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A COMPLETE LOGIC FOR REASONING ABOUT PROGRAMS VIA NONSTANDARD MODEL THEORY II*

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In Part I of the present paper we defined the first order dynamic language DL_d (of type d). In Definition 13 we defined a decidable proof concept $\langle \vdash^N, \text{Prn}\rangle$ for DL_d, and in Theorem 2 we proved that $\langle \vdash^N, \text{Prn}\rangle$ is a strongly complete inference system for DL_d. That is, for every theory Th and formula φ of first order dynamic language we have $\text{Th} \models \varphi$ iff $\text{Th} \vdash^N \varphi$. By *Dynamic Logic* of type d we understand $\langle DL_d, \langle \vdash^N, \text{Prn}\rangle\rangle$.

Here we investigate further properties of our Dynamic Logic, its expressive power, how it can be used for various purposes, how it can be adapted to various situations. Then we investigate Floyd's method using the framework of DL. A complete characterization of the amount of information implicitly contained in Floyd's method will be found but several questions remain open in this line. The proof method $\vdash^N$ is proved to be strictly stronger than Floyd's method in Section 6. Different semantics of programming are compared in Section 7 within the framework of DL. Comparisons with several approaches related in several ways are given in Sections 7–9.

5. Properties of DL_d

5.1. Methods of proving properties of programs

The proof concept $\langle \vdash^N, \text{Prn}\rangle$ introduced in Definition 13 is also a new *method* of proving properties of programs. E.g. $\vdash^N$ can be used to prove partial correctness, total correctness, termination etc. of programs, see the example after Definition 9 in Part I. The proof method $\vdash^N$ is complete by Theorem 2. In Andréka–Csirmaz–Németi–Sain [1] the proof method $\vdash^N$ was compared with the Floyd–Hoare method of proving partial correctness and it was found that $\vdash^N$ is strictly stronger, i.e. there are correct programs provable by $\vdash^N$ but not provable by the Floyd–Hoare method. (See Theorem 10 here.) We shall return to this point later, in Section 6.

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There is a second, Hilbert style, definition of the proof concept $\vdash^N$. Then $\vdash^N$ is defined by a decidable set $Lx \subseteq DF_d$ of logical axioms and a decidable set $R \subseteq (DF_d)^* \times DF_d$ of proof rules. Both Lx and R are defined by finite schemes of formulas. Then an $\vdash^N$-proof is defined to be a finite string w of elements of DF_d such that if $w = \langle \varphi_i : i < n \rangle$ for some $n \in \omega$, then for all $i < n$ either $\varphi_i \in Lx$ or there is $\langle s, \varphi_i \rangle \in R$ such that $s \in \{\varphi_j : j < i\}^*$. This definition of $\vdash^N$ is available from the authors.

5.2. About choosing axioms to express properties of time

To execute programs in arbitrary elements of M_{td} might look counter-intuitive. However, we may replace M_{td} by Mod(Ax) for a certain fixed set $Ax \subseteq F_{td}$ of axioms expressing all the intuitive requirements about *time* and about *processes* 'happening in time'. After having done so, there is nothing wrong with executing programs in models $\mathfrak{M} \in M_{td}$ of Ax since Ax does contain all our intuitive ideas about time, processes etc. It is important, however, to keep Ax to be recursively enumerable.

To illustrate these here, we define a set $Ax \subseteq F_{td}$ of axioms of the above kind. Roughly speaking, Ax will be nothing but the Peano Axioms for the sort t. However, in our present syntax F_{td} variables of sort t may occur in formulas which contain symbols of sort d and i as well. The induction axioms will be stated for these formulas 'of mixed sort', too. The axiom system IA defined below originates from B. Biró.

Definition 14 (the theories PA, OA, IA, Ax_0, Ax_e, Ax). Let d be a similarity type. Then td, F_{td} and Z were defined in Definitions 4 and 6 in Section 2. Let $z \in Z$ be arbitrary. Let $\varphi \in F_{td}$.

We define the induction formula, φ_z^+ as follows:

$$\varphi_z^+ \overset{\text{df}}{=} ([\varphi(0) \wedge \forall z(\varphi \rightarrow \varphi(z+1))] \rightarrow \forall z\, \varphi),$$

where $\varphi(0)$ and $\varphi(z+1)$ denote the formulas obtained from φ by replacing every free occurrence of z in φ by 0 and $z+1$ respectively. The *induction axioms* are:

$$\text{IA} \overset{\text{df}}{=} \{\varphi_z^+ : \varphi \in F_{td} \text{ and } z \in Z\}.$$

Clearly $\text{IA} \subseteq F_{td}$ since if $\varphi(z) \in F_{td}$ and $z \in Z$, then $\varphi(0)$, $\varphi(z+1) \in F_{td}$ because 0 and $z+1$ are terms of sort t.

It is important to stress here that $\varphi(z)$ may contain *other free* variables of all sorts. All the free variables of $\varphi(z)$ are also free in φ_z^+ except for z. They are the 'parameters' of the induction φ_z^+.

The theory IA says that if a 'property' $\varphi(z)$ changes during time T, then it must change 'some time', i.e. there is a time point $b \in T$ when $\varphi(z)$ is just changing.

We define

$$\text{IA}^+ \overset{\text{df}}{=} \text{IA} \cup \{j \neq k : j, k \in \text{Lab and } j \neq k\}.$$

Notations. We define the abbreviations $<$ and $\prec$ as follows:

$$z_0 < z_1 \Leftrightarrow [z_0 \leqslant z_1 \wedge z_0 \neq z_1]$$

and

$$z_0 \prec z_1 \Leftrightarrow [z_0 < z_1 \wedge \forall z_2(z_0 < z_2 \rightarrow z_1 \leqslant z_2)].$$

The finite set $\mathrm{OA} \subseteq F_{\mathrm{td}}$ of *order axioms* is defined as follows:

$$\mathrm{OA} \stackrel{\mathrm{df}}{=} \{\forall z_0(z_0 \prec z_0 + 1), \forall z_0(0 \leqslant z_0 \wedge [0 = z_0 \vee \exists z_1(z_1 + 1 = z_0)]),$$
$$\forall z_0 \forall z_1 \forall z_2([z_0 \leqslant z_1 \vee z_1 \leqslant z_0] \wedge [z_0 \leqslant z_1 \leqslant z_2 \rightarrow z_0 \leqslant z_2])\}.$$

Let PA denote the set of *Peano Axioms* for the sort t (see e.g. Example 1.4.11 in [8]).

Now we define the theories Ax_0, Ax_{e}, Ax:

$$\mathrm{Ax}_0 \stackrel{\mathrm{df}}{=} \mathrm{OA} \cup \mathrm{IA}^+.$$

Ax_{e} denotes Ax_0 together with the *axiom of extensionality*, i.e.

$$\mathrm{Ax}_{\mathrm{e}} \stackrel{\mathrm{df}}{=} \mathrm{Ax}_0 \cup \{\forall y_0 \forall y_1(\forall z_0[\mathrm{ext}(y_0, z_0) = \mathrm{ext}(y_1, z_0)] \rightarrow y_0 = y_1)\},$$

$$\mathrm{Ax} \stackrel{\mathrm{df}}{=} \mathrm{Ax}_0 \cup \mathrm{PA}. \quad \square$$

Note that Ax_0, Ax_{e}, $\mathrm{Ax} \subseteq F_{\mathrm{td}}$ and $\mathrm{OA} \subseteq F_t$, $\mathrm{PA} \subseteq F_t$. Recall the similarity type d' and the standard model $\mathfrak{N} \in M_{\mathrm{td}'}$ from Definition 6 in Part I. Let $d = d'$. Then $\mathfrak{N} \models \mathrm{Ax}_{\mathrm{e}} \cup \mathrm{PA}$.

Remark. The reason for introducing Ax_0 is that all the results in this paper remain true if we replace the type t by a single binary relation symbol $\leqslant$, i.e. if we replace the structure $\boldsymbol{T}$ by an ordering $\langle T, \leqslant\rangle$ and replace the relation $z_1 = z_0 + 1$ by $z_0 \prec z_1$ in all the definitions and theorems. The modified OA is then a complete axiomatization of $\mathrm{Th}(\langle\omega, \leqslant\rangle)$.

Theorem 3 (uniqueness of traces). *Let $p \in P_d$ and $\mathfrak{M} \in \mathrm{Mod}(\mathrm{Ax}_{\mathrm{e}})$. Let $k \in {}^{\omega}D$. Then p has at most one trace of input k in $\mathfrak{M}$.*

Proof. Let $\bar{s} = \langle s_0, \ldots, s_c\rangle$ and $\bar{r} = \langle r_0, \ldots, r_c\rangle$ be two traces of p in $\mathfrak{M}$ such that $(\forall j < c)\ \mathrm{ext}(s_j, 0) = \mathrm{ext}(r_j, 0)$. (I.e. $\bar{s}$ and $\bar{r}$ are of the same input.)

We define

$$\varphi(z_0) \stackrel{\mathrm{df}}{=} (\mathrm{ext}(s_0, z_0) = \mathrm{ext}(r_0, z_0) \wedge \cdots \wedge \mathrm{ext}(s_c, z_0) = \mathrm{ext}(r_c, z_0)).$$

(Here $s_0, \ldots, s_c, r_0, \ldots, r_c$ are the parameters of the induction $\overset{+}{\varphi}_{z_0}$.)

$\mathfrak{M} \models \varphi(0)$ by our assumption. $\mathfrak{M} \models \forall z_0(\varphi(z_0) \rightarrow \varphi(z_0 + 1))$ because $\bar{s}$ and $\bar{r}$ are traces of the *same* program p and since $\mathfrak{M} \models (j \neq k)$ for every two distinct $i, j \in \mathrm{Lab}$ (by $\mathrm{IA}^+ \subseteq \mathrm{Ax}_{\mathrm{e}}$).

By $IA \subseteq Ax_e$ and $\mathfrak{M} \models Ax_e$ we have

$$\mathfrak{M} \models [\varphi(0) \wedge \forall z_0(\varphi(z_0) \rightarrow \varphi(z_0+1))] \rightarrow \forall z_0\, \varphi(z_0).$$

Therefore $\mathfrak{M} \models \forall z_0\, \varphi(z_0)$, i.e. $(\forall j<c)(\forall b \in T)\, \text{ext}^{\mathfrak{M}}(s_j, b) = \text{ext}^{\mathfrak{M}}(r_j, b)$. Then $\bar{s} = \bar{r}$ by the axiom of extensionality. □

The following theorem says that if a trace terminates sometime in $\text{Mod}(Ax_0)$ then it cannot run again any later time. Moreover if the trace $\bar{s}$ stops sometime, then there is an *earliest* time $m \in T$ such that $\bar{s}$ stops at time m and from that time on $\bar{s}$ remains unchanged.

Theorem 4 (uniqueness of termination and output). *Let $p \in P_d$ and $\mathfrak{M} \in \text{Mod}(Ax_0)$. Let $\bar{s}$ be a trace of p in $\mathfrak{M}$ and assume that $\bar{s}$ terminates p at a time.*

Then there is $m \in T$ such that for every $b \in T$ conditions (i)–(iii) *below are equivalent*:

(i) $b \geq m$,

(ii) *$\bar{s}$ terminates p at time b in $\mathfrak{M}$*,

(iii) $\text{ext}(\bar{s}, b) = \text{ext}(\bar{s}, m)$.

Proof. Let $p \in P_d$, $\mathfrak{M} \in \text{Mod}(Ax_0)$, and let $\bar{s} = \langle s_0, \ldots, s_c \rangle$ be a trace of p in $\mathfrak{M}$.

Suppose $\bar{s}$ terminates p in $\mathfrak{M}$ at time $b_0 \in T$. Then $p = \langle (i_0: u_0), \ldots, (i_n: \textbf{halt}) \rangle$ and $\text{ext}^{\mathfrak{M}}(s_c, b_0) = i_n$.

Let $H = \{b \in T: \text{ext}^{\mathfrak{M}}(s_c, b) = i_n\}$. We have to show that

$$(\exists m \in T)(H = \{b \in T: b \geq m\} \text{ and } (\forall b \in H)\, \text{ext}(\bar{s}, b) = \text{ext}(\bar{s}, m)).$$

(1) Let $\varphi \stackrel{\text{df}}{=} \varphi(z_0, y_0) \stackrel{\text{df}}{=} (\text{ext}(y_0, z_0) \neq i_n)$. Then $\varphi(z_0, y_0) \in F_{td}$ since i_n is a term of sort d by definition. Now the induction formula $\varphi^{+}_{z_0}$ is

$$[\varphi(0, y_0) \wedge \forall z_0(\varphi(z_0, y_0) \rightarrow \varphi(z_0+1, y_0))] \rightarrow \forall z_0\, \varphi(z_0, y_0).$$

By $\mathfrak{M} \models Ax_0$ and $IA \subseteq Ax_0$ we have $\mathfrak{M} \models \forall y_0(\varphi^{+}_{z_0})$.

$\mathfrak{M} \not\models \forall z_0\, \varphi(z_0, s_c)$ since $\text{ext}^{\mathfrak{M}}(s_c, b_0) = i_n$. Therefore

$$\mathfrak{M} \not\models [\varphi(0, s_c) \wedge \forall z_0(\varphi(z_0, s_c) \rightarrow \varphi(z_0+1, s_c))].$$

Hence either $\text{ext}\,(s_c, 0) = i_n$ or $\text{ext}(s_c, b) \neq i_n$ and $\text{ext}(s_c, b+1) = i_n$ for some $b \in T$.

Let $m \stackrel{\text{df}}{=} 0$ or $m \stackrel{\text{df}}{=} b+1$ for the above b. Then $m \in H$ and either $m = 0$ or $m = b+1$ for some $b \notin H$. Let this m be fixed for the rest of the proof.

(2) Next we prove $(\forall b \in H)(\forall a \geq b)\, \text{ext}(\bar{s}, a) = \text{ext}(\bar{s}, b)$.

Let $\bar{y} = \langle y_0, \ldots, y_c \rangle$. Let

$$\psi(z_0, \bar{y}) \stackrel{\text{df}}{=} \forall z_1[(z_1 \leq z_0 \wedge \text{ext}(y_c, z_1) = i_n) \rightarrow \bigwedge_{j \leq c} \text{ext}(y_j, z_0) = \text{ext}(y_j, z_1)].$$

We shall show that $\mathfrak{M} \models \forall z_0\, \psi(z_0, \bar{s})$. $\mathfrak{M} \models \psi[0, \bar{s}]$ is obvious.

Assume $\mathfrak{M} \models \psi[b, \bar{s}]$. We show that $\mathfrak{M} \models \psi[b+1, \bar{s}]$.

Case 1: $(\forall a \leq b)\, \mathrm{ext}(s_c, a) \neq i_n$. Then for every $a \leq b+1$ *either* $a = b+1$ and then $\mathrm{ext}(\bar{s}, a) = \mathrm{ext}(\bar{s}, b+1)$ *or* $a < b+1$ and then $a \leq b$ and hence $\mathrm{ext}(s_c, a) \neq i_n$. Thus $\mathfrak{M} \models \psi[b+1, \bar{s}]$.

Case 2: $(\exists a \leq b)\, \mathrm{ext}(s_c, a) = i_n$. Then $\mathrm{ext}(s_c, b) = i_n$ by our assumption $\mathfrak{M} \models \psi[b, \bar{s}]$. Thus $\mathrm{ext}(\bar{s}, b+1) = \mathrm{ext}(\bar{s}, b)$ by the definition of a trace and hence $\mathfrak{M} \models \psi[b+1, \bar{s}]$.

Cases 1–2 prove that $\mathfrak{M} \models \forall z_0(\psi(z_0, \bar{s}) \rightarrow \psi(z_0+1, \bar{s}))$. Then by $\mathfrak{M} \models \forall \bar{y}(\psi(z_0, \bar{y}))^{+}_{z_0}$ and by $\mathfrak{M} \models \psi[0, \bar{s}]$ we have $\mathfrak{M} \models \forall z_0\, \psi(z_0, \bar{s})$. I.e. we have

$$\mathfrak{M} \models \forall z_0 \forall z_1[(\mathrm{ext}(s_c, z_1) = i_n \wedge z_1 \leq z_0) \rightarrow \mathrm{ext}(\bar{s}, z_0) = \mathrm{ext}(\bar{s}, z_1)].$$

(3) Now we prove $H = \{b \in T : b \geq m\}$ and $(\forall b \in H)\, \mathrm{ext}(\bar{s}, b) = \mathrm{ext}(\bar{s}, m)$.

By (2) we have that $(\forall a \geq m)\, \mathrm{ext}(\bar{s}, a) = \mathrm{ext}(\bar{s}, m)$ and therefore $H \supseteq \{b \in T : b \geq m\}$.

Therefore it is enough to prove $H \subseteq \{b \in T : b \geq m\}$. We shall use that $\mathfrak{M} \models \mathrm{OA}$ by $\mathrm{OA} \subseteq \mathrm{Ax}_0$.

If $m = 0$, then $\{b \in T : b \geq m\} = T$ by $\mathfrak{M} \models \mathrm{OA}$. Suppose $m = b_1 + 1$ and $b_1 \notin H$. Let $b \in H$ be arbitrary. Then $b \not\leq b_1$ by (2) and $b_1 \notin H$. Then $b > b_1$ and therefore $b \geq b_1 + 1 = m$ by $\mathfrak{M} \models \mathrm{OA}$.

We have seen that there is an *earliest* time m when $\bar{s}$ stops and from that time on $\bar{s}$ remains unchanged. □

Corollary 5. *Let* $p \in P_d$. *Then statements* (i)–(iii) *below hold*:

(i) *Let* $\mathfrak{M} \models \mathrm{Ax}_0$ *and let* $k \in {}^{\omega}D$. *Then there is at most one output of* p *with input* k *in* $\mathfrak{M}$, *i.e.* p *is deterministic.*

(ii) $\mathrm{Ax}_0 \models [\Diamond(p, \psi) \rightarrow \Box(p, \psi)]$ *for every* $\psi \in \mathrm{DF}_d$.

(iii) $\mathrm{Ax}_0 \models (\forall x_0 \cdots x_{2c-1})\left[\Diamond\left(p, \bigwedge_{j<c} x_j = x_{c+j}\right) \rightarrow \Box\left(p, \bigwedge_{j<c} x_j = x_{c+j}\right)\right]$.

Proof. (iii) is a special case of (ii) and (ii) follows from (i) which is an immediate corollary of Theorem 4. □

Note that the formula $\neg(\Diamond(p, \psi) \rightarrow \Box(p, \psi))$ means that there is an input such that to this fixed input there are two different outputs of p such that one output satisfies ψ while the other does not. See row 8 in Table 1 in Part I.

Definition 15 (the sets Pe, IA^{q}, and IA^{f} of axioms).

$$\mathrm{Pe} \stackrel{\mathrm{df}}{=} \{0 \neq z_0+1,\ z_0 \neq 0 \rightarrow \exists z_1(z_1+1 = z_0),\ z_0+1 = z_1+1 \rightarrow z_0 = z_1,$$
$$z_0 \neq z_0+1,\ z_0 \neq (z_0+1)+1, \ldots, z_0 \neq (\cdots(z_0+1)\cdots+1), \ldots\}.$$

$$\mathrm{IA}^{\mathrm{f}} \stackrel{\mathrm{df}}{=} \{\varphi \in \mathrm{IA}^{+} : \varphi \text{ contains no free variable of sort } t \text{ or } d\}.$$

$$\text{Lax} \stackrel{\text{df}}{=} \{j \neq k : j, k \in \text{Lab and } j \neq k\}.$$

$$\text{IA}^{\text{q}} \stackrel{\text{df}}{=} \{\varphi_z^+ : z \in Z, \varphi \in F_{\text{td}} \text{ and no variable of sort } t \text{ is quantified in } \varphi\} \cup \text{Lax}.$$

That is

$$\text{IA}^{\text{q}} = \{\varphi_z^+ : z \in Z, \varphi \in F_{\text{td}} \text{ and for all } i \in \omega \text{ the symbol '}\exists z_i\text{' does not occur in } \varphi\} \cup \text{Lax}. \quad \square$$

Proposition 6 (Andréka–Csirmaz). *Statements* (i)–(iv) *below hold*:

(i) $\text{IA}^+ \cup \text{Pe} \not\models \{\Diamond(p, \psi) \to \Box(p, \psi): p \in P_d, \psi \in F_d$ *has one free variable*$\}$,

(ii) $(\text{IA}^{\text{q}} \cap \text{IA}^{\text{f}}) \cup \text{PA} \not\models \{\Diamond(p, \psi) \to \Box(p, \psi): p \in P_d, \psi \in F_d$ *has one free variable*$\}$,

(iii) $\text{IA}^{\text{q}} \cup \text{OA} \models \{\Diamond(p, \psi) \to \Box(p, \psi): p \in P_d, \psi \in DF_d\}$,

(iv) $\text{IA}^{\text{f}} \cup \text{OA} \models \{\Diamond(p, \psi) \to \Box(p, \psi): p \in P_d, \psi \in DF_d\}$.

Proof. To prove (iv) it is enough to observe that all the induction axioms used in the proof of Theorem 4 were ones without parameters, i.e. they were members of IA^{f}. (iii) was proved in [1]. (i) and (ii) can be proved from the results in Section 5 of [14] using the proof of Proposition 12 in the present paper. A direct proof of (i) can be obtained by using ultraproducts. $\square$

L. Csirmaz proved that

$$\text{OA} \cup \{\varphi \in \text{IA}^+ : \varphi \text{ contains no free variable of sort } t\} \not\models \text{IA}.$$

Thus Proposition 6(iv) is strictly stronger than $\text{Ax}_0 \models \Diamond(p, \psi) \to \Box(p, \psi)$.

In many situations, the following set Ex of axioms does belong to the intuitively natural assumptions about processes happening in time.

Definition 16 (the set Ex of axioms).

Notation. "$\exists! x_0$" means that "there exists a *unique* x_0 such that", i.e.

$$\exists! x_0\, \psi \stackrel{\text{df}}{\iff} \exists x_0(\psi \wedge \forall x_k(\exists x_0(x_k = x_0 \wedge \psi) \to x_k = x_0)),$$

where x_k does not occur in ψ.

$$\text{Ex} \stackrel{\text{df}}{=} \{([\forall z_0 \exists! x_0\, \varphi] \to \exists y_i \forall z_0 \forall x_0[\text{ext}(y_i, z_0) = x_0 \leftrightarrow \varphi]): \varphi \in F_{\text{td}} \text{ and } y_i \text{ does not occur in } \varphi\}.$$

Note that φ may contain free variables and therefore the formulas in Ex written out in more detail are as follows:

Let z_0, x_0, and y_i not occur in $\bar{z}$, $\bar{x}$, and $\bar{y}$. Let $\varphi(z_0, \bar{z}, x_0, \bar{x}, \bar{y})$ contain no other variables than indicated. Then the 'existence-formula' belonging to $\varphi(z_0, \bar{z}, x_0, \bar{x}, \bar{y})$

is

$$\forall \bar{z}\forall \bar{x}\forall \bar{y}(\forall z_0 \exists! x_0\, \varphi(z_0, \bar{z}, x_0, \bar{x}, \bar{y}) \rightarrow \exists y_i \forall z_0 \forall x_0(\mathrm{ext}(y_i, z_0) = x_0 \leftrightarrow \varphi(z_0, \bar{z}, x_0, \bar{x}, \bar{y}))). \quad \square$$

The set Ex of axioms is useful when proving formulas of kind $\Diamond(p, \psi)$. Here we illustrate this by Theorem 7.

Let $d \stackrel{\mathrm{df}}{=} \langle\{+', -, 0', 1'\}, \{(+', 3), (-, 3), (0', 1), (1', 1)\}\rangle$.

We shall use the following abbreviations: $2'$ abbreviates $(1' +' 1')$ and $3'$ abbreviates $(2' +' 1')$.

Let $p \in P_d$ be the following program:

$$p \stackrel{\mathrm{df}}{=} \langle (0'\colon \textbf{if } x_0 = 0' \textbf{ goto } 3'),$$
$$(1'\colon x_0 \leftarrow x_0 - 1'),$$
$$(2'\colon \textbf{if true goto } 0'),$$
$$(3'\colon \textbf{halt})\rangle.$$

(Here $n = 3$ and $c = 1$.)

Next we define the set DIA of induction axioms for the data.

Let $\varphi \in F_{\mathrm{td}}$. Then $\varphi(0')$ and $\varphi(x_0 +' 1')$ denote the formulas obtained from φ by replacing every free occurrence of x_0 in φ by $0'$ and $x_0 +' 1'$ respectively.

We define the induction formula φ^x as follows:

$$\varphi^x \stackrel{\mathrm{df}}{=} ([\varphi(0') \wedge \forall x_0(\varphi \rightarrow \varphi(x_0 +' 1'))] \rightarrow \forall x_0\, \varphi),$$

$$\mathrm{DIA} \stackrel{\mathrm{df}}{=} \{\varphi^x : \varphi \in F_{\mathrm{td}}\}.$$

Theorem 7. *Let p and* DIA *be as defined above. Let*

$$\mathrm{Th} \stackrel{\mathrm{df}}{=} \mathrm{Ex} \cup \mathrm{DIA} \cup \{\forall x_0((x_0 +' 1') - 1' = x_0)\} \cup \mathrm{OA}.$$

Then

$$\mathrm{Th} \models \Diamond(p, \textbf{true}).$$

I.e. p terminates for every input in every model of Th.

Proof. Recall the function θ from Definition 11 in Part I.

Consider the formula $\theta(\Diamond(p, \textbf{true})) \in F_{\mathrm{td}}$. Note that the only free variable of $\theta(\Diamond(p, \textbf{true}))$ is x_0. Let $\mathfrak{M} \in M_{\mathrm{td}}$ be such that $\mathfrak{M} \models \mathrm{Th}$. We shall use that fact that $\mathfrak{M} \models \theta(\Diamond(p, \textbf{true}))^x$.

First we show that $\mathfrak{M} \models \theta(\Diamond(p, \textbf{true}))(0')$. By Lemma 1 in the proof of Theorem 2 and by the meaning of $\Diamond(p, \textbf{true})$ we have to show that there is a trace $\langle s_0, s_1\rangle$ of p in $\mathfrak{M}$ which terminates and which is of input $0'$.

Let ψ_0 denote the formula $x_0 = 0'$. Then $\psi_0 \in F_{td}$ and $\mathfrak{M} \models \forall z_0 \exists! x_0 \psi_0$. Then by $\mathfrak{M} \models$ Ex we have $\mathfrak{M} \models \exists y_0 \forall z_0 \forall x_0 (\text{ext}(y_0, z_0) = x_0 \leftrightarrow \psi_0)$ i.e. $\mathfrak{M} \models \exists y_0 \forall z_0 \, \text{ext}(y_0, z_0) = 0'$.

Let $s_0 \in I$ be such that $(\forall b \in T)\, \text{ext}(s_0, b) = 0'$. Similarly, let ψ_1 denote the formula $(z_0 = 0 \to x_0 = 0') \wedge (z_0 \neq 0 \to x_0 = 3')$. Then $\psi_1 \in F_{td}$ and $\mathfrak{M} \models \forall z_0 \exists! x_0 \, \psi_1$. Hence

$$\mathfrak{M} \models \exists y_0 (\text{ext}(y_0, 0) = 0' \wedge (\forall z_0 \neq 0)\, \text{ext}(y_0, z_0) = 3').$$

Let $s_1 \in I$ be an intension such that $\text{ext}(s_1, 0) = 0'$ and $\text{ext}(s_1, b) = 3'$ for every $b \in T$, $b \neq 0$.

Now it is easy to check that $\langle s_0, s_1 \rangle$ is a trace of p in $\mathfrak{M}$, with input $0'$ and which terminates (at time 1). Therefore $\mathfrak{M} \models \theta(\Diamond(p, \mathbf{true}))(0')$.

Let φ denote $\theta(\Diamond(p, \mathbf{true}))$. Next we show that $\mathfrak{M} \models (\varphi \to \varphi(x_0 +' 1'))$.

Let $a \in D$ and suppose $\mathfrak{M} \models \varphi[a]$, i.e. suppose that in $\mathfrak{M}$ there is a trace $\langle s_2, s_3 \rangle$ of p which terminates and which is of input a. We have to show that there is a trace $\langle s_4, s_5 \rangle$ of p in $\mathfrak{M}$ which terminates and which is of input $a +' 1'$.

Let ψ_4 be the formula

$$(z_0 \leq 1 \to x_0 = a +' 1') \wedge (z_0 = 2 \to x_0 = a) \wedge \forall z_1 (z_0 = z_1 + 3 \to x_0 = \text{ext}(s_2, z_1)).$$

Let ψ_5 be the formula

$$(z_0 = 0 \to x_0 = 0') \wedge (z_0 = 1 \to x_0 = 1') \wedge (z_0 = 2 \to x_0 = 2')$$
$$\wedge \forall z_1 (z_0 = z_1 + 3 \to x_0 = \text{ext}(s_3, z_1)).$$

Then by $\mathfrak{M} \models$ PA we have $\mathfrak{M} \models (\forall z_0 \exists! x_0 \, \psi_4 \wedge \forall z_0 \exists! x_0 \, \psi_5)$.

Then by $\mathfrak{M} \models$ Ex we have two intensions $s_4, s_5 \in I$ such that $\langle s_4, s_5 \rangle$ is a trace of p since $\langle s_2, s_3 \rangle$ is a trace of p (and by $\mathfrak{M} \models (x_0 +' 1') - 1' = x_0$), $\langle s_4, s_5 \rangle$ terminates since $\langle s_2, s_3 \rangle$ terminates, and clearly $\langle s_4, s_5 \rangle$ is of input $a +' 1'$.

We have seen that $\mathfrak{M} \models \varphi(0') \wedge \forall x_0 (\varphi \to \varphi(x_0 +' 1'))$. Then $\mathfrak{M} \models \forall x_0 \, \varphi$ by $\mathfrak{M} \models \varphi^x$. I.e. $\mathfrak{M} \models \theta(\Diamond(p, \mathbf{true}))$. Then by Lemma 1 in the proof of Theorem 2 in Part I we have $\mathfrak{M} \models \Diamond(p, \mathbf{true})$. □

As a contrast we note that according to the *standard semantics* (see Definition 18), the set $\{\varphi \in F_d : \text{Th} \models \varphi\}$ of axioms does not imply termination of p, e.g. by [22].

6. Naur–Floyd–Hoare inductive assertions proof method

The set HF_d of Floyd–Hoare statements of type d is an important sublanguage of DF_d.

$$\text{HF}_d \stackrel{\text{df}}{=} \{(\varphi \to \Box(p, \psi)) : p \in P_d \text{ and } \varphi, \psi \in F_d\}.$$

Clearly $\text{HF}_d \subseteq \text{DF}_d$.

Properties of the Floyd–Hoare languages $\langle \text{HF}_d, M_{td}, \models \rangle$ and $\langle \text{HF}_d \cup F_{td}, M_{td}, \models \rangle$ were investigated in several papers, e.g. in [1, 3–7, 10–14, 20, 21, 25].

In Definition 17 below we recall the Naur–Floyd–Hoare proof concept ($\vdash^{F}$, Prf) for the language HF_d.

Note that $F_d \subseteq HF_d$ is practically true since φ is semantically equivalent to $(\mathbf{true} \to \Box(\langle(i_0\colon \mathbf{halt})\rangle, \varphi))$ under very mild hypotheses (namely if $i_0 \in \text{Lab}$ and $\exists y \forall z(\text{ext}(y, z) = i_0)$).

Recall the classical proof concept ($\vdash$, Prc) from Definition 12. We shall use Definitions 10 and 12 of Part I.

Definition 17 (Floyd–Hoare proof concept ($\vdash^{F}$, Prf)). The set Prf of all Floyd–Hoare proofs of type d is defined as follows:

$w \in \text{Prf}$ iff $w = \langle H, r, (\varphi \to \Box(p, \psi))\rangle$ for some $(\varphi \to \Box(p, \psi)) \in HF_d$ such that conditions (i) and (ii) below hold:

(i) $H \subseteq F_d$ and $|H| < \omega$.

(ii) $r = \langle\langle \pi_0, \ldots, \pi_{n+1}\rangle, \langle \Phi_0, \ldots, \Phi_n\rangle\rangle$ such that conditions (1)–(4) below hold for every $m \leq n$. (Recall from Convention 1 that $p = \langle (i_0\colon u_0), \ldots, (i_n\colon \mathbf{halt})\rangle$.)

(1) $\Phi_m \in F_d$ and $\langle H, \pi_{n+1}, (\varphi \to \Phi_0)\rangle \in \text{Prc}$.

(2) If $u_m =$ "$x_i \leftarrow \tau$", then

$$\langle H, \pi_m, (\Phi_m \to \Phi_{m+1}(x_i/\tau))\rangle \in \text{Prc},$$

where $\Phi_{m+1}(x_i/\tau)$ denotes the formula obtained from Φ_{m+1} by replacing x_i everywhere by τ.

(3) If $u_m =$ "**if** χ **goto** v", then

$$\langle H, \pi_m, (((\Phi_m \wedge \neg \chi) \to \Phi_{m+1}) \wedge ((\Phi_m \wedge \chi) \to \Phi_v))\rangle \in \text{Prc}.$$

(4) If $u_m =$ "**halt**", then

$$\langle H, \pi_m, (\Phi_m \to \psi)\rangle \in \text{Prc}.$$

By these we have defined the set $\text{Prf} \subseteq (HF_d)^*$. Clearly Prf is a *decidable* subset of $(HF_d)^*$.

Let $\text{Th} \subseteq F_d$ and let $\rho \in HF_d$. Then we define

$$\text{Th} \vdash^{F} \rho \text{ iff } (\exists \langle H, w, \rho\rangle \in \text{Prf}) H \subseteq \text{Th}.$$

By this we have defined the proof concept ($\vdash^{F}$, Prf) on the language HF_d in accordance with Definition 10 in Part I. □

Proposition 8. ($\vdash^{F}$, Prf) *is a decidable proof concept on* HF_d.

Proof. The proof is straightforward by using the fact that the set Prc of classical first order proofs is a decidable subset of $(F_d)^*$. □

Recall OA and IA^q from Definitions 14 and 15 respectively.

Theorem 9 (semantic characterization of Floyd's method). *Let* $\mathrm{Pres} \subseteq \mathrm{PA}$ *be Presburger's arithmetic, i.e.* Pres *is the theory of* $\langle \omega, 0, 1, + \rangle$, *and* $\mathrm{Pres} \subseteq F_t$. *Let* $\mathrm{Th} \subseteq F_d$ *and* $\rho \in \mathrm{HF}_d$ *be arbitrary.*

Consider statements (i)–(v) *below*:

(i) $\mathrm{Th} \vdash^{\mathrm{F}} \rho$,

(ii) $\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \models \rho$,

(iii) $\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{OA} \models \rho$,

(iv) $\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{Pres} \models \rho$,

(v) $\mathrm{Th} \cup \mathrm{IA} \models \rho$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), (i) $\not\Leftarrow$ (iv) *and* (i) $\not\Leftarrow$ (v). *Moreover,* (i)–(iii) *are equivalent, but there are* d, *a finite* $\mathrm{Th} \subseteq F_d$ *and* $\rho \in \mathrm{HF}_d$ *such that*

$$\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{Pres} \models \rho \quad \textit{and} \quad \mathrm{Th} \cup \mathrm{IA} \models \rho \quad \textit{but} \quad \mathrm{Th} \nvdash^{\mathrm{F}} \rho.$$

Proof. Let $\mathrm{Th} \subseteq F_d$ and $\rho \overset{\mathrm{df}}{=} (\varphi \to \Box(p, \psi)) \in \mathrm{HF}_d$.

Proof of (i) $\Rightarrow$ (ii): Assume $\mathrm{Th} \vdash^{\mathrm{F}} \rho$. Let $\mathfrak{M} \models \mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}}$. We have to show $\mathfrak{M} \models (\varphi \to \Box(p, \psi))$.

Let $\langle g, k, r \rangle$ be an arbitrary evaluation of the variables. Suppose $\mathfrak{M} \models \varphi[g, k, r]$, i.e. $\mathbf{D} \models \varphi[k]$. We have to show $\mathfrak{M} \models \Box(p, \psi)[g, k, r]$.

Let $\bar{s} = \langle s_0, \ldots, s_c \rangle$ be a trace of p in $\mathfrak{M}$ with input k. We have to show $\mathbf{D} \models \psi[q]$ for every possible output q of $\bar{s}$. $\mathrm{Th} \vdash^{\mathrm{F}} \rho$ means that $\langle H, r, \rho \rangle \in \mathrm{Prf}$ for some $H \subseteq \mathrm{Th}$ and for some r.

Let $r = \langle \langle \pi_0, \ldots, \pi_{n+1} \rangle, \langle \Phi_0, \ldots, \Phi_n \rangle \rangle$. Let $\bar{y} \overset{\mathrm{df}}{=} \langle y_0, \ldots, y_{c-1} \rangle$ and define the formula γ as

$$\gamma \overset{\mathrm{df}}{=} \gamma(z_0, y_0, \ldots, y_c) \overset{\mathrm{df}}{=} \left[\bigwedge_{m=0}^{n} (\mathrm{ext}(y_c, z_0) = i_m \to \Phi_m(\mathrm{ext}(\bar{y}, z_0))) \right].$$

Clearly $\gamma \in F_{td}$ and then $\gamma^{+}_{z_0} \in \mathrm{IA}^{\mathrm{q}}$ since no quantifier of sort t occurs in γ.

By Definition 17(ii)(1) we have that $\langle H, r, \rho \rangle \in \mathrm{Prf}$ and $H \subseteq \mathrm{Th}$ imply $\mathrm{Th} \vdash (\varphi \to \Phi_0)$, and then $\mathbf{D} \models \Phi_0[k]$ by $\mathbf{D} \models \varphi[k]$ and $\mathfrak{M} \models \mathrm{Th}$. Thus $\mathfrak{M} \models \gamma(0, \bar{s})$ (by $\mathfrak{M} \models \{i_m \neq i_l$ for $m < l \leq n\}$).

By conditions (ii)(2)–(3) in Definition 17, by the soundness of the classical proof concept, and by the facts that $\langle H, r, \rho \rangle \in \mathrm{Prf}$, $\mathfrak{M} \models H$ and $\bar{s}$ is a trace of p in $\mathfrak{M}$ we obtain that $\mathfrak{M} \models \forall z_0 (\gamma(z_0, \bar{s}) \to \gamma(z_0 + 1, \bar{s}))$. Then by $\mathfrak{M} \models \gamma^{+}_{z_0}$ we conclude $\mathfrak{M} \models \forall z_0\, \gamma(z_0, \bar{s})$.

Let $b \in T$ and assume that $\bar{s}$ terminates at time b with output $q \in {}^{\omega}D$. Then $\mathbf{D} \models \Phi_n[q]$ since $\mathfrak{M} \models \gamma(b, \bar{s})$. By condition (ii)(4) in Definition 17 we have $\langle H, \pi_n, (\Phi_n \to \psi) \rangle \in \mathrm{Prc}$ and hence $H \models (\Phi_n \to \psi)$. Thus $\mathbf{D} \models (\Phi_n \to \psi)$ and therefore $\mathbf{D} \models \psi[q]$ by $\mathbf{D} \models \Phi_n[q]$.

We have proved $\mathfrak{M} \models (\varphi \to \Box(p, \psi))$.

(ii) $\Rightarrow$ (iii) is obvious.

Proof of (iii) $\Rightarrow$ (i): One can prove the present implication by using [13], see Theorem 4 of [12]. In doing this the methods of [10] and [14] can be useful.

We have proved (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof of (v)$\not\Rightarrow$(i): It can be proved that $\mathrm{IA} \models \forall z_0 \exists z_1 (z_0 = 0 \vee z_0 = z_1 + 1)$.

Let d consist of the symbols $0'$, suc, $\leqslant'$. Let $\mathrm{Th} \subseteq F_d$ be the theory of $\langle \omega, 0, \mathrm{suc}, \leqslant \rangle$. It is known that Th is finitely axiomatizable that is we may assume $|\mathrm{Th}| < \omega$ and Th is complete, see [8].

We have to find $\rho \in \mathrm{HF}_d$ such that $\mathrm{Th} \cup \mathrm{IA} \models \rho$ but $\mathrm{Th} \nvdash^{\mathrm{F}} \rho$. Let

$$p \stackrel{\mathrm{df}}{=} \langle (0\colon x_0 \leftarrow \mathrm{suc}\, x_0), (1\colon x_1 \leftarrow \mathrm{suc}\, x_1),$$
$$(2\colon \mathbf{if}\ x_0 \neq x_2\ \mathbf{goto}\ 0), (3\colon \mathbf{halt}) \rangle.$$

Let $\varphi(z_0, y_0, y_1)$ be the formula

$$\forall z_1 [\mathrm{ext}(y_0, z_0) = \mathrm{ext}(y_0, z_1) \rightarrow \mathrm{ext}(y_1, z_0) = \mathrm{ext}(y_1, z_1)].$$

Then $\varphi(z_0, y_0, y_1)^{+}_{z_0} \in \mathrm{IA}$. Hence from $\mathrm{Th} \cup \mathrm{IA}$ it is $\vdash^{\mathrm{N}}$-provable that if $\langle y_0, \ldots, y_3 \rangle$ is a trace of p and $\mathrm{ext}(y_0, 0) = 0'$, then $\forall z_0\, \varphi(z_0, y_0, y_1)$.

Let $\rho \in \mathrm{HF}_d$ consist of a program which executes the above p twice with the output condition of ρ stating that the results of the two executions of p coincide. Let the input condition of ρ be $x_0 = 0'$. With the kind of induction, used while proving $\forall z_0\, \varphi(z_0, y_0, y_1)$ above, one can prove $\mathrm{Th} \cup \mathrm{IA} \vdash^{\mathrm{N}} \rho$. For a hint see the proof of (iv) $\not\Rightarrow$ (i) below. By Theorem 2 (our completeness theorem for $\vdash^{\mathrm{N}}$) then (v) holds for Th and ρ.

The easiest way of proving $\mathrm{Th} \nvdash^{\mathrm{F}} \rho$ goes by using ultraproducts. Assume $\mathrm{Th} \vdash^{\mathrm{F}} \rho$. Let Φ_1 be the inductive assertion in the first execution of p and Φ_2 be the one in the second execution of p. Look at the relations which Φ_1 and Φ_2 are supposed to define in the model $\langle \omega, 0, \mathrm{suc}, \leqslant \rangle$. By an easy ultraproduct construction one can see that these relations are not definable. Hence $\mathrm{Th} \nvdash^{\mathrm{F}} \rho$.

Proof of (iv) $\not\Rightarrow$ (i): Let d, Th and ρ be as in the proof of ((v) $\not\Rightarrow$ (i)). Then $\mathrm{Th} \nvdash^{\mathrm{F}} \rho$ by the same ultraproduct construction as above.

The idea of the proof of $\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{Pres} \models \rho$ is the following:

If we have Pres postulated about time structure, then we can perform addition on time. Then we can say that "if φ is true at time $z_0 < z_1$ and χ is true at time z_1, then φ is true at time $z_1 + z_0$ too", i.e. if e.g. we execute the same program twice and z_1 is the time of the first termination, then we can say that "if φ holds at time $z_0 < z_1$, then it will hold exactly z_0 time after z_1 again". Another way of proving these is by using Theorem 4 in [12]. □

Part ((i) $\Leftrightarrow$ (iii)) of Theorem 9 implies that the language $\langle \mathrm{DF}_d, \mathrm{Mod}(\mathrm{Ax}_0), \models \rangle$ is reasonable enough, it contains *no impossible* models. I.e. the models of Ax_0 do not contradict the Floyd–Hoare proof rules for programs.

Part ((i) $\Leftrightarrow$ (ii)) of Theorem 9 is a kind of semantic characterization of the information implicitly contained in the Floyd-method. It appears that this information content of Floyd-method is IA^{q}. Theorem 9 also says that if we can reason about time as being ordered, i.e. use $\mathrm{OA} \subseteq F_t$, then our reasoning ability is not beyond the power of Floyd's method. But if we can perform addition on time (note that

Pres$\models$OA) or if we can quantify over time points (IA), then our reasoning ability is definitely beyond the power of Floyd's method. Note that quantifying over time is roughly the same as using time-modalities.

It is interesting to compare the powers of different proof methods. The following Theorem 10 says that the proof method ($\vdash^{N}$, Prn) is strictly stronger than the Floyd–Hoare method ($\vdash^{F}$, Prf).

Definition 18 (the standard dynamic language $\langle DF_d, STM_d, \models^{\omega}\rangle$). Let $\mathfrak{M} = \langle \mathbf{T}, \mathbf{D}, I, \text{ext}\rangle \in M_{td}$.

$\mathfrak{M}$ is said to be *standard* iff conditions (i)–(iii) below hold:

(i) $\mathbf{T} = \langle \omega, \leq, +, \cdot, 0, 1\rangle$,

(ii) $I = {}^{T}D$.

(iii) $(\forall s \in {}^{T}D)(\forall b \in T)\ \text{ext}(s, b) = s(b)$.

The class of all standard elements of M_{td} is denoted by STM_d.

Let $\text{Th} \subseteq DF_d$ and $\varphi \in DF_d$. Then we define

$$\text{Th} \models^{\omega} \varphi \Leftrightarrow (\forall \mathfrak{M} \in \text{Mod(Th)} \cap STM_d)\ \mathfrak{M} \models \varphi. \quad \square$$

Note that $STM_d \models \text{Ax}$ and $\mathfrak{N} \in STM_d$, where $\mathfrak{N}$ and d' were defined in Definition 6 in Part I.

Theorem 10 (Biró–Csirmaz). *There are a similarity type d and a finite theory* $\text{Th} \subseteq F_d$ *such that for some Floyd–Hoare statement* $(\varphi \to \Box(p, \psi)) \in HF_d$ *conditions* (i)–(iii) *below hold*:

(i) $\text{Th} \cup \text{Ax}_0 \vdash^{N} (\varphi \to \Box(p, \psi))$,

(ii) $\text{Th} \nvdash^{F} (\varphi \to \Box(p, \psi))$,

(iii) $\text{Th} \models^{\omega} (\varphi \to \Box(p, \psi))$.

Proof. In the proof of part ((v)$\not\Rightarrow$(i)) of Theorem 9 a finite $\text{Th} \subseteq F_d$ and $\rho \in HF_d$ were constructed such that $\text{Th} \nvdash^{F} \rho$ but $\text{Th} \cup \text{IA} \vdash^{N} \rho$. By $\text{IA} \subseteq \text{Ax}_0$ we proved (i) and (ii). By $STM_d \models \text{Ax}$ obviously (i) always implies (iii) but validity of (iii) can be checked directly by looking at our concrete Th and ρ.

Note that we do not need the full power of the proof of Theorem 9 here since the ultraproduct construction proving (ii) is clear, and to prove (i) we have the full power of $\text{Ax}_0 = \text{IA}^{+} \cup \text{PA}$ at our disposal. $\square$

Problem. Do there exist d, $\text{Th} \subseteq F_d$ and $\rho \in HF_d$ such that

$$\text{Ax} \cup \text{Th} \models \rho \quad \text{and} \quad \text{Ax}_0 \cup \text{Th} \not\models \rho?$$

Definition 19 (the sets PA', PA_d of axioms about data). Recall from Definition 6 that the similarity type d' consists of the binary relation symbol $\leq'$ and the operation symbols $+'$, $\cdot'$, $0'$, $1'$ with arities 2, 2, 0, 0 respectively. Note that d' is disjoint from t, actually d' is a disjoint copy of t. PA' denotes the set of Peano axioms formulated in $F_{d'}$.

Note that $\mathfrak{N} \models PA' \cup PA$ where $\mathfrak{N}$ was introduced in Definition 6. Also note that $PA \subseteq F_t^Z$ while $PA' \subseteq F_{d'}$.

Let d be an arbitrary similarity type containing d'. Then $F_d \supseteq F_{d'}$ but possibly $F_d \neq F_{d'}$.

We define PA_d as

$$PA_d \overset{\text{df}}{=} PA' \cup \{([\varphi(0') \wedge \forall x(\varphi(x) \to \varphi(x +' 1'))] \to \forall x\, \varphi(x)):$$
$$\varphi(x) \in F_d\}.$$

Clearly $d = d'$ iff $PA_d = PA'$. □

Theorem 11 (Andréka–Csirmaz–Németi–Paris). *Let the similarity type d contain d'. Let* $Th \subseteq F_d$ *and* $\rho \in HF_d$. *Assume* $PA' \subseteq Th$. *Then* (i)–(iii) *below hold*:

(i) $Th \vdash^F \rho \Leftrightarrow Th \cup Ax_0 \models \rho$,

(ii) $Th \vdash^F \rho \Leftrightarrow Th \cup Ax \models \rho$,

(iii) $PA' \vdash^F \rho \Leftrightarrow PA' \cup Ax \models \rho$.

Proof. Proofs of (iii) can be found in [3, 7]. The proof of (i) can be found in [1] as Theorem 6 there.

(ii) was proved in [1] as Theorem 6 there under the additional assumption that $Th \supseteq PA_d$. The condition $Th \supseteq PA_d$ was eliminated from the proof of (ii) by Jeff B. Paris (Manchester) and L. Csirmaz recently. □

About (ii) of Theorem 11 above we would like to emphasize that if $PA' \subseteq Th$, then d may contain symbols for which the induction axioms are not postulated, moreover, it is allowed that for some $\varphi(x) \in F_d$ we have

$$(\varphi(0) \wedge \forall x[\varphi(x) \to \varphi(x +' 1')] \wedge \exists x \neg \varphi(x)) \in Th.$$

Of course in this case $\varphi(x) \in F_d$ but $\varphi(x) \notin F_{d'}$. To be able to appreciate the difference between the conditions $PA_d \subseteq Th$ and $PA' \subseteq Th$ see the concrete example constructed in the proof of Proposition 13.

Note that by Theorem 10 the condition $PA' \subseteq Th$ is necessary in Theorem 11(i) and (ii).

7. Connections with some other branches of explicit time semantics of programming

We use the names "Explicit Time Semantics", "Nonstandard Semantics", "Nonstandard Time Semantics" as synonyms. For works in this field see [1, 3, 5–7, 10–14, 20, 21].

In Definition 20 below we recall the Continuous Traces Language CL_d of type d from [3, 4, 10–14].

Definition 20 (the Continuous Traces Language $\mathrm{CL}_d = \langle \mathrm{HF}_d, M_d, \models^c \rangle$). Let $\varphi \in F_d$. Let $m \in \omega$ and let $\bar{y} = \langle y_0, \ldots, y_m \rangle$ and $\bar{x} = \langle x_0, \ldots, x_m \rangle$. Let $\bar{x} = \mathrm{ext}(\bar{y}, z_0)$ denote the formula $\bigwedge_{i \le m} x_i = \mathrm{ext}(y_i, z_0)$.

We define the formula $\varphi_m \in F_{\mathrm{td}}$ to be $\forall \bar{x}(\bar{x} = \mathrm{ext}(\bar{y}, z_0) \to \varphi)$.

Remark. $\varphi_m = \varphi_m(z_0, \bar{y}) \in F_{\mathrm{td}}$ and φ_m is equivalent with $\varphi(\mathrm{ext}(y_0, z_0), \ldots, \mathrm{ext}(y_m, z_0), x_{m+1}, \ldots, x_w)$ if $\varphi = \varphi(x_0, \ldots, x_w)$.

We define

$$IA_0 \overset{\mathrm{df}}{=} \{(\varphi_m)^+_{z_0} : \varphi \in F_d,\ m \in \omega\} \cup \{(i \neq j) : i, j \in \mathrm{Lab} \text{ and } i \neq j\}.$$

Recall the set $\mathrm{Pe} \subseteq F_{\mathrm{td}}$ of 'successor axioms' from Definition 15. The Continuous Traces Axioms are $\mathrm{Ctax} \overset{\mathrm{df}}{=} \mathrm{IA}_0 \cup \mathrm{Pe}$.

Let $\mathbf{E} \in M_d$ and $\varphi \in \mathrm{DF}_d$. Then we define

$$\mathbf{E} \models^c \varphi \Leftrightarrow (\forall \mathfrak{M} \in \mathrm{Mod}(\mathrm{Ctax}))[\mathbf{D} = \mathbf{E} \Rightarrow \mathfrak{M} \models \varphi].$$

The Continuous Traces Language is

$$\mathrm{CL}_d \overset{\mathrm{df}}{=} \langle \mathrm{HF}_d, M_d, \models^c \rangle. \quad \square$$

Let $\mathrm{Th} \subseteq \mathrm{HF}_d$ and $\varphi \in \mathrm{HF}_d$. We shall use $\mathrm{Th} \models^c \varphi$ in the usual sense, i.e.

$$\mathrm{Th} \models^c \varphi \text{ iff } (\forall \mathbf{E} \in M_d)[\mathbf{E} \models^c \mathrm{Th} \Rightarrow \mathbf{E} \models^c \varphi].$$

It is easy to check that $\mathrm{Th} \models^c \varphi$ iff $\mathrm{Th} \cup \mathrm{Ctax} \models \varphi$.

The Continuous Traces Semantics (or Language) $\mathrm{CL}_d = \langle \mathrm{HF}_d, M_d, \models^c \rangle$ was introduced in [4] and further refined and investigated in [3, 10–14]. In [3, 4] $\models^c$ was denoted by $\models^{\mathrm{pc}}$.

Recall IA^{q} from Definition 15.

Proposition 12 (semantic characterization of Continuous Traces Semantics). *Let* $\mathrm{Th} \subseteq F_d$ *and* $\rho \in \mathrm{HF}_d$. *Then statements* (i)–(iii) *below are equivalent*:

(i) $\mathrm{Th} \models^c \rho$,
(ii) $\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \models \rho$,
(iii) $\mathrm{Th} \cup \mathrm{IA}_0 \models \rho$.

Proof. Assume (i). Then by [10] $\mathrm{Th} \vdash^{\mathrm{F}} \rho$. Then by the proof of part ((i) $\Rightarrow$ (ii)) of Theorem 9 we have $\mathrm{Th} \cup \mathrm{IA}_0 \models \rho$. To see this observe that in the proof of Theorem 9((i) $\Rightarrow$ (ii)) the only elements of IA we used were of the form $\varphi\,(\mathrm{ext}(\bar{y}, z_0))^+_{z_0}$ for some $\varphi(\bar{x}) \in F_d$, φ containing no other free variables than $\bar{x}$. This proves (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii) is obvious since $\mathrm{IA}_0 \subseteq \mathrm{IA}^{\mathrm{q}}$.

(iii) $\Rightarrow$ (i) holds by $\mathrm{IA}_0 \subseteq \mathrm{Ctax}$.

Assume (ii). Then by Theorem 4 in [12] $\mathrm{Th} \vdash^{\mathrm{F}} \rho$. Then by the proof of Theorem 9((i) $\Rightarrow$ (ii)) we have $\mathrm{Th} \cup \mathrm{IA}_0 \models \rho$. I.e. (iii) holds. $\square$

Remark. Let $d \supseteq d'$, PA′ and PA_d be as in Definition 19. Then for every $Th \subseteq F_d$ such that $PA_d \subseteq Th$ the Continuous Traces Semantics $\langle HF_d, Mod(Th), \models^c \rangle$ is equivalent with the *Definable Traces Semantics* of Gergely–Úry [20, 21] w.r.t. Th.

However, there are $d \supseteq d'$ and a decidable $Th \subseteq F_d$ such that $PA' \subseteq Th$ and $\langle HF_d, Mod(Th), \models^c \rangle$ is *not* equivalent with the Definable Traces Semantics of [20, 21] w.r.t. Th.

Proposition 13. *Let d' and PA′ be as in Definition* 19. *Then there are $d \supseteq d'$ and a decidable theory $Th \subseteq F_d$ and $p \in HF_d$ such that $PA' \subseteq Th$ and $Th \models p$ and $Th \models^c p$ and $Th \vdash^F p$ but Th does not imply p w.r.t. Definable Traces Semantics of* [20, 21].

Proof. Let d consist of d' together with a new unary relation symbol R and a new constant symbol c. Let $Th \overset{df}{=} PA' \cup \{R(0'), [R(x) \to R(x +' 1')]\}$. Let the program p be $\langle (0'\colon x_0 \leftarrow 0'), (1'\colon \textbf{if } x_0 = c \textbf{ goto } 4'), (2'\colon x_0 \leftarrow x_0 +' 1'), (3'\colon \textbf{if true goto } 1'), (4'\colon \textbf{halt}) \rangle$. Then $Th \vdash^F \Box(p, R(x_0))$ and clearly $Th \models^c \Box(p, R(x_0))$.

Let $\mathbf{D}$ be a model of Th such that $\mathbf{D} \models \neg R(c)$. Such a $\mathbf{D}$ exists since if $\langle D, +, \cdot, 0, 1 \rangle$ is any nonstandard model of PA′ and $R \subseteq D$ is the set of all standard numbers and $c \in D$ is any nonstandard number, then $\mathbf{D} \models Th$ and $\mathbf{D} \models \neg R(c)$.

Then $\Box(p, R(x_0))$ is not true in $\mathbf{D}$ w.r.t. definable traces since there is a definable trace of p in $\mathbf{D}$ which terminates with output c (of course this definable trace is not continuous). At the same time $\mathbf{D} \models^c \Box(p, R(x_0))$ and $\mathbf{D} \models \Box(p, R(x_0))$ since no continuous trace of p terminates in $\mathbf{D}$. $\Box$

In this context see Section 10 at the end of this paper.

8. Connections with related approaches in programming theory

Let y be a variable in a program scheme. Sometimes such a y is called an 'identifier' instead of 'variable'. We use the word 'variable'. What we call an 'intension' of y is called an 'L-value' of y in '*Scott–Strachey semantics*' *of programming languages* (see [23, p. 202]), while an extension ext(y, timepoint) is called an 'R-value' of y there. Intensions for y are often called '*addresses* or locations in a computer corresponding to the identifier y' and ext(y, timepoint) is often called the '*content* of the address' (mentioned above) at the point 'timepoint' of time. See the "Temporal Notions" of the Milne–Strachey book [23] on programming semantics.

In '*operational semantics*', abstract machines with registers are often used. Then to a variable (or identifier) y which occurs in a program, a register of the machine is associated. During the 'computation' or 'execution of the program' the register associated to y remains the same, it does *not* change. However, the *content* of the register *may change* many times. The register associated to y is the intension of y while the content of this register at time z is the extension of y at time z and the present paper denotes it by ext(y, z). See e.g. Section 3 of Cook [9].

In *other approaches* to programming semantics, e.g. in VDL, the concepts '*environment*' and '*state*' (or 'store' or 'memory') do correspond to our intensions and extensions. See e.g. [23, p. 203]. Namely an environment maps the variables $\{y_w : w \in \omega\}$ to 'locations' and a state maps the locations to data values, i.e. to elements of D. Thus environments are like our traces, *locations correspond to our intensions* and a state corresponds to the function $\text{ext}(-, z) : I \to D$ where $z \in T$ is a parameter. I.e. *states* are like elements of T, they *correlate extensions to the intensions.*

In short: our intension–extension duality corresponds to the usual locations–values duality as described e.g. in [23, pp. 202–203]. During the execution of a program, the intension associated to a variable y does not change. Analogously, the location associated to y does not change. I.e. the command "$y = y+1$" is a meaningful statement about the location or intension associated to y but it is not so meaningful if we *try* to interpret it as a statement about the value or extension associated to y.

9. Connections with related approaches in nonclassical model theory, philosophical logic and semantics of natural languages

The above mentioned problem was raised and studied not only in 'program semantics theory' but also in a broader theory of Semantics of Languages in general. A frequently used and well-developed branch of the latter is 'Intensional Model Theory', see e.g. [24, 18]. Intensional Logic and Model Theory was elaborated by R. Montague (a student of Tarski) and his followers, see [24, 18]. Our way of using these notions in programming theory is explained in [25], [16, Section 2, pp. 3–4], and in [15] with several nice drawings on pp. 33–34.

In Intensional Model Theory frequently used examples are the sentences "The temperature rises", "The price of sugar rises" etc., see [24, p. 268], [19]. They correspond to "y rises" where y is a variable standing for intensions. At a valuation k of the variables into $\mathfrak{M} = \langle \mathbf{T}, \mathbf{D}, I, \text{ext}\rangle$ such that $k(y) = s \in I$, the statement "y rises" is true in $\mathfrak{M}$ iff $\text{ext}(s, n) < \text{ext}(s, n+1)$ holds in $\mathfrak{M}$. I.e. "y rises" is meaningful only if y denotes an '*intension*', i.e. a function from some set T of time points to some set D of possible values. This 'motivating example' sentence: "y rises" of intensional model theory is quite similar to the 'programming language sentence' $y = y+1$. The latter is true if $\text{ext}(s, n+1) = \text{ext}(s, n) + 1$ holds in $\mathfrak{M}$ at the valuation $k(y) = s \in I$.

Another intensional logic example: "The president changes in every four years". I.e. "y changes in every four years". This is true at valuation $k(y) = s \in I$ in $\mathfrak{M}$ iff $\mathfrak{M} \models (\text{ext}(s, n) \neq \text{ext}(s, n+4))$. Compare with "$y$ changes during the execution of a program".

The model theoretic treatments of *time* in [17, p. 188 Definition 3.1] and [24, p. 258] are similar to ours in the respect that a 'generalized model or interpretation' in both cases contains a structure $\mathbf{T} = \langle T, \leq, \ldots\rangle$ called time structure, see [24, pp. 37–38, 98–101, 258]. In [24, p. 258] an 'interpretation' is a tuple $\langle A, I, J, \leq, F\rangle$

where $\langle J, \leq \rangle$ is the same as our **T**; A is our 'D' and ${}^{J}A$ which is denoted there by S_{eAIJ} is our 'I'. ${}^{J}A$ is called there the 'set of intensions' exactly as we do here. Our operation symbol 'ext' is denoted there by '$\vee$'.

In [17, p. 188] our $\mathfrak{M}$ is '$\mathfrak{A}$' and our **T** is '$\langle T, < \rangle$'. However, there are no intensions there thus the analogy stops here. It is true that the meaning of a constant symbol 'c' in $\mathfrak{A}$ is an element of ${}^{T}A$ where A is $\bigcup\{A_t : t \in T\}$ but there are no variables ranging over ${}^{T}A$ in [17], while in [24] the variables of type $\langle s, e \rangle$ are just doing that.

An essential *difference* between intensional models (M_{td} of this paper or in [24]) and 'Kripke-style' models ([2] or [17]) is that in a Kripke model $\langle \mathbf{T}, \langle \mathbf{D}_t : t \in T \rangle \rangle$ the elements of ${}^{T}D$ do not form a separate universe or 'sort' to speak about while in an intensional model $\mathfrak{M} = \langle \mathbf{T}, \mathbf{D}, {}^{T}D, \mathrm{ext} \rangle$ they definitely do so. See [2, p. 3].

For further considerations on the subject of Sections 8 and 9 see [26].

10. Recent developments and problems

To motivate the problems below, first we formulate some results. The notations d' and $\mathrm{PA}' \subseteq F_{d'}$ were introduced in Definition 19. **N**, $\models^{\omega}$, OA, STM_d and IA^{q} were introduced in Definitions 6, 18, 14 and 15.

Theorem 14 (Plotkin–Németi). *Let* $\mathrm{Th} \subseteq F_d$ *be recursively enumerable. Assume* $\mathrm{Th} \supseteq \mathrm{PA}'$ *and that* $\mathrm{ZFC} \models$ "Th *is consistent*". *Then there is* $\rho \in \mathrm{HF}_d$ *such that*

$$\mathrm{ZFC} \vdash \text{“}\mathrm{Th} \models^{\omega} \rho\text{”} \quad \textit{and} \quad \mathrm{Th} \nvdash^{\mathrm{F}} \rho.$$

Theorem 15. *There is a decidable set* $\mathrm{Tax} \subseteq F_t^Z$ *of time axioms such that* $\mathbf{N} \models \mathrm{Tax}$ *and for some* $\rho \in \mathrm{HF}_d$, *we have*

$$\mathrm{PA}' \cup \mathrm{IA}^{+} \cup \mathrm{Tax} \vdash^{\mathrm{N}} \rho \quad \textit{and} \quad \mathrm{PA}' \nvdash^{\mathrm{F}} \rho.$$

We define

$$\mathrm{IA}^{1} \stackrel{\mathrm{df}}{=} \{\varphi^{+}_{z_0} : \varphi \in F_{\mathrm{td}} \text{ and} (\forall i > 0)(z_i \text{ does not occur in } \varphi \text{ neither free nor bound})\} \cup \mathrm{Lax}.$$

Theorem 16. *There is a finite* $\mathrm{Th} \subseteq F_d$ *and* $\rho \in \mathrm{HF}_d$ *such that*

$$\mathrm{Th} \cup IA^{1} \cup \mathrm{Pe} \vdash^{\mathrm{N}} \rho \quad \textit{and} \quad \mathrm{Th} \nvdash^{\mathrm{F}} \rho.$$

Cf. Theorems 9 and 10. Theorem 16 above says that any fragment of DL_d in which time modalities (always, sometime) are still available has a greater reasoning power than that of Floyd–Hoare method.

Theorems 17, 18 below are not recent developments, they are included here to motivate Problem 1 below and to show that the dynamic logic $\langle \mathrm{DL}_d, \vdash^{\mathrm{N}} \rangle$ developed in the present paper is complete in the sense of [9] and Harel w.r.t. standard time models. About these notions see the introduction of Part I. Note that in the axiom systems Numb and Sth(K) below no dynamic formulas occur, they contain only classical formulas from F_{td}.

Theorem 17 (arithmetical completeness in the sense of Harel). *Let* $K \subseteq M_d$.

$$\mathrm{Sth}(K) \stackrel{\mathrm{df}}{=} \{\varphi \in F_{td}\colon \langle \mathbf{N}, \mathbf{D}, {}^{\omega}D, \text{value of}\rangle \models \varphi \text{ for all } \mathbf{D} \in K\}.$$

I.e.

$$\mathrm{Sth}(K) = \{\varphi \in F_{td}\colon (\forall \mathfrak{M} \in \mathrm{STM}_d)[\mathbf{D} \in K \Rightarrow \mathfrak{M} \models \varphi]\}.$$

Let $\rho \in \mathrm{DF}_d$. *Then*

$$\mathrm{Sth}(K) \vdash^{\mathrm{N}} \rho \text{ iff } K \models^{\omega} \rho.$$

Corollary 18 (completeness w.r.t. oracles in the sense of [9]). *Let*

$$\mathrm{Numb} \stackrel{\mathrm{df}}{=} \{\varphi \in F_{td'}\colon \mathfrak{N} \models \varphi\}.$$

Let $\rho \in \mathrm{DF}_{d'}$. *Then*

$$\mathrm{Numb} \vdash^{\mathrm{N}} \rho \text{ iff } \mathfrak{N} \models \rho.$$

Problem 1. Does there exist $\mathrm{St} \subseteq F_{td}$ such that for every $\mathrm{Th} \subseteq F_d$ and $\rho \in \mathrm{HF}_d$ we have

$$\mathrm{Th} \models^{\omega} \rho \Leftrightarrow \mathrm{St} \cup \mathrm{Th} \models \rho?$$

Problem 2. Do there exist d, $\mathrm{Th} \subseteq F_d$ and $\rho \in \mathrm{HF}_d$ such that

$$\mathrm{Ax} \cup \mathrm{Th} \models \rho \quad \text{and} \quad \mathrm{Ax}_0 \cup \mathrm{Th} \not\models \rho?$$

Cf. Definitions 14 and 18, Theorem 10, and Proposition 6.

Problem 3. What is the answer to Problem 2 above under the additional restriction that Th be recursively enumerable?

Problem 4. Let $\mathrm{Pres} \subseteq F_t$ be as in Theorem 9. By Theorem 9 there are a finite $\mathrm{Th} \subseteq F_d$ and $\rho \in \mathrm{HF}_d$ such that

$$\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{Pres} \models \rho \quad \text{but} \quad \mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{OA} \not\models \rho.$$

Note that roughly speaking $\mathrm{OA} \subseteq \mathrm{Pres} \subseteq \mathrm{PA}$ and they are the theories of $\langle \mathrm{suc}, \leqslant \rangle$, $\langle \mathrm{suc}, \leqslant, + \rangle$ and $\langle \mathrm{suc}, \leqslant, +, \cdot \rangle$ respectively.

Are there a decidable $\mathrm{Th} \subseteq F_d$ and $\rho \in \mathrm{HF}_d$ for some d such that

$$\mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{PA} \models \rho \quad \text{but} \quad \mathrm{Th} \cup \mathrm{IA}^{\mathrm{q}} \cup \mathrm{Pres} \not\models \rho?$$

In other words: Theorem 9 says that the ability of performing addition on time increases the reasoning power of dynamic logic. Does the ability of performing multiplication on time affect the reasoning power in any similar way?

Problem 5. Let IA^q and IA^f be as in Definition 15 and Proposition 6. Let $Th \subseteq F_d$ and $\rho \in HF_d$ be arbitrary. Is it true that

$$Th \vdash^F \rho \text{ iff } Th \cup (IA^q \cap IA^f) \cup PA \models \rho?$$

Cf. Theorems 9 and 10.

Problem 6. Let $Th \subseteq F_d$ and $\rho \in HF_d$. Assume $Th \supseteq PA'$ and $Th \cup Ax \cup Ex \models \rho$. Is then $Th \vdash^F \rho$ true?

Problem 7. Let d and DIA be as in Theorem 7. Let $\rho \in HF_d$. Assume $F_d \supseteq Th \supseteq PA'$ and $Th \cup Ax \cup Ex \cup DIA \models \rho$.

Is then $Th \vdash^F \rho$ true?

Problem 8. Continue the investigation started in Definition 16 and Theorem 7!

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