Logic of space-time and relativity theory

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1 Introduction

Our goal is to make relativity theory accessible and transparent for any reader with logical background. The reader does not have to "believe" anything. The emphasis is on the logic-based approach to relativity theory. The purpose is giving insights as opposed to mere recipes for calculations. Therefore proofs will be visual geometric ones, efforts will be made to replace computational proofs with suggestive drawings.

Relativity theory comes in (at least) two versions, special relativity (Einstein 1905) and general relativity (Einstein, Hilbert 1915). They differ in scope, the scope of general relativity is broader. Special relativity is a theory of motion and light propagation in vacuum far away from any gravitational object. I.e. special relativity does not deal with gravity. Also, special relativity is a "prelude" for general relativity, it provides a foundation or starting point for the general theory. General relativity unifies special relativity and the theory of gravitation. In some sense, general relativity is an "extension" of special relativity putting also gravity into the picture. General relativity can be used as a foundation for cosmology, e.g. it is a suitable framework for discussing the (evolution, properties of the) whole universe (expanding or otherwise). Special relativity, on the other hand, is not rich enough for this purpose. General relativity also provides the theory of black holes, wormholes, timewarps etc. Special relativity shows us that there is no such thing as space in itself, instead, a unified space-time exists. This inseparability of space and time becomes more dramatic in general relativity. Namely, general relativity shows us that gravity is nothing but the curvature of space-time. It is extremely difficult, if not impossible, to explain gravity without invoking the curvature (i.e. geometry) of space-time. The crucial point is that curvature of space is not enough (by far), it is space-time whose curvature explains gravity.[1] From a different angle: general relativity is a "geometrization" of much of what we know about the world surrounding us. E.g. it provides a full geometrization of our understanding of gravity and related phenomena like motion and light signals.

In Sec. 2 we study special relativity, in Sec. 3 we do the same for general relativity, in Sec. 4 we apply the so obtained tools to black holes, wormholes,

[1]If we took into account the curvature of space only, then apples would no more fall down from trees. Gravitational attraction as such would disappear. On the other hand, if we keep the temporal aspects of curvature but ignore curvature of pure space, then gravity would not disappear, instead, this would cause only minor discrepancies in predicting trajectories of very fast moving bodies (relative to the source of gravity, e.g. the Earth or a black hole).

timewarps. The emphasis is on the space-time aspects. In Sec. 5 we briefly discuss the literature.

2 Special relativity

In this section, among others, we give a first-order logic (FOL) axiom system for special relativity such that we use only a handful of simple, streamlined axioms. In our approach, axiomatization is not the end of the story, but rather the beginning. Namely: axiomatizations of relativity are not ends in themselves (goals), instead, they are only tools. Our goals are to obtain simple, transparent, easy-to-communicate insights into the nature of relativity, to get a deeper understanding of relativity, to simplify it, to provide a foundation for it. Another aim is to make relativity theory accessible for many people (as fully as possible). Further, we intend to analyze the logical structure of the theory: which assumptions are responsible for which predictions; what happens if we weaken/fine-tune the assumptions, what we could have done differently. We seek insights, a deeper understanding. We could call this approach "reverse relativity" in analogy with "reverse mathematics".

2.1 Motivation for special relativistic kinematics in place of Newtonian kinematics

Why should we replace Newtonian Kinematics with such an exotic or counter-common-sense theory as special relativity? The Newtonian theory proved very successful for 200 years. By now, the Newtonian picture of motion has become the same as the current common-sense picture of motion. Hence the question is why we have to throw away our common-sense understanding of motion.[2]

The answer is that there are several independently good reasons for replacing the Newtonian worldview with relativity. These reasons are really good and decisive ones. They are so compelling, that any one of them would be sufficient for justifying and motivating our replacing the Newtonian worldview with relativity. We will mention a few of these reasons, but for simplicity of presentation, we will base this work on a fixed one of these reasons, namely on the outcome of the Michelson-Morley experiment. We will call this outcome of the Michelson-Morley experiment the Light Axiom. There are deeper, more philosophical reasons for replacing the Newtonian

[2]A second, equally justified question would ask why exactly those postulates/axioms are assumed in relativity which we will assume. We will deal with both questions.

worldview with relativity theory, which might convince readers who are not experimentally minded, i.e. who are not easily convinced by mere facts about how results of certain experiments turned out. These philosophical reasons (under the name "principles of relativity") are intimately intertwined with issues which were significantly present through the last 2500 years of the history of our culture; see p. 63 and [Bar89].

We now turn to the Light Axiom which will play a central role in this work. The first test of the Light Axiom was the Michelson-Morley experiment in 1887 and it has been tested extremely many times and in many radically different ways ever since. As a consequence, the Light Axiom has been generally accepted. An informal, intuitive formulation of the axiom is the following. (Later we will present this axiom in more formal, more precise terms, too, see AxPh in Sec. 2.3.)

Light Axiom: The speed of light is finite and direction independent, in the worldview of any inertial observer.

In other words, the Light Axiom means the following. Imagine a (huge) spaceship drifting through outer space in inertial motion. (Inertial here means that the rockets of the spaceship are switched off, and that the spaceship is not spinning.) Assume a scientist in this inertial spaceship is making experiments. The claim is that if the scientist measures the speed of light, he will find that this speed is the same in all directions and that it is finite. It is essential here that this is claimed to hold for all possible inertial spaceships irrespective of their velocities relative to the Earth or the Sun or the center of our galaxy or whatever reference system would be chosen. The point is that no matter which inertial spaceship we choose, the speed of light in that spaceship is independent of the direction in which it was measured, i.e. it is "isotropic".

In the technical language what we called "inertial spaceship" above is called an inertial reference frame, and the scientist in the spaceship making the experiments is called an "observer". Later "observer" and reference frame tend to be identified.[3]

Let us notice that the Light Axiom is surprising, it is in sharp contrast with common-sense. Namely, common-sense says that if we send out a light signal from Earth, and a spaceship is racing with this light signal moving with almost the speed of the signal in the same direction as the signal does, then the velocity of the signal relative to the spaceship should be very small.

[3]However, it is good to keep in mind that some thought-experiments are carried out by a team of observers (and if the members of this team do not move relative to each other then they are called, for simplicity, a single observer).

Hence, one would think that the astronaut in the spaceship will observe the motion of the light signal as very slow. With the same kind of reasoning, the astronaut should observe light signals moving in the opposite direction very fast. But the Light Axiom states that light moves with the same speed in all directions for the astronaut in the spaceship, too. Hence, the Light Axiom flies in the face of common-sense. This gives us a hint/promise that very interesting, surprising things might be in the making. See also Fig. 18 on p. 40.

In fact, if we add the Light Axiom to Newtonian Kinematics, then we obtain a logical contradiction. I.e. (Newtonian Kinematics + Light Axiom) is an inconsistent theory in the usual sense of logic as we will outline soon (cf. Prop. 2.1). Seeing this contradiction, Einstein did the natural thing. He weakened Newtonian Kinematics (NK for short) to a weaker theory NK^- such that NK^- became consistent with the Light Axiom. Then the theory (NK^- + Light Axiom) became known as Special Relativistic Kinematics (SRK for short). We will study this theory under the name $\mathbf{Specrel}_0$ to be introduced in a logical language soon. We represent the above outlined process by the following diagram:

(NK + Light Axiom) **leads to Contradiction** (!)

$\Downarrow$

NK **gets replaced with the weaker** NK^-

$\Downarrow$

(NK^- + Light Axiom) **receives the name** Special Relativity (SRK).

SRK **is consistent** (this will be proved in Cor. 2.2, p.43).

To see the above process more clearly, let us invoke a possible axiomatization of NK, still on the intuitive level.

Preparation for NK: If we want to represent motion of "particles" or "bodies" or "mass-points", it is natural to use a 4-dimensional Cartesian coordinate system $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers), with one time dimension t and three space dimensions x,y,z. A three-dimensional part of this is depicted in Fig. 1. The time-axis t is drawn vertically. Representing the motion of a body, say b, in a 4-coordinate system can be done by specifying a function f which to each time instance $t \in \mathbb{R}$ tells us the space coordinates x, y, z where the body b is found at time t. Hence a function f : Time $\rightarrow$ Space specifies motion of a particle in this sense. The function f representing motion of b is called the worldline or lifeline of b. Fig. 1 represents motion of bodies, in this spirit. Besides the coordinate axes, we have represented the worldlines of inertial bodies b_1, b_2 and b_3 in Fig. 1. The

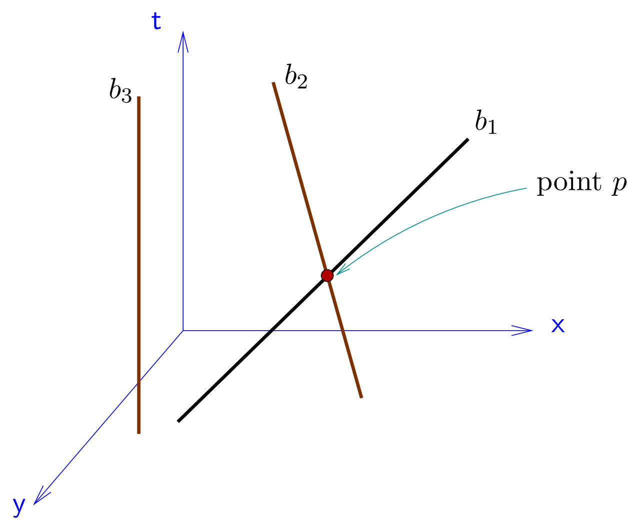


Figure 1: A space-time diagram. Wordlines of bodies b_1, b_2, b_3 represent motion. (Coordinate z is not indicated in the figure.) b_3 is motionless and b_1 moves faster than b_2.

straight line labeled by b_1 is the worldline of b_1. The slope of the worldline of b_1 is greater than that of b_2 which means that b_1 moves faster than b_2 does. The worldline of the third body b_3 is parallel with the time-axis, this means that b_3 is motionless. Bodies b_1 and b_2 meet at space-time point $p = \langle t, x, y, z \rangle$. Such a meeting (of two or more bodies) is called an event. We will extensively refer to such 4-dimensional coordinate systems and such worldlines of bodies and events.

The axioms of NK are summarized as (i)-(v) below.

(i) Each observer "lives" in a 4-coordinate system as described above. The observer in his own coordinate system is motionless in the origin, i.e. his worldline is the time-axis.[4]

(ii) Inertial motion is straight: Let o be an arbitrary inertial observer and let b be an inertial body. Then in o's 4-coordinate system the worldline of b is a straight line. I.e. in an inertial observer's worldview or 4-coordinate system all worldlines of inertial bodies appear as straight lines.

As we said, an observer in his metaphorical "spaceship" is inertial if his rockets are turned off and the spaceship is not spinning. In special relativity,

[4]It is sufficient to assume that his worldline is parallel with the time-axis.

we discuss only inertial motion, hence in our axiomatization the adjective "inertial" could be omitted. (Of course, then we need a general claim that only inertial things/objects will be studied.)

(iii) Motion is permitted: In the worldview or 4-coordinate system of any inertial observer it is possible to move through any point p in any direction with any finite speed.

(iv) Any two observers "observe" the same events. I.e. if according to o_1 bodies b_1 and b_2 have met, then the same is true in the 4-coordinate system of any o_2. We postulate the same for triple meetings e.g. of b_1, b_2, b_3.

(v) Absolute time: Any two observers agree about the amount of time elapsed between two events. (Hence temporal relationships are absolute.)

So, now, NK is defined as the theory axiomatized by (i)-(v) above.

It is easy to see that (NK + Light Axiom) is inconsistent. Einstein's idea was to check which ones of (i)-(v) are responsible for contradicting the Light Axiom and to throw away or weaken the "guilty" axioms of NK. We will see that (v) is guilty and that part of (iii) is suspicious. Hence we throw away (v) and weaken (iii) to a safer form (iii$^-$) where (iii$^-$) is the following.

(iii$^-$) Slower-than-light motion is possible: in the worldview of any inertial observer, through any point in any direction it is possible to move with any speed slower than that of light (here, light-speed is understood as measured at that place and in that direction where we want to move).

In the formal part we will carefully study whether all of these modifications are really needed and to what extent (cf. Thm.s 2.3, 2.5). We define NK$^-$ as the remaining theory:

$$\mathrm{NK}^- := \{(\mathrm{i}), (\mathrm{ii}), (\mathrm{iii}^-), (\mathrm{iv})\}$$

and Special Relativistic Kinematic is defined as

$$\mathrm{SRK} := (\mathrm{NK}^- + \textsf{Light Axiom}).$$

The formalized version of this SRK will appear later as the theory $\mathbf{Specrel}_0$. We will prove that $\mathbf{Specrel}_0$ is consistent (i.e. contradiction-free) and will study its properties. Therefore SRK is also consistent, since, as we said, $\mathbf{Specrel}_0$ is a formalized version of SRK. Actually, the whole process of arriving from NK and the Light Axiom (or some alternative for the latter) to SRK will be subjected to logic-based conceptual analysis in Sec. 2.5.

Before turning to formalizing (and studying) Special Relativity SRK in logic, let us prove (informally only) on the present level of precision why absolute time (i.e. axiom (v)) is excluded by the Light Axiom, or more precisely, it is excluded if we want to keep a fragment of our intuitive picture of the world, i.e. if we want to keep (i), (ii), (iv) of NK. We will prove:

$$(\mathrm{NK}^- + \textsf{Light Axiom}) \ \vdash \ \text{Negation of (v)},$$

where we use turnstile " $\vdash$ " as the symbol of logical provability or derivability. I.e. $A \vdash B$ means that from statement A one can prove, rigorously, statement B.

Actually, we will prove something stronger and stranger from the Light Axiom (and a fragment of NK^-). We will prove that the time elapsed between two events may be different for different observers even in the special case when this elapsed time is zero for one of the observers. I.e. the very question whether two events happened at the same time or not will depend on the observer: two events A and B may happen at the same time for me, while event A happened much later than event B for the Martian in his spaceship. We will refer to this phenomenon by saying that " simultaneity is not absolute ". Moreover, we will see later (Cor. 2.1) that the temporal order of some events may be switched: event A may precede event B for me, while for the Martian in his spaceship, event B precedes event A.

We say that events e and e' are simultaneous for observer O if in O's coordinate system the two events e, e' happen at the same time.

Proposition 2.1. (Simultaneity is not absolute) *Assume* SRK. *Moving clocks get out of synchronism, i.e.: Assume that a spaceship S is in uniform motion relative to another one, say E, and assume that two events e, e' happen simultaneously at the rear and at the nose of the spaceship S according to the spaceship S. Then e and e' take place at different times in E's coordinate system.*

I.e., the time elapsed between e and e' is zero as "seen" from the spaceship S, but the time elapsed between e and e' is nonzero as "seen" from E. See Fig. 3.

Intuitive proof. Assume that we are in spaceship E, and let us call E "Earth". Assume that spaceship S – let us call it "Spaceship" – moves away from us in a uniform motion with, say, 0.9 light-speed. The captain of Spaceship positions his brothers called Rear, Middle, and Nose at the rear, middle and nose of the spaceship, respectively, and asks Rear and Nose to switch on their flashlights towards Middle exactly at the same time. Then the light signals (photons[5]) Ph1 and Ph2 from the two flashlights arrive to Middle at the same time, because Middle is exactly in the middle of the spaceship, and because the speed of Ph1 sent by Rear is the same as the speed of Ph2 sent by Nose (by the Light Axiom). See Fig. 2.

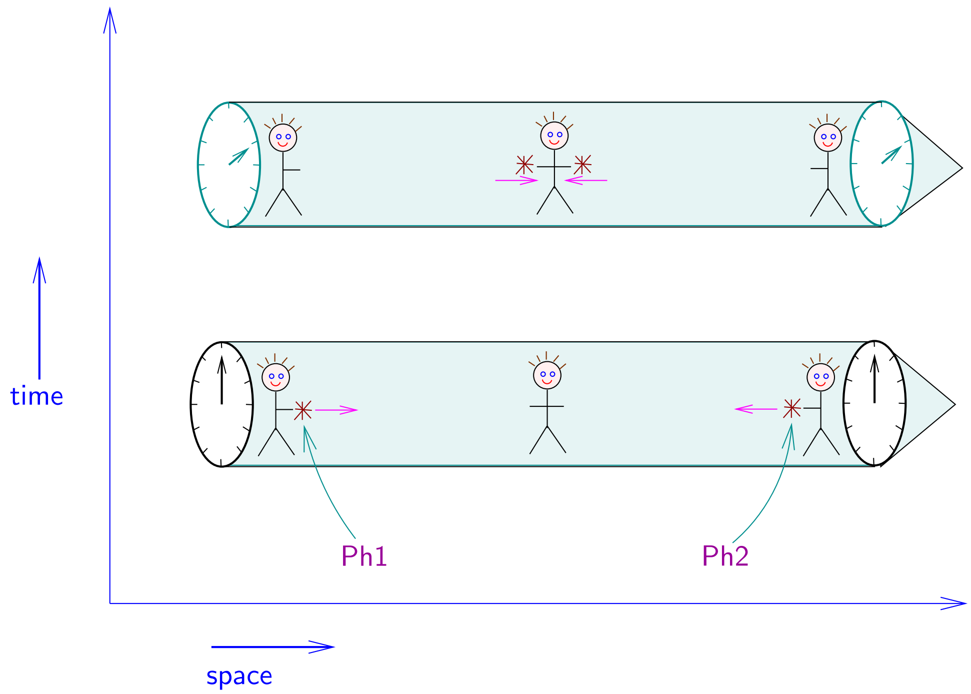


Figure 2: Seen from Spaceship, the two light-signals (i.e. photons) Ph1 and Ph2 are sent out at the same time, and meet in the middle. This is indicated by the clocks at the rear and at the nose of the spaceship. Notice that time in this figure is running upwards! I.e., this figure is similar to drawings in cartoons in that a sequence of scenes is represented in it. However, here the temporal order of the scenes is switched: the scene at the bottom took place earliest. The reason for this convention is our seeking compatibility with the usual space-time diagrams like Fig. 1.

How do we see all this from the Earth? We see that Rear and Nose send

[5] We use the word "photon" as a synonym for light signal. It tacitly refers to the corpuscular conception of light. In this work we do not need the quantum-mechanical definition of photons. (That will be needed only in the final, as yet nonexistent, generalization of general relativity called quantum gravity.)

light signals (or photons) Ph1 and Ph2 towards Middle, and we also see that Ph1, Ph2 arrive to Middle at the same time (because this is a 3-meeting of bodies/entities and axiom (iv) in NK^-). (Spaceship's hull is missing, we can imagine it having only a grid of metal rods for keeping it together or something to this effect.) However, by the Light Axiom, the speeds of Ph1 and Ph2 are the same for us on the Earth, too. Since Spaceship moves away from us (with 0.9 light-speed), we see Ph1 crawl very slowly along the hull of Spaceship because the ship is "running away" from us (and from Ph1, too). On the other hand, the other photon, Ph2, flashes along the hull of the spaceship towards us with enormous relative speed (relative to the hull of the spaceship). Because of this difference of their speeds relative to Spaceship, according to Earth, Ph1 and Ph2 either meet close to the rear of the spaceship, or if they meet in the middle, then Nose had to switch on his flashlight much later than Rear did. See Fig. 3.

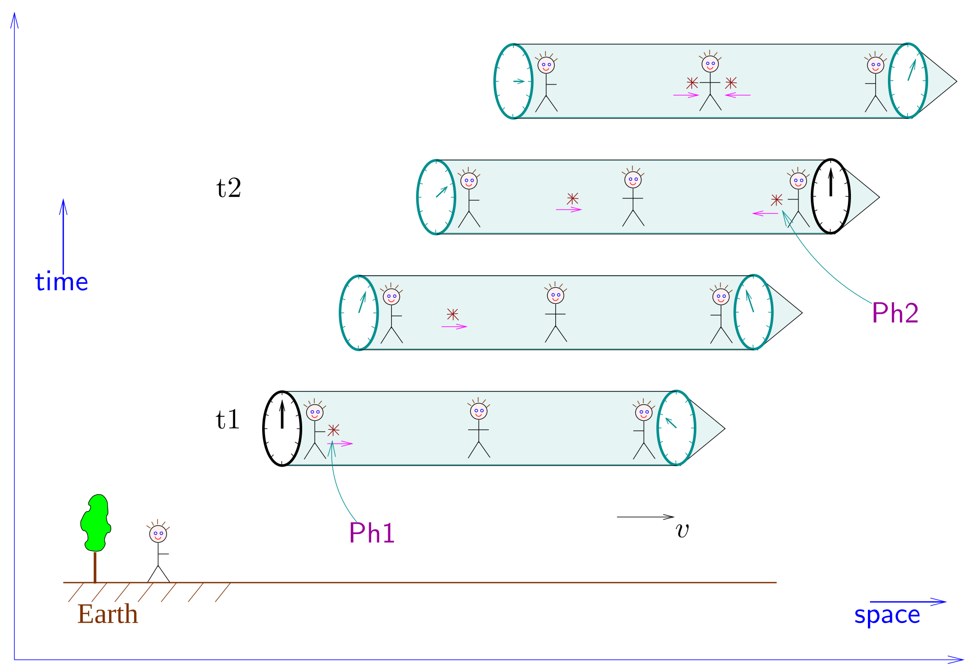


Figure 3: Seen from the Earth, the photon Ph2 had to be sent out later in order that it arrive in the middle at the same time as Ph1 does. But seen from Spaceship, they were sent out at the same time. Hence the clocks at the nose and the rear are out of synchronism, as seen from the Earth.

So far we proved that at least one of two things cannot be absolute. These are (a) being in the middle of the spaceship, and (b) simultaneity. Here, (a) means that Spaceship observes Middle in the middle of the ship, while Earth observes that Middle is not in the middle of the ship; and (b) means that emissions of photons Ph1 and Ph2 are simultaneous for Spaceship but not for Earth.

The first possibility is that Middle stands closer to the rear of Spaceship as seen from the Earth, i.e. that he is not in the middle of the ship according to Earth observers, while he is in the middle according to the ship observers. Here is a thought-experiment which shows that this is not possible. Let us ask the captain to give mirrors to Rear and Nose, and order Middle to send photons Ph3, Ph4 at the same time to these two mirrors. Since Middle is exactly in the middle of the ship, the bounced-back photons arrive to him at the same time, by the Light Axiom. By (iv) in NK^-, we on the Earth also see that the two photons Ph3, Ph4 meet again at Middle after bouncing back, so they traveled their round-trips in the same amount of time. We will show that, as seen from the Earth, the time needed for the round-trip is proportional to the covered distance: if, say, Nose is twice as far from Middle as Rear is, then the time needed for Ph4 for the round-trip Middle-Nose-Middle is twice as much as the time needed for Ph3 for the round-trip Middle-Rear-Middle, even in a fast-moving spaceship. From the Earth we see that the round-trip took the same time for Ph3 and for Ph4, therefore we have to infer that Middle is really in the middle of the ship. See Fig. 4.

We now prove that the time needed for the round-trip is proportional to the covered distance. Indeed, assume that the distance Middle-Nose is twice as much as the distance Middle-Rear. We will show that the round-trip Middle-Nose-Middle takes twice as much time for a photon Ph4 as the round-trip Middle-Rear-Middle for a photon Ph3. Let us watch from the Earth how the two photons Ph3 and Ph4 move relative to the spaceship (as in Fig. 4, but now Middle standing closer to Rear). We will see that Ph3 covers the segment Middle-Rear fast, traveling towards us, and then covers the segment Rear-Middle slowly, moving away from us. The same way, Ph4 covers the segment Middle-Nose slowly, moving away from us, while Ph4 covers the segment Nose-Middle fast, moving towards us. The relative speed of Ph4 in the "towards-us" segment Nose-Middle is the same as the relative speed of Ph3 in the "towards-us" segment Middle-Rear; hence this part of the trip takes twice as much time for Ph4 as for Ph3 because we assumed that the distance Nose-Middle is twice as much as the distance Middle-Rear. The situation is completely analogous for the "away-from-us" segments, so the trip Middle-Nose takes twice as much time for Ph4 as the trip Rear-Middle for Ph3. Summarizing the segments, the round-trip takes twice as much time for Ph4 as for Ph3.

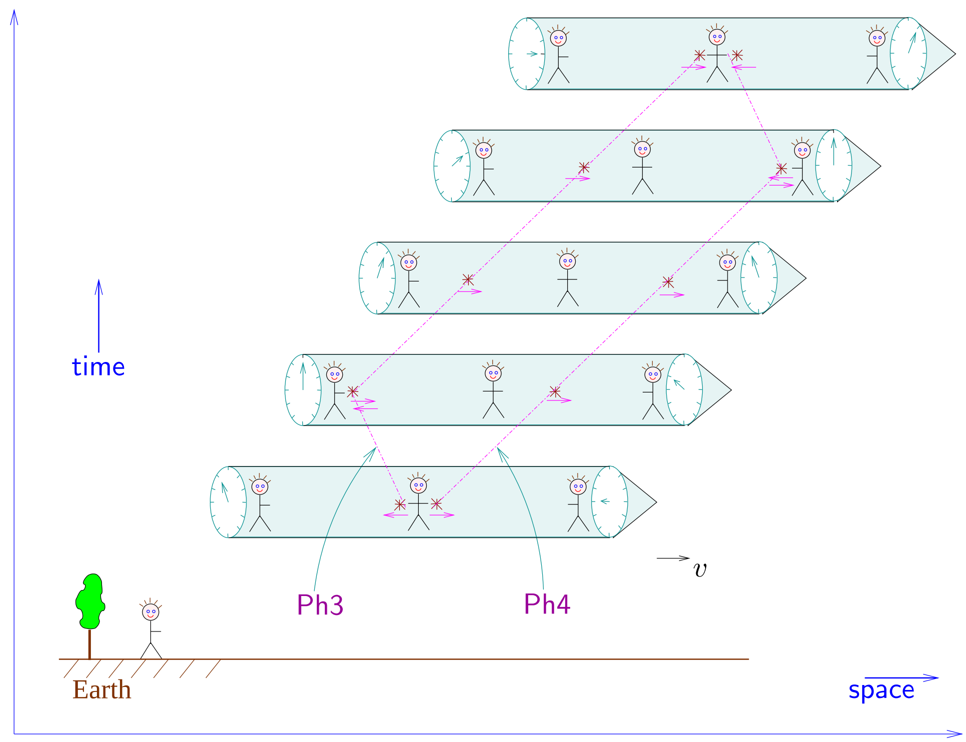


Figure 4: The round-trip for Ph3 takes the same time as for Ph4, seen both from Spaceship and from the Earth. Hence Earth infers that Middle is indeed in the middle of the ship.

As we said earlier, we observe from the Earth that Ph3, Ph4 and Middle meet in a single event. Therefore, since we observe that Ph3 arrives to Middle exactly when Ph4 arrives to Middle after their round-trips, we have to infer, on the Earth, that Middle really stands exactly in the middle of Spaceship. There remains only the possibility that Nose sent out his photon Ph2, which we see as fast-moving along the hull of the space ship, much later than Rear sent Ph1 which we see as slowly-moving along the hull of the spaceship. Thus, as seen from the Earth, the clocks at the nose and at the rear of the spaceship show different times (at the same Earth-moment). This is what we mean when we say that the clocks of the spaceship get out of synchronism.

Summing up: Let e and e' be the events when Rear sends his photon Ph1 towards Middle, and when Nose sends his photon Ph2 towards Middle, respectively. Then these two events took place at the same time as seen from Spaceship, while as seen from the Earth, e' took place later than e did. This finishes the proof of Prop. 2.1. $\square$

Let us notice that Prop. 2.1 above is a far reaching claim. It implies that one of the most basic words of natural language refers to an illusion only and carries no real meaning. The word in question is the word "now".

In order to be able to carry out the proof of Prop. 2.1 and other similar chains of thought in the "safe", precise setting of mathematical logic, in the next subsection we introduce a first-order logic language in which we formalize our axioms, statements, and proofs.

2.2 Language

Motivation for language. We want to talk about space-time as relativity theory conceives it. We will talk about space-time as experienced through motion. Though we discuss here kinematics (theory of motion) only, one can derive (logically) dynamic predictions of relativity, too (i.e. phenomena involving forces, energy) using the same approach/axioms.[6] We will use first-order logic, and the most important decision is to choose the language (vocabulary), i.e. what objects and what relations between them will belong to the language. We will specify our first-order language by specifying its models.

We want to axiomatize motion. What moves? Bodies. Hence our model has a universe B for bodies. What does it mean to move? To move means changing location in time. We will have coordinate systems, or reference frames in other words, for marking locations and time, and we will use quantities in setting up these coordinate systems. So our model has another universe Q for quantities. We will think of quantities as real numbers, so Q together with the operations $+, *, <$ will form a linearly ordered field. We will think of coordinate systems as belonging to special bodies called observers. Hence Ob is a one-place relation on B (picking out a subset of B). We will have another kind of special bodies, photons, too. Hence Ph is another one-place relation on B. The heart of our model is the so-called worldview-relation W. This is a 6-place relation connecting bodies

[6] For the spirit of this we refer to the relativity textbook [Rin01, Sec. 6 "Relativistic particle mechanics"]. [Rin01, Sec. 6.2] is particularly relevant here.

and quantities. We think of $W(o,b,t,x,y,z)$ as the statement that the body b is in location xyz at time t in observer o's coordinate system. We will simply pronounce this as

o sees the body b at $txyz$

though this has no connection with optical seeing, instead, it is an act of coordinatizing only. With this intuition in mind we now fix the language of our theories of special relativity.

The language. We fix a natural number $n > 1$, it will be the number of space-time dimensions. In most works $n = 4$, i.e. one has 3 space-dimensions and one time-dimension. Recent generalizations of general relativity in the literature indicate that it might be useful to leave n as a variable (e.g. string-theory uses 11 dimensions).

We declare two sorts of objects. One sort is for "quantities", it will be denoted by Q. (This is the same as "real numbers" in other treatments.) We have two-place (i.e. binary) operation symbols $+, *$ and a 2-place predicate symbol $<$ of sort "quantities". To avoid misunderstandings, we emphasize that, in this work, Q is not the set of rational numbers. (The letter Q abbreviates "quantities". It is a coincidence that the same letter is used in the literature to denote the rationals. We do not follow that convention.)

The other sort, B, is for entities which do the "moving". We will call these "*bodies*". (We call the moving entities "bodies" whatever they may be, in reality they can be e.g. coordinate systems or electromagnetic waves, or centers of mass.) We have two kinds of special bodies, observers and photons. Thus Ob and Ph are one-place predicate symbols of sort B.

We have a relation which connects these two sorts, the $(n+2)$-place relation W which is of sort $\mathsf{B}\times\mathsf{B}\times\mathsf{Q}\times\cdots\times\mathsf{Q}$. The sentence "observer o observes body b at space-time location $p_1, ..., p_n$" is denoted as $\mathsf{W}(o, b, p_1, ..., p_n)$, or as $\mathsf{W}(o, b, p)$ in short.

Summing up, a model $\mathfrak{M}$ of our language is of form

$\langle Q, +, *, <; \quad B, Ob, Ph; \quad W\rangle$

where $\langle Q, +, *, <\rangle$ is a structure similar to ordered fields, Ob, Ph are subsets of B, and $W \subseteq B \times B \times Q \times \cdots \times Q$.

2.3 Axiomatization Specrel of special relativity in first-order logic

In this subsection we formalize the axioms we talked about on an intuitive level in Sec. 2.1, by using the first-order logic language introduced in the previous subsection. The formalized version of (NK^- + Light Axiom) will be called $\mathbf{Specrel}_0$.

Axiom 1 (AxField) . The quantities behave like real numbers do in the sense that $\langle Q, +, *, < \rangle$ is a linearly ordered field in which every positive member has a square root. Such fields are called "quadratic".

For an axiom system for linearly ordered fields we refer to e.g. [CK73, p. 41, below item 18]. We will often simply say "field" or "ofield" or "ordered field" instead of "linearly ordered field". We recall that if $\langle Q, +, *, < \rangle$ is a field, then 0 and 1 denote the neutral elements of $+$ and $*$ respectively, and an element $x \in Q$ is called positive iff $x > 0$. Further, y is called a square root of x iff $y * y = x$ and $y \geq 0$, we denote this by writing $y = \sqrt{x}$. With this notation, the absolute value $|y|$ of y is $|y| := \sqrt{y^2}$ (as usual, y^2 denotes $y * y$). The inverses of the operations $+$ and $*$ will be denoted by $-$ and $/$, respectively. Thus a linearly ordered field is quadratic (or Euclidean) iff $(\forall x > 0)(\exists y) x = y * y$ is true in it. According to the usual practice, we will often omit $*$ from an expression, e.g. we write td in place of $t * d$.

On AxField: In most physics books, the set of quantities is taken to be the set of real numbers (with $+, *$ as addition, multiplication of the real numbers). Some, fancy, books use also imaginary numbers for quantities. We know from mathematics that much complexity is tied to the real numbers. Hence in our axiomatic approach, we single out those properties of the quantities that we rely on in the investigation in question. In special relativity only the quadratic ordered field-structure of the quantities is presupposed, but we could do much even with assuming only the ring-structure. In particular, the ordering and the existence of square roots are used mostly in order to be able to formulate results in a simpler way. (E.g. we use square roots in expressing the distance between two coordinate points. Without the use of square roots we always would have to talk about the square of distance. This would cause only an inconvenience but not an impossibility.) As a pay-off of this explicit way of handling the quantities, we can build models of special relativity with a finite field as structure of quantities, or we can use fields with infinitesimally small numbers. In general relativity, in addition to AxField it will suffice to use an axiom-schema called CONT, see Sec. 3.6.

By a coordinate point , or space-time location , we understand an n-tuple (i.e. a sequence of length n) $p = \langle p_1, \ldots, p_n \rangle$ of elements of Q, the set of all these n-tuples is denoted by Q^n. If $p \in \mathsf{Q}^n$, then $p_1, \ldots, p_n$ are its components, i.e. $p = \langle p_1, \ldots, p_n \rangle$. We call $\overline{0} := \langle 0, \ldots, 0 \rangle \in \mathsf{Q}^n$ the (n-dimensional) origin, and we call $\overline{t} := \{\langle x, 0, \ldots, 0 \rangle \in \mathsf{Q}^n : x \in \mathsf{Q}\}$ the (n-dimensional) time-axis. By the worldline (or lifeline, or history) of a body b as observed by the observer m we mean the set of space-time locations where m observes b to be present,

$$\mathsf{wline}_m(b) := \{p \in \mathsf{Q}^n : \mathsf{W}(m, b, p)\}.$$

Axiom 2 **(AxSelf)** An observer m in his own coordinate system is motionless in the origin (of space), i.e. his worldline is the time-axis: $\mathsf{wline}_m(m) = \overline{t}$. As a formula of the FOL language this axiom is

$$(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{Q}^n)[\,\mathsf{W}(m, m, p) \leftrightarrow p_2 = \cdots = p_n = 0\,].$$

Having a field in our language makes it possible to talk about straight lines. We recall that the straight line going through $p, q \in \mathsf{Q}^n, p \neq q$ is the set $\{p + x * (p - q) : x \in Q\}$. In the latter formula, $+, -$ and $*$ denote operations of Q^n as a vector-space. We will often say just "line" for "straight line".

Axiom 3 **(AxLine)** The motion of an observer as observed by any observer is uniform, i.e. such that both the "spatial direction" and the "pace" of the motion are constant (and "longest possible" with this property). In geometrical terms this means that in each observer's coordinate system, the worldline of an observer is a straight line, i.e. $\mathsf{wline}_m(k)$ is a straight line for all $m, k \in \mathsf{Ob}$. Formally,

$$\begin{aligned}&(\forall m, k \in \mathsf{Ob})(\exists p, q \in \mathsf{Q}^n)(\mathsf{W}(m, k, p) \wedge \mathsf{W}(m, k, q) \wedge p \neq q \wedge \\ &\quad (\forall r \in \mathsf{Q}^n)[\mathsf{W}(m, k, r) \quad \leftrightarrow \quad (\exists x \in \mathsf{Q}) r = p + x * (p - q)]).\end{aligned}$$

On AxLine: AxLine is a formalized version of postulate (ii) in Sec. 2.1. Later we will consider non-uniform motions, too. We will call those motions "accelerated" ones. Newton's First Law of Motion states that "an object moves with constant, uniform motion until acted on by a force". A body is called "inertial" if no force acts on it. Hence AxLine indicates that Ob denotes the set of inertial observers when using AxLine.

We introduce the speed of a uniform motion. In geometric terms, this is the "slant" or "slope" of the straight line (representing the motion). For $p, q \in \mathsf{Q}^n$, let $\mathsf{space}(p, q)$ and $\mathsf{time}(p, q)$ denote the spatial distance and the time-distance between p and q, respectively:

$$\mathsf{space}(p, q) := \sqrt{(p_2 - q_2)^2 + \ldots + (p_n - q_n)^2}, \quad \text{and}$$

$\mathsf{time}(p,q) := |p_1 - q_1| = \sqrt{(p_1 - q_1)^2}$.

Now $\mathsf{speed}(p,q)$ denotes the speed necessary to reach q from p (or p from q):

$\mathsf{speed}(p,q) := \mathsf{space}(p,q)/\mathsf{time}(p,q)$ when $\mathsf{time}(p,q) \neq 0$.

Axiom 4 (AxPh) For every observer, the speed of light is 1, and moreover, photons move uniformly along straight lines and in each location in each direction it is possible to send out a photon. In geometrical terms this means that the worldlines of photons are exactly the straight lines of slope 1. Formally this is:

$$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph})[\,(\mathsf{wline}_m(\mathsf{ph}) \text{ is a straight line}) \wedge (\forall p,q \in \mathsf{Q}^n)$$
$$p \neq q \Rightarrow (\mathsf{speed}(p,q){=}1 \text{ iff } (\exists \mathsf{ph} \in \mathsf{Ph})[\mathsf{W}(m,\mathsf{ph},p) \wedge \mathsf{W}(m,\mathsf{ph},q)])].$$

On AxPh: This is the formal version of the Light Axiom used in Sec. 2.1. It expresses that the speed of light is finite (nonzero) and isotropic, i.e. direction-independent. We formulated the Light Axiom in a seemingly stronger form, namely such that we require the speed of light to be 1. This way we are freed from having to deal with always adjusting everything to the actual speed of light. Instead, we adjust the units of measurement to the speed of light: we measure distances with “light-years” if we measure time in “years”. We emphasize that assuming that the speed of light is 1 instead of some finite direction independent number (which might depend on the observer) is not a “physical” assumption but instead a merely “linguistic” one. It would be sufficient (for our results) to use a more literal formalization of the Light Axiom in Sec. 2.1. That such a weaker version of AxPh is sufficient for our results in shown in [AMN02], [Mad02, p. 121].

In Sec. 2.1 we talked about photons bounced back from a mirror. When using AxPh, we will simulate this “bouncing back” by treating the out-going and the bounced-back photons as two different photons that have met at the mirror (see e.g. Fig. 8).

The next axiom states that each observer can make thought-experiments in which he assumes the existence of “slowly moving” observers. This is the formalized version of postulate (iii$^-$) in Sec. 2.1.

Axiom 5 (AxThEx) For each observer $m \in \mathsf{Ob}$, in each space-time location, in each direction, with any speed smaller than that of the light it is possible to "send out" an observer:

$$(\forall m \in \mathsf{Ob})(\forall p, q \in \mathsf{Q}^n)[\mathsf{space}(p,q) < \mathsf{time}(p,q) \rightarrow$$
$$(\exists k \in \mathsf{Ob})(\mathsf{W}(m,k,p) \wedge \mathsf{W}(m,k,q))].$$

In geometric terms this means that each line in the coordinate system with slant smaller than 1 is the worldline of a (potential) observer, in m's worldview.

The next axiom is the formalized version of postulate (iv) in Sec. 2.1.

Axiom 6 (AxEvent) If an observer observes three bodies at the same space-time location, then all other observers observe that these three bodies meet:

$$(\forall m, k \in \mathsf{Ob})(\forall b, b', b'' \in \mathsf{B})(\forall p \in \mathsf{Q}^n)(\exists p' \in \mathsf{Q}^n)$$
$$[\mathsf{W}(m,b,p) \wedge \mathsf{W}(m,b',p) \wedge \mathsf{W}(m,b'',p) \rightarrow$$
$$(\mathsf{W}(k,b,p') \wedge \mathsf{W}(k,b',p') \wedge \mathsf{W}(k,b'',p'))].$$

On AxEvent: In Sec. 2.1 we talked about "events". E.g. "Rear sent light signal Ph1" was called an event, another event was that "Nose sent light signal Ph2", and a third event was that "Middle, Ph1, and Ph2 meet". In the axiom AxEvent above, we talk about "3-meetings". We will reserve the word "event" for the set of all bodies present at a space-time location. Let us call $\mathsf{AxEvent}^+$ the axiom we obtain from AxEvent by replacing "3-meetings" with "events" in it, in this latter sense. We will see at the end of this subsection (cf. Thm. 2.1) that, in our approach, AxEvent is equivalent with this seemingly stronger axiom.

$\mathbf{Specrel}_0 := \{\mathsf{AxField}, \mathsf{AxSelf}, \mathsf{AxLine}, \mathsf{AxThEx}, \mathsf{AxEvent}, \mathsf{AxPh}\}.$

$\mathbf{Specrel}_0$ is the formalized version of SRK = NK^-+Light Axiom introduced in Sec. 2.1. Most of the interesting predictions of special relativity can be proved (in the rigorous manner of first-order logic) from $\mathbf{Specrel}_0$. However, some of the predictions have a little bit more complicated forms because different observers may use different "units of measurement". The last axiom brings the units of measurement of two observers to a common "platform".

For an observer m and space-time location $p \in \mathsf{Q}^n$, $\mathsf{ev}_m(p)$ denotes the "full event" happening in m's coordinate system at p,

$$\mathsf{ev}_m(p) := \{b \in \mathsf{B} : \mathsf{W}(m,b,p)\}.$$

We call the next axiom the *Axiom of Simultaneous Distance*.

Axiom 7(AxSim) Any two observers agree on the spatial distance between two events, if these two events are simultaneous for both of them:

$$(\forall m, k \in \mathsf{Ob})(\forall p, q, p', q' \in \mathsf{Q}^n)[\,\mathsf{ev}_m(p) = \mathsf{ev}_k(p') \wedge \mathsf{ev}_m(q) = \mathsf{ev}_k(q') \wedge \mathsf{time}(p, q) = \mathsf{time}(p', q') = 0 \quad \rightarrow \quad \mathsf{space}(p, q) = \mathsf{space}(p', q')\,].$$

Specrel := $\mathbf{Specrel}_0 \cup \{\mathsf{AxSim}\}$
$= \{\mathsf{AxField}, \mathsf{AxSelf}, \mathsf{AxLine}, \mathsf{AxThEx}, \mathsf{AxEvent}, \mathsf{AxPh}, \mathsf{AxSim}\}$.

In Sec. 2.5 we will prove that **Specrel** is consistent, and hence the weaker $\mathbf{Specrel}_0$ is also consistent. This will show that we have succeeded in eliminating the contradiction from (NK+Light Axiom): there is no statement A such that from the new theory (NK$^-$+Light Axiom) we can derive both A and its negation $\neg A$. In the next subsection we will prove that $\mathbf{Specrel}_0$ implies (in the rigorous manner of first-order logic) the negations of (v) and (iii), i.e. the negations of "absolute time" and "all motion is possible". In the next subsection we will also begin to investigate what the world looks like assuming SRK, in which ways it is different from our common-sense Newtonian world. Before doing this, we show two simple properties of $\mathbf{Specrel}_0$.

An important theme will be to establish which things all the observers perceive ("see") the same way, and which things they perceive differently. The things that they see the same way will be called " absolute ", the things that they see differently will be called " relative ". Whence the name "relativity theory". First we show that all observers see the same "events" to occur, and not only they see the same 3-meetings to occur.
Let $\mathsf{AxEvent}^+$ denote the statement that if an observer observes an event, then all other observers observe this event:

$$(\forall m, k \in \mathsf{Ob})(\forall p \in \mathsf{Q}^n)(\exists p' \in \mathsf{Q}^n)(\forall b \in \mathsf{B})[\mathsf{W}(m, b, p) \;\leftrightarrow\; \mathsf{W}(k, b, p')].$$

The symbol $\models$ denotes the semantic consequence relation of FOL. Before discussing the details, we note that in the case of FOL, $\models$ coincides with FOL-provability $\vdash$. If $\mathfrak{M}$ is a possible model and φ is a FOL formula, then $\mathfrak{M} \models \varphi$ abbreviates the statement "formula φ is valid in model $\mathfrak{M}$". For a set Ax of formulas, $Ax \models \varphi$ means that for every possible model $\mathfrak{M}$, if $\mathfrak{M} \models Ax$, then $\mathfrak{M} \models \varphi$.

Theorem 2.1. $\{\mathsf{AxEvent}, \mathsf{AxPh}, \mathsf{AxField}\} \models \mathsf{AxEvent}^+$.

Proof. Assume $\mathfrak{M} = \langle Q, \dots, W\rangle \models \{\mathsf{AxEvent}, \mathsf{AxPh}, \mathsf{AxField}\}$ and let $m, k \in Ob$, $p \in Q^n$. There are (at least) two distinct lines ℓ_1, ℓ_2 of slope 1 going through p, e.g. $\ell_1 = \{p + x * \langle 1, -1, 0, \ldots, 0\rangle : x \in Q\}$ and $\ell_2 = \{p + x * \langle 1, 1, 0, \ldots, 0\rangle : x \in Q\}$ are such. (We used AxField here.) There are photons $\mathsf{ph}_1, \mathsf{ph}_2$ "living on ℓ_1, ℓ_2" respectively, by AxPh. (I.e. $\ell_i = \mathsf{wline}_m(\mathsf{ph}_i)$ for $i = 1, 2$.) Let us consider the worldlines of these photons in k's worldview. By AxPh, these are straight lines of slope 1. We are going to show that they intersect in a unique point.

We have that $\mathsf{wline}_m(\mathsf{ph}_1) \neq \mathsf{wline}_m(\mathsf{ph}_2)$. Let $q \in \ell_1,\ q \notin \ell_2$ and let ℓ_3 be the straight line of slope 1 going through q and parallel with ℓ_2. (I.e. $\ell_3 = \{q + x * \langle 1, 1, 0, \ldots, 0\rangle : x \in Q\}$.) Let $\mathsf{ph}_3 \in \mathsf{Ph}$ be such that $\mathsf{wline}_m(\mathsf{ph}_3) = \ell_3$. Now, m "sees" 3-meetings of $\{\mathsf{ph}_1, \mathsf{ph}_2, \mathsf{ph}_2\}$, $\{\mathsf{ph}_1, \mathsf{ph}_3, \mathsf{ph}_3\}$ but m does not see a 3-meeting of $\{\mathsf{ph}_2, \mathsf{ph}_3, \mathsf{ph}_3\}$. By AxEvent, the same must hold for k. Thus $\mathsf{wline}_k(\mathsf{ph}_1)$ and $\mathsf{wline}_k(\mathsf{ph}_2)$ must meet but must not coincide and hence they intersect in a unique point.

Let p' be their intersection point, i.e. $\{p'\} = \mathsf{wline}_k(\mathsf{ph}_1) \cap \mathsf{wline}_k(\mathsf{ph}_2)$. Now, it is easy to show by using AxEvent again that $\mathsf{ev}_m(p) = \mathsf{ev}_k(p')$. (Indeed, let $b \in B$ be arbitrary. Then m sees a 3-meeting of $\mathsf{ph}_1, \mathsf{ph}_2, b$ iff $b \in \mathsf{ev}_m(p)$, and the same for m, p replaced with k, p'.) $\square$

Theorem 2.2. *No observer observes the same event at two different space-time locations in models of* $\{\mathsf{AxPh}, \mathsf{AxField}\}$.

Proof. Let $p, q \in \mathsf{Q}^n$, $p \neq q$. There is a straight line ℓ of slope 1 through p which avoids q (because through each point p there are at least 2 distinct lines of slope 1). By AxPh, ℓ is the worldline of a photon $\mathsf{ph} \in \mathsf{Ph}$ (in m's worldview). Then $\mathsf{ph} \in \mathsf{ev}_m(p)$ while $\mathsf{ph} \notin \mathsf{ev}_m(q)$, showing that $\mathsf{ev}_m(p) \neq \mathsf{ev}_m(q)$. $\square$

Thm.s 2.1 and 2.2 imply that in each observer's worldview, the space-time locations and the events observed by any observer are in one-one correspondence. Thus, in a given observer's worldview, we can speak of events as if they were space-time locations. E.g. we can quantify over events, meaning that we have in fact quantified over space-time locations. By the same token, we can apply any function defined on space-time locations to events. Specifically, let $\mathsf{loc}_m(e)$ denote the location of event e in m's worldview, then

$$\mathsf{loc}_m(e) = p \quad \text{iff} \quad \mathsf{ev}_m(p) = e.$$

By the time-distance between two events as seen by an observer we will mean the time-distance between the space-time locations where the observer sees the two events, and similarly for spatial distance. Formally, with $p = \mathsf{loc}_m(e)$, $p' = \mathsf{loc}_m(e')$ we have

$$\mathsf{time}_m(e,e') := \mathsf{time}(p,p'), \qquad \mathsf{space}_m(e,e') := \mathsf{space}(p,p'),$$

$\mathsf{time}_m(e) := p_1$ denotes the time where m sees event e happen, and $\mathsf{space}_m(e) := \langle 0, p_2, \ldots, p_n \rangle$ denotes the space-location where m sees event e happen.

2.4 Characteristic differences between Newtonian and special relativistic kinematics

The most frequently quoted predictions of special relativity are the following three paradigmatic effects. (1) moving clocks slow down, (2) moving meter-rods shrink, and (3) moving pairs of clocks get out of synchronism. These three effects are easily formulated in the first-order logic language introduced so far.

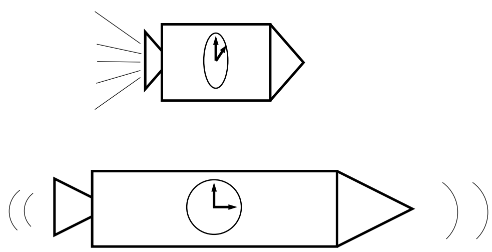


Figure 5: Moving clocks slow down and moving spaceships shrink.

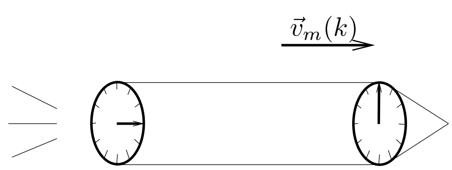


Figure 6: Moving clocks get out of synchronism.

Let m, k be observers in a model of our language. By the direction of spatial separation of two events e, e' in m's worldview we mean the natural thing, i.e. we mean the straight line connecting the "spatial projections"

$\langle 0, p_2, \ldots, p_n \rangle$ and $\langle 0, q_2, \ldots, q_n \rangle$ if p and q are the space-time locations m sees e and e' at, respectively (or the point $\langle 0, p_2, \ldots, p_n \rangle$ if these two points are the same). The spatial direction of motion of a body b in m's worldview is the direction of spatial separation of two distinct events in $\mathsf{wline}_m(b)$, whenever the latter is a straight line. (In order to deal with the "degenerate" situations in the next theorem, we say that a point is both parallel and orthogonal to a line or to another point.) We say that e, e' are simultaneous in m's worldview iff $\mathsf{time}_m(e, e') = 0$. Let $v_m(k)$ denote the speed of k as seen by m, i.e. $v_m(k)$ is the slope of the worldline of k in m's worldview. We note that $\mathbf{Specrel}_0 \not\models (\forall m, k \in \mathsf{Ob}) v_m(k) = v_k(m)$ while $\mathbf{Specrel} \models (\forall m, k \in \mathsf{Ob}) v_m(k) = v_k(m)$ (see Cor. 2.3).

Thm. 2.3 below implies that Absolute Time (i.e. (v) of NK) is inconsistent with SRK (i.e. with NK$^-$+$\mathsf{Light\ Axiom}$), hence it was necessary to omit it from NK. Thm. 2.3 says that simultaneity of events is not absolute. Actually, it implies something more surprising, more exotic: the question of what happened earlier and what later is not absolute either (see Cor. 2.1 after the theorem). Fig. 7 illustrates the statements in Thm. 2.3. Thm. 2.3 is a more detailed version of Prop. 2.1.

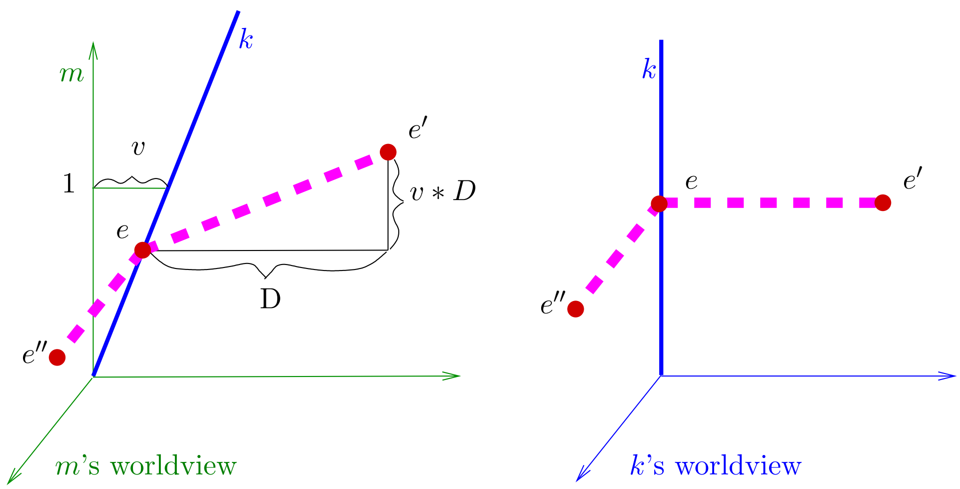


Figure 7: Illustration for Thm. 2.3. Simultaneity of events is not absolute. Events e, e', e'' are simultaneous for k, but e, e' are not simultaneous for m.

Theorem 2.3. (simultaneity of events is not absolute) *Assume* $\mathbf{Specrel}_0$ *and let* m, k *be observers. Statements (i) and (ii) below hold.*

(i) *Assume that in m's worldview the spatial separation of events e, e' is parallel with the direction of motion of k. Then*

e, e' *are simultaneous in k's worldview*

iff

$\mathsf{time}_m(e, e') = v_m(k) * \mathsf{space}_m(e, e')$.

(ii) *Assume that e, e'' are simultaneous both in k's worldview and in m's worldview. Then in m's worldview the spatial separation of e, e'' is orthogonal to the direction of motion of k.*

Proof. The proof of Thm. 2.3 follows the structure and ideas of the intuitive proof of Prop. 2.1.

Proof of (i). Let e, e' be simultaneous events in k's worldview. Let $p = \mathsf{loc}_k(e)$, $p' = \mathsf{loc}_k(e')$ and let $q = (1/2) * (p + p')$, their "middle-point". Let ℓ_1, ℓ_2, ℓ_3 be straight lines parallel with the time-axis and going through p, p', q respectively and let m_1, m_2, m_3 be observers with $\ell_i = \mathsf{wline}_k(m_i)$ $(i = 1, 2, 3)$. See Fig. 8.

Let $\delta := |p - q| = \sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2}$. This exists since the field $\langle Q, +, *, \leq\rangle$ is quadratic by $\mathsf{AxField}$. By using δ now we can construct straight lines of slope 1 connecting ℓ_1, ℓ_2, ℓ_3 as follows. Let $u = \langle 1, 0, \ldots, 0\rangle$ (the "time unit-vector"), $p1 = q + \delta * u$, $p2 = p + 2\delta * u$, $p3 = p' + 2\delta * u$, $p4 = p1 + 2\delta * u$. The straight lines connecting the points $pp1$, $p'p1$, $p2p4$, and $p3p4$ all have slope 1 by construction, hence by AxPh there are photons $\mathsf{ph}_1, \ldots, \mathsf{ph}_4$ whose worldlines these are, respectively. Let $e_i = \mathsf{ev}_k(pi)$ for $i = 1, \ldots, 4$. See Fig. 8.

We will think of the pattern constructed so far as representing the two thought-experiments in Prop. 2.1. We will think of k, m_1, m_2, m_3 as the Spaceship, Rear, Nose, and Middle respectively; e is the event when Rear sent ph_1 towards Middle, e' is the event when Nose sent ph_2 towards Middle, and e_1 is the event when these two reached Middle. The upper part of the arrangement (events $e_1, \ldots, e_4$ and photons $\mathsf{ph}_1, \ldots, \mathsf{ph}_4$) represents the experiment of Middle by which he tested that he indeed was standing in the middle (the two photons $\mathsf{ph}_2, \mathsf{ph}_1$ sent towards Rear and Nose arrived back, after bouncing back at the mirrors, at the same time in event e_4).

Switch now to the worldview of m! See Fig. 9. The worldlines of $m_1, m_2, m_3, \mathsf{ph}_1, \ldots, \mathsf{ph}_4$, respectively, are all straight lines, the last four of slope 1, by AxLine, AxPh. The meeting points of these lines are exactly as those of the corresponding worldlines in k's worldview, by $\mathsf{AxEvent}$. The

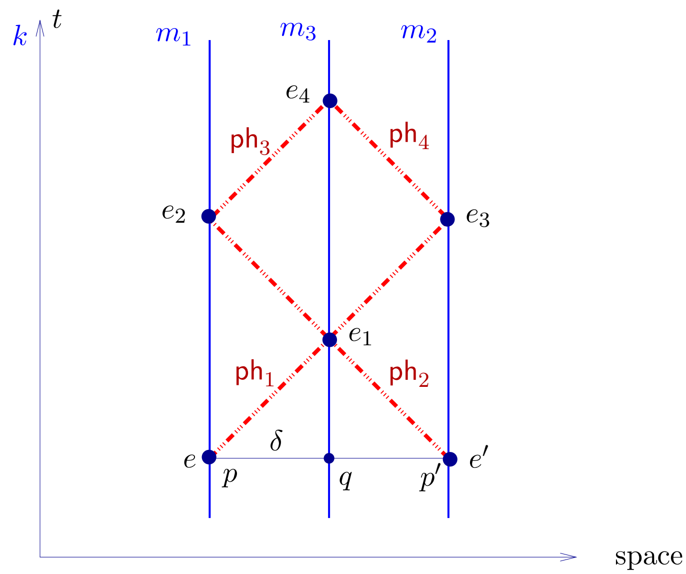


Figure 8: Illustration for the proof of Thm. 2.3(i).

worldlines of m_1, m_2, m_3, k are parallel in m's worldview because they are so in k's worldview. (In more detail, e.g. for m_1, m_2: their worldlines do not meet, by $\mathsf{AxEvent}$. Their worldlines are not skew, because one can construct photons $\mathsf{ph}_1, \mathsf{ph}_2$ with meeting points e, e', e_1, e_2, e_3 as in Fig. 8, and this ensures that they are in one plane (i.e. in the plane determined by e, e', e_1).) Now assume that the spatial separation of e, e' is parallel with the spatial direction of movement of k, in m's worldview. This means that there is a plane P containing the whole configuration (the worldlines of $m_1, \dots, \mathsf{ph}_4$), and it is vertical, i.e. it contains a line parallel with the time-axis. The worldlines of $\mathsf{ph}_1, \mathsf{ph}_2$ are parallel with those of $\mathsf{ph}_3, \mathsf{ph}_4$, respectively, in m's worldview, because they are so in k's worldview (and because they are all in one plane). Thus the distance of events e_2 and e_1 is the same as the distance of events e_4 and e_3 according to m, i.e. with the notation $r_i = \mathsf{loc}_m(e_i)$ we have $|r_2 - r_1| = |r_4 - r_3|$. Similarly, since the lines connecting e_4, e_1 and e_3, e' are also parallel, we get $|r_1 - r'| = |r_3 - r_4|$ (where $r' = \mathsf{loc}_m(e')$). Thus $|r_2 - r_1| = |r_1 - r'|$. For this reason, the distance between e and C in Fig. 9 is the same as the distance between C and E, i.e. C is the middle-point of e and E. Thus m also sees that m_3 is positioned exactly in the middle of m_1 and m_2.

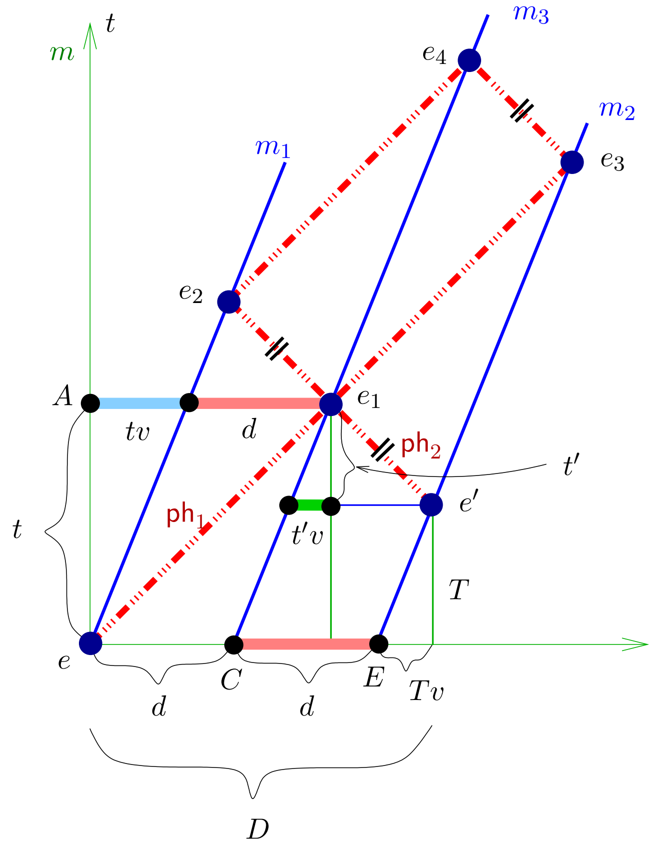


Figure 9: Illustration for the proof of Thm. 2.3(i).

We now recall the intuitive chain of thought in the proof of Prop. 2.1 taking into account the quantitative aspects, too. We think of m as Earth. Let us measure time in "seconds", and let us assume that the length of the ship as Earth sees it is $2d$. Now, Earth sees that inside the spaceship the photon ph_1 covered distance d with velocity $1 - v$, thus it took $t = d/(1-v)$ seconds for ph_1 to reach the middle of the ship. Similarly ph_2 covered distance d with velocity $1+v$, so it took $t' = d/(1+v)$ seconds for it to reach the middle of the ship. Hence Nose had to send ph_2 exactly $T = t - t' = [d/(1-v)] - [d/(1+v)] = 2dv/(1-v^2)$ seconds later in order that they meet in the middle. Since the photon was sent T seconds later, the ship covered $T * v$ distance during this time, thus the spatial distance of event e (which is sending out photon ph_1) and event e' (which is sending out photon ph_2) is $D = 2*d + T*v = ([2d(1-v^2)] - [2dv^2])/(1-v^2) = 2d/(1-v^2)$. Hence $T = v*D$, as was to be shown. This computation can be faithfully reconstructed in the settings of the present Thm. 2.3, see Fig. 9.

This proves the "only if" part in (i). The "if" part in (i) can be proved by taking an event e'' in k's worldview which is simultaneous with e and which takes place at the same place as e', i.e. $\mathsf{space}_k(e'', e') = 0$; now we can use the previously proven part for e, e'' and then use $\mathsf{time}_m(e'', e') \neq 0$.

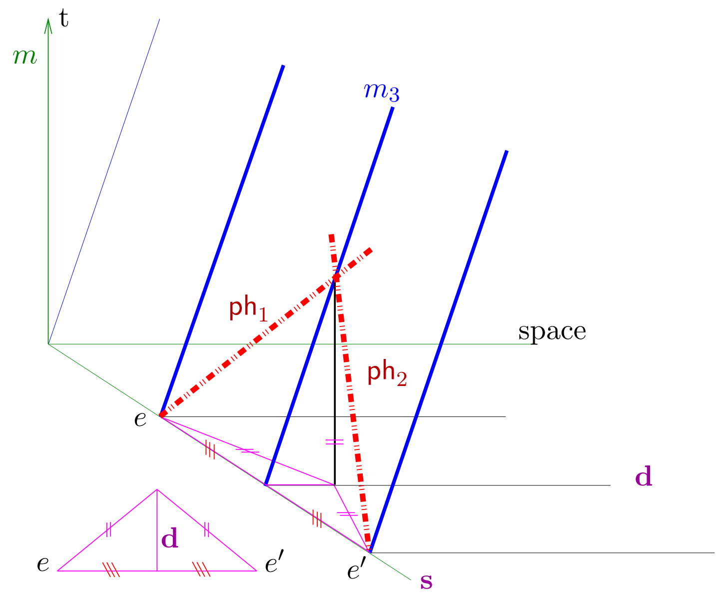


Figure 10: Illustration for the proof of Thm. 2.3(ii). If $\mathsf{ph}_1, \mathsf{ph}_2, m_3$ meet, then the spatial separation s of e, e' must be orthogonal to the spatial direction d of movement. This figure shows how m "sees" the thought-experiment illustrated in Fig. 8, but conducted in a spatial direction not necessarily parallel with the direction d of motion of k.

The **proof of (ii)** is similar to that of (i), we include Fig. 10 for illustration. □

Remark. The first-order logic axiomatization of relativity theories we are describing here is a very good place for applying Tarski's first-order logic axiomatization of Euclidean geometry. We try to illustrate this claim. In the proof of Thm. 2.3 we used several geometrical properties of the Euclidean geometry G built on the field $\langle Q, +, *\rangle$ in place of the reals. E.g.

we used that "for any two distinct points there is a unique (straight) line connecting them", or "through any point there is a unique line ℓ' parallel with ℓ", we used the notions of planes, being parallel etc. The properties of G we used in the proof are easy to check directly by using the axioms of a quadratic ordered field and the (analytic) definitions of a straight line etc. There is another way, though. Instead of directly checking in the geometry G validity of each statement which arises in the proof, we could use the axioms in a first-order logic axiomatization of (synthetic) Euclidean geometry and derive everything from those axioms (or just rely on the existing theorems and definitions of this area of research). We recall that [Hil77] axiomatized Euclidean geometry over the reals by using second-order logic axioms, and [Tar59] gave a first-order logic axiom system for this geometry. This also made possible to replace the field of reals with arbitrary fields and investigate what algebraic properties of the field correspond to what geometrical properties. This subject—which is highly relevant in the approach presented in this paper—is quite rich, see e.g. [Tar59], [Gol87], works of Schwabhäuser, Szmielev, Szczerba and Tarski [SST83], [Szc70], [Szm74], [AvB02]. [Ax78], [Gol87] and [Mun86] make use of Tarski's axiom system for Euclidean geometry in their axiomatizations of Special Relativity Theory. Actually, Fig. 11 shows that by using the methods of axiomatic Euclidean geometry, the proof of Thm. 2.3 could be made simpler and more transparent. □

Corollary 2.1. (The temporal order of events is not absolute) *Assume* $\mathbf{Specrel}_0$. *For all observers m, k not at rest relative to each other there are events e, e' such that e happens earlier than e' according to m while e happens later than e' according to k. Formally:*

$$\mathbf{Specrel}_0 \models (\forall m, k \in \mathsf{Ob})[\, v_m(k) \neq 0 \rightarrow (\exists e, e')[\, \mathsf{time}_m(e) < \mathsf{time}_m(e') \wedge \mathsf{time}_k(e) > \mathsf{time}_k(e')\,]\,].$$

Proof. Assume that $v_m(k) \neq 0$. Let e, e' be distinct events in the life of k (i.e. $k \in e \cap e',\ e \neq e'$) and assume $\mathsf{time}_m(e) < \mathsf{time}_m(e')$. If $\mathsf{time}_k(e) > \mathsf{time}_k(e')$ then we are done. So assume $\mathsf{time}_k(e) < \mathsf{time}_k(e')$. Let P be the "plane of movement of k", i.e. let P be a plane parallel to the time-axis and which contains $\mathsf{wline}_m(k)$. By Thm. 2.3(i), the events on P which are simultaneous according to k with e form a straight line ℓ which is not "horizontal", see Fig. 12. Therefore there is an event e'' on ℓ such that $\mathsf{time}_m(e'') > \mathsf{time}_m(e')$. Now $\mathsf{time}_k(e'') = \mathsf{time}_k(e) < \mathsf{time}_k(e')$, and we are done. □

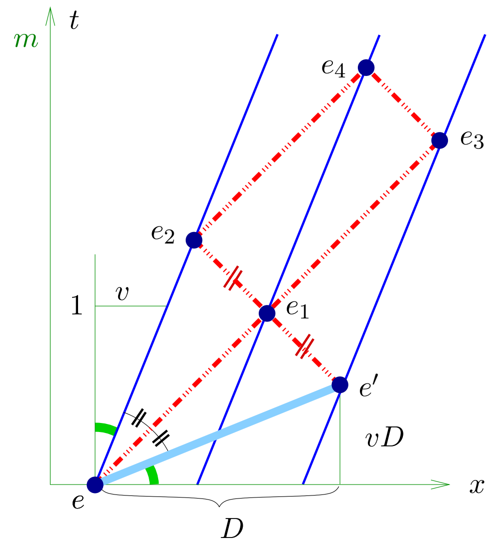


Figure 11: Simpler proof for Thm. 2.3(i) by using synthetic geometry: By using the parallelograms $e_2 - e_1 - e_3 - e_4$ and $e_1 - e' - e_3 - e_4$ we get that the distance between e_2 and e_1 is the same as the distance between e_1 and e'. The triangles $e - e_2 - e_1$ and $e - e' - e_1$ are congruent since the two photon-lines are orthogonal to each other. Since the slope of the light-line $e - e_1$ is 1, the angle between the time-axis $\bar{t}$ and $e - e_2$ is therefore the same as the angle between the "space axis" $\bar{x}$ and $e - e'$, yielding the desired result.

In the next theorem we formalize the three paradigmatic effects of SRK and prove them from **Specrel**. Fig.s 13,14 illustrate the statement of Thm. 2.4 in cartoon and in space-time diagram respectively, while Fig. 17 at the end of the proof summarizes in one picture how two observers "see" each other's coordinate systems. Fig. 18 gives a geometric illustration and explanation for the three paradigmatic effects.

Theorem 2.4. (the three paradigmatic effects) *Assume* **Specrel** *and* $n \geq 3$, *and let* m, k, k' *be observers with* $v := v_m(k)$, *and* $v_k(k') = 0$. *Assume that* k*'s spaceship, the rear and nose of which are marked by observers* k, k', *moves forwards (i.e.* $\mathsf{wline}_m(k), \mathsf{wline}_m(k')$ *are contained in a plane parallel with the time-axis and for some* e'', e^* *with* $k \in e''$, $k' \in e^*$ *and* $\mathsf{space}_m(e'', e^*) = 0$ *we have* $\mathsf{time}_m(e^*) < \mathsf{time}_m(e'')$*), time flows forwards for* k *as seen by* m *(i.e.* $\mathsf{time}_m(\mathsf{ev}_k(\bar{0})) < \mathsf{time}_m(\mathsf{ev}_k(\mathbf{1}_t))$*). Assume fur-*

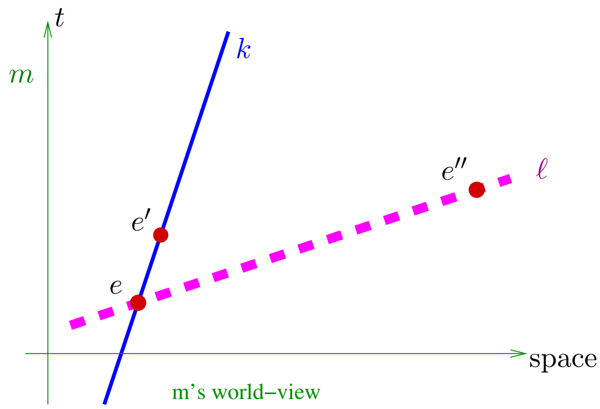


Figure 12: Illustration for the proof of Cor. 2.1. In m's worldview, e happens earlier than e' and e'' happens later than e'. However, in k's worldview e is simultaneous with e''.

ther that, according to k, the clocks in the ship are synchronized (i.e. $\mathsf{time}_k(\mathsf{ev}_{k'}(\overline{0})) = 0$) and the length of the ship is D (i.e. $|\mathsf{space}_k(e')| = D$), see Fig. 14). Then (1)-(3) below hold.

(1) *(moving pairs of clocks get out of synchronism) According to m, the clock-readings at the nose of k's spaceship are $v * D$ less than the simultaneous readings at the rear of the ship. (The clocks in the nose are late relative to those in the rear. See Fig. 13.) Formally:*

$$(\forall e, e')[\, k \in e \ \wedge \ k' \in e' \ \wedge \ \mathsf{time}_m(e, e') = 0 \rightarrow \mathsf{time}_{k'}(e') = \mathsf{time}_k(e) - (v * D)\ ]\,.$$

(2) *(moving clocks slow down (called "time-dilation")) Any process that lasts t seconds in the ship, lasts for $t/\sqrt{1-v^2}$ seconds as seen by m. Formally:*

$$(\forall e, e')[\, k \in e \cap e' \rightarrow \mathsf{time}_m(e, e') = \mathsf{time}_k(e, e')/\sqrt{1-v^2}\,]\,.$$

(3) *(moving ships get shorter (called "length-contraction")) According to m, the length of k's ship is only $D * \sqrt{1-v^2}$ (and not D as k states). Formally:*

$$(\forall e, e')[\, k \in e \wedge k' \in e' \wedge \mathsf{time}_m(e, e') = 0 \rightarrow \mathsf{space}_m(e, e') = D * \sqrt{1-v^2}\,]\,.$$

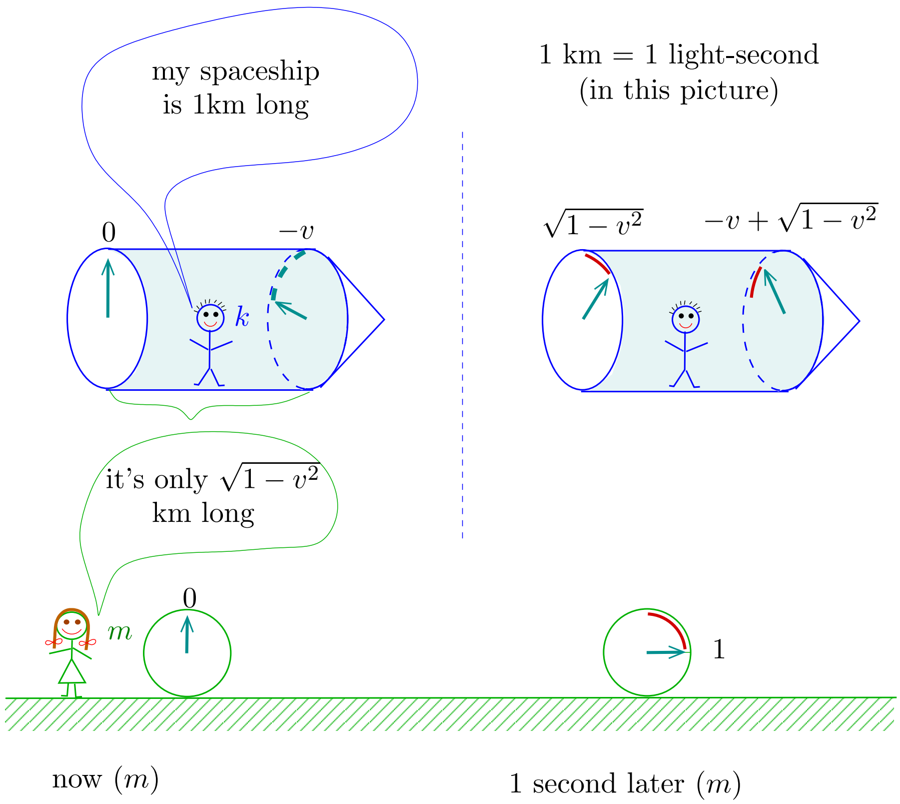


Figure 13: Illustration for Thm. 2.4. According to m, the length of the spaceship is d km, it is 1 km wide and tall, and the clocks in the nose show $dv/\sqrt{1-v^2}$ less time than those in the rear. According to k, the length of the ship is $D = d/\sqrt{1-v^2}$, it is 1 km wide and tall, and the clocks in the nose and the ones in the rear all show the same time. In the picture we chose $d = \sqrt{1-v^2}$ and $D = 1$. Compare this picture with Fig. 3. As v increases, the spaceship becomes squat: it becomes shorter while its width and height remain the same. Cf. also Fig.s 5,6.

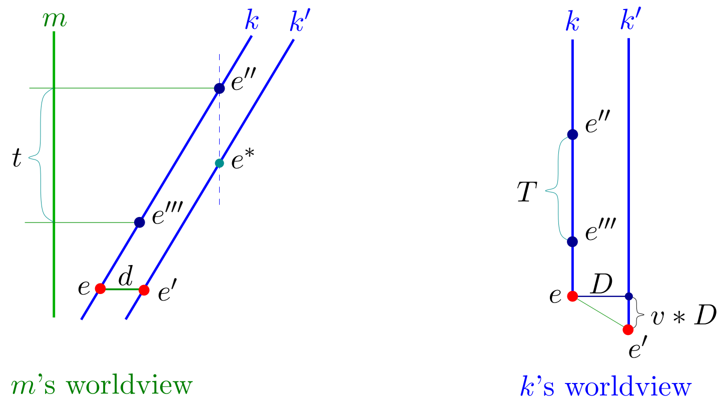


Figure 14: Illustration for Thm. 2.4, which states that $t = T/\sqrt{1-v^2}$ and $d = D * \sqrt{1-v^2}$.

> *Hence, moving spaceships become 'squat": They get shorter but their width and height do not change by* AxSim. *So they get distorted. This distortion effect remains true in weaker fragments of* **Specrel**, *e.g. in* **Specrel**$_0$, *and for arbitrary n, as shown in [AMN02, Sec. 4.8, esp. p. 653].*

Proof. To prove Thm. 2.4, we will use two new thought-experiments analogous to the one in Prop. 2.1. Indication for the proof in geometrical flavor is in the caption of Fig. 18. Let m, k, k', v be as in the hypothesis part of the theorem.

The thought-experiment for proving time-dilation uses *Einstein's light-clock*, cf. Fig. 15. This light-clock consists of two mirrors and a photon which bounces back and forth between the two mirrors. The two mirrors, M_1 and M_2, are positioned at the rear of k's spaceship so that their spatial separation is orthogonal to the movement of the ship (as seen by m) and their spatial distance is 1 light-second. Thus for the photon ph from one mirror M_1 to the other M_2 lasts for 1 second; one tick of the clock lasts 1 second as seen from the ship k. Let e, e' be the events when ph leaves mirror M_1 and reaches mirror M_2, respectively. According to m, the spatial distance between e and e' is not 1 (as seen from k's ship) but $\sqrt{1+x^2}$ where x is the distance the second mirror M_2 covers while the photon reaches it. If t is the time elapsed between e and e' as m sees it, then $x = t * v$, and thus $\mathsf{space}_m(e, e') = \sqrt{1+x^2}$, hence $t = \sqrt{1+x^2}$ because the speed of ph is 1 in

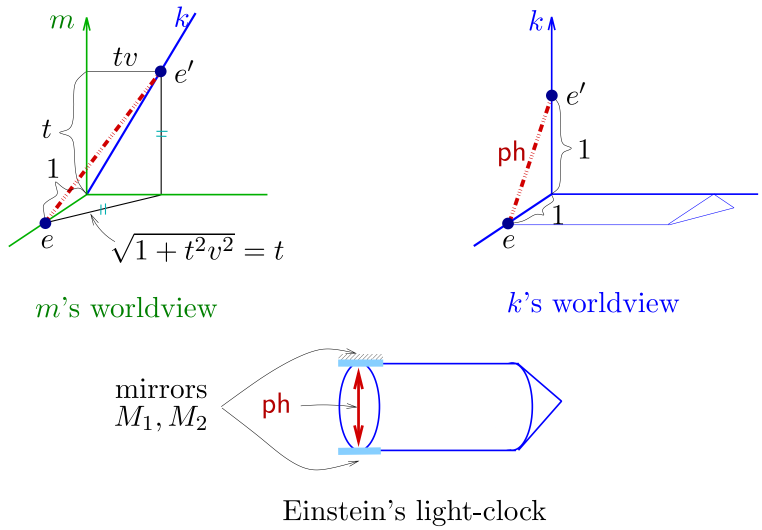


Figure 15: Illustration for the proof of Thm. 2.4(2) (time-dilation). Einstein's light-clock consists of two mirrors and a photon ph bouncing between them. One tick lasts $t = (1/\sqrt{1-v^2})$ seconds in m's worldview, if one tick lasts 1 second in the ship.

m's worldview, too. Now from $t^2 = 1 + t^2v^2$ we get $t = 1/\sqrt{1-v^2}$ (which is greater than 1). We obtained the desired rate of time-dilation.

The thought-experiment for proving length-contraction (Thm. 2.4(3)) uses a so-called "two-dimensional light-clock", this is the following. See Fig. 16. There are two pairs of mirrors and two photons bouncing between them. The first pair of mirrors M_1, M_2 and the photon ph bouncing between them is as in Einstein's light-clock. The second pair of mirrors M_3, M_4 and ph′ are like M_1, M_2, ph with the difference that M_3, M_4 are separated in the direction of movement of the ship. Thus if ph, ph′ leave mirrors M_1, M_3 in the same event e (we may assume that M_3 is positioned where M_1 is), then after bouncing they will be back in the same event e' again. The whole scene in m's worldview is as follows. Photon ph behaves exactly as in Einstein's light-clock, so $t = \mathsf{time}_m(e,e') = 2/\sqrt{1-v^2}$, as before. Let us see what the "tick" made by ph′ looks like in m's worldview. By the arguments in the proof of Thm. 2.3(i), if $d = \mathsf{space}_m(e,e')$, then $t = d/(1-v) + d/(1+v) = 2d/(1-v^2)$. Hence $d = \sqrt{1-v^2}$, the distance between mirrors M_3, M_4 is $\sqrt{1-v^2}$ (which is smaller than 1) as seen by m and not 1 as seen by k. We obtained the desired rate of length-contraction.

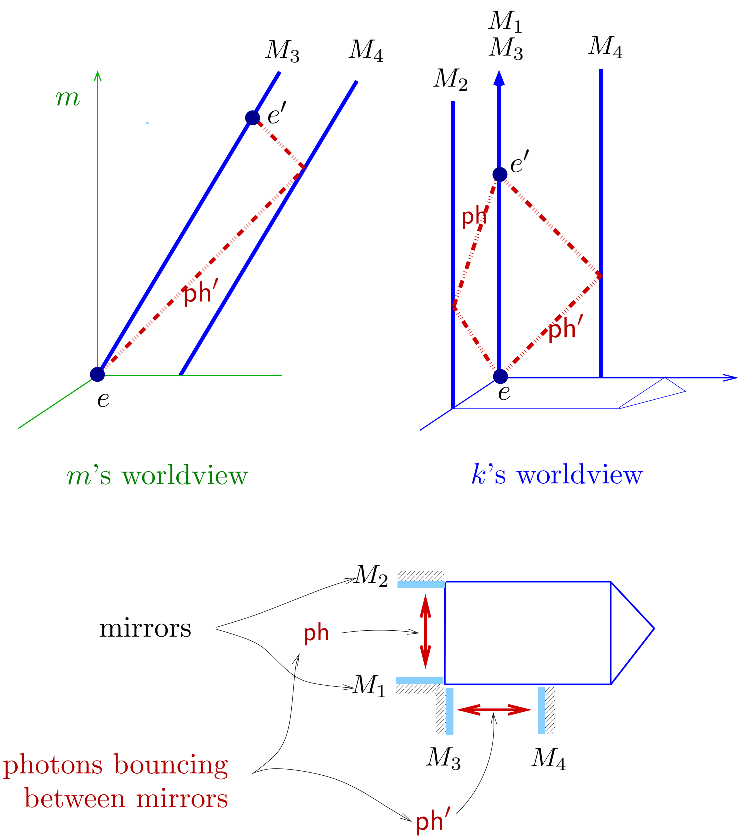


Figure 16: Illustration for the proof of Thm. 2.4(3) (length-contraction). The "two-dimensional" light-clock consists of two pairs of mirrors and two photons ($\mathsf{ph}, \mathsf{ph}'$) bouncing between them. The two photons' bouncing-time is the same in k's worldview, thus it has to be the same in m's worldview, too.

We get Thm. 2.4(1) by combining Thm. 2.3(i) and Thm. 2.4(2) (cf. e.g. Fig. 17). $\square$

Thm. 2.5 below implies that the statement "Motion with every finite speed is possible" (i.e. (iii) of NK) is inconsistent with $\mathbf{Specrel}_0$. This justifies the step of weakening (iii) to (iii)$^-$ in NK$^-$. Thm. 2.5 below also shows that we do not have to postulate as an axiom that no observer can move faster than light; as an axiom this would be difficult to motivate. Luckily, "no faster-than-light observer" turns out to be a corollary of the well-motivated axioms in $\mathbf{Specrel}_0$. Putting it more succinctly: "no faster-

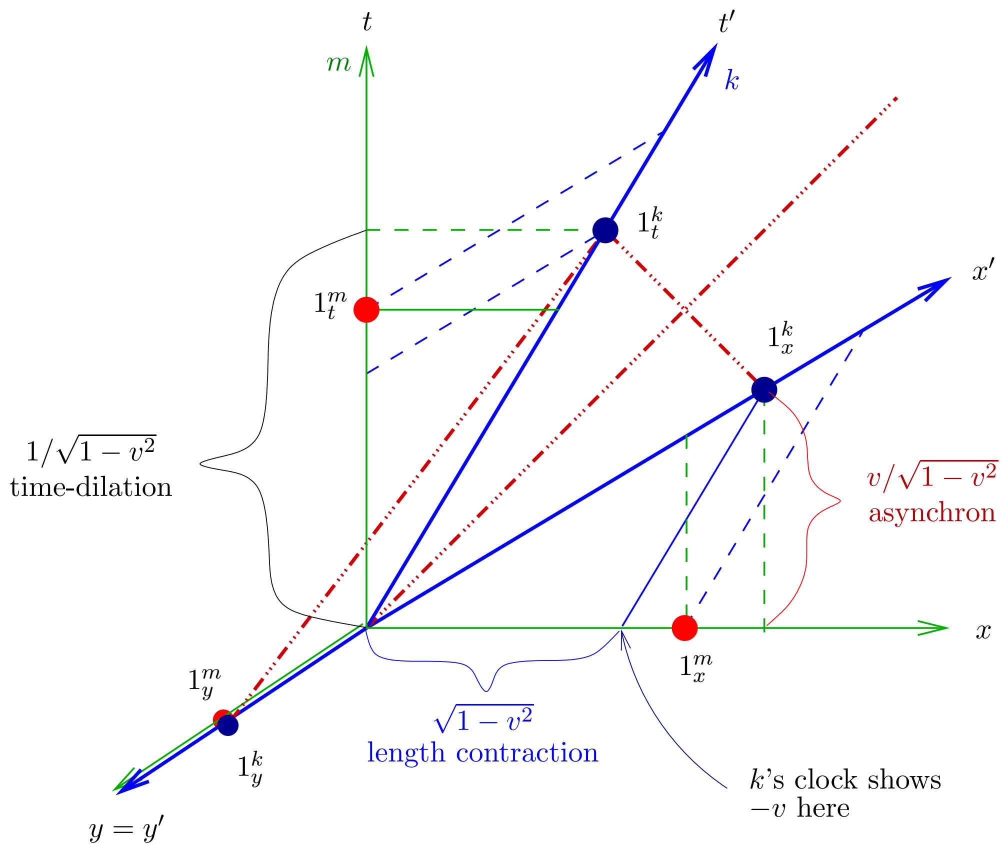


Figure 17: Illustration for Thm. 2.4. The three photon-worldlines illustrate, in some sense, the three thought-experiments for proving time-dilation, length-contraction, and getting out of synchronism, respectively. This figure also illustrates Lorentz transformations to be defined soon (in Def. 2.1).

than-light observer" (No FTL for short) is a theorem only in our approach and not an axiom.

Theorem 2.5. (no faster-than-light motion) *Assume* **Specrel**$_0$.

(i) *No observer can move with the speed of light, i.e.* $v_m(k) \neq 1$ *for all observers* m, k.

(ii) *Assume* $n \geq 3$. *Then* $v_m(k) < 1$ *for all observers* m, k, *i.e. if* $n > 2$ *then no observer can move faster than light. If* $n = 2$, *then* $v_m(k) > 1$ *for some observers* m, k *is possible.*

For **proof** see e.g. [AMN99, Prop. 1, Thm. 3], [Mad02, 2.3.5, 2.8.25, 3.2.13], [MNT04, Thm. 3, Thm. 5]. A proof can also be reconstructed from the proof of Thm. 2.8, p. 39. □

For a conceptual analysis of the above No FTL theorem we refer to Sec. 2.7. There we will address the question "why No FTL?", e.g. which parts of **Specrel**$_0$ are responsible for No FTL, etc.

In Newtonian Kinematics, NK, spatial distance of events is not absolute (e.g. two events that took place in the dining car of a moving train at different times, took place at different places for someone not on the train), but the time elapsed between two events is the same for any two observers, moving relative to each other or not. Thm. 2.4(ii) says that in SRK the time elapsed between two events is not absolute, either. In this respect, SRK is a more symmetric theory than NK. But is there anything left that the observers see the same way? Curiously, a "mix" of time and space does remain absolute (as opposed to being relative like time and space are).

Theorem 2.6. (relativistic distance) *Assume* **Specrel** *and* $n \geq 3$, *and let* m, k *be observers,* e, e' *be events. Then*

$$\mathsf{time}_m(e,e')^2 - \mathsf{space}_m(e,e')^2 = \mathsf{time}_k(e,e')^2 - \mathsf{space}_k(e,e')^2. \quad \square$$

The above theorem is the starting point for building Minkowski geometry, which is the "geometrization" of SRK. It also indicates that time and space are intertwined in SRK.

Let us denote the quantity that is the same for all observers as stated in Thm. 2.6 above by

$$\boldsymbol{\mu}(p,q) := \mathsf{time}(p,q)^2 - \mathsf{space}(p,q)^2 \ .$$

The letter $\boldsymbol{\mu}$ refers to Minkowski distance (also called relativistic distance). We will see that every coordinate property that the observers observe the same way about events (in a model of **Specrel**) can be defined from this relativistic distance $\boldsymbol{\mu}$ (Cor. 2.4). More importantly, the whole structure $\mathfrak{M}$ can be retrieved from relativistic distance $\boldsymbol{\mu}$ (provided we disregard irrelevant properties of observers and photons like e.g. "there are several distinct photons on ph's worldline"). This means that we can redefine photons, observers, and even the field-operations on quantities Q from knowing only relativistic distance $\boldsymbol{\mu}$ (Thm. 2.11). This indicates that there is an "observer-independent reality" which is behind the different worldviews of the observers. This observer-independent reality is often called "objective" or absolute (as opposed to being "subjective" or relative like relative motion). We will explore these ideas in Sec. 2.6.

2.5 Explicit description of all models of Specrel, basic logical investigations

An advantage of having axiomatic theories is that we can use the benefits of the well-developed syntax-semantics duality of first-order logic. Namely, if we want to check whether a formula φ follows from **Specrel**, instead of making a rigorous syntactic derivation, we can check whether in all models of **Specrel** the formula φ holds or not. Specifically, we can *prove* that φ does not follow from **Specrel** by exhibiting a model of **Specrel** in which φ fails. In this subsection we give an explicit description of all models of **Specrel** and $\mathbf{Specrel}_0$. Based on this, then we will give a sample of logical investigations such as consistency, completeness, categoricity, decidability, and independence of axioms.

Thm.s 2.3-2.5 in the previous subsection provide all the important ingredients for describing the models of our theories $\mathbf{Specrel}_0$ and **Specrel**. The "heart" of this description is the description of the so-called worldview transformations. Let m, k be observers. The worldview transformation w_{mk} relates the worldview of m with that of k, it relates those space-time locations where m and k observe the same events. I.e.

$$\mathsf{w}_{mk} := \{\langle p, q\rangle \in \mathsf{Q}^n \times \mathsf{Q}^n : \mathsf{ev}_m(p) = \mathsf{ev}_k(q)\}.$$

The worldview transformation is defined to be a binary relation on space-time locations, but under very mild assumptions it is a transformation of Q^n indeed and $\mathsf{ev}_k = \mathsf{ev}_m \circ \mathsf{w}_{km}$, hence the name "worldview transformation". (Here, and later, $f \circ g$ denotes the composition of functions f and g, i.e. $(f \circ g)(x) = f(g(x))$.) In fact, the worldview transformation $\mathsf{w}_{mk} : \mathsf{Q}^n \to \mathsf{Q}^n$

is the natural coordinate-transformation between the coordinate systems of m and k. It shows how the worldview of one observer m is distorted in the eye of another observer k.

Thm.s 2.3, 2.4 give quite a lot of information on the worldview transformations in models of **Specrel**. They imply that w_{mk} is a Lorentz transformation as defined below, up to a suitable choice of coordinate directions.

It will be convenient to use the so-called unit-vectors. Let $1 \le i \le n$. The i-th unit-vector is

$$\mathbf{1}_i := \langle 0, \dots, 0, 1, 0 \dots, 0 \rangle \quad \text{where the 1 stands in the } i\text{-th place.}$$

We will also use the names $\mathbf{1}_t, \mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_z$ for the first four unit-vectors. From now on we fix a quadratic ordered field $\mathfrak{Q} = \langle Q, +, *, \le \rangle$.

Definition 2.1. (Lorentz transformation) *Let* $-1 < v < 1,\ v \in Q$. *By the Lorentz transformation (or boost) of velocity* v *and over* $\mathfrak{Q}$ *we understand a linear mapping* $f : \mathsf{Q}^n \to \mathsf{Q}^n$ *for which*

$$f(\mathbf{1}_t) = \langle 1/\sqrt{1-v^2},\ v/\sqrt{1-v^2},\ 0, \dots, 0 \rangle,$$

$$f(\mathbf{1}_x) = \langle v/\sqrt{1-v^2},\ 1/\sqrt{1-v^2},\ 0, \dots, 0 \rangle, \qquad \textit{and}$$

$$f(\mathbf{1}_i) = \mathbf{1}_i \quad \textit{for all } 3 \le i \le n.$$

Fig. 17 illustrates Lorentz transformations. A Lorentz transformation as a coordinate transformation usually is written as

$$t' = (t - vx)/\sqrt{1-v^2}, \quad x' = (x - vt)/\sqrt{1-v^2}, \quad y' = y, \quad z' = z.$$

The usual Newtonian (or Galilean) coordinate transformation is $t' = t,\ x' = x - vt,\ y' = y,\ z' = z$. Comparing the two transformations reveals that in SRK time and space are treated in a symmetric way, while in NK they are treated differently. In the formula for Lorentz transformations, the divisors $/\sqrt{1-v^2}$ represent time-dilation and length-contraction, while "$t - vx$" in place of "t" in the first part represents "getting out of synchronism".

By a space-isometry (over $\mathfrak{Q}$) we understand a Euclidean isometry (i.e. a mapping that preserves Euclidean distance between space-time locations) which takes the time-axis to a line parallel to the time-axis. These are affine mappings, i.e. linear mappings composed with translations.

Theorem 2.7. (description of worldview transformations of **Specrel**) *Assume* $n \ge 3$, *let* $\mathfrak{Q} = \langle Q, +, *, \le \rangle$ *be a quadratic ordered field and let* $f : Q^n \to Q^n$. *The following are equivalent.*

(i) *f is a worldview transformation in a model of* **Specrel** *with field-reduct* $\mathfrak{Q}$.

(ii) *$f = \sigma \circ \lambda \circ \sigma'$ for some Lorentz transformation λ and space-isometries σ, σ' (over $\mathfrak{Q}$).*

(iii) *f is a bijection and preserves relativistic distance, i.e.* $(\forall p, q \in \mathsf{Q}^n)\boldsymbol{\mu}(p, q) = \boldsymbol{\mu}(f(p), f(q))$.

On the proof. (i)⇒(ii): By Thm. 2.3(i) we know that w_{mk} is like a Lorentz transformation on the "plane of motion", i.e. on the vertical plane containing $\mathsf{wline}_m(k)$, and it takes the subspace of Q^n orthogonal to this plane to itself, by Thm. 2.3(ii). AxSim then states that w_{mk} is a Euclidean isometry on this orthogonal subspace. The proof of (i)⇒(ii) from here on is not difficult. (ii)⇒(iii): A possibility for proving this is that one checks by a computation that both space-isometries and Lorentz transformations preserve relativistic distance. However, we would like to provide more insight here concerning (ii)⇒(iii). Namely, showing that Lorentz transformations preserve lines of slope 1 (i.e. that they preserve $\boldsymbol{\mu}(p, q) = 0$) is the most important step in proving that **Specrel** is consistent. Fig. 18 illustrates a non-computational, geometric proof for this crucial part of the proof. (iii)⇒(i): If f preserves $\boldsymbol{\mu}$, then f preserves lines of slope 1, preserves lines of slope < 1, and also "it satisfies AxSim". The rest follows from the construction we give after Thm. 2.8. □

By a space-dilation (over $\mathfrak{Q}$) we understand a Euclidean dilation (i.e. a mapping that "dilates" Euclidean distances between space-time locations with a given ratio $r \in Q$) and takes the time-axis to a line parallel to the time-axis. These are affine mappings. By a field-automorphism-induced mapping over $\mathfrak{Q}$ we understand the natural extension of an automorphism of $\mathfrak{Q}$ to Q^n. These are not necessarily affine mappings, but they are collineations, i.e. they take straight lines to straight lines.

Theorem 2.8. (description of worldview transformations of $\mathbf{Specrel}_0$) *Assume $n \geq 3$, let $\langle Q, +, *, \leq \rangle$ be a quadratic ordered field and let $f : Q^n \to Q^n$. The following are equivalent.*

(i) *f is a worldview transformation in a model of* $\mathbf{Specrel}_0$.

(ii) *$f = \delta \circ \lambda \circ \delta' \circ \alpha$ for some Lorentz transformation λ, space-dilations δ, δ', and field-automorphism-induced mapping α.*

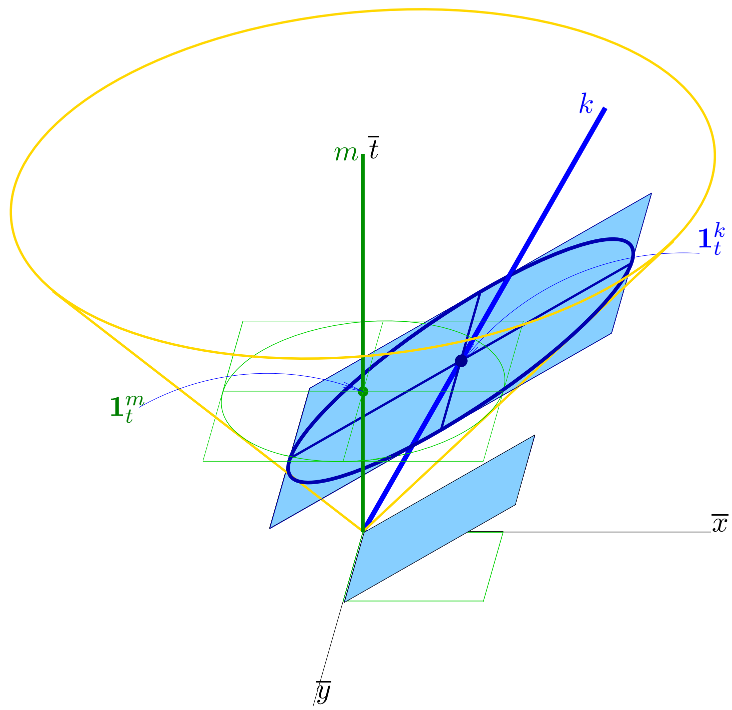


Figure 18: The Light Axiom states that if we switch on a light source for a moment, we will observe a light-sphere expanding away from us with the speed of light, and that we are all the time in the center of this light-sphere. Assume that observers m, k are present in the event of switching on the light source and that they are moving relative to each other. Then both observers m and k have to observe that they are in the center of the photon-sphere! How is this possible? This figure illustrates how. Let $\overline{x}$ be in the direction of movement of k in m's worldview, and let $\overline{y}$ be any spatial direction orthogonal to $\overline{x}$. The expanding light-sphere in space-time when concentrating on the 3-dimensional subspace determined by $\overline{t}, \overline{x}, \overline{y}$ is a cone. k's plane of simultaneity is tilted just so that k is in the center of the ellipse that is the intersection of this plane with the light-cone. For this, the "long axis" of the ellipse is tilted just the amount that it is symmetric to the worldline of k (w.r.t. a photon-line as in Fig. 17), and the "small axis" of the ellipse is parallel with $\overline{y}$. This implies "k's clocks getting out of synchronism" (Thm. 2.3). The time-unit $\mathbf{1}_t^k$ of k is on k's worldline exactly so "high" that the length of the "short axis" of the ellipse is 1, when AxSim is true (and arbitrary otherwise). This implies that k's time flows slowly as seen by m when AxSim is true. (Paradigmatic effects in Thm. 2.4!) The other space-unit $\mathbf{1}_x^k$ of k is chosen so that the length of the "long axis" of the ellipse counts as 1, too. With this choice of the units of measurement, k sees the ellipse as a circle, hence k thinks that he is in the center of the light-cone. There is enough room for everyone in the center of the expanding lightsphere!

(iii) *f is a bijection and preserves relativistic distance* 0, *i.e.*
$(\forall p, q \in \mathsf{Q}^n)[\boldsymbol{\mu}(p,q) = 0 \quad \textit{iff} \quad \boldsymbol{\mu}(f(p), f(q)) = 0]$.

On the proof. (ii)⇔(iii) is a variant of the Alexandrov-Zeeman theorem, see e.g. [Gol87, App. 2]. (i)⇒(iii) follows from AxPh, and (ii)⇒(i) follows from the construction we give soon, because (ii) implies that f preserves lines of slope 1 and lines of slope < 1. □

Remark: We could have proved Thm.s 2.3 - 2.5 by first deriving the properties of the worldview transformations as in Thm.s 2.7, 2.8, and then deriving the paradigmatic effects (i)-(iii) from their properties. We think that deriving the predictions of relativity theory directly from the axioms is more illuminating. For the student, Lorentz transformations appear as non-observation oriented theoretical constructions not explaining why we are doing what we are doing. In our approach, stating the axioms in **Specrel** and then deriving the three paradigmatic effects motivate the introduction of Lorentz transformations. In some sense, **Specrel** can be considered as an "implicit definition" of the Lorentz transformations.

We now turn to an exhaustive description of all models of $\mathbf{Specrel}_0$ and **Specrel**.

Given an arbitrary quadratic ordered field $\mathfrak{Q}$, let PLines and TLines denote the set of all straight lines in Q^n of slope 1 and of slope < 1, respectively. Let WT denote the set of all transformations of Q^n which preserve both PLines and TLines. Then WT forms a group, and Thm. 2.8 describes the members of WT.

Consider $\mathfrak{M} = \langle \mathfrak{Q};\ B, Ob, Ph;\ W \rangle \models \mathbf{Specrel}_0$ in which $Ob \neq \emptyset$. Then Ph and Ob are disjoint, by Thm. 2.5(i). We let

$$B_1 := B - (Ph \cup Ob).$$

In the discussion below we will see that (after having chosen $\mathfrak{Q}$), the "heart" of a $\mathbf{Specrel}_0$ model is a subgroup $\mathsf{WT}_0 \subseteq \mathsf{WT}$ such that $\{f^{-1}[\bar{t}] : f \in \mathsf{WT}_0\} = \mathsf{TLines}$. Here $f[H] := \{f(a) : a \in H\}$ for any function f and subset H of the domain $\mathsf{Dom}(f)$ of f, as usual. Having chosen the heart WT_0 of our model, we still have to decorate it with "observer names" (Ob), "photon names" (Ph), and with extra bodies B_1 not necessarily in $\mathsf{Ob} \cup \mathsf{Ph}$. This decorating or labelling gives rise to an extra plurality of (nonisomorphic) possible models for $\mathbf{Specrel}_0$ in addition to the possible choices of WT_0. (These choices will be formally specified in items (i)-(v) below.) Below we present the details giving precise meanings to what we understand by the above, e.g. by "labelling", "heart" etc. The reader not interested in the details might skim over them just to have an impression and continue serious reading with Cor. 2.2.

Let us return to our $\mathfrak{M} = \langle \mathfrak{Q};\ B, Ob, Ph;\ W\rangle \models \mathbf{Specrel}_0$. For any observer $m \in Ob$ let us define the "worldview of m" as a structure

$$\mathsf{w}_m := \langle Q^n, \mathsf{wline}_m(b) : b \in B\rangle$$

where Q^n is the carrier set and $\mathsf{wline}_m(b)$ is a one-place relation (or unary predicate) with relation symbol b (denoting this subset of Q^n) for each $b \in B$ (recall that $\mathsf{wline}_m(b) \subseteq \mathsf{Q}^n$, and cf. Fig. 1 on p.7). The set of all worldviews contains exactly the information content of $W \cap Ob \times B \times Q^n$. Let $W_1 := W - Ob \times B \times Q^n$. When we want to define a model of $\mathbf{Specrel}_0$, we have to define the 6-place relation W. Instead of defining W directly, often it is easier to define the set of worldviews along with W_1.

All these worldviews are isomorphic with each other, actually the worldview transformations are isomorphisms between the worldviews, i.e. $\mathsf{w}_{mk} : \mathsf{w}_m \to \mathsf{w}_k$ is an isomorphism for any $m, k \in Ob$, by the definition of w_{mk}. What do the worldviews w_m look like? The carrier set is Q^n, the photons are distributed surjectively on the PLines by AxPh; the observers are distributed surjectively on the TLines by AxLine, AxThEx, Thm. 2.5; $\mathsf{wline}_m(m) = \bar{t}$ by AxSelf. Instead of specifying all the worldviews one-by-one, we can specify one "Platonic" (or generic) worldview

$$\mathfrak{P} := \langle Q^n, \pi(b) : b \in B\rangle, \qquad \text{where}$$

$$\pi \restriction Ph : Ph \twoheadrightarrow \mathsf{PLines} \text{ is surjective,}$$

$$\pi \restriction Ob : Ob \twoheadrightarrow \mathsf{TLines} \text{ is surjective,} \quad \text{and}$$

$$\pi \restriction B_1 : B_1 \to \{Y : Y \subseteq Q^n\},$$

and then for each observer $m \in Ob$ we can specify an element of WT which describes how m realizes this Platonic worldview, i.e. we specify a function

$$w : Ob \to \mathsf{WT} \quad \text{such that } w(m)[\pi(m)] = \bar{t};$$

and then we define w_m as the image of $\mathfrak{P}$ by the function $w(m)$. The information content of $\pi \restriction Ob : Ob \twoheadrightarrow \mathsf{TLines}$ can be recovered from this last function w (by $\pi(m) = w(m)^{-1}[\,\bar{t}\,]$), so we may skip specifying $\pi \restriction Ob$.

With the above intuition in mind, we can construct a model of $\mathbf{Specrel}_0$ by specifying a quintuple $\langle \mathfrak{Q}, w, \pi, \beta, W_1\rangle$ with the following properties:

(i) $\mathfrak{Q}$ is a quadratic ordered field,

(ii) w, π, β are functions with disjoint domains Ob, Ph, B_1 respectively,

(iii) $w : Ob \to \mathsf{WT}$ is such that $\{w(m)^{-1}[\bar{t}] : m \in Ob\} = \mathsf{TLines}$,

(iv) $\pi : Ph \twoheadrightarrow \mathsf{PLines}$ is surjective,

(v) $\beta : B_1 \to \{Y : Y \subseteq Q^n\}$, and $W_1 \subseteq (B_1 \cup Ph) \times (B_1 \cup Ph \cup Ob) \times Q^n$.

Let us call a quintuple satisfying the above conditions a pre-model. From any pre-model $\langle \mathfrak{Q}, w, \pi, \beta, W_1 \rangle$ we can construct a model of $\mathbf{Specrel}_0$ by defining

$\mathfrak{M}(\mathfrak{Q}, w, \pi, \beta, W_1) := \langle \mathfrak{Q};\ B, Ob, Ph;\ W \rangle$ where

$Ob := \mathsf{Dom}(w)$, $Ph := \mathsf{Dom}(\pi)$, $B := \mathsf{Dom}(\beta) \cup Ob \cup Ph$; and W is defined the natural way

$W := \{\langle m, b, p\rangle : p \in w(m)[\beta(b)], m \in Ob\} \cup \{\langle m, \mathsf{ph}, p\rangle : p \in w(m)[\pi(\mathsf{ph})], m \in Ob\} \cup \{\langle m, k, p\rangle : p \in w(m)[w(k)^{-1}[\bar{t}]], m \in Ob\} \cup W_1$.

It can be checked that all models constructed from pre-models are models of $\mathbf{Specrel}_0$; and conversely, all models of $\mathbf{Specrel}_0$ with nonempty observer-part arise this way from pre-models.

We described the models of $\mathbf{Specrel}_0$ with $\mathsf{Ob} \neq \emptyset$. The description when $\mathsf{Ob} = \emptyset$ is easy. The description of the models of **Specrel** is exactly like above with the only change that in place of WT we use its subset WT^+ characterized in Thm. 2.7.

Corollary 2.2. (Consistency) **Specrel** *is a consistent theory, i.e. for no formula* φ *can both* φ *and its negation* $\neg\varphi$ *be derived from* **Specrel**. *Moreover,* $\mathbf{Specrel} + (\mathsf{Ob} \neq \emptyset)$ *is also consistent.*

Proof. To construct a model of $\mathbf{Specrel} + (\mathsf{Ob} \neq \emptyset)$ we have to show that there exists at least one pre-model $\langle \mathfrak{Q}, w, \pi, \beta, W_1 \rangle$ with $w : Ob \to \mathsf{WT}^+$. Of the conditions (i)-(v) in the definition of a pre-model, only condition (iii) is not trivial to satisfy. However, Thm. 2.7(ii) shows that for all $\ell \in \mathsf{TLines}$ there is $w \in \mathsf{WT}^+$ which takes $\bar{t}$ to ℓ, and we are done. □

Cor. 2.2 above completes justification of the move NK $\mapsto$ NK$^-$. It shows that by this move, we indeed got rid of all contradictions between NK and the Light Axiom.

Having a description of all models of **Specrel** and $\mathbf{Specrel}_0$ at hand makes it easy to see which statements follow from **Specrel** and $\mathbf{Specrel}_0$ and which do not. As an example we include the following.

Corollary 2.3. *Let* $n \geq 3$. $\mathbf{Specrel} \vdash (\forall m, k \in \mathsf{Ob}) v_m(k) = v_k(m)$ *while* $\mathbf{Specrel}_0 \nvdash (\forall m, k \in \mathsf{Ob}) v_m(k) = v_k(m)$.

Hint for proof. Field-automorphism-induced mappings α can occur in worldview-transformations in models of $\mathbf{Specrel}_0$, but not in models of **Specrel**. □

Theorem 2.9. (independence of the axioms) *Assume* $n \geq 3$.

(i) $(\mathbf{Specrel}_0 - \{\mathsf{AxLine}, \mathsf{AxThEx}\}) \vdash \mathsf{AxLine}$.

(ii) $\mathbf{Specrel} - \{\mathsf{AxLine}\}$ *is an independent axiom system, i.e.* $(\mathbf{Specrel} - \{\mathsf{Ax}\}) \nvdash \mathsf{Ax}$ *for any element* Ax *in* **Specrel** *different from* AxLine.

(iii) *Every model of* $\mathbf{Specrel}_0 - \{\mathsf{AxThEx}\}$ *can be extended to a model of* $\mathbf{Specrel}_0$. *Hence if formula* η *is universally quantified in the sort* B *and* $\mathbf{Specrel}_0 \vdash \eta$, *then* $(\mathbf{Specrel}_0 - \{\mathsf{AxLine}, \mathsf{AxThEx}\}) \vdash \eta$. *The same holds for* **Specrel** *in place of* $\mathbf{Specrel}_0$. □

By Thm. 2.9(iii) above, all our paradigmatic effects can be proved in the more economical fragment $\mathbf{Specrel} - \{\mathsf{AxLine}, \mathsf{AxThEx}\}$ of **Specrel**. (This is so because the paradigmatic effects can be reformulated as sentences universally quantified in sort B, by using Thm. 2.1.) On the other hand, Thm.s 2.3, 2.4 do not hold if we omit any one of the axioms of $\mathbf{Specrel}_0 - \{\mathsf{AxLine}, \mathsf{AxThEx}\}$, and Thm. 2.4 does not hold if we omit AxSim. Such investigations of economy asking which axioms are needed for proving what theorem are called "reverse relativity theory" motivated by the highly successful branch of mathematics called "reverse mathematics" and is pursued in [AMN02]. We will return to this important subject in Sec. 2.7.

Specrel has many non-elementarily equivalent models over any quadratic ordered field. We show that **Specrel** can be extended to a theory $\mathbf{Specrel} \cup \mathbf{Comp}$ which is categorical over any quadratic ordered field, it can be extended to a complete and decidable theory, and **Specrel** can also be extended to a hereditarily undecidable theory. Both extensions are natural. (Cf. Thm. 2.10 below.)

Recall that from any pre-model $\langle \mathfrak{Q}, w, \pi, B_1, W_1 \rangle$ we can construct a model of $\mathbf{Specrel}_0$. The most natural pre-models for **Specrel** are $\langle \mathfrak{Q}, \mathsf{Id} \restriction \mathsf{WT}^+, \mathsf{Id} \restriction \mathsf{PLines}, \emptyset, \emptyset \rangle$. We will call the models constructed from these standard models for **Specrel**. Thus, in the standard models we include all the possible kinds of observers, but otherwise we are as "economic" as possible. There is exactly one standard model of dimension n over any quadratic ordered field $\mathfrak{Q}$. We are going to give a complete axiom system for these standard models.

$\mathsf{AxCoord}$ $(\forall m \in \mathsf{Ob})(\forall\,\text{space-isometry } S \text{ of } \mathsf{Q}^n)(\exists k \in \mathsf{Ob})\mathsf{w}_{mk} = S.$

AxExt^{ob} $(\forall m, k \in \mathsf{Ob})(\mathsf{w}_{mk} = \mathsf{Id} \to m = k).$

AxExt^{ph} $(\forall \mathsf{ph}, \mathsf{ph}' \in \mathsf{Ph})(\forall m \in \mathsf{Ob})(\mathsf{wline}_m(\mathsf{ph}) = \mathsf{wline}_m(\mathsf{ph}') \to \mathsf{ph} = \mathsf{ph}').$

$\mathsf{AxNobody}$ $\mathsf{B} = \mathsf{Ob} \cup \mathsf{Ph}$ and $\mathsf{W} \subseteq \mathsf{Ob} \times \mathsf{B} \times \mathsf{Q}^{n-2}$.

Comp $:= \{\mathsf{AxCoord}, \mathsf{AxExt}^{ob}, \mathsf{AxExt}^{ph}, \mathsf{AxNobody}\}.$

The above are natural axioms which hold in all standard models of **Specrel**. AxCoord expresses that each observer can "re-coordinatize" his worldview with a space-isometry. There is a quantifier ranging over space-isometries in this formula. Nevertheless, this axiom can be expressed with a first-order logic formula because space-isometries are affine mappings and hence can be "coded" with the images of the n unit-vectors $\mathbf{1}_i$. The next two axioms in **Comp** say, intuitively, that of each kind of observers and photons we have only one copy (or, in other words, according to Leibniz's principle, if we cannot distinguish two observers or photons with some observable properties expressible in our language, then we treat them as equal). Hence we call them extensionality principles, whence the abbreviation AxExt. The example of AxExt^{ph} reveals that here we consider only space-time-oriented properties of photons, hence two photons of different "color" but same worldline are not distinguished in the theory **Comp**. AxExt^{ob} also expresses that we really identify observers with coordinate systems. These axioms are natural to assume, we did not include them in **Specrel** because these "simplifying axioms" are not needed for proving the theorems. The last axiom in **Comp** says that every body is an inertial observer or photon. This is a real restriction that we usually do not want to make when we use AxLine. The main reason is that we can treat accelerated observers in **Specrel** if we do not make this restriction AxNobody, see Sec. 3.1. Treating accelerated observers in **Specrel** is important for the transition from special relativity theory to general relativity theory, as we shall see. AxNobody excludes accelerated bodies. So we do "pay a physical price" for assuming AxNobody.

By a theory we understand an arbitrary set of first-order logic formulas (i.e. we will not assume that a theory contains all its semantical consequences). We call a theory Th decidable (or undecidable respectively) if the set of all first-order logic semantical consequences of Th is decidable (or undecidable respectively). We call Th complete if it implies either φ or $\neg\varphi$ for each first-order logic formula φ without free variables (of its language). It is known that the theory of quadratic ordered fields is undecidable. A quadratic ordered field is called real-closed if every polynomial of odd degree has zero as a value. This last requirement can be expressed with the infinite set $RC := \{\phi_{2n+1} : n \in \omega\}$ of first-order logic formulas, where ϕ_n denotes the following sentence

$$\forall x_0 \dots \forall x_n \exists y (x_n \neq 0 \to x_0 + x_1 \cdot y + \dots + x_n \cdot y^n = 0).$$

Tarski proved that the theory of real-closed fields is complete and decidable (cf., e.g., [Hod93, Thm. 2.7.2, p. 67, p. 92]). The above suggests that if we want to obtain interesting and relevant decidability-theoretic results, then we have to concentrate on real-closed fields; or at least include a decidable theory of field-axioms into our theories.

Theorem 2.10. *Let $n \geq 3$.*

(i) *The models of* **Specrel** $\cup$ **Comp** *are exactly the models isomorphic to standard ones.*

(ii) **Specrel** $\cup$ **Comp** $\cup\, TF$ *is complete and decidable, for any complete, decidable theory TF of quadratic ordered fields.*

(iii) **Specrel** $\cup$ (**Comp** $-$ {Ax}) $\cup\, RC$ *can be extended to a hereditarily undecidable theory Th for any* Ax $\in$ **Comp**, *in the sense that no consistent extension of Th is decidable.*

For **proof** of Thm. 2.10 and for related results we refer to [AMN99, §7], [AMN04, §7].

In the standard models of **Specrel** there is no orientation for time, "reversing time" is an automorphism of these models. In relativity theory, both special and general, we sometimes use the fact that "time has a direction", i.e. that Ax↑ below is assumed.

Ax↑ $\quad (\forall m, k \in \mathsf{Ob})(\forall p, q \in \bar{t})[\,(v_m(k) < 1 \wedge p_t \leq q_t) \Rightarrow \mathsf{w}_{km}(p)_t \leq \mathsf{w}_{km}(q)_t\,].$

Axiom $\mathsf{Ax}\uparrow$ expresses that every observer sees the time of another slowly moving observer "flow forwards", i.e. m sees k's clocks ticking "forwards" and not "backwards". All our theorems so far are true for **Specrel** + $\mathsf{Ax}\uparrow$ in place of **Specrel** with minor modifications.

2.6 Observer-independent geometries in relativity theory; duality and definability theory of logic

According to the approach taken so far, each observer observes the world through the "looking-glass" or "spectacles" of his own coordinate system. The question comes up: Is there an observer-independent, "absolute" reality which the individual observers observe through their respective coordinate systems, or is the set of worldviews of the different observers just an ad-hoc collection of subjective personal views? (The philosophy of subjective idealism contra the assumption of the existence of an objective external world.) We will see that relativistic space-time (or relativistic geometry) provides such an observer-independent reality. Namely, in the present subsection we show that the observers (and their coordinate systems) can be defined from relativistic distance $\boldsymbol{\mu}$ as defined at the end of Sec. 2.5, in models of **Catrel** := **Specrel** $\cup$ **Comp**. If we regard the worldviews of the various observers as "subjective" (in some sense), then $\boldsymbol{\mu}$ is "objective" in the sense that $\boldsymbol{\mu}$ is the same for all observers, in **Specrel**. Thus relativistic distance $\boldsymbol{\mu}$ provides such an observer-independent, absolute reality. This statement will be made more tangible in the definition of the observer-independent geometry $\mathsf{Mg}(\mathfrak{M})$ associated to **Specrel** models $\mathfrak{M}$ below.

We already saw that in models of $\mathbf{Specrel}_0$, all observers observe the same events. Let Events denote the set of events observed by some (or equivalently by each) observer,

$$\mathsf{Events} := \{\mathsf{ev}_m(p) : m \in \mathsf{Ob},\ p \in \mathsf{Q}^n\}.$$

In models of **Specrel**, we can define relativistic distance $\boldsymbol{\mu}$ of events, by Thm. 2.6, as

$$\boldsymbol{\mu}(e, e') := \boldsymbol{\mu}(\mathsf{loc}_m(e), \mathsf{loc}_m(e')), \quad \text{for any observer } m \in \mathsf{Ob}.$$

Relativistic distance of events is a function $\boldsymbol{\mu} : \mathsf{Events} \times \mathsf{Events} \to \mathsf{Q}$. Given $\mathfrak{M} \models$ **Specrel** we define its metric-geometry (or Minkowski geometry) $\mathsf{Mg}(\mathfrak{M})$ as a two-sorted structure as follows:

$$\mathsf{Mg}(\mathfrak{M}) := \langle \mathsf{Events}, \boldsymbol{\mu};\ \mathsf{Q}, 1 \rangle.$$

$\mathsf{Mg}(\mathfrak{M})$ is also referred to as the space-time of $\mathfrak{M}$. The two sorts of $\mathsf{Mg}(\mathfrak{M})$ are Events and Q; $\boldsymbol{\mu}$ is a function of sort $\mathsf{Events} \times \mathsf{Events} \to \mathsf{Q}$ and 1 is a constant of sort Q. We want to state a strong equivalence between $\mathsf{Mg}(\mathfrak{M})$ and $\mathfrak{M}$. These two structures have different vocabularies (or signatures, or languages).

The part of logic that connects structures and theories on different vocabularies is called *definability theory.* The strongest kind of connection between two theories is *definitional equivalence* of theories. When two theories are definitionally equivalent, we say that they are lexicographical variants of the same theory, the only difference being that they use different concepts of the theory as basic ones. We will need definitional equivalence of first-order logic theories on vocabularies that have different *sorts* (or universes), hence the definitionally equivalent models will have different kinds of universes. This amounts to defining new "entities" in a model, not only new relations or functions on already existing entities as in "standard" one-sorted definability theory of first-order logic. Definability of new sorts is important in the kind of definability that arises in relativity theory; therefore we worked out such a definability theory in [AMN02, §6.3] and [Mad02, §4.3]. Below we recall the elements of definability theory that we are going to use.

Let L be a vocabulary, possibly many-sorted, and let L' be an expansion of L, i.e. L' may contain new sorts, and new relation and function symbols. The L-reduct of a structure $\mathfrak{M}'$ on vocabulary L' is the obvious thing (we "forget" the interpretations of symbols not in L). Let Th and Th' be theories on vocabularies L and L', respectively. We say that Th' is a definitional expansion of Th iff the following (i)-(ii) hold:

(i) The models of Th are exactly the L-reducts of models of Th'.

(ii) For any two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of Th' with the same L-reduct there is a unique isomorphism between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ that is the identity on this common reduct.

Let Th_1 and Th_2 be arbitrary theories (possibly on completely different vocabularies). We say that Th_1 and Th_2 are definitionally equivalent when they have a joint definitional expansion Th_3. (More precisely, definitional equivalence is the transitive closure of the notion just defined. In this subsection we will not need to take transitive closure.)

Intuitive explanation for definitional equivalence of theories: By "Th' is a definitional expansion of Th" we mean that each sort, relation and function in the vocabulary of Th' that is not present in the vocabulary of Th is actually defined in Th' in the following sense. Properties (i)-(ii) above express that we consider a new relation (i.e. one in L' but not in L) on an existing sort as defined (in Th') if it can be "put" on every model of Th in a unique way so that it satisfies Th', and we consider a new sort (and relations on it) as defined if it can be added to each model of Th in a unique way, up to a unique isomorphism.

In definability theory of logic, there are a semantical and a syntactical approach to definability, and of course the interesting thing is to state their equivalence (this is Beth's theorem in the usual definability theory of first-order logic). We presented here the notions of the semantic approach; when Th' is a definitional expansion of Th we can say that Th' is an "implicit, or semantical definition" of the symbols not occurring in Th. In [AMN02, §6.3] and [Mad02, §4.3] we worked out the "syntactical" counterpart of this definability, i.e. we gave concrete prescriptions for what an "explicit, syntactical definition" of a new element of the vocabulary can look like. Then we proved the analogue of Beth's theorem (stating that a new element of the vocabulary has an implicit definition exactly when it has an explicit definition). When two theories are definitionally equivalent, there is a computable meaning-preserving translation function between their languages (see [Mad02, 4.3.27 and 4.3.29]). For more on definability theory we refer to e.g. [Mak93].

We now proceed to state definitional equivalence between theories occurring in relativity theory. From now on, in the present subsection, we assume $n \geq 3$.

The formula $\boldsymbol{\mu}(p,q) = (p_1 - q_1)^2 - (p_2 - q_2)^2 - \cdots - (p_n - q_n)^2$ is referred to as the (squared) Minkowski metric and the structure $\langle \mathsf{Q}^n, \boldsymbol{\mu};\ \mathsf{Q}, 1\rangle$ is the (metric) Minkowski geometry over $\mathfrak{Q}$. For a class K of structures, IK denotes the class of all structures isomorphic to elements of K.

Theorem 2.11. (definitional equivalence between metric-geometries and **Catrel** models) *Assume $n \geq 3$.*

(i) **Catrel** *is definitionally equivalent to the first-order logic theory of its metric-geometries, i.e.* **Catrel** *and Th_m in (ii) below are definitionally equivalent.*

(ii) $\mathsf{I}\{\langle \mathsf{Events}, \boldsymbol{\mu};\ \mathsf{Q}, 1\rangle : \mathfrak{M} \models \textbf{Catrel}\} =: \mathsf{MG}$ *is axiomatizable by finitely many formulas, i.e. there is a finite axiom system Th_m such that* MG *is the class of all models of Th_m.*

(iii) $\mathsf{I}\{\langle \mathsf{Q}^n, \boldsymbol{\mu};\ \mathsf{Q}, 1\rangle : \mathsf{Q}$ *is a quadratic ofield*$\} = \mathsf{MG}$. *I.e., the class of Minkowski geometries (over quadratic ordered fields) coincides with the class of metric-geometric models of* **Catrel**.

Outline of proof. **(i):** We define a joint definitional expansion Th_3. The vocabulary of Th_3 contains all the symbols occurring either in **Specrel** or in $\mathsf{Mg}(\mathfrak{M})$, plus one new $n+2$-place relation symbol J of type $\mathsf{B} \times \mathsf{Events}^{n+1}$. Let $\mathfrak{M} = \langle \mathsf{Q}, +, *, \leq;\ \mathsf{B}, \mathsf{Ob}, \mathsf{Ph};\ \mathsf{W}\rangle \models \mathbf{Specrel}_0$ be given, and define

$$J_{\mathfrak{M}} := \{\langle m, \mathsf{ev}_m(\overline{0}), \mathsf{ev}_m(\mathbf{1}_t), \ldots, \mathsf{ev}_m(\mathbf{1}_n)\rangle : m \in \mathsf{Ob}\} \cup \{\langle \mathsf{ph}, e_1, \ldots, e_{n+1}\rangle : \mathsf{ph} \in \mathsf{Ph}, e_1, \ldots, e_{n+1} \in \mathsf{Events}, \mathsf{ph} \in e_1, \ldots, \mathsf{ph} \in e_{n+1}\}, \quad \text{and the expansion of } \mathfrak{M}$$

$$F(\mathfrak{M}) := \langle \mathsf{Q}, +, *, \leq, 0, 1;\quad \mathsf{B}, \mathsf{Ob}, \mathsf{Ph};\quad \mathsf{Events}, \boldsymbol{\mu};\quad \mathsf{W}, J_{\mathfrak{M}}\rangle,$$

Th_3 is the set of all formulas valid in $\{F(\mathfrak{M}) : \mathfrak{M} \models \mathbf{Catrel}\}$.

Clearly, $J_{\mathfrak{M}}$ gives an "interpretation" of observers and photons in Events, and so makes a connection between the two "alien" sorts B and Events. The following are the main ideas in showing that Th_3 is a definitional expansion of Th_m (the theory of metric-geometries of **Catrel**). We have $\mathsf{Events}, \boldsymbol{\mu}, \mathsf{Q}, 1$ at our disposal and we have to "define" (or recover) $+, *, \leq, 0, \mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{W}, J$. First we define $0 := \boldsymbol{\mu}(e, e)$, then we define "lightlike collinearity" on Events by using $\boldsymbol{\mu}$ as follows: e_1, e_2, e_3 are *lightlike collinear* iff $\boldsymbol{\mu}(e_i, e_j) = 0$ for all $i, j = 1, 2, 3$. From lightlike collinearity then we define usual *collinearity* as in the proof of the Alexandrov-Zeeman theorem in [Gol87, App. 2], or in [AMN99], [AMN04]. For the idea of this part of the proof see Fig. 19. A proof for (a generalization of) the Alexandrov-Zeeman theorem using a different, elegant idea is in [Hor05]. A definability-theoretic analysis of the Alexandrov-Zeeman theorem in an axiomatic setting can be found in [Pam06]. From collinearity and $0, 1, \boldsymbol{\mu}$ we define the *field-operations* $+, *$ by using Hilbert's coordinatization technique (see e.g. [Gol87, pp. 23-27] or [AMN02, §6.5.2]). Since the original field was quadratic and ordered, we can recover the *ordering* $\leq$, too. From collinearity and $\boldsymbol{\mu}$ we can define the so-called relativistic (or Minkowski) *orthogonality* relation (see Fig. 22), and from $\boldsymbol{\mu}$ again then we can define the $n+1$-tuples of events $\langle e_0, e_1, \ldots, e_n\rangle$ that correspond exactly to $\langle \mathsf{ev}_m(\overline{0}), \mathsf{ev}_m(\mathbf{1}_t), \ldots, \mathsf{ev}_m(\mathbf{1}_n)\rangle$ for some observer m by requiring that $\boldsymbol{\mu}(e_0, e_i) = 1$ and e_0, e_i is orthogonal to e_0, e_j for all $i, j = 1, \ldots, n$, $i \neq j$. We can use these to define $\mathsf{Ob}, \mathsf{Ph}, \mathsf{B}, J$ and W. In the above we made use of the fact that all the constructions can be tracked with first-order logic formulas. Showing that Th_3 is a definitional expansion of **Catrel** is the easier direction, for a proof see [Mad02, p. 241]. **(ii):** In the above construction, we defined the operations of **Catrel** by using $\boldsymbol{\mu}, 1$, thus we can express the finitely many axioms defining **Catrel** by using $\boldsymbol{\mu}, 1$ and we are done. **(iii):** The functions ev_m and loc_m define isomorphisms between the structures $\langle \mathsf{Q}^n, \boldsymbol{\mu};\ \mathsf{Q}, 1\rangle$ and $\langle \mathsf{Events}, \boldsymbol{\mu};\ \mathsf{Q}, 1\rangle$. □

From the proof of Thm. 2.11 we actually can construct a finite theory Th_m axiomatizing the metric-geometries MG. This Th_m, however, is complicated and not really illuminating. It would be nice to find a streamlined, finite axiom system Th axiomatizing MG which contains few and easy-to-understand, illuminating axioms about $\boldsymbol{\mu}, 1$.

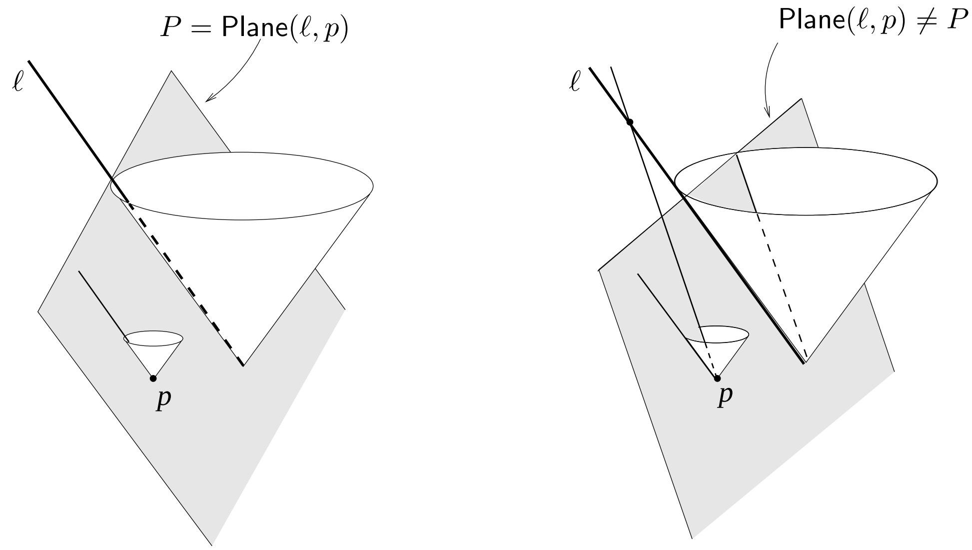


Figure 19: Collinearity can be defined from lightlike collinearity, as follows. Assume $n = 3$. Given a lightline ℓ, the plane P tangent to the light-cone and containing ℓ is the set of those points p through which no lightline intersecting ℓ goes. The reason for this is illustrated in the two parts of the figure. Then we get all spacelike lines as intersections of tangent planes. Then each timelike plane can be defined by a pair of intersecting lightlines and the spacelike lines connecting them; timelike lines then are the "new" intersections of timelike planes. In the above, spacelike, timelike lines, and lightlines are straight lines that lie outside, inside, and on the light-cone, respectively. A plane is timelike if it contains a timelike line. The case $n > 3$ is similar.

We called a property absolute if every observer "sees" it the same way. This "absolute" means also "observer-independent" or "coordinate-independent". Thm. 2.6 states that relativistic distance $\boldsymbol{\mu}$ is such an absolute property. Clearly, every formula expressible by the use of $\boldsymbol{\mu}, 1$ is absolute, too. The corollary below says that these are all the absolute coordinate properties of events, in **Specrel**. By a coordinate property of events we understand a property expressible in terms of the coordinates $\mathsf{loc}_m(e)$ of events e; more concretely by a coordinate property of events $e_1, \ldots, e_r$ we understand a formula either in the form $(\forall m \in \mathsf{Ob})\psi(\mathsf{loc}_m(e_1), \ldots, \mathsf{loc}_m(e_r))$ or in the form $(\exists m \in \mathsf{Ob})\psi(\mathsf{loc}_m(e_1), \ldots, \mathsf{loc}_m(e_r))$ where ψ is a formula in the vocabulary of the field-reduct $\langle Q, +, *, \leq\rangle^n$.

Corollary 2.4. *Every relation definable (in FOL) on* Events *in a model of* **Catrel** *can be defined from* $\boldsymbol{\mu}$ *and* 1. *Every coordinate-property of events in a model of* **Specrel** *can be defined from* $\boldsymbol{\mu}$ *and* 1. □

Thm. 2.11 can be interpreted as saying that metric-geometries are the observer-independent, "absolute realities" corresponding to models of **Specrel**. If we abstract from the concrete values of the metric-properties in the metric-geometries, we get the so-called causal-geometries, to be defined below. These correspond to the "absolute realities" of models of $\mathbf{Specrel}_0$. We are going to elaborate these ideas.

Let us call events e, e' causally separated, in symbols $e \sim_c e'$, iff there is either an observer or a photon that participates both in e and in e', i.e. iff $e \cap e' \cap (\mathsf{Ob} \cup \mathsf{Ph}) \neq \emptyset$. We call two space-time locations $p, q \in \mathsf{Q}^n$ causally separated, in symbols $p \sim_c q$, iff $\mathsf{time}(p, q) \geq \mathsf{space}(p, q)$, i.e. iff $(p_1 - q_1)^2 \geq (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2$. Visually, $p \sim_c q$ means that q is inside or on the light-cone emanating from p. In $\mathbf{Specrel}_0$ we have $\langle \mathsf{Events}, \sim_c\rangle \cong \langle \mathsf{Q}^n, \sim_c\rangle$, in fact loc_m and ev_m are isomorphisms between these structures, for any $m \in \mathsf{Ob}$.

Let $\mathfrak{M} \models \mathbf{Specrel}_0$ and define its causal-geometry as

$$\mathsf{Cg}(\mathfrak{M}) := \langle \mathsf{Events}, \sim_c\rangle.$$

Ants and elephants may use different units of measurement (e.g., their feet). The following axiom expresses that we abstract from the value of the units of measurement. We do so by requiring that all kinds of units of measurement be there. (In connection with the intuition/philosophy related to the following "ant-elephant" axiom cf. the Incredible Shrinking Man in [Nic82, pp. 194-5].)

AxDil $\quad (\forall m \in \mathsf{Ob})(\forall \lambda > 0)(\exists k \in \mathsf{Ob})(\forall p \in \mathsf{Q}^n)\, \mathsf{w}_{mk}(p) = \lambda p.$

$$\mathbf{Catrel}_0 := \mathbf{Specrel}_0 \cup \mathbf{Comp} \cup \{\mathsf{AxDil}\} = \mathbf{Catrel} - \{\mathsf{AxSim}\} \cup \{\mathsf{AxDil}\}.$$

Assume $\mathfrak{M} \models \mathbf{Catrel}_0$. Then every part of $\mathfrak{M}$ can be recovered from $\mathsf{Cg}(\mathfrak{M})$, except 0 and 1. By this we mean that $\mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{W}, \mathsf{Q}, \leq$ all can be defined (or recovered) from $\mathsf{Cg}(\mathfrak{M})$, but instead of $+, *$, which are not definable, we can define their affine ternary versions $+_3, *_3$ where

$$+_3(x,y,z) := x + y - z, \qquad *_3(x,y,z) := x * y/z.$$

Alternately, $\mathfrak{M}$ can be defined over $\mathsf{Cg}(\mathfrak{M})$ parametrically only, i.e. if we add two (arbitrary) constants to $\mathsf{Cg}(\mathfrak{M})$. So we have here an analogue of Thm. 2.11 working between $\mathbf{Catrel}_0$ and its causal-geometries of form $\mathsf{Cg}(\mathfrak{M})$. In particular, $\mathfrak{M}$ and $\mathsf{Cg}(\mathfrak{M})$ are definitionally equivalent in the parametric sense of definability.

Instead of stating the precise analogue of Thm. 2.11 for this intimate connection between causal-geometries and $\mathbf{Catrel}_0$, we turn to the question of what the definable relations in causal-geometries are.

Algebraic logic is a branch of logic that investigates the structure of the definable concepts in a theory, or in a model of a theory. Let us consider $\mathsf{Cg}(\mathfrak{M})$ for an arbitrary $\mathfrak{M} \models \mathbf{Catrel}_0$. The definable relations in $\mathsf{Cg}(\mathfrak{M})$ are exactly those absolute properties of events which do not involve concrete values of the metric $\boldsymbol{\mu}$. We will call these relations causal-relations.

The unary (i.e. one-place) causal-relations are Events and $\emptyset$. What are the binary (i.e. two-place) causal-relations?

We call events e, e' timelike (lightlike, spacelike) separated, in symbols $e \sim_t e'$ ($e \sim_\ell e'$, $e \sim_s e'$) iff [$e \neq e'$ and $e \cap e' \cap \mathsf{Ob} \neq \emptyset$ ($e \cap e' \cap \mathsf{Ph} \neq \emptyset$, $e \cap e' \cap (\mathsf{Ob} \cup \mathsf{Ph}) = \emptyset$, respectively)]. On the "coordinate-side", we call space-time locations p, q timelike (lightlike, spacelike) separated, in symbols $p \sim_t q$ ($p \sim_\ell q$, $p \sim_s q$) iff [$p \neq q$ and $\mathsf{time}(p,q) > \mathsf{space}(p,q)$ ($\mathsf{time}(p,q) = \mathsf{space}(p,q)$, $\mathsf{time}(p,q) < \mathsf{space}(p,q)$, respectively)]. These are corresponding properties via the bijections loc_m and ev_m for $m \in \mathsf{Ob}$, as before.

Timelike, lightlike, and spacelike separability of events are all causal-relations, i.e. they can be defined from $\sim_c$. In fact, all these four relations can be defined from each other.

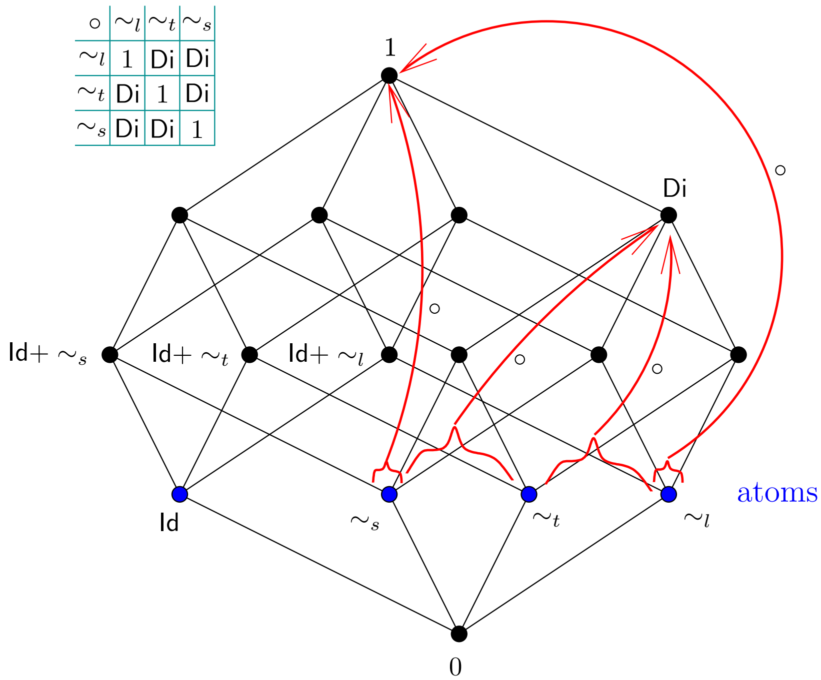
Below we sketch how $\sim_t$ can be defined from $\sim_c$. The argument is easier to follow in the isomorphic structure $\langle \mathsf{Q}^n, \sim_c \rangle$. The points causally separated from a point x are in the "solid" light-cone emanating from x. This light-cone consists of two separate parts, the "upward" and the "downward" parts. We cannot distinguish with a formula the two separate parts of the light-cone, but we can express that "y, z are in the same half-cone of x", in symbols $y \equiv_x z$, as follows.

$$y \equiv_x z \Leftrightarrow [x \sim_c y \wedge x \sim_c z \wedge \exists w (x \sim_c w \wedge \neg w \sim_c y \wedge \neg w \sim_c z)].$$

From this then we can define timelike separability as follows:

$$x \sim_t y \Leftrightarrow \exists zw(x \sim_c z \sim_c y \wedge \neg x \equiv_z y \wedge x \sim_c w \sim_c y \wedge \neg x \equiv_w y \wedge \neg z \sim_c w).$$

We used four variables in the above definition. By using 3 variables only, $\sim_t$ is not definable from $\sim_c$. This can be proved by using the techniques of algebraic logic, as follows. The four relations $\sim_t, \sim_\ell, \sim_s, \mathsf{Id}$ form the atoms of the Boolean algebra they generate. All these relations are symmetric, i.e. they are their own converses. Moreover, the relational composition of any distinct two is $\mathsf{Di} := -\mathsf{Id}$, while the relational composition of any non-identity one with itself is the unit $\mathsf{Events} \times \mathsf{Events}$ of the Boolean algebra. Hence they form a relation algebra. Relation algebras are introduced and briefly discussed in Ch. **??**, Sec. 2.3 (see also [HMT85], [HH02], [ANS01]). The elements of a concrete relation algebra are binary relations, and the operations are the Boolean ones together with relational composition of binary relations, taking converse of a binary relation, and the relation Id as a constant. The binary causal-relations then form a relation algebra. We



$\circ$	$\sim_l$	$\sim_t$	$\sim_s$
$\sim_l$	1	Di	Di
$\sim_t$	Di	1	Di
$\sim_s$	Di	Di	1

Figure 20: The binary causal-relations (in $\mathbf{Specrel}_0$) form a relation algebra.

can check that in this relation algebra $\sim_t$ is not in the subalgebra generated by $\sim_c$, hence $\sim_t$ cannot be defined from $\sim_c$ by using only three variables, by Tarski's theorem [TG87] or Ch. **??**, Prop. 2.4. That $\sim_t$ can be generated from $\sim_c$ by using 4 variables is equivalent to the fact that, in the so-called 4-dimensional cylindric algebra of 4-placed causal-relations, $\sim_t$ is indeed in the subalgebra generated by $\sim_c$. For cylindric algebras we refer to [HMT85], [HMT$^+$81], [ANS01].

One can prove that all the binary causal-relations are the ones occurring in the above relation algebra, which is represented in Fig. 20. Thus we have described all the binary causal-relations. Similar definability results for the special case $\mathsf{Q} =$"the rational numbers" are in [vB83, pp. 23-30].

There are infinitely many ternary (i.e. 3-place) causal-relations. The most often used ternary and 4-place causal relations are the following ones (again, it is easier to define their "coordinate-versions" in $\langle \mathsf{Q}^n, \sim_c \rangle$).

Collinearity of 3 space-time-locations is a causal relation, let $\mathsf{coll}\,(p, q, r)$ denote that p, q, r are collinear, i.e. they lie on a straight line.

Betweenness: $\mathsf{Bw}\,(p, q, r)$ iff "$\mathsf{coll}(p, q, r)$ and q is between p and r".

Equidistance: $\mathsf{Eq}\,(p, q, r, s)$ iff "$\boldsymbol{\mu}(p, q) = \boldsymbol{\mu}(r, s)$". Minkowski-circles (or Minkowski-spheres) can be defined from equidistance, cf. Fig. 21.

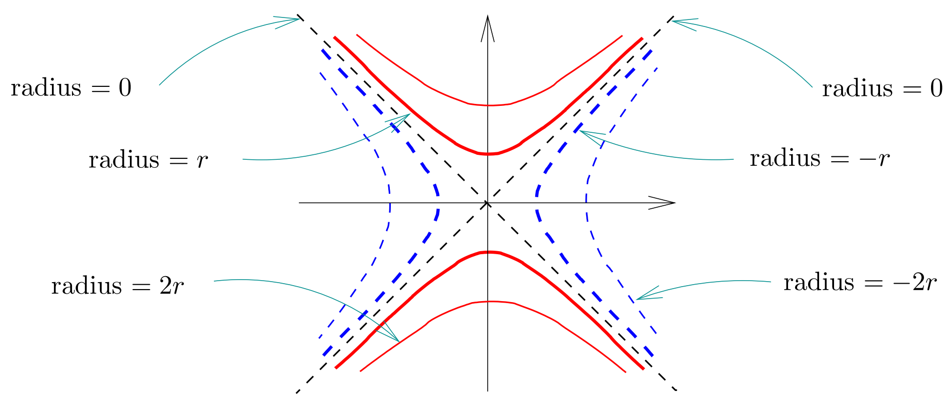


Figure 21: Minkowski circles.

Orthogonality: $\mathsf{Ort}\,(p, q, r, s)$ iff "the line connecting p, q is Minkowski-orthogonal to the line connecting r, s", i.e. iff $(p_1-q_1)(r_1-s_1)-(p_2-q_2)(r_2-s_2) - \cdots - (p_n - q_n)(r_n - s_n) = 0$. For illustration see Fig. 22.

Betweenness with ratio ρ: Let ρ be any rational number. Then $\mathsf{Bw}_\rho(p, q, r)$ iff "$\mathsf{Bw}(p, q, r)$ and $\boldsymbol{\mu}(p, q) = \rho * \boldsymbol{\mu}(q, r)$".

The above are all definable from $\sim_c$. The last example shows that there are infinitely many ternary causal-relations.

If we have time-orientation, i.e. in models of $\mathbf{Specrel}_0 + \mathsf{Ax}\!\uparrow$, the following important relation can also be defined. Event e causally precedes event e', in symbols $e \prec_c e'$ iff ($e \sim_c e'$ and $(\exists m \in \mathsf{Ob})\mathsf{time}_m(e) \le \mathsf{time}_m(e')$). In $\mathbf{Specrel}_0 + \mathsf{Ax}\!\uparrow$, $e \prec_c e'$ is equivalent with the simpler formula $(\forall m \in \mathsf{Ob})\mathsf{time}_m(e) \le \mathsf{time}_m(e')$, assuming $n > 2$. It can be proved (analogously to Cor. 2.1) that the corresponding property in space-time locations is: $p \prec_c q$ iff ($p \sim_c q$ and $p_t \le q_t$). $\prec_c$ is also called causality relation, or after, the first axiomatization of special relativity in [Rob14] axiomatized this causality relation. The general relativistic version of $\prec_c$ is quite important, too, and is more intricate than the $\mathbf{Specrel}_0$ version; cf., e.g., works of Penrose, Malament, Buseman.

Finally, we list some absolute relations that can be defined in $\mathsf{Mg}(\mathfrak{M})$, but cannot be defined in $\mathsf{Cg}(\mathfrak{M})$, for $\mathfrak{M} \in \mathbf{Catrel}$. Clearly, $\boldsymbol{\mu}$ is such.

Minkowski scalar-product: $\mathsf{g}_4(p,q,r,s) := (p_1 - q_1) * (r_1 - s_1) - (p_2 - q_2) * (r_2 - s_2) - \cdots - (p_n - q_n) * (r_n - s_n)$. We note that $\mathsf{g}_4(p,q,p,q) = \boldsymbol{\mu}(p,q)$ and $\mathsf{g}_4(p,q,r,s) = 0$ iff $\mathsf{Ort}(p,q,r,s)$. Hence, g_4 "codes" both Minkowski-distance and "relativistic angle".

Relativistic (non-squared) distance of causally separated points:
$\mathsf{rd}(p,q) := \sqrt{(p_1 - q_1)^2 - (p_2 - q_2)^2 - \cdots - (p_n - q_n)^2}$. This is a partial function defined exactly when the expression in the argument of the square root is nonnegative, i.e. when $p \sim_c q$. $\mathsf{rd}(e,e') > 0$ means proper time elapsed between e and e', for any observer who takes part in both e and e' (and that there is such an observer), i.e. any observer who takes part in both e, e' measures that the elapsed time between e and e' as $\mathsf{rd}(e,e')$ (and there is such an observer).

We note that $\boldsymbol{\mu}, \mathsf{g}_4, \mathsf{rd}$ are definable from each other. Actually, we will make use of this in our section on general relativity, in Sec. 3.4, where we will use the binary version g of g_4 which is defined by

$$\mathsf{g}(p,q) := \mathsf{g}_4(p, \overline{0}, q, \overline{0}).$$

For more concrete definitions and for intuition for the above relations we refer to e.g. [Gol87], or [Mad02, §4.2].

By Thm. 2.11 and the analogue for $\mathbf{Catrel}_0$, if we want to study special relativity in the form of $\mathbf{Catrel}$ or $\mathbf{Catrel}_0$, then this can be done *equivalently* by studying the simple metric-geometries and causal-geometries $\langle \mathsf{Events}, \boldsymbol{\mu}; \mathsf{Q}, 1\rangle$ and $\langle \mathsf{Events}, \sim_c\rangle$, using their finitely axiomatized theories Th_m and Th_c respectively. (A finite Th_c can be found in [Lat72].) The possibility of switching to the geometries instead of the original models

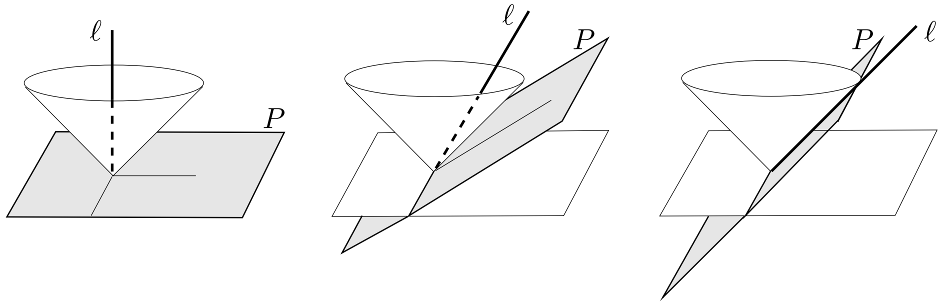


Figure 22: ℓ is Minkowski-orthogonal to each straight line in P.

(without losing information) is useful e.g. because the transition from special relativity to general relativity (GR) is quite smooth on the level of geometries. The study of GR on this causal-geometric (axiomatic) level of abstraction is promoted e.g. in [KP67]. [Bus67] uses the generalization of rd in his approach to general relativity space-times. The study of GR on the metric-geometric level using the generalization of g is most common, see e.g. [Wal84], [HE73], [Rin01] (but in this "g-oriented" cases the linguistic economy of the Kronheimer-Penrose approach is usually sacrificed).

One of the most useful and most interesting branches of mathematical logic is, in our opinion, definability theory. Definability theory is strongly related to relativity theory, in fact its existence was initiated by Hans Reichenbach in 1924 [Rei69] motivated by relativity theory. Reichenbach in his works emphasized the need of definability theory and made the first steps in creating it. It was Alfred Tarski who later (1930) founded and established this branch of mathematical logic.

Very briefly, the reason for the need of definability theory (of logic) in relativity theory is as follows. When one sets up a physical theory Th, one wants to use only so-called observational concepts, like e.g., "meeting of two particles".[7] While investigating the theory Th, one defines new, so-

[7] The concepts potentially usable in scientific theories (such as e.g. relativity) have been partially ordered in the literature as being more observable (and less "theoretical") or less observable and more theoretical. Here "observable" also means primary or empirical. This observable/theoretical distinction, or rather hierarchy, is recalled from the literature (of relativity theory) in e.g. [Fri83, pp. 4–5]. This observable/theoretical hierarchy is not perfectly well defined and is known to be problematic, but as Friedman puts it, it is still better than nothing. E.g. the motion of the oceans called tides are more observable (or closer to be observable) than the pull of gravity of the Moon which, we think, is causing them. That is, the gravitational force field of a mass-point (like the moon) is a more

called "theoretical" concepts, like e.g. "relativistic distance of events". Some defined concepts then prove to be so useful that one builds a new theory Th' based on the most useful theoretical concepts, and investigates this new theory Th' in its own merits.

The new theory Th' usually is simple, streamlined, elegant - built so that we satisfy our aesthetic desires. The original theory Th contains its own interpretation, because we defined it so. The physical interpretation of the new streamlined theory Th' is provided by its connection with Th. The strongest known relationship between two theories is definitional equivalence. When Th and Th' are definitionally equivalent, in the rigorous sense of definability theory of first-order logic, the observational oriented theory Th can be recaptured completely from the theoretical-oriented streamlined theory Th'; and vice versa, the theoretical concepts of Th' can be defined (justified) over the observational Th. Looser relationships between Th and Th' are also very useful, these kinds of relationships between theories are called interpretability and duality theories. Cf. [vB82] for more on logic, definability theory, model theory for empirical theories.

In Sec.s 2.1-2.3 when we formalized special relativity in first-order logic, we tried to choose the basic concepts of our language as observational as possible; and we introduced the more theoretical concepts of relativity as definitions at later stages, when development of the theory justified them. Eventually, in Sec. 2.6, this process led to the introduction of a new theory with new basic concepts (new vocabulary, like $\sim_c$, $\prec_c$). This is a natural way of theory development, theory "understanding", theory analysis. In modern approaches to logic, theories are considered as *dynamic objects* as opposed to the more classical "eternally frozen" idea of theories. For approaches to the dynamic trend in mathematical logic cf. [vB96].

Theories form a rich structure when we investigate their interconnections. Algebraic logic establishes a duality between hierarchies of theories (on different vocabularies) and between classes of algebras, cf. e.g., [HMT85, §4.3] or [ANS01, Part II]. Investigating a theory via investigating the hierarchy of its different perspectives and subtheories is like investigating a 3-dimensional object from all sides. This leads us to the subject of the next

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theoretical concept than the motion of a body (e.g. ocean's shore-line). Actually the gravitational force field might turn out to be a "wrong concept" and we may have to replace it with something else like the curvature of space-time. Probably the motion of the ocean's shore-line will be less questionable as a "something" which one can talk about. As [Fri83, p. 4] points out, the observational/theoretical distinction is not an absolute one. E.g. what is an observational concept at a certain stage of theory development might turn out to be a theoretical one later.

subsection.

2.7 Conceptual analysis and "reverse relativity"

In the previous subsections we strengthened **Specrel** to a complete theory **Catrel**+RC. But in reality, when working with a theory, we do not want to make our axioms generate a complete theory. Our purpose is just the opposite: we want to make our axioms as weak, simple, and intuitively acceptable and convincing as possible while still strong enough for proving interesting theorems of relativity theory. Similar striving for economy of assumptions is e.g. in [Sza06],[Sza02], [Ax78]. The reasons for wanting to study weak theories as opposed to strong ones are, among others, the desire for answering the "why-type questions", and seeking a conceptual analysis of the theory. For more on this we refer to [AMN02, §1.1]. Further reasons for striving for weak physical theories having many models are presented in [vB82]. Namely, e.g. "small" mechanical systems like our solar system or another one or our galaxy can be regarded as many different "small models" of e.g. Newtonian mechanics.

Among other things, we can use logic to find out which axioms are responsible for certain surprising predictions of relativity theory like e.g. "no observer can move faster than the speed of light", "the twin paradox" or issues concerning the possibility of time travel. We can call such studies "reverse relativity" alluding to the analogy with the highly successful direction called reverse mathematics, cf. e.g. [Sim05], [Fri04].

In reverse relativity, we single out an interesting prediction of relativity theory like "observers cannot move faster than light (NoFTL)", or one of our paradigmatic effects in Thm. 2.4 and ask ourselves which axioms (of e.g. **Specrel**) are responsible for the prediction in question, cf. Thm. 2.5. Let φ denote the prediction in question. So typically we know that **Specrel** $\vdash \varphi$ and that φ "is interesting". Then we ask ourselves whether the whole of **Specrel** is needed for proving φ. (Recall, the axioms of **Specrel** are "assumptions", hence they cost money so to speak.) A further question is to ask which fragment of **Specrel** is needed/sufficient for proving φ. This type of research has been carried through for several interesting choices of φ, e.g. in [AMN02], [Mad02], [AMN04].

Let us take as an example for φ the prediction NoFTL (i.e. that no observer can move faster-than-light relative to another) established as Thm. 2.5(ii), p. 36. Certainly, NoFTL is an interesting prediction, indeed, many thinkers tried to get rid of NoFTL either by using "tachions" or by circumnavigating it by using wormholes, cf. Sec. 4 for the latter. An instructive "saga" of such efforts is provided by relativist Kip Thorne in [Tho94]. The analysis of NoFTL tells us that **Specrel** can be considerably weakened without losing NoFTL. By Thm. 2.5 in this work, the assumption $n > 2$ is needed, however. The two key axioms of **Specrel** are the Light Axiom, AxPh, and AxEvent (in some sense). It turns out that both of these are needed for NoFTL. However, both of them can be weakened considerably without losing NoFTL. In case of AxPh, isotropy is not needed for NoFTL. Of AxPh, it is enough to assume that photons are not like bullets, they do not race with each other, and they can be sent from each point in each direction; i.e. that for any observer m in each direction d there is a number $c_m(d) \in \mathsf{Q}$ representing the speed of light for m in direction d. Of AxEvent, it is enough to assume that if m sees an event on the worldline of k, then k also sees that event; and that if m sees an event that k sees then m sees all events in a neighborhood (in k's coordinate system) of this event. Some reflection reveals that this is a more natural, milder assumption than AxEvent was. As it turns out, the rest of the axioms of $\mathbf{Specrel}_0$ can also be weakened without losing NoFTL.

Careful analysis of the noFTL prediction can be found in [Mad02, 2.8.25, 3.2.13, 3.2.14], [MNT04, Thm. 3, Thm. 5], [AMN02]. Similar pieces of conceptual analysis, analysing predictions similarly interesting (like NoFTL) can be found in [AMN02, §4.2], [AMN04, MNT04, MNS06b, MNS06a]. Predictions that have been analyzed in these works include the twin paradox, the paradigmatic effects, the effect of gravity on clocks.

3 General relativistic space-time

In this section we extend our logic-based study of relativity from special relativity to general relativistic space-time (GR space-time). In particular, in Sec. 3.6 we present a purely first-order logic axiomatization **Genrel** for GR space-time. Thm. 3.3 is a kind of completeness theorem for **Genrel**. Besides providing a first-order logic axiomatization of GR space-times (analogously to Sec. 2) and comparing it with **Specrel**, we will put extra emphasis on discussing the exotic properties of various distinguished examples of GR space-times in Sec. 4. One of the reasons for this is that these exotic GR space-times are at the center of attention nowadays, e.g. because of their fantastic properties and because astronomers have been discovering examples of these, e.g. finding observational evidence for huge black holes in the last 15 years. Another reason for putting emphasis on examples is that while **Specrel** has basically one intended model, general relativity (GR) has many different intended models (e.g. various kinds of exotic black holes, wormholes, timewarps, models for the expanding universe, the Big Bang, to mention a few). This contrast between special relativity and GR motivates our shifting the emphasis to distinguished models in what comes below. This kind of motivation is further elaborated in the book [TW00].

A motivation for the logical analysis of GR is that, in principle, GR space-times permit such counter-common-sense arrangements as is time travel (in one form or another). This was discovered by Kurt Gödel during his co-operation with Einstein. But the so-called paradoxes of time travel offer themselves for a logical analysis, since these kinds of circularity are the "bread-and-butter" of the logician ever since Gödel's incompleteness proof or since the first logical analysis of the liar paradox and its variants. Even if we would want to exclude time travel by some axiom like one or another form of the so-called Cosmic Censor Hypothesis, it remains a question how to find and justify a natural axiom to this effect without making unjustified assumptions. This dilemma is illustrated by the debates about the various forms of the Cosmic Censor Hypothesis and related assumptions discussed e.g. in [Ear95].

3.1 Transition to general relativity: accelerated observers in special relativity

In **Specrel**, we restricted attention to inertial observers. It is a natural idea to generalize the theory to including accelerated observers as well. Actually, when creating general relativity, Einstein emphasized that accelerated observers should be included, cf. [Ein61, pp. 59-62]. Indeed, the usual transition from special relativity to the general theory of relativity goes as follows. First special relativity is generalized to accommodate accelerated observers, and then one introduces Einstein's principle of equivalence (EPE) which states that the phenomena of acceleration and gravity are equivalent (in a carefully specified concrete sense). Then, at this point, our language is rich enough to talk about gravity in the form of acceleration. After this point, one refines the theory, arriving at GR, and then it all hangs together to form a worldview broader than special relativity and also broader than Newtonian gravitation theory. The above is illustrated by e.g. [Ein61], the classic general relativity book [MTW70, pp. 163-165], [Rin01, e.g. p. 72, §3.8, §12.4, pp. 267-272]. Even works intending to venture to the unknown beyond GR use the above "methodology" of starting by accelerated observers, cf. e.g. [Smo01, pp. 77-80].

The same is done in the research area reported in the present work. Namely, in Sec. 2 and in related works, the logical analysis of special relativity is done, yielding e.g. the hierarchy of theories containing **Specrel**$_0$, **Specrel**, **Catrel**. The next stage extends **Specrel** by considering new kinds of entities called accelerated observers and stating further axioms governing their behavior. This yields a new theory **Accrel** which can be regarded as an extension and refinement of **Specrel**. Gravity can be studied in **Accrel** in form of acceleration; this is done e.g. in [MNS06a], in the spirit outlined above. The works [AMN06b], [MNS06b], [MNS06a] which study **Accrel** stay inside the purely first-order logic based approach represented by **Specrel** in Sec. 2 in this work. Using the experience and motivation gained by studying **Accrel**, in Sec. 3.6 we introduce a first-order logic theory **Genrel** for general relativistic space-time. All this converges to a logical analysis of GR.

Instead of recalling **Accrel**, which is very similar in spirit to the axiom system **Genrel** in Sec. 3.6, we summarize its main features relevant to **Genrel**. In **Accrel**, "accelerated observer" means "not necessarily inertial observer", and "observer" means a body that has a worldview, i.e. which occurs in the domain of the worldview relation W. Thus, an accelerated observer has a worldview. Roughly, the worldview of an accelerated observer k is obtained from the worldview of an inertial one, m, by re-coordinatizing it along a smooth bijection w_{mk} with an open subset of Q^n as its domain. Thus k may use only part of Q^n for coordinatizing events, and more importantly, the worldlines of inertial observers and photons are no longer straight lines in an accelerated observer's worldview. I.e., AxEvent and AxLine cease to hold. They hold in generalized, weaker forms only. Specifically, an accelerated observer can recognize worldlines of inertial bodies as so-called "geodesic curves", this is the motivation for Sec. 3.3 and for the axiom AxLine^- in **Genrel**.

The key axiom of accelerated observers states that at each moment of his life, each accelerated observer sees the nearby world for a short while as an inertial observer does. Technically, in **Accrel** we formulate this as stating that at each moment of the life of an accelerated observer k there is a so-called co-moving inertial observer m such that the linear approximation (i.e. the differential) of the worldview transformation w_{mk} at this space-time point is the identity function. This axiom, called AxAcc in the quoted works, is the connecting point between the worldviews of inertial and non-inertial observers. The counterpart of AxAcc in the present work will be discussed next.

We can think of an accelerated observer in special relativity as a spaceship which uses fuel for accelerating (in a space where there is no gravity). When the drive is switched off, the ship will transform into an inertial ship—this is the co-moving inertial observer at the event of switching off the ship-drive. Or, equivalently, we can think of the co-moving inertial observer as a spaceprobe which was let go from the ship—a metaphorical apple dropped in space. The ship can measure its own acceleration by measuring the acceleration of the spaceprobe; just as here on Earth we can measure gravity by measuring the acceleration of a dropped apple. EPE then implies that the spaceship can interpret "falling of the metaphorical apples" either by thinking that he is accelerating in an empty space, or by thinking that he is suspended in a space where there is gravity, and dropped apples fall because they are no longer suspended. By EPE, the worldview of an accelerated observer in special relativity is similar to a "suspended" observer in a space-time where there is gravity. The gravitational counterpart, by EPE, of AxAcc is Einstein's Locally Special Relativity Principle which we recall at the beginning of Sec. 3.2; it will be our starting point in defining GR space-times.

Summing up: by EPE, investigation of gravity can be reduced to the investigation of the worldlines of inertial bodies in a GR space-time.

Let us turn to the reasons of why the transition from special relativity to general relativity goes via accelerated observers and EPE. In Sec. 2, we chose to derive special relativity from the outcome of the famous Michelson-Morley experiment, i.e. from the Light Axiom. However, as we already mentioned, relying on the Light Axiom is not really necessary. As Einstein always emphasized (e.g. in [Ein61]), relativity can be derived from a deep philosophical principle called the special principle of relativity (SPR). SPR has been around in our culture for 2500 years (roughly), hence it is well understood and it blends nicely with our best understanding of the world. Roughly, SPR says that the Laws of Nature are the same for all inertial observers. The modern form of SPR was articulated by the Normann-French Nicole d'Oresme around 1300 (Paris) and (a bit more thoroughly) by Galileo Galilei (around 1600). After Olaf Roemer, James Bradley and followers discovered that the speed of light is finite (and related issues were clarified), SPR could have been used[8] to show inconsistency with the Newtonian worldview and then to derive special relativity (analogously to the train of thought we used in Sec. 2). This in turn would have predicted the outcome of the Michelson-Morley experiment.[9] Einstein elaborated this idea in detail, and in particular, emphasized that special relativity can be derived from SPR (in

[8] With hindsight, this possibility was there around the 1830's or so. Roemer made the discovery around 1680, but it was not generally accepted until 1750 approx.

[9] Of course, for this, light propagation needs to be regarded as Law of Nature, but as Einstein points out, this is absolutely natural.

place of the Light Axiom). The same kind of philosophical taste lead Einstein to asking why SPR is restricted to inertial observers only. Why aren't the Laws of Nature the same for all observers (not only the inertial ones)? After all, we ourselves sitting on the surface of the Earth (and fighting gravity all the time) are not inertial observers according to the definition used in SPR.

So, Einstein started working towards GR by generalizing SPR to the general principle of relativity (PR) which says that the Laws of Nature are the same for all observers, including accelerated ones. This move creates some extra tasks to handle, because accelerated observers experience the existence of a new "force-field", namely gravity. So Einstein introduced his principle of equivalence EPE unifying acceleration created by gravity with "ordinary gravity". Now, to uphold PR we have to regard gravity (and light propagation of course) as part of what constitute Laws of Nature. This creates some extra tasks (mentioned above) since in **Specrel** properties of gravity were not part of the picture. As we will see in the next section, this extra work can be handled leading to a unification broader than that provided by special relativity. The new theory GR unifies space, time, motion, light-propagation, and gravitation into a single purely geometrical perspective.

3.2 Einstein's "locally special relativity principle"

Einstein's locally special relativity principle saying that General Relativity is locally Special Relativity is the following. Let p be a point in a GR space-time. Then if we drop a small enough spaceship, put an experimental scientist in the spaceship who lives for a short enough time, the experimentalist will find special relativity true in the spaceship. This holds true even on the event horizon of a spinning black hole or wherever you want. Of course, it is crucial that the spaceship is small enough and that its life is considered only for a small enough time interval. This is Einstein's locally special relativity principle. See Fig. 23. These local tiny spaceships will appear later as "local reference frames" LFR's. They play the same role in GR as co-moving observers did in AxAcc.

Next we implement Einstein's locally special relativity principle formulated above for formalizing GR space-times. In this and in the next few subsections, for simplicity, we will use $\mathbb{R}$ in place of an arbitrary linearly ordered quadratic field Q, and also we will use $n = 4$. Later, in Sec. 3.6 we will return to the generality of Q and $n \geq 2$.

For general relativity, we will use global coordinate frames, GFR's. A global coordinate frame is based on an open subset of $\mathbb{R}^4$. So a global

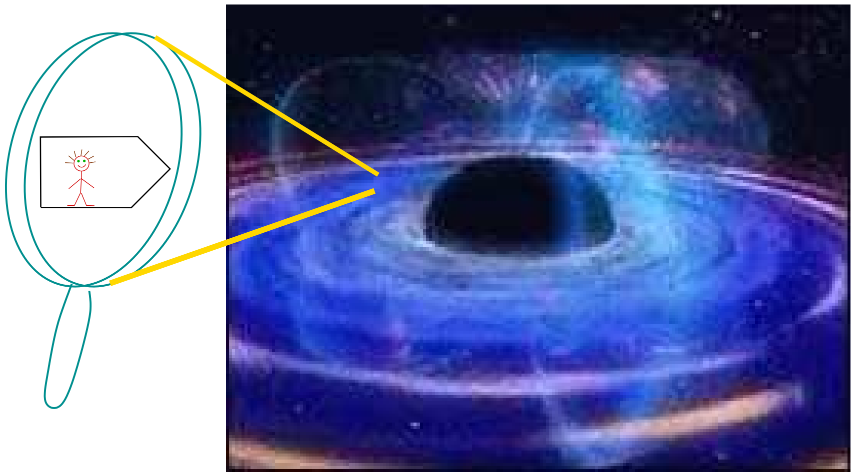

Figure 23: Einstein's locally special relativity principle: where-ever we drop a small enough spaceship, for a short enough time it will experience special relativity.

coordinate frame looks like a special relativity frame, we even call one of the coordinates time, the others space etc. The difference is that in a global frame the coordinates do not carry any physical or intuitive meaning.[10] They serve only as a matter of convention in gluing the local special relativity frames, LFR 's, together. For simplicity, at the beginning we will pretend that the coordinate system of our global frame is the whole of $\mathbb{R}^4$. Later we will refine this to saying that the global frame is an open subset of $\mathbb{R}^4$. And even later, in Sec. 3.6, we will generalize this to be a manifold. Since the differences are extremely minor and secondary from the point of view of the basic notions we are going to introduce now, let us first pretend that the global frame is $\mathbb{R}^4$.

Imagine a general relativistic coordinate system, a GFR, representing the whole universe, with a black hole in the middle etc. So we are looking at the bare coordinate grid of $\mathbb{R}^4$ intending to represent the whole of space-time. What is the first thing we want to specify for our readers about the points of this grid $\mathbb{R}^4$? Well, it is how the local tiny little special relativistic space-times are associated to the points p of $\mathbb{R}^4$, in accordance with Einstein's locally special relativity principle formulated at the beginning of this subsection. Thus, to every point p of $\mathbb{R}^4$ we want to specify how the

[10]To be precise, the topology of the global frame will be relevant.

local special relativity space-time at point p is squeezed into the local neighborhood of p. The point is in specifying how the clocks of the LFR slow down or speed up at p, and which axis of the local LFR points in what direction and is distorted (shortened/lengthened) in what degree. The LFR at p corresponds to the metaphorical spaceship dropped at p as in Fig. 23.

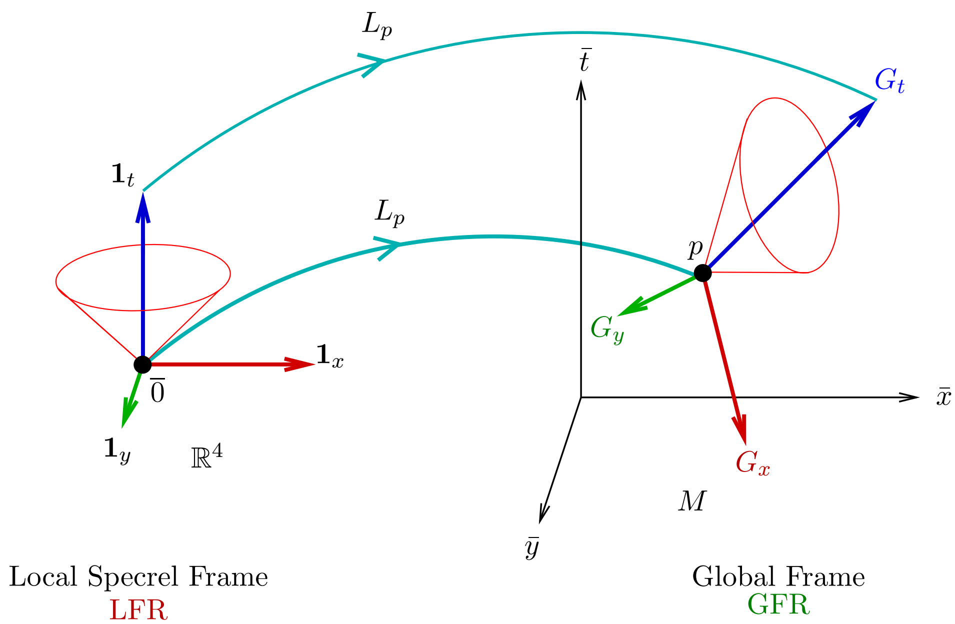

Figure 24: The local frame at p is an affine mapping L_p of $\mathbb{R}^4$ to $\mathbb{R}^4$ taking $\overline{0}$ to p. We will use L_p in small neighborhoods of $\overline{0}$ only.

How do we specify the local frames LFR's? A local frame L_p at p will be a bijective mapping $L_p : \mathbb{R}^4 \to \mathbb{R}^4$ such that $L_p(\overline{0}) = p$. We will think of the first $\mathbb{R}^4$ as the coordinate system of special relativity, or of the Minkowski space represented by LFR, and of the second $\mathbb{R}^4$ as the global frame GFR upon which we want to build our GR space-time. Thus we write L_p : LFR $\to$ GFR, where formally $\text{LFR} := \text{GFR} := \mathbb{R}^4$. Since we want to use our local frames to specify how the tiny clocks slow down in the "linear limit" (roughly, in an "infinitesimally small" neighborhood of p), we will choose these L_p's to be affine mappings.

So, the key device in building our GR space-time is associating to each point p of our global coordinate grid GFR an affine transformation L_p mapping the Minkowski space represented by LFR to the global frame GFR.

Definition 3.1. (general relativistic space-time) *By a general relativistic space-time we understand a pair* $\langle M, L\rangle$ *where*

$M \subseteq \mathbb{R}^4$ *is open and*

L *is a function defined on* M *such that* $L(p) : \mathbb{R}^4 \to \mathbb{R}^4$ *is a bijective affine mapping which takes the origin* $\overline{0}$ *to* p*, for each* $p \in M$*. Further, we require that* L *(as a function of* $p \in \mathbb{R}^4$*) be infinitely many times continuously differentiable, in short smooth.*

For better readability, we will write L_p to denote $L(p)$. We will use local special relativity frames (the LFR's) for importing the notions of special relativity to our GR space-times. We will use the inverse mapping L_p^{-1} of the affine transformation L_p to translate our general relativistic problems to special relativity, and we will use L_p to bring back the answers special relativity gives us. Though L_p^{-1} is defined on the whole of $\mathbb{R}^4$, we will use it only in small enough neighborhoods of p (i.e. we will use it in the limit, more and more accurately as we close on p). Restricting attention to small neighborhoods of p is what is meant by saying that GR can be *locally* reduced to special relativity, but only locally. If we want to solve a problem at a point q farther away from p , then we will have to use the mapping L_q associated to q in place of using L_p.

We can specify the local frame L_p by the images of the four unit-vectors $\mathbf{1}_i$, as follows: $G(p) = \langle G_t(p), ..., G_z(p)\rangle$ is a 4-tuple of vectors (i.e. elements of $\mathbb{R}^4$) such that $L_p : \mathbb{R}^4 \to \mathbb{R}^4$ is the affine transformation which maps the origin $\overline{0}$ to p, and $\mathbf{1}_i$ to $G_i(p)+p$, for $i \in \{t, x, y, z\}$, see Fig. 24. Thus, we can specify a GR space-time $\langle M, L\rangle$ by simply specifying four vector-fields. Here we use the word "field" as in analysis and not as in algebra. Thus "field" in "vector-field" means that we have a vector at each point of $M \subseteq \mathbb{R}^4$. This gives us the following equivalent definition for a GR space-time.

Definition 3.2. (general relativistic space-time in vector-fields form) *By a general relativistic space-time in vector-fields form we understand a four-tuple* $\langle G_t, G_x, G_y, G_z\rangle$ *of vector-fields such that*

each $G_i : M \to \mathbb{R}^4$ *is smooth, where* $M = \mathsf{Dom}(G_i) \subseteq \mathbb{R}^4$ *is open (*$i \in \{t, x, y, z\}$*) and*

the vectors $G_t(p), ..., G_z(p)$ *at each point* $p \in M$ *are linearly independent in the usual sense. (This means that the affine mapping they specify is a bijection.)*

An advantage of Def. 3.2 is that it is simple, and that it is in this form that we can "draw" or visualize a general relativistic space-time. See Fig.s 30, 31, 33, 40, 34.

Now, how do we use a GR space-time, i.e. such a 4-tuple of vector-fields, for representing some aspects of reality? Very, very roughly, the information content of a GR space-time $G = \langle G_t, ..., G_z \rangle$ can be visualized as follows. The vector tetrad $G_t(p), ..., G_z(p)$ at point p tells us how the measuring instruments (clocks, meter-rods) of the tiny little inertial observer we imagine as being dropped at p go crazy (go wrong) from the point of view of the big, global general relativistic coordinate grid GFR we are using in G. This information is very subjective, since as we said, the big global coordinate grid carries no physical meaning, it is conventional. But some objective content can be extracted from this subjective information. As we said, for a fixed p, the vector tetrad $G_t(p), ..., G_z(p)$ wants to represent how the local frame is glued into the holistic picture of the global frame. The important point is, however, how the individual local frames are distorted, rotated etc w.r.t. *each other*, the big global frame grid is only a theoretical, conventional device to serve as a common denominator in arranging the little local frames relative to each other.

What can be described in terms of the 4-tuple $\langle G_t, ..., G_z \rangle$ of vector-fields? Well, we can describe the (potential) worldlines (parameterized with wristwatch times) of inertial observers and the worldlines of photons. We will see that knowing what the potential worldlines of inertial observers and photons are tells us everything important about a GR space-time. Gravity, curvature can be defined explicitly from knowing the above mentioned worldlines.

3.3 Worldlines of inertial observers and photons in a general relativistic space-time

The worldlines of inertial observers will be described mathematically as time-like geodesic curves. We now turn to defining these. In this subsection we fix a general relativistic space-time $\langle M, L \rangle$, and we let

$\mathrm{GFR} := M, \quad \mathrm{LFR} := \mathbb{R}^4$.

By a (smooth) curve f we understand a smooth mapping $f : I \to \mathbb{R}^4$, where I is an open interval of $\mathbb{R}$. By a point of the curve we mean a point in its range.

Intuitively, the curve $f : I \to \mathrm{GFR}$ is called timelike at point p iff the local frame at p "sees" an observer co-moving with the curve at p. In more detail, the curve f is called timelike at p iff the speed of f as seen by the local frame at p is smaller than 1. This means that the tangent of the curve $L_p^{-1} \circ f$ has slope smaller than 1, at the origin; geometrically this means that the tangent straight line at the origin is within the light-cone. The curve f is called timelike iff the curve f is timelike at each of its points p . (Note: this is independent of how the curve f is parameterized.) See Fig. 25.

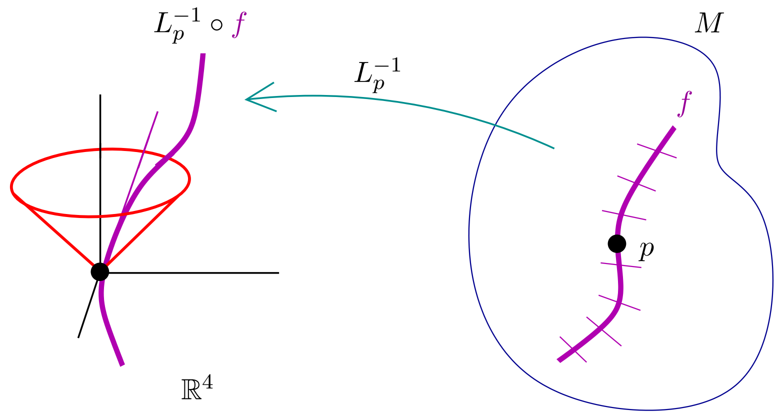

Figure 25: The curve f is timelike at point p.

Note that talking about the tangent of $L_p^{-1} \circ f : I \to$ LFR involves nothing "fancy", since (at this step) we are in a special relativity space-time and we are using its Euclidean geometry over $\mathbb{R}^4$ and we are looking at a smooth curve in it.

We think of a timelike curve as a curve that in principle can be the worldline of a (perhaps accelerated) observer.

When is a timelike curve $f : I \to$ GFR called a timelike geodesic? First we have to check whether the curve f represents (or measures) relativistic time correctly. Here, the parametrization will be important. In what follows, $[a, b]$ denotes the closed interval of $\mathbb{R}$ with endpoints a, b, i.e. $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$. For the definition of relativistic distance $\mathsf{rd} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ see Sec. 2.6, p. 56.

Definition 3.3. *We say that f represents time correctly if the following statement holds. For every $t \in I$, and for every positive ε there is a positive δ such that for all $s \in [t - \delta, t + \delta]$ it holds that $|s - t|$ agrees with what $\mathsf{rd}(f(s), f(t))$ is as measured by the local frame determined by $L_{f(t)}$ up to an error bound by $\varepsilon * |s - t|$. Here "$\mathsf{rd}(f(s), f(t))$ as measured by the local frame" is the relativistic (Minkowski) distance between $L_p^{-1}(f(s))$ and $L_p^{-1}(f(t))$ understood in special relativity.) Formally this is:*

$$(\forall t \in I)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall s \in [t-\delta, t+\delta])$$
$$|\mathsf{rd}(L_p^{-1}(f(s)), L_p^{-1}(f(t))) - |s-t|| < \varepsilon * |s-t|.$$

We call a curve time-faithful iff it is timelike and represents relativistic time correctly. Intuitively, a timelike curve is time-faithful iff at each point p of the curve, the local frame at p "sees" an observer co-moving with the curve such that the parametrization of the curve "agrees" with how time passes for this co-moving observer. See Fig. 26.

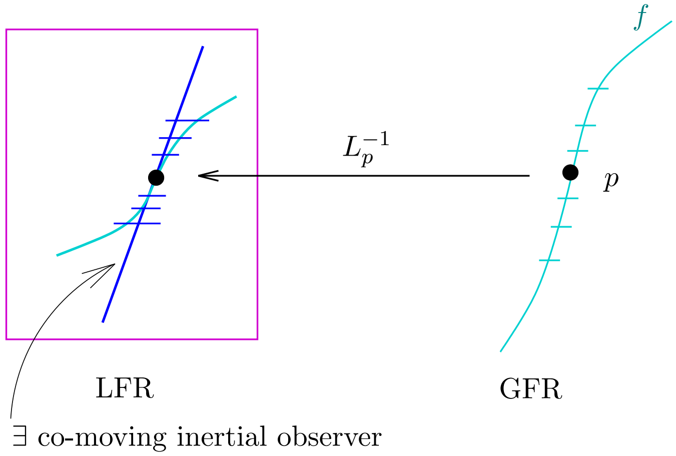

Figure 26: A time-faithful curve at p.

We imagine that a time-faithful curve f is the worldline of an observer b such that the parameter t measures proper time of b; or in other words, t shows the time on the wristwatch of b. We imagine the motion of b such that $f(t)$ in GFR is the space-time location of observer b at his wristwatch time t. The condition in Def. 3.3 serves to ensure that wristwatch time t of observer b (whose motion is represented by the curve f) agrees with the relativistic time interval measured by the relativistic metric rd at the local frame which is situated at the location $f(t)$. More precisely, small time intervals on the wristwatch of b agree with the relativistic time interval measured by the rd's of the local special relativity frames. Thus f is the general relativistic analogue of the special relativistic $\mathsf{wline}_m(b)$ + parametrization with "proper time" or "inner time" of b.

Put differently, we think that a timelike curve can be the worldline of (the mass-center of) a spaceship which uses fuel (i.e. uses its ship-drive) for accelerating and decelerating. The curve is time-faithful if the parametrization of the curve agrees with "inner time" of the spaceship.

Definition 3.4. (timelike geodesic) *By a timelike geodesic we understand a time-faithful curve* $f : I \to \mathrm{GFR}$ *satisfying the following. For any* t *in* I *there is a neighborhood* S *of* $f(t)$ *(understood in* $\mathbb{R}^4$*) such that inside* S*,* f *is a "straightest possible" curve in the following sense: For any two points* p, q *of* S *connected by* f*, the distance of* p *and* q *as measured by* f *is maximal among the distances measured by "competing" time-faithful curves inside* S*.*

Formally, this maximality condition is expressed by the following. Assume that h *is a time-faithful curve with range inside* S*. Assume that* $p = f(s) = h(s')$*,* $q = f(r) = h(r')$ *and* $h(t')$ *is in* S *for all* t' *which are between* s' *and r'. Then* $|s - r| \geq |s' - r'|$*. See Fig. 27.*

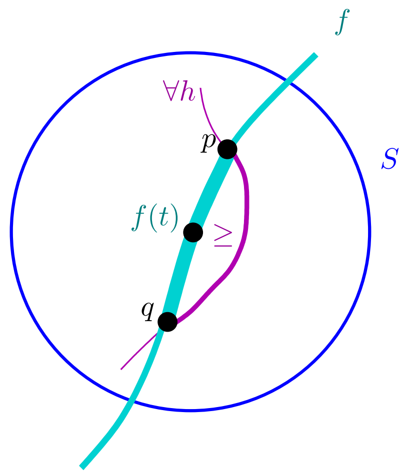

Figure 27: f is a timelike geodesic curve.

If f is a timelike geodesic, then we imagine that an inertial observer b can move along it. By an inertial observer b we imagine (the mass-center of) a spaceship with ship-drive switched off, i.e. a spaceship which does not use fuel for influencing its motion.

The above definition of timelike geodesics is the natural reformulation of the Euclidean notion of geodesics (straight curves in a possibly curved surface of Euclidean 3-space) with "minimal" replaced by "maximal" (cf. [HE73, Prop. 4.5.3, p. 105]). If one thinks about this maximality condition, one will find that it is strongly connected to the Twin Paradox of special relativity. Indeed, the Twin Paradox is exactly the statement that in special relativity, worldlines of inertial observers are timelike geodesics in the sense of Def. 3.4, cf., e.g., [MNS06b, Thm. 3.1].

By the above definition of timelike geodesics, we can express what worldlines of inertial observers are in GR space-times $\langle M, L\rangle$. The definition of photonlike geodesics is analogous, and goes as follows.

The curve f is called photonlike at p iff the speed of f as seen by the local frame at p equals 1. In more detail, this means that the tangent of the curve $L_p^{-1} \circ f$ at the origin has slope 1. The curve f is called photonlike iff the curve f is photonlike at each of its points p.

We imagine that a photonlike curve can be the worldline of a photon perhaps directed (diverted) by suitably many mirrors.

A photonlike geodesic is a photonlike curve f with the property that each point in the curve has a neighborhood in which f is the unique photonlike curve through any two points of f. (In more detail, let F denote the range of f. Then any point in F has a neighborhood S such that whenever f' is a photonlike curve connecting two points of $F \cap S$ and such that F'=*the range of* f' is inside S, we have that $F' \subseteq F$, cf. [HE73, Prop. 4.5.3].)

We imagine that photonlike geodesics are worldlines of photons. Let us notice at this point that a GR space-time $\langle M, L\rangle$ determines "inertial motion" and also determines how photons move.

3.4 The global grid seen with the eyes of the local grids: general relativistic space-time in metric-tensor field form

In the previous subsection, we imported special relativistic notions by using the local frames, the L_p's. We used the inverse L_p^{-1} of the affine transformation L_p to translate our general relativistic "problems", or "questions", to special relativity, and then we used L_p to bring back the answers special relativity gave us. Since we will use the notions of special relativity this way all the time, it is useful to "transport", via L_p, the most useful notions of special relativity themselves to our general relativistic frame GFR to be ready for use when we need them. Thus, in each point p of GFR, we can "store", e.g., relativistic squared distance $\boldsymbol{\mu}$ of special relativity transported via L_p; we will denote this by $\boldsymbol{\mu}_p$ and we will call it the"local special relativistic squared distance $\boldsymbol{\mu}$ at p". This way we will get "fields" of notions, where we use the word "field" as in analysis and not as in algebra (as mentioned on p. 67). Special relativistic squared distance (i.e. Minkowski distance) $\boldsymbol{\mu}$ and scalar product g_4 were defined on p. 36, p. 56. Here we use the simplified version $\mathsf{g}(p, q) = \mathsf{g}_4(p, \overline{0}, q, \overline{0})$ of g as introduced on p. 56.

Definition 3.5. (fields of "transported" special relativity notions) *Let* $\langle M, L\rangle$ *be a general relativistic space-time. For any* $p \in M$ *we define* $\boldsymbol{\mu}_p : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ *as follows: for any* $q, r \in \mathbb{R}^4$

$\boldsymbol{\mu}_p(q,r) := \boldsymbol{\mu}(L_p^{-1}(q), L_p^{-1}(r))$; *and similarly we define*

$\mathsf{g}_p(q,r) := \mathsf{g}(L_p^{-1}(q), L_p^{-1}(r))$. *See Fig. 28.*

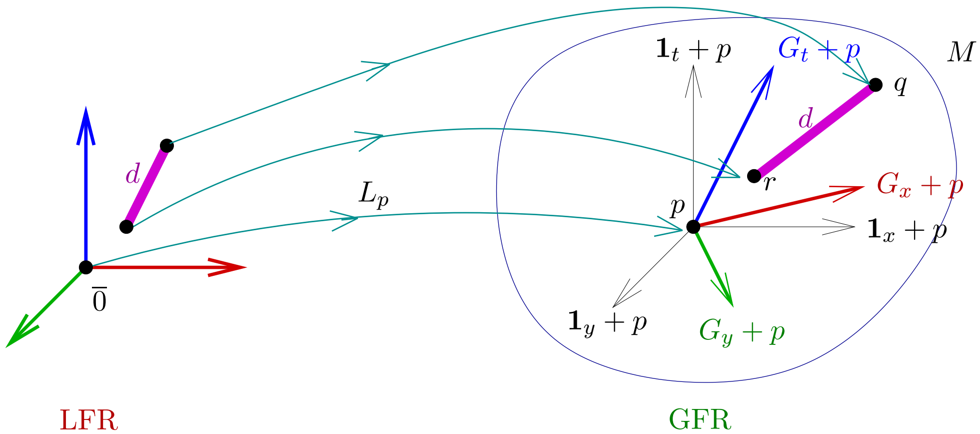

Figure 28: The metric $\boldsymbol{\mu}_p$ of local special relativity at p. $d = \boldsymbol{\mu}_p(q,r)$.

In the literature, the most often used form for specifying a GR spacetime is by transporting the Minkowski scalar product g to each point p of GFR. The reason for this is that it is easy to make calculations with these data. From $\langle M, L\rangle$ then we get $\langle M, \mathsf{g}_p\rangle_{p\in M}$, usually just written as $\langle M, \overline{\mathsf{g}}\rangle$. We call $\overline{\mathsf{g}} := \langle \mathsf{g}_p : p \in M\rangle$ the metric-tensor field of $\langle M, L\rangle$, and we call $\langle M, \overline{\mathsf{g}}\rangle$ the metric-tensor field form of $\langle M, L\rangle$.

Since g is linear in its two arguments, the most convenient way of specifying $\mathsf{g}_p : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ is to specify $\mathsf{g}_p(\mathbf{1}_i + p, \mathbf{1}_j + p)$ for all $1 \le i, j \le 4$:

$\mathsf{g}_{ij}(p) := \mathsf{g}_p(\mathbf{1}_i + p, \mathbf{1}_j + p) := \mathsf{g}(L_p^{-1}(\mathbf{1}_i + p), L_p^{-1}(\mathbf{1}_j + p))$, for all $1 \le i, j \le 4$.

Then we can specify $\langle M, \overline{\mathsf{g}}\rangle$ by associating the 4 by 4 matrix $(\mathsf{g}_{ij}(p) : 1 \le i, j \le 4)$ to each point $p \in M$. What is the meaning of the $\mathsf{g}_{ij}(p)$'s? Well, $\sqrt{|\mathsf{g}_{ii}(p)|}$ tells us how long the i-th unit-vector of the big grid GFR is in the eye of the local special relativity frame at p. On the other hand, $\mathsf{g}_{ij}(p)$ for $i \neq j$ tells us what "angle" between the unit-vectors $\mathbf{1}_i$ and $\mathbf{1}_j$ of the big GFR is as seen by the local special relativity at p. If $\mathsf{g}_{ij}(p) = 0$, then the local special relativity at p (also) thinks that $\mathbf{1}_i$ and $\mathbf{1}_j$ are orthogonal to each other. If $\mathsf{g}_{tt}(p) = 1$, then time at the local special relativity at p flows just as in the GFR grid. If, say, $\mathsf{g}_{tt}(p) = 4$, then two hours pass in the local special relativity at p while in the big grid only one hour passes, hence local special relativity LFR-time at p is twice as fast as coordinate GFR-time. In general, $\mathsf{g}_{tt}(p) > 0$ tells us how the local special relativity sitting at p sees "time of the coordinate grid GFR" to flow (how much slower or faster). If $\mathsf{g}_{tt}(p)$ is negative, then the local special relativity at p "sees" the time-axis of the GFR as a spatial direction, and not as a "temporal direction". This means that in the local special relativity at p, no observer can "move/live" in the space-time direction $\mathbf{1}_t$ of the GFR. If $\mathsf{g}_{xx}(p) = -1$, then the local special relativity at p sees that spatial distance along the x-axis behaves like the one in the big GFR-grid. Also, $\mathsf{g}_{ii}(p) > 0$ iff $\mathbf{1}_i$ of GFR is timelike as seen by the LFR. Etc.

By definition, $\boldsymbol{\mu}_p(q, r)$ is the relativistic squared distance between q and r as seen by the LFR at p, and $\boldsymbol{\mu}_p$ can be expressed using the g_{ij}'s as follows:

$$\boldsymbol{\mu}_p(q, r) = \sum\{\mathsf{g}_{ij}(p) * (q_i - p_i) * (r_j - p_j) : 1 \le i, j \le 4\}.$$

The "infinitesimal version" of the above formula is called the line-element, $(\star)$ below. In this work, we will use the line-element only as an economic linguistic device for specifying the matrix $(\mathsf{g}_{ij}(p) : 1 \le i, j \le 4)$. Namely, for $p \in M$, the line-element at p is

$$(\star) \quad \mathsf{d}s^2(p) = \sum\{a_{ij}\, \mathsf{d}i\mathsf{d}j : 1 \le i \le j \le 4\}.$$

In the above, we consider $\mathsf{d}s^2$, $\mathsf{d}1, \ldots, \mathsf{d}4$ as "specific linguistic markers", the information content of the line-element $(\star)$ above is

$\mathsf{g}_{ii}(p) = a_{ii}$ for $1 \le i \le 4$, and

$\mathsf{g}_{ij}(p) = \mathsf{g}_{ji}(p) = \frac{1}{2}a_{ij}$ for $1 \le i < j \le 4$.

E.g. if at $p \in M$ the line-element is $\mathsf{d}s^2(p) = \mathsf{d}t^2 - \mathsf{d}x^2 - \mathsf{d}y^2 - \mathsf{d}z^2$, then $\mathsf{g}_{tt}(p) = 1$, $\mathsf{g}_{xx}(p) = \mathsf{g}_{yy}(p) = \mathsf{g}_{zz}(p) = -1$, and $\mathsf{g}_{ij}(p) = 0$ for $i \ne j$. For more examples see Sec. 4. In Sec. 4 we will use the line-element in specifying a given space-time $\langle M, L\rangle$, not only because of its economy, but also in order to keep comparability with the literature. We want to emphasize that, in this work, the line-element is just a convenient linguistic way of specifying g_p, we will not attach independent meanings to $\mathsf{d}s^2$, $\mathsf{d}1, \ldots, \mathsf{d}4$.

Let $\langle M, L\rangle$ be a GR space-time and let $G = \langle G_t, \ldots, G_z\rangle$ and $\langle M, \overline{\mathsf{g}}\rangle$ be its vector-fields and metric-tensor field forms, respectively. Now, G contains the same information as $\langle M, L\rangle$, but $\langle M, \overline{\mathsf{g}}\rangle$ contains slightly less information. However, as we will see in the next section, the really relevant "information" of a space-time $\langle M, L\rangle$ is what is contained in $\langle M, \overline{\mathsf{g}}\rangle$. We use

GR space-times in the form $\langle M, L\rangle$ or G because these are easy to draw (visualize), see Fig.s 30, 31, 33, 40, **??**; while a GR space-time in the form of $\langle M, \overline{\mathsf{g}}\rangle$ is not so convenient to draw.

From the perspective of the present subsection, in a GR space-time $\langle G_t, \dots, G_z\rangle$, the tetrad $\langle G_t(p), \dots, G_z(p)\rangle$ tells us how the big GFR sees the local special relativity unit-vectors of an arbitrarily chosen observer in the local special relativity space-time that "sits" at p. On the other hand, the matrix $(\mathsf{g}_{ij}(p) : 1 \leq i, j \leq 4)$ tells us how the local special relativity space-time sitting at p "sees" the unit-vectors of the big global frame GFR!

3.5 Isomorphisms between general relativistic space-times

We said that the big global frame carries no physical meaning, and only timelike and photonlike geodesics carry physical meanings, everything else (e.g. gravity) can be defined from these geodesics. We give "meaning" to this statement (or claim) in the form of defining what isomorphisms of GR space-times are.

Definition 3.6. *Let $G = \langle M, L\rangle$ and $G' = \langle M', L'\rangle$ be two general relativistic space-times. An isomorphism between these two GR space-times is a bijection $Iso : M \to M'$ such that (i)-(iii) below hold.*

(i) *Both Iso and the inverse of Iso are smooth.*

(ii) *Iso preserves timelike geodesics. In more detail, for any curve $f : I \to M$, f is a timelike geodesic in G iff $f \circ Iso$ is a timelike geodesic in G'.*

(iii) *Iso preserves photonlike geodesics (in the above sense).*

We note that we could omit (iii), because one can prove that it follows from (i)-(ii) above.

By using Def. 3.6, it is difficult to check whether a bijection $Iso : M \to M'$ is an isomorphism if we know only L and L' and we did not compute what the geodesics are in $\langle M, L\rangle$ and in $\langle M', L'\rangle$. Below we give an equivalent definition that uses only the "building blocks" L and L' of the general relativistic space-times.

We will use the notion of the differential of a differentiable function.: When $f : \mathbb{R}^4 \to \mathbb{R}^4$ is differentiable, the differential of f at $p \in \mathbb{R}^4$ is the affine transformation $\mathsf{D}(f)_p$ which is closest to f at p. Of the properties of a differential we will mostly use that f and $\mathsf{D}(f)_p$ take p to the same point, and they take a curve g passing through p to tangent curves, i.e. $f(g)$ and $\mathsf{D}(f)_p(g)$ are tangent at $f(p)$.

Theorem 3.1. (Equivalent form of definition of isomorphisms) *Let* $G = \langle M, L\rangle$ *and* $G' = \langle M', L'\rangle$ *be general relativistic space-times and let* $\langle M, \overline{\mathsf{g}}\rangle$ *and* $\langle M', \overline{\mathsf{g}}'\rangle$ *be their metric-tensor field forms, respectively. Let* $Iso : M \to M'$ *be a smooth bijection such that its inverse is also smooth. Then (i)-(iii) below are equivalent.*

(i) *Iso is an isomorphism between* G *and* G' *in the sense of Def. 3.6.*

(ii) *For any* $p \in M$*, the differential of* $L'(Iso(p))^{-1} \circ Iso \circ L(p)$ *at the origin is a Lorentz transformation (on LFR) perhaps composed with a space-isometry. See Fig. 29.*

(iii) *For any* $p, q, r \in M$ *we have that* $\mathsf{g}_p(q, r) = \mathsf{g}'_{Iso(p)}(\mathsf{D}(Iso)_p(q), \mathsf{D}(Iso)_p(r))$.

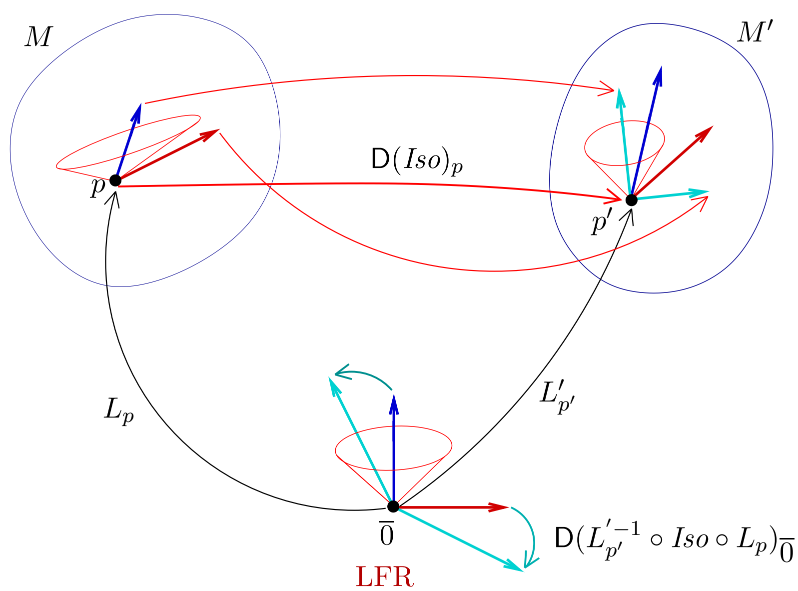

Figure 29: Isomorphism between general relativistic space-times.

In connection with Thm. 3.1(ii) above we note the following. Here, the general pattern is $L'(Iso(p))^{-1} \circ Iso \circ L(p) : LFR \to LFR$ because $L(p) : LFR \to M$, $Iso : M \to M'$, and $L'(Iso(p))^{-1} : M' \to LFR$. (More precisely, $L(p) : LFR \to \mathbb{R}^4 \supseteq M$ etc., but that does not matter here.) Recall that $LFR = \mathbb{R}^4$ refers to the special relativistic frame we are using (in the present section).

By the above, we have the basic building blocks of General Relativity at place and we can start working, we can start discussing black holes, wormholes, the Expanding Universe, etc. So, from here the reader can jump right to Sec. 4 which discusses examples of GR space-times, e.g. black holes. In the next subsection we present a FOL axiomatization of GR space-times which is analogous to **Specrel** and which can serve as a starting point for logic-oriented investigations of GR similar to the ones in Sec. 2.

In Sec. 3.6 we will use the following.

Definition 3.7. *A Lorentz manifold is a system of GR space-times connected by commuting partial isomorphisms. I.e. a Lorentz manifold is a system $\langle\langle G_m, \psi_{mk}\rangle : m, k \in J\rangle$ such that for all $m, k \in J$ the following hold.*

(i) $G_m = \langle M_m, \overline{\mathsf{g}}_m\rangle$ *is the metric-tensor field form of a GR space-time $\langle M_m, L_m\rangle$ in the sense of Def. 3.5, except that we do not assume smoothness of L, we only assume that $\overline{\mathsf{g}}_m$ is smooth.*

(ii) $\langle \psi_{mk} : m, k \in J\rangle$ *is a commuting system of smooth functions with open domains. Commuting means that $\psi_{kh} \circ \psi_{mk} \subseteq \psi_{mh}$ for all $m, k, h \in J$.*

(iii) ψ_{mk} *is a partial isomorphism between G_m and G_k. This means that ψ_{mk} is an isomorphism between G_m and G_k restricted to the domain and the range of ψ_{mk}, respectively, in the sense of Thm. 3.1(iii).*

Thus, a Lorentz manifold could be called an "organized system of GR space-times" or a "patched space-time". It is not difficult to show that the above definition of Lorentz manifolds is equivalent with the usual definition in the literature, e.g. with the one in [Wal84, pp. 12,23].

3.6 Axiomatization Genrel of general relativistic space-time in first-order logic

In this subsection we give an axiom system called **Genrel** to general relativistic space-times. This axiom system is formulated in first-order logic and is analogous to **Specrel**, even at the level of the individual axioms.

The vocabulary (or language) of GR space-time models is the same as that in the case of special relativity, with the same intuition as in Sec. 2.1; the motivation for the axioms in **Genrel** is in Sec.s 3.2,3.3. In the present subsection we will use arbitrary ordered fields and arbitrary dimensions $n \geq 2$ for the space-time models as in Sec. 2 (and not only $\mathbb{R}$ and $n = 4$). We will use the same notation as in Sec. 2, e.g. we will use ev_m, PLines, etc.

<u>Convention:</u> The elements of Ob are called inertial observers. The elements in the domain $\mathsf{Dom}(\mathsf{W}) = \{b : (\exists k, p)\mathsf{W}(b, k, p)\}$ of W are called (ordinary) observers.

We will no longer require that an observer m uses the whole coordinate grid Q^n for coordinatizing the events; the part he uses is denoted by $\mathsf{Cd}(m)$, and is defined as $\mathsf{Cd}(m) := \{p \in \mathsf{Q}^n : \mathsf{ev}_m(p) \neq \emptyset\}$. If $p, q \in \bar{t}$, then $[p,q] := \{r \in \bar{t} : p_t \leq r_t \leq q_t\}$.

AxSelf^- An observer m in his own coordinate system is motionless in the origin, and his worldline is connected, i.e.

$$(\forall m \in \mathsf{B})[\,\mathsf{wline}_m(m) = \bar{t} \cap \mathsf{Cd}(m) \wedge (\forall p, q \in \mathsf{wline}_m(m))[p,q] \subseteq \mathsf{wline}_m(m)\,].$$

We formalize when two subsets h and g of Q^n are tangent at $p \in \mathsf{Q}^n$:

$\mathsf{tangent}(h, g, p)$ means that

$$p \in h \cap g \text{ and } (\forall \varepsilon > 0)(\exists \delta > 0)(\forall s \in [p_t - \delta, p_t + \delta])(\forall q \in h)(\forall r \in g) [\,q_t = r_t = s \;\Rightarrow\; |q - r| \leq \varepsilon * |s - p_t|\,].$$

AxPh^- An inertial observer m at the origin, where he stands, sees photons move in each direction with speed 1, and each photon meeting m moves with speed 1, i.e.

$$(\forall m \in \mathsf{Ob})[\,(\forall \ell \in \mathsf{PLines})(\forall p \in \ell \cap \mathsf{wline}_m(m))(\exists \mathsf{ph} \in \mathsf{Ph}) \mathsf{tangent}(\ell, \mathsf{wline}_m(\mathsf{ph}), p) \wedge (\forall \mathsf{ph} \in \mathsf{Ph})(\forall p \in \mathsf{wline}_m(\mathsf{ph}) \cap \mathsf{wline}_m(m))(\exists \ell \in \mathsf{PLines})\mathsf{tangent}(\ell, \mathsf{wline}_m(\mathsf{ph}), p)\,].$$

AxThEx^- An inertial observer m at the origin, where he stands, sees inertial observers move in each direction with speeds < 1, and sees at least one inertial observer in each event, i.e.

$$(\forall m \in \mathsf{Ob})(\forall \ell \in \mathsf{TLines})(\forall p \in \ell \cap \mathsf{wline}_m(m))(\exists k \in \mathsf{Ob}) \mathsf{tangent}(\ell, \mathsf{wline}_m(k), p) \quad \text{and} \quad (\forall p \in \mathsf{Cd}(m))(\exists k \in \mathsf{Ob}) k \in \mathsf{ev}_m(p).$$

$\mathsf{AxSelf}^-, \mathsf{AxPh}^-, \mathsf{AxThEx}^-$ express that an inertial observer experiences special relativity in the space-time location where he is. Next we formalize a generalization of AxLine to general relativity. It will say that in each inertial observer's worldview, the worldlines of inertial observers and photons are timelike and photonlike geodesics, respectively. In formulating this axiom, we will follow the definitions given in Sec. 3.3. We quantified over curves in the definition of geodesics. Since we want to use the language of first-order logic, instead of arbitrary (smooth) curves, we will quantify over bodies representing special curves; namely 3 times continuously differentiable curves which can be defined by first-order logic formulas. This will be the only difference in the definition.

Let us call a curve r-smooth if it is r-times continuously differentiable. In general relativity, it is enough to use 3-smooth curves in place of arbitrarily smooth curves. E.g. for defining curvature, the Riemann-tensor etc., one needs only 3-smooth ingredients in place of smooth ingredients.

Let ψ be a first-order logic formula in our present vocabulary, and assume that the free variables of ψ are among $t, x_1, \ldots, x_n, y_1, \ldots, y_r$ where $t, x_1, \ldots, x_n$ are variables of sort Q. We will denote this assumption as $\psi = \psi(t, \overline{x}, \vec{y}\,)$. We can easily express in first-order logic that, at parameter $\vec{y}$, the formula ψ defines a 3-smooth curve, we will denote this formula by $\mathsf{curve}(\psi)$. To give a flavor for this definition, we start formulating $\mathsf{curve}(\psi)$.

$\mathsf{fn}(\psi)$ denotes the formula $\forall t(\exists \overline{x}\psi(t, \overline{x}, \vec{y}\,) \rightarrow \exists! \overline{x}\psi(t, \overline{x}, \vec{y}\,))$. This expresses that ψ defines a (partial) function at parameter $\vec{y}$. When this is the case, we will denote by $\psi(t)$ the value of this function at t. In a similar way, we can express that the domain of the function defined by ψ is an open interval.

$\mathsf{vel}(\psi, t) = \overline{v}$ denotes the formula $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall s \in [t - \delta, t + \delta])(\,|[(\psi(s) - \psi(t))/(s - t)] - \overline{v}| < \varepsilon\,)$. This formula expresses that the velocity vector of the function defined by ψ at t is $\overline{v} \in \mathsf{Q}^n$. Then we can express that the velocity vector changes with t continuously, i.e. the function defined by ψ is 1-smooth. Similarly, we can express that it is 3-smooth. Let $\mathsf{curve}(\psi)$ denote that ψ defines a 3-smooth function, and the domain of this function is an open interval. We note that $\mathsf{vel}(\psi, t)$ is the tangent-vector of the curve ψ at t. The length of this vector depends on the parametrization of the curve.

By the worldcurve of observer k in m's worldview we understand the worldline $\mathsf{wline}_m(k)$ parameterized with the wristwatch time of k. We can define this worldcurve by the formula $\gamma_{mk} = \gamma_{mk}(t, \overline{x}) = \gamma(t, \overline{x}, m, k)$ as follows:

$\gamma_{mk} := \gamma(t, \overline{x}, m, k)$ denotes the formula "$\mathsf{w}_{km}(t, \overline{0}) = \overline{x} \wedge \mathsf{W}(k, k, t, \overline{0})$".

Intuitively, γ_{mk} holds for $t, \overline{x}$ iff m sees k present at coordinates $\overline{x}$ such that k's wristwatch shows t when k is present at $\overline{x}$. Below we use $\gamma_{mk}(t)$ as a function of t.

Finally, we express that $\psi = \psi(t, \overline{x}, \vec{y}\,)$ defines a time-faithful curve, or that ψ defines a photonlike curve, respectively as

$\mathsf{timef}(\psi)$ denotes the formula "$\mathsf{curve}(\psi) \land \forall t(\exists \overline{x}\psi(t,\overline{x}) \rightarrow$
$(\exists k \in \mathsf{Ob})(\exists s \in \mathsf{Q})[\,\psi(t) = \gamma_{mk}(s) \land \mathsf{vel}(\psi,t) = \mathsf{vel}(\gamma_{mk}, s)\,])$".

$\mathsf{phot}(\psi)$ denotes the formula "$\mathsf{curve}(\psi) \land \forall t(\exists \overline{x}\psi(t,\overline{x}) \rightarrow$
$(\exists \mathsf{ph} \in \mathsf{Ph})[\,\psi(t) \in \mathsf{wline}_m(\mathsf{ph}) \land \mathsf{tangent}(\exists t\psi, \mathsf{wline}_m(\mathsf{ph}), \psi(t))\,])$"

To be able to quantify conveniently over parametrically defined timelike and photonlike curves, we will use the following axiom schema which is an analogue of the Comprehension axiom schema in Set Theory. Below, we are thinking of $\psi(t,\overline{x})$ as defining a curve, hence $t \in \mathsf{Dom}\psi$ abbreviates the formula $\exists \overline{x}\psi(t,\overline{x})$, or equivalently, $\exists \overline{x}\psi(t,\overline{x},\vec{y}\,)$. We systematically do not indicate $\vec{y}$ because in $\exists \overline{x}\psi(t,\overline{x},\vec{y}\,)$, the variables in $\vec{y}$ are free variables, they remain free variables while we use ψ in building up new formulas like $\mathsf{Ax}\exists_\psi$ below and eventually in postulating the axioms, all free variables become universally quantified. Hence e.g. $\mathsf{Ax}\exists_\psi$ looks like $\forall \vec{y}\,(\dots\psi(t,\overline{x},\vec{y}\,)\dots)$.

In each inertial observer's worldview, the parametrically definable timefaithful curves are worldcurves of (not necessarily inertial) observers; and the photonlike curves are worldlines of bodies. Formally: Let $\psi(t,\overline{x},\vec{y}\,)$ be a formula. Then

$\mathsf{Ax}\exists_\psi$ $(\mathsf{timef}(\psi) \rightarrow (\forall m \in \mathsf{Ob})(\exists b \in \mathsf{B})(\forall t \in \mathsf{Dom}\psi)\psi(t) = \gamma_{mb}(t)) \land$
$(\mathsf{phot}(\psi) \rightarrow (\forall m \in \mathsf{Ob})(\exists b \in \mathsf{B})\{\psi(t) : t \in \mathsf{Dom}\psi\} = \mathsf{wline}_m(b))$.

COMPR $:= \{\mathsf{Ax}\exists_\psi : \psi$ is a formula of our vocabulary$\}$.

For any $p \in \mathsf{Q}^n$ and $\varepsilon > 0$ let $S(p,\varepsilon) := \{q \in \mathsf{Q}^n : |q-p| < \varepsilon\}$, the open ball (or sphere) of radius ε with center p. We now can formulate

"γ_{mk} is a timelike geodesic" iff

$\mathsf{timef}(\gamma_{mk}) \land (\forall p \in \mathsf{wline}_m(k))(\exists \varepsilon > 0)(\forall q, r \in \mathsf{wline}_m(k) \cap S(p,\varepsilon))$
$(\forall b \in \mathsf{B})[\,\mathsf{timef}(\gamma_{mb}) \land \mathsf{wline}_m(b) \subseteq S(p,\varepsilon) \land q = \gamma_{mb}(t') = \gamma_{mk}(t) \land r = \gamma_{mb}(s') = \gamma_{mk}(s) \quad \Rightarrow \quad |t-s| \geq |t'-s'|\,]$.

"$\mathsf{wline}_m(\mathsf{ph})$ is a photonlike geodesic" can be expressed analogously (see p. 72 where photonlike geodesics were defined for $\langle M, L\rangle$).

AxLine^- In each inertial observer's worldview, the worldlines of inertial observers and photons are geodesics. Formally:

$(\forall m, k \in \mathsf{Ob})$"$\gamma_{mk}$ is a timelike geodesic" and

$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph})$"$\mathsf{wline}_m(\mathsf{ph})$ is a photonlike geodesic".

Now we turn to formulating a generalization of AxEvent. It expresses that if m observes k participate in an event, then k himself "sees" this event. Further, if k sees an event that m sees, then k sees all events which occur "near" this event in m's worldview.

AxEvent$^-$

$$(\forall m,k \in \mathsf{Ob})(\forall p \in \mathsf{Q}^n)(k \in \mathsf{ev}_m(p) \;\Rightarrow\; (\exists q \in \mathsf{Q}^n)\mathsf{ev}_k(q) = \mathsf{ev}_m(p)) \wedge (\mathsf{Dom}(\mathsf{w}_{mk}) \text{ is open and } \mathsf{w}_{mk} \text{ is a 3-smooth function})).$$

To formulate a generalization of AxSim, we will use a variant of AxSim that works for $n = 2$, too. For more on this variant we refer to [AMN02, Sec.s 2.8, 3.9], [AMN06b, p.162], [AMN99].

AxSim$^-$ Any two inertial observers see each other's wristwatches run slow with the same ratio when they meet:

$$(\forall m,k \in \mathsf{Ob})(\forall t,s \in \mathsf{Q})[\,\mathsf{ev}_m(\gamma_{mk}(t)) = \mathsf{ev}_k(\gamma_{km}(s)) \quad\Rightarrow\quad |\mathsf{vel}(\gamma_{mk},t)| = |\mathsf{vel}(\gamma_{km},s)|\,].$$

To be able to use the notions of continuity and differentiability etc. in arbitrary fields in place of $\mathbb{R}$ properly, we need the axiom schema of Continuity. Reasons and details for this can be found e.g. in [MNS06b], [Gol87], [vB83, p. 29].

AxSup$_\psi$ is a formula expressing that every subset of Q defined by $\psi(t,\vec{y}\,)$ with parameter $\vec{y}$ has a supremum if it is non-empty and bounded. Formally AxSup$_\psi$ is: $\exists t'\forall t[\psi(t,\overline{y}) \to t < t'] \to \exists t'(\forall t[\psi(t,\overline{y}) \to t < t'] \wedge \forall t''[\forall t(\psi(t,\overline{y}) \to t < t'') \to t'' \geq t'])$.

CONT $:= \{\mathsf{AxSup}_\psi : \psi$ is a formula in our vocabulary$\}$.

The formula schema CONT above is a variant of Tarski's first-order logic version of Hilbert's axiom of continuity in his axiomatization of Euclidean geometry. It is also strongly related to the induction axiom schema in the dynamic logic of actions in the sense of [Sai86], [ANS82].

Genrel $:= \{\mathsf{AxSelf}^-, \mathsf{AxLine}^-, \mathsf{AxThEx}^-, \mathsf{AxPh}^-, \mathsf{AxEvent}^-, \mathsf{AxSim}^-\} \cup \{\mathsf{AxField}\} \cup \mathsf{CONT} \cup \mathsf{COMPR}.$

As a first theorem we state that special relativity is the special case of general relativity where the worldlines of all observers and photons are straight lines; and where the Light Axiom holds. This is stated in the next theorem. We conjecture that a much weaker axiom suffices in place of the Light Axiom.

Theorem 3.2. **Genrel** $\cup \{\mathsf{AxLine}, \mathsf{AxPh}\}$ *is equivalent to* **Specrel** $\cup \mathsf{CONT} \cup \mathsf{COMPR}$ *in the sense that they have the same models, if* $n \geq 3$.

Proof outline. Assume **Genrel** + AxLine + AxPh. First one proves that $(\forall m \in \mathsf{Ob})\mathsf{Cd}(m) = \mathsf{Q}^n$, by AxLine. Then AxEvent can be proved from AxLine, AxPh and $\mathsf{AxEvent}^-$ along the lines of the proof in [AMN02, pp. 98-100]. Now the axiom AxEOb used in [MNT04] holds, so one gets that the worldview transformations are affine mappings, by [MNT04, Thm. 1]. It is proved in [AMN02, Thm. 3.9.11] that AxSim^- implies AxSim when the worldview transformations are affine. This proves one direction of Thm. 3.2. In the other direction we have to prove in an axiomatic setting that timelike and photonlike straight lines are timelike and photonlike geodesics, respectively. This is done, basically, in [MNS06b, Thm. 3.1]. □

Next we show that the metric-geometric forms of the models of **Genrel** are Lorentz manifolds and each Lorentz manifold is the metric-geometric form of a model of **Genrel**. This will be the analogue of Thm. 2.11 in Sec. 2.6.

Definition 3.8. *Let* $\mathfrak{Q} = \langle \mathsf{Q}, +, *, \leq \rangle$ *be an ordered quadratic field. By a 3-smooth n-dimensional Lorentz manifold over* $\mathfrak{Q}$ *we understand* $\langle \langle M_m, \overline{\mathsf{g}}_m, \psi_{mk}, \mathfrak{Q} \rangle : m, k \in J \rangle$ *where* $\langle \langle \langle M_m, \overline{\mathsf{g}}_m \rangle, \psi_{mk} \rangle : m, k \in J \rangle$ *is a Lorentz manifold in the sense of Def. 3.7 with the following changes:*

(a) *in place of* $\mathbb{R}^4$ *we use* $\mathfrak{Q}^n$, *and*

(b) *in place of requiring the metric-tensor fields* g_m *and the partial isomorphisms* ψ_{mk} *to be smooth, we only require that they be 3-smooth.*

The following theorem says that the models of **Genrel** are exactly the 3-smooth Lorentz manifolds over ordered real-closed fields, up to ignoring some "decorations" on the **Genrel** side. Thm. 3.3 can be considered as a completeness theorem for **Genrel**.

Theorem 3.3. (completeness theorem for **Genrel**) *Assume* $n \geq 3$.

(i) *There is a theory* $\mathbf{Comp}^-$ *analogous to* **Comp** *such that* **Genrel** $\cup$ $\mathbf{Comp}^-$ *is definitionally equivalent with the class* LM *of all 3-smooth Lorentz manifolds over ordered real-closed fields.*

(ii) *There is a definable function* Lm *that maps the class of all models of* **Genrel** *onto the class* LM. *Moreover, if* $\mathfrak{M} \models$ **Catrel** *then* $\mathsf{Lm}(\mathfrak{M})$ *is definitionally equivalent with the Minkowski geometry* $\mathsf{Mg}(\mathfrak{M})$ *of* $\mathfrak{M}$.

The proof of Thm. 3.3 is based on the following proposition, which states that the models of **Genrel** are "locally special relativistic".

Proposition 3.1. *Assume that* $\mathfrak{M} \models$ **Genrel** *and* $m, k \in \mathsf{Ob}$. *If* $p \in \mathsf{wline}_m(k) \cap \bar{t}$ *then* $\mathsf{D}(\mathsf{w}_{mk})_p$ *exists and it preserves relativistic (Minkowski) distance* $\boldsymbol{\mu}$.

Proof outline. Assume $p \in \mathsf{wline}_m(k) \cap \bar{t}$. Then $k, m \in \mathsf{ev}_m(p)$ by AxSelf^-, and so $p \in \mathsf{Dom}(\mathsf{w}_{mk})$ and $F := \mathsf{D}(\mathsf{w}_{mk})_p$ exists by $\mathsf{AxEvent}^-$. By AxPh^-, and the definition of w_{mk}, F takes PLines and only PLines going through p to PLines going through $p' := \mathsf{w}_{mk}(p)$. Thus F is a bijection and by AxSim^- it preserves $\boldsymbol{\mu}$. □

To give an idea for the proof of Thm. 3.3, we define $\mathsf{Lm}(\mathfrak{M})$ for $\mathfrak{M} \models$ **Genrel**. Let us fix a model $\mathfrak{M} = \langle \mathsf{Q}, +, *, \leq;\ \mathsf{B}, \mathsf{Ob}, \mathsf{Ph};\ \mathsf{W} \rangle \models$ **Genrel** and let $\mathfrak{Q} = \langle \mathsf{Q}, +, *, \leq \rangle$. Let $m \in \mathsf{Ob}$ and $p \in \mathsf{Cd}(m)$. Let $k \in \mathsf{ev}_m(p) \cap \mathsf{Ob}$ be arbitrary (such a k exists by AxThEx^-) and let $p' := \mathsf{w}_{mk}(p)$. We define

$L_{p,k} := \mathsf{D}(\mathsf{w}_{mk})_{p'} \circ \tau(p')$, where $\tau(p') : \mathsf{Q}^n \to \mathsf{Q}^n$ is "translation with p'", and

$\mathsf{g}_{p,k}$ is the metric-tensor field belonging to this $L_{p,k}$ as defined in Def. 3.5, i.e. $\mathsf{g}_{p,k}(q, r) := \mathsf{g}(L^{-1}_{p,k}(q), L^{-1}_{p,k}(r))$, for all $q, r \in \mathsf{Q}^n$.

It follows from Prop. 3.1 that, though $L_{p,k}$ depends on how we choose $k \in \mathsf{ev}_m(p)$, the metric-tensor $\mathsf{g}_{p,k}$ does not depend on how we choose $k \in \mathsf{ev}_m(p)$. Therefore we will omit the index k from the notation:

$\mathsf{g}_p := \mathsf{g}_{p,k}$, $\quad \overline{\mathsf{g}}_m := \langle \mathsf{g}_p : p \in \mathsf{Cd}(m) \rangle$, $\quad G_m := \langle \mathsf{Cd}(m), \overline{\mathsf{g}}_m \rangle$, and

$\mathsf{Lm}(\mathfrak{M}) := \langle \langle \mathsf{Cd}(m), \overline{\mathsf{g}}_m, \mathsf{w}_{mk}, \mathfrak{Q} \rangle : m, k \in \mathsf{Ob} \rangle$.

laim 3.1. *Assume* $\mathfrak{M} \models$ **Genrel**. *The following (i),(ii) hold.*

(i) $\mathsf{Lm}(\mathfrak{M})$ *is a 3-smooth Lorentz manifold over* $\mathfrak{Q}$, *and* $\mathfrak{Q}$ *is an ordered real-closed field.*

(ii) *The worldlines* $\mathsf{wline}_m(k), \mathsf{wline}_m(\mathsf{ph})$ *of observers and photons in m's worldview are timelike and photonlike geodesics in* G_m. *Conversely, any timelike geodesic* ℓ *is a worldline of an observer locally, i.e.* $(\forall p \in \ell)(\exists \varepsilon > 0)(\exists k \in \mathsf{Ob})\ell \cap S(p, \varepsilon) \subseteq \mathsf{wline}_m(k)$. *The same converse holds for photonlike geodesics, too.*

We outlined above that the geometry of each model of **Genrel** is a Lorentz 3-smooth manifold. The converse is also true, each Lorentz 3-smooth manifold over an ordered real-closed field is isomorphic (as a manifold) to the geometry $\mathsf{Lm}(\mathfrak{M})$ of a model $\mathfrak{M}$ of **Genrel**.

The extension **Comp**$^-$ for **Genrel** is completely analogous with **Comp**. A typical axiom in **Comp**$^-$ states that if the worldlines of two photons ph_1 and ph_2 coincide for all observers, then $\mathsf{ph}_1 = \mathsf{ph}_2$. Further, the worldline of a photon is a maximal photonlike geodesic. The point in the axioms in **Comp**$^-$ is to eliminate things like the multiplicity of otherwise undistinguishable objects (like ph_1 and ph_2 above) which cannot be defined over LM because they are undistinguishable. In some sense these statements are incarnations of Occam's razor.

We omit the rest of the idea for proof of Thm. 3.2. $\square$

What we call the worldview of an inertial observer $m \in \mathsf{Ob}$ in **Genrel** corresponds to "spaceships with ship-drive switched off": the worldline of the center of mass is a geodesic, but we did not care about whether the spaceship rotates or not. One can base an axiomatization of GR on worldviews of the so-called "local inertial frames" (LIF's, cf. [Rin01, pp. 177-179]) which correspond to nonrotating inertial spaceships. LIF's reflect the local special relativity more closely (e.g. they do not rotate). However, LIF's can be defined in models of **Genrel** and the axiomatization based on LIF's would provide the same geometrical entities $\mathsf{Lm}(\mathfrak{M})$ behind the models as the present **Genrel** does. The role of Einstein's field equations in interpreting **Genrel** is touched upon in Sec. 4.5.

4 Black holes, wormholes, timewarp. Distinguished general relativistic space-times

In Sec. 3.6 we introduced the first-order logic theory **Genrel** of general relativity. In such a situation, the next natural thing to do is to turn to the intended models of the theory in question, and to discuss what these models look like. Indeed, we will do this in the present section, we will study some of the intended models of the theory **Genrel**. A difference between special relativity and GR is that while the special theory had basically one intended model (namely Minkowski space-time), the general theory has many nonisomorphic intended models, as we will see below.

It will be convenient for us to study the models of **Genrel** in their geometric forms. Hence we will speak about GR space-times $\langle M, L\rangle$, but it is important to remember that in Sec. 3.6 we saw that such a space-time $\langle M, L\rangle$ is equivalent with a **Genrel** model $\mathfrak{M} \models$ **Genrel**. So each one of the GR space-times in this section represents a distinguished **Genrel** model, and discussing these will shed some light on the semantic aspects of **Genrel**.

For discussing the models of **Genrel** we leave the realm of first-order logic and then we work in mathematics proper, the reason for which is that by Tarski's theorem one cannot satisfactorily describe the semantics of a language $\mathcal{L}$ inside the framework of $\mathcal{L}$ itself. To study the semantics of $\mathcal{L}$, we have to rise above $\mathcal{L}$ and use the meta-language of $\mathcal{L}$. In our case, this metalanguage is ordinary mathematics (or equivalently Set Theory, say ZF).

4.1 Special relativity as special case of general relativity

Minkowski space-time is $\mathsf{G}_{sr} = \langle M, L\rangle$ where $M = \mathbb{R}^4$ and $L(p)$ is translation with p (i.e. $L_p(q) = p+q$ for any $q \in \mathbb{R}^4$), for all $p \in \mathbb{R}^4$. In vector-fields form this is $\langle G_1, \ldots, G_4\rangle$ where $G_i(p) = \mathbf{1}_i$ for all $p \in \mathbb{R}^4$ and $1 \le i \le 4$, see upper part of Fig. 30. In metric-tensor field form Minkowski space-time is $\langle M, \overline{\mathsf{g}}\rangle$ where the 4 by 4 matrix $(\mathsf{g}_{ij}(p) : 1 \le i, j \le 4)$ at $p \in \mathbb{R}^4$ is

$$\mathsf{g}(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

i.e. the line-element is $\mathsf{d}s^2 = \mathsf{d}t^2 - \mathsf{d}x^2 - \mathsf{d}y^2 - \mathsf{d}z^2$. The timelike and photonlike geodesics in this space-time are the straight lines of slope < 1 and of slope 1, respectively. The automorphisms (i.e. isomorphisms onto itself) of Minkowski space-time G_{sr} are exactly the possible worldview transformations in a model of **Specrel** (where $\langle \mathsf{Q}, \ldots\rangle = \langle \mathbb{R}, \ldots\rangle$), cf. Thm. 2.7.

Worldview of a uniformly accelerated observer in Specrel:

We let $n = 2$, for simplicity. Consider the following space-time $\mathsf{G}_{ua} = \langle M, L\rangle$ where $M = \{\langle t, x\rangle \in \mathbb{R}^2 : x > 0\}$ and $G_t(p) = \langle 1/p_2, 0\rangle$, $G_x(p) = \langle 0, 1\rangle$ for all $p = \langle p_1, p_2\rangle \in M$, cf. the lower part of Fig. 30. Thus the line-element is

$\mathsf{d}s^2 = x^2\mathsf{d}t^2 - \mathsf{d}x^2$, and the g_{ij}-matrix is

$$\mathsf{g}(t,x) = \begin{pmatrix} x^2 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this space-time, as we approach the origin, local LFR clocks tick slower and slower beyond any limit compared with coordinate GFR time, and local LFR clocks tick faster and faster beyond any limit compared with coordinate GFR time as we move away from the origin. On the other hand, local meter-rods do not change along the x-axis, local LFR spatial distances agree with coordinate GFR spatial distances. This space-time looks different from Minkowski space-time G_{sr}, but in fact it is isomorphic to a sub-space-time of 2-dimensional G_{sr}. The isomorphism denoted by *Iso* is represented in Fig. 30, it maps M of G_{ua} bijectively onto $\{\langle t,x\rangle : |t| < x\}$. The space-time G_{ua} is the worldview (or space-time) of a uniformly accelerated observer k who lives in Minkowski space-time, with uniform (relativistic) acceleration $a = 1$. (The space-time for arbitrary uniform acceleration a is given by $G_t(p) = \langle 1/(ap_2), 0\rangle$, $G_x(p) = \langle 0, 1\rangle$.) One can think of G_{sr} as the worldview of an inertial observer m in special relativity, and then *Iso* is the worldview transformation w_{km} between the worldview G_{ua} of accelerated k and the worldview G_{sr} of m.

G_{ua} is rather similar to the exterior of the (2-dimensional tr-slice of the) simplest black hole G_{sb} below, which, in turn, is no longer partially isomorphic to any open part of G_{sr}. Studying the simple space-time G_{ua} of accelerated observers can lead to a deeper understanding of the space-time G_{sb} of the important Schwarzschild black hole to which we turn now. This connection is elaborated e.g. in the textbook [Rin01, Sec.s 3.7, 12.4].

4.2 The Schwarzschild black hole

There is an overwhelmingly mounting observational evidence for the existence of large black holes in our universe. In the last 15 years astronomers have observed them. At the same time, black holes have really fantastic properties. Black hole physics is at the cutting edge of modern science.

There are many kinds of black holes, the Schwarzschild black hole is the simplest one. We will recall more exotic ones in Sec. 4.3. Why is the Schwarzschild black hole important? Here are four reasons for this: it is the simplest form of relativistic gravity (all the mass is in one point), it is an idealization of the gravitational space-time of our own Sun, it is a typical general relativistic space-time, and many other GR space-times build on it.

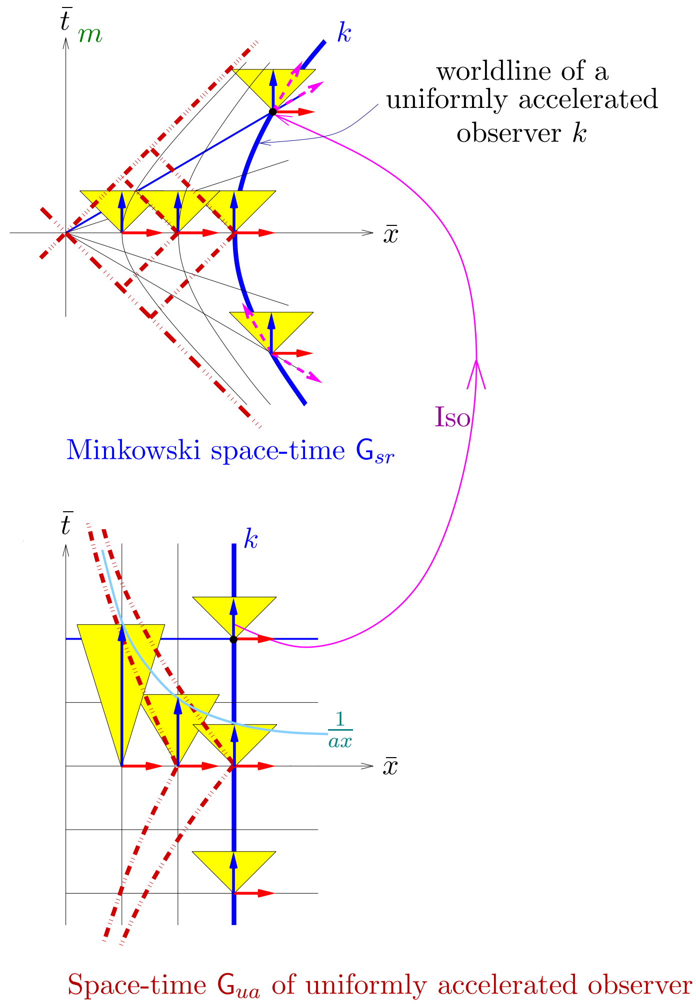

Figure 30: Isomorphism from the worldview of a uniformly accelerated observer to Minkowski space-time. (The example in the text is given for uniform acceleration $a = 1$, the figure is for $a = 1/3$.) $Iso(t, x)$, for $t > 0$, is the point p on the Minkowski-circle $M(\overline{0}, x)$ of radius x and with center the origin such that the relativistic arc-length of $M(\overline{0}, x)$ from $\langle 0, x\rangle$ to p is axt. $\frac{1}{ax}$ is the length of G_t at x.

Worldview of a suspended observer far away from the black hole:

The Schwarzschild (black hole) space-time is $\mathsf{G}_{sb} := \langle M, L\rangle$ where

$M = \{p \in \mathbb{R}^4 : |\langle p_2, p_3, p_4\rangle| \neq 0, 1\}$ and L is given as follows. For any $p = \langle p_1, p_2, p_3, p_4\rangle \in \mathbb{R}^4$ let $\mathbf{r} := \mathbf{r}(p) := \langle 0, p_2, p_3, p_4\rangle$, $r := |\mathbf{r}|$, and $\mathbf{1}_r := \mathbf{1}_r(p) := \frac{1}{r} * \mathbf{r}$. Here, "r" stands for "radius". Now L is specified by the following four vector-fields

For $r > 1$:
$G_t(p) = \sqrt{\frac{r}{r-1}} * \mathbf{1}_t$, $G_x(p) = \sqrt{\frac{r-1}{r}} * \mathbf{1}_r$, the lengths of $G_y(p)$ and $G_z(p)$ are 1, and $G_t(p), G_x(p), G_y(p), G_z(p)$ are pairwise orthogonal.

For $r < 1$:
$G_t(p) = \sqrt{\frac{1-r}{r}} * \mathbf{1}_r$, $G_x(p) = \sqrt{\frac{r}{1-r}} * \mathbf{1}_t$, the lengths of $G_y(p)$ and $G_z(p)$ are 1, and $G_t(p), G_x(p), G_y(p), G_z(p)$ are pairwise orthogonal.

See Fig. 31.

G_{sb} in metric-tensor form is the following. We use cylindric-polar coordinates because they are more convenient (by spatial spherical symmetry of the space-time). The line-element is

$$\mathsf{d}s^2 = (1 - \tfrac{1}{r})\mathsf{d}t^2 - (1 - \tfrac{1}{r})^{-1}\mathsf{d}r^2 - r^2\mathsf{d}\varphi^2,$$

where $\mathsf{d}\varphi$ represents two coordinates the usual Euclidean way. Namely, φ represents "space-angle", i.e. φ is the usual Euclidean combination of two polar coordinates θ (longitude) and η (latitude). Formally, $\mathsf{d}\varphi^2 = \mathsf{d}\theta^2 + \sin(\theta)^2\mathsf{d}\eta^2$ (metric on Euclidean unit 2-sphere). We note that by defining the line-element we also defined the metric-tensor field $\overline{\mathsf{g}}$ of the Schwarzschild space-time G_{sb}.

In the general form of the Schwarzschild space-time, there is a parameter that we chose to be 1 in the above. Namely, the general form of the line-element for Schwarzschild black hole is

$$\mathsf{d}s^2 = (1 - \tfrac{\mathsf{M}}{r})\mathsf{d}t^2 - (1 - \tfrac{\mathsf{M}}{r})^{-1}\mathsf{d}r^2 - r^2\mathsf{d}\varphi^2,$$

the parameter $\mathsf{M} \in \mathbb{R}$ is thought of as the "mass" of the black hole. M is also called the radius or size of the black hole. (For historical reasons $2m$ is used for M in the literature, but this does not matter when one wants to understand the "logic" of the black holes.) Similarity with the accelerated observer can be discovered by choosing $x = r - 1$. Then the accelerated line-element (i.e. that of G_{ua}) becomes $\mathsf{d}s^2 = (r-1)^2\mathsf{d}t^2 - \mathsf{d}r^2$. For lack of space we do not discuss this more here, but we note that there is more in this direction in [AMN06a].

The set of coordinate points $p \in \mathbb{R}^4$ with $r = 0$ is called the singularity (this coincides with the time-axis), and the set of coordinate points $p \in \mathbb{R}^4$ with $r = 1$ is called the event horizon (EH). These are disjoint from

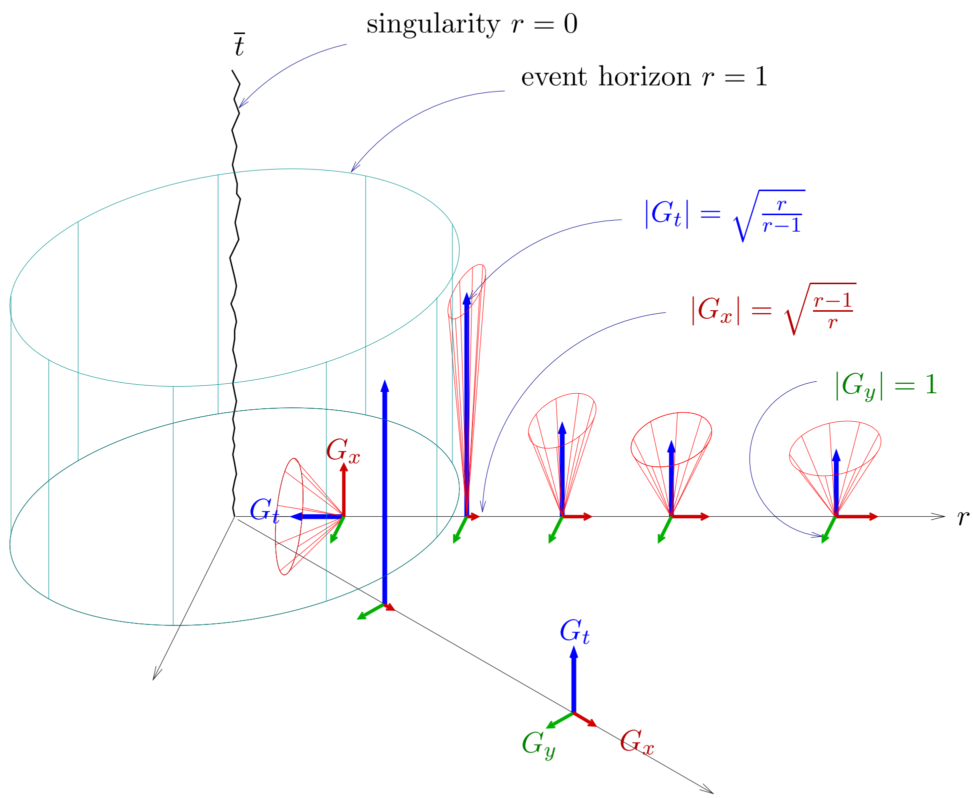

Figure 31: Illustration for the Schwarzschild space-time. G_t gets longer as we approach $r = 1$ from above, i.e. local time runs slower and slower as we approach $r = 1$. G_x gets shorter as we approach $r = 1$, i.e. there is more and more space (compared to "coordinate-space") as we approach $r = 1$. The length of G_y stays 1, this means that spatial distances orthogonal to the radius agree with coordinate-distances. Time and (radial) space are interchanged in the interior of the black hole, this means that, in the interior, the r-axis is the worldline of an observer, but lines parallel to the time-axis are not possible worldlines. The singularity is in the future of an observer inside the black hole.

the domain of G_{sb}, i.e. we did not associate local clocks and meter-rods to these points. Thinking in terms of the global coordinate frame, the event horizon is a sphere of radius 1 (or M in the more general case), and the singularity is the center of this sphere. Loosely speaking, we will refer to the EH and its interior as the black hole (BH). When we want to be more careful, we will refer to the part outside of the EH as the exterior of the BH and we will refer to the inside part of the EH as the interior of the BH. Later we will see that the interior and the exterior of the BH behave, in some sense, like two different universes connected by a one-way membrane, namely by the EH. Here one-way membrane means that an observer may fall through the EH into the interior of the BH, but nothing, not even light can come out from the interior. This effect is not yet clear from our present space-time diagram (Fig. 32) but it will be clear after we apply the so-called Eddington-Finkelstein re-coordinatization (and extension) to it, cf. Fig. 35.

Let us think for a while in terms of the global coordinate system, and let us see what the exterior of the BH looks like. Infinitely far away everything is normal, the farther away we are from the EH, the more "normal life is", e.g. both local time and local meter-rods agree with the global GFR coordinate ones asymptotically. Space-times with this property are called asymptotically flat. Asymptotically flat means asymptotically Minkowski (or asymptotically special relativistic), namely as we move away from the BH, space-time becomes more and more like Minkowski space-time is. This can be formalized by saying that as $r(p)$ tends to infinity, so the metric-tensor $\mathsf{g}(p)$ tends to "Minkowski g".

Convention: by coordinate properties (e.g. coordinate time) we always mean the global, GFR-coordinate properties.

Let us now approach the EH from far away. As we get close, local clocks begin to tick radically slower (compared to the global coordinate time), beyond any limit; so metaphorically local clocks "stop" or freeze at the EH. (This is only metaphorical because we did not associate local clocks and meter-rods to the points of the EH. However, this will be helped after the upcoming Eddington-Finkelstein re-coordinatization and then time will really freeze on the EH.) At the same time, local meter-rods in the radial direction get smaller (beyond any limit) as we approach the EH, but local meter-rods orthogonal to the radius of the EH continue to agree with the coordinate meter-rods.

Far away from the BH, GFR-coordinate-speed of light tends to be the same, namely 1, in all directions, but as we approach the EH, the coordinate-speed of light in the radial direction gets radically smaller compared to the coordinate-speed of light in directions orthogonal to the radius (this coordinate "anisotropy" is the reason why the light-cones in Fig. 31 close to the EH but in the exterior get "flattened out"); and as we get closer to the EH, the coordinate-speed of light tends to 0 in all directions. We note that the fact that $\mathsf{g}_{ti}(p) = 0$ for $i \neq t$ everywhere means that at each event, the coordinate-speed of light in a spatial direction d and in its opposite direction

$-d$ is the same, if measured by the global frame. The above means that the so-called light-cones get infinitely narrow towards the EH but they do not get tilted as seen by the GFR, cf. Fig. 31, Fig. 32.

Since a timelike curve must stay inside the local light-cones, this means that observers stay for all the coordinate time outside the EH, they never "enter" the EH, as seen via the global coordinate grid. We will see that this is only an "artifact" of Schwarzschild's particular choice of global coordinate system (GFR), similar to the "artifact" mentioned earlier and also in Fig. 30. Avoiding of this artifact will be done via the so-called Eddington-Finkelstein re-coordinatization which will be presented soon.

Let us see what the worldlines of observers and photons in the Schwarzschild space-time look like in the exterior of the EH.

Consider, for an example, the curve $f : \mathbb{R} \to \mathbb{R}^4$ where $f(\tau) = \langle\sqrt{1.5} * \tau, 3, 0, 0\rangle$ for all $\tau \in \mathbb{R}$. (This f is represented in Fig. 32 as "suspended observer".) This is a time-faithful curve, so we can imagine an observer k whose worldcurve this f is; the worldline of k (in G_{sb}) is the vertical line going through $\langle 0, 3, 0, 0\rangle$ and k's wristwatch shows τ at the coordinate point $\langle\sqrt{1.5} * \tau, 3, 0, 0\rangle$. Is this curve f a geodesic? Local clocks at coordinate points farther away from the EH tick faster, so one can "gain time" by moving outwards a little, clocking up a lot of wristwatch time while out there, and then coming back. Special relativistic time-dilation dampens this effect somewhat since the clocks of a fast moving observer slow down. But it is not hard to show that by moving outwards and then coming back with small velocities all the time, one gains time, and there is an optimum "outward-bulging" with maximum gain of time. Hence, vertical lines are not geodesics and the timelike geodesics always "bulge a little outwards", i.e. they accelerate (or "turn"), as seen in GFR, towards the BH. So, f is not a geodesic.

Radial timelike geodesics are similar to the worldlines of pebbles thrown up into the air here on the Earth; with "upward" replaced by "outward"; a geodesic curve which begins to move away from the EH looses (coordinate) speed, eventually it stops and "falls back" towards the EH. According to GR, things thrown up fall back not because gravity of the Earth pulls them with myriad small invisible "hands", but because time ticks slower near the Earth, and faster farther away from the Earth. Newton's apple falls from the tree because of the "gravitational time-dilation" (known also as gravitational red-shift)! This is the first explanation for gravity since its behavior has been described by Newton.

There is a similar reason for a photonlike geodesic with a velocity in nonradial direction to bend toward the EH as if the EH "pulled" the geodesic

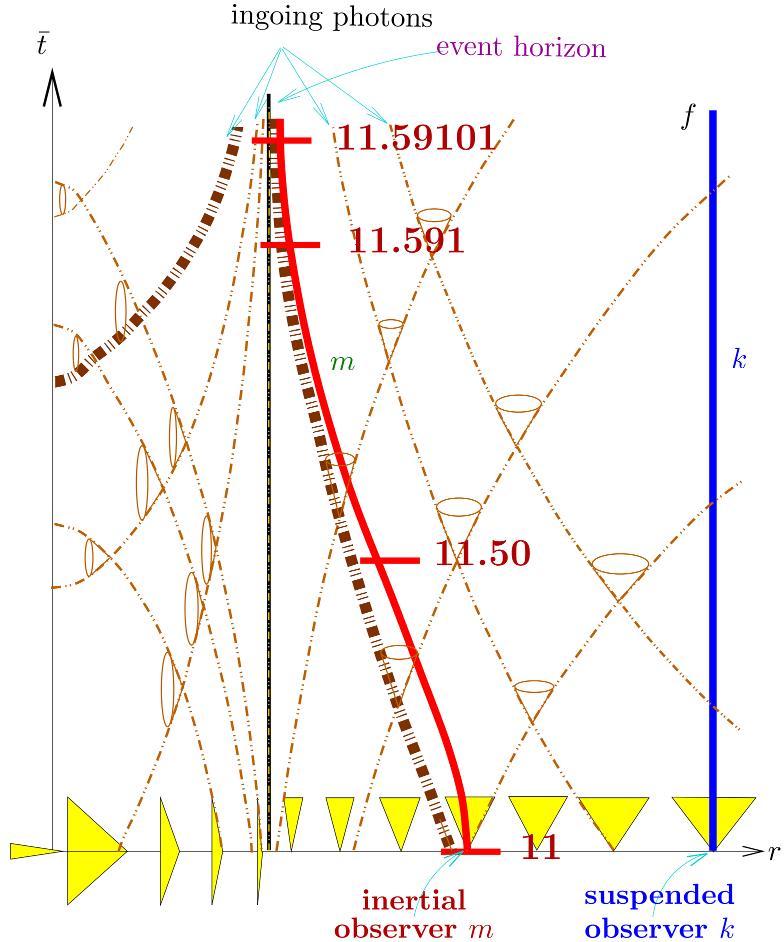

Figure 32: The "tr-slice" of the Schwarzschild black hole. This is a geodesically closed 2-dimensional sub-space-time of G_{sb}. The worldline of m is a geodesic, it "bulges" outward because m can maximize his time by bulging outwards. The worldline of k is not a geodesic (because it does not "bulge" outwards). Photons and inertial observers moving in radially "freeze" to the EH. The wristwatch of an in-falling inertial observer slows down "infinitely", and will show times which are bounded.

towards itself. So, even photons "fall" (gravitational "light-bending" effect).

Since the BH attracts, if an observer wants to stay at a constant distance from the EH, he has to use fuel for accelerating away from it. We call an observer like k above a "suspended observer". Suspended observer means that the worldline of the observer is parallel with the time-axis in the present Schwarzschild GFR coordinate system. However, the notion of "suspended observer" is coordinatization-independent, i.e. observer-independent, because these "vertical" worldlines can be defined by a first-order logic formula in the language of local clocks and meter-rods (i.e. in the language of **Genrel** in Sec. 3.6). In other words, there is an experiment with which an observer can check whether he is suspended or not. In still other words, being suspended is an observational concept.

Let us consider a (timelike or photonlike) geodesic curve that starts out towards the EH in a radial direction. By cylindrical symmetry of the space-time, there is "no reason" for the geodesic to bend right or left, since "right" and "left" are completely alike by symmetry. This implies that a geodesic curve which starts in a radial direction, will always move in this radial direction, i.e. the (range of the) curve will be a subset of a tr-plane. Thus a tr-plane is a geodesically closed sub-space-time of G_{sb}.

Let us review briefly the interior of the EH. In the interior of the EH, time and (radial) space are interchanged, the r-axis is in fact the worldline of a possible inertial observer. Hence the global r-axis is the local time-axis for some LFR "living" in the interior of the BH. Similarly, the $\bar{t}$-axis of the GFR is a spatial direction for this LFR. The singularity is no more a "place" for this observer (or for any observer in the interior), instead, it is like a future time instance like the Big Crunch in usual cosmology, something that will happen in the distant future but not "present" in the "now" of the in-falling observer. Similarly, for this inertial observer inside the BH, the EH is like a time instance or a "time-slice of space-time" which happened sometime in the past like the Big Bang in cosmology. We see the light coming from the Big Bang or from the EH in our past but we cannot influence it causally because it is in our past. Local time at the EH (in the interior) is "infinitely fast" compared to coordinate-time and local meter-rods in the $\bar{t}$-direction are "infinitely long". As we move towards the singularity, local time "slows down" (compared to GFR) beyond any limit, and local meter-rods in the $\bar{t}$-direction get shorter, approaching zero coordinate-length at the singularity. Thus local light-cones in the interior of the EH are "infinitely wide" at the EH, and get "infinitely narrow" at the singularity, see Fig. 32.

We could say that the space-time G_{sb} is the worldview of an observer k suspended far away from the black hole. How far away? We measure "far" by multiples of the radius of the EH (which we chose to be 1 presently), and the farther away the suspended observer is, the more he will experience special relativistic space-time on his own worldline. He sees that inertial observers fall towards the black hole, but he never sees them reach the black hole, for him (k) they stay outside for the eternity of k. As they fall towards the black hole, they move slower and slower as they approach the event-horizon, and eventually they "freeze" onto the event-horizon. Also the wristwatches of the inertial observers tick slower and slower towards the EH, and "stop altogether at the EH". The same happens to the photons sent towards the EH in a radial direction: the photons slow up as they approach the EH, and eventually, and metaphorically, they "freeze" onto the event horizon. Our suspended observer k observes things this way both via photons (i.e. visually by his eyes) and via his coordinate system.

Since an in-falling inertial observer m lives for an infinite GFR-time, it is possible in theory that there is no upper bound for the time his wristwatch shows, i.e. that m also will experience that all his infinite time passes outside the EH. However, this is not the case: the wristwatch readings of an in-falling inertial observer m are bounded, e.g. the wristwatch of m may approach 12 as he falls in, tick slower and slower and never reach 12 o'clock, cf. Fig. 32. So what happens in the in-falling inertial observer's own worldview or spaceship when his wristwatch reaches 12 o'clock?

We will see that he falls through the EH into the interior of the BH. For our suspended observer k, the interior of the BH is not visible by photons, he cannot get information about the inside of the BH while suspended. However, he may wonder what might be inside of that "big black ball", i.e. the EH. While suspended he cannot find out the answer. But, in principle, he may choose to fall in, and because of this, the interior of the BH is a "reality" for him.

In many ways this worldview G_{sb} of the suspended observer k is similar to the worldview of G_{ua} of an accelerated observer in special relativity. However, the following is an important and significant difference between G_{ua} and G_{sb}. Assume two inertial observers m, h fall radially into the black hole, in the same "path", i.e. in the same direction, and they started to fall at the same GFR-time, and close to each other. During their fall, one of them, say m, measures the distance between them by sending photons to h which it mirrors back to m. Then m measures constantly the time it takes for the photon to get mirrored back. The result of this measurement is called radar-distance. In G_{ua} this radar-distance remains the same, since the distance of parallel geodesics in G_{sr} does not change (this means that it is "flat"). However, in G_{sb}, m will find that the radar-distance between him and h is growing; and since G_{sb} is the relativistic version of a gravitational source, this is expected to be so because in Newtonian gravity, things closer to a gravitational source fall faster than things more distant. Technically speaking, timelike geodesics that started out in a parallel way, will increase their distance from each other in the tr-plane; the space-time G_{sb} is curved.[11] This shows that there is no partial isomorphism between G_{sb} and G_{sr} with an open domain.

Worldview of an observer falling into the black hole:

In the worldview of an observer falling into the black hole, the worldline of the in-falling observer m ought to be a straight line parallel with the time-axis. Instead of aiming for this, it is more convenient to "re-coordinatize" G_{sb} in such a way that the worldline of a radially ingoing photon will be a straight line of slope 1. There are no essential differences between such a worldview and a worldview where the worldline of m would be a straight line.[12] In fact, since the EH and the singularity in its middle are special "marked" places in this worldview, it is quite natural to make a worldview where these "do not move" in the GFR.

In G_{sb}, the worldline of a radially in-falling photon in the tr-plane and outside the EH is $\{\langle -r - \ln(r-1) + \mathsf{constant}, r\rangle : r > 1\}$, and the worldline of a photon inside the EH is $\{\langle -r - \ln(1-r) + \mathsf{constant}, r\rangle : r > 1\}$. (This is not difficult to show by using the definition of a photonlike curve given in Sec. 3.3, and by knowing that the worldline is a subset of the tr-plane.) Thus the following simple (partial) function $Iso : \mathbb{R}^4 \to \mathbb{R}^4$ will take these photon worldlines to straight lines of slope 1:

$$Iso\,(t,x,y,z) := \langle t + \ln|r-1|, x, y, z\rangle \quad \text{where } r = \sqrt{x^2+y^2+z^2}.$$

Let us look for the isomorphic image of G_{sb} along Iso. By Thm. 3.1(ii) (p. 76), the simplest way of defining the isomorphic image $\mathsf{G}^0_{ef} = \langle M', L'\rangle$ of $\mathsf{G}_{sb} = \langle M, L\rangle$ is by letting $L'(Iso(p)) = \mathsf{D}(Iso)_p \circ L_p$ for all $p \in M$. By doing so we get the following definition (see Fig. 33).

The Schwarzschild black hole in Eddington-Finkelstein coordinates is $\mathsf{G}^0_{ef} = \langle M', L'\rangle$ where $M' = \{p \in \mathbb{R}^4 : r(p) \neq 0, 1\}$, and L' is specified by the vector-tetrad $G'(p) = \langle G'_t(p), G'_x(p), G'_y(p), G'_x(p)\rangle$ where

$$G'_x(p) = \sqrt{\tfrac{1}{r(r-1)}}\mathbf{1}_t + \sqrt{\tfrac{r-1}{r}}\mathbf{1}_r, \quad G'_t(p) = G_t(p) \ \text{ for } r > 1,$$

[11] Parallel geodesics diverge means negative curvature. Hence the tr-plane of G_{sb} is negatively curved. This is an instance of the so-called tidal forces which G_{sb} (and, in general, GR) inherited from the Newtonian theory of gravity.

[12] "parallel with the time-axis" is inessential here, since it is easy to rotate a picture

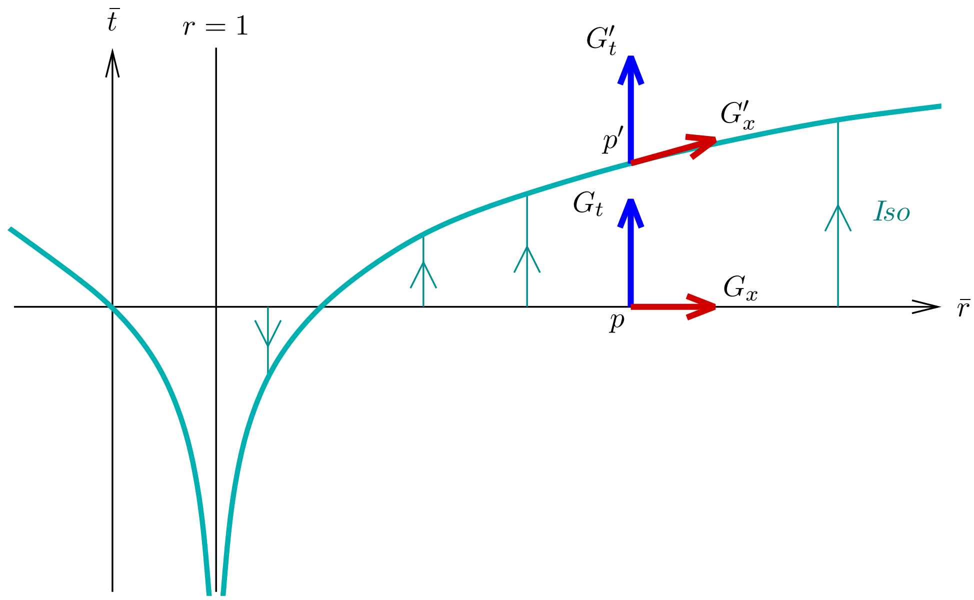

Figure 33: Eddington-Finkelstein re-coordinatization as an isomorphism between space-times

$$G'_t(p) = \sqrt{\tfrac{1}{r(1-r)}}\mathbf{1}_t - \sqrt{\tfrac{1-r}{r}}\mathbf{1}_r, \quad G'_x(p) = G_x(p) \text{ for } r < 1, \text{ and}$$

$$G'_y(p) = G_y(p), \quad G'_z(p) = G_z(p) \text{ for all } r.$$

(A few words on how we got $G'(p)$: Assume $r = r(p) > 1$ and let $p' = \mathsf{Iso}(p)$. Now $\mathsf{D}(Iso)_p$ takes $\mathbf{1}_t + p$ to $\mathbf{1}_t + p'$ and $\mathbf{1}_r + p$ to $\mathbf{1}_r + \frac{1}{r-1}\mathbf{1}_t + p'$. It can be seen in Fig. 33 that we get $G'_x(p)$ by considering its "slope (relative to $\mathbf{1}_x$)" and by considering the length of its "$\bar{r}$-projection". The slope is given by the derivative of Iso, thus it is $\frac{1}{r-1}$, and the $\bar{r}$-projection of $G'_x(p)$ is $G_x(p) = \frac{r-1}{r} * \mathbf{1}_r$. We obtain $\frac{1}{r-1}|G_x| = \frac{1}{r(r-1)}$ as the t-component of G'_x. The case $r < 1$ is analogous. G'_t, G'_y, G'_z are obtained similarly.) Thus, $G'(p)$ specifies the local LFR frame at p, see Fig. 34.

To see what the metric-tensor field $\overline{\mathsf{g}}_{ef}$ of G_{ef} is we have to "look at the coordinate unit vectors $\mathbf{1}_i$" with the eye of this LFR. This was explained in Sec. 3.4, on p. 73.

In the coordinate system specified by $G'(p)$, the coordinates of $\mathbf{1}_t$ and $\mathbf{1}_r$ are $\langle \sqrt{\frac{r-1}{r}}, 0, 0, 0\rangle$ and $\langle -\sqrt{\frac{1}{r(r-1)}}, \sqrt{\frac{r}{r-1}}, 0, 0\rangle$ respectively. Hence for $r = r(p) > 1$,

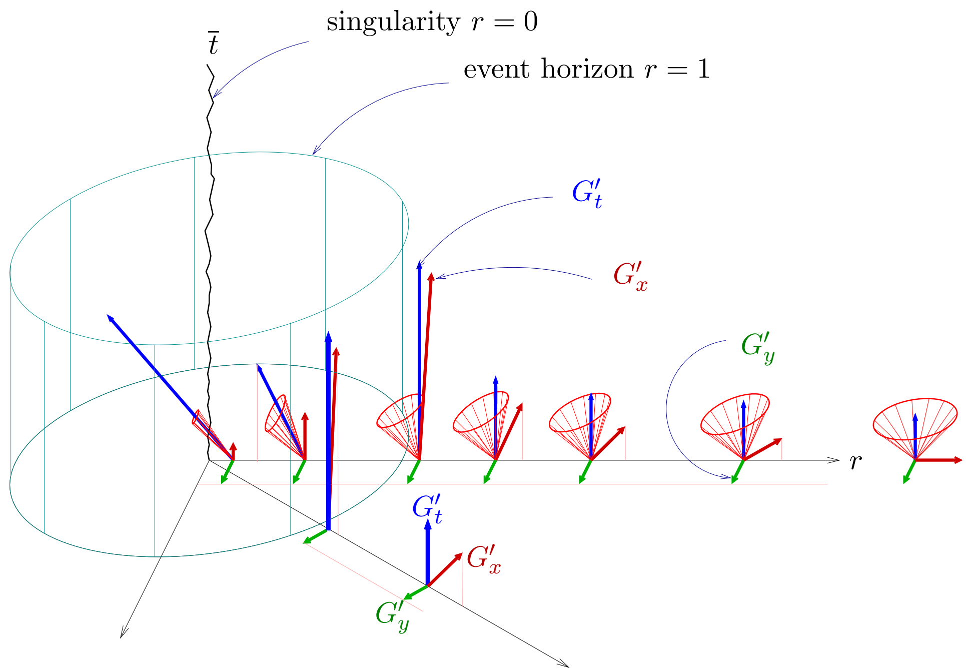

Figure 34: The Schwarzschild space-time in Eddington-Finkelstein coordinates.

$$\mathsf{g}_{tt}(p) = \mathsf{g}_p(\mathbf{1}_t, \mathbf{1}_t) = (\sqrt{\tfrac{r-1}{r}}\,)^2 = \tfrac{r-1}{r},$$

$$\mathsf{g}_{rr}(p) = \mathsf{g}_p(\mathbf{1}_r, \mathbf{1}_r) = (-\sqrt{\tfrac{1}{r(r-1)}}\,)^2 - (\sqrt{\tfrac{r}{r-1}}\,)^2 = -(1 + \tfrac{1}{r}), \quad \text{and}$$

$$\mathsf{g}_{tr}(p) = \mathsf{g}_{rt}(p) = \mathsf{g}_p(\mathbf{1}_t, \mathbf{1}_r) = \sqrt{\tfrac{r-1}{r}} * -\sqrt{\tfrac{1}{r(r-1)}} = -\tfrac{1}{r}.$$

For $r < 1$ we obtain the same final values for $g_{ij}(p)$. For this reason, the metric-tensor field $\overline{\mathsf{g}}_{ef}$ belonging to G^0_{ef} is given by the following line-element

$$\mathsf{d}s^2 = (1 - \tfrac{1}{r})\mathsf{d}t^2 - \tfrac{2}{r}\mathsf{d}t\mathsf{d}r - (1 + \tfrac{1}{r})\mathsf{d}r^2 + r^2\mathsf{d}\varphi^2.$$

In this metric-tensor, $\mathsf{g}_{tr} \neq 0$ because the coordinate unit vectors $\mathbf{1}_t, \mathbf{1}_r$ are not orthogonal in the eye of the local LFR specified by the vector-tetrad $G'(p)$, see Fig.s 33,34. This, $\mathsf{g}_{tr} \neq 0$, means that the light-cones in the tr-planes are tilted as illustrated in Fig.s 34,35.

We can see that the above $\overline{\mathsf{g}}_{ef}$ can smoothly be extended to the event horizon, i.e. to $EH = \{p \in \mathbb{R}^4 : r(p) = 1\}$. The reason for this is that the above formula for $\overline{\mathsf{g}}_{ef}$ is not degenerate for $r = 1$. Hence we can extend G^0_{ef} to EH, and this way we get Eddington-Finkelstein space-time, in short EF-space-time, $\mathsf{G}_{ef} = \langle M_{ef}, \overline{\mathsf{g}}_{ef} \rangle$ where $M_{ef} = \{p \in \mathbb{R}^4 : r(p) \neq 0\}$. This is an extension of G^0_{ef} and *Iso* is a partial isomorphism between G_{sb} and G_{ef}. The event horizon EH is part of the space-time here; and in fact the EH in a tr-plane is the worldline of a photon! This extended G_{ef} explains what happens on the event horizon and shows how the inside of the BH can be connected to the outside. (We note that the above given G' cannot be smoothly extended to EH, but one can smoothly change G' to G'' which gives the same metric-tensor field and which can be smoothly extended to EH.)

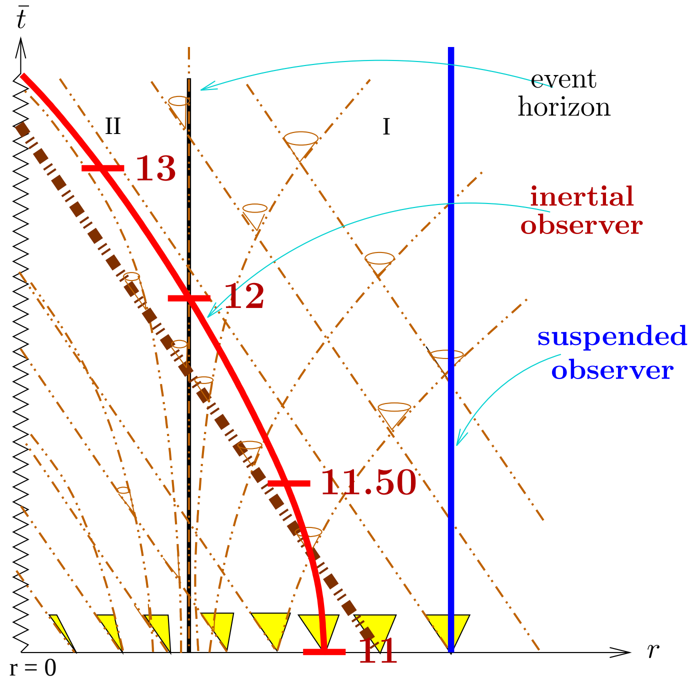

Figure 35: The "tr-slice" of space-time of the Schwarzschild black hole in Eddington-Finkelstein coordinates.

What will an in-falling inertial observer experience in G_{ef}? Throughout we assume that the BH in question is big enough so that the tidal forces on the EH and also well inside the EH are negligible. The present "animation" is based largely on G_{ef} in Fig. 35, but also Fig. 32, Fig. 36 are taken into account. So, it is useful to consult these figures with an emphasis on Fig. 35 before reading on. For visualizability, we assume that the BH is like the ones in centers of galaxies in that there are some stars (suns) orbiting our BH. So there is the EH, outside that there are the nearby suns orbiting the EH, and far away, there are the distant galaxies. As explained in [Rin01, §12.5, pp. 267-271, Fig. 12.6], it is observationally possible to decorate the exterior of our BH with a latticework or "scaffolding" consisting of suspended observers (spaceships using fuel for maintaining their latitude) which surround the EH (only the exterior) and which maintain constant radar-distance from each other by using rockets. Gyroscopes are used to avoid rotation. We will think of these suspended observers as milestones, telling our in-falling observer where he is and what his speed is. It is impossible to have suspended observers on the EH or inside the EH, so one sign telling the in-falling observer that he is already inside will be the nonexistence or disappearance of the milestones.

As the in-falling observer m approaches the EH, he will see the milestones flashing by him, so to speak, faster and faster approaching the speed of light. If there were a milestone on the EH, then it would flash by (i.e. move relative to the observer) with the speed of light. However, this milestone cannot be realized by an observer. When all the milestones have flashed by our in-falling m, he will notice that there are no more milestones and even the BH has disappeared. Then m finds himself in basically empty flat-looking space[13] with no BH and no singularity in any of his spatial directions. More precisely, if he knows what to look for, then he can still observe some traces of the EH just as we can "see" our Big Bang (via the cosmic microwave background radiation) but it is all in the past, gone so to speak, and not influenceable causally. Like history is not changeable. The nearby suns (say, of different colors for fun) are also visible, even moving as m watches them, but they are like ghosts, their light comes from *before* the (Big Bang like) EH and they are causally not "touchable", since they all are in the distant past. All of the outside world, even the future of the suspended observers outside the EH, are in the past for m inside the BH, moving, living, dynamical, and changing but in the distant past before the Big Bang like EH, causally unreachable. What is interesting about this is that it is tempting to say that for m safely inside the EH the exterior of the EH does not exist. He is in a different universe, period. But that would not be the complete truth. Namely, the nearby suns circling the BH are still visible for m, they are just not influenceable causally. For more on this experience of seeing several

[13]Space may become flat (inside EH, of course), space-time remains curved.

universes via a BH we refer to the professional physics movie "Falling into a BH" [Ham01], and the Prologue of [Tho94]. What we described so far is nothing but a decoding of the space-time diagrams Fig. 32-36 and of the metric of G_{ef}. This is the meaning of the mathematical expression of saying "space and time gets interchanged". Soon we will discuss more subtle BH's where m can avoid the fate of eventually hitting the singularity.

If we want to concentrate on the causal structure of a space-time, e.g. of the Eddington-Finkelstein black hole in Fig. 35, then that can be represented more compactly by a so-called conformal diagram (or Penrose diagram) of G_{ef}. Such a conformal diagram (of Fig. 35) is represented in Fig. 36. In a conformal diagram, photonlike geodesics are represented as straight lines of slope 1 and local time flows upwards.

G_{sb} and G_{ef} are two worldviews of the Schwarzschild black hole, connected by *Iso*. With an analogous way as we obtained G_{ef} we can obtain a re-coordinatization where the worldlines of the outgoing photons will be straight lines of slope 1. In this worldview, the interior of EH will behave like a so-called white hole: things can come out of the interior but cannot move inside. If we go on completing the worldlines of observers when we can we will also obtain a so-called hypothetical dual universe. All of these fit into one world-view called the Kruskal-Szekeres space-time, whose conformal structure is illustrated in Fig. 36.

4.3 Double black holes, wormholes

After an observer falls into a Schwarzschild black hole, he has only a finite time to live inside, and he must meet the singularity. There are many more friendly kinds of black holes, where he can live for an infinite time inside the black hole, he can avoid meeting a singularity, and he can even come out into an asymptotically flat region. (The expression "wormhole" refers to this last property.) We briefly describe here two such black holes, the electrically charged black hole and the rotating black hole.

Electrically charged black holes

This black hole is also called Reissner-Nordström black hole in the literature. Its line-element is

$$\mathsf{d}s^2 = (1 - \tfrac{1}{r} + \tfrac{e}{r^2})\mathsf{d}t^2 - (1 - \tfrac{1}{r} + \tfrac{e}{r^2})^{-1}\mathsf{d}r^2 - r^2\mathsf{d}\varphi^2$$

where $0 \le e < \frac{1}{4}$. (Notice the strong analogy with the Schwarzschild space-time G_{sb}.) Here, e represents the square of the electric charge. In this space-time $r = 0$ is the singularity, and there are two event horizons at

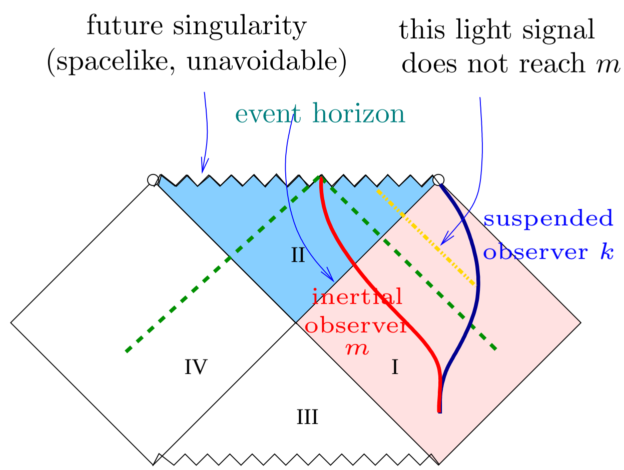

Figure 36: Conformal or Penrose diagram of "completed" Schwarzschild black hole. The shaded area consisting of blocks I, II is conformal diagram of EF-black hole. Region I is the exterior of the BH, region II is the interior of BH, region III is the white hole, and region IV is the dual universe. In some sense, regions III, IV may or may not exist but regions I, II have a stronger ontological status, they probably exist.

$$r^- = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - e} \qquad \text{and} \qquad r^+ = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - e}\ .$$

The exterior of the outer event horizon is similar to the Schwarzschild black hole: the light-cones get narrower towards the outer EH, and they are "infinitely narrow" at the EH. Inside the outer EH, the space-time remains similar to G_{sb} till about halfway towards the inner EH: time and space get interchanged and as we move inwards, local time gets faster and faster. But after a while, local time begins to slow down again, and local time "stops" at the inner event horizon, where time and space get interchanged once more. The innermost part, the interior of the inner event horizon, is similar somewhat to the exterior of the outer EH, but time runs faster and faster towards the singularity, beyond any limit. The singularity can be avoided in the inside of the black hole, the in-falling observer can "live forever".[14] The coordinatization represented by the above line-element is analogous to the Schwarzschild coordinatization of simple black hole, where the events of the in-falling inertial observers' entering the outer EH are not included. An

[14]Actually, it is extremely difficult to go near the singularity (because of the repelling effect), so the in-falling observer is safe, will not be hurt by the BH.

Eddington-Finkelstein-type re-coordinatization of the space-time where the worldlines of the ingoing photons are straight lines of slope 1 is illustrated in Fig. 37.

The conformal diagram of the electrically charged BH is shown in Fig. 39. An observer falling into this BH may come out to an asymptotically flat region (after crossing the EH's) as indicated in Fig. 39.

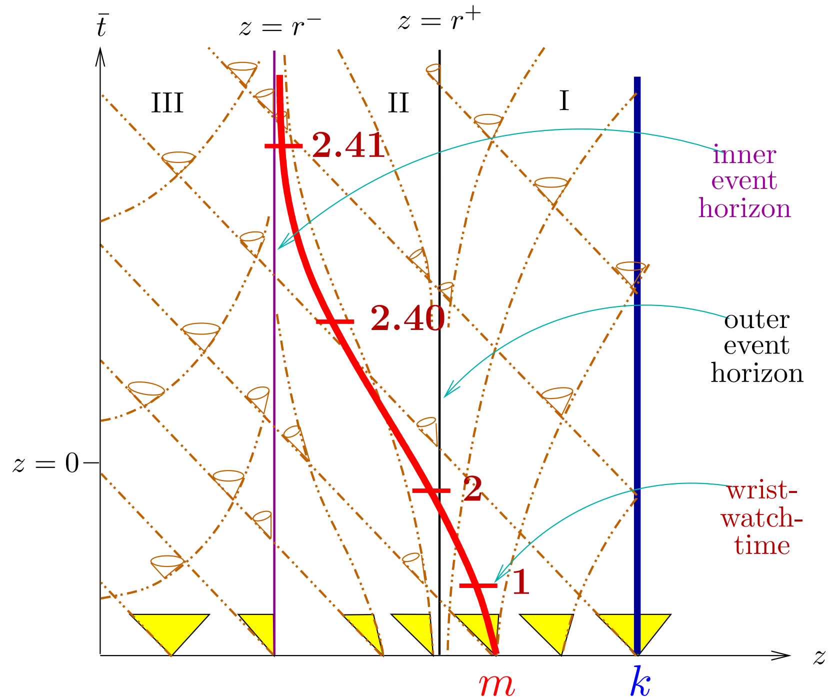

Figure 37: The "tr-slice" of electrically charged black hole. (Also the "tz-slice" of space-time of slowly rotating black hole in coordinates where z is the axis of rotation of black hole.) r^+ is the outer event horizon, r^- is the inner event horizon, $z = 0$ is the "center" of the black hole. The tilting of the light-cones indicates that not even light can escape through these horizons. That there is an outward push counteracting gravity can be seen via the shape of the light-cones in region III (central region of the black hole). The time measured by m is finite (measured between an event outside the inner EH and the event when m meets the inner event horizon) while the time measured by k is infinite.

Rotating (spinning) black holes

The space-times of slowly rotating black holes, called slow Kerr space-times in the literature, are similar to the electrically charged ones in that there are two EH's. We can think that the second, inner EH is the result of a repelling force overtaking the attraction of gravity. In the case of electrically charged black holes, the repelling force can be thought of, roughly, as the result of an excess of electrons (or protons) "in the BH", cf., e.g., [d'I83, pp. 239-244] or [HE73, p.156] for a more careful explanation. In the case of rotating black holes, the repelling force can be thought of as the centrifugal force of rotation. The metric-tensor $\mathsf{g}(p)$ of Kerr black hole at $p = (t, r, \varphi, \vartheta)$ is given by the 4 by 4 matrix

$$\mathsf{g}_{Kerr} = \begin{pmatrix} -1+\mu & 0 & -\mu a \sin^2\vartheta & 0 \\ 0 & \Sigma/\Delta & 0 & 0 \\ -\mu a \sin^2\vartheta & 0 & \mathsf{g}_{\varphi\varphi} & 0 \\ 0 & 0 & 0 & \Sigma \end{pmatrix},$$

where $\Sigma = r^2 + a^2 \cos^2\vartheta$, $\Delta = r^2 - \mathsf{M}r + a^2$, $\mu = \mathsf{M}r/\Sigma$, and $\mathsf{g}_{\varphi\varphi} = (r^2 + a^2 + \mu a^2 \sin^2\vartheta)\sin^2\vartheta$. We used the so-called Boyer–Lindquist coordinates $(t, r, \varphi, \vartheta)$ where $(t, r, \varphi, \vartheta)$ are kind of polar-cylindric coordinates, r being radius (to be precise, r is the logarithm of the radius) and φ, ϑ being angular coordinates like η and θ were on p. 88. In the Kerr metric, g_{Kerr}, $\mathsf{M}, a \in \mathbb{R}$ are parameters, M corresponding to mass and a to the angular momentum of the rotating singularity. Indeed, choosing $a = 0$ yields the metric of the simple Schwarzschild BH. A two-dimensional slice of a slowly rotating black hole is very similar to the one in Fig. 37, and a "spatial" representation is in Fig. 38.

We meet two interesting features in these black holes. The first interesting feature is that there are so-called Malament-Hogarth events in these space-times. An event e is called a Malament-Hogarth event if in the causal past of e there is a time-faithful curve which is infinite in the future direction. The words "past" and "future" are important here, these refer to a time-orientation of the space-time, as follows. A time-orientation on a GR space-time $\langle M, L\rangle$ is a smooth vector-field each member of which is timelike (formally, a time-orientation is a smooth $T : M \to \mathbb{R}^4$ such that $\boldsymbol{\mu}_p(T(p) - p) > 0$ for all $p \in M$). All the GR space-times mentioned in Sec. 4 have natural time-orientations. Given a time-orientation, the notion of a future-oriented timelike curve can be defined. The causal past of an event e is defined to be the set of events e' which can be connected with e by a future-oriented timelike curve such that e' is "earlier" in this curve than e is.

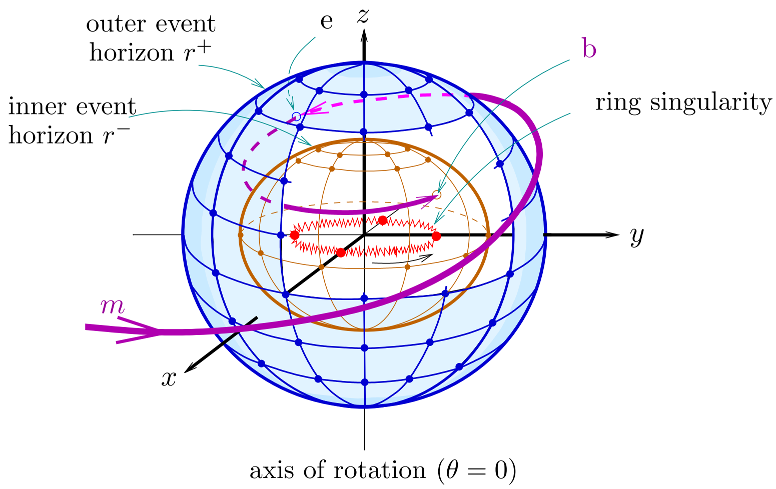

Figure 38: A slowly rotating (Kerr) black hole has two event horizons and a ring-shape singularity (the latter can be approximated/visualized as a ring of extremely dense and thin "wire"). The ring singularity is inside the inner event horizon in the "equatorial" plane of axes x, y. Time coordinate is suppressed. Fig. 37 is a space-time diagram of this with x, y suppressed. Rotation of ring is indicated by an arrow. Orbit of an in-falling observer m is indicated, it enters outer event horizon at point e, and meets inner event horizon at point b. For more on the basics of this figure cf. [[O'N95, Fig. 2.2, p. 63]].

One could think that in Malament-Hogarth events "actual infinity" is an observable physical reality. This phenomenon raises lots of intriguing questions to think over and has consequences even for the foundation of mathematics. For more on this we refer to [ND06], [NA06], and to [EN02]. There are Malament-Hogarth events in both the charged and the rotating black holes, see Fig. 39.

Another intriguing feature is the presence of closed timelike curves (CTC 's) in Kerr space-time. CTC's raise the question of time-travel into the past, and offer themselves for a logical treatment like the Liar Paradox. For more on this we refer to [Ear95]. There are CTC's in the space-time of a rotating black hole, see e.g. [O'N95, pp. 76-77, Prop. 2.4.7], [Wüt99], [ANW06]. There are many other kinds of space-times with CTC's, e.g. Tipler- van Stockum's rotating cylinder, Gödel's universe, the ones described in [Tho94] and [Nov98], to mention a few. Cf. Fig. 40.

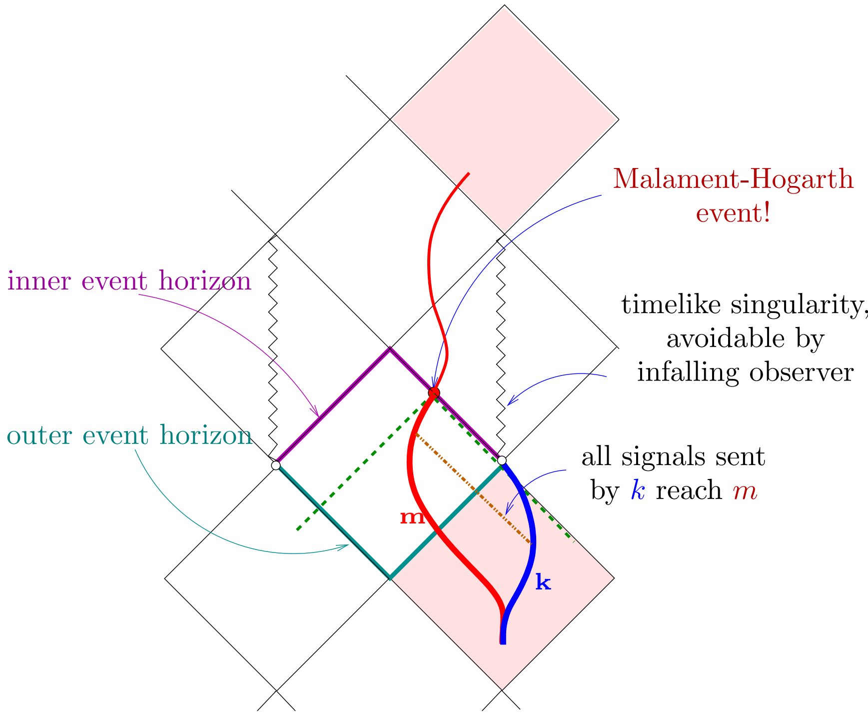

Figure 39: Penrose diagram of electrically charged black hole (and also of slowly rotating black hole). The red line represents a segment of the life-line of an in-falling inertial observer m, and the blue line represents the life-line of a suspended observer k. The time passed on the red line is finite, while the time passed on the blue line, i.e. for the suspended observer, is infinite. In principle, the in-falling observer has access in a finite wristwatch-time of his to all of the future history of the suspended observer k (an ultimate effect of "slow time" caused by BH's).

4.4 Black holes with antigravity (i.e. with a cosmological constant Λ). Triple black holes

One can combine the idea of a BH with a universe in which the vacuum regions have a nonzero curvature characterized by Einstein's cosmological term $\Lambda \in \mathbb{R}$. Λ may be positive or negative, but $|\Lambda|$ is small. Recent cosmological observations suggest that Λ or something like it might be out there, i.e. might be important for understanding the acceleration of our expanding universe. The line-element is a generalization of the one of the

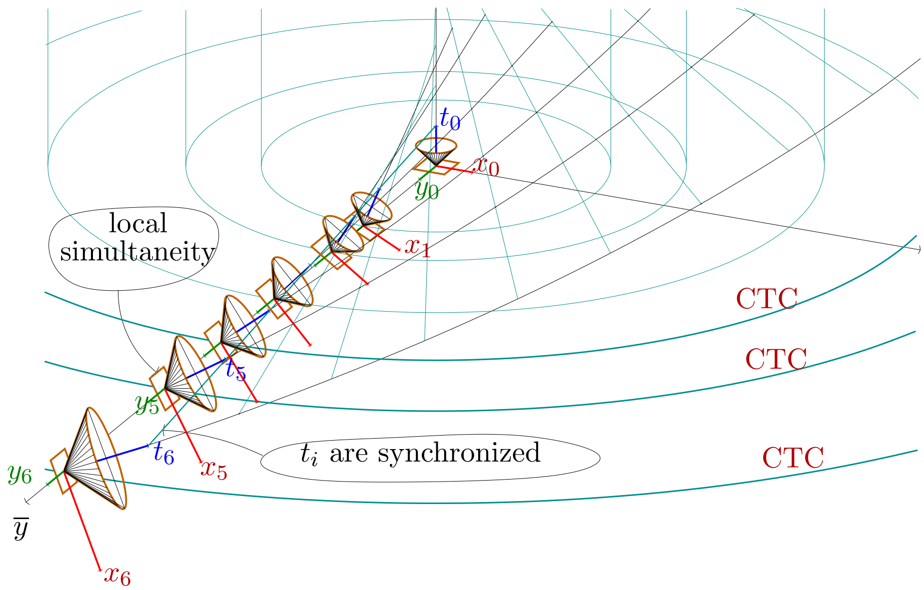

Figure 40: Starting point for Gödel's rotating cosmological model. This is a GR space-time, the vector fields and local light-cones representing the local special relativity frames (LFR's) are indicated. CTC's can be seen in the figure.

charged BH (generalizing G_{sb}) on p. 100

$$\mathsf{d}s^2 = (1 - \frac{\mathsf{M}}{r} + \frac{e}{r^2} - \Lambda r^2)\mathsf{d}t^2 - (1 - \frac{\mathsf{M}}{r} + \frac{e}{r^2} - \Lambda r^2)^{-1}\mathsf{d}r^2 - r^2\mathsf{d}\varphi^2.$$

Here M is the mass of the BH, e is (square of) its electric charge, and Λ represents the hypothetical antigravitational property of intergalactic vacuum. The parameters M, e, Λ can be chosen independently of each other obtaining various kinds of special cases. $\Lambda > 0$ causes the timelike geodesics outside the outer EH behave as if an antigravitational force would be pushing them outwards, away from the EH, cf. [Rin01, §14.4, pp. 304-306]. If $\Lambda = 0$, we are in asymptotically flat universe, $\Lambda > 0$ means negative curvature, hence what is called de-Sitter universe, while $\Lambda < 0$ means positive curvature (for our vacuum), and so-called anti-de-Sitter universe.

The choice $\Lambda < 0$ causes distant clocks speed up[15] (via the $-\Lambda r^2$ term in g_{tt}), while small $\Lambda > 0$ causes them to run slow (assuming $\Lambda r^2 \leq 1$; at

[15]Hence Malament-Hogarth computers breaking the Turing Barrier become possible, cf. [ND06], [Hog04].

$\Lambda r^2 = 1$ there is a "coordinate-event horizon"[16] if $\mathsf{M} = e = 0$). The behavior (speed) of distant clocks determine the behavior of geodesics (gravitation) according to the same "logic" as explained at the Schwarzschild BH on pp. 91-92. The choice of $\mathsf{M} = e = 0$ yields de-Sitter and anti-de-Sitter space-times respectively, depending on the sign of Λ.

4.5 Einstein's field equations

In the present work we concentrate on the space-time aspects of general relativity (GR). One of the reasons for this choice is that to study GR, it is reasonable to start with studying GR space-time, this enables one to study advanced and exotic examples of GR space-times like black holes, wormholes, cosmological models etc. as done e.g. in the GR textbook [TW00], and then turn to studying Einstein's field equations *EFE* and the rest of GR together with its "borderlines". This order is followed in, e.g., Penrose's recent book [Pen04]. About this possible continuation of studies we note the following. *EFE* is not a new axiom in the language of GR space-times restricting these. Instead, *EFE* is a definitional expansion of GR space-times in the sense of definability theory of mathematical logic (described in Sec. 2.6). *EFE* comes in two versions, a more flexible version, EFE^+, permitting the use of an extra parameter Λ for "fine-tuning" our space-time, and a less flexible one, *EFE*, in which $\Lambda = 0$ is assumed (or equivalently Λ is not used).

First we consider the $\Lambda = 0$ version. In this case, *EFE* is an explicit definition associating a tensor field denoted as $\langle T_{ij}(p) : p \in M\rangle$, or briefly T_{ij}, to every GR space-time $G = \langle M, \mathsf{g}\rangle$. From the logical point of view, T_{ij} is a brand new symbol not occurring in **Genrel** (or in the language of $\langle M, \mathsf{g}\rangle$ or $\langle M, L\rangle$). Hence **Genrel** + *EFE* can be regarded as a new theory expanding **Genrel** with new kinds of entities not mentioned in **Genrel** (or in its manifold oriented forms). Since *EFE* is an explicit definition (over **Genrel**) of the new entities called T_{ij}, the new theory **Genrel** + *EFE* is a conservative extension of **Genrel** (in the logical sense). This is the reason why we said earlier that *EFE* does not restrict the generality of **Genrel**,

[16] the event horizon at $r^2 = 1/\Lambda$ (i.e. where g_{tt} becomes 0 because of Λ) is in many respects like a huge Schwarzschild BH turned inside out, cf. [Rin01, p. 306, lines 11-22]. An electric BH in a de-Sitter space-time possesses three distinct event horizons, the innermost one caused, roughly, by e, the middle one caused, roughly, by M, and far out the outermost one cased by Λ. As we move away from the singularity lying on the time-axis $\bar{t}$, in the positive r direction, time and space get interchanged at crossing each one of the three event horizons.

though it introduces a new linguistic (or conceptual) device to add such restrictions later if/when wanted and justified.

The physical role of *EFE* is the following. *EFE* helps us in elaborating the connections between **Genrel** and other physical theories (such as e.g. electrodynamics, or e.g. mechanics). This is so because the new concept T_{ij} (or new property T_{ij} of $\langle M, \mathsf{g}\rangle$) can be interpreted in the various physical theories as representing typical physical quantities like mass-energy-momentum density at points $p \in M$. In other words, the "new" tensor field T_{ij} can be regarded as associating various physical properties (or entities) to points p of the space-time under investigation. It is in this connection that T_{ij} makes it possible for related theories of physics to induce restrictions on the models of **Genrel** via **Genrel** + *EFE*.

The more flexible theory **Genrel**+ EFE^+ is also a conservative extension of **Genrel** but in EFE^+ we introduce two new concepts, T_{ij} and Einstein's cosmological parameter Λ . What is Λ? It intends to specify the curvature of vacuum.[17] What do we mean by referring to the vacuum, in **Genrel** there was no such concept as the vacuum. Again, using the concept of vacuum is connected to the physical interpretations of the theory. Roughly, vacuum consists of those points p of the space-time where $T_{ij}(p) = 0$. The assumption $\Lambda = 0$ amounts to assuming that the curvature of vacuum is the same as the curvature of Minkowski space-time, i.e. as that of special relativity. Intuitively, **Genrel** + EFE^+ permits us to choose an arbitrary but fixed value $\Lambda \in \mathbb{R}$ for the whole space-time. Usually $|\Lambda|$ is small. It was this extra flexibility which made it possible for Kurt Gödel to specify his rotating universe [Göd49] as a universe containing only pressureless dust.

EFE^+ can be written in the following form:

$$(EFE^+)\ \ T_{ij} - \Lambda * \mathsf{g}_{ij} = \text{expression of}(\mathsf{g}_{ij} \text{ and derivatives of } \mathsf{g}_{ij}).$$

Cf. e.g. [Rin01, item 14.15, p. 303] or [Wal84, item 5.2.17, p. 99]. In (EFE^+) we suppressed the constants deriving from units of measurement. By inspecting (EFE^+) above, one can see that instead of determining T_{ij} (matter-energy-momentum-etc density), it determines only the difference of T_{ij} and Λ, more precisely, it tells us the value of $T_{ij} - \Lambda\mathsf{g}_{ij}$ (from knowing g_{ij} and its behavior). Hence (EFE^+) leaves us a certain degree of freedom for distributing effects between T_{ij} and Λ. Further, g_{ij} occurs on both sides of the equation, hence (EFE^+) is only an implicit circumscription, not an explicit definition.

[17] The curvature of a GR space-time $G = \langle M, \mathsf{g}\rangle$ is a definable property of G. In more detail, the curvature tensor field of G is defined from the behavior of the geodesics of G.

For completeness, we note that EFE^+ can also be used for a kind of classification of space-times, roughly, in terms of what they may "contain". An example is "vacuum space-times" which refer to space-times compatible with $T_{ij} = 0$ (uniformly). A complication here is that in principle the "division of labor" between T_{ij} and Λ is up to the interpreter's mind to choose. E.g. de-Sitter space-time (having a constant negative curvature) can be classified as a vacuum space-time with $\Lambda \neq 0$, or equivalently as one with $\Lambda = 0$ and T_{ij} nonzero. This classification can be further stretched to associate "realisticity" or "physicality" to space-times but such judgements often turn out to be subjective later. For illustration we note that Minkowski space-time is vacuum and so are G_{sb}, G_{ef}, the rotating BH's space-time, but the electrically charged BH's space-time is not vacuum (because the presence of electrical field at p implies $T_{ij}(p) \neq 0$).

5 Connections with the literature

Elaborating the logical foundations of relativity goes back to Hilbert's 6-th Problem. Most of the (logic oriented) work we are aware of concentrate on special relativity or on its fragments. Probably the first axiomatization for special relativity was given by Alfred Arthur Robb in 1914 [Rob14], and his work is the model or starting point of many later axiomatizations. There are many works in which an axiom system for special relativity is given, a small sample (which is far from being complete) of these is: [Rob14], [Rei69], [Car24], Alexandrov and his school starting with 1950 ([Ale74], [Gut82]), Suppes and his school starting with 1959 ([Sup59], [Sup68], [Sup72]), [Sze68], [Win77], [Ax78], [Fri83], [Mun86], [Gol87], [Sch97], [Lat72]. Of these, only [Ax78] and [Gol87] are in first-order logic. These works usually stop with a kind of completeness theorem for their axiomatizations. What we call the analysis of the logical structure of relativity theory begins with proving such a completeness theorem but the real work comes afterwards, during which one often concludes that we have to change the axioms. Very roughly, one could phrase this as "we start off where the others stopped (namely, at completeness)". Most of this literature concentrate on what we call $\mathbf{Specrel}_0$, namely the causal fragment of special relativity without its metric aspect (which is present in **Specrel**). We note that there are interesting works connecting modal logic with special relativity, e.g. [Gol80], [vB83, p. 4, pp. 22-29], [Cas02], [SS03].

As a contrast with special relativity, we know only of a few attempts for providing a logical analysis of general relativity. Examples are [Bas66],

[KP67], [Bus67], Ehlers, Pirani and Shild [EPS72], [Wal59]. None of these examples tries to stay within the framework of first-order logic (or even something like that, say, second-order logic) or attempts proving something like a completeness theorem. In Sec. 3.6 of the present work we propose a relatively simple first-order logic axiomatization **Genrel** for general relativistic space-times, and in Thm. 3.3 we formulate a completeness theorem for **Genrel**. What remains as a future research task is doing "reverse mathematics" for **Genrel**, i.e. providing a conceptual analysis for **Genrel** which would be analogous to the conceptual analysis provided for **Specrel** in Sec. 2 and in [AMN02]. Of course, a related future research task remains to push the present logic based conceptual analysis to the not yet existing theories conjectured to exist beyond general relativity like quantum gravity.

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