Omitting types for finite variable fragments and complete representations of algebras

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To the memory of Leon Henkin, teacher co-author and friend.

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Abstract . We give a novel application of algebraic logic to first order logic. A new, flexible construction is presented for representable but not completely representable atomic relation and cylindric algebras of dimension n (for finite $n > 2$) with the additional property that they are one-generated and the set of all n by n atomic matrices forms a cylindric basis. We use this construction to show that the classical Henkin-Orey omitting types theorem fails for the finite variable fragments of first order logic as long as the number of variables available is > 2 and we have a binary relation symbol in our language. We also prove a stronger result to the effect that there is no finite upper bound for the extra variables needed in the witness formulas. This result further emphasizes the ongoing interplay between algebraic logic and first order logic.

1 Introduction

Daniele Mundici [41] initiated the following type of investigations in first order logic (FOL). Concerning various positive results like Craig's Interpolation Theorem or Beth's theorem, Mundici suggested to ask how resource sensitive the positive result is. E.g. if Craig's theorem says that to $\varphi \to \psi$ there exists an interpolant θ with $\varphi \to \theta$ and $\theta \to \psi$, how complicated (i.e. how "expensive") θ is relative to φ and ψ. Recent work measures expensiveness with the number of variables needed for θ (or for whatever is being claimed to exist by the "positive theorem" in question). An example for such an investigation is

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Monk's result [39] saying that for any bound $k \in \omega$ there is a valid 3-variable formula φ which cannot be proved by using only k variables. In this paper we apply this "resource-oriented" kind of investigation to the Henkin-Orey omitting type theorem [17], [43].

For a finite number n, $\mathcal{L}_n$ denotes the n-variable fragment of FOL. More concretely, the formulas of $\mathcal{L}_n$ are those formulas of FOL which involve at most n individual variables. A systematic study of the fragments $\mathcal{L}_n$ via cylindric algebras was initiated by Leon Henkin [19]. The above outlined issue of "resource-sensitivity" is often addressed in the following form. We ask ourselves if certain distinguished positive properties of FOL are inherited by $\mathcal{L}_n$. Examples of such distinguished properties studied in the literature for $\mathcal{L}_n$ include interpolation, Beth definability, submodel preservation, and completeness theorems, cf. [3], [5], [16], [28], [29], [39]. A general first impression might be that, usually, positive properties turn out to be resource sensitive in such a strong way that a goal formulatable in $\mathcal{L}_n$ cannot be solved in $\mathcal{L}_{n+k}$ for any finite k. However, this is not true in such generality, e.g. by [20, Thm.1.5.14 and the discussions preceding and following it], some natural properties of substitutions in $\mathcal{L}_n$ which are not provable in $\mathcal{L}_n$ are provable in $\mathcal{L}_{n+2}$, cf. also [44]. A further "counterexample" is provided by the guarded fragment of FOL introduced in [4]. The main point of the guarded fragment (and its variants e.g. the packed fragment) is that if we work inside the guarded fragment then we are kind of safe of the above quoted "complexity explosion" phenomenon undermining the positive results of FOL. Cf. [15],[31],[32].

We say that a logic $\mathcal{L}$ has the Omitting Type Property (OTP) if the usual formulation of the Omitting Type Theorem remains true after all occurrences of $\mathcal{L}_{\omega\omega}$ are replaced with $\mathcal{L}$ in it. In brief, the OTP for $\mathcal{L}$ says that if T is a complete theory, and $\Sigma(\overline{x})$ is a type which is realized in each model of T, then there is a "witness formula" for this, i.e. there is $\psi(\overline{x}) \in \mathcal{L}$ that ensures $\Sigma(\overline{x})$ in T. For concrete detail see section 1.1 below.

In this paper we obtain the following new results. Let $n > 2$ be finite. Then $\mathcal{L}_n$ strongly fails the OTP in the following sense: (i) one binary relation symbol suffices in the language. (ii) There is no bound on the number of extra variables needed for writing up the missing witness formula. (iii) The type in question uses only one free variable. This result was conjectured (but not proved) in [49]. Algebraic results: A flexible, new kind of construction is given for relation and cylindric algebras suitable e.g. to construct representable but not completely representable atomic algebras. Such algebras were first constructed by Robin Hirsch and Ian Hodkinson [23], [27], [30]. By the use of the new construction we improve their result and construct one-generated representable but not completely representable atomic algebras with the further property that the relation algebras have n-dimensional cylindric bases and the cylindric algebras are neat-reducts and are constructed from the n-dimensional cylindric bases.

The desirability of such a construction is pointed out in [23, p.836, lines 11-22]. We need these extra properties in showing strong failure of the OTP for $\mathcal{L}_n$. Hodkinson proved that RCA_n is not closed under completion. We refine this result by showing that even $RCA_n \cap Nr_nCA_{n+k}$ is not closed under completion for all finite $n > 2$ and $k \geq 0$.

The layout of this paper is as follows: In section 1.1 we formulate our main result in logical form, in section 1.2 we formulate our main result in algebraic form and prove the logical result (strong failure of the OTP for $\mathcal{L}_n$) modulo the algebraic result. In section 2 we recall the necessary algebraic machinery from the literature. In sections 3,4 we present the new construction for how to construct representable but not completely representable relation and cylindric algebras and state the necessary conditions for our desired properties. In section 5 we give concrete algebras that fit our bill and with this we prove our main result in algebraic form.

1.1 The main result in logical form

We work in usual first order logic. In the process, we use standard notation. In particular, $\models$ denotes the usual satisfiability relation. For a formula φ and a first order structure $\mathcal{M}$ in the language of φ we write $\varphi^{\mathcal{M}}$ to denote the set of all assignments that satisfy φ in M, i.e.

$$\varphi^{\mathcal{M}} = \{s \in {}^{\omega}M : \mathcal{M} \models \varphi[s]\}.$$

For example if $\mathcal{M} = (\mathbb{N}, <)$ and φ is the formula $x_1 < x_2$ then a sequence $s \in {}^{\omega}\mathbb{N}$ is in $\varphi^{\mathcal{M}}$ iff $s_1 < s_2$. Let Γ be a set of formulas (Γ may contain free variables). We say that Γ is *realized* in $\mathcal{M}$ if $\bigcap_{\varphi\in\Gamma} \varphi^{\mathcal{M}} \neq \emptyset$. Let φ be a formula and T be a theory. We say that φ *ensures* Γ in T if $T \models \varphi \to \gamma$ for all $\gamma \in \Gamma$.

The classical Henkin-Orey Omitting Types Theorem, OTT for short, [11, Theorem 2.2.9], or rather the contrapositive thereof, states that if T is a complete, consistent theory in a countable language $\mathcal{L}$ and $\Gamma(x_1, \ldots, x_n) \subseteq \mathcal{L}$ is realized in all models of T, then there is a formula $\varphi \in \mathcal{L}$ such that φ ensures Γ in T and $T \models (\exists x_1 \ldots x_n)\varphi$. The formula φ is called a T*-witness* for Γ.

The question we address here is: *Can we always guarantee that the witness uses the same number of variables as T and Γ, or do we need extra variables? If we do need extra variables, is there perhaps an upper bound on the number of extra variables needed?* In other words, let $\mathcal{L}_n$ denote the set of formulas of $\mathcal{L}$ which are built up by using only n variables. The question is: if $T \cup \Gamma \subseteq \mathcal{L}_n$, can we guarantee that the witness stays in $\mathcal{L}_n$, or do we occasionally have to "step outside" $\mathcal{L}_n$?

Assume that $T \subseteq \mathcal{L}_n$. We say that T is n-complete iff for all sentences $\varphi \in \mathcal{L}_n$ we have that either $T \models \varphi$ or $T \models \neg\varphi$. We say that T is n-atomic iff for all $\varphi \in \mathcal{L}_n$ there is $\psi \in \mathcal{L}_n$ such that $T \models \psi \to \varphi$ and for all $\chi \in \mathcal{L}_n$ either $T \models \psi \to \chi$ or $T \models \psi \to \neg\chi$.

Theorem 1.1 . *Assume that $\mathcal{L}$ is a countable first order language containing a binary relation symbol. For all $n > 2$ and $k \geq 0$ there are a consistent n-complete and n-atomic theory T using only n variables, and a set $\Gamma(x_1)$ using only 3 variables (and one free variable x_1) such that Γ is realized in all models of T, but each T-witness for Γ uses more than $n+k$ variables.*

The proof of Theorem 1.1 uses methods from algebraic logic, and indeed, Theorem 1.1 has an algebraic formulation (Theorem 1.2) which we are going to state now. We note that the algebraic form is stronger than the logical one.

1.2 The main result in algebraic form

For undefined terminology in the coming theorem the reader is referred to [20], [27], or [36]. All such notions, however, will be recalled below in section 2. The novelty in the following Theorem 1.2 is (ii)-(iv) and (vi). Relation and cylindric algebras satisfying (i) and (v) were given by Hirsch and Hodkinson in [23], and by Hodkinson in [30]. See also [27, Chapter 17.2] and [26]. In the next section we give an entirely new construction for Theorem 1.2 (i) and (v), too.

Theorem 1.2 . *Let $n > 2$ and $k \geq 0$ be finite. There is a countable symmetric, simple, integral, atomic relation algebra $\mathcal{R}$ such that the following hold:*

(i) $\mathcal{R}$ is representable, but not completely representable.

(ii) $\mathcal{R}$ is generated by a single element.

(iii) The set B_n of all n by n basic matrices over $\mathcal{R}$ constitutes an n-dimensional cylindric basis in the sense of Maddux [36, Def. 4]. Thus B_n is a cylindric atom structure and the full complex algebra $\mathcal{C}m(B_n)$ with universe the power set of B_n is an n-dimensional cylindric algebra.

(iv) The subalgebra $\mathcal{C}$ of $\mathcal{C}m(B_n)$ generated by the n by n basic matrices is representable, but $\mathcal{C}m(B_n)$ is not representable.

(v) Hence $\mathcal{C}$ is a simple, atomic, representable but not completely representable CA_n.

(vi) Further, $\mathcal{C}$ is generated by a single 2-dimensional element g, the relation algebraic reduct of $\mathcal{C}$ does not have a complete representation and is also generated by g (as a relation algebra), and $\mathcal{C}$ is a sub-neatreduct of some simple representable $\mathcal{D} \in CA_{n+k}$ such that the relation algebraic reducts of $\mathcal{C}$ and $\mathcal{D}$ coincide with $\mathcal{R}$.

Proof of Theorem 1.1 modulo Theorem 1.2. We may assume that we have one binary relation symbol in our language $\mathcal{L}$. First we give T, Γ such that Γ uses two free variables x, y, and not only one. Then we will "code" the two variables x, y into one variable. Some preliminaries: Assume that $\mathcal{M} = \langle M, G\rangle$ is a model. Then $\mathcal{C}_n(\mathcal{M}) = \{\varphi^{\mathcal{M}} : \varphi \in \mathcal{L}_n\}$ denotes the algebra of all n-place definable relations in $\mathcal{M}$. The operations of $\mathcal{C}_n(\mathcal{M})$ are the Boolean set operations together with the so-called cylindrifications $C_i(\varphi^{\mathcal{M}}) = (\exists v_i\varphi)^{\mathcal{M}}$ and the diagonal constants $D_{ij} = (v_i = v_j)^{\mathcal{M}}$. Thus we have finitely many unary operations (C_i) and constants (D_{ij}) in addition to the Boolean operations $(i, j < n)$. For any term $\tau(x)$ in the algebraic language of $\mathcal{C}_n(\mathcal{M})$ with one free variable x there is a formula $\varphi \in \mathcal{L}_n$, and vice versa, such that $\tau^{\mathcal{C}}(G) = \varphi^{\mathcal{M}}$ for any model $\mathcal{M} = \langle M, G\rangle$ where $\mathcal{C} = \mathcal{C}_n(\mathcal{M})$.

We are ready to define T, Γ. Let $g, \mathcal{C}, \mathcal{D}$ be as in Thm.1.2(vi). Then g generates C and g is 2-dimensional in $\mathcal{C}$. We can write up a theory $T \subseteq \mathcal{L}_n$ such that for any model $\mathcal{M}$ we have

$\mathcal{M} = \langle M, G\rangle \models T$ iff ($\mathcal{C}_n(\mathcal{M})$ is isomorphic to $\mathcal{C}$ such that G corresponds to g via this isomorphism).

For example, we can take $T = \{\varphi \in \mathcal{L}_n : \tau_\varphi(g) = 1 \text{ in } \mathcal{C}\}$, where τ_φ is the cylindric algebraic term corresponding to the formula φ. Now, $T \subseteq \mathcal{L}_n$, T is consistent and n-complete, n-atomic because $\mathcal{C}$ is simple and atomic. We now specify our $\Gamma(x, y)$. For any atom $a \in At^+$ let τ_a be a relation algebraic term such that $\tau_a(g) = a$ in $\mathcal{R}$, the relation algebra reduct of $\mathcal{C}$. For each τ_a there is a formula $\gamma_a(x, y) \in \mathcal{L}_3$ such that $\tau_a(g) = \gamma_a^{\mathcal{M}}$, see e.g. [59] or [8, Thm.9(ii), p.151]. We define $\Gamma(x, y) = \{\neg\gamma_a : a \in At^+\}$. We now show that Γ is realized in each model of T. Let $\mathcal{M} \models T$ be any model. Then $\mathcal{C}_n(\mathcal{M}) \cong \mathcal{C}$, hence $\mathcal{M}$ gives a representation for $\mathcal{R}$ because $\mathcal{R}$ is the relation algebraic reduct of $\mathcal{C}_n(\mathcal{M})$. But $\mathcal{R}$ does not have a complete representation, which means that $X = \bigcup\{\gamma_a^{\mathcal{M}} : a \in At^+\} \subsetneq M \times M$, say $(u, v) \in M \times M - X$. This means that Γ is realized at (u, v) in $\mathcal{M}$. We have seen that Γ is realized in each model of T. Assume now that $\varphi \in \mathcal{L}_{n+k}$ is such that $T \models \exists\overline{x}\varphi$. We may assume that φ has only two free variables, say, x, y. Take the representable $\mathcal{D} \in CA_{n+k}$ from Theorem 1.2(vi). Recall that $g \in C \subseteq D$ and $\mathcal{D}$ is simple. Let $\mathcal{M} = \langle M, g\rangle$ where M is the base set of $\mathcal{D}$. Then $\mathcal{M} \models T$ because $\mathcal{C}$ is a subreduct of $\mathcal{D}$ generated by g. By $T \models \exists\overline{x}\varphi$ then $\varphi^{\mathcal{M}} \neq \emptyset$. Also, $\varphi^{\mathcal{M}} \in D$ and $\varphi^{\mathcal{M}}$ is 2-dimensional, hence $\varphi^{\mathcal{M}} \in R$ because $\mathcal{R}$ is the relation algebraic reduct of $\mathcal{D}$, too. But $\mathcal{R}$ is atomic, hence $\varphi^{\mathcal{M}} \cap \gamma_a \neq \emptyset$ for some $a \in At^+$. This shows that $\mathcal{M} \not\models \varphi \to \neg\gamma_a$ where $\neg\gamma_a \in \Gamma$, thus φ is not a T-witness for Γ.

Now we modify T, Γ so that Γ uses only one free variable. We use the technique of so-called partial pairing functions. Let $g, \mathcal{C}, \mathcal{D}$ be as in Thm.1.2(vi) with $\mathcal{D} \in CA_{2n+2k}$. We may assume that g is disjoint from the identity $1'$ because $1'$ is an atom in the relation algebraic reduct of $\mathcal{C}$. Let U be the base set of $\mathcal{C}$. We may assume that U and $U \times U$ are disjoint. Let $M = U \cup (U \times U)$, let $G = g \cup \{\langle u, (u,v)\rangle : u, v \in U\} \cup \{\langle (u,v), v\rangle : u, v \in U\} \cup \{\langle (u,v),(u,v)\rangle : u, v \in U\}$, and let $\mathcal{M} = \langle M, G\rangle$. From G we can define $U \times U$ as $\{x : G(x,x)\}$, and from $U \times U$ and G we can define the projection functions between $U \times U$ and U, and g. (This means that $\mathcal{M}$ is a definitional extension of $\langle U, g\rangle$ with a new sort or "universe" $U \times U$, in the sense of [34, sec. 4.3].) All these definitions use only 3 variables. Thus for all $t \geq 3$ and for all $\varphi(x,y) \in \mathcal{L}_t$ there is a $\psi(x) \in \mathcal{L}_t$ such that $\psi^{\mathcal{M}} = \{(u,v) \in U \times U : \varphi^{\langle U,g\rangle}(u,v)\}$. For any $a \in At^+$ let $\psi_a(x)$ be the formula corresponding to $\gamma_a(x,y)$ this way. Conversely, for any $\psi \in \mathcal{L}_t$ there is $\varphi \in \mathcal{L}_{2t}$ such that "the projection of $\psi^{\mathcal{M}}$ to U" is $\varphi^{\langle U,g\rangle}$. Let us define now T as the $\mathcal{L}_n$-theory of $\mathcal{M}$, and let us define $\Gamma(x) = \{\neg\psi_a(x) : a \in At^+\}$. These T, Γ will do. ■

2 Definitions

2.1 Algebras of relations, duality with relational structures

Algebras of relations arise naturally in logic, e.g. the algebra of n-place definable relations of a model is very useful. Let ω denote the smallest infinite ordinal. Let U be a set and $n < \omega$. Then $\mathcal{R}el_n(U)$ denotes the algebra whose carrier-set (or *universe*) is the set of all n-place relations on U, i.e. the powerset of ${}^nU$, the set of all U-termed sequences of length n, and whose operations are the Boolean set operations together with the so-called *cylindrifications* C_i and *diagonal elements* D_{ij} for $i, j < n$:

$D_{ij} = \{s \in {}^nU : s_i = sj\}$, and for $x \subseteq {}^nU$

$C_i x = \{s(i/u) : s \in x,\ u \in U\}$, where

$s(i/u)$ denotes the sequence s modified at place i to u.

For a model $\mathcal{M}$ with base set U if $x = \varphi^{\mathcal{M}}$, then $C_i x = (\exists x_i \varphi)^{\mathcal{M}}$ and $D_{ij} = (x_i = x_j)^{\mathcal{M}}$. For binary relations of U, we often use a natural expansion $\mathcal{R}el(U)$ of $\mathcal{R}el_2(U)$. We get $\mathcal{R}el(U)$ from $\mathcal{R}el_2(U)$ by adding the operations of forming *converse* and *composition* of binary relations x, y,

$x^{\smile} = \{(u,v) : (v,u) \in x\}$, and

$x;y = \{(u,v) : (\exists z)[(u,z) \in x \text{ and } (z,v) \in y]\}$.

In $\mathcal{R}el(U)$ we denote the constant D_{01} by $1'$ and we call it *the identity* constant, and we omit the cylindrifications C_0, C_1 because they can be defined from composition: $C_0x = 1; x,\ C_1x = x; 1$ where 1 denotes the *unit* of the algebra, $U \times U$.

These algebras are Boolean algebras with additional operations that are additive w.r.t. the operation $+$ of forming "union". Such algebras are called *Boolean algebras with operators*, in short *BAO*'s. We denote the operations of a Boolean algebra as $+, \cdot, -, 0, 1$ (corresponding to forming "union", "intersection", "complement", "empty set", and "biggest set" respectively). A natural duality between *BAO*'s and relational structures is worked out in Jónsson-Tarski [33], as follows. Let $\mathcal{M} = \langle U, R_i \rangle_{i\in I}$ be any relational structure. Then its *complex algebra*, $\mathcal{C}m(\mathcal{M})$ is defined as follows. The universe of $\mathcal{C}m(\mathcal{M})$ is the powerset of U. The operations of $\mathcal{C}m(\mathcal{M})$ are the Boolean set operations together with the operations f_i that arise from the relations R_i as follows:

$$f_i(X_1, \dots, X_n) = \{u \in U : (\exists u_1 \in X_1, \dots, u_n \in X_n) R_i(u_1, \dots, u_n, u)\},$$

where $X_1, \dots, X_n \subseteq U$ and f_i is an n-place operation if R_i is an $n+1$-place relation in $\mathcal{M}$. The subsets X_i are often called "complexes" of U, hence the name "complex algebra". The complex algebras are atomic *BAO*'s. Conversely, assume that $\mathcal{C} = \langle C, +, -, f_i \rangle_{i\in I}$ is an atomic *BAO*. Then $At(\mathcal{C})$ denotes the set of all atoms of $\mathcal{C}$. The *atom structure* $\mathcal{A}t(\mathcal{C})$ of $\mathcal{C}$ is defined as follows. The universe of $\mathcal{A}t(\mathcal{C})$ is $At(\mathcal{C})$, and the relations of $\mathcal{A}t(\mathcal{C})$ are

$$R_i(a_1, \dots, a_n, a) \quad \text{iff} \quad a \leq f_i(a_1, \dots, a_n) \text{ in } \mathcal{C}.$$

Viewed as operators, $\mathcal{C}m$ and $\mathcal{A}t$ are in some way *dual* to each other. If we apply $\mathcal{A}t\mathcal{C}m$ to an atom structure, we get back the original atom structure. As for $\mathcal{C}m\mathcal{A}t$ applied to a given atomic Boolean algebra with operators, we may not get the original algebra but possibly a bigger one, in fact we get its *minimal* completion in the sense of Monk [40]. We recall that the minimal completion is the smallest algebra containing the original algebra and closed under arbitrary suprema.

The class of n-dimensional cylindric algebras is denoted as CA_n while that of relation algebras is denoted as RA. CA_n and RA are classes of algebras defined by equations valid in all $\mathcal{R}el_n(U)$ and in all $\mathcal{R}el(U)$, respectively. In this paper we will deal with *simple* algebras only, i.e. we will deal with algebras that have no proper congruences. (We note that simple algebras correspond to models while arbitrary algebras correspond to classes of models in algebraic logic.) A *cylindric algebra* $\mathcal{C} \in CA_n$ is simple iff $x > 0 \to C_0 \dots C_{n-1}x = 1$ is valid in it, and a *relation algebra* $\mathcal{R} \in RA$ is simple iff $x > 0 \to 1; x; 1 = 1$ is valid in it. A relation algebra is called *symmetric* iff $x = x^{\smile}$ is valid in it, and it is called *integral* iff $1'$ is an atom in it.

All the algebras $\mathcal{R}el_n(U), \mathcal{R}el(U)$ are simple. We call a simple algebra in CA_n or RA *representable* iff it is embeddable in $\mathcal{R}el_n(U)$ or $\mathcal{R}el(U)$ for some U. A *representation* of an algebra $\mathcal{C}$ is an embedding *rep* into $\mathcal{R}el_n(U)$ or

$\mathcal{R}el(U)$. A representation of $\mathcal{C}$ is called *complete* iff it takes all, even infinite, suprema of elements of C to unions, i.e. if we have

$$rep(\sum\{x_i : i \in S\}) = \bigcup\{rep(x_i) : i \in S\} \quad \text{whenever } \{x_i : i \in S\} \subseteq C.$$

An algebra is called *completely representable* if it has a complete representation. It is proved in [23] that a Boolean algebra has a complete representation iff it is atomic. A representation of a relation or cylindric algebra $\mathcal{C}$ is complete iff the union of the representations of the atoms is the *unit*, i.e. the biggest set (of $\mathcal{R}el_n(U)$ or of $\mathcal{R}el(U)$). Complete representability of $\mathcal{C}$ implies representability of $\mathcal{C}m\mathcal{A}t\mathcal{C}$.

2.2 Connection between algebras of relations; neat reducts and cylindric bases

In an algebra of n-ary relations we can recover the algebra of k-ary relations for $k \leq n$ by identifying $R \subseteq {}^kU$ with $R \times {}^{n-k}U$. Abstractly, if $\mathcal{C} \in CA_n$, then let

$$Nr_k\mathcal{C} = \{x \in C : x = C_kC_{k+1}\dots C_{n-1}x\},$$

and let $\mathcal{R}d_k\mathcal{C}$ denote the reduct of $\mathcal{C}$ where we "forget" the operations C_i and D_{ij} if $i \geq k$ or $\{i, j\} \nsubseteq k$ resp. Then $Nr_k\mathcal{C}$ is closed under the operations of $\mathcal{R}d_k\mathcal{C}$, hence we define $\mathcal{N}r_k\mathcal{C}$ as the subalgebra of $\mathcal{R}d_k\mathcal{C}$ with universe $Nr_k\mathcal{C}$. Now, $\mathcal{R}el_k(U) \cong \mathcal{N}r_k\mathcal{R}el_n(U)$ for any $k \leq n < \omega$. We can recover the operations of conversion and composition on binary relations in $\mathcal{R}el_n(U)$ for $n \geq 3$ as follows (cf. [20, Def.5.3.7]). Let $\mathcal{C} \in CA_n$ and $x, y \in Nr_2\mathcal{C}$. Then we define

$$x; y = C_2(C_1(D_{10} \cdot x) \cdot C_0(D_{02} \cdot y)),$$

$$x^{\smile} = C_2(D_{20} \cdot C_0(D_{01} \cdot C_1(D_{12} \cdot x))).$$

These definitions imitate the first-order logic definitions of relation composition and conversion, respectively. The *relation algebra reduct* $\mathcal{R}a\mathcal{C}$ of $\mathcal{C}$ is defined as the expansion of $\mathcal{N}r_2\mathcal{C}$ with the above $^{\smile}$ and ; (and then forgetting C_0, C_1, D_{10}). We have $\mathcal{R}el(U) \cong \mathcal{R}a\mathcal{R}el_n(U)$ for any $n \geq 3$. In general, $\mathcal{R}a\mathcal{C}$ is a relation algebra if $\mathcal{C} \in CA_n$ and $n > 3$.

The algebra $\mathcal{N}r_k\mathcal{C}$ is called the *neat k-reduct* of $\mathcal{C}$. Nr_kCA_n denotes the class of all neat k-reducts of CA_n's. Neat reducts were introduced by Leon Henkin in [18, p.40]. A classical result of Henkin, the so called Neat Embedding Theorem, or NET for short [20] says that the class of representable cylindric algebras coincides with the class of algebras which embed into a neat-reduct of a cylindric algebra with infinitely many extra dimensions. Variations on the NET give results as to which representable algebras are completely representable [49]. Other variations on the NET give classes of representable cylindric algebras that have the so called strong amalgamation property [35], [53], [56]. Neat reducts proved immensely fruitful not only for representation theory but also for such seemingly remote areas as positive solutions of the finitization problem in algebraic logic, see [42], [45], [46]. Neat reducts is a venerable old notion in cylindric algebras that is gaining some momentum lately, see e.g. [37],[42],[1],[12],[45],[24],[25],[56],[47],[51],[57],[13].

We have seen that we can construct a relation algebra $\mathcal{R}a\mathcal{C}$ from each $\mathcal{C} \in CA_n$ if $n > 3$. Conversely, we can build CA_n's from relation algebras in special cases, via so-called cylindric bases. We recall the definitions from Maddux [36]. Let $\mathcal{R} \in RA$. Let $n > 2$. $B_n = B_n\mathcal{R}$ is the set of all n by n matrices of atoms of $\mathcal{R}$ which satisfy the following conditions for all $i, j, k < n$.

$m_{ii} \leq 1'$, $\quad m_{ij} = m_{ji}^{\smile}$, $\quad$ and $\quad m_{ij} \leq m_{ik}; m_{kj}$.

The matrices in B_n are called *atom matrices* or *basic matrices.* Two atom matrices m and m' in B_n are said to *agree up to k* if $m_{ij} = m'_{ij}$ whenever $i, j \in n - \{k\}$. For any $i, j < n$, let

$T_i = \{(m, m') \in B_n \times B_n : m$ and m' agree up to i $\}$,

$E_{ij} = \{m \in B_n : m_{ij} \leq 1'\}$, $\quad$ and for every $M \subseteq B_n$ we let

$\mathcal{C}aM = \mathcal{C}m\langle M, T_i, E_{ij}\rangle_{i,j<n}$.

We say that $M \subseteq B_n$ is an *n-dimensional cylindric basis for $\mathcal{R}$*, if the following hold.

(1) If $a, b, c \in At\mathcal{R}$, and $a \leq b; c$ then there is a basic matrix $m \in M$ such that $m_{01} = a$, $m_{02} = b$ and $m_{21} = c$.

(2) If $m, m' \in M$, $i, j < n$, $i \neq j$ and m agrees with m' up to i, j then there is some $m'' \in M$ such that m'' agrees with m up to i and m'' agrees with m up to j, i.e $(m, m'') \in T_i$ and $(m'', m') \in T_i$, or simply $(m, m') \in T_i; T_j$.

(3) If $m \in M$ and $i, j < n$ then $m[i/j] \in M$, $\quad$ where $m[i/j]$ is the unique element of E_{ij} which agrees with m up to i.

We recall the following theorem from Maddux [36], which says how one obtains cylindric algebras from relation algebras possessing cylindric bases.

Theorem ([36, Thm.10]) . *Assume $\mathcal{R}$ is an atomic relation algebra, $2 < n$ is finite and M is an n-dimensional cylindric basis for $\mathcal{R}$. For all $x \in R$, let*

$$h(x) = \{m \in M : m_{01} \leq x\}.$$

Then $\mathcal{C}aM$ is a complete atomic CA_n and h is an embedding of $\mathcal{R}$ into the relation algebra reduct of $\mathcal{C}aM$.

3 Construction of relation algebras

In this paper we will deal with symmetric relation algebras only. We restrict ourselves to symmetric relation algebras for convenience only, everything what we do in this paper works for non-symmetric relation algebras, too. Working with symmetric relation algebras simplifies the investigations because we do not have to deal with the operation of conversion.

In a simple, symmetric relation algebra the identity $1'$ is always an atom. The reason is that $(x;1;y)^{\smile} = y;1;x \neq x;1;y$ if $x, y \leq 1'$ are disjoint nonzero subidentity elements. Hence a simple, symmetric relation algebra is always integral.

Let $T \subseteq U \times U \times U$ be a ternary relation on U. We will write $T(a,b,c)$ for $(a,b,c) \in T$. We call T *symmetric* iff $T(a,b,c)$ implies $T(a',b',c')$ for all permuted versions (a',b',c') of (a,b,c). Atomic, simple, symmetric RA's arise from atom structures of form

$\langle H, T, 1'\rangle$, where $1' \in H$ and T is a symmetric ternary relation on H such that $T(1',a,b)$ iff $a = b$ and $(\forall a,b,a',b',c \in H)(\exists d \in H)[(T(a,b,c) \wedge T(a',b',c)) \rightarrow (T(a,a',c) \wedge T(b,b',d)]$.

If x is a set, then $\mathsf{Cof}(x)$ denotes the set of all finite and cofinite subsets of x, and $\mathsf{Cof}^{\infty}(x)$ denotes the set of infinite elements of $\mathsf{Cof}(x)$, i.e.

$\mathsf{Cof}(x) = \{y \subseteq x : |y| < \omega \text{ or } |x - y| < \omega\}$.

General construction: "blow up and blur"

Assume that $\mathfrak{M}$ is a simple, symmetric, atomic relation algebra. We will replace each non-identity atom of $\mathfrak{M}$ with an infinite set of new atoms, and we will define the operations on the new atoms such that the structure of the original $\mathfrak{M}$ cannot be seen on the level of the new atoms, yet $\mathfrak{M}$ will still be there on the global level. By this we mean that $\mathfrak{M}$ will be a subalgebra of the completion of the new algebra, but in general $\mathfrak{M}$ will not be a subalgebra of the new algebra, see Theorem 3.2.

To "hide" the structure of $\mathfrak{M}$ on the level of the new atoms, we will use entities that we call "blur"s. The existence of such "blur"s will be shown in section 5. Conditions (J1)-(J4)$_n$ and (E1)-(E4) in Def.3.1 below are tailored for our specific purposes in the present paper. (For different purposes, different conditions will suffice.)

Definition 3.1. (blur)

(i) Let I denote the set of all non-identity atoms of $\mathfrak{M}$. Let U, V, W be subsets of I. We say that (U, V, W) is *safe*, in symbol $\mathsf{safe}(U, V, W)$, if $a \leq b; c$

in $\mathfrak{M}$ for all $a \in U,\ b \in V,\ c \in W$. We say that (U, V, W) is *unsafe* if it is not safe.

(ii) Let J be a subset of the powerset of I, and let $n \in \omega$. We say that J is an *n-complex-blur* for $\mathfrak{M}$ if conditions (J1)-(J4)$_n$ below are satisfied.

(J1) Each element of J is nonempty and finite.

(J2) $\bigcup J = I$, i.e. $(\forall a \in I)(\exists W \in J) a \in W$.

(J3) $(\forall P \in I)(\forall W \in J) P; W \supseteq I$.

(J4)$_n$ $(\forall V_2, \ldots, V_n, W_2, \ldots, W_n \in J)(\exists T \in J)(\forall 2 \le i \le n)\mathsf{safe}(V_i, W_i, T)$.

(iii) Let E be a ternary relation on ω. We say that E is an *index-blur* if conditions (E1)-(E4) below are satisfied.

(E1) E is symmetric, i.e.
$(\forall i, j, k \in \omega)(E(i, j, k) \to [E(j, i, k) \text{ and } (E(i, k, j) \text{ and } E(k, j, i)]).$

(E2) $(\forall i \in \omega)(\exists s \in \omega)(\forall j \ge i + s)(\exists k) E(i, j, k)$.

(E3) $\{k : E(i, j, k)\}$ is finite for all $i, j \in \omega$.

(E4) $(\forall i, n \in \omega)(\exists j \ge n, k \ge n) E(i, j, k)$.

(iv) We say that (J, E) is an *n-blur* for $\mathfrak{M}$ iff J is an n-complex-blur for $\mathfrak{M}$ and $E = \langle E(U, V, W) : U, V, W \in J \rangle$ such that $E(U, V, W)$ is an index-blur for $\mathfrak{M}$, for all $U, V, W \in J$. □

We will often use the following corollary of (J1),(J2) and (J4)$_n$:

(J4c)$_n$ $(\forall P_2, \ldots, P_n, Q_2, \ldots, Q_n \in I)(\exists W \in J) W \cap P_2; Q_2 \cap \cdots \cap P_n; Q_n \ne \emptyset$.

Indeed, for each P_i, Q_i ($2 \le i \le n$) let $V_i, W_i \in J$ be such that $P_i \in V_i$ and $Q_i \in W_i$, such V_i, W_i exist by (J2). let $T \in J$ be such that $\mathsf{safe}(V_i, W_i, T)$ for all i. Such a T exists by (J4)$_n$. Then $T \ne \emptyset$ by (J1), and $T \cap P_2; Q_2 \cap \cdots \cap P_n; Q_n \ne \emptyset$ by the definition of a safe triple.

For simplicity, we shall often write $E(i, j, k)$ in place of $E(U, V, W)(i, j, k)$ when U, V, W are clear from context. We will use the flexibility that $E(U, V, W)$ may be different for different U, V, W's only in achieving one-generation of our algebras in section 5. (For all the other purposes in the present paper, we could have assumed the $E(U, V, W)$'s equal with each other.)

We are ready to define the "blow-up-and-blur" relation algebra $\mathfrak{M}^* = \mathsf{Bb}(\mathfrak{M}, J, E)$. Let $\mathfrak{M}$ be a simple, symmetric, atomic relation algebra and let (J, E) be a 3-blur for $\mathfrak{M}$. The non-identity atoms of $\mathfrak{M}^*$ are defined as

$$At = \{(i, P, W) \in \omega \times I \times J : P \in W\}.$$

Then the atoms of $\mathfrak{M}^*$ will be $At^+ = At \cup \{1'\}$, see Figure 1. For any $a = (i, P, W) \in At$ we define

$$\nu(a) = i, \quad I(a) = P, \quad J(a) = W.$$

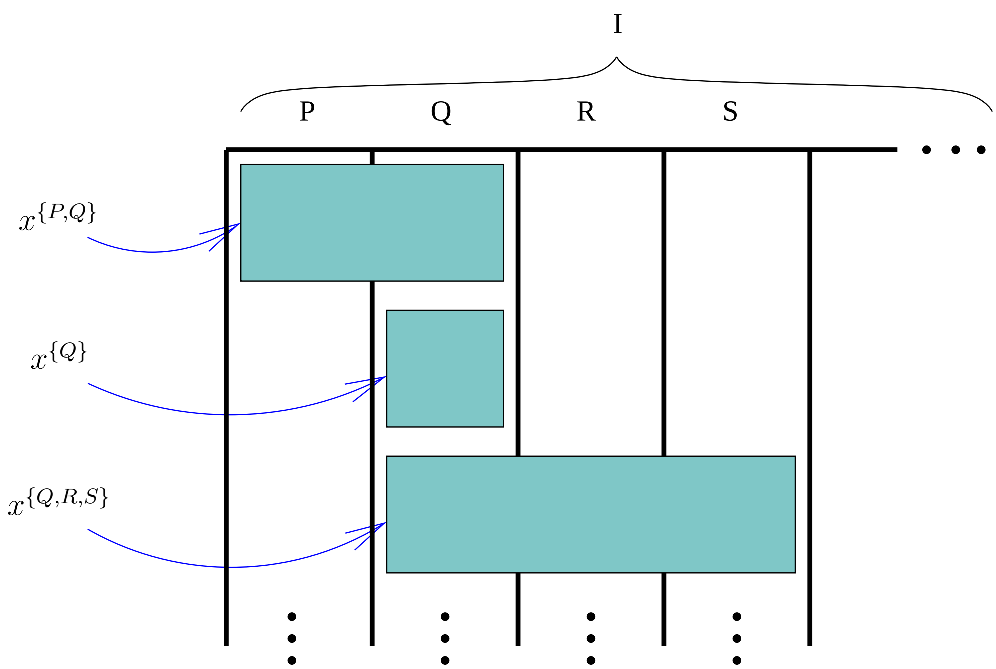

Figure 1: The non-identity atoms of the blow-and-blur relation algebra.

To define composition ; on the non-identity atoms of $\mathfrak{M}^*$, we let T to be the following ternary relation on At:

(TD) $T(a, b, c)$ iff $[\mathsf{safe}(J(a), J(b), J(c))$ or
$(I(a) \leq I(b); I(c)$ in $\mathfrak{M}$ and $E(J(a), J(b), J(c))(\nu(a), \nu(b), \nu(c)))]$.

The ternary relation T^+ on At^+ is defined by ($T^+(a, b, c)$ iff $[T(a, b, c)$ or one of $\{a, b, c\}$ is $1'$ and the other two are equal]. We define composition ; on the power-set of At^+ by $x; y = \{c \in At^+ : (\exists a \in x)(\exists b \in y)T^+(a, b, c)\}$. For all $W \in J$ we define $x^W = \{a \in At : J(a) = W\}$. We define

$$M^* = \{x \subseteq At^+ : (\forall W \in J)(x \cap x^W) \in \mathsf{Cof}(x^W)\}.$$

Theorem 3.2. *Let $\mathfrak{M}$ be a symmetric, atomic relation algebra and assume that (J, E) is a 3-blur for $\mathfrak{M}$. Then the following (i)-(iii) hold.*

(i) $\mathfrak{M}^*$ is a simple, atomic, symmetric representable relation algebra. The ultrafilter (or canonical) extension of $\mathfrak{M}^*$ is completely representable if $\mathfrak{M}$ is finite.

(ii) $\mathfrak{M}$ *is a subalgebra of the completion of* $\mathfrak{M}^*$*.*

(iii) The set B_n *of all* n *by* n *basic matrices for* $\mathfrak{M}^*$ *is a cylindric basis for* $\mathfrak{M}^*$ *if* $(\mathrm{J4})_n$ *holds.*

Proof. First we show that M^* is closed under the relation algebraic operations. By definition, $\langle x^W : W \in J\rangle$ is a partition of At. This, together with the definition of M^*, implies that M^* is closed under the Boolean operations. Next we show that M^* is closed under composition ;. We will prove this by proving statements (1a)-(1c) below.

(1a) Assume that $a, b \in At$. Then $|a;b \cap x^W| < \omega$ if $(J(a), J(b), W)$ is unsafe, and $a;b \supseteq x^W$ otherwise.

Proof of (1a). Assume that $(J(a), J(b), W)$ is unsafe. Then by (TD), $a;b \cap x^W \subseteq \{(k, P, W) : E(\nu(a), \nu(b), k),\ I(a) \le I(b); P \text{ in } \mathfrak{M},\ P \in W\}$. This last set is finite by (E3) and (J1). ∎

(1b) Assume that $a \in At,\ X \in \mathsf{Cof}^\infty(x^W)$. Then $(a; X \cap x^V) \in \mathsf{Cof}^\infty(x^V)$ for all $V \in J$.

Proof of (1b). If $\mathsf{safe}(J(a), W, V)$ then $a; X \supseteq x^V$ by (TD) and we are done. Assume therefore that $(J(a), W, V)$ is unsafe. Let $n \in \omega$ be such that $(\forall i \ge n)(\forall P \in W)(i, P, W) \in X$. Such an n exists because X is co-finite in x^W and W is finite by (J1). Let $Q \in V$ be arbitrary, let $i = \nu(a),\ P = I(a)$, i.e. $a = (i, P, J(a))$. Then $Q \in P; W$ by (J3), let $S \in W$ be such that $Q \le P; S$ in $\mathfrak{M}$. Let $N = \{k : (\exists j \le n) E(i, j, k)\}$. Now N is finite by (E3), let m be an upper bound for N. Let $k > m + i + s$ be arbitrary, where i satisfies (E2) with s. There is j such that $E(i, j, k)$, by (E2),(E1). Now, this j is bigger than n by $k \notin N$ and (E1). Hence $(j, S, W) \in X$ and so $(k, Q, V) \in a; X$ for all $k > m + i + s$. Since V is finite, we are done. ∎

(1c) Assume that $X \in \mathsf{Cof}^\infty(x^V),\ Y \in \mathsf{Cof}^\infty(x^W)$. Then $X; Y \supseteq At$.

Proof of (1c). Let $U \in J$ be arbitrary. If $\mathsf{safe}(V, W, U)$ then $X; Y \supseteq x^U$. Assume therefore that (V, W, U) is unsafe. Let $P \in U$ and $i \in \omega$ be arbitrary. Let n be a bound for X, Y, i.e. $(\forall k \ge n)(\forall P \in V)(k, P, V) \in X$ and similarly for Y. Let $j, k \ge n$ be such that $E(i, j, k)$. Such j, k exist by (E4). Let $R \in X,\ S \in Y$ be such that $P \le R; S$ in $\mathfrak{M}$. Such R, S exist by (J3). Now, $(i, P, U) \in (j, R, V); (k, S, W)$ by the definition of T, and since $(j, R, V) \in X,\ (i, S, W) \in Y$. ∎

Statements (1a)-(1c) show that M^* is closed under composition ;. Thus, $\mathfrak{M}^*$ is indeed an algebra. Associativity of ; is implied by $(\mathrm{J4})_3$. Thus, $\mathfrak{M}^*$ is an atomic, symmetric relation algebra. It is simple because $1;x;1=1$ is valid in it by $(J4)_3$.

Now we show that $\mathfrak{M}$ is a subalgebra of the completion of $\mathfrak{M}^*$. The completion of $\mathfrak{M}^*$ is the complex algebra $\mathcal{C}$ of the atom structure of $\mathfrak{M}^*$. The universe of $\mathcal{C}$ is the powerset of At^+. Let us define $h : M \to C$ by

$$h(x) = \{a \in At : I(a) \leq x\} \cup (x \cap 1').$$

We prove that h is an embedding. Let $P \in I$ be arbitrary. Then $h(P) = \{a \in At : I(a) \leq P\} = \{a \in At : I(a) = P\} \neq \emptyset$ by (J2). Clearly, $\langle h(P) : P \in I\rangle$ is a partition of At. By definition, h is completely additive. Since $\mathfrak{M}$ is atomic and I is the set of non-identity atoms of $\mathfrak{M}$, the above show that h is indeed a Boolean embedding. It remains to show that h respects composition. Since h is completely additive, it is enough to check this for the non-identity atoms of $\mathfrak{M}$. Let $P, Q \in I$ be arbitrary, $P \neq Q$. Now, $h(P);h(Q) = \{a \in At : I(a) \leq P\};\{a \in At : I(a) \leq Q\} \subseteq \{a \in At : I(a) \leq P;Q\} = h(P;Q)$, by the condition on $I(a), I(b), I(c)$ in the definition of $T(a, b, c)$. On the other hand, let $a \in At$ be arbitrary such that $I(a) \leq P;Q$, say, let $a = (m, R, U)$. Let V, W be arbitrary such that $P \in V,\ Q \in W$. Let i, j be such that $E(i, j, m)$. Such i, j exist by (E4). Now, $T((i, P, V), (j, Q, W), (m, R, U))$ by (TD), and $(i, P, V) \in h(P)$, $(j, Q, W) \in h(Q)$, so $a = (m, R, U) \in h(P);h(Q)$. Since $a \in h(P;Q)$ was chosen arbitrarily, this shows $h(P;Q) \subseteq h(P);h(Q)$. We have proved that $\mathfrak{M}$ is isomorphic to a subalgebra of the completion of $\mathfrak{M}^*$.

Next we show that $\mathfrak{M}^*$ is representable. For any $a \in At^+$ and $W \in J$ we set

$$U^a = \{x \in M^* : a \in x\} \quad \text{and} \quad U^W = \{x \in M^* : |x \cap x^W| \geq \omega\}.$$

Then the principal ultrafilters of $\mathfrak{M}^*$ are exactly U^a for $a \in At^+$ and U^W are non-principal ultrafilters for $W \in J$. Let

$$\mathsf{Uf} = \{U^a : a \in At^+\} \cup \{U^W : W \in J\}.$$

We note that when $\mathfrak{M}$ is finite, Uf is the set of all ultrafilters of $\mathfrak{M}^*$, but if $\mathfrak{M}$ is infinite, $\mathfrak{M}^*$ may have other ultrafilters, too. For $F, G, K \in \mathsf{Uf}$ we define $F;G = \{X;Y : X \in F,\ Y \in G\}$. The triple (F, G, K) is said to be *consistent* iff $F;G \subseteq K,\ F;K \subseteq G$ and $G;K \subseteq F$. We already proved as (1a)-(1c) the following:

(i) (U^a, U^b, U^W) is consistent whenever $a, b \in At$ and $\mathsf{safe}(J(a), J(b), W)$.

(ii) (F, G, K) is consistent whenever at least two of F, G, K are non-principal and $F, G, K \in \mathsf{Uf} - \{U^{Id}\}$.

Definition 3.3. (consistent colored graph)

(1) We call (G, l) a *consistent colored graph* if G is a set, and $l : G \times G \to \mathsf{Uf}$ such that for all $x, y, z \in G$ the following hold:
$l(x, y) = U^{Id}$ iff $x = y$, $\quad l(x, y) = l(y, x)$, $\quad$ and
the triple $(l(x, y), l(x, z), l(y, z))$ is consistent.

(2) We say that the consistent colored graph (G, l) is *complete* if for all $x, y \in G$, and $F, K \in \mathsf{Uf}$, whenever $(l(x, y), F, K)$ is consistent, there is a node z such that $l(z, x) = F$ and $l(z, y) = K$. $\square$

Consistent colored graphs can be used to build representations either in a step by step manner or by games [27]. We will build a complete consistent graph step-by-step. So assume (inductively) that (G, l) is a consistent colored graph and $(l(x, y), F, K)$ is a consistent triple. We shall extend (G, l) with a new point z such that $(l(x, y), l(z, x), l(z, y)) = (l(x, y), F, K)$. Let $z \notin G$. We define $l(z, p)$ for $p \in G$ as follows:

$l(z, x) = F$, $\quad l(z, y) = K$, $\quad$ and

if $p \in G - \{x, y\}$, then $l(z, p) = U^W$ for some $W \in J$ such that both $(U^W, F, l(x, p))$ and $(U^W, K, l(y, p))$ are consistent. Such a W exists by $(\text{J4c})_3$.

Condition (ii) above Def.3.3 guarantees that this extension is again a consistent colored graph.

We now show that any non-empty complete colored graph (G, l) gives a representation for $\mathfrak{M}^*$. For any $x \in M^*$ define

$rep(X) = \{(u, v) \in G \times G : X \in l(u, v)\}$.

We show that *rep* is a representation for $\mathfrak{M}^*$. *rep* is a Boolean homomorphism because all the labels $l(u, v)$ are ultrafilters. $rep(1') = \{(u, u) : u \in G\}$, and $rep(X)^{-1} = rep(X)$ for all $X \in M^*$. The latter follows from the second condition in the definition of a consistent colored graph. From the third condition in the definition of a consistent colored graph we have:

$$rep(X); rep(Y) \subseteq rep(X; Y).$$

Indeed, let $(u, v) \in rep(X)$, $(v, w) \in rep(Y)$. I.e. $X \in l(u, v)$, $Y \in l(v, w)$. Since $(l(u, v), l(v, w), l(u, w))$ is consistent, then $X; Y \in l(u, w)$, i.e. $(u, w) \in rep(X; Y)$. On the other hand, we have

$$rep(X; Y) \subseteq rep(X); rep(Y),$$

because (G, l) is complete and because (i) and (ii) hold. Indeed, let $(u, v) \in rep(X; Y)$. Then $X; Y \in l(u, v)$. We show that there are $F, K \in \mathsf{Uf}$ such that

$X \in F$, $Y \in K$ and $(l(u,v), F, K)$ is consistent. We distinguish between two cases:

Case 1. $l(u,v) = U^a$ for some $a \in At$. By $X;Y \in U^a$ we have $a \in X;Y$. Then there are $b \in X$, $c \in Y$ with $a \leq b;c$. Then (U^a, U^b, U^c) is consistent.

Case 2. $l(u,v) = U^W$ for some $W \in J$. Then $|X;Y \cap x^W| \geq \omega$ by $X;Y \in U^W$. Now if both X and Y are finite, then there are $a \in X$, $b \in Y$ with $|a;b \cap x^W| \geq \omega$. Then (U^W, U^a, U^b) is consistent by (i). Assume that one of X, Y, say X is infinite. Let $S \in J$ be such that $|X \cap x^S| \geq \omega$ and let $a \in Y$ be arbitrary. Then (U^W, U^S, U^a) is consistent by (ii) and $X \in U^S, Y \in U^a$.

Finally, rep is one-to-one because $rep(a) \neq \emptyset$ for all $a \in At$. Indeed $(u,u) \in rep(1')$ for any $u \in G$. Let $a \in At$. Then $(U^{1'}, U^a, U^a)$ is consistent, so there is a $v \in G$ with $l(u,v) = U^a$. Then $(u,v) \in rep(a)$. This finishes the proof of representability of $\mathfrak{M}^*$.

When $\mathfrak{M}$ is finite, Uf is the set of all ultrafilters of $\mathfrak{M}^*$. Then Uf is the set of atoms of the canonical (or ultrafilter) extension of $\mathfrak{M}^*$, and in this algebra for any $F, K, U \in \mathsf{Uf}$ we have that $F \leq K;U$ iff (F,K,U) is consistent. Hence the above constructed representation is a complete representation for the canonical extension because $\bigcup\{rep(F) : F \in \mathsf{Uf}\} = G \times G$.

We show that B_n is a cylindric basis, if $(\mathrm{J4})_n$ holds. To show that condition (1) in the definition of a cylindric basis holds, we show that each $m' \in B_3$ can be extended to an $m \in B_n$. Indeed, let $m \in B_n$ be defined as follows: $m_{ij} = m'_{ij}$ if $i,j < 3$, $m_{ij} = m'_{i2}$ if $i < 3, j \geq 3$, $m_{ij} = m'_{2j}$ if $i \geq 3, j < 3$, and $m_{ij} = 1'$ if $i,j \geq 3$. To show that condition (2) in the definition of a cylindric basis holds, let $m, m' \in B_n$ and $i,j < n$, $i \neq j$ be such that m agrees with m' up to i,j. Let $a \in \cap\{m'_{il}; m_{lj} : l \in n - \{i,j\}\}$ be arbitrary. There is such an a by $(\mathrm{J4c})_{n-1}$ if all the $m'_{i\ell}, m_{\ell j}$ are non-identity atoms, otherwise, if say $m_{i\ell} = 1'$ then $a = m_{\ell j}$ will do because of our hypotheses on m, m'. We now define $m'' \in B_n$ by defining $m''_{ij} = m''_{ji} = a$. The other members of m'' are defined as necessary, as follows: $m''_{kl} = m_{kl} = m'_{kl}$ if $\{k,l\} \cap \{i,j\} = \emptyset$, $m''_{il} = m''_{li} = m'_{il}$, $m''_{jl} = m''_{lj} = m'_{jl}$ for $l \in n - \{i,j\}$ and $m''_{ii} = 1'$. Then $m'' \in B_n$ and it has the desired properties. Condition (3) in the definition of a cylindric basis holds because $m[i/j] \in B_n$ always. We have shown that B_n is an n-dimensional cylindric basis. **QED(Theorem 3.2)**

4 Going to higher dimensions, cylindric algebras

In this section we show that the subalgebra of $\mathcal{C}mB_n$ generated by the atoms is representable, if we impose further conditions on $\mathfrak{M}$ and E, J. Throughout this section, $\mathfrak{M}$ is a simple, symmetric (hence integral) atomic relation algebra and we denote $\mathfrak{M}^*$ as $\mathcal{R}$. Thus $R = M^*$ and $\mathcal{R} = \mathfrak{M}^*$ from now on. We also fix a natural number $n \in \omega,\ n > 2$ and we assume that (J, E) is an n-blur for $\mathfrak{M}$.

Definition of the blow-up-and-blur cylindric algebra

We begin with giving a description of a subalgebra of $\mathcal{C}mB_n$ which contains all the atoms. We will assume that the relations $E(U, V, W)$ are defined by formulas in the first order language $\mathcal{L}(\omega, <)$ of the structure $\langle \omega, < \rangle$. We recall that there is a quantifier-elimination result for this theory, cf. [58, p.375]. In this quantifier-elimination, a basic formula with free variables $x_i : i < k$ orders 0 and the free variables linearly, and states about the distance of neighboring ones in this order either $= k$ or $> k$ for $k \in \omega$. E.g. a basic formula in this quantifier-elimination may state $3 = x_1 = x_2 < x_2 + 1 < x_3$. (For example, one can express $x + 2 < y$ by $\exists zw(x < z < w < y)$.) We say that the distance between x_i and x_j is *indefinite* according to φ when $\varphi \not\to (x_i = x_j + k$ or $x_j = x_i + k)$ for any $k \in \omega$. In the previous example, the distance between x_3 and x_1 is indefinite, while the distances between $0, x_2, x_1$ are *definite*. If $\varphi \in \mathcal{L}(\omega, <)$, then $\varphi(x_i : i < k)$ denotes that all the free variables of φ are among $\{x_i : i < k\}$ and if $a_0, \ldots, a_k \in \omega$, then $\varphi(x_i/a_i)_{i<k}$ denotes that the formula φ holds in $\langle \omega, < \rangle$ when the variables x_i are evaluated to a_i. We use von Neumann's convention that $n = \{0, 1, \ldots n-1\}$. We extend the notations $\nu(a),\ I(a)$ from $a \in At$ to all $a \in At^+$ by defining $\nu(1') = 0,\ I(1') = 1'$.

Definition 4.1. (diagram) Let $K \subseteq n$.

(i) By a *K-diagram* we understand a pair $\langle \varepsilon, \varphi \rangle$ where $\varepsilon : K \times K \to R$ and $\varphi(x_{ij} : i, j \in K) \in \mathcal{L}(\omega, <)$. By a *diagram* we understand an n-diagram.

(ii) The K-diagram $\langle \varepsilon, \varphi \rangle$ denotes an element $e(\varepsilon, \varphi) \in \mathcal{C}mB_n$ as follows
$$e(\varepsilon, \varphi) = e_n(\varepsilon, \varphi) = \{m \in B_n : (\forall i, j \in K)[m_{ij} \leq \varepsilon_{ij} \text{ and } \varphi(x_{ij}/\nu(m_{ij}))]\}.$$
$\square$

Let $a, b, c \in At^+ \cup \{x^W : W \in J\}$. Then $\mathsf{min}(a, b, c)$ denotes that either $a, b, c \in At^+$ and $a \leq b; c$, or else at least two of $\{a, b, c\}$ are not atoms and $b = c$ if $a = 1'$. This is equivalent to $(U(a), U(b), U(c))$ being consistent where $U(a) = U^a$ if $a \in At^+$ and $U(x^W) = U^W$ for $W \in J$.

We say that $\langle \varepsilon, \varphi \rangle$ is a *normal K-diagram* if it is a K-diagram and conditions (D1)-(D5) below hold for all $i, j, k \in K$.

(D1) $\varepsilon_{ij} \in At^+ \cup \{x^W : W \in J\}$.

(D2) $\mathsf{min}(\varepsilon_{ij}, \varepsilon_{ik}, \varepsilon_{kj}),\ \ \varepsilon_{ij} = \varepsilon_{ji},\ \ \varepsilon_{ii} = 1'$.

(D3) $\varphi \to x_{ij} = \nu(\varepsilon_{ij})$ whenever $\varepsilon_{ij} \in At^+$,
$\varphi \to x_{ij} = x_{ik}$ whenever $\varepsilon_{jk} = 1'$, and
$\varphi \to (x_{ii} = 0 \wedge x_{ij} = x_{ji})$.

(D4) $\varphi \to E(U,V,W)(x_{ij}, x_{jk}, x_{ik})$ whenever $\varepsilon_{ij} \leq x^U$, $\varepsilon_{jk} \leq x^V$, $\varepsilon_{ik} \leq x^W$ and (U,V,W) is unsafe. We say that i,j,k *is ε-unsafe* iff (U,V,W) is unsafe where U,V,W are as above.

(D5) φ is a basic formula according to the quantifier-elimination of the first-order theory of $\langle \omega, < \rangle$. Further, in the order induced by φ, variables denoting atoms are all at the beginning, and between them and the first non-atom variable there is an indefinite distance. We will call such formulas *ε-basic formulas*, or just *basic formulas.*

ND_K denotes the set of all normal K-diagrams, $\mathsf{ND} = \mathsf{ND}_n$. We note that each $m \in B_n$ defines a normal diagram in a natural way, i.e. if $\varepsilon_{ij} = m_{ij}$ for all $i,j < n$ and φ is $\bigwedge\{x_{ij} = \nu(m_{ij}) : i,j < n\}$, then $e(\varepsilon, \varphi) = \{m\}$. We will identify $m \in B_n$ with this normal diagram.

Lemma 4.2. *Assume that $\mathfrak{M}$ is finite, (J,E) is an n-blur for $\mathfrak{M}$, and each $E(U,V,W)$ is definable in $\langle \omega, < \rangle$. Let $K \subseteq n$. Then each K-diagram is a finite sum of normal K-diagrams. I.e., for any K-diagram $\langle e, \varphi \rangle$ there are $d_1, \ldots, d_t \in \mathsf{ND}_K$ such that $e(\varepsilon, \varphi) = e(d_1) \cup \cdots \cup e(d_t)$.*

Proof. Since $\mathfrak{M}$ is finite, I is finite, too. Hence each element of R is a finite sum of atoms and elements of form $x^{W,k} = \{(i,P,W) : i \geq k, P \in W\}$. Thus a K-diagram $d = \langle \varepsilon, \varphi \rangle$ is a finite sum of K-diagrams of form $\langle \eta, \psi \rangle$ where each η_{ij} is an atom or of form x^W, and ψ is a conjunction of φ with formulas $x_{ij} \geq k$ if $\varepsilon_{ij} = x^{W,k}$ for some $k \in \omega$. Thus we may assume that condition (D1) holds for d. We can replace each ε_{ii} with $1' \cap \varepsilon_{ii}$ and each ε_{ij} with $\varepsilon_{ij} \cap \varepsilon_{ji}$, and get the same set of matrices. If $1' \cap \varepsilon_{ii} \neq 1'$ then $e(\varepsilon, \varphi) = \emptyset$, so we may assume that $\varepsilon_{ii} = 1'$ and $\varepsilon_{ij} = \varepsilon_{ji}$ for all $i,j \in K$. If i,j,k is ε-unsafe, and two of $\varepsilon_{ij}, \varepsilon_{jk}, \varepsilon_{ik}$ are atoms, then we may assume that the third one is an atom, too, by (E3) and (TD). If $\varepsilon_{jk} = 1'$ then $e(\varepsilon, \varphi) = e(\varepsilon, \varphi \wedge x_{ij} = x_{jk})$ for any $i < n$, so let us conjunct $\bigwedge\{x_{ij} = x_{ik} : i < n\}$ to φ when $\varepsilon_{jk} = 1'$. To satisfy (D4), let us conjunct formulas defining $E(U,V,W)$ to φ whenever i,j,k is ε-unsafe and U,V,W are the elements of J "occurring on the sides of i,j,k" (i.e. $\varepsilon_{ij} \leq x^U$ etc). Also, let us conjunct formulas $x_{ii} = 0 \wedge x_{ij} = x_{ji}$ to φ. Then the element denoted by the diagram does not change and (D4) will be satisfied. Now, under these conditions, if either (D3) or $\mathsf{min}(\varepsilon_{ij}, \varepsilon_{ik}, \varepsilon_{kj})$ for some $i,j,k \in K$ is not satisfied, then $e(d) = \emptyset$. Thus we may assume that (D2),(D3) are satisfied. So far we may assume that (D1)-(D4) hold for d. By the quantifier-elimination result for $\langle \omega, < \rangle$, φ is a disjunction of basic formulas for the quantifier-elimination, and by (D3),(D4) we may assume that all the atoms are at the beginning and there is an indefinite distance between them and the non-atom ε_{ij}'s. **QED(Lemma 4.2)**

In the following, we will use a further condition $(J5)_n$. It is a stronger version of $(J4c)_n$ where "$(\exists W \in J)$" is replaced by "$(\forall W \in J)$". The existence of blurs satisfying $(J5)_n$ will be shown in section 5.

$(J5)_n$ $(\forall P_2, \ldots, P_n, Q_2, \ldots, Q_n \in I)(\forall W \in J) W \cap P_2; Q_2 \cap \cdots \cap P_n; Q_n \neq \emptyset$.

Lemma 4.3. *Assume that $\mathfrak{M}$ is finite, (J, E) is an n-blur for $\mathfrak{M}$, $(J5)_n$ holds, and each $E(U, V, W)$ is definable in $\langle \omega, < \rangle$. Let*

$$C = \{e(d_1) \cup \cdots \cup e(d_k) : d_1, \ldots, d_k \text{ are diagrams } \}.$$

Then C is closed under the operations of $\mathcal{C}m(B_n)$.

Proof. Let $d = \langle \varepsilon, \varphi \rangle$ be a diagram. For all $i, j < n$ let η^{ij} be defined as $\eta^{ij}_{ij} = -\varepsilon_{ij}$ and $\eta^{ij}_{kl} = 1$ if $kl \neq ij$. Let $u_{ij} = 1$ for all $i, j < n$. Then

$$-e(\varepsilon, \varphi) = \sum \{e(\eta^{ij}, \varphi) : i, j < n\} \cup e(u, \neg\varphi).$$

$$D_{ij} = e(\varepsilon, TRUE), \text{ where } \varepsilon_{ij} = 1' \text{ and } \varepsilon_{kl} = 1 \text{ if } kl \neq ij.$$

Since C is clearly closed under finite union, all what remains to show is that $C_i e(d) \in C$ when d is a diagram. By Lemma 4.2 we may assume that $d = \langle \varepsilon, \varphi \rangle$ is normal.

(2) $C_i e(\varepsilon, \varphi) = e(\varepsilon', \varphi')$ where $\varepsilon'_{kl} = 1$ if $i \in \{k, l\}$ and $\varepsilon'_{kl} = \varepsilon_{kl}$ otherwise, and φ' is $(\exists x_{i0} \ldots x_{in-1})\varphi$, whenever $\langle \varepsilon, \varphi \rangle$ is normal.

Proof of (2). First we show $C_i e(\varepsilon, \varphi) \subseteq e(\varepsilon', \varphi')$. Let $m \in e(\varepsilon, \varphi)$ and $m' \in B_n$ be such that m' agrees with m up to i. We will show that $m' \in e(\varepsilon', \varphi')$. We have $m'_{kl} = m_{kl} \in \varepsilon_{kl} = \varepsilon'_{kl}$ if $i \notin \{k, l\}$. Let $k < n$. Then $m'_{ik} \in 1 = \varepsilon'_{ik}$. Hence $m'_{kl} \in \varepsilon'_{kl}$ for all $k, l < n$. Since m agrees with m' up to i, we have $\varphi'(x_{kl}/\nu(m'_{kl}))_{k,l \neq i}$. Thus $m' \in e(\varepsilon', \varphi')$ indeed. To show the other inclusion, $e(\varepsilon', \varphi') \subseteq C_i e(\varepsilon, \varphi)$, we will use that $\langle \varepsilon, \varphi \rangle$ is normal. Let $m' \in e(\varepsilon', \varphi')$ be arbitrary. We have to show the existence of an $m \in e(\varepsilon, \varphi)$ such that m agrees with m' up to i. If $\varepsilon_{ij} = 1'$ for some $j \neq i$ then we define m as $m[i/j]$, then $m \in e(\varepsilon, \varphi)$ by $\langle \varepsilon, \varphi \rangle \in \mathsf{ND}$. Assume therefore that $\varepsilon_{ij} \neq 1'$ for all $j \neq i$. By the definition of φ', there are $\nu_{ik} \in \omega$ for all $k < n$ such that $\varphi(x_{ik}/\nu_{ik}, x_{kl}/\nu(m_{kl}))$. Let $k_2, \ldots, k_n$ be a listing of $n - \{i\}$ such that all the atoms are at the beginning of this listing, i.e. there is ℓ such that ($\varepsilon(i, k_j)$ is an atom iff $j < \ell$), for all $j < n$. For $j < \ell$ define $m_{ik_j} = m_{k_j i} = \varepsilon(i, k_j)$, and $P_j = I(m(i, k_j))$. Then $\nu_{ik_j} = \nu(m_{ik_j})$ by (D3). If $\ell < n$, then $\varepsilon(i, k_l) = x^W$ for some $W \in J$. Let $P_\ell \in W$ be such that $P_\ell \in \bigcap \{P_j; I(m(k_j, k_\ell)) : j < \ell\}$. There is such a P_ℓ by $(J5)_n$ because $P_j \neq 1'$ and $m(k_j, k_\ell) \neq 1'$ for all $j < \ell$ by our assumptions made so far. If $\ell + 1 = q < n$, then $\varepsilon(i, k_q) = x^W$ for some $W \in J$. If $m(k_j, k_q) = 1'$ for some $j < q$, then let $P_q = P_j$, otherwise let $P_q \in W$ be such that $P_q \in \bigcap\{P_j; I(m(k_j, k_q)) : j < q\}$. And so on, till we reach n. For all $\ell \geq j < n$ define $m_{ik_j} = m_{k_j i} = (\nu_{ik_j}, P_j, W_j)$ where $m(i, k_j) = x^{W_j}$. We have defined m. We now show that $m \in e(\varepsilon, \varphi)$. First, $m \in B_n$ because we chose the P's so that in a "triangle" they "commute", and in an ε-critical triangle the indices ν_{kl} satisfy E since they satisfy φ which satisfies (D4). Then $m \in e(\varepsilon, \varphi)$ because we defined m so that $m_{kl} \leq \varepsilon_{kl}$ for all $k, l < n$ and $\varphi(x_{kl}/\nu(m_{kl}))_{kl}$. ■

Lemma 4.3 has been proved. **QED(Lemma 4.3)**

Let $\mathsf{Bb}_n(\mathfrak{M}, J, E)$ denote the subalgebra of $\mathcal{C}m(B_n)$ with universe C, and let us call it the *blow-up-and-blur n-dimensional cylindric algebra.*

Theorem 4.4. *Assume that $\mathfrak{M}$ is finite, (J, E) is a n-blur for $\mathfrak{M}$, (J5)$_n$ holds, and E is definable in $\langle\omega, <\rangle$. Then the neat t-reduct of $\mathsf{Bb}_n(\mathfrak{M}, J, E)$ is isomorphic to $\mathsf{Bb}_t(\mathfrak{M}, J, E)$ for all $t \leq n$, and the relation algebraic reduct of $\mathsf{Bb}_n(\mathfrak{M}, J, E)$ is isomorphic to $\mathsf{Bb}(\mathfrak{M}, J, E)$.*

Proof. First we prove a claim that will be useful later, too.

Claim 4.5. Assume $\langle\varepsilon, \varphi\rangle \in \mathsf{ND}$ and $\varphi(x_{ij}/s_{ij})_{i,j<n}$. Assume further that $k \leq n$ and there is a k-basic matrix $m' \in B_k$ over $\mathcal{R}$ such that $\nu(m'_{ij}) = s_{ij}$ and $m'_{ij} \leq \varepsilon_{ij}$ for all $i, j < k$. Then there is an extension $m \in e(\varepsilon, \varphi)$ of m' such that $\nu(m_{ij}) = s_{ij}$ for all $i, j < n$.

Proof. We define $m_t = \langle m_{ij} : i, j < t\rangle$ by induction on $k \leq t \leq n$ such that m_t is a t-dimensional basic matrix, $m_{ij} \leq \varepsilon_{ij}$ and $\nu(m_{ij}) = s_{ij}$ for all $i, j < t$. Assume now that we have our t-dimensional matrix. We want to define m_{tj} for $j < t$ in an appropriate way. Assume that $K \subseteq t$ is such that ε_{tj} are atoms for $j \in K$, and they are not atoms for $j \notin K$. Let us define $m_{tj} = \varepsilon_{tj}$ for $j \in K$. Then the relevant necessary conditions are satisfied because $\langle\varepsilon, \varphi\rangle$ is normal. If $\varepsilon_{tj} = m_{tj} = 1'$ for some $j \in K$ then we define $m_{t\ell} = m_{j\ell}$ for all $\ell \in K$. So we may assume $m_{tj} \neq 1'$ for all $j \in K$. Let $\ell < t$, $\ell \notin K$ be arbitrary. If $m_{j\ell} = 1'$ for some $j \in K$ then we define $m_{t\ell} = m_{j\ell}$. Otherwise $\varepsilon_{t\ell} = x^W$ and let $P \in W$ be such that $P \leq \bigcap\{I(m_{tj}); I(m_{j\ell}) : j \in K\}$. There is such a P by (J5)$_n$. Define $m_{t\ell} = (s_{t\ell}, P, W)$. Then the relevant necessary conditions hold; and we can proceed till we exhaust n. ■

Now let $2 < t < n$, we are going to prove

(3) $\mathsf{Bb}_t(\mathfrak{M}, J, E) \cong \mathcal{N}r_t\mathsf{Bb}_{t+1}(\mathfrak{M}, J, E)$.

For $m \in B_{t+1}$ let $m \restriction t$ denote the restriction of m to t. Then $(m \restriction t) \in B_t$. Define for $x \subseteq B_{t+1}$

$$h(x) = \{m \restriction t : m \in x\}.$$

Then $h : \mathcal{R}d_t\mathcal{C}m(B_{t+1}) \to \mathcal{C}m(B_t)$ is an isomorphism, because we have seen (when proving that B_n is a cylindric basis, or by Claim 4.5 above) that $(\forall m \in B_t)(\exists m' \in B_{t+1})m = m' \restriction t$. Let $x \subseteq \mathsf{Bb}_{t+1}(\mathfrak{M}, J, E)$, $C_t x = x$. Then $x = e(d_1) \cup \cdots \cup e(d_k)$ for some $d_1, \ldots, d_k \in \mathsf{ND}_{t+1}$, by Lemma 4.2. By (2) in the proof of Theorem 4.3, there are $d'_1, \ldots, d'_k \in \mathsf{ND}_t$ such that $C_t e(d_i) = e(d'_i)$, then $h(C_t e(d_i)) = e_t(d'_i)$, so $h(x) \in \mathsf{Bb}_t(\mathfrak{M}, J, E)$. Conversely, for any $d' \in \mathsf{ND}_t$ there is $d \in \mathsf{ND}_{t+1}$ such that $h(C_t e(d)) = e_t(d')$, so h maps the C_t-closed elements of $\mathsf{Bb}_{t+1}(\mathfrak{M}, J, E)$ onto $\mathsf{Bb}_t(\mathfrak{M}, J, E)$, and we are done. For $t = 2$ we have that the range of h is R by Claim 4.5, thus h is an isomorphism between $\mathcal{R}$ and the relation algebraic reduct of $\mathsf{Bb}_3(\mathfrak{M}, J, E)$ by the theorem of Maddux quoted at the end of section 2. **QED(Theorem 4.4)**

Theorem 4.6. *Assume that $\mathfrak{M}$ is finite, let (J, E) be an n-blur for $\mathfrak{M}$ such that (J5)$_n$ holds, Assume further that there is an $s \in \omega$ such that $x + s < y = z$ implies $E(U, V, W)(x, y, z)$ and $E(U, V, W)$ is definable in $\langle \omega, < \rangle$ for all $x, y, z \in \omega$ and $U, V, W \in J$. Then C is representable.*

Proof. Let $\langle \omega^+, < \rangle$ be a nonprincipal ultrapower of $\langle \omega, < \rangle$. We say that (G, ℓ, σ) is a *consistent colored edge-ordered graph* (or simply: consistent graph) if G is a set, $\ell : G \times G \to R$, $\sigma : G \times G \to \omega^+$ such that (G1)-(G3) below hold for all $i, j, k \in G$.

(G1) $\ell_{ij} \in At^+ \cup \{x^W : W \in J\}$.

(G2) $\mathsf{min}(\ell_{ij}, \ell_{ik}, \ell_{kj})$, $\ell_{ij} = \ell_{ji}$, $(\ell_{ij} = 1'$ iff $i = j)$.

(G3) $\ell_{ij} \in At^+ \to \sigma_{ij} = \nu(\ell_{ij})$, $\sigma_{ij} = \sigma_{ji}$, and $\sigma_{ii} = 0$.

Let $K \subseteq n$, $g : K \to G$, and $\langle \varepsilon, \varphi \rangle \in \mathsf{ND}_K$. We say that $\langle \varepsilon, \varphi \rangle$ *is of type* g, in symbols $\mathsf{type}_g(\varepsilon, \varphi)$, iff for all $i, j, k, l \in K$ we have the following:

$$\begin{aligned} &\varepsilon_{ij} = \ell(g_i, g_j) \qquad \text{and} \quad \text{for all } s \in \omega \\ &\qquad \varphi \to x_{ij} + s = x_{kl} \quad \text{iff} \quad \sigma(g_i, g_j) + s = \sigma(g_k, g_l), \quad \text{and} \\ &\qquad \varphi \to x_{ij} \leq x_{kl} \quad \text{iff} \quad \sigma(g_i, g_j) \leq \sigma(g_k, g_l). \end{aligned}$$

Intuitively, the latter means that the order of variables induced by φ agrees with that of σ, if the distance between two variables is finite according to φ, then it is finite and the same according to σ, and if the distance between two variables is indefinite according to φ, then it is infinite according to σ. Assume now $g \in {}^nG$. Then we define

$$U^g = \{x \in C : (\exists \langle \varepsilon, \varphi \rangle \in \mathsf{ND})(e(\varepsilon, \varphi) \subseteq x \wedge \mathsf{type}_g(\varepsilon, \varphi))\}.$$

$$g \Vdash x \text{ iff } x \in U^g.$$

We say that (G, ℓ, σ) is *complete* if $g \Vdash C_i y$ implies $(\exists u \in G) g(i/u) \Vdash y$, for all $g \in {}^nG,\ i < n,\ y \in C$.

Claim 4.7. U^g is an ultrafilter whenever $g \in {}^nG$ and (G, ℓ, σ) is consistent.

Proof. $\emptyset \notin U^g$ because $e(\varepsilon, \varphi) \neq \emptyset$ for all $(\varepsilon, \varphi) \in \mathsf{ND}$ by Claim 4.5 with taking $k = 0$. If $\mathsf{type}_g(\varepsilon, \varphi)$, $\mathsf{type}_g(\varepsilon', \varphi')$, then $\varepsilon = \varepsilon'$ and $\mathsf{type}_g(\varepsilon, \varphi \wedge \varphi')$. Thus U^g is closed under intersection. It is upward closed by its definition. It remains to show that $(\forall x \in C)(x \in U^g$ or $-x \in U^g)$. It is enough to show this latter for $x = \langle \varepsilon, \varphi \rangle \in \mathsf{ND}$. We prove a more general statement because it will be useful later.

(4) Let $K \subseteq n$, $\langle \varepsilon, \varphi \rangle \in \mathsf{ND}_K$, and $g : K \to G$. Then $e(\eta, \delta) \subseteq e(\varepsilon, \varphi)$ or $e(\eta, \delta) \subseteq -e(\varepsilon, \varphi)$ for some $(\eta, \delta) \in \mathsf{ND}_K$ of type g.

To prove (4), first we introduce some notation. Let $j \in K \times K$ and $s \in \omega$. We define the formula $\mathsf{shift}(\delta, j, s)$ the following way. Search for the infinite gap immediately preceding x_j (according to the ordering induced by δ). If there is no such gap, then we define $\mathsf{shift}(\delta, j, s)$ to be δ. Otherwise, assume that this infinite gap is at x_k, i.e. x_k is the biggest variable below x_j such that the distance between x_k and the next variable, say x_l, is indefinite (of course, everything is understood according to the ordering induced by δ). We then define $\mathsf{shift}(\delta, j, s)$ to be $\delta \wedge x_k + s < x_l$. Then the type of $\langle \eta, \mathsf{shift}(\delta, j, s) \rangle$ is the same as that of $\langle \eta, \delta \rangle$, and $\mathsf{shift}(\delta, j, s) \to s < x_j$ if the distance between x_{ii} and x_j is indefinite according to δ. Let φ be a basic formula and let $s \in \omega$. We say that $\mathsf{dist}_\varphi(x_j, x_k) = s$ if $\varphi \to x_j + s = x_k$, and we say that $\mathsf{dist}_\varphi(x_j, x_k) = s^>$ iff ($\varphi \to x_j < x_k$ and s is the largest element in ω such that $\varphi \to x_j + s < x_k$).

Let $\langle \eta, \delta \rangle \in \mathsf{ND}_K$ be arbitrary such that $\mathsf{type}_g(\eta, \delta)$. Let $j, k \in K \times K$ be arbitrary. Assume that $\mathsf{dist}_\varphi(x_j, x_k) = s \in \omega$. If $\mathsf{dist}_\delta(x_j, x_k) = t \in \omega$ and $t \neq s$, then $e(\eta, \delta) \subseteq -e(\varepsilon, \varphi)$ and we are done. If $\mathsf{dist}_\delta(x_j, x_k) = t^>$, then $e(\eta, \mathsf{shift}(\delta, k, s + 1)) \subseteq -e(\varepsilon, \varphi)$ and we are done. Thus we may assume

(4a) $\mathsf{dist}_\varphi(x_j, x_k) = s$ implies $\mathsf{dist}_\delta(x_j, x_k) = s$, for all $s \in \omega$.

Assume now that $\mathsf{dist}_\varphi(x_j, x_k) = s^>$. If $\mathsf{dist}_\delta(x_j, x_k) = t$ and $t \leq s$, then $e(\eta, \delta) \subseteq -e(\varepsilon, \varphi)$ and we are done. If $\mathsf{dist}_\delta(x_j, x_k) = t^>$, then $\mathsf{dist}_{\delta'}(x_j, x_k) = r^>$ with $r \geq s$, for some shifted version δ' of δ. Thus we may assume

(4b) $\mathsf{dist}_\varphi(x_j, x_k) = s^>$ implies ($\mathsf{dist}_\delta(x_j, x_k) = t > s$ or $\mathsf{dist}_\delta(x_j, x_k) = t^>$ and $t > s$).

From (4a),(4b) we get $\delta \to \varphi$. If $\eta_j \cap \varepsilon_j = \emptyset$ for some $j \in K \times K$, then $e(\eta, \delta) \subseteq -e(\varepsilon, \varphi)$ and we are done. Assume therefore that

(4c) $\eta_j \cap \varepsilon_j \neq \emptyset$ for all $j \in K \times K$.

If $\varepsilon_j \in At^+$, then $\mathsf{dist}_\varphi(x_{ii}, x_j) = \nu(\varepsilon_j)$, hence $\mathsf{dist}_\delta(x_{ii}, x_j) = \nu(\varepsilon_j)$ by (4a), so $\eta_j = \varepsilon_j$. Assume $\varepsilon_j = x^W$ for $W \in J$. Then $\eta_j \leq \varepsilon_j$ by $\eta_j \in At^+ \cup \{x^W : W \in J\}$ and $\eta_j \cap \varepsilon_j \neq \emptyset$. Thus we have $\eta_j \leq \varepsilon_j$ for all $j \in K \times K$. We also have $\delta \to \varphi$, so $e(\eta, \delta) \subseteq e(\varepsilon, \varphi)$ and we are done. This proves (4), and thus finishes the proof of Claim 4.7. ■

Claim 4.8. Let $g \in {}^nG$ and let $\langle \varepsilon, \varphi \rangle \in \mathsf{ND}$ be of type g. Assume $e(\varepsilon, \varphi) \subseteq C_i y$. Let g' be the restriction of g to $n - \{i\}$. Then there is $d \in \mathsf{ND}$ such that $e(d) \subseteq y$ and d is of type g'.

Proof. Let $K = \{(j,k) \in n \times n : i \notin \{j,k\}\}$. By (2) in the proof of of Theorem 4.3 we have that for any $d \in \mathsf{ND}$ there is $d' \in \mathsf{ND}_K$ such that $C_i e(d) = e(d')$. By Lemma 4.2, $y = \bigcup\{e(d) : d \in H\}$ for some finite $H \subseteq \mathsf{ND}$. Then $C_i y = \bigcup\{e(d') : d \in H\}$. Let $g' \Vdash x$ denote that there is $d \in \mathsf{ND}_K$ of type g' such that $e(d) \subseteq x$. Then $g' \Vdash \bigcup\{e(d') : d \in H\}$ and by (4) in the proof of Claim 4.7 we have that either $g' \Vdash e(d')$ or $g' \Vdash -e(d')$ for all $d \in H$. Since the intersection of finitely many elements of ND_K of type g' is again an element of ND_K of type g', we have that $g' \Vdash e(d')$ for some $d \in H$. Now $d \in \mathsf{ND}$, $e(d) \subseteq y$, and $\mathsf{type}_{g'}(d)$, and so we are done. ■

We shall build a consistent, complete colored edge-ordered graph, and we shall see that such graphs give rise to representations of C.

Claim 4.9. There is a consistent complete edge-ordered graph (G, ℓ, σ) such that $(\forall m \in B_n)(\exists g \in {}^nG) g \Vdash \{m\}$.

Proof. The extension step: Assume that (G, ℓ, σ) is countable, consistent, $g \in {}^nG$, $i < n$, $g \Vdash C_i y$. We shall extend the graph with an additional element u such that the extended graph is consistent and $g(i/u) \Vdash y$ in it. By $g \Vdash C_i y$, there is $\langle \varepsilon, \varphi \rangle \in \mathsf{ND}$ such that $g \Vdash e(\varepsilon, \varphi)$ and $e(\varepsilon, \varphi) \subseteq C_i y$. By Claim 4.8, there is $\langle \varepsilon', \varphi' \rangle \leq y$ of type g'', the restriction of g to $n - \{i\}$. We shall extend the graph such that $g(i/u) \Vdash e(\varepsilon', \varphi')$. If $\varepsilon'_{ij} = 1'$ for some $j < n, j \neq i$ then we let $u = g_j$. Otherwise let $u \notin G$. Let us define $\ell(u, g_j) = \ell(g_j, u) = \varepsilon'(i,j)$ if $j \neq i$, and $\ell(u,u) = 1'$. Assume that $v \in G - \{g_j : j < n, j \neq i\}$. Let $W \in J$ be such that $(J(\varepsilon'_{ij}), J(\ell(g_j, v)), W)$ is safe for all $j < n,\ j \neq i$. Such a W exists by $(\mathrm{J4})_n$. Define $\ell(u,v) = \ell(v,u) = x^W$. By this, we extended ℓ to $G \cup \{u\}$ such that $(\forall i,j < n) \ell(g'_i, g'_j) = \varepsilon'_{ij}$ where $g' = g(i/u)$. Further, $(\forall j,k \in K, t \in \omega)(\varphi' \to x_j = x_k + t \text{ iff } \sigma(j) = \sigma(k) + t)$. We now define $\sigma(u,v) \in \omega^+$ for all $v \in G$. By $g \Vdash e(\varepsilon, \varphi) \leq C_i e(\varepsilon', \varphi')$ we can choose $\sigma(u, g_j) \in \omega^+$ such that φ' holds. Let us define $\sigma(u, g_j)$ to be any such choice. Let $z \in \omega^+$ be such that z is "infinitely high" above all $\sigma(i,j),\ i,j \in G$. There is such a z because G is countable and ω^+ is ω-saturated. Let us define $\sigma(u,v) = z$ for all $v \in G - \{g_j : j < n, j \neq i\}$. It is not difficult to check that this extension is consistent, and $g(i/u) \Vdash e(\varepsilon', \varphi')$ in it. Hence $g \Vdash y$ as was desired, by $e(\varepsilon', \varphi') \subseteq y$. Since there are countably many $g, i, y, \varepsilon, \varphi, \varepsilon', \varphi'$, we can continue with this kind of extension until we get a complete graph.

The first step: Let $m \in B_n$ be arbitrary. Let $G = n$, $\ell_{ij} = m_{ij}$, $\sigma_{ij} = \nu(m_{ij})$ for all $i, j < n$. Then (G, ℓ, σ) is consistent, and $g = \langle i : i < n \rangle \in {}^nG$ is such that $g \Vdash m$. Let $m' \in B_n$ be arbitrary. Then $m \leq C_0 C_1 \ldots C_{n-1} m'$, therefore in a complete extension of this graph there is g such that $g \Vdash m'$. ■

Claim 4.10. Assume that (G, ℓ, σ) is a complete, consistent graph and $(\forall m \in B_n)(\exists g \in {}^nG) g \Vdash m$. Define for all $x \in C$

$$rep(x) = \{g \in {}^nG : g \Vdash x\}.$$

Then *rep* is a representation for C.

Proof. *rep* is a Boolean homomorphism by Claim 4.7, i.e. because U^g is an ultrafilter for all $g \in {}^nG$. It is a Boolean isomorphism because $rep(m) \neq \emptyset$ for all $m \in B_n$.

Let $i, j < n$. We are going to show $rep(D_{ij}) = \{g \in {}^nG : g_i = g_j\}$. By definition, $g \Vdash D_{ij}$ iff $(\exists \langle \varepsilon, \varphi \rangle \in \mathsf{ND})[\, e(\varepsilon, \varphi) \leq D_{ij} \wedge \mathsf{type}_g(\varepsilon, \varphi)\,]$. The latter condition implies that $\varepsilon_{ij} \leq 1'$ and thus $g_i = g_j$. This proves one inclusion. To show the other inclusion, assume that $g \in {}^nG$ is such that $g_i = g_j$. Let us define $\langle \varepsilon, \varphi \rangle$ as follows. For all $k, l < n$ define $\varepsilon_{kl} = \ell(g_k, g_l)$ and let φ be the conjunction of the following formulas where $j, k \in n \times n$: $x_j = \sigma(j)$ if $\sigma(j) \in \omega$, $x_j = x_k + t$ if $\sigma_j = \sigma_k + t$, $x_j > x_k$ if $\sigma_j > \sigma_k$ and the distance between σ_j, σ_k is infinite. Now, $g \Vdash e(\varepsilon, \varphi)$, $\mathsf{type}_g(\varepsilon, \varphi)$ and $\varepsilon_{ij} = 1'$.

We are going to show $rep(C_i y) = C_i rep(y)$. Assume $g \in rep(C_i y)$. Then $g \Vdash C_i y$, hence $(\exists u \in G) g(i/u) \Vdash y$, by completeness of the graph. This shows $g \in C_i rep(y)$. To show the other inclusion, assume $g \Vdash y$, we have to show that $g' = g(i/u) \Vdash C_i y$ for any $u \in G$. Let $\langle \varepsilon, \varphi \rangle \in \mathsf{ND}$ be such that $e(\varepsilon, \varphi) \leq y$ and $\mathsf{type}_g(\varepsilon, \varphi)$. Define $\langle \varepsilon', \varphi' \rangle$ such that $\varepsilon'_{ik} = \ell(u, g_k)$ and $\mathsf{type}_{g'}(\varphi')$, and "$\varphi'$ and φ agree on $x_{kl} : k, l < n, k, l \neq i$". Then $g' \Vdash e(\varepsilon', \varphi')$ and $e(\varepsilon', \varphi') \subseteq C_i e(\varepsilon, \varphi) \subseteq C_i y$, hence $g' \Vdash C_i y$. ■

By this we have proved Theorem 4.6 **QED(Theorem 4.6)**

5 Concrete examples, generation with one element, and proof of Theorem 1.2

Let k be a finite or infinite cardinal, and let $\mathcal{E}_k = \mathcal{E}_k(2,3)$ denote the relation algebra which has k non-identity atoms and in which $a_i \leq a_j ; a_l$ iff $|\{i, j, l\}| \in \{2, 3\}$ for all non-identity atoms a_i, a_j, a_l. (This means that all "triangles are allowed" except the "monochromatic" ones.) These algebras were defined by Maddux, e.g. in [36].

Let k be finite, let I be the set of all non-identity atoms of $\mathcal{E}_k(2,3)$ and let $P_0, P_1, \ldots, P_{k-1}$ be an enumeration (i.e. listing without repetition) of the elements of I. Let $\ell \in \omega, \ell \geq 2$ and let J_ℓ denote the set of all subsets of I of cardinality ℓ. Let $w_0, w_1, \ldots, w_N$ be an enumeration of $J = J_\ell$ such that neighboring members in this listing intersect (i.e. $w : |J| \to J$ is a bijection and $w_i \cap w_{i+1} \neq \emptyset$ for all $i+1 < |J|$). Assume further that this listing begins as follows: $w_j = \{P_0, P_1 \ldots, P_{\ell-2}, P_{\ell+j-1}\}$ for $j \leq k - \ell$. There is such a listing of J, e.g. consider each element of J as an ℓ-tuple of elements of I with increasing indices (for example, identify $\{P_1, P_0, P_3\}$ with $\langle P_0, P_1, P_3 \rangle$) and then order these ℓ-tuples according to the lexicographic order.

We are ready to define our E. For all $U, V, W \in J$ let $E(U, V, W)$ be the smallest (according to inclusion $\subseteq$) symmetric ternary relation on ω that satisfies the following:

(S1) $E(W, W, W) \supseteq \{(i, i, i) : i \in \omega\}$, for all $W \in J$.

(S2) $E(W, W, W) \supseteq \{(i, i, i+1) : i \in \omega\}$, for all $W \in J$.

(S3) $E(W, W, V) \supseteq \{(i, i, i) : i \in \omega\}$, for all $W, V \in J$ such that V is the successor of W in the listing w.

(S4) $E(U, V, W) \supseteq \{(i, i+t, i+t) : i \in \omega\}$, for all $t \geq \ell$, $U, V, W \in J$.

Lemma 5.1. *Assume that $n > 2$, $\ell \geq 2n-1$, $k \geq (2n-1)\ell$, $k \in \omega$. Let $\mathfrak{M} = \mathcal{E}_k(2,3)$, let $J = J_\ell$ be the set of all ℓ-element subsets of the set I of non-identity atoms of $\mathfrak{M}$, and let E be as defined above by (S1)-(S4). Then $\mathfrak{M}, J, E$ satisfy all the conditions of Theorems 3.2,4.4,4.6 and Lemmas 4.2,4.3. Further, $\mathcal{R} = \mathsf{Bb}(\mathfrak{M}, J, E)$ is generated by a single element.*

Proof. $\mathfrak{M} = \mathcal{E}_k(2,3)$ is a simple, symmetric, finite, atomic relation algebra. $J = J_\ell$ satisfies (J1) because $0 \neq \ell < \omega$, and J satisfies (J2) by $k \geq \ell$. (J3) is satisfied, by the definition of $\mathfrak{M}$ and by $\ell \geq 2$: if $P \in I, W \in J$ then let $Q \in W - \{P\}$, now $P;Q = I$ in $\mathfrak{M}$. $(\mathrm{J4})_n$ is satisfied, by $k \geq (2n-l)\ell$: let $V_2, \ldots, V_n, W_2, \ldots, W_n \in J$ be arbitrary. Then $U = \bigcup\{V_i \cup W_i : 2 \leq i \leq n\}$ has cardinality at most $(2n-2)\ell$, hence the cardinality of $I - U$ is $\geq k-(2n-2)\ell$, which is $\geq \ell$ by $k \geq (2n-1)\ell$. Hence there is $T \subseteq I - U$, $|T| = \ell$. Now $T \in J$ and $\mathsf{safe}(V_i, W_i, T)$ because $V_i \cap W_i \cap T = \emptyset$ for all $2 \leq i \leq n$. J satisfies $(\mathrm{J5})_n$ by $\ell \geq 2n-1$: Let $P_2, \ldots, P_n, Q_2, \ldots, Q_n \in I$ be arbitrary, then $U = \{P_2, \ldots, Q_n\}$ has cardinality $\leq 2n-2$, and so each $W \in J$ contains an $S \notin U$ by $\ell \geq 2n-1$, now $S \leq P_q;Q_2 \cap \cdots \cap P_n;Q_n$ by the definition of $\mathfrak{M}$.

Let us turn to E. By definition, E is symmetric. For any $U, V, W \in J$ we have that $E(U, V, W)$ is definable in $\langle \omega, < \rangle$ because of the following. Let ψ_1, ψ_2, ψ_4 denote the following formulas, respectively: $x = y = z$, $(x = y \wedge z = y+1) \vee (y = z \wedge x = z+1) \vee (z = x \wedge y = x+1)$, and $(x = y \geq z+\ell) \vee (y =$

$z \geq x + \ell) \vee (z = x \geq y + \ell)$. Now, $E(W, W, W)$ is defined by $\psi_1 \vee \psi_2 \vee \psi_4$, $E(W, W, V)$ is defined by $\psi_1 \vee \psi_4$ if V is the "successor" of W, and in all other cases $E(U, V, W)$ is defined by ψ_4. Since ψ_1, ψ_2, ψ_4 are definable in $\langle \omega, < \rangle$, we are done with showing that the $E(U, V, W)$'s are definable in $\langle \omega, < \rangle$. (E2) is satisfied with $s = \ell$ by (S4), (E3) is satisfied because $E(i, j, q) \to q \in \{i, j\}$, and (E4) is satisfied because of (S4). Also by (S4) we have $x + \ell < y = z \to E(U, V, W)$ for all $U, V, W \in J$.

We now show that there is $g \in R$ such that $\mathcal{R}$ is generated by g. Let $W = w_0$ and let

$$g = (0, P_0, W) + (1, P_0, W) + (2, P_1, W) + \cdots + (\ell - 1, P_{\ell-2}, W).$$

For any $U \in J$ and $i \in \omega$ let us define $r_i^U = \sum\{(i, P, U) : P \in U\}$ and $c^U = \sum\{e^V : V \cap U = \emptyset\}$. We call r_i^U the "*i-th row in the block U*" and we call c^U the "*safe complement of U*".

First we aim at generating the first row r_0^W and the safe complement c^W from g (where $W = w_0$).

(1) $(0, P_0, W) = g - g; g$.

Proof. In computing $g \cap g; g$, only (S1),(S2) can play a role, (S4) cannot play a role, because the difference $|\nu(a) - \nu(b)|$ between the indices of any two atoms a, b below g is less than ℓ. Thus $(0, P_0, W) \not\leq g; g$ because only $(0, P_0, W); (0, P_0, W)$ or $(0, P_0, W); (1, P_0, W)$ could "bring" it in, but $P_0 \not\leq P_0; P_0$ in $\mathfrak{M}$, so these products do not "produce" $(0, P_0, W)$. On the other hand, e.g. $(1, P_0, W) \leq (1, P_0, W); (2, P_1, W) \leq g; g$, and similarly for the other atoms below g. ■

With similar arguments we get

(2) $r_0^W + c^W = (0, P_0, W); (g - (0, P_0, W)) + (0, P_0, W)$.

(3) $c^W = (r_0^W + c^W) \cap [(g - (0, P_0, W)); (g - (0, P_0, W))]$.

(4) $r_0^W = (r_0^W + c^W) - c^W$.

We now get all the atoms below g as follows:

(5) $(1, P_0, W) = g \cap (r_0^W; r_0^W)$,

(6) $(2, P_1, W) = g \cap (1, P_0, W); (1, P_0, W)$,

(7) $(3, P_2, W) = g \cap (2, P_1, W); (2, P_1, W)$, and so on.

Let $V = w_1$. We now generate all the rows r_i^W, r_i^V in the blocks W and V. We already have r_0^W.

(8) $r_i^W + r_{i+1}^W + r_i^V = r_i^W ; r_i^W - c^W, \quad$ for all $i \in \omega$,

(9) $r_1^W = (1, P_0, W) + [(1, P_0, W); (1, P_0, W) \cap (r_0^W ; r_0^W) - c^W]$.

We get the rows by repeated uses of (8),(9). For example,

$$r_0^V = (r_0^W ; r_0^W) - r_0^W - r_1^W - c^W,$$

$$r_2^W = (r_1^W ; r_1^W) - r_1^W - c^W - (r_0^V ; r_0^V),$$

$$r_1^V = (r_1^W ; r_1^W) - r_1^W - r_2^W - c^W, \quad \text{and so on.}$$

Now that we have all the elements of g and all the rows r_i^W, we generate all the atoms in the ℓ'th row r_ℓ^W. By using

(10) $(i, P, U) = r_i^U - (i-1, P, U); (i-1, P, U) \quad$ for all $U \in J,\ P \in U,\ i \geq 1$

we get

(11) $(i, P_0, W) = r_i^W - (i-1, P_0, W); (i-1, P_0, W) \quad$ for $i > 0$, from (5),

(12) $(i, P_1, W) = r_i^W - (i-1, P_1, W); (i-1, P_1, W) \quad$ for $i \geq 2$, from (6), and so on.

With this we get (ℓ, P_j, W) for $j < \ell - 1$, and we get the last element of r_ℓ^W by $(\ell, P_{\ell-1}, W) = r_\ell^W - \{(\ell, P_j, W) : J < \ell - 1\}$. Now we generate all the elements of the first row r_0^W by using (S4):

(13) $(0, P, W) = r_0^W - (\ell, P, W); (\ell, P, W) \quad$ for all $P \in W$,

and we get all elements of x^W from the elements of the first row by using (10) repeatedly. We now have all the elements of x^W, and we have c^W and r_i^V for all $i \in \omega$. Next we generate all the elements of the first $k - \ell$ blocks. Recall that here neighboring elements of J intersect in exactly $\ell - 1$ elements, by our condition on w.

Assume that $W = w_q,\ V = w_{q+1}, |V \cap W| = \ell - 1$, and g generates all atoms (i, P, W) for $i \in \omega, P \in W$, and all rows r_i^V for $i \in \omega$, and c^W. We show that g generates all elements (i, P, V) for $i \in \omega,\ P \in V$, and it also generates c^V and all rows r_i^U for $U = w_{q+2}$.

(14) $(i, P, V) = r_i^V - (i, P, W); (i, P, W) \quad$ for $i \in \omega,\ P \in V \cap W$,

(15) $(i, Q, V) = r_i^V - \{(i, P, V0 : P \in V \cap W\} \quad$ for $Q \in V - W$,

(16) $c^V = (r_0^V ; r_0^V) \cap (r_2^V ; r_2^V)$,

(17) $r_i^U = (r_i^V ; r_i^V) - r_i^V - r_{i+1}^V - c^V$.

Thus far we have showed that g generates all the elements of the first $k - \ell$ blocks, and by our assumption on w this implies that for all $P \in I$ there is $W \in J$ such that g generates (ℓ, P, W). By

(18) $(0, P, U) = r_0^U - (\ell, P, W); (\ell, P, W) \quad \text{if } P \in U \cap W$

we get all elements in r_0^U, for any $U \in J$. Now, assuming by induction that g generates r_i^U for all $i \in \omega$, we get all elements of x^U by using (10), and we get c^U, r_i^V for the "successor" V of U (i.e. $V = w_{j+1}$ if $U = w_j$) by using (16),(17). Finally,

(19) $x^W = \bigcap\{c^V : V \cap W = \emptyset\}$,

and so we showed that g generates all elements of $\mathcal{R}$. **QED(Lemma 5.1)**

Proof of Theorem 1.2. Let $n > 2$, $k \geq 0$. Let $N = n+k$, $\ell \geq 2N-1$, $K \geq (2N-1)\ell$, $K < \omega$. Let $\mathfrak{M} = \mathcal{E}_K(2,3)$, $J = J_\ell$ and let E be as in Lemma 5.1. Then $\mathfrak{M}$, J, E satisfy all the conditions of Theorems 3.2,4.4,4.6 and Lemmas 4.2,4.3, by Lemma 5.1. Let $\mathcal{R} = \mathsf{Bb}(\mathfrak{M}, J, E)$, let $\mathcal{C}$ be the subalgebra of $\mathsf{Bb}_n(\mathfrak{M}, J, E)$ generated by its atoms, and let $\mathcal{D} = \mathsf{Bb}_N(\mathfrak{M}, J, E)$. Then $\mathcal{R}$ is countable, symmetric, simple, integral and atomic. **(i)**: $\mathcal{R}$ is representable by Thm.3.2(i). However, $\mathcal{R}^+ = \mathcal{C}mAt\mathcal{R}$ is not representable by the following. $\mathcal{R}^+$ is infinite, hence it can have a representation on an infinite set only. By Thm.3.2(ii), any representation of $\mathcal{R}^+$ gives a representation for $\mathfrak{M}$, too, on the same base set. However, by Ramsey's theorem, $\mathfrak{M} = \mathcal{E}_K(2,3)$ can have a representation on a finite set only if K is finite. This shows that $\mathcal{C}mAt\mathcal{R}$ is not representable, hence $\mathcal{R}$ does not have a complete representation. **(ii)**: $\mathcal{R}$ is generated by a single element g by Lemma 5.1. **(iii)** follows from Thm.3.2(iii), **(iv)** follows from Theorem 4.6. **(v)**: By definition of $\mathcal{C}$ we have $C_0C_1 \ldots C_{n-1}x = 1$ if $x \neq 0$ in $\mathcal{C}$, this implies that $\mathcal{C}$ is simple, and it is atomic by its definition. **(vi)**: By the definition of $\mathcal{C}$, the relation algebra reduct of $\mathcal{C}$ contains all atoms of $\mathcal{R}$, hence it contains g (as defined in the proof of Lemma 5.1), hence it contains $\mathcal{R}$. On the other hand, $\mathcal{R}a\mathcal{C} \subseteq \mathcal{R}$ by Thm. 4.4 because $\mathcal{C} \subseteq \mathsf{Bb}_n(\mathfrak{M}, J, E)$. Thus $\mathcal{R}$ is (isomorphic to) the relation algebraic reduct of $\mathcal{C}$. Each $m \in B_n$ is generated from R by the cylindric algebraic operations, hence each $m \in B_n$ is generated in $\mathcal{C}$ by $g \in R$. Since $\mathcal{C}$ is generated by the matrices, we have that $\mathcal{C}$ is generated by a single 2-dimensional element. $\mathcal{C}$ is a sub-neatreduct of a representable CA_{n+k}, because $\mathcal{C} \subseteq \mathsf{Bb}_n(\mathfrak{M}, J, E) \cong Nr_n\mathcal{D}$ by Theorem 4.4 and $\mathcal{D}$ is representable by Theorem 4.6. $\mathcal{D}$ is simple because its relation algebra reduct, $\mathcal{R}$, is simple. We have that $Nr_2C = Nr_2D$ by Theorem 4.4, since the relation algebra reduct of both is $\mathcal{R}$. Theorem 1.2 has been proved. **QED(Theorem 1.2)**

Theorem 5.2. *$RCA_n \cap Nr_nCA_{n+k}$ is not closed under completions, for each $n > 2$ and $k \geq 0$, where, RCA_n denotes the class of all representable cylindric algebras of dimension n.*

Proof. This follows immediately from Thm.s 4.4,4.6, and Lemma 5.1. **QED**

Concluding remarks: The results in this paper concerning strong failure of the omitting type property for $\mathcal{L}_n$ were announced in [9], relying on an unpublished construction in [2]. Later, a weaker version of failure of the omitting type property of $\mathcal{L}_n$ was proved in [49], that paper relied on constructions of Hirsch and Hodkinson in [23]. The present paper contains an improved version of the construction in [2] upon which strong failure of the omitting type property of $\mathcal{L}_n$ is fully proved here. Contrasting positive results on omitting types can be found in [48],[53]. We should point out that from the present Theorem 1.1, one can easily prove that the omitting types property fails for any first order definable expansion of $\mathcal{L}_n$ as defined in [10]. The result applies also to richer extensions of first order logic, like the ones dealing with transitive closure due to Maddux [36]. In contrast, the Omitting Types Property holds for $\mathcal{L}_1$, and more generally, for countable $\mathcal{L}_n$ theories, $n \leq \omega$, with only *unary* relation symbols. Two-variable logic $\mathcal{L}_2$ does not have the Omitting Type Property, this is proved in [7]. However, there is a difference between $\mathcal{L}_2$ and $\mathcal{L}_n$ for $n \geq 3$ in this respect. Namely, $\mathcal{L}_2$ does have the OTP restricted to atomic theories T, while in the present paper the theories constructed to show failure of OTP for $\mathcal{L}_n$ $(3 \leq n < \omega)$ are all atomic. It would be interesting to know whether the guarded fragment of FOL has the omitting type property (in some form). It remains an open problem to find a theorem analogous to the solution of problem 2.12 in [20] for the present subject, i.e. for the number of variables needed for a witness. Problem 2.12 of [20] asks whether for all $2 < n < \omega$ there is a $k < n$ such that $SNr_nCA_{n+k} = SNr_nCA_{n+k+1}$. (Here, SK denotes the class of all subalgebras of elements of K.) Hirsch, Hodkinson, and Maddux [29] proved that the answer to this problem is in the negative in the strongest sense, i.e. for all $2 < n < \omega$ and for all $k < \omega$ we have $SNr_nCA_{n+k} \neq SNr_nCA_{n+k+1}$. The analogous result in the omitting types context would be to find for all $2 < n < \omega$ and $k < \omega$ an n-complete theory $T \subseteq \mathcal{L}_n$ and a tye $\Sigma(\overline{x}) \subseteq \mathcal{L}_n$ such that Σ is realized in each model of T, there is a witness for this in $\mathcal{L}_{n+k+1}$, but there is no such witness in $\mathcal{L}_{n+k}$.

The proof of the omitting type theorem can be regarded as a strengthening of Henkin's proof of the completeness theorem for FOL. Since the nonfinitizability results (e.g. in [39]) can be interpreted as signifying the failure of completeness (e.g. for $\mathcal{L}_n$), the present strong negative results on the omitting type property can be regarded as improvements or strengthening of the nonfinitizability results of Monk, Jónsson and others. This view induces a kind of parallel between the present work and the framework of Problem 2.12 in [20].

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