, let us turn $\mathfrak{M}$ to another model $\mathfrak{N}$ of **Bax** by stretching the meter rods of any observer m by the factor $f(m)$. (The interested reader should formalize this construction.)

Clearly, $\mathfrak{N} \models$ **Ax(syt$_0$)**, because $\mathfrak{M} \models$ **Ax(syt$_0$)** and we did not change the clock of any observer when turning $\mathfrak{M}$ to $\mathfrak{N}$. Nor did we invalidate **Ax(||)** because of our restriction on f. Thus $\mathfrak{N} \models \mathbf{Bax} + \mathbf{Ax(syt)}$, but f can be chosen so that $c_m = c_k$ becomes false.

The reader is invited to check whether this idea is sound. ∎

The rest of this sub-section is devoted to exploring the consequences of $\mathbf{Ax(syt_0)}$. In particular, we shall examine the conditions under which $\mathbf{Ax(syt_0)}$ follows from the previously introduced $\mathbf{Ax(symm)}$, $\mathbf{Ax\triangle 1}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\square 1}$ or $\mathbf{Ax\square 2}$, and vice versa.

PROPOSITION 3.9.43 $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax(syt_0)} \to \mathbf{Ax(eqtime)}$.

Proof: Assume $\mathbf{Bax}^- + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ })$. Let $m, k \in Obs$ be such that $tr_m(k) = \bar{t}$. We have $\mathsf{f}_{mk} \in Aftr$ by Prop. 3.9.50(i). By $\mathbf{Ax(syt_0)}$ one obtains

$$|\mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t| = |\mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t|.$$

Let $\mathsf{f}_{mk} = g \circ \tau$ for some $g \in Linb$ and translation τ. Then

$$|g(1_t)_t| = |g^{-1}(1_t)_t|. \tag{250}$$

We have $g[\bar{t}] = \bar{t}$ by assumption and $g[S] = S$ follows from Cor. 3.8.30. Then $g(1_t)_t = 1/g^{-1}(1_t)_t$, and by (250) we have $|g(1_t)_t| = 1$. Hence g preserves distance between points in $\bar{t}$, and so does $\mathsf{f}_{mk} = g \circ \tau$. ∎

PROPOSITION 3.9.44 $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax(syt_0)} \to \mathbf{Ax}(\|)$.

Proof: Assume $\mathbf{Basax} + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ })$. Let $m, k \in Obs$ be such that $tr_m(k) \parallel \bar{t}$. We have $\mathsf{f}_{mk} \in Aftr$ by Prop. 3.9.50(i). First we shall show that $\mathsf{f}_{mk}, \mathsf{f}_{km}$ preserve the distance between points in $\bar{t}$.

Case 1: $m \uparrow k$. See Figure 117 for the elements of the argument. Let $p \in \bar{t}$ be such that p is simultaneous with m's origin for k. Formally,

$$\mathsf{f}_{mk}(\bar{0})_t = p_t. \tag{251}$$

Such a p exists and is unique. Since f_{km} takes parallel lines to parallel lines,

$$\mathsf{f}_{km}(p)_t = 0 \tag{252}$$

holds. Similarly, let $q \in \bar{t}$ be such that

$$q_t = \mathsf{f}_{km}(p + 1_t)_t. \tag{253}$$

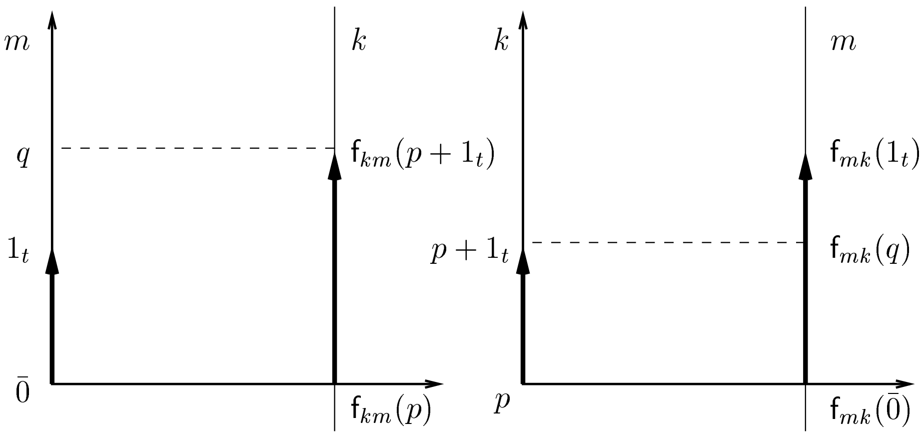


Figure 117: Illustration for the proof of Prop. 3.9.44.

Then

(254) $$\mathsf{f}_{mk}(q)_t = p_t + 1.$$

Now, suppose that f_{km} does not preserve the distance between points in $\bar{t}$. Suppose, e.g. $\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t > 1$. Then $q_t > 1$ by $\mathsf{f}_{km} \in Aftr$.

Axiom $\mathbf{Ax(syt_0)}$, $\mathsf{f}_{km} \in Aftr$ and $m \uparrow k$ imply

(255) $$\mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t = \mathsf{f}_{km}(p+1_t)_t - \mathsf{f}_{km}(p)_t.$$

Consider function f_{mk}. It takes the collinear points $\bar{0}, 1_t, q$ to collinear points in $tr_k(m)$. We have $\mathsf{f}_{mk}(q)_t > \mathsf{f}_{mk}(\bar{0})_t$ by $m \uparrow k$. We claim that $\mathsf{f}_{mk}(q)_t < \mathsf{f}_{mk}(1_t)_t$. Using (251) to (255),

$$\begin{aligned} \mathsf{f}_{mk}(q)_t &= p_t + 1 \\ &< p_t + q_t \\ &= \mathsf{f}_{mk}(\bar{0})_t + \mathsf{f}_{km}(p+1_t)_t - \mathsf{f}_{km}(p)_t \\ &= \mathsf{f}_{mk}(1_t)_t. \end{aligned}$$

Thus f_{mk} does not preserve betweenness. This contradicts $\mathsf{f}_{mk} \in Aftr$.

Case 2: $m \downarrow k$. This case can be treated in a similar fashion. (It is enough to use "$p - 1_t$" whenever we used "$p + 1_t$" above.) We omit the details.

Up to this point we have that proven that $|\mathsf{f}_{mk}(1_t) - \mathsf{f}_{mk}(\bar{0})| = 1$. Compare f_{mk} to a translation τ that takes $\bar{t}$ to $tr_k(m)$. They are both in PT and

$$|\mathsf{f}_{mk}(1_t)_t - \mathsf{f}_{mk}(\bar{0})_t| = 1 = |\tau(1_t)_t - \tau(\bar{0})_t|.$$

Then by Lemma 3.8.50 $\mathsf{f}_{mk} = N \circ \tau$ for some isometry N. Since isometries form a group, f_{mk} is an isometry. ∎

Remark 3.9.45 By contrast, $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) \not\models \mathbf{Ax(syt_0)} \to \mathbf{Ax(\|)}$. The reason for this is contained in the idea explained for Conj. 3.9.42 above.

The following theorem is an analogon to Thm. 2.8.6(iii).

THEOREM 3.9.46

$$\mathbf{Basax} + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ }) \models (\forall m, k \in Obs) v_m(k) = v_k(m).$$

Proof: The proof follows by Prop. 3.9.50 and Thm. 2.8.6(ii). ∎

PROPOSITION 3.9.47 *The following items hold.*

(i) $\mathbf{Basax} \models \mathbf{Ax(symm)} \to \mathbf{Ax(syt_0)}$,

(ii) *If* $n \geq 3$, *then* $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t) \models \mathbf{Ax(syt_0)} \leftrightarrow \mathbf{Ax(symm)}$.

Proof of item (i): Assume $\mathbf{Basax}+\mathbf{Ax(symm)}$. Let $m, k \in Obs$ be arbitrary. By $\mathbf{Ax(symm)}$ we have $m', k' \in Obs$ such that $tr_m(m') = tr_k(k') = \bar{t}$ and $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$. By Prop. 2.3.3(x) we have

$$\mathsf{f}_{mk} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km} \circ \mathsf{f}_{mm'}. \tag{256}$$

Let $p, q \in \bar{t}$ be arbitrary. Let us calculate the following expression:

$$\begin{aligned} |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t| &= |\mathsf{f}_{k'm'}(p)_t - \mathsf{f}_{k'm'}(q)_t| \\ &= |(\mathsf{f}_{k'k} \circ \mathsf{f}_{km} \circ \mathsf{f}_{mm'})(p)_t - (\mathsf{f}_{k'k} \circ \mathsf{f}_{km} \circ \mathsf{f}_{mm'})(q)_t| \\ &= |(\mathsf{f}_{k'k} \circ \mathsf{f}_{km})(p)_t - (\mathsf{f}_{k'k} \circ \mathsf{f}_{km})(q)_t|, \end{aligned}$$

by (256) and $\mathbf{Ax(eqtime)}$ applied for m and m'.

Now Lemma 3.9.21 implies that $\mathsf{f}_{k'k}$ is an isometry and $\mathsf{f}_{k'k}[\bar{t}] = \bar{t}$. Then $(\forall x \in \bar{t})\mathsf{f}_{k'k}(x) = \pm x + t$ for some $t \in \bar{t}$ (i.e., $\mathsf{f}_{k'k}$ might reflect x to the origin and then translate it along $\bar{t}$). On the other hand, we have $\mathsf{f}_{km} = \widetilde{\varphi} \circ g$ for some $\varphi \in Aut(\mathbf{F})$. Then

$$\begin{aligned} |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t| &= |\mathsf{f}_{km}(\pm p + t)_t - \mathsf{f}_{km}(\pm q + t)_t| \\ &= |g(\widetilde{\varphi}(\pm p + t))_t - g(\widetilde{\varphi}(\pm q + t))_t| \\ &= |g(\widetilde{\varphi}(\pm p) + \widetilde{\varphi}(t))_t - g(\widetilde{\varphi}(\pm q) + \widetilde{\varphi}(t))_t| \\ &= |g(\widetilde{\varphi}(\pm p))_t - g(\widetilde{\varphi}(\pm q))_t| \\ &= |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(q)_t|, \end{aligned}$$

as required by $\mathbf{Ax(syt_0)}$.

Proof of item (ii): It is enough to prove the "$\rightarrow$" direction because of item (i). Assume $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax(syt_0)}$ for $n \geq 3$. By Prop. 3.9.51(ii) $\mathbf{Ax\triangle 2}$ holds. Then by Thm. 3.9.27(i) $\mathbf{Ax(symm)}$ holds, too. ∎

COROLLARY 3.9.48 $\mathbf{Basax} \models \mathbf{Ax(symm)} \rightarrow \mathbf{Ax(syt)}$.

Proof: This corollary follows by Prop. 3.9.47 and Prop. 3.9.44. ∎

QUESTION 3.9.49 *Does the conclusion of Prop. 3.9.47(ii) above remain true if we omit* $\mathbf{Ax}(\sqrt{\ })$ *or* $n \geq 3$ *(where the latter is assumed to exclude FTL observers)?*

The following proposition clarifies the conditions under the which world-view transformations f_{mk} are "nice". Recall that by Thm. 2.9.4(i) every world-view transformation $\mathsf{f}_{mk} = \mathit{poi} \circ \mathit{exp} \circ \widetilde{\varphi}$, for some $\mathit{poi} \in \mathit{Poi}$, $\mathit{exp} \in \mathit{Exp}$ and $\varphi \in \mathit{Aut}(\mathfrak{F})$. This may look disappointing, since while poi is intuitively expected and exp is understood as responsible for asymmetries, $\widetilde{\varphi}$ can hardly be assigned any physical meaning. Having tried to eliminate the expansion exp by some symmetry axiom, we still have to make automorphism φ vanish by additional postulates if need be. In this context the reader should compare Prop. 3.9.50 to Thm. 2.9.5 and Cor's 3.9.28 and 3.9.30.

PROPOSITION 3.9.50 *The following items hold.*

(i) $\mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)} \models \mathsf{f}_{mk} \in \mathit{Aftr}$.

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)} \models \mathsf{f}_{mk} \in \mathit{Poi}$.

Proof of item (i): Assume $\mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)}$. Let $m, k \in \mathit{Obs}$. Then we have

$$\mathsf{f}_{mk} = \widetilde{\varphi} \circ A, \tag{257}$$

for some $A \in \mathit{Aftr}$ and $\varphi \in \mathit{Aut}(\mathfrak{F})$ by Thm. 4.3.11 and Lemma 3.1.6. By Prop. 2.3.3(x) we have

$$\mathsf{f}_{km} = (\widetilde{\varphi} \circ A)^{-1} = A^{-1} \circ \widetilde{\varphi}^{-1} = \widetilde{\varphi}^{-1} \circ B, \tag{258}$$

for some $B \in \mathit{Aftr}$.

In order not to lose the actual insight, we are going to describe the idea of the proof informally first. Throughout this explanation and the formal proof consult Figure 118. Our field $\mathfrak{F}$ can be identified with the time axis of either observer. The

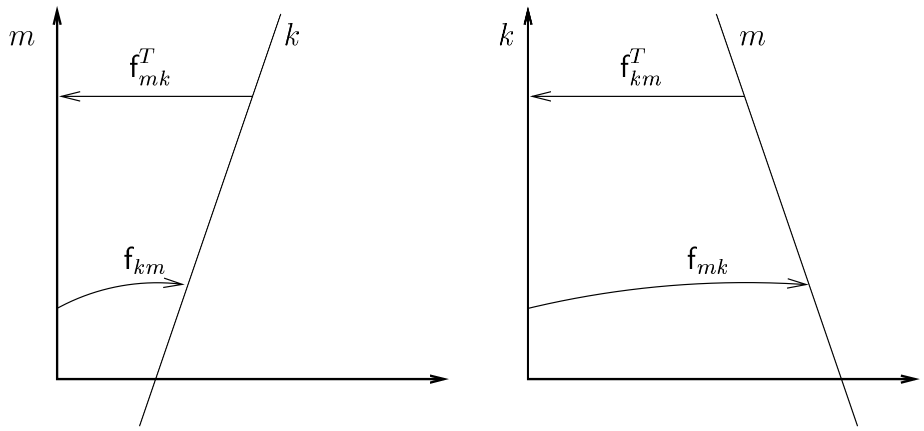


Figure 118: Illustration for the proof of Prop. 3.9.50.

function f_{km} maps k's axis to $tr_m(k)$, and, similarly, f_{mk} maps m's axis to $tr_k(m)$. Thus observer m can "observe" the field, mapped onto itself by φ^{-1}, by measuring how k's clock ticks. Analogously, k observes the field, identified by m's time axis, mapped onto itself by φ. But $\mathbf{Ax(syt_0)}$ says that the way m can see k's clock tick is indistinguishable from the way k can see m's clock tick, which, in turn, implies that φ^{-1} cannot differ from φ. (If φ moves a (non-rational) point x, then φ^{-1} must move it similarly, $\varphi(x) = \varphi^{-1}(x)$.) Since there cannot be nontrivial automorphisms of finite order in an *ordered* field, $\varphi^2 = \mathrm{Id}$ implies $\varphi = \mathrm{Id}$.

Now let us work out this idea formally. Let us define $\mathsf{f}^T_{mk}, \mathsf{f}^T_{km} : F \to F$ as follows:

$$\begin{aligned} \mathsf{f}^T_{mk}(x) &= \mathsf{f}_{mk}(x1_t)_t, \qquad \text{and} \\ \mathsf{f}^T_{km}(x) &= \mathsf{f}_{km}(x1_t)_t. \end{aligned}$$

By (257) it is straightforward to check that f^T_{mk} is a composition of φ and a linear function, i.e., there are $a, b \in F$ such that

$$\mathsf{f}^T_{mk}(x) = a\varphi(x) + b. \tag{259}$$

Similarly, by (257) there are $c, d \in F$ such that

$$\mathsf{f}^T_{km}(x) = c\varphi(x) + d. \tag{260}$$

By $\mathbf{Ax(syt_0)}$ we have

$$(\forall x \in F)|\mathsf{f}^T_{mk}(x) - \mathsf{f}^T_{mk}(0)| = |\mathsf{f}^T_{km}(x) - \mathsf{f}^T_{km}(0)|. \tag{261}$$

Now, by (259), (260) and (261) we get

$$(\forall x \in F)|a||\varphi(x)| = |b||\varphi^{-1}(x)|.$$

Since $\varphi(1) = 1$, we have $|a| = |b|$. Therefore $(\forall x \in F)|\varphi(x)| = |\varphi^{-1}(x)|$. Since φ is order-preserving, this implies $(\forall x \in F)\varphi(x) = \varphi^{-1}(x)$. Hence $\varphi^2 = \mathrm{Id}$. Thus $\varphi = \mathrm{Id}$ by φ preserving order.

Proof of item (ii): Assume **Basax** + $\mathbf{Ax(syt_0)}$ + $\mathbf{Ax}(\sqrt{\ })$. Let $m, k \in Obs$ be arbitrary. We have $\mathsf{f}_{mk} = poi \circ exp \circ \widetilde{\varphi}$, for some $poi \in Poi$, $exp \in Exp$ and $\varphi \in Aut(\mathfrak{F})$ by Thm. 2.9.4(i). On the other hand, $\mathsf{f}_{mk} \in Aftr$ by item (i). Then $\widetilde{\varphi} \in Aftr$ and $\varphi = \mathrm{Id}$ by Lemma 3.8.36. To sum up, we have

$$\mathsf{f}_{mk} = poi \circ exp. \tag{262}$$

Let exp be an expansion by $\lambda > 0$, i.e. $(\forall p \in {}^nF)exp(p) = \lambda p$. Let $p, q \in \bar{t}$ be arbitrary. By $\mathbf{Ax(syt_0)}$ and (262) we get

$$\begin{aligned} |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t| &= |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(q)_t|, \\ \lambda|poi(p)_t - poi(q)_t| &= (1/\lambda)|poi^{-1}(p)_t - poi^{-1}(q)_t|. \end{aligned} \tag{263}$$

We intend to prove $\lambda = 1$.

We know that $poi, poi^{-1} \in Poi$ preserve the square of Minkowski-distance, where the latter notion was introduced in Def. 2.9.1. Introducing the following abbreviations,

$$\begin{aligned} T_{poi} &\stackrel{\text{def}}{=} (poi(p)_t - poi(q)_t)^2, \\ S_{poi} &\stackrel{\text{def}}{=} \sum_{i=1}^{n-1}(poi(p)_i - poi(q)_i)^2, \end{aligned}$$

and $T_{poi^{-1}}, S_{poi^{-1}}$ similarly, the mentioned property of poi, poi^{-1} implies

$$|T_{poi} - S_{poi}| = |p - q| = |T_{poi^{-1}} - S_{poi^{-1}}|. \tag{264}$$

On the other hand, by Thm. 3.9.46 we have

$$\frac{S_{poi}}{T_{poi}} = v_k(m) = v_m(k) = \frac{S_{poi^{-1}}}{T_{poi^{-1}}}. \tag{265}$$

A simple calculation from (264) and (265) results in

$$T_{poi} = T_{poi^{-1}}.$$

Comparing this with (263) one gets $\lambda = 1$, $exp = \mathrm{Id}$. Therefore $\mathsf{f}_{mk} = poi \in Poi$. ∎

Having shown the previous crucial proposition, we shall deal with the relationship between $\mathbf{Ax(syt_0)}$ and $\mathbf{Ax}(\omega)$. Prop. 3.9.50 will be exploited.

PROPOSITION 3.9.51 *The following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax\triangle 2} \to \mathbf{Ax(syt_0)}$,

(ii) *If* $n \geq 3$, *then* $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax\triangle 2} \leftrightarrow \mathbf{Ax(syt_0)}$.[330]

To prove Prop. 3.9.51(ii), one needs the following lemma.

LEMMA 3.9.52

$$\mathbf{Basax} + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ }) \models (h \textit{ is a median observer for } m, k) \to |\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|.$$

Proof: The proof is very close to that of Lemma 3.9.16. Note that the essential step in Case 1 of the proof of Lemma 3.9.16 was reached via $\mathbf{Ax(syt_0)}$: the premises of Lemma 3.9.16 implied $\mathbf{Ax(syt_0)}$ by Prop. 3.9.51(i), and (201) followed by $\mathbf{Ax(syt_0)}$. Indeed, Case 1 and Case 2 can be repeated using $\mathbf{Ax(syt_0)}$ directly, otherwise unchanged. We shall omit this part of the proof.

Now, for the general case, assume $\mathbf{Basax} + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ })$. Let $m, k \in Obs$ be arbitrary. By **Ax5** there is an observer k' such that $tr_h(m) \cap tr_h(k) \neq \emptyset$. Let $p \in tr_h(m) \cap tr_h(k)$. Again by **Ax5** there is a $h' \in Obs$ such that $p \in tr_h(h') \parallel \bar{t}$.

It is easy to check that h' is a median observer for m and k'. Using Case 2 for m, k', h' we get

(266) $$|\mathsf{f}_{mh'}(1_t)_t - \mathsf{f}_{mh'}(\bar{0})_t| = |\mathsf{f}_{k'h'}(1_t)_t - \mathsf{f}_{k'h'}(\bar{0})_t|.$$

On the other hand, $\mathbf{Ax(\|)}$ holds by Prop. 3.9.44. Then $\mathsf{f}_{h'h}$ is an isometry and, of course, $\mathsf{f}_{h'h}\bar{t} \parallel \bar{t}$. This yields

(267) $$|\mathsf{f}_{mh'}(1_t)_t - \mathsf{f}_{mh'}(\bar{0})_t| = |\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t|,$$

(268) $$|\mathsf{f}_{kh'}(1_t)_t - \mathsf{f}_{kh'}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|.$$

Combining items (266) to (268) yields the proof of the general case. ∎

[330] The condition $n \geq 3$ is necessary because, assuming $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$, $\mathbf{Ax\triangle 2}$ excludes the existence of faster than light observers, while $\mathbf{Ax(syt_0)}$ does not. Cf. Thm. 3.9.8 items (ii) and (iii).

Proof of Prop. 3.9.51(i): Assume $\mathbf{Basax} + \mathbf{Ax}\triangle\mathbf{2} + \mathbf{Ax}(\sqrt{\ })$. We have to show $\mathbf{Ax(syt_0)}$.

Let $m, k \in Obs$ be arbitrary. By $\mathbf{Ax}\triangle\mathbf{2}$ there is an isometry N such that $\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N$ and $N[\bar{t}] \parallel \bar{t}$. Lemma 3.9.5 entails $N \in Aftr$. Let $N = A \circ \tau_a$ for some $A \in Linb, a \in {}^nF$.

Let $p, q \in \bar{t}$. Then

$$\begin{aligned}
|\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t| &= |N(\mathsf{f}_{km}(N(p)))_t - N(\mathsf{f}_{km}(N(q)))_t| \\
&= |\mathsf{f}_{km}(N(p))_t - \mathsf{f}_{km}(N(q))_t|
\end{aligned}$$

because $N[\bar{t}] \parallel \bar{t}$ and N preserves distance.

We have $\mathsf{f}_{km} \in Aftr$ by Lemma 3.9.9. Let $\mathsf{f}_{km} = B \circ \tau_b$ for some $B \in Linb, b \in {}^nF$. Then

$$\begin{aligned}
|\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(q)_t| = |\mathsf{f}_{km}(N(p))_t - \mathsf{f}_{km}(N(q))_t| &= |B(N(p))_t - B(N(q))_t| \\
&= |B(\pm p + a)_t - B(\pm q + a)_t| \\
&= |B(p)_t - B(q)_t| \\
&= |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(q)_t|,
\end{aligned}$$

as required by $\mathbf{Ax(syt_0)}$.

Proof of Prop. 3.9.51(ii): Assume $\mathbf{Basax}(n) + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ })$ for $n \geq 3$. We have to show that $\mathbf{Ax}\triangle\mathbf{2}$ is implied.

Let $m, k \in Obs$ be arbitrary. Faster than light observers are excluded by $n \geq 3$ and Thm. 3.4.1. Then by Thm. 3.8.25 there is a median observer h for m and k. Thus we have

(269) $$tr_h(k) = \sigma_{\bar{t}}[tr_h(m)].$$

On the other hand, by Lemma 3.9.52 we have that h sees the clocks of m and k tick with the same rate:

(270) $$|\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t| = |\mathsf{f}_{kh}(1_t)_t - \mathsf{f}_{kh}(\bar{0})_t|.$$

By Prop. 3.9.50 $\mathsf{f}_{mh}, \mathsf{f}_{kh}, \mathsf{f}_{mk} \in PT$.

Case 1: $m \uparrow k$. Consider transformations f_{mh} and $\mathsf{f}_{kh} \circ \sigma_{\bar{t}}$. Both map $\bar{t}$ axis to the same line, $tr_h(m)$ by Prop. 2.3.3 and (269). Since $m \uparrow k$, $\sigma_{\bar{t}} \restriction \bar{t} = \mathrm{Id}$, (270) implies

(271) $$\mathsf{f}_{mh}(1_t)_t - \mathsf{f}_{mh}(\bar{0})_t = (\mathsf{f}_{kh} \circ \sigma_{\bar{t}})(1_t)_t - (\mathsf{f}_{kh} \circ \sigma_{\bar{t}})(\bar{0})_t.$$

Hence by Lemma 3.8.50 there is a $N \in Triv$ such that

(272) $$N \circ \mathsf{f}_{mh} = \mathsf{f}_{kh} \circ \sigma_{\bar{t}}.$$

From this point the proof follows the same steps as that of Prop. 3.9.13 from (210).

Case 2: $m \downarrow k$. This case is similar to Case 1, but we should consider $\mathsf{f}_{kh} \circ \sigma_\ell$ instead of $\mathsf{f}_{kh} \circ \sigma_{\bar{\ell}}$, where ℓ is the line used in the proof of Prop. 3.9.13 Case 2, defined on p. 367. (Transformation σ_ℓ reverses the arrow of time, thereby ensuring that a relation analogous to (271) holds.) The reader is invited to fill in the details. ∎

For brevity, we shall only collect the implications of the previously shown propositions regarding the connection of **Ax(syt$_0$)** with the rest of **Ax(ω)** (i.e. **Ax□1**, **Ax□2** and **Ax△1**). In some cases the relationship could be asserted more sharply. The interested reader is advised to explore the details herself/himself.

COROLLARY 3.9.53 Assume $n \geq 3$. Then the following items hold.

(i) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models \mathbf{Ax\Box 2} \leftrightarrow \mathbf{Ax}(\mathbf{syt_0})$,

(ii) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax5}^+ + \mathbf{Ax(eqtime)} \models \mathbf{Ax\Box 1} \leftrightarrow \mathbf{Ax}(\mathbf{syt_0})$,

(iii) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax(eqtime)} \models \mathbf{Ax\triangle 1} \leftrightarrow \mathbf{Ax}(\mathbf{syt_0})$.

Proof: Direction "$\to$" of item (i) follows by Thm. 3.9.29(i) and Prop. 3.9.47(i); direction "$\leftarrow$" follows by Prop. 3.9.47(ii) and Thm. 3.9.29(ii).

Direction "$\to$" of item (ii) follows by Thm. 3.9.31(i) and Prop. 3.9.47(i); direction "$\leftarrow$" follows by Prop. 3.9.47(ii) and Thm. 3.9.31(ii).

Direction "$\to$" of item (iii) follows by Thm. 3.9.26(i) and Prop. 3.9.47(i); direction "$\leftarrow$" follows by Prop. 3.9.47(ii) and Thm. 3.9.26(ii). ∎

3.9.3 Connection between Ax(symm) and Ax(eqm)

We are going introduce **Ax(eqm)**, a fairly strong symmetry axiom that will be useful in Chapter 6, devoted to observer independent geometry. In the present context we only aim at discussing the intuitive meaning of **Ax(eqm)** and its relation to the other symmetry principles.

Ax(eqm) $(\forall m, k \in \mathit{Obs})(\forall i, j \in n)(\forall p \in \bar{x}_i)(\forall q \in \bar{x}_j)$
$[((w_m(\bar{0}) = w_k(\bar{0}) \quad \wedge \quad w_m(p) = w_k(q)) \to |p_i| = |q_j|)]$.[331]

[331] Of course, the quantification $(\forall i, j \in n)$ can be substituted by a disjunction, thus it does not violate first order logic.

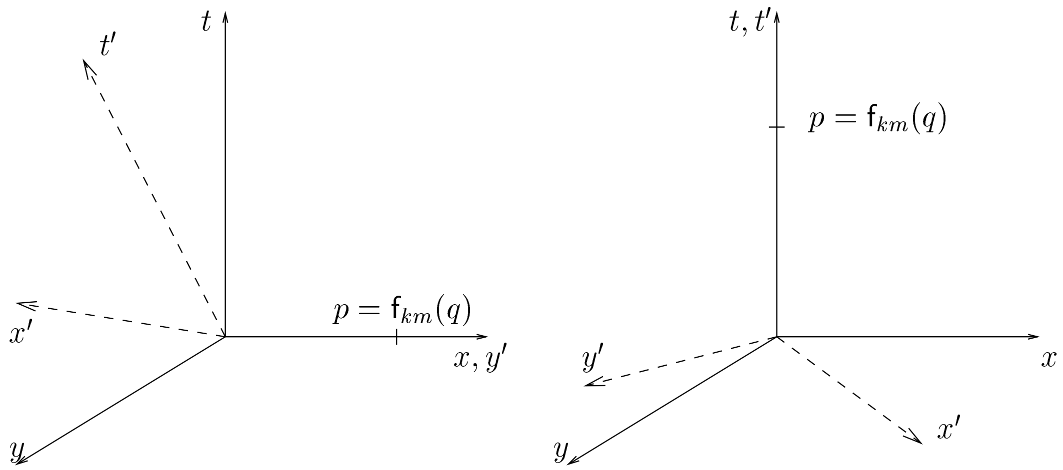


Figure 119: Illustration for **Ax(eqm)**.

Intuitively, **Ax(eqm)** states that observers agree on distances. More precisely, observers whose origin coincides agree on distances measured on the coordinate axes. See Figure 119.

Next we state some propositions and conjectures concerning the connections between **Ax(eqm)** and other symmetry axioms. Note that **Ax(Gal)** is an auxiliary axiom introduced in Chapter 6, in the spirit of discussion of geometry.

PROPOSITION 3.9.54 *If $n \geq 3$, then*

$$\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax(eqm)} \models \mathbf{Ax}(\omega^{-}).$$

Proof: We shall fill in the proof later. ∎

QUESTION 3.9.55 *Exactly which parts of* $\mathbf{Ax}(\omega^{-})$ *follow from the assumptions* $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax(eqm)}$ *and $n \geq 3$?*

In connection with this question we conjecture the following.

Conjecture 3.9.56 *If $n \geq 3$ then the following items hold.*

(i) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax(eqm)} \models \mathbf{Ax\Box 2} \wedge \mathbf{Ax\triangle 2}$,

(ii) $\mathbf{Basax}(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax(eqm)} \not\models \mathbf{Ax\Box 1} \vee \mathbf{Ax\triangle 1}$.

PROPOSITION 3.9.57 $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\omega) \models \mathbf{Ax(eqm)}$.

Proof: We shall fill in the proof later. ∎

QUESTION 3.9.58 *How much of* $\mathbf{Ax}(\omega)$ *is needed for the conclusion of the previous proposition?*

In connection with this question we state Proposition 3.9.59 and Conjectures 3.9.60 and 3.9.61.

PROPOSITION 3.9.59 *The following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax}\triangle\mathbf{1} \not\models \mathbf{Ax(eqm)}$,

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax}\square\mathbf{1} \not\models \mathbf{Ax(eqm)}$.

Proof: We shall fill in the proof later. ∎

Conjecture 3.9.60 *The following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax}\triangle\mathbf{2} \models \mathbf{Ax(eqm)}$,

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax}\square\mathbf{2} \models \mathbf{Ax(eqm)}$.

Conjecture 3.9.61 *We suppose that the following items hold.*

(i) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax}\triangle\mathbf{1} + \mathbf{Ax}(\|) \models \mathbf{Ax(eqm)}$,

(ii) $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(Gal)} + \mathbf{Ax}\square\mathbf{1} + \mathbf{Ax}(\|) \models \mathbf{Ax(eqm)}$.

3.9.4 Isotropy

The claim that no spatial direction is distinguished from the others, usually referred to as the *isotropy of space*, should also be considered as a (yet unformalized) instance of SPR. We may expect further insight from investigating to which extent our distinguished axiom systems, from $\mathbf{Bax}^-$ to $\mathbf{Basax}$, comply with this claim.

First we give a rather strong formalization of isotropy:

Ax(isotropy) $(\forall m, k, k_1 \in Obs)\Big[v_m(k) = v_m(k_1) \rightarrow$

$(\exists k_2 \in Obs)\Big(tr_m(k_1) = tr_m(k_2) \quad \wedge \quad$ [for all isometry h on ${}^n\mathbf{F}$

$((\forall p \in \bar{t})h(p) = p \quad \wedge \quad h[tr_m(k)] = tr_m(k_1)) \rightarrow h \circ \mathsf{f}_{mk} = \mathsf{f}_{mk_2} \circ h]\Big)\Big]$.

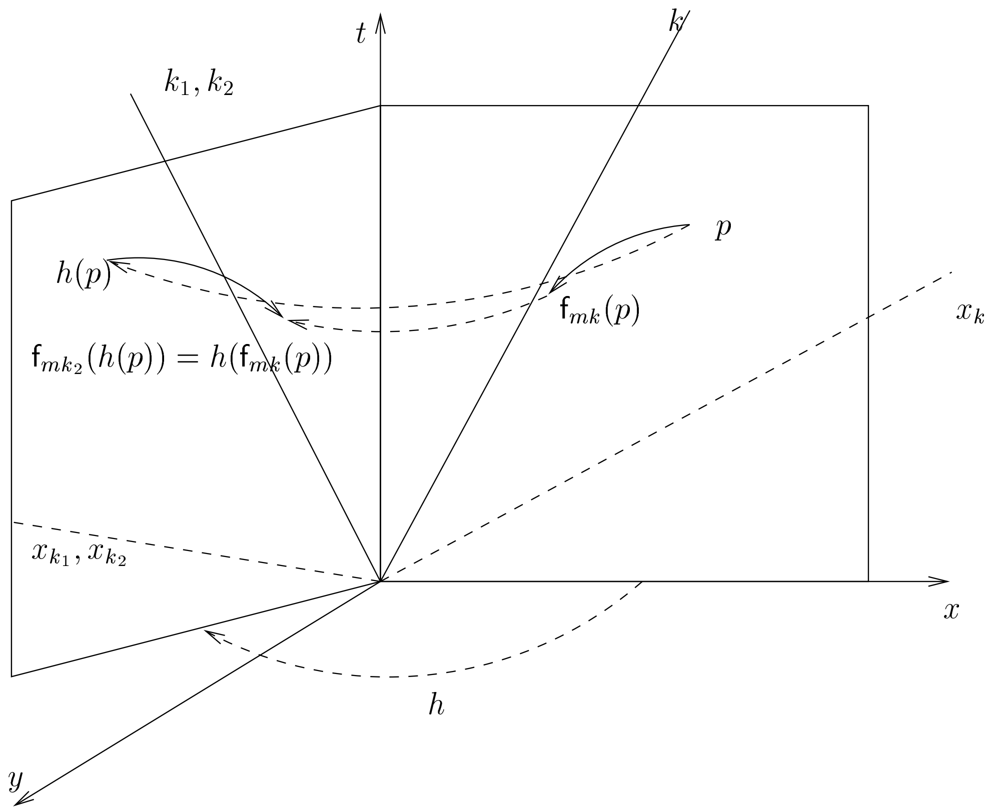


Figure 120: Illustration for the definition of **Ax(isotropy)**.

For an illustration of the idea behind **Ax(isotropy)** the reader is asked to consult Figure 120. Intuitively, **Ax(isotropy)** compares the two directions in which observers k and k_1, respectively, move in the world-view of an observer m. They move with the same speed relative to m. This implies that there is a congruence transformation that takes $tr_m(k)$ to $tr_m(k_1)$. Now, **Ax(isotropy)** aims to say that the way k and k_1 see events differs only by this transformation. This would be, however, too strong: Observer k_1 might "look in the wrong direction" thereby spoiling the simple relationship of world-views. That's the reason why we take a brother k_2 of k_1 and relate f_{mk} to f_{mk_2}.

Remark 3.9.62 Note that **Ax(isotropy)** could have been formulated more concisely:

Ax(isotropy') $(\forall m, k, k_1 \in Obs)(\exists k_2 \in Obs)$

$$\Big(tr_m(k_1) = tr_m(k_2) \quad \land \quad \text{[for all isometry } h \text{ on } {}^n\mathbf{F}$$
$$((\forall p \in \bar{t})h(p) = p \quad \land \quad h[tr_m(k)] = tr_m(k_1)) \to h \circ \mathsf{f}_{mk} = \mathsf{f}_{mk_2} \circ h]\Big).$$

We omitted the condition "$v_m(k) = v_m(k_1)$". If this fails, h is chosen from the empty set. We preferred the original form of **Ax(isotropy)** because it might be understood more easily.

An apparently weaker isotropy principle is the following:

Ax(isotropy$^-$) $(\forall m, k \in Obs)(\exists k' \in Obs)\Big(tr_m(k) = tr_m(k') \quad \land$

$$\Big[\text{ for all isometry } h \text{ on } {}^n\mathbf{F} \ (\forall p \in \bar{t} \cup tr_m(k))h(p) = p \to h \circ \mathsf{f}_{mk'} = \mathsf{f}_{mk'} \circ h\Big]\Big).$$

We feel this axiom weaker because it states isotropy only in a special configuration. It might be true even in the Reichenbachian versions of our theories. The following proposition and conjecture elaborate the connection of **Ax(isotropy)** and **Ax(isotropy$^-$)**.

THEOREM 3.9.63 *Let $\mathfrak{M}^M_{\mathfrak{F}}$ be a Minkowski model associated to an Euclidean ordered field $\mathfrak{F}$, as specified in Def. 3.8.42. Then the following items hold.*

(i) $\mathfrak{M}^M_{\mathfrak{F}} \models$ **Ax(isotropy$^-$)**,

(ii) $\mathfrak{M}^M_{\mathfrak{F}} \models$ **Ax(isotropy)**.

Proof: Item (i) follows from Thm. 3.9.64(i) below. Checking item (ii) is left to the reader.[332] ∎

THEOREM 3.9.64 *Assume* $\mathfrak{M} \models$ **Basax** + **Ax(*Triv*)** + **Ax($\sqrt{\ }$)**. *Assume*

(i) either $(\forall m, k \in Obs)\mathsf{f}_{mk} \in Aftr$,

(ii) or that $\mathfrak{F}^{\mathfrak{M}}$ has no nontrivial automorphism.

Then $\mathfrak{M} \models$ **Ax(isotropy$^-$)**.

[332] We are under the impression that Attila Andai has actually checked item (ii).

We note that only the case $n > 3$ is nontrivial. Note that item (i) is in first order logic, while item (ii) is not. Item (i) could be written as $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }) + \mathsf{f}_{mk} \in \mathit{Aftr} \models \mathbf{Ax(isotropy^-)}$.

Proof: Assume $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ })$. Let $m, k \in \mathit{Obs}$. In both cases (items (i) and (ii)) we have $\mathsf{f}_{mk} \in \mathit{Aftr}$. We are considering the case of standard configuration first. We shall deal with the general case later on.

Case 1: Assume m and k are in standard configuration: $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$, $tr_m(k), tr_k(m) \in \mathsf{Plane}(\bar{t}, \bar{x})$. Let $h : {}^nF \to {}^nF$ be an isometry with $(\forall p \in \bar{t} \cup tr_m(k))h(p) = p$. Then $(\forall p \in \mathsf{Plane}(\bar{t}, \bar{x}))h(p) = p$ follows by $h \in \mathit{Aftr}$. It is enough to show that $h \circ \mathsf{f}_{mk} = \mathsf{f}_{mk} \circ h$.

Claim 3.9.65 Let $e \in \mathit{Exp}$. Then $h \circ e = e \circ h$.[333]

We omit the trivial proof. (Claim 3.9.65) ∎

Claim 3.9.66 Let $\mathit{slor} \in \mathit{SLor}$. Then $h \circ \mathit{slor} = \mathit{slor} \circ h$.

This claim follows by the fact that slor "has no effect" in directions orthogonal to $\mathsf{Plane}(\bar{t}, \bar{x})$, while $h \restriction \mathsf{Plane}(\bar{t}, \bar{x}) = \mathrm{Id}$. Formally, let $p \in {}^nF$. Let us choose $p', p'' \in {}^nF$ so that $p' \in \mathsf{Plane}(\bar{t}, \bar{x})$, $p'' \perp \mathsf{Plane}(\bar{t}, \bar{x})$ and $p = p' + p''$. Then

$$\begin{aligned} \mathit{slor}(h(p)) &= \mathit{slor}(h(p' + p'')) = \mathit{slor}(p' + h(p'')) = \mathit{slor}(p') + h(p'') \\ &= h(p'' + \mathit{slor}(p')) = h(\mathit{slor}(p'' + p')) = h(\mathit{slor}(p)). \end{aligned}$$

(Claim 3.9.66) ∎

Now, $\mathsf{f}_{mk} = \mathit{slor} \circ \mathit{exp}$, for some $\mathit{slor} \in \mathit{SLor}$ and $\mathit{exp} \in \mathit{Exp}$ by Thm. 2.9.4(ii), m and k being in standard configuration, and because of $\mathsf{f}_{mk} \in \mathit{Aftr}$. Then

$$\begin{aligned} h \circ \mathsf{f}_{mk} &= h \circ \mathit{slor} \circ \mathit{exp} \\ &= \mathit{slor} \circ h \circ \mathit{exp} \\ &= \mathit{slor} \circ \mathit{exp} \circ h = \mathsf{f}_{mk} \circ h, \end{aligned}$$

using Claims 3.9.66 and 3.9.65 in turn. Note that we did not have to choose another observer to replace k.

Case 2: Let m, k be arbitrary (i.e., they are not necessarily in standard configuration). By $\mathbf{Ax}(\mathit{Triv})$ and $\mathbf{Ax}(\sqrt{\ })$, there are $m', k' \in \mathit{Obs}$ such that m' and k'

[333] This remains true even if e is an affine transformation effecting different expansions in the time direction and in spatial directions. I.e., $e(1_t) = \lambda_0 1_t$, $(\forall 0 < i < n)e(e_i) = \lambda_1 e_i$, $\lambda_0 \neq \lambda_1$. This may be relevant for **Bax**.

are in standard configuration, and $\mathsf{f}_{mm'}, \mathsf{f}_{kk'} \in \mathit{Triv}$. (By $\mathbf{Ax}(\mathit{Triv})$, m and k can be translated in order to bring their origin to the same point; by $\mathbf{Ax}(\mathit{Triv})$ and $\mathbf{Ax}(\sqrt{\ })$ their world-view can be rotated so that they see one another move in $\mathsf{Plane}(\bar{t}, \bar{x})$.)

Again, $\mathsf{f}_{m'k'} = \mathit{slor} \circ \mathit{exp}$, for some $\mathit{slor} \in \mathit{SLor}$ and $\mathit{exp} \in \mathit{Exp}$. By Prop. 2.3.3(x) we have

$$\mathsf{f}_{mk} = \mathsf{f}_{mm'} \circ \mathsf{f}_{m'k'} \circ \mathsf{f}_{k'k}.$$

Now,

$$\begin{aligned} h \circ \mathsf{f}_{mk} &= h \circ \mathsf{f}_{mm'} \circ \mathsf{f}_{m'k'} \circ \mathsf{f}_{k'k} \\ &= \mathsf{f}_{mm'} \circ h' \circ \mathsf{f}_{m'k'} \circ \mathsf{f}_{k'k} \\ &= \mathsf{f}_{mm'} \circ \mathsf{f}_{m'k'} \circ h' \circ \mathsf{f}_{k'k}, \end{aligned} \tag{273}$$

for some h' isometry such that $(\forall p \in \bar{t} \cup \bar{x}) h'(p) = p \wedge h \circ \mathsf{f}_{mm'} = \mathsf{f}_{mm'} \circ h'$. (It can be checked that such an h' exists.) In the second step we used Case 1.

Our next aim is to find an observer k'' for which $h' \circ \mathsf{f}_{k'k''} = \mathsf{f}_{k'k''} \circ h$. Recall that

$$h \circ \mathsf{f}_{mm'} = \mathsf{f}_{mm'} \circ h'.$$

Then, by the bijectivity of all the functions involved,

$$h' \circ \mathsf{f}_{m'm} = \mathsf{f}_{m'm} \circ h.$$

Now, by $\mathbf{Ax}(\mathit{Triv})$ there is $k'' \in \mathit{Obs}$ such that $\mathsf{f}_{k'k''} = \mathsf{f}_{m'm}$. Hence

$$h' \circ \mathsf{f}_{k'k''} = \mathsf{f}_{k'k''} \circ h.$$

Recall that by (273) we have $h \circ \mathsf{f}_{mk} = \mathsf{f}_{mk'} \circ h' \circ \mathsf{f}_{k'k}$. Then

$$\begin{aligned} h \circ \mathsf{f}_{mk''} &= h \circ \mathsf{f}_{mk} \circ \mathsf{f}_{kk''} \\ &= \mathsf{f}_{mk'} \circ h' \circ \mathsf{f}_{k'k} \circ \mathsf{f}_{kk''} \\ &= \mathsf{f}_{mk'} \circ h' \circ \mathsf{f}_{k'k''} \\ &= \mathsf{f}_{mk'} \circ \mathsf{f}_{k'k''} \circ h \\ &= \mathsf{f}_{mk''} \circ h, \end{aligned}$$

as required. ∎

Conjecture 3.9.67 *Statements analogous to Thm. 3.9.64 above can be formulated for* **Bax** *and* **Reich**(**Basax**). *More precisely, assume that $\mathfrak{F}$ has no nontrivial automorphism. Then the following items should hold.*

(i) If $\mathfrak{M} \models$ **Bax**$(\mathfrak{F})$ + **Ax(***Triv***)** + **Ax**$(\sqrt{\ })$, *then* $\mathfrak{M} \models$ **Ax(isotropy$^-$)**.

(ii) If $\mathfrak{M} \models$ **Reich**(**Basax**)$(\mathfrak{F})$+**Ax(***Triv***)**+**Ax**$(\sqrt{\ })$, *then* $\mathfrak{M} \models$ **Ax(isotropy$^-$)**.

Idea for the proof of item (i): The proof of Theorem 3.9.64 should go through in **Bax**, too. Cf. footnote 333. ∎

Questions for future research 3.9.68

1. It would be interesting to know what the logical relationship between **Ax(isotropy)** and **Ax(isotropy$^-$)** is.[334] One's first intuitive impression is that **Ax(isotropy$^-$)** should be weaker. Indeed, Thm. 3.9.64 seems to point in the direction that **Ax(isotropy$^-$)** is a relatively week assumption.

2. Under what set Th of axioms is Th + **Ax(isotropy)** $\models$ **Ax(isotropy$^-$)** true? (Certainly Th = **Basax**+**Ax**$(\sqrt{\ })$+**Ax(***Triv***)** is enough, but this is a vacuous truth, because for this choice of Th we have $Th \models$ **Ax(isotropy$^-$)**. Hence we are interested in such a Th that is strictly weaker than **Basax** + **Ax**$(\sqrt{\ })$ + **Ax(***Triv***)**.) A probable candidate might be **Bax**$^-$ + $c_m < \infty$ + **Ax**$(\sqrt{\ })$.

As usual with axioms expressing instances of SPR, it is interesting which distinctions among weak theories are blurred by adding them to the theories in question. In this connection we have the following expectations.

Conjecture 3.9.69 *The following items hold.*

(i) **Bax**$^-$ + **Ax(isotropy)**+ *some natural auxiliary axioms* $\models$ **Bax**,

(ii) **Reich**(**Bax**) + **Ax(isotropy)**+ *some natural auxiliary axioms* $\models$ **Bax**,

(iii) **Ax(isotropy)** *does not make* **Bax** *significantly*[335] *stronger,*

(iv) **Reich**(**Bax**) + **Ax(isotropy$^-$)** *is essentially*[336] *weaker than* **Bax**.

We did not have enough time to check these conjectures. The relationship between **Ax(isotropy)** and our collection of symmetry axioms might also be interesting. The reason why we have not used isotropy assumptions in this study is that they were not necessary to prove our main theorems.

[334] They are *both* true in standard Minkowski models, therefore standard relativity theory does not establish any connection between them.

[335] In the sense that the main pecularity of **Bax**, namely that observers can disagree about the speed of light, remains possible.

[336] By this we mean that the real point in **Reich**(**Bax**) is not affected: simultaneity remains arbitrary.

Remark 3.9.70 An alternative approach to isotropy would be the following. Let some observer k move in direction $\bar{x}$ in the world-view of observer m. Formally, $tr_m(k) \in \mathsf{Plane}(\bar{t}, \bar{x})$ and $v_m(k) \neq 0$. Let $h : {}^{n-1}F \to {}^{n-1}F$ be a rotation that leaves $\bar{x}$ pointwise fixed (the class of rotations can easily be defined in first order logic). Moreover, let $\widetilde{h}$ be the natural extension of h from ${}^{n-1}F$ to ${}^{n}F$ (if $p \in {}^{n}F$, then $\widetilde{h}(p) = \{p_0\} \times h(\langle p_1, p_2, \ldots p_{n-1}\rangle))$. Then $\widetilde{h}$ commutes with f_{mk}, i.e. $\widetilde{h} \circ \mathsf{f}_{mk} = \mathsf{f}_{mk} \circ \widetilde{h}$.

It is easy to generalize this approach to cases in which the spatial projection of $tr_m(k)$ does not fall on $\bar{x}$. We challenge the reader to formalize isotropy along these lines.

3.9.5 Homogenity

The assumption that physical laws do not distinguish any point of space to any other is usually referred to a *homogenity of space*. Similarly, one can assume that there is no distinguished time coordinate value (for any observer). Combining both ideas, one could speak about the homogenity of *space-time*. However, following the tradition of the literature, we shall prefer the expression *affinity of space-time*. We can look at this principle as capturing an instance of SPR; that's why we feel justified to discuss it here.

Let us tentatively formalize the principle asserting that space-time is affine:

Ax(homogenity) $(\forall m, k \in Obs)(\forall p \in {}^{n}F)(\exists k' \in Obs)$
$(\forall q \in {}^{n}F)[\mathsf{f}_{mk'}(q) = \mathsf{f}_{mk}(q) + p]$.

The traditional formulation of this principle says that "physical laws" (i.e., in our case, f_{mk} functions) do not depend on the choice of the point p of space-time where we want to apply them, or, more concretely, where our moving observer k is; however, the only means by which we can characterize k's location is the point where m sees k's origin.

We note that we shall never use **Ax(homogenity)** in its own right because it is nothing but a special case of our frequently used auxiliary axiom **Ax(*Triv*)**.

In this connection we note that **Ax($\|$)** can also be considered as a homogenity (or space-time affinity) principle. Recall that **Ax($\|$)** says that parallel observers agree on spatial distances. Axiom **Ax($\|$)** is part of **Ax(syt)** but it is also used independently as an auxiliary axiom. Moreover, in any reasonable relativity theory, **Ax(homogenity)** and **Ax($Triv_t$)** together imply **Ax(*Triv*)**. Note that **Ax($Triv_t$)** can be regarded as a very weak isotropy principle.

In passing we note that **Ax(isotropy)** and **Ax(homogenity)** seem to play an important role in the literature. Cf. e.g. [90], p. 160, line 6 (and Gyula Dávid personal communication, 2 Nov. 1998).

QUESTION 3.9.71 *Consider the following axiom system:*

$$\begin{aligned}\mathbf{Bax}^{-\star} \;&=\; \mathbf{Bax}^{-} \cup \{\mathbf{Ax(\|)}, \mathbf{Ax(ext)}, \mathbf{Ax(}\mathit{Triv}\mathbf{)},\\ &\qquad \mathbf{Ax(isotropy)}, \mathbf{Ax(homogenity)}, B = Ib\}.\end{aligned}$$

It would be interesting to study what kinds of "nonstandard models" (models essentially different from those of **Bax***)* $\mathbf{Bax}^{-\star}$ *has. Examples of these are the "nonstandard" models of* **Reich**(**Bax**)*. Are there other kinds of nonstandard models in* $\mathsf{Mod}(\mathbf{Bax}^{-\star})$*?*

3.9.6 Further ideas

Remark 3.9.72 Consider the following potential axiom:

Ax△3 $(\forall m,k \in Obs)\,[(\mathsf{f}_{mk}(\bar 0) = \bar 0 \quad\wedge\quad tr_m(k) \subseteq \mathsf{Plane}(\bar t, \bar x)) \to |\mathsf{f}_{mk}(1_y)| = 1]$.

Intuitively: Meter rods orthogonal to the direction ($\bar x$) of movement do not shrink; in other words, if k does not move in direction $\bar y$ then distances in direction $\bar y$ do not change. Still more intuitively, if there is no movement in direction $\bar y$ then nothing happens (changes) in direction $\bar y$.

It might be interesting to see under what conditions is **Ax△3** $\leftrightarrow$ **Ax△1** (or **Ax△3** $\leftrightarrow$ **Ax(symm)**) true. Of course, $n \geq 3$ is a necessary condition.

Remark 3.9.73 For completeness we note that **Ax(eqspace)**, introduced in §2.8, can also be considered as a symmetry axiom. Recall that **Ax(eqspace)** states the following:

Ax(eqspace) $(\forall m,k \in Obs)(\forall p,q \in {}^nF)$
$$\Big((p_t = q_t \;\wedge\; \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \to |p-q| = |\mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q)|\Big).$$

That is, if two events are simultaneous for both m and k, then m and k agree on the spatial distance between them.

The relationship between **Ax(eqspace)** and **Ax(symm)** has been clearified by Theorem 2.8.16, which says that assuming **Basax** + $\mathbf{Ax}(\sqrt{\ })$ + $\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}$, $n \geq 3$, **Ax(symm)** $\leftrightarrow$ **Ax(eqspace)** holds.

PROPOSITION 3.9.74 *The following items hold.*

(i) **Basax** + $\mathbf{Ax}(\sqrt{\ })$ $\models$ $\mathbf{Ax(syt_0)}$ $\leftrightarrow$ **Ax(eqspace)**,

(ii) **Bax** + $\mathbf{Ax}(\sqrt{\ })$ $\not\models$ $\mathbf{Ax(syt_0)}$ $\leftrightarrow$ **Ax(eqspace)**.

Proof: We shall fill in the proof later. ∎

Remark 3.9.75 As we have seen at several stages in this study, to prove the famous twin paradox from some axiom system of relativity we always need to introduce some sort of symmetry principle. Turning the point of view around, we can regard $\mathbf{Ax(Twinp_0)}$ and $\mathbf{Ax(Twinp)}$ themselves, the formulae expressing the twin paradox, as symmetry principles in their own right. (These variants of $\mathbf{Ax(TwP)}$ (see p. 140) will be introduced in §**??** (p. **??**) below. We note $\mathbf{Ax(Twinp_0)}$ does not exclude FTL observers in certain situations while $\mathbf{Ax(Twinp)}$ does. Assume $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ })$ for example.) What makes $\mathbf{Ax(Twinp_0)}$ interesting is that it describes bidirectional movement along the same path, which suggests that it might be a natural symmetry principle for our Reichenbachian theories like **Reich**(**Basax**) or **Reich**(**Bax**). (Other candidates for the role of adequate symmetry axiom for **Reich**(**Basax**) etc. are $\mathbf{Ax(syt)}$ and $\mathbf{Ax(syx)}$.)

PROPOSITION 3.9.76 $\mathbf{Basax} + \mathbf{Ax}(\|) < \mathbf{Basax} + \mathbf{Ax}(\|) + \mathbf{Ax(Twinp_0)} < \mathbf{Basax} + \mathbf{Ax}(\|) + \mathbf{Ax}(\omega)$.

Idea of the proof: Intuitively, $\mathbf{Basax} + \mathbf{Ax}(\|) < \mathbf{Basax} + \mathbf{Ax}(\|) + \mathbf{Ax(Twinp_0)}$ holds because when building a model of **Basax**, we are free to choose the length of each observers's unit vectors in another's world-view. See the model construction methods in §§3.2, 3.5, 3.6. Axiom $\mathbf{Ax}(\|)$ restricts this freedom only inside the equivalence classes generated by the relation

$$R(m,k) \stackrel{\text{def}}{\Longleftrightarrow} (\exists h \in Obs) tr_h(m) \parallel tr_h(k).$$

Thus we can build a model in which $\mathbf{Ax(Twinp_0)}$ fails for three particular observer.

Further, $\mathbf{Basax} + \mathbf{Ax}(\|) + \mathbf{Ax(Twinp_0)} < \mathbf{Basax} + \mathbf{Ax}(\|) + \mathbf{Ax}(\omega)$ holds at least for a "bookkeeping" reason. By this we mean that $\mathbf{Ax}\triangle\mathbf{1}$ and $\mathbf{Ax}\Box\mathbf{1}$ quantify over observers in order admit the possibly disturbing orientation of observers that should see eachother the same way. Now, $\mathbf{Ax}\triangle\mathbf{1}$ or $\mathbf{Ax}\Box\mathbf{1}$ can fail in models in which there are "not enough" observers. ∎

QUESTION 3.9.77 *Do* $\mathbf{Bax}^- + \mathbf{Ax(Twinp_0)} + \mathbf{Ax}(\|)$, *perhaps together with some more auxiliary axioms, imply* $\mathbf{Ax}(\omega^-)$*?*

3.10 Groups of Relativity

3.10.1 Group Properties of the World-View Transformations

In this sub-section we shall investigate the collection of world-view transformations that are associated to frame models in a straightforward manner.

Recall that if a model $\mathfrak{M}$ satisfies certain axioms, e.g. $\mathfrak{M} \models$ **Bax**, then its world-view transformations, members of $\{\mathsf{f}_{mk} : m, k \in Obs^{\mathfrak{M}}\}$, are bijections. This means that this set together with the usual function composition and inverse forms a partial group. In what follows we try to find the necessary and sufficient conditions under which this structure becomes a group. In particular, we shall examine the connection between our symmery axioms introduced in §3.9 and the group property of world-view transformations. Moreover, we try to identify possibly interesting research areas, e.g. the relation of this structure to other structures defined in connection with relativity models. Examples are the geometrical structures defined in Chapter 6 and the velocity group which will be defined and discussed in a later version of this study.

Definition 3.10.1 Recall that $\mathsf{FM}(n)$ is the class of n-dimensional frame models. $\mathsf{FM}(n)$ is a special case of $\mathsf{Mod}_{\mathrm{OFG}}(Th(n))$, where Th is a theory in our frame language like **Basax**, **Newbasax**, **Bax** etc. Let $\mathfrak{M} \in \mathsf{FM}$. Then

1. The *Poincaré Group of* $\mathfrak{M}$ is defined as $\mathsf{PG}_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle PG_{\mathfrak{M}}, \circ, \mathrm{Id}, ^{-1} \rangle$, where

$$PG_{\mathfrak{M}} \stackrel{\text{def}}{=} \{\mathsf{f}_{mk} : m, k \in Obs^{\mathfrak{M}}\}.$$

2. The *Lorentz Group of* $\mathfrak{M}$ is $\mathsf{LG}_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle LG_{\mathfrak{M}}, \circ, \mathrm{Id}, ^{-1} \rangle$, where

$$LG_{\mathfrak{M}} \stackrel{\text{def}}{=} \{\mathsf{f}_{mk} : m, k \in Obs^{\mathfrak{M}} \wedge \mathsf{f}_{mk}(\bar{0}) = \bar{0}\}.$$

3. The *Standard Lorentz Group of* $\mathfrak{M}$ is $\mathsf{SL}_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle SL_{\mathfrak{M}}, \circ, \mathrm{Id}, ^{-1} \rangle$, where

$$\begin{aligned} SL_{\mathfrak{M}} \stackrel{\text{def}}{=} & \{\mathsf{f}_{mk} : m, k \in Obs^{\mathfrak{M}} \wedge \mathsf{f}_{mk}(\bar{0}) = \bar{0} \wedge \\ & (\forall p \in \mathsf{Plane}(\bar{t}, \bar{x}))\mathsf{f}_{mk}(p) \in \mathsf{Plane}(\bar{t}, \bar{x})\}. \end{aligned}$$

4. Let Th be a theory in our frame language. Then

$$\mathsf{PG}(Th, \mathbf{n}) \stackrel{\text{def}}{=} \{\mathsf{PG}_{\mathfrak{M}} : \mathfrak{M} \in \mathsf{Mod}(Th(\mathbf{n}))\}.$$

   $\mathsf{PG}(Th, \mathbf{n})$ is called the *class of the Poincaré groups of theory Th* in n dimensions.

5. Similarly, $\mathsf{LG}(Th, \mathbf{n}) \stackrel{\text{def}}{=} \{\mathsf{LG}_{\mathfrak{M}} : \mathfrak{M} \in \mathsf{Mod}(Th(\mathbf{n}))\}$ is called the *class of the Lorentz groups of theory Th* in n dimensions.

6. Finally, let $\mathsf{SL}(Th, \mathbf{n}) \stackrel{\text{def}}{=} \{\mathsf{SL}_{\mathfrak{M}} : \mathfrak{M} \in \mathsf{Mod}(Th(\mathbf{n}))\}$ be called the *class of the standard Lorentz groups of theory Th* in n dimensions.

Remark 3.10.2 1. Note that $\mathsf{SL}_{\mathfrak{M}} \subseteq \mathsf{LG}_{\mathfrak{M}} \subseteq \mathsf{PG}_{\mathfrak{M}}$.

2. $\mathsf{PG}_{\mathfrak{M}}$, $\mathsf{LG}_{\mathfrak{M}}$, $\mathsf{SL}_{\mathfrak{M}}$ are always partial groups in the sense of Andréka & Németi [], but they need *not* be closed under the operations.

3. What is usually called the Poincaré Group in literature is $\mathsf{PG}_{\mathfrak{N}}$, where $\mathfrak{N}$ is the standard (or generic) model of special relativity. In our terminology, this is the Minkowski-model $\mathfrak{M}^M_{\mathfrak{F}}$ defined in Def. 3.8.42. The following statements are corollaries of Prop. 3.8.63:

$$\begin{aligned}
\mathsf{PG}_{\mathfrak{M}^M_{\mathfrak{F}}} &= \{f \in Poi : f(1_t)_t > 0\},\\
\mathsf{LG}_{\mathfrak{M}^M_{\mathfrak{F}}} &= \{f \in Lor : f(1_t)_t > 0\},\\
\mathsf{SL}_{\mathfrak{M}^M_{\mathfrak{F}}} &= \{f \in SLor : f(1_t)_t > 0\}.
\end{aligned}$$

4. Let us notice that $\mathsf{PG}(Th)$ is *not* an abstract class of groups, but rather a class of concrete groups. By this we mean that for $\mathsf{PG}_{\mathfrak{M}}$ a set ${}^nF$ is given so that the elements of $\mathsf{PG}_{\mathfrak{M}}$ are transformations of ${}^nF$. In other words an element of $\mathsf{PG}(Th)$ is a *represented group*, i.e. a group whose elements are permutations of some fixed set. Hence $\mathsf{PG}(Th) \neq \mathbf{I}\mathsf{PG}(Th)$, i.e. an isomorphic copy of a Poincaré group (of theory Th) is not necessarily a Poincaré group (of the same theory). In this sense Poincaré groups of relativity are analogous to Boolean set algebras or cylindric algebras, as opposed to abstract Boolean algebras or abstract cylindric algebras.

Now we are going to examine the question: What are the conditions that make $\mathsf{PG}_{\mathfrak{M}}$ a group? We shall prove that not even the strongest of our "core" theories, **Basax**, is enough to ensure this. Although the addition of some symmetry axioms introduced in §3.9 is sufficient, these axioms are too strong (not necessary) assumptions in general. This means that we can add an extra axiom to a core theory, say **Newbasax**, which states the group property of $\mathsf{PG}_{\mathfrak{M}}$, and assymetric models still remain admitted.

First let us define a formula asserting that the world-view transformations produce a group.

Ax(group$^-$) $(\forall m, k, m', k' \in Obs)(\exists m'', k'' \in Obs)\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{m''k''}$

Let us recall some proposed axioms from §3.4. The property required by these formulae is, of course, different from, but analogous to, being a group. Because of the apparent similarities we insert these issues here.

Ax(group) $(\forall m, k, m', k' \in Obs)(\exists k'' \in Obs)\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{mk''} \circ exp \circ N \circ \tilde{\varphi}$, for some $N \in Triv$, $exp \in Exp$, $\varphi \in Aut(\mathfrak{F})$.

This axiom characterizes how far the composition of two world-view transformation can be approximated in general by a third world-view transformation. The restriction on the choice of the approximant ($\mathsf{f}_{mk''}$), namely that one of its "feet" must be m, is probably unimportant. Notice that exp comes from the possible asymmetry of (some) models, N is present because observers with the "right" orientation of spatial axes are allowed to be missing, and φ is a general burden on world-view tranformations unless one eliminates them by axioms (cf. Thm. 2.9.5, Cor's 3.9.28 and 3.9.30, and Prop. 3.9.50).

Ax(group$^+$) $(\forall m, k, m', k' \in Obs)(\exists k'' \in Obs)\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{mk''}$

Axiom **Ax(group$^+$)** is stronger than **Ax(group)** in that it postulates the equality of a composition of two arbitrarily chosen world-view transformations to one that connects the world of m to some k''. That is, the r.h.s. is chosen from the set $\{\mathsf{f}_{mh} : h \in Obs\}$, where m is fixed.

Remark 3.10.3 **Bax$^-$** $\models$ **Ax(group$^+$)** $\Rightarrow$ **Ax(group)** $\wedge$ **Ax(group$^-$)**.

PROPOSITION 3.10.4 *The following items hold.*

(i) **Basax** $\models$ **Ax□1** $\Rightarrow$ **Ax(group$^-$)**.

(ii) **Basax** $\models$ **Ax(group$^+$)** $\Rightarrow$ **Ax□1**.

Proof of item (i): Assume **Basax** + **Ax□1**. Let $m, k, m', k' \in Obs$ be arbitrary. By **Ax□1** there exists $m'' \in Obs$ such that $\mathsf{f}_{km} = \mathsf{f}_{k'm''}$. Therefore

$$\mathsf{f}_{mk} = \mathsf{f}_{m''k'} \tag{274}$$

by Prop. 2.3.3(x). Again by **Ax□1**, there is $k'' \in Obs$ such that:

$$\mathsf{f}_{m'k'} = \mathsf{f}_{k'k''}. \tag{275}$$

By Prop. 2.3.3(x), (274) and (275) we get

$$\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{m''k'} \circ \mathsf{f}_{k'k''} = \mathsf{f}_{m''k''},$$

as desired.

Proof of item (ii): Assume **Basax** + **Ax(group$^+$)**. Let $m_0, k_0, m_1 \in Obs$. Applying **Ax(group$^+$)** for $m = k = m_1, m' = m_0, k' = k_0$ one obtains that there is $k_1 \in Obs$ such that

$$\mathsf{f}_{m_1k_1} = \mathsf{f}_{m_1m_1} \circ \mathsf{f}_{m_0k_0} = \mathsf{f}_{m_0k_0}.$$

∎

The following proposition derives **Ax(group$^+$)** from symmetry axioms, together with auxiliary axioms like **Ax(*Triv*)**, **Ax5$^+$** etc. We shall also need **Ax(↑)**, which was introduced among the "essential axioms" of **BaCo** in §3.8, and it states that the clocks of "observer brothers" run in the same direction.

PROPOSITION 3.10.5 *Let $n \geq 3$. Let* **Ax** *be one of the following axioms:*

$$\{\mathbf{Ax(symm)}, \mathbf{Ax(syt_0)}, \mathbf{Ax\triangle 2}, \mathbf{Ax\square 2},$$
$$\mathbf{Ax\triangle 1} \wedge \mathbf{Ax(eqtime)}, \mathbf{Ax\square 1} \wedge \mathbf{Ax(eqtime)}\}.$$

Then $\mathbf{Basax}(n) + \mathbf{Ax} + \mathbf{Ax}(Triv_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(\uparrow)} + \mathbf{Ax5^+} \models \mathbf{Ax(group^+)}$.

Proof: Assume $\mathbf{Basax}(n) + \mathbf{Ax} + \mathbf{Ax}(Triv_t) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax5^+}$ for $n \geq 3$. We have $(\forall m, k \in Obs)\mathsf{f}_{mk} \in Poi$ for every choice of **Ax**. We must refer to

- Thm. 2.9.5, if $\mathbf{Ax} = \mathbf{Ax(symm)}$,
- Prop. 3.9.50(ii), if $\mathbf{Ax} = \mathbf{Ax(syt_0)}$,
- Cor. 3.9.28, if $\mathbf{Ax} = \mathbf{Ax\triangle 2}$,
- Cor. 3.9.30, if $\mathbf{Ax} = \mathbf{Ax\square 2}$,
- Thm. 3.9.23 and Cor. 3.9.28, if $\mathbf{Ax} = \mathbf{Ax\triangle 1} \wedge \mathbf{Ax(eqtime)}$,
- Thm. 3.9.20 and Cor. 3.9.30, if $\mathbf{Ax} = \mathbf{Ax\square 1} \wedge \mathbf{Ax(eqtime)}$.

Let $f = \mathsf{f}_{k'm'} \circ \mathsf{f}_{km}$, $\ell = f[\bar{t}]$. We have $ang^2(\ell) < 1$ by Thm. 3.4.1. We are looking for an observer k_0 such that $tr_m(k_0) = \ell$ and

$$f \text{ reverses the arrow of time} \iff \mathsf{f}_{k_0m} \text{ reverses the arrow of time.}$$

More formally,

$$f(1_t)_t - f(\bar{0})_t > 0 \iff \mathsf{f}_{k_0m}(1_t)_t - \mathsf{f}_{k_0m}(\bar{0})_t > 0.$$

We have two cases.

Case 1: $(m \uparrow k \wedge m' \uparrow k') \vee (m \downarrow k \wedge m' \downarrow k')$. Then $f(1_t)_t - f(\bar{0})_t > 0$. By **Ax5**$^+$ there is $k_0 \in Obs$ such that $tr_m(k_0) = \ell$ and $m \uparrow k_0$. Thus $\mathsf{f}_{k_0 m}(1_t)_t - \mathsf{f}_{k_0 m}(\bar{0})_t > 0$.

Case 2: $(m \uparrow k \wedge m' \downarrow k') \vee (m \downarrow k \wedge m' \uparrow k')$. Then $f(1_t)_t - f(\bar{0})_t < 0$. One of k, m', k'—let us call it $k^\star$—is such that $k^\star \downarrow m$. We must apply **Ax5**$^+$ in the world-view of $k^\star$. So there is $k_0 \in Obs$ such that $tr_{k^\star}(k_0) = \mathsf{f}_{mk^\star}[\ell]$ and $k^\star \uparrow k_0$. By Prop. 2.3.3 then we have $tr_m(k_0) = \ell$ and $m \downarrow k_0$ Thus $\mathsf{f}_{k_0 m}(1_t)_t - \mathsf{f}_{k_0 m}(\bar{0})_t < 0$.

Now compare $\mathsf{f}_{k_0 m}$ and f. They both map $\bar{t}$ to ℓ, and by $\mathsf{f}_{k_0 m}, f \in Poi$ and the above definition of k_0 we have

$$\mathsf{f}_{k_0 m}(1_t)_t - \mathsf{f}_{k_0 m}(\bar{0})_t = f(1_t)_t - f(\bar{0})_t.$$

Then by Lemma 3.8.50 there is some $N \in Triv$ such that $N[\bar{t}] = \bar{t}$ and

$$f = N \circ \mathsf{f}_{k_0 m}.$$

By **Ax(**$Triv_t$**)** there is such a $k'' \in Obs$ that $\mathsf{f}_{k'' k_0} = N$. Hence $f = \mathsf{f}_{k'' m}$ and by Prop. 2.3.3(x) we have

$$\mathsf{f}_{mk''} = f^{-1} = \mathsf{f}_{mk} \circ \mathsf{f}_{m'k'}.$$

∎

PROPOSITION 3.10.6 *The following items hold.*

(i) **Basax** + **Ax△2** + **Ax(**$Triv_t$**)** + **Ax(**$\sqrt{\ }$**)** + **Ax(↑)** + **Ax5**$^+$ $\models$ **Ax(group**$^+$**)**.

(ii) **Basax** + **Ax△1** + **Ax(eqtime)** + **Ax(**$Triv_t$**)** + **Ax(**$\sqrt{\ }$**)** + **Ax(↑)** + **Ax5**$^+$ $\models$ **Ax(group**$^+$**)**.

Proof: The proof of Prop. 3.10.5 goes through, because we used $n \geq 3$ only to exclude faster than light observers, and the premises of both items have the same effect. See Cor. 2.7.6 and Thm. 3.9.8(ii). ∎

The following proposition asserts that **Ax(group**$^-$**)** is not implied by **Basax**, and therefore it does not follow from our weaker axioms systems either.

PROPOSITION 3.10.7 **Basax**$(n) \not\models$ **Ax(group**$^-$**)**, *for any* $n > 1$.

Proof: We shall fill in the proof later. ∎

Now let us show that **Ax(group**$^-$**)** is weaker than our symmetry axioms introduced in §3.9.

PROPOSITION 3.10.8 $\mathbf{Basax} + \mathbf{Ax(group^-)} \not\models (\mathbf{Ax\Box 2} \vee \mathbf{Ax\triangle 2} \vee \mathbf{Ax(syt_0)})$.

Proof: We shall fill in the proof later. ∎

QUESTION 3.10.9 *Does* $\mathbf{Ax\Box 1}$ *or* $\mathbf{Ax\triangle 1}$ *follow from* $\mathbf{Basax} + \mathbf{Ax(group^-)}$*?*

PROPOSITION 3.10.10 *If* $n \geq 3$*, then* $\mathbf{Basax}(n) \models \mathbf{Ax(group)}$.

Proof: We shall fill in the proof later. ∎

Note that it is an open issue whether $\mathbf{Bax} \models \mathbf{Ax(group)}$, $\mathbf{Basax} + \mathbf{Ax(group)} \models \mathbf{Ax\triangle 1}$ or $\mathbf{Basax} + \mathbf{Ax(group^+)} \models \mathbf{Ax}(\omega)$.

Future research task 3.10.11

1. Investigate $\mathsf{IPG}(\mathbf{Bax})$, $\mathsf{IPG}(\mathbf{Newbasax})$, $\mathsf{IPG}(\mathbf{Newbasax} + \mathbf{Ax}(\omega))$ from an algebraic point of view.

2. How do these groups relate to $Poi, Lor, SLor$?

Future research task 3.10.12 Assume $\mathfrak{M} \models \mathbf{Bax}^{--}$. Consider the automorphism group $Aut(\mathfrak{G}_{\mathfrak{M}})$ of the geometry defined in Chapter 6 above.

1. How does $\mathsf{PG}_{\mathfrak{M}}$ relate to $Aut(\mathfrak{G}_{\mathfrak{M}})$? Is for example $\mathsf{PG}_{\mathfrak{M}} \subset Aut(\mathfrak{G}_{\mathfrak{M}})$ true?

2. How does $Aut(\mathfrak{G}_{\mathfrak{M}})$ relate to the automorphism groups of its reducts?

3.10.2 Duality Theory for Poincaré Groups

We shall see in Chapter 6 (Observer Independent Geometry) that a duality theory between models $\mathfrak{M}$ and geometries $\mathfrak{G}_{\mathfrak{M}}$ can be developed. It can be depicted by the schema:

($\star$) Models $\rightleftarrows$ Geometries.

We would like to note that a duality theory between models $\mathfrak{M}$ and Poincaré groups $\mathsf{PG}_{\mathfrak{M}}$ can be elaborated in an analogous fashion. It will be of the general pattern

($\star\star$) Models $\rightleftarrows$ Poincaré groups, analogous with ($\star$). Similarly to the case of our geometric duality theory, one can single out strong enough relativity theories such that on the level

$$\mathsf{Mod}(Th) \rightleftarrows \mathsf{PG}(Th)$$

the duality theory works well (exactly in the same sense as we used this expression in the geometry chapter). Among others, this means that for nice enough models, the model $\mathfrak{M}$ is recoverable from its Poincaré group $\mathsf{PG}_{\mathfrak{M}}$.

Of course, one can combine the duality theories of patterns ($\star$) and ($\star\star$) to obtain dualities between geometries and concrete groups.

The just indicated ideas about extending our duality theory from geometries to Poincaré groups will be further discussed in a later version of this work.

3.10.3 The Group of Velocities

In this sub-section we shall discuss the following question:

($\star$) Given some theory of kinematics, in which sense can we speak about the addition of velocities; how should we formalize this operation (if it turns out to meaningful), and what sorts of nice algebraic structures (if any) can be associated to this idea?

Clearly, we are interested only in vectorial velocities introduced below Def. 2.2.2. Recall that the function we are referring to, $\vec{v}_m(b)$, associates a member of ${}^{n-1}F$ to each observer m and inertial body b. That is, velocity, like location, speed, simultaneity etc., is an observer-dependent concept; we can speak of relations of velocities only in terms of observers (or reference frames). Accepting this, we can consider $u \in {}^{n-1}F$ the sum of $v \in {}^{n-1}F$ and $w \in {}^{n-1}F$ only if there are observers that measure it to be so (in a sense to be specified below). That is, any definition of addition of velocities must refer to entities whose properties are stated in the "relativistic part" of our axioms. (This is unlike the case of operations $+^V, -^V$ defined on the vector field ${}^n\mathbf{F}$, which only refer to the ordered field $\mathfrak{F}$ and hence their definition only depends on $\mathbf{Ax}_{\rm OF}$ and $\mathbf{Ax}_{\rm G}$, which are always assumed.)

Let us try to find a formal notion of velocity addition. We want a function that answers a question of the following pattern: If I observe your spaceship move off by velocity v, and a cabin leaves your spaceship by velocity w (as see by you), what is the cabin's velocity as measured by me? More formally, we would like to say the following:

($\star\star$) The relativistic sum of $v, w \in {}^{n-1}F$ is $u \in {}^{n-1}F$, only if there are $m, k, k_1 \in Obs$ such that $v = \vec{v}_m(k)$, $w = \vec{v}_k(k_1)$, and $u = \vec{v}_m(k_1)$.

Undoubtedly, this is what one has in mind when discussing the "velocity transformation" (cf. [224, §§2.15, 1.3]).

But if one intends to define an operation on *velocities* (as opposed to the issue of how velocities transform when changing form one inertial frame to another[337]), $(\star\star)$ would be insufficient. Given the velocity vectors $v, w \in {}^{n-1}F$ (but assume $\|v\| < 1 \wedge \|w\| < 1$), in models of any "reasonable" theory one can find $m, k \in Obs$ such that $v = \vec{v}_m(k)$; but the orbit and, consequently, the velocity of the third observer (or inertial body) k_1 as seen by m would depend on the orientation of k's spatial coordinate axes. Thus we have to make significant restrictions on the choice of k (restricting the admissible orientation of k's spatial coordinate axes).[338]

There are a number of ways to restrict k's free choice of spatial orientation in order to make the velocity addition a function; but we also aim at making this operation the foundation of a *group*. Hence we also have the purpose of making addition associative. We are not sure whether the set of restrictions exposed below is the only way to achieve that.

Let us first consider the textbook case $v = \langle v_0, 0, \dots, 0\rangle$ (cf. [224, §§2.15]). Then we can require that $m, k \in Obs$ must be chosen so that $v = \vec{v}_m(k)$ and m and k are in *standard configuration*. That is,

$$\mathsf{f}_{mk}[\bar{t}], \mathsf{f}_{mk}[\bar{x}] \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \quad \text{and} \quad (\forall 1 < i \in n)\mathsf{f}_{mk}[\overline{\bar{0}e_i}] = \overline{\bar{0}e_i}.$$

Observer k still has some degrees of freedom: (i) it can choose to point its unit vector along the $\bar{x}$ axis *in* the direction of its movement or *against* it ("towards" m or in the opposite direction), (ii) k's clock may run forwards of backwards as seen by m (with earlier notation, $m \uparrow k$ or $m \downarrow k$). This freedom must be eliminated, in order to have a well-defined notion of velocity addition. Thus we require the following condition on m, k in the definition under preparation:

$$(\star\star\star) \qquad (\forall i \in n)\mathsf{f}_{mk}(1_i)_i > 0.$$

By these restrictions on k (m, k in standard configuration and $(\star\star\star)$) we have at least a chance of defining a unique sum for $v, w \in {}^{n-1}F$ if v is chosen in a special way. (Whether this expectation is fulfilled or not depends on the relativity theory assumed. The point is that without the above conditions we would have no chance of finding a function even in the nicest models.)

But we aim at a definition that does not restrict the v's direction. The restrictions we are making on the choice of k should reduce to the above conditions if $v = \langle v_0, 0, \dots, 0\rangle$, and should provide an unambiguous definition on all velocities that

[337] From the viewpoint of definition, this would be a trivial task.

[338] Of course, this argument applies to the classical Galilean-Newtonian kinematics, too.

can be realized by observers. For this purpose we shall introduce the notion of a weakly standard configuration.

Definition 3.10.13 Assume $\mathbf{Bax}^-$. We say that $m, k \in Obs$ are in *weakly standard configuration* if and only if the following conditions hold.

(i) $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$.

(ii) There is a median observer h for m and k.

(iii) Consider the plane $P \stackrel{\text{def}}{=} \mathsf{f}_{km}[S]$ (i.e. members P coordinatize events that are simultaneous with $\bar{0}$ for k). For any $0 < i < n$, let $P_i = \mathsf{Plane}(tr_m(h), \overline{\bar{0}e_i})$. Informally, P_i is the plane that contains both the i-th spatial coordinate axis $\overline{\bar{0}e_i}$ and the median observer h's trace as seen by m. Then $\mathsf{f}_{km}[\overline{\bar{0}e_i}] \subseteq P \cap P_i$.

(iv) If $i \in n$, then $\mathsf{f}_{km}(e_i)_i > 0$.

See Figure 121.

Since Def. 3.10.13 prescribes the coordinate axes of observer k, we must show that its requirements are not impossible to satisfy. More specifically, we have to show that the prescribed axes are indeed Minkowski-orthogonal, i.e. they are the coordinate axes of a possible observer. The following proposition asserts this.

PROPOSITION 3.10.14 *Assume* $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathsf{f}_{mk} \in Aftr$. *Let* $m, k, h \in Obs$ *be such that* h *is a median observer for* m *and* k. *Let* P *and* P_i *be defined as in item (iii) of Def. 3.10.13 above. Then the following items hold.*

(i) The lines $\{\mathsf{f}_{mk}[P \cap P_i] : i \in n\}$ *are pairwise orthogonal. Moreover, they fall in* S.

(ii) Further, if we assume $\mathbf{Ax(}Triv_t\mathbf{)} + \mathbf{Ax5}^+$, *then*

$$(\exists k' \in Obs)(tr_m(k') = tr_m(k) \ \wedge \ (\forall i \in n)\mathsf{f}_{k'm}[\overline{\bar{0}e_i}] \subseteq P \cap P_i).$$

To show Prop. 3.10.14 we shall need the following couple of lemmas.

LEMMA 3.10.15 *Let* $A \in PT$. *If* $A[\bar{t}] = \bar{t}$, *then* $A = h \circ N$, *for some* $exp \in Exp$ *and isometry* N *such that* $N[\bar{t}] = \bar{t}$.

Proof: The proof follows by Lemma 3.6.20 in §3.6. ∎

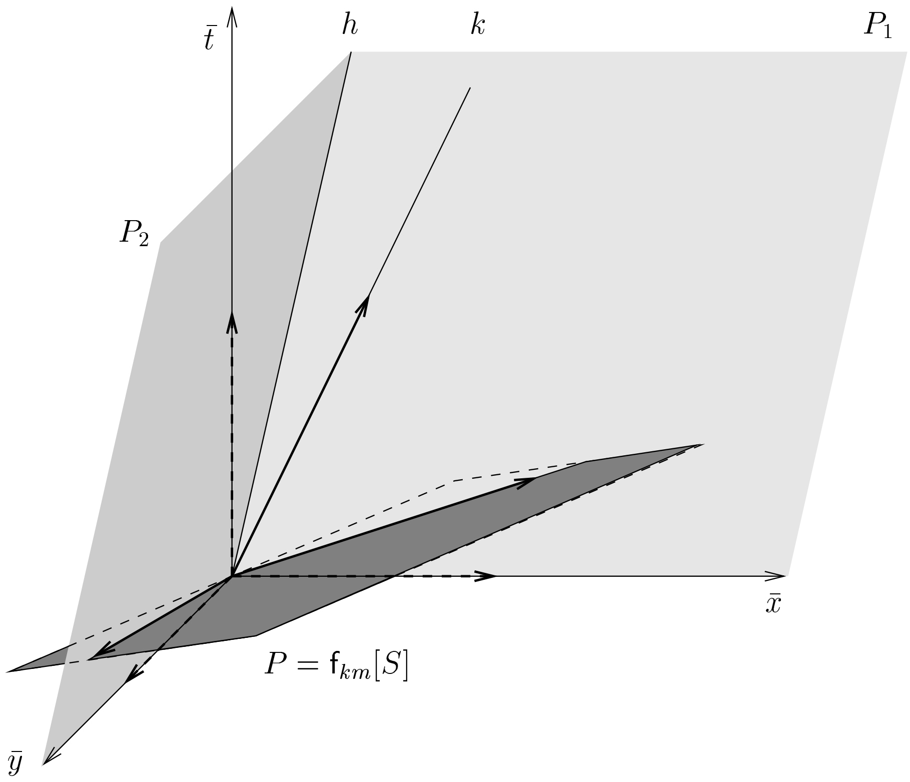


Figure 121: Illustration for Def. 3.10.13. Observers m and k are in weakly standard configuration. The unit vectors of k are drawn with solid lines, those of m are dashed.

LEMMA 3.10.16 *Let $\ell \in \mathsf{Eucl}$ and $g_1, g_2 \in PT$ with $g_1[\ell] = g_2[\ell] = \bar{t}$. Then*

$$g_1 = g_2 \circ exp \circ N, \textit{ for some } exp \in Exp \textit{ and isometry } N \textit{ such that } N[\bar{t}] = \bar{t}.$$

Proof: $g_2^{-1} \circ g_1[\bar{t}] = \bar{t}$. Applying Lemma 3.10.15 completes the proof. ∎

Proof of Prop. 3.10.14(i): Let $\ell_i = \mathsf{f}_{mk}[P \cap P_i]$ for any $i \in n$. One obtains $\ell_i \subseteq S$ easily:

$$\mathsf{f}_{mk}[P \cap P_i] \subseteq \mathsf{f}_{mk}[P] = \mathsf{f}_{mk}[\mathsf{f}_{km}[S]] = S.$$

Next we have to show that $\ell_i \perp_e \ell_j$ for $i \neq j$. We shall work in the world view of the median observer h. Consider f_{hm} and $\sigma_{\bar{t}} \circ \mathsf{f}_{hk}$. Both map the line $tr_h(m)$ to $\bar{t}$. Using **Basax** $+ (\forall m, k \in Obs)\mathsf{f}_{mk} \in Aftr$ we get $\mathsf{f}_{hm}, \sigma_{\bar{t}} \circ \mathsf{f}_{hk} \in PT$. Then by Lemma 3.10.16

$$\mathsf{f}_{hm} = \sigma_{\bar{t}} \circ \mathsf{f}_{hk} \circ exp \circ N, \tag{277}$$

for some $exp \in Exp$ and $N \in Triv$ such that $N[\bar{t}] = \bar{t}$.

Now, let $t_i \stackrel{\text{def}}{=} \mathsf{f}_{mh}[\overline{\bar{0}e_i}]$ (i.e. t_i is the image of m's i-th axis in h's world view). Let $s_i \stackrel{\text{def}}{=} \mathsf{f}_{mh}[P \cap P_i]$. We shall use the following statement:

Claim 3.10.17 $t_i = \sigma_{\bar{t}}[s_i]$.

We shall postpone the proof of Claim 3.10.17. Its implications are illustrated in Figure 122.

Let $i, j \in n$ be such that $i \neq j$. We have $\mathsf{f}_{hm}[t_i] \perp_e \mathsf{f}_{hm}[t_j]$ by definition. Using (277) and Claim 3.10.17 one obtains

$$\begin{aligned} (\sigma_{\bar{t}} \circ \mathsf{f}_{hk} \circ exp \circ N)[\sigma_{\bar{t}}[s_i]] \quad &\perp_e \quad (\sigma_{\bar{t}} \circ \mathsf{f}_{hk} \circ exp \circ N)[\sigma_{\bar{t}}[s_j]], \\ (\mathsf{f}_{hk} \circ exp \circ N)[s_i] \quad &\perp_e \quad (\mathsf{f}_{hk} \circ exp \circ N)[s_j]. \end{aligned}$$

Since both exp and N map perpendicular lines to perpendicular lines, this implies

$$\mathsf{f}_{hk}[s_i] \perp_e \mathsf{f}_{hm}[s_j].$$

But $\mathsf{f}_{hk}[s_i] = \mathsf{f}_{hk}[\mathsf{f}_{mh}[P \cap P_i]] = \ell_i$. Thus $\ell_i \perp_e \ell_j$.

Proof of Prop. 3.10.14(ii): We shall fill in the proof later. ∎

PROPOSITION 3.10.18 *Assume* **Basax**. *Let $m, k \in Obs$ be in weakly standard configuration. Assume $tr_m(k) \in \mathsf{Plane}(\bar{t}, \bar{x})$. Then $m, k \in Obs$ are in standard configuration.*

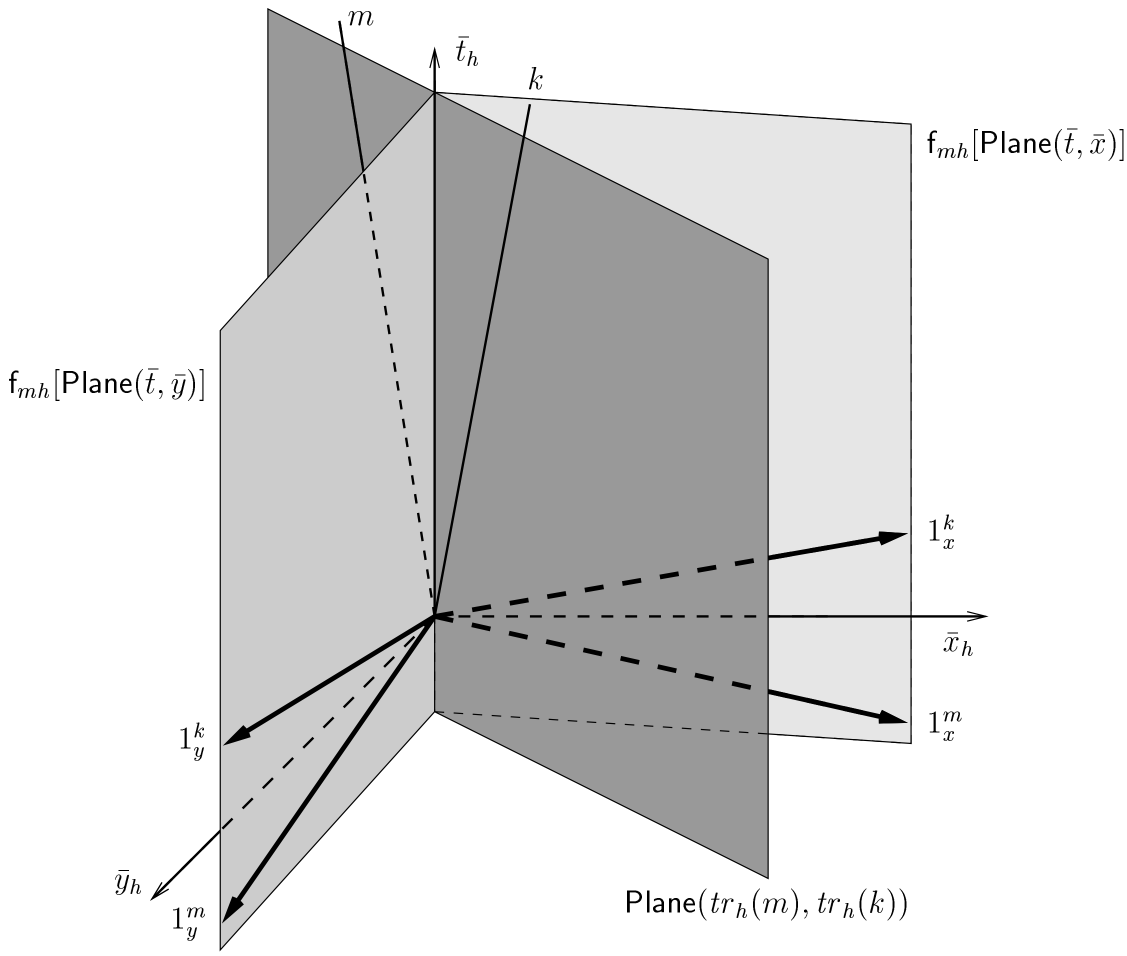


Figure 122: Illustration for the proof of Prop. 3.10.14(i).

Proof: We shall fill in the proof later. ∎

Now we feel ready to define the algebraic structure of velocities which might be a group in certain frame models.

Definition 3.10.19 Let $\mathfrak{M}$ be a frame model. The *velocity structure* $\mathsf{VL}_{\mathfrak{M}}$ is the algebraic structure $\langle VL_{\mathfrak{M}}, \oplus^{\mathfrak{M}}, \ominus^{\mathfrak{M}}, \bar{0}\rangle$, where

1. $VL_{\mathfrak{M}} = \{\vec{v}_m(k) : m, k \in Obs^{\mathfrak{M}}\}$,

2. Operation $\oplus^{\mathfrak{M}} : VL_{\mathfrak{M}} \times VL_{\mathfrak{M}} \to VL_{\mathfrak{M}}$, called the *relativistic addition of velocities*, is defined as follows. Let $v, w \in VL_{\mathfrak{M}}$. By the relativistic sum of v and w, $v \oplus^{\mathfrak{M}} w$, we mean the following set:

$$\begin{aligned} v \oplus^{\mathfrak{M}} w \;\; = \;\; & \Big\{\vec{v}_m(k_1) : m, k, k_1 \in Obs^{\mathfrak{M}} \;\wedge\; v = \vec{v}_m(k) \;\wedge\; w = \vec{v}_k(k_1) \;\wedge \\ & m, k \text{ are in weakly standard configuration } \wedge \\ & (\forall i \in n \setminus 1)(\mathsf{f}_{mk}(e_i) - \mathsf{f}_{mk}(\bar{0})_i \geq 0)\Big\}. \end{aligned}$$

3. $\ominus^{\mathfrak{M}} v = \Big\{\langle v, u\rangle : v \oplus^{\mathfrak{M}} u = \{\bar{0}\}\Big\}$.

Remark 3.10.20

(i) We shall drop the superscript $^{\mathfrak{M}}$ from $\oplus$ and $\ominus$ whenever the context excludes misunderstandings.

(ii) Let $\mathfrak{M} \models \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}\}$. Then $\mathsf{VL}_{\mathfrak{M}}$ is a multigroupoid.

(iii) An alternative definition of the addition of velocities would be the following. Let $v, w \in VL_{\mathfrak{M}}$ be arbitrary for some $\mathfrak{M} \models \mathbf{Bax}^{-}$. Then

$$\begin{aligned} v \boxplus^{\mathfrak{M}} w \;\; = \;\; & \Big\{\vec{v}_m(k_1) : m, k, k_1 \in Obs^{\mathfrak{M}} \;\wedge\; v = \vec{v}_m(k) \;\wedge\; w = \vec{v}_k(k_1) \;\wedge \\ & (\exists P \in \mathsf{Plane}(\mathbf{F}))(tr_m(k), tr_k(m), \bar{t} \subseteq P \;\wedge \\ & (\forall i \in n)(\mathsf{f}_{mk}(e_i) - \mathsf{f}_{mk}(\bar{0})_i \geq 0)\Big\}. \end{aligned}$$

The intuitive idea behind the definition of $\boxplus$ is the following. If $v = \langle v_0, 0, \ldots, 0\rangle$, then v is realized by some m and k in standard configuration. Then $tr_m(k), tr_k(m), \bar{t} \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$. In this case $P = \mathsf{Plane}(\bar{t}, \bar{x})$. Moreover, we must restrict k's freedom of directing its axes to either of the two remaining directions in order to obtain a possibly unambiguous definition, as explained above. On the other hand, if $v \neq \langle v_0, 0, \dots, 0\rangle$, then we try to replace $\mathsf{Plane}(\bar{t}, \bar{x})$ by some other plane P, and to require similar restrictions on the orientation of k's axes.

We conjecture that $\oplus$ and $\boxplus$ are equivalent under some mild assumptions.

Conjecture 3.10.21 *We guess that under very mild conditions on $\mathfrak{M}$, $\mathsf{VL}_{\mathfrak{M}}$ turns out to be a multigroup.*

THEOREM 3.10.22 *Let $n \geq 3$. Assume* $\mathbf{Basax}(n) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{Ax5}^+ + \mathbf{Ax}(\sqrt{\ }) + \mathsf{f}_{mk} \in \mathit{Aftr}$. *Then $\mathsf{VL}_{\mathfrak{M}}$ is a group.*

Proof: We shall fill in the proof later. ∎

Question for future research 3.10.23 How far can the previous theorem be pushed into the domain of weaker theories like **Bax**? In other words, under what weaker conditions on $\mathfrak{M}$ will $\mathsf{VL}_{\mathfrak{M}}$ still remain a group?

PROPOSITION 3.10.24 *Let $n \geq 3$.*

(i) $\mathbf{Basax}(n) \models (\oplus$ *is non-commutative*$)$,

(ii) $\mathbf{Basax}(2) \models (\oplus$ *is commutative*$)$.

Proof: We shall fill in the proof later. ∎

Let us turn to the issue of connections between $\mathsf{VL}_{\mathfrak{M}}$, and the algebraic structures of the world-view transformations defined above, $\mathsf{PG}_{\mathfrak{M}}$, $\mathsf{LG}_{\mathfrak{M}}$ and $\mathsf{SL}_{\mathfrak{M}}$. First we shall define an equivalence relation $\equiv_v$ on the Poincaré group $\mathsf{PG}_{\mathfrak{M}}$. Here $\equiv_v$ is called *velocity-equivalence.*

Definition 3.10.25 Let $\mathfrak{M} \models \mathbf{Bax}^-$. Then

$$\mathsf{f}_{mk} \equiv_v \mathsf{f}_{m'k'} \overset{\text{def}}{\iff} v_m(k) = v_{m'}(k'),$$

for any $m, k, m', k' \in \mathit{Obs}$.