(iv) *There is an $\mathfrak{M}$ as in (i) such that there are distinct maximal geodesics ℓ, ℓ_1 such that $\ell \cap \ell_1$ is a half-line (moreover $\ell_1 \in L^T$). Intuitively this means that an unfinished geodesic can be continued in two different ways, cf. Figure 354.*

Further if we omit the condition $\mathbf{Ax}(Triv)$ *then one can choose ℓ and ℓ_1 such that both are continuous and derivable in some natural sense which we do not formalize here.*

$\triangleleft$

We include Figures 353, 354 as a hint for the idea of a possible proof.

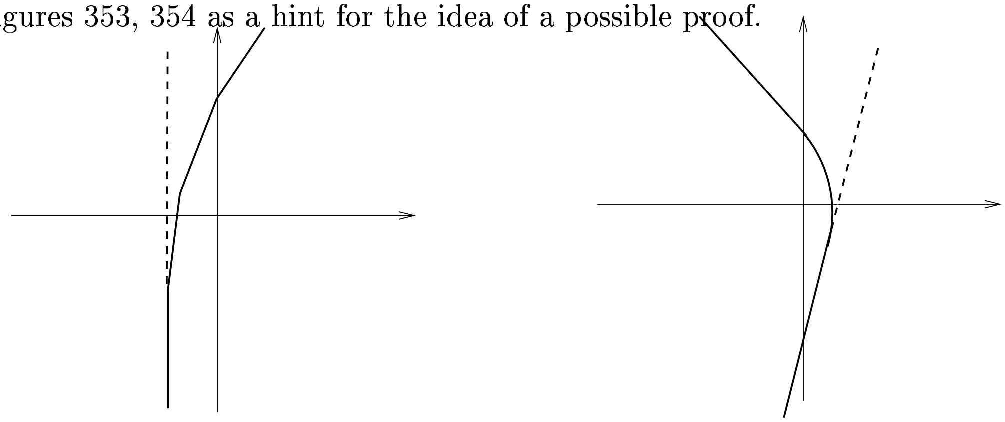

Figure 354: Illustration for the idea of a possible proof of (iv) of Conj.6.8.35.

Question for future research 6.8.36 Let $\mathfrak{G}$ be a relativistic geometry. Intuitively, we say that in $\mathfrak{G}$, at every point $e \in Mn$, in every direction there is exactly one maximal geodesic iff $(\star)$ below holds.

$(\star)$ Every short geodesic ℓ_0 is contained in a single maximal geodesic.

(So, in this formulation of the above intuitive principle, ℓ_0 plays the role of "coding" or representing a *direction*.)

It would be interesting to see natural collections Th of our axioms for relativity such that $(\star)$ is valid in $\mathsf{Ge}(Th)$.

$\triangleleft$

Summary of §6.8: Assume $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqm)}$. Then all the elements of L turn out to be geodesics (cf. Prop.6.8.7). If in addition we assume $n > 2$, $\mathbf{Basax} +$

$\mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ })$ and that $\mathfrak{F}$ is Archimedean, then the set L of lines coincides with the set of maximal geodesics (cf. Corollary 6.8.33, p.1204). We conjecture that the condition $n > 2$ is needed in the previous sentence, namely we conjecture that there is a model $\mathfrak{M}$ of $\mathbf{Basax}(2) + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax(eqm)}$ with $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ the ordered field of reals in which some maximal time-like geodesics of $\mathfrak{G}_{\mathfrak{M}}$ are not in $L_{\mathfrak{M}}$ (cf. Conjecture 6.8.35, p.1205). Further, we conjecture that for any $n \geq 2$, maximal time-like geodesics are not necessarily lines even if we assume **Basax** and $\mathfrak{F} = \mathfrak{R}$ (cf. Conjecture 6.8.35). As a contrast, if $n > 2$ and if we assume $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)}$ or $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{R}(\mathbf{Ax\ syt_0})$ together with some auxiliary axioms and that $\mathfrak{F}$ is Archimedean, then the set of maximal time-like geodesics coincides with the set L^T of time-like lines, cf. Thm.6.8.24 (p.1200) and Corollary 6.8.27 (p.1202). The latter condition (i.e. that $\mathfrak{F}$ is Archimedean) is needed even if we assume $\mathbf{BaCo} + \mathbf{Ax(rc)}$, cf. Thm.6.8.16 (p.1193) and Figure 348 (p.1194). Assuming $\mathbf{Reich(Bax)}^{\oplus} + \mathbf{Ax(diswind)}$, the maximal photon-like geodesics are exactly the members of L^{Ph} (cf. item (v) of Prop.6.8.8).

[1268] This figure is from Hawking-Ellis [126].

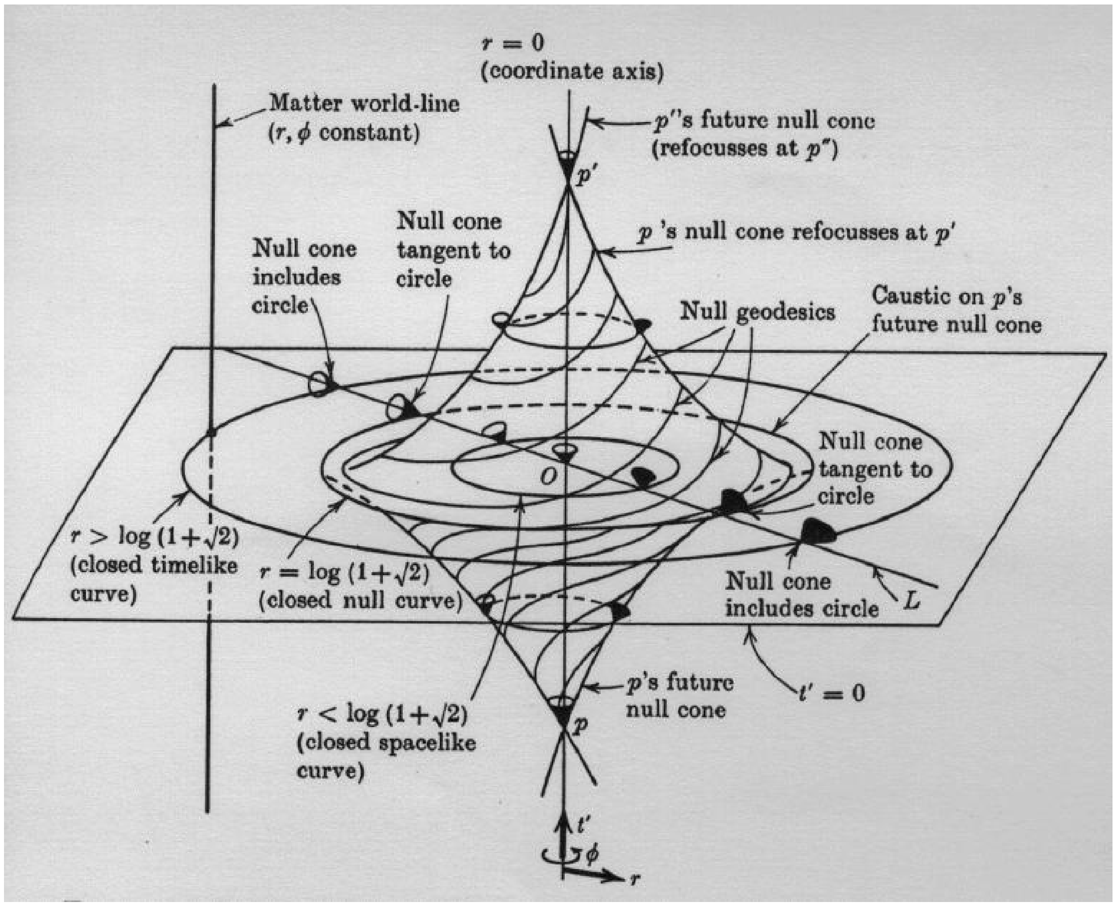

Figure 355: Gödel's rotating universe with the irrelevant coordinate z suppressed. The space is rotationally symmetric about any point; the diagram represents correctly the rotational symmetry about the axis $r = 0$, and the time invariance. The light cone opens out and tips over as r increases (see line L) resulting in closed time-like curves. The diagram does not correctly represent the fact that all points are in fact equivalent.[1268]

6.9 On what we learned (so far) about choosing our first order language for relativity

We have learned that there is an *observer-independent* "geometric" structure $\mathfrak{G}_{\mathfrak{M}}$ inside every frame model $\mathfrak{M}$ (and that a large part of $\mathfrak{M}$ is recoverable [even definable] from $\mathfrak{G}_{\mathfrak{M}}$, under some assumptions on $\mathfrak{M}$). Let us think about how the presence of $\mathfrak{G}_{\mathfrak{M}}$ might suggest to refine (or enrich, or improve) our first-order frame language for relativity.

Originally a model was

$$\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G; \in, W\rangle$$

with G a "geometry" on ${}^nF$ (i.e. $G \subseteq \mathcal{PP}({}^nF)$). Now, we could add g to G and even split G to G^T, G^{Ph}, G^S (where G^T, G^{Ph}, G^S are the natural counterparts of L^T, L^{Ph}, L^S). But the question is where should G and g "live". Originally they were assumed to "live" on the "coordinate-system" or vector-space ${}^n\mathbf{F}$. But now, it might be more natural to "shift" G and g to the observer-independent universe $Mn_{\mathfrak{M}}$ of events. In the discussion below, we ignore splitting G into G^T, G^{Ph}, G^S because the important things can be decided by concentrating on G and g. After shifting the "lines" to the observer-independent universe $Mn_{\mathfrak{M}}$ we will use L, L^T, etc. instead of G, G^T, etc.

A possible choice for the new language would be the following. A *new-model* would be a tuple

$$\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, L;\ g, \equiv^T, \equiv^{Ph}, \in, W\rangle \tag{400}$$

where $L, g, \equiv^T, \equiv^{Ph}$ are defined on $Mn_{\mathfrak{M}}$ instead of ${}^nF$. Of course realizing such a plan takes *some coding* because $Mn_{\mathfrak{M}}$ is not a universe (or sort) of $\mathfrak{M}$. All the same, this idea can be carried through[1269] because $Mn_{\mathfrak{M}}$ is first-order definable over $\mathfrak{M}$ by Prop.6.3.18.

Let us assume that new-models are defined the above way. Recall that for each observer m, in Def.6.2.76 (p.880), we defined $L_m, L^T_m, L^{Ph}_m, g_m$ (living in ${}^nF$) as inverse images of L, L^T, L^{Ph}, g along w_m. For each observer m, $\equiv^T_m \subseteq {}^nF \times {}^nF$ and $\equiv^{Ph}_m \subseteq {}^nF \times {}^nF$ are defined in the obvious way, i.e. $p \equiv^T q \overset{\text{def}}{\iff} (\exists \ell \in L^T_m) p, q \in \ell$ and $p \equiv^{Ph} q \overset{\text{def}}{\iff} (\exists \ell \in L^{Ph}_m) p, q \in \ell$. This way we obtain an *observer oriented* geometry:

$$\mathbf{G}_m \overset{\text{def}}{:=} \langle {}^nF, \mathbf{F_1}, L_m;\ g_m, \equiv^T_m, \equiv^{Ph}_m\rangle$$

[1269] similarly to the style in which we defined G to be understood on ${}^nF$ despite to the fact that ${}^nF$ is not a universe, or we defined $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ despite of $\mathcal{P}(B)$ not being a universe for the coordinate-system ${}^nF$ of each observer $m \in Obs$, cf. Def.6.2.76. Since the above definitions of L_m, g_m etc. can be carried out in the first-order language of $\mathfrak{M}$, we do not have to add the "personalized" geometries $\mathbf{G}_m$ to new-models $\mathfrak{M}$ as defined above, because they are already there as definable objects.

This observation enables us to view a new-model $\mathfrak{M}$ as in (400) as an extended tuple

$$\langle (B;\ Obs, Ph, Ib), \mathfrak{F}, L;\ g, \equiv^T, \equiv^{Ph}, (L_m;\ g_m, \equiv^T_m, \equiv^{Ph}_m), \in, w_m \rangle_{m \in Obs}.$$

For purely technical (and historical) reasons in some of the related works[1270] we might suppress/suppressed the observer-independent $L, g, \equiv^T, \equiv^{Ph}$ part of new-models and define new-new-models as e.g.

$$\langle (B;\ Obs, Ph, Ib), \mathfrak{F}, L_m;\ g_m, \equiv^T_m, \equiv^{Ph}_m, \in, w_m \rangle_{m \in Obs}. \tag{401}$$

(In the "Accelerated observers" chapter of [24] we use the notation G_m in place of L_m. In principle this may happen in the continuation of the present work on accelerated observers [23], too.) If/when this happens, we would like to emphasize that (400) is the "real" (ideally nice) definition and (401) arose only as a compromise of notational convenience.

Before closing this section let us return to way (400) of defining new models. Then

$$\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, L;\ g, \equiv^T, \equiv^{Ph}, \in, W \rangle$$

with $g : Mn_{\mathfrak{M}} \times Mn_{\mathfrak{M}} \overset{\circ}{\longrightarrow} F$ etc. This definition can be improved (from the logical point of view[1271]) by adding an extra universe (or sort) M of abstract events.[1272] Then a new-new-new-model is of the form

$$\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, M, L;\ g, \equiv^T, \equiv^{Ph}, \in, W \rangle \tag{402}$$

where M is considered as the universe of "abstract events", $c : M \longrightarrow \mathcal{P}(B)$ is the function associating the <u>*content*</u> $c(e)$ of event e to e.

Now, $w_m : {}^nF \overset{\circ}{\longrightarrow} M$ and

$$\langle M, L;\ g, \equiv^T, \equiv^{Ph} \rangle$$

[1270] e.g. in "Accelerated observers" chapter of [24]

[1271] I.e. from the point of view of easy checkability that $\mathfrak{M}$ is indeed a well defined first-order model of some language.

[1272] We note that this new universe M is already definable in our frame-models (by Prop.6.3.18 on p.957), hence the change is not as big as might seem to be.

is a "geometry" in the sense of Def.6.2.2 above.[1273] That is in particular $g : M \times M \overset{\circ}{\longrightarrow} F$.

This definition is much nicer than e.g. (400) above (not to mention the convenience driven definition (401)) but it has a minor cost: If we want to define new-models (402)-style then we will need extra axioms forcing M to be as close to Mn ($\subseteq \mathcal{P}(B)$) as possible (or rather as convenient). E.g. probably we will want $c : M \longrightarrow \mathcal{P}(B)$ to be injective (unless a reason emerges for relaxing this).

At this point we note that our earlier writing $c : M \longrightarrow \mathcal{P}(B)$ can be translated to the purely first-order language of $\mathfrak{M}$ by stipulating that $c \subseteq M \times B$ is a binary relation (between universes M and B) and that we understand $\bar{c}(e)$ as an abbreviation $\bar{c}(e) = \{b \in B : \langle e, b\rangle \in c\}$. Therefore if we want to be precise then $c \subseteq M \times B$ is a relation and $\bar{c} : M \longrightarrow \mathcal{P}(B)$ is the function understood in definition (402) above.

We close this section by observing that we feel that on the long run style (402) might prove to be the most fruitful extension of our language (e.g. for studying accelerated observers among other generalizations). On the short run, style (401) might be the easiest to follow. Indeed in the "Accelerated observers" chapter of [24] (401) was followed. The same may happen in the continuation of the present work [23].

All the axioms which we formulated earlier governing the behavior of $w_m : {}^nF \overset{\circ}{\longrightarrow} \mathcal{P}(B)$ would be now re-formulated to speak about composite function $w_m \circ c : {}^nF \overset{\circ}{\longrightarrow} \mathcal{P}(B)$ (since ${}^nF \overset{w_m}{\longrightarrow} M \overset{c}{\longrightarrow} \mathcal{P}(B)$ is our new schema of coordinatizing $\mathcal{P}(B)$). E.g. $tr_m(b)$ would be defined as

$$tr_m(b) := \{p \in {}^nF \ : \ b \in c(w_m(p))\}.$$

One of the purposes of improving our frame language as described in the present section is to prepare the road to generalizing our theory to accommodate *accelerated observers* too (besides inertial ones). The subject of accelerated observers will be studied in the continuation of this work on accelerated observers [23] (cf. also the "Accelerated observers" chapter of [24]) and we will start the study of accelerated observers by refining our frame language (in accordance with the considerations we made above).

[1273]Cf. footnote 198 on p.188 for the idea of w_m or w_m^- being a <u>partial</u> function ${}^nF \overset{\circ}{\longrightarrow} M$.