write

$$\{\langle Fm(Th), Th\rangle \;:\; Th \text{ is a theory}\} \quad \overset{\mathsf{Mod}}{\underset{\mathsf{Th}}{\rightleftarrows}} \quad \{\,\mathsf{K} \;:\; \mathsf{K} \text{ is a class of similar models}\,\}.$$

This was already explained except for the case when K is arbitrary (on the right-hand side). Let K be arbitrary, then $\mathsf{Th}(\mathsf{K})$ is a theory and $\mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ is the closure of K in the category of classes of models. To get the morphisms in the category of models we consider K as a topological space $\mathbf{K} = \langle \mathsf{K}, \mathcal{O}_{\mathsf{K}}\rangle$ where

$$\mathcal{O}_{\mathsf{K}} = \{\,\mathsf{K} \setminus \mathsf{Mod}(Th) \;:\; Th \text{ is a theory}\,\}.$$

Now, the morphisms between $\mathbf{K}$ and $\mathbf{K}_1$ are exactly the continuous functions.

In our next item we will discuss a restriction (or "sub-structure") of the (syntax, semantics)-duality, under the name $(\mathsf{Mod}, \mathsf{Th})$-duality.[1027]

(IV)

Analogy with the operators Mod *and* Th *in model theory. The* $(\mathsf{Mod}, \mathsf{Th})$*-duality:*[1028]

Let Fm and M be, respectively, the set of all formulas and the class of all models of an arbitrary first-order vocabulary. Recall that the functions

$$\mathsf{Mod} : \mathcal{P}(Fm) \longrightarrow \mathcal{P}(\mathsf{M}) \quad \text{and} \quad \mathsf{Th} : \mathcal{P}(\mathsf{M}) \longrightarrow \mathcal{P}(Fm)$$

were defined on p.28.

Now, our functions $\mathcal{G}$ and $\mathcal{M}$ can be put in analogy with the functions Mod and Th. To make the analogy with $(\mathcal{M}, \mathcal{G})$-duality sharper, we consider $\langle \mathcal{P}(\mathsf{M}), \subseteq\rangle$ and $\langle \mathcal{P}(Fm), \supseteq\rangle$ as the two worlds connected by the duality $(\mathsf{Th}, \mathsf{Mod})$. I.e. we

[1027] The name $(\mathsf{Mod}, \mathsf{Th})$-duality is not extremely fortunate for this restriction, since the general functors in the (syntax, semantics)-duality could also be called Mod and Th. Perhaps we should have used the expression "small $(\mathsf{Mod}, \mathsf{Th})$-duality" or "poset-$(\mathsf{Mod}, \mathsf{Th})$-duality". Actually, what we call $(\mathsf{Mod}, \mathsf{Th})$-duality below was called "syntax-semantics duality" in Chapter 4, p.453. (The explanation for this is that, as we said, $(\mathsf{Mod}, \mathsf{Th})$-duality is a part of the (syntax, semantics)-duality cf. Fig.314.) We hope context will help.

[1028] To see that the $(\mathsf{Mod}, \mathsf{Th})$-duality as discussed here is a restriction to a single vocabulary of the more general (syntax, semantics)-duality discussed in (III) above, we note the following: Let *Voc* be an arbitrary but fixed vocabulary. Now, if we restrict (syntax, semantics)-duality to *Voc* then we obtain the $(\mathsf{Mod}, \mathsf{Th})$-duality. More precisely when writing up the more general duality, instead of $\{\,\mathsf{K} \;:\; \mathsf{K} \text{ is a class of similar structures}\,\}$ we used the more special $\{\,\mathsf{Mod}(Th) \;:\; Th \text{ is a theory}\,\}$ on the right-hand side. The only reason for this was to save space, but cf. p.1026 where we indicated the more general formulation. Because of this connection between $(\mathsf{Mod}, \mathsf{Th})$-duality and the (syntax, semantics)-one, from other parts of this work sometimes we refer to $(\mathsf{Mod}, \mathsf{Th})$-duality with the name "syntax-semantics duality", cf. p.453.

changed the ordering on $\mathcal{P}(Fm)$ to make the similarity with our original duality more obvious. Theorem schema (A) concludes $\mathfrak{M} \rightarrowtail \mathcal{M}(\mathcal{G}(\mathfrak{M}))$. The counterpart here says $\mathsf{K} \subseteq \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$. On the other side we had $\mathfrak{G} \longleftarrow\!\prec \mathcal{G}(\mathcal{M}(\mathfrak{G}))$. The counterpart here says T "$\geq$" $\mathsf{Th}(\mathsf{Mod}(T))$ where "$\geq$" is $\subseteq$.

The closure operator induced on $\mathcal{P}(Fm)$ by this duality is an important one. We denote it as follows:

$$\mathsf{T} :\stackrel{\text{def}}{=} \mathsf{Mod} \circ \mathsf{Th},$$

i.e. for $Axi \subseteq Fm$, $\mathsf{T}(Axi) = \mathsf{Th}(\mathsf{Mod}(Axi))$ is the *theory generated* by the axiom system Axi.

(V) *Analogy with Galois theory of Cylindric algebras:* Let us take the Galois theory of Cylindric algebras as an example, cf. Andréka-Comer-Németi [9, 10] and Comer [61]. Here, $\mathfrak{M}$ corresponds to an RCA, say $\mathfrak{A}$, and $\mathcal{G}(\mathfrak{M})$ corresponds to the Galois group of $\mathfrak{A}$. Then $\mathcal{M}(\mathcal{G}(\mathfrak{M}))$ corresponds to the Galois closure $\mathfrak{A}^+$ of $\mathfrak{A}$, for which it is true that $\mathfrak{A}^{++} = \mathfrak{A}^+ \supseteq \mathfrak{A}$. So in a sense $\mathcal{M}(\mathcal{G}(\mathfrak{M}))$ is a kind of "Galois closure" of the original model $\mathfrak{M}$ (which will contain extra observers whose existence is kind of suggested by the observers already existing in $\mathfrak{M}$). We note that the Galois theory of cylindric algebras is strongly analogous with the Galois theory of fields, cf. item (I) above.

(VI) *Analogy with algebraic logic* will be discussed in §6.6.7. Algebraic logic can be regarded as a very important duality theory (actually it is a system or collection of duality theories). Connections with *Galois connections* and *adjoint functors* will be discussed in §§ 6.6.5, 6.6.6.

(VII) For further uses of Galois theories and duality theories (e.g. in connection with differential equations) cf. Janelidze [142, p.369]. For further duality theories in physics we refer to Varadarajan [270], but cf. also Lawvere-Schanuel [163, pp. 5–6, pp. 76–77]. Important additional information is in Remark 6.6.61 ("Motivation for Galois connections") item (II) footnote 1077 (p.1079). Duality theories involving C^*-algebras, and Laplace transform are on pp. 1098–1105.

(VIII) Further examples for duality theories (in and outside of physics) will be given on pp. 1096–1105.

This concludes Remark 6.6.4 (Galois theories, Galois connections, duality theories all over mathematics, in analogy with the ones in the present work).

◁

* * *

For stating our first theorems (of schema (A)–(I)) we introduce two new axioms **Ax(Bw)**, **Ax(∞ph)** and the new axiom system **Pax**$^+$.

Ax(Bw) $(\forall m, k \in Obs)[m \overset{\odot}{\to} k \ \Rightarrow\ (\mathsf{f}_{mk}$ is betweenness preserving) [1029]].

Ax(∞ph) $(\forall m \in Obs)(\forall ph, ph' \in Ph)\Big([\,\bar{0} \in tr_m(ph) \cap tr_m(ph') \ \wedge\ (\,ph$ and ph' move in the same direction as seen by $m\,) \wedge v_m(ph) = \infty\,] \ \to\ v_m(ph') = \infty \Big)$.

Intuitively, no observer can emit simultaneously in the same direction two photons one with infinite speed and the other one with finite speed.

In connection with **Ax(Bw)** and **Ax(∞ph)** we state Propositions 6.6.5, 6.6.9 which will be needed later. Recall that **Pax** is weaker than **Bax**$^-$, cf. p.482 in §4.3. The proposition below says that **Pax**$^\oplus$ + **Ax**($\sqrt{\ }$) implies **Ax(Bw)** and that if $n > 2$ **Bax**$^\oplus$ implies **Ax(Bw)**.

PROPOSITION 6.6.5

(i) **Pax** + **Ax**($\sqrt{\ }$) $\models$ **Ax(Bw)**.

(ii) *Assume $n > 2$. Then* **Bax**$^\oplus$ $\models$ **Ax(Bw)**.

Proof: Item (i) follows from Thm.4.3.13 on p.482 saying that the word-view transformations are bijective collineations in all models of **Pax**, and from Lemma 3.1.6 on p.163 saying that a line preserving bijection is an affine transformation composed by a field automorphism. Item (ii) follows from Thm.3.4.40 on p.241 saying that **Bax** implies that $\mathsf{f}_{mk} = \widetilde{\varphi} \circ f$, for some $f \in Aftr$ and $\varphi \in Aut(\mathbf{F})$, from Thm.3.4.19 on p.221 which says that **Bax** does not allow FTL observers, and from Lemma 6.6.6 below. ∎

LEMMA 6.6.6 *Let $\mathfrak{F} = \langle \mathbf{F}, \le \rangle$ be an ordered field. Let $\varphi \in Aut(\mathbf{F})$ be such that $(\forall x \in F)\ (|x| < 1 \ \Rightarrow\ |\varphi(x)| < 1)$.*

Then we have $\varphi \in Aut(\mathfrak{F})$, i.e. φ is order preserving.

We omit the **proof**. ∎

QUESTION 6.6.7 *Assume $n > 2$. Does* **Bax**$^{-\oplus}$ $\models$ **Ax(Bw)** *hold?*

◁

[1029] This can be formalized as $(\forall p, q, r \in {}^nF)(\mathsf{Betw}(p,q,r) \ \Rightarrow\ \mathsf{Betw}(\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r))$.

Remark 6.6.8 Many of the theorems of the present work remain true if we replace the assumption $\mathbf{Ax}(\sqrt{\ })$ with the "weaker" $\mathbf{Ax(Bw)}$. An example for such a theorem is Thm.4.3.24 saying that if $n > 2$ then $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ })$ excludes FTL observers. There are similar examples almost in every chapter. By replacing $\mathbf{Ax}(\sqrt{\ })$ with $\mathbf{Ax(Bw)}$, usually we obtain theorems stronger than the original one, since usually $\mathbf{Pax}$ is assumed and then Prop.6.6.5(i) implies that the new theorem is stronger (or equivalent).

$\lhd$

PROPOSITION 6.6.9 $\quad \mathbf{Bax}^- \models \mathbf{Ax}(\infty ph)$.

We omit the easy **proof**. ∎

Definition 6.6.10 $\quad \mathbf{Pax}^+ \stackrel{\text{def}}{:=} \mathbf{Pax} + \mathbf{AxE_{01}} + \mathbf{Ax(Bw)} + \mathbf{Ax}(\infty ph) + \Big([\,\mathbf{Ax(eqtime)} \ \wedge\ (\forall m,k \in Obs)(\forall 0 < i \in \omega)\ tr_m(k) \neq \bar{x}_i\,] \quad \vee \quad \mathbf{Ax(eqm)}\Big)$.[1030]

$\lhd$

If we replace $\mathbf{Ax(Bw)}$ by $\mathbf{Ax}(\sqrt{\ })$ in $\mathbf{Pax}^+$ then[1031] we get a stronger axiom system than $\mathbf{Pax}^+$.

The theory $\mathbf{Pax}^+$ above is designed to be weak, just strong enough for defining the function $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$.[1032] This is why $\mathbf{Pax}^+$ is so artificial. Our next proposition shows that in our definitions, and statements the assumption $\mathbf{Pax}^+$ can be replaced by more natural (but stronger) theories. In passing we note that $\mathbf{Pax}^+(2)$ allows $\mathbf{Basax} + \mathbf{Ax(symm)}$ models with FTL observers.

PROPOSITION 6.6.11 *Assume $n > 2$. Then (i)–(iii) below hold.*

(i) $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(Bw)} + \mathbf{Ax(eqtime)} \ \models\ \mathbf{Pax}^+$.

(ii) $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)} \ \models\ \mathbf{Pax}^+$.

(iii) $\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqtime)} \ \models\ \mathbf{Pax}^+$.

[1030] Instead of $\mathbf{Ax(eqtime)}$ we could use the weaker axiom $\mathbf{Ax(eqtime)} \vee \mathbf{Ax(eqspace)}^{\oplus}$. Then we would obtain a weaker axiom system $\mathbf{Pax}^{+-}$. The theorems of the present sub-section (i.e. §6.6.1) remain true if we replace $\mathbf{Pax}^+$ with $\mathbf{Pax}^{+-}$ in them. For an even more general duality theory cf. Remark 6.6.51 (p.1065).

[1031] by Thm.4.3.13 (p.482), Lemma 3.1.6 (p.163) and Remark 3.6.7 (p.268)

[1032] That functor $\mathcal{M}$ will be defined later (beginning with p.1052).

Proof: Assume $n > 2$. Then, by the proof of Thm.4.3.24 (p.497), $\mathbf{Bax}^{-\oplus}+\mathbf{Ax(Bw)}$ excludes FTL observers. Further, $\mathbf{Bax}^- \models \mathbf{Ax(}\infty ph\mathbf{)}$ by Prop.6.6.9. Therefore item (i) of the proposition holds. Item (ii) follows by (i) and by Prop.6.6.5(i). Item (iii) follows by Thm.3.4.19 (p.221) and Prop.6.6.5(ii). ∎

Below we state a theorem corresponding to the theorem schemas (C) and (D) on p.1009 way above. The theorem below implies that $\mathsf{Mod}(Th) \equiv^w_\Delta \mathsf{Ge}(Th)$, if we assume that Th satisfies **Ax(diswind)** and condition ($\star$) in the theorem. We will see that more than this is true, namely Thm.6.6.13 says that $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$ under the same conditions.

THEOREM 6.6.12
There is a first-order definable meta-function $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ *such that (i)–(iii) below hold, for any Th satisfying condition ($\star$) way below.*

- **(i)** $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$ *(and of course* $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$*).*
- **(ii)** *Both* $\mathcal{M} \circ \mathcal{G}$ *and* $\mathcal{G} \circ \mathcal{M}$ *have strong fixed-point property in the sense that for any* $\mathfrak{G} \in \mathsf{Ge}(Th)$ *and* $\mathfrak{M} \in \mathsf{Mod}(Th)$

$$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G} \quad \textit{and} \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M},$$

  moreover there is an isomorphism between $\mathfrak{G}$ *and* $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$ *which is the identity map on* F*, and the analogous statement holds for* $\mathfrak{M}$ *and* $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$*, see Figure 315 and pictures* (C), (D) *in Figure 311 (p.1010).*
- **(iii)** *Moreover,* $\mathcal{G}$ *and* $\mathcal{M}$ *are* <u>*first-order definable*</u> *meta-functions, assuming* $Th \models$ **Ax(diswind)**.
- ($\star$) $n > 2$ *and* $Th \models \mathbf{Bax}^\oplus + \mathbf{Ax(}Triv_t\mathbf{)}^- + \mathbf{Ax(}\|\mathbf{)}^- + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ })$.

Proof: The theorem follows by Thm.6.6.46 (p.1061) way below. ∎

Our next theorem states a very strong connection between our frame-models $\mathsf{Mod}(Th)$ and our observer-independent geometries $\mathsf{Ge}(Th)$. The methodological importance of these kinds of theorems (from the point of view of physics) was discussed in the introduction of §6.2.2 (p.806) and in the introduction to the present chapter (§6.1). The theorem below says that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent under some assumptions. But if two theories (or axiomatizable classes of models) are definitionally equivalent then this means that, basically, *they are the same* theory "represented" in two different ways; cf. Remark 6.3.31 (p.973) and the

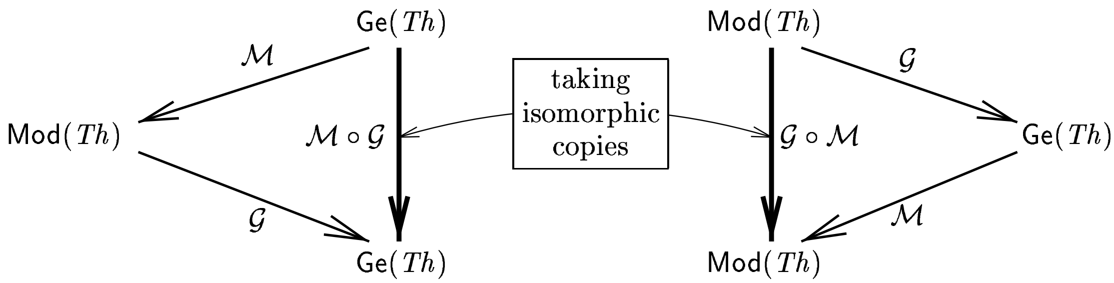


Figure 315: $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G}$ and $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M}$.

discussion on p.972 (in §6.3). The same applies to classes of models (like $\mathsf{Ge}(Th)$ and $\mathsf{Mod}(Th)$) in place of theories. Therefore our next theorem can be interpreted as saying that our observational world $\mathsf{Mod}(Th)$ is basically the same as our theoretical world $\mathsf{Ge}(Th)$. The theorem implies that our theoretical concepts are already available in $\mathsf{Mod}(Th)$ as "abbreviations" or "shorthands"[1033]; and that in the other direction, our observational concepts (like observer, coordinate system etc.) are present in our theoretical world $\mathsf{Ge}(Th)$ as "abbreviations".

THEOREM 6.6.13 $\mathsf{Mod}(Th)$ *and* $\mathsf{Ge}(Th)$ *are definitionally equivalent, in symbols*

$$\boxed{\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th),}$$

assuming $n > 2$ *and* $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\|)^- + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$.

In the proof of Thm.6.6.13 we will use Lemma 6.6.14 below. Therefore the proof of Thm.6.6.13 comes below the lemma.

The subject matter of the following lemma belongs to definability theory, i.e. to §6.3. For a similar lemma cf. Lemma 6.5.4 (p.994).

LEMMA 6.6.14 *Let* K, L *and* K^+ *be classes of models. Then (i) and (ii) below hold.*

(i) *Assume that* IK *is axiomatizable and that* K^+ *is rigidly definable over both* K *and* L. *Then*

$$\mathsf{K} \equiv_\Delta \mathsf{L}.$$

[1033]This direction can be interpreted as concluding that our theoretical concepts are acceptable (or well chosen) from the point of view of Machian-Einsteinian philosophy of theory making.

(ii) *Assume that* K^+ *is rigidly definable over* K, $\mathbf{I}\mathsf{L}$ *is closed under taking ultraproducts,* $Voc\mathsf{K}^+ \cap Voc\mathsf{L} = Voc\mathsf{K} \cap Voc\mathsf{L}$, *and* $\mathsf{K} \equiv_\Delta \mathsf{L}$. *Then* $\mathsf{K}^+ \equiv_\Delta \mathsf{L}$.

Proof: To prove item (i) assume K, L, K^+ satisfy the assumptions in (i). Then, by Lemma 6.5.4 on p.994, to prove that $\mathsf{K} \equiv_\Delta \mathsf{L}$ it is enough to prove that $\mathbf{I}\mathsf{K}^+$ is closed under taking ultraproducts. Since K^+ is definable over K, there is a definitional expansion K^{++} of K such that K^+ is a reduct of K^{++}. Let such a K^{++} be fixed. Thus $\mathbf{I}\mathsf{K}^{++}$ is a definitional expansion of $\mathbf{I}\mathsf{K}$ and $\mathbf{I}\mathsf{K}^+$ is a reduct of $\mathbf{I}\mathsf{K}^{++}$. Hence, $\mathbf{I}\mathsf{K}^{++}$ is axiomatizable (because $\mathbf{I}\mathsf{K}$ is axiomatizable by our assumption in (i)). Thus, $\mathbf{I}\mathsf{K}^{++}$ is closed under taking ultraproducts. Since $\mathbf{I}\mathsf{K}^+$ is a reduct of $\mathbf{I}\mathsf{K}^{++}$, we have that $\mathbf{I}\mathsf{K}^+$ too is closed under taking ultraproducts. This completes the proof of item (i). We omit the proof of item (ii). ∎

Outline of proof of Thm.6.6.13: Assume $n > 2$ and that Th satisfies the assumption of the theorem. Let $\mathsf{Ge}^-(Th)$ be the topology free reduct of $\mathsf{Ge}(Th)$. Let $\mathsf{Ge}^-(Th)+T_0$ denote the expansion of $\mathsf{Ge}^-(Th)$ with the subbase T_0 (of the topology) and the membership relation $\in_{Mn\times T_0}$ as indicated in Prop.6.3.19 on p.959. Hence, the models $\mathsf{Ge}^-(Th) + T_0$ are of the form $\langle \mathfrak{G}, T_0; \in_{Mn\times T_0}\rangle$ with $\mathfrak{G} \in \mathsf{Ge}^-(Th)$ and $T_0, \in_{Mn\times T_0}$ as indicated on p.959. By the proof of Prop.6.3.19, $\mathsf{Ge}^-(Th)+T_0$ is rigidly definable over $\mathsf{Ge}^-(Th)$. By this and by Lemma 6.6.14(ii), we conclude that it is sufficient to prove $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}^-(Th)$ for proving $\mathsf{Mod}(Th) \equiv_\Delta (\mathsf{Ge}^-(Th) + T_0)$. According to our convention below ($\star\star$) on p.809 we consider the latter sufficient for proving $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$. Therefore to prove the present theorem it enough to prove $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}^-(Th)$. We will do just this.

To prove $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}^-(Th)$, by Lemma 6.6.14(i), it is enough to find a class M such that M is rigidly definable both over $\mathsf{Mod}(Th)$ and $\mathsf{Ge}^-(Th)$. Now, we turn to constructing such an M. First, we define the vocabulary of M. (The common vocabulary of $\mathsf{Mod}(Th)$ and $\mathsf{Ge}^-(Th)$ consists of the sort symbol F and relation/function symbols $+, \cdot, \leq$). $Voc\,\mathsf{M} :=$ "$Voc\,\mathsf{Mod}(Th) + Voc\,\mathsf{Ge}^-(Th) +$ (relation symbols O and P, where the rank of O is $\langle B, \underbrace{Mn, \ldots, Mn}_{(n+1)\text{-times}}\rangle$, and the rank of P is $\langle B, L\rangle$)". Now,

$$\mathsf{M} :\stackrel{\text{def}}{=} \mathbf{I}\Big\{ \langle \mathfrak{M}, \mathfrak{G}_{\mathfrak{M}}; O, P\rangle \;:\; \mathfrak{M} \in \mathsf{Mod}(Th),$$
$$O = \{ \langle m, w_m(\bar{0}), w_m(1_0), \ldots, w_m(1_{n-1})\rangle \;:\; m \in Obs^{\mathfrak{M}} \}$$
$$P = \{ \langle ph, \{ e \in Mn \;:\; ph \in e \}\rangle \;:\; ph \in Ph^{\mathfrak{M}} \} \Big\}.$$

By the proof of Prop.6.3.18 (p.957) and Thm.6.3.22 (p.961) it is not hard to see that M is rigidly definable over $\mathsf{Mod}(Th)$. By Def.6.6.41 (p.1054), Prop.6.6.44 (p.1059), Remark 6.2.66 (ii) (p.867), Propositions 6.2.88 (p.895) and 6.2.92 (p.901) it is not hard to see that M is rigidly definable over $\mathsf{Ge}^-(Th)$. ∎

Conjecture 6.6.15 *We conjecture that in the above theorem* **Ax(diswind)** *is needed (because we conjecture that* $\perp_r$ *is not first-order definable in* $\mathsf{Mod}(Th \setminus \{\mathbf{Ax(diswind)}\})$*, where Th is as in Thm.6.6.13 above), cf. Figure 316.*
◁

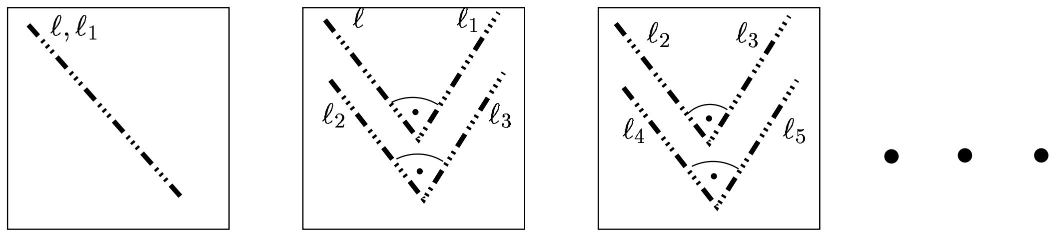


Figure 316: We conjecture that **Ax(diswind)** is needed in Thm.6.6.13, i.e. that without assuming **Ax(diswind)** $\perp_r$ is not definable. (Hint: $\ell, \ell_1, \ldots \in L^{Ph}$, $\ell \perp_r \ell_1$ by closing $\perp_r$ up under limits; and $\ell \perp_r \ell_1 \;\Rightarrow\; \ell_2 \perp_r \ell_3 \;\Rightarrow\; \ell_4 \perp_r \ell_5 \;\Rightarrow\; \ldots$, by closing $\perp_r$ up under parallelism.)

The following theorem implies that the sentences in our frame language can be translated (in a meaning preserving way) to sentences in the language of our observer independent geometries and vice-versa, under some assumptions. Cf. the text above Thm.6.6.13, Remark 6.3.31, introduction of §6.2.2 and the text above Prop.6.4.8 (p.987). In connection with the following theorem we note that F is a common sort of $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$.

THEOREM 6.6.16 *Let* $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ *be a first-order definable meta-function such that for this choice of* $\mathcal{M}$ *the conclusions of Thm.6.6.12 above hold. Assume* $n > 2$ *and that Th is as in Thm.6.6.13 above. Then there are "natural" translation mappings*

$$T_{\mathcal{M}} : Fm(\mathsf{Mod}(Th)) \longrightarrow Fm(\mathsf{Ge}(Th)) \quad \text{and} \quad T_{\mathcal{G}} : Fm(\mathsf{Ge}(Th)) \longrightarrow Fm(\mathsf{Mod}(Th))$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{Mod}(Th))$, $\psi(\bar{y}) \in Fm(\mathsf{Ge}(Th))$ *with all their free variables belonging to sort* F, $\mathfrak{M} \in \mathsf{Mod}(Th)$ *and* $\mathfrak{G} \in \mathsf{Ge}(Th)$, *and evaluations* $\bar{a}, \bar{b}$ *of* $\bar{x}, \bar{y}$, *respectively (in* F *of course), (i)–(iv) below hold.*[1034]

[1034] We note that the formulas φ and $T_{\mathcal{M}}(\varphi)$ have the same free variables (therefore (i) below makes sense). Similarly for $T_{\mathcal{G}}$ etc.

(i) $\mathcal{M}(\mathfrak{G}) \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathfrak{G} \models T_{\mathcal{M}}(\varphi)[\bar{a}]$ *and* $\mathcal{G}(\mathfrak{M}) \models \psi[\bar{b}] \;\Leftrightarrow\; \mathfrak{M} \models T_{\mathcal{G}}(\psi)[\bar{b}]$.

(ii) $\mathfrak{M} \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathcal{G}(\mathfrak{M}) \models T_{\mathcal{M}}(\varphi)[\bar{a}]$ *and* $\mathfrak{G} \models \psi[\bar{b}] \;\Leftrightarrow\; \mathcal{M}(\mathfrak{G}) \models T_{\mathcal{G}}(\psi)[\bar{b}]$.

(iii) $\mathfrak{M} \models \varphi(\bar{x}) \leftrightarrow T_{\mathcal{G}}(T_{\mathcal{M}}(\varphi))(\bar{x})$ *and* $\mathfrak{G} \models \psi(\bar{y}) \leftrightarrow T_{\mathcal{M}}(T_{\mathcal{G}}(\psi))(\bar{y})$.

(iv) $\mathsf{Mod}(Th) \models \varphi \;\Leftrightarrow\; \mathsf{Ge}(Th) \models T_{\mathcal{M}}(\varphi)$ *and* $\mathsf{Ge}(Th) \models \psi \;\Leftrightarrow\; \mathsf{Mod}(Th) \models T_{\mathcal{G}}(\psi)$.

Proof: The theorem follows from Theorems 6.6.12 and by Prop.6.4.8 on p.987 (and by noticing that Thm.6.6.12 implies that $\mathsf{Mod}(Th) \equiv^{w}_{\Delta} \mathsf{Ge}(Th)$).

∎

Below we state a theorem corresponding to the theorem schemas (A), (C)–(H) on p.1009 way above. In connection with the formulation of the next theorem we note that for any Th, $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ by the definition of $\mathcal{G}$. (Hence, in particular $\mathcal{G} : \mathsf{Mod}(\mathbf{Pax}^{+}) \longrightarrow \mathsf{Ge}(\mathbf{Pax}^{+})$.)

THEOREM 6.6.17
There is a first-order definable meta-function $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^{+}) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^{+})$ *such that (i)–(iv) below hold.*

(i) *The members of the range of* $\mathcal{M}$ *are fixed-points of* $\mathcal{G} \circ \mathcal{M}$*, formally: For any* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^{+})$

$$(\mathcal{G} \circ \mathcal{M})(\mathcal{M}(\mathfrak{G})) \cong \mathcal{M}(\mathfrak{G}),$$

see picture (F) *in Figure 311 (p.1010).*

(ii) *Both* $\mathcal{G} \circ \mathcal{M}$ *and* $\mathcal{M} \circ \mathcal{G}$ *have fixed-point property in the sense that for any* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^{+})$ *and* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^{+})$

$$(\mathcal{G} \circ \mathcal{M})^2(\mathfrak{M}) \cong (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \quad \textit{and} \quad (\mathcal{M} \circ \mathcal{G})^2(\mathfrak{G}) \cong (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}),$$

see Figure 317 and pictures (G) *and* (H) *in Figure 311 (p.1010).*

(iii) $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^{+} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^{+} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit)$ *and for any* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^{+} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit)$
$\mathfrak{M}$ *is embeddable into* $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$*, i.e.*

$$\mathfrak{M} \rightarrowtail (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}),$$

see Figure 318 and picture (A) *in Figure 311 (p.1010).*

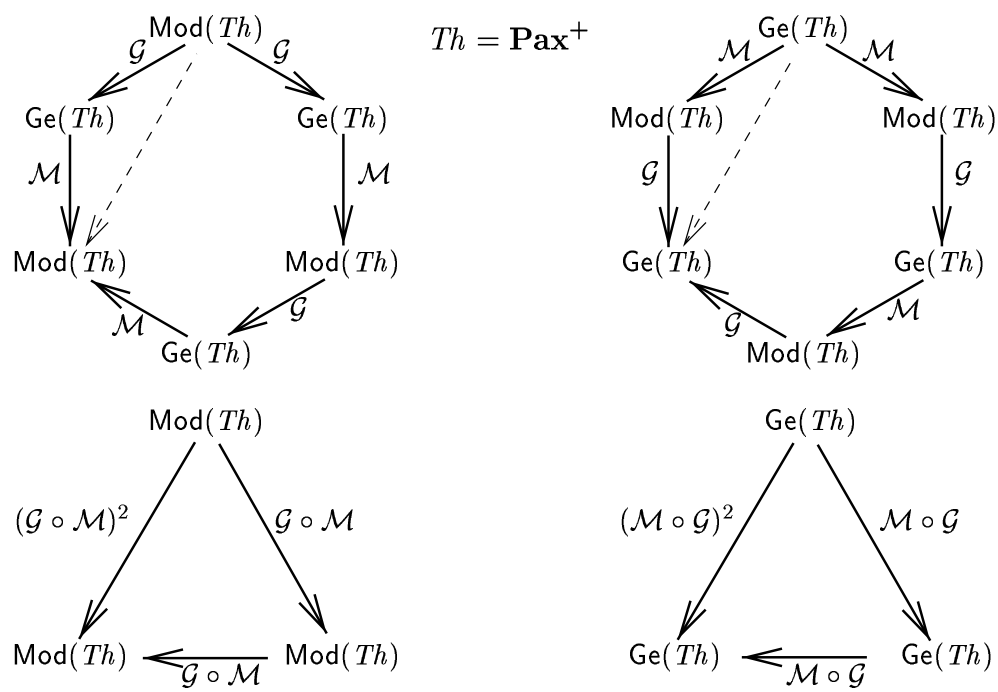


Figure 317: These diagrams commute up to isomorphism.

$Th = \mathbf{Pax}^{+} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit$

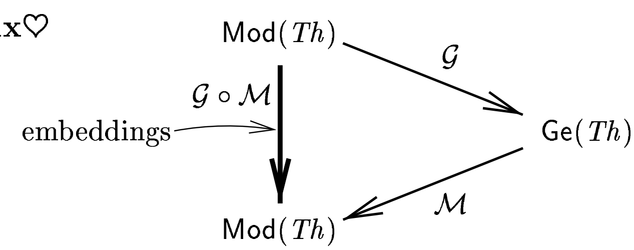


Figure 318: $\mathfrak{M} \rightarrowtail (\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$

(iv) $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)}) \longrightarrow \mathsf{Mod}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$ *(and of course* $\mathcal{G} : \mathsf{Mod}(\mathbf{Pax^+} + \mathbf{Ax(eqm)}) \longrightarrow \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$*), and*

$\mathcal{M} \circ \mathcal{G}$ *has a strong fixed-point property in the sense that for any* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$

$$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G},$$

(cf. the left-hand side of Fig.315 and picture (D) *in Fig.311).*

Further, the members of the range of $\mathcal{G}$ *are fixed-points of* $\mathcal{M} \circ \mathcal{G}$*, formally: For any* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax^+} + \mathbf{Ax(eqm)})$

$$(\mathcal{M} \circ \mathcal{G})(\mathcal{G}(\mathfrak{M})) \cong \mathcal{G}(\mathfrak{M}),$$

cf. picture (E) *in Figure 311 (p.1010).*

Proof: The theorem follows by Thm.6.6.46 (p.1061) way below. ∎

Assume, for $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax^+}) \longrightarrow \mathsf{Mod}(\mathbf{Pax^+})$ that the conclusions of Thm.6.6.17 hold and $\mathcal{M}$ is a first-order definable meta-function. Let

$$Th := \mathbf{Pax^+} + \mathbf{Ax(eqm)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit.$$

Then, by Thm.6.6.17, $\mathcal{G} \circ \mathcal{M}$ and $\mathcal{M} \circ \mathcal{G}$ are closure operators on $\langle \mathsf{Mod}(Th), \subseteq_{\mathsf{w}} \rangle$ and $\langle \mathsf{Ge}(Th), {}_{\mathsf{w}}\supseteq \rangle$ up to isomorphism, respectively (cf. p.1013), assuming $\mathcal{G} \circ \mathcal{M}$ and $\mathcal{M} \circ \mathcal{G}$ preserve $\subseteq_{\mathsf{w}}$. Further, $\mathcal{M} \circ \mathcal{G}$ is the "identity operator" on $\mathsf{Ge}(Th)$ up to isomorphism, i.e. for any $\mathfrak{G} \in \mathsf{Ge}(Th)$, $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G}$. The analogous statement for $\mathcal{G} \circ \mathcal{M}$ does not hold in general, i.e. there is $\mathfrak{M} \in \mathsf{Mod}(Th)$ such that $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \ncong \mathfrak{M}$. This asymmetry is caused by our choice of $\mathcal{G}$, i.e. by the fact that $\mathcal{G}$ is surjective in the sense that $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism. We will have a duality theory for the $(g, \mathcal{T})$-free reduct of our geometries in §6.6.4 which will be more symmetric.

Further theorems in this line (duality theories, Galois connections etc.) will follow after we elaborate the definitions of e.g. the function $\mathcal{M}$. For that definition we will need some preparation e.g. coordinatization of our geometries summarized in §6.6.2 below.

Our next sub-section is on coordinatization. For applications of this kind of coordinatization in physics cf. e.g. Varadarajan [270].

6.6.2 Coordinatization of geometries by ordered fields

In the present sub-section our geometries, in most of the cases, are of the form $\langle Mn; Bw\rangle$, where Mn is the set of points and Bw is a ternary relation (of betweenness) on Mn. We do not assume that our geometries $\langle Mn; Bw\rangle$ are reducts of relativistic geometries. It is known from elementary geometry that if a geometry $\langle Mn; Bw\rangle$ satisfies certain axioms, then it can be coordinatized by an ordered field and this ordered field is unique up to isomorphism (cf. e.g. Hilbert [134] or Goldblatt [108] or Schwabhäuser-Szmielew-Tarski [237]). We will recall this coordinatization procedure from the literature (cf. [108, 134, 237]) in a slightly modified form. Before recalling the coordinatization we collect some axioms obtaining the axiom system **opag** which will be sufficient for the coordinatization[1035] of $\langle Mn; Bw\rangle$ by an ordered field. The "geometrical theory" **opag** and the theory of ordered fields will turn out to be weakly definitionally equivalent, cf. Prop.6.6.29 (p.1045).

Roughly speaking, **opag** is an axiomatization of affine geometry. Affine geometry has been thoroughly studied in the literature, and several axiomatizations for affine geometry are available in the literature, cf. Remark 6.7.17 on p.1148. (So **opag** is not particularly new, it has been put together to suit our purposes in the present work.)

Beside the geometry $\langle Mn; Bw\rangle$ we will also discuss the geometry $\langle Mn; coll\rangle$. In the case of "$\langle Mn; Bw\rangle$" *coll* is a defined relation, i.e. we use the abbreviation *coll* over $\langle Mn; Bw\rangle$ exactly as it was introduced in item 6.2.12 on p.818.

The new sort *lines* of $\langle Mn; coll\rangle$ as well as of $\langle Mn; Bw\rangle$ together with the incidence relation $\in\ \subseteq Mn\times lines$ are explicitly defined (in the sense of §6.3.2) as follows. (Recall that in the case of "$\langle Mn; Bw\rangle$" *coll* is a defined relation.) First we define

$$R := \{\, \langle a,b\rangle \ :\ (\exists c\in Mn)\, coll(a,b,c),\ a\neq b \,\}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1. Intuitively, the elements of U will code the elements of *lines*. We define a kind of incidence relation E' between Mn and U as follows. Let $e\in Mn$ and $\ell\in U$.

[1035]The coordinatizations (by Hilbert and others) of (synthetic) geometries mentioned above are related to the subject matter of the present section because observer m coordinatizes Mn by the world-view function w_m, i.e. $w_m : {}^nF \longrightarrow Mn$ is a coordinatization of Mn. In passing we note that the coordinatization methods of Hilbert, von Neumann, von Staudt (cf. in [13]), and others are applied in pure logic e.g. in Andréka-Givant-Németi [13, pp. 16–19]. (The reference to von Neumann can be found in [13].) Tarski's school call such coordinatization results representation theorems. The idea is that we represent an abstract axiomatic geometry as a concrete (analytic) geometry in the Cartesian spirit. Cf. Remark 6.6.87 (p.1106).

Then

$$e\ E'\ \ell \quad \overset{\text{def}}{\Longleftrightarrow} \quad coll(pj_0(\ell), pj_1(\ell), e).$$

Then we define the equivalence relation $\equiv$ on U as follows. Let $\ell, \ell' \in U$. Then

$$\ell \equiv \ell' \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall e \in Mn)(e\ E'\ \ell\ \leftrightarrow\ e\ E'\ \ell').$$

We define the new sort $lines := U/\!\equiv$ together with $\in_{U,U/\equiv}\ \subseteq U \times U/\!\equiv$. Finally, the incidence relation $\mathrm{E} \subseteq Mn \times lines$ is defined as follows. Let $e \in Mn$ and $\ell \in lines$. Then

$$e\ \mathrm{E}\ \ell \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\exists \ell' \in \ell)\, e\ E'\ \ell'.$$

Since the axiom of extensionality holds for the incidence relation E we identify E with the real set theoretic membership relation $\in$. More precisely, without loss of generality we may assume that $lines \subseteq \mathcal{P}(Mn)$ and that E coincides with the set theoretic $\in$, so we will do this from now on.[1036] This completes the explicit definition of the two sorted geometry $\langle Mn, lines;\ \in, coll\,\rangle$ over the one-sorted geometry $\langle Mn;\ coll\rangle$, and the explicit definition of the two sorted geometry $\langle Mn, lines;\ \in, Bw, coll\,\rangle$ over the one-sorted geometry $\langle Mn;\ Bw\rangle$. For the connection of $lines$ with L of $\mathfrak{G}_{\mathfrak{M}}$ cf. Item 6.6.39 on p.1052.

Next, we introduce axioms $\mathbf{A_0}$–$\mathbf{A_4}$, $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{Pa}$. Though these axioms will be in the two-sorted language of $\langle Mn, lines;\ \in, coll\,\rangle$, by Thm.6.3.26 (p.962), they can be translated to the one-sorted languages of both $\langle Mn;\ coll\rangle$ and $\langle Mn;\ Bw\rangle$.

$\mathbf{A_0}$ $(\forall a, b, c \in Mn)[\, coll(a, b, c) \leftrightarrow (\exists \ell \in lines)\, a, b, c \in \ell\,]$.

Intuitively, a, b, c are collinear iff there is a line that contains a, b, c.

$\mathbf{A_1}$ $(\forall a, b \in Mn)(\, a \neq b\ \rightarrow\ (\exists! \ell \in lines)\, a, b \in \ell\,)$.

Informally, any two distinct points lie on exactly one line.[1037]

Though axioms $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$ below are not first-order formulas in their present form, they can be easily reformulated in the first-order languages of both $\langle Mn;\ Bw\rangle$ and $\langle Mn;\ coll\,\rangle$. Throughout $n \geq 2$ is the dimension of our geometry. If $H \subseteq Mn$ then we will use the definition of $Plane'(H)$ exactly as it was introduced in Def.6.2.15(ii) (p.820). Intuitively, $Plane'(H)$ is the n-long closure of H under $coll$. Recall that the definition of $Plane'(H)$ is a first-order one over both structures $\langle Mn;\ coll, H\rangle$ and $\langle Mn;\ Bw, H\rangle$.

[1036]For more detail on why and how we can do this (with "$\in$", E and $lines$) we refer to Appendix ("Why fist-order logic?").

[1037]Cf. axiom AS1 in Golblatt [108, p.112] and axioms I_1 and I_2 in Hilbert [134, §2].

$\mathbf{A_2}$ Intuitively, if H is a less than $n+2$ element subset of Mn then the "n-long closure" $Plane'(H)$ of H under $coll$ will be closed under $coll$, hence the plane $Plane(H)$ generated by H coincides with $Plane'(H)$ (cf. Def.6.2.15, p.819), formally:

$$(\forall H \subseteq Mn)$$
$$\Big(\big(\,|H| \leq n+1 \ \wedge\ a,b \in Plane'(H) \ \wedge\ coll(a,b,c)\,\big) \ \rightarrow\ c \in Plane'(H)\Big).$$

For introducing axioms $\mathbf{A_3}$ and $\mathbf{A_4}$ we need the following definition.

Definition 6.6.18 Consider a geometry $\langle Mn; Bw\rangle$.

(i) Let $H \subseteq Mn$. Then H is called *independent* iff $(\forall e \in H)\, e \notin Plane'(H \setminus \{e\})$.

(ii) Let $P \subseteq Mn$. Then P is called an i-dimensional plane iff there is an $i+1$ element independent subset H of Mn such that $Plane'(H) = P$.

◁

$\mathbf{A_3}$ Intuitively, if $i \leq n$ and H is an $i+1$ element independent subset of Mn then there is exactly one i-dimensional plane that contains H, formally:

$$(\forall H, H' \subseteq Mn)\Big(\big(\,|H| = |H'| \leq n{+}1 \ \wedge\ (\text{both } H \text{ and } H' \text{ are independent}) \ \wedge$$
$$H \subseteq Plane'(H')\,\big) \ \rightarrow\ Plane'(H) = Plane'(H')\Big).$$

$\mathbf{A_4}$ Mn is an n-dimensional plane.

Our next two axioms $\mathbf{P_1}$ and $\mathbf{P_2}$ concern "parallel lines". For these axioms we need the notion of parallelism.

Definition 6.6.19 Informally, two lines are parallel if they are in the same 2-dimensional plane, they do not meet or they coincide, formally: Let $\ell, \ell' \in lines$. Then ℓ and ℓ' are *parallel*, in symbols $\ell \parallel \ell'$, iff $(\exists a,b,c \in Mn)\ell,\ell' \subseteq Plane'(\{a,b,c\})$ and $(\ell \cap \ell' \neq \emptyset$ or $\ell = \ell')$.[1038]

◁

[1038] If we apply these definitions (i.e. the def. of $lines$ and $\parallel$) to $\mathfrak{G}_{\mathfrak{M}}$ then (assuming **Pax** + **Ax(diswind)**):
(i) $lines$ and L are potentially different with $L \subseteq lines$, further
(ii) $\parallel$ and $\parallel_{\mathfrak{G}}$ are potentially different with $\parallel_{\mathfrak{G}}$ being the restriction of $\parallel$ to L. Cf. Item 6.6.39 on p.1052.

P₁ $(\forall \ell \in \mathit{lines})(\forall a \in \mathit{Mn})(\exists! \ell' \in \mathit{lines})(a \in \ell' \;\wedge\; \ell \parallel \ell')$.

Informally, if we are given a line ℓ and a point a, then there is exactly one line ℓ' that passes through point a and is parallel to line ℓ.[1039] This axiom is called Euclid's axiom in the literature.

P₂ $(\ell \parallel \ell' \;\wedge\; \ell' \parallel \ell'') \;\rightarrow\; \ell \parallel \ell''$.

I.e. the relation of parallelism is transitive.[1040]

Definition 6.6.20

(i) $\mathbf{ag} :\overset{\text{def}}{=} \{\mathbf{A_0}, \mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_4}, \mathbf{P_1}, \mathbf{P_2}\}$.

(ii) If $\langle \mathit{Mn}; \mathit{coll}\,\rangle \models \mathbf{ag}$ then we say that $\langle \mathit{Mn}; \mathit{coll}\,\rangle$ is an *affine geometry*.

$\lhd$

An algebraic structure $\mathbf{D} = \langle D;\, +, \cdot\rangle$ whith binary operations $+$ (addition) and $\cdot$ (multiplication), is called a *division ring* iff 1–3 below hold.

1. $\langle D;\, +\rangle$ is an Abelian (i.e. commutative) group. We let 0 denote its neutral (i.e. identity) element.
2. $\langle D \setminus \{0\};\, \cdot\rangle$ is a group.
3. The distributive laws

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad (y + z) \cdot x = y \cdot x + z \cdot x$$

hold for all $x, y, z \in D$.

We note that a division ring in which the multiplication is commutative ($x \cdot y = y \cdot x$) is a field.

Assume $\mathbf{D} = \langle D;\, +, \cdot\rangle$ is a division ring. Then the set of lines $\mathsf{Eucl}(n, \mathbf{D}) \subseteq \mathcal{P}({}^nD)$ of the "coordinate system ${}^nD$" is defined completely analogously to the case of fields on p.45. Further, $\mathit{coll}_{\mathbf{D}}$ is a ternary relation on ${}^nD$ defined as

$$\mathit{coll}_{\mathbf{D}} :\overset{\text{def}}{=} \{\, \langle p, q, r\rangle \in {}^nD \times {}^nD \times {}^nD \;:\; (\exists \ell \in \mathsf{Eucl}(n, \mathbf{D}))p, q, r \in \ell \,\}\,.$$

The following fact (known from geometry) says that a geometry is an affine one iff it can be coordinatized by a division ring.

[1039] Cf. axiom AS3 in Goldblatt [108, p.113], and axiom IV in Hilbert [134, §7].
[1040] Cf. axiom AS4 in Goldblatt [108, p.113].

FACT 6.6.21 *Assume $n > 2$. Then*

$$\langle Mn;\ coll \rangle \models \mathbf{ag}$$
$$\Updownarrow$$
$$(\textit{there is a division ring } \mathbf{D} = \langle D;\ +, \cdot \rangle \textit{ such that } \langle Mn;\ coll \rangle \cong \langle {}^nD;\ coll_{\mathbf{D}} \rangle).$$

On the proof: A proof can be recovered from Goldblatt [108, pp. 23-27, 71, 114] and Hilbert [134, §24]. Cf. also the proof of Fact 6.6.28 (p.1044). ∎

Fact 6.6.21 above gives hints how one can try to find relativistic models behind geometries. It also gives an idea for a possible generalization of our approach, namely in our frame theory for relativity instead of requiring that $\mathfrak{F}$ is an ordered field we could require only that $\mathfrak{F}$ is an ordered division ring.

Our theorem below implies that the theory of division rings and the theory of affine geometries are weakly definitionally equivalent. Therefore, by Prop.6.4.8 (p.987), there are meaning preserving translation mappings between the two theories such that these translation mappings are inverses of each other in some sense. Cf. the discussion of weak definitional equivalence on pp. 984–987 for more intuition for the next theorem.

THEOREM 6.6.22 *Assume $n > 2$. Then*

$$\text{(the class of division rings)} \ \equiv^{w}_{\Delta} \ \{ \langle Mn;\ coll \rangle : \langle Mn;\ coll \rangle \models \mathbf{ag} \}, \quad \textit{but}$$
$$\text{(the class of division rings)} \ \not\equiv_{\Delta} \ \{ \langle Mn;\ coll \rangle : \langle Mn;\ coll \rangle \models \mathbf{ag} \},$$

i.e. the theory of division rings and the theory of affine geometries (if $n > 2$) are weakly definitionally equivalent, but they are not definitionally equivalent.

On the proof: We omit the proof but cf. the proof of Thm.6.6.29. ∎

It is interesting that by the above theorem the "one-sorted" class of division rings is weakly definitionally equivalent with the geometries $\langle Mn, lines;\ \in, coll \rangle$ satisfying **ag**.

To make our division ring **D** in Fact 6.6.21 commutative (i.e. to make it a field) we introduce a new axiom **Pa** called Pappus-Pascal Property in the literature, cf. e.g. Hilbert [134] or Goldblatt [108, p.21]. In the axiom **Pa** we will use the following abbreviation.

Notation 6.6.23 Let $a, b, c, d \in Mn$. Then

$$\langle a, b\rangle \parallel \langle c, d\rangle \overset{\text{def}}{\iff} \Big(a \neq b \ \wedge \ c \neq d \ \wedge \ (\exists \ell, \ell' \in \mathit{lines})(\ell \parallel \ell' \ \wedge \ a, b \in \ell \ \wedge \ c, d \in \ell')\Big).$$

◁

Pa $(\forall \ell, \ell' \in \mathit{lines})(\forall a, b, c \in \ell \setminus \ell')(\forall a', b', c' \in \ell' \setminus \ell)$

$$[\,(\,\langle a, b'\rangle \parallel \langle a', b\rangle \quad \wedge \quad \langle a, c'\rangle \parallel \langle a', c\rangle\,) \quad \to \quad \langle b, c'\rangle \parallel \langle b', c\rangle\,],$$

see Figure 319.

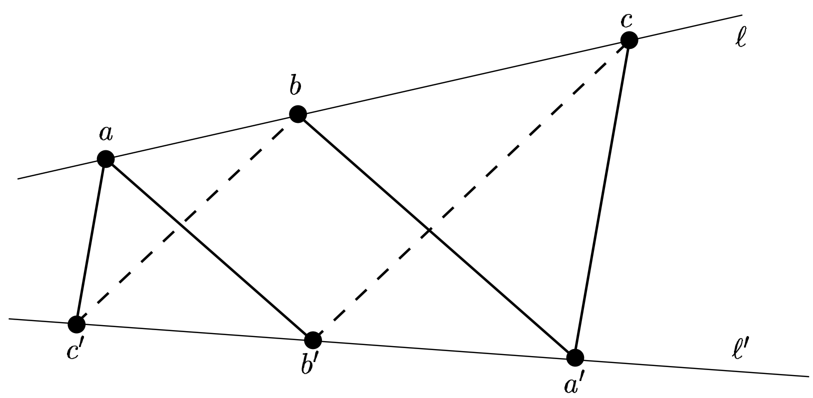


Figure 319: Pappus-Pascal Property.

Definition 6.6.24

(i) $\mathbf{pag} \overset{\text{def}}{=} \mathbf{ag} + \mathbf{Pa}$.

(ii) If $\langle Mn; \mathit{coll}\,\rangle \models \mathbf{pag}$ then we say that $\langle Mn; \mathit{coll}\,\rangle$ is a *Pappian affine geometry*.

◁

The following fact (known from geometry) says that a geometry is a Pappian affine one iff it can be coordinatized by a field.

FACT 6.6.25

$$\langle Mn;\ coll\rangle \models \mathbf{pag}$$
$$\Updownarrow$$
$$(\textit{there is a field } \mathbf{F} \textit{ such that } \langle Mn;\ coll\rangle \cong \langle {}^nF, coll_{\mathbf{F}}\rangle).$$

On the proof: A proof can be recovered from Goldblatt [108, pp. 23-27, 71, 114] and Hilbert [134, §24]. Cf. also the proof of Fact 6.6.28 (p.1044). ∎

THEOREM 6.6.26

$$\begin{array}{lll} \text{(the class of fields)} & \equiv^w_\Delta & \{\langle Mn;\ coll\rangle : \langle Mn;\ coll\rangle \models \mathbf{pag}\}, \quad \textit{but} \\ \text{(the class of fields)} & \not\equiv_\Delta & \{\langle Mn;\ coll\rangle : \langle Mn;\ coll\rangle \models \mathbf{pag}\}, \end{array}$$

i.e. the theory of fields and the theory of Pappian affine geometries are weakly definitionally equivalent, but they are not definitionally equivalent.

On the proof: We omit the proof but cf. the proof of Thm.6.6.29. ∎

To make our field an ordered field in Fact 6.6.25 we need a few further axioms. These further axioms concern betweenness Bw, and they are in the language of $\langle Mn;\ Bw\rangle$. (*coll* is a defined relation.)

$\mathbf{B_1}$ $Bw(a,b,c) \ \rightarrow\ (\,a \neq b \neq c \neq a \ \wedge\ Bw(c,b,a) \ \wedge\ \neg Bw(b,a,c)\,)$.

Intuitively, if b lies between a and c then a,b,c are distinct points and b lies between c and a. Further, for any three points a,b,c at most one of them lies between the other two.[1041]

$\mathbf{B_2}$ $a \neq b \ \rightarrow\ (\exists c) Bw(a,b,c)$.

Informally, for any two distinct points a,b there is at least one point c such that b lies between a and c.[1042]

Axiom $\mathbf{B_3}$ below is called Pasch's Law in the literature.

$\mathbf{B_3}$ Intuitively, if a line ℓ lies in the plane determined by a triangle abc, and passes between a and b but not through c, then ℓ passes between a and c, or between b and c,[1043] formally:

$(\,\neg coll(a,b,c) \ \wedge\ \ell \subseteq Plane'(\{a,b,c\}) \ \wedge\ (\exists d \in \ell) Bw(a,d,b)\,) \ \rightarrow$
$(\exists e \in \ell)(Bw(a,e,c) \ \vee\ Bw(b,e,c))$, see Figure 320.

[1041] Cf. axioms B1, B3 in Goldblatt [108, pp. 70-71] and axioms II_1 and II_3 in Hilbert [134, §3].
[1042] Cf. axiom B2 in Goldblatt [108, p.70] and axiom II_2 in Hilbert [134, §3].
[1043] Cf. axiom B4′ in Goldblatt [108, p.136] and axiom II_4 in Hilbert [134, §3].

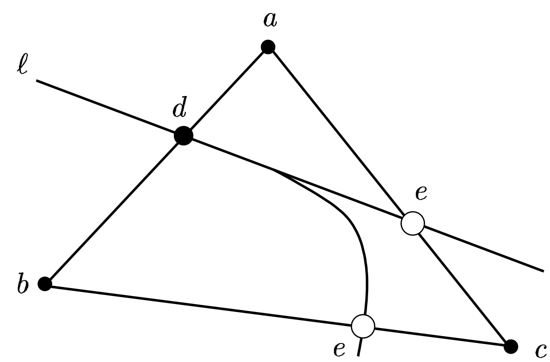


Figure 320: Pasch's Law.

So far it was clear what we meant when we wrote $\langle Mn; coll \rangle \models \mathbf{pag}$. Now, beside *coll* we want to use *Bw* too, and we want to write $\langle Mn; coll, Bw \rangle \models \mathbf{pag} +$ (some new axioms [concerning *Bw*]). Since *coll* is *definable* from *Bw*, we will write $\langle Mn; Bw \rangle \models$"..." instead of $\langle Mn; coll, Bw \rangle \models$"...". We hope that the similarity between the expressions $\langle Mn; coll \rangle$ and $\langle Mn; Bw \rangle$ will create no confusion[1044] (because context will help).

Definition 6.6.27

(i) $\mathbf{opag} = \mathbf{pag} + \{\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}\}$.

(ii) If $\langle Mn; Bw \rangle \models \mathbf{opag}$ then we say that $\langle Mn; Bw \rangle$ is an *ordered Pappian affine geometry*.

◁

The following fact (known from geometry) says that a geometry is an ordered Pappian affine one iff it can be coordinatized by an ordered field.

FACT 6.6.28

$$\langle Mn, Bw \rangle \models \mathbf{opag}$$
$$\Updownarrow$$
$$(\textit{there is an ordered field } \mathfrak{F} \textit{ such that } \langle Mn, Bw \rangle \cong \langle {}^nF, \mathsf{Betw} \rangle).$$

[1044] Cf. Convention 6.3.1 on p.931.

Proof: Proof of direction "$\Uparrow$" goes by checking the axioms, while direction "$\Downarrow$" follows from Prop.6.6.38 (p.1052) way below. (Cf. also Def.6.6.37 on p.1051). ∎

THEOREM 6.6.29

$$\text{(the class of ordered fields)} \ \equiv^w_\Delta \ \{ \langle Mn;\, Bw\rangle : \langle Mn;\, Bw\rangle \models \mathbf{opag} \}, \quad \textit{but}$$
$$\text{(the class of ordered fields)} \ \not\equiv_\Delta \ \{ \langle Mn;\, Bw\rangle : \langle Mn;\, Bw\rangle \models \mathbf{opag} \},$$

i.e. the theory of ordered fields and the theory of ordered Pappian affine geometries are weakly definitionally equivalent, but they are not definitionally equivalent.

On the proof: A proof for the "$\equiv^w_\Delta$" part can be obtained by Def.6.6.31, Prop.6.6.32, Def.6.6.34, Prop.6.6.35 and Examples 6.3.16 (p.954).

A proof for the "$\not\equiv_\Delta$" part can be obtained by using item (6) on p.972 and Fact 6.6.28 as follows. It can be seen that $\langle {}^nF, \mathsf{Betw}\rangle$ has many non-trivial automorphisms for any ordered field $\mathfrak{F}$. (E.g. $x \mapsto 2x$ induces such an automorphism of $\langle {}^nF, \mathsf{Betw}\rangle$.) Thus any ordered Pappian affine geometry has many non-trivial automorphisms, in particular, the automorpism group has more than one element, by Fact 6.6.28. On the other hand, there are ordered fields with one-element automorphism groups (e.g. the ordered field $\mathfrak{R}$ of real numbers is such). Then (6) on p.972 implies that the class of ordered fields cannot be definitionally equivalent ($\equiv_\Delta$) with the class of ordered Pappian affine geometries. By this, the "$\not\equiv_\Delta$" part of our theorem is proved, too. ∎

Assume $\langle Mn;\, Bw\rangle \models \mathbf{opag}$. In Def 6.6.31 for every $o, e \in Mn$ with $o \neq e$ we will define an "ordered field" $\mathfrak{F}_{oe}$ corresponding to o, e. Prop.6.6.32 says that $\mathfrak{F}_{oe}$ is indeed an ordered field. In Prop.6.6.33 we will see that the ordered field $\mathfrak{F}_{oe}$ does not depend on the particular choice of o, e. Thus, there is a unique ordered field $\mathfrak{F}$ behind the geometry $\langle Mn;\, Bw\rangle$. In Def.6.6.34 we will define this ordered field $\mathfrak{F}$ explicitly over $\langle Mn;\, Bw\rangle$. Finally, in Def.6.6.37 we will define a coordinatization of the geometry $\langle Mn;\, Bw\rangle$ by $\mathfrak{F} = \langle F, \ldots\rangle$ which will be proved to be an isomorphism between $\langle Mn;\, Bw\rangle$ and $\langle {}^nF;\, \mathsf{Betw}\rangle$ as Prop.6.6.38.

Notation 6.6.30 Let $\langle Mn; Bw\rangle$ be a geometry, and $o, e \in Mn$. Then the <u>half-line</u> $[oe$ with origin o and containing e is defined as follows.

$$[oe \ \stackrel{\text{def}}{:=} \ \{ a \in Mn : coll(o,e,a) \ \wedge \ \neg Bw(a,o,e) \}.^{1045}$$

◁

[1045] We note that we had a slightly different notion of a half-line denoted as $\vec{\ell}_{oe}$ in §6.2.6, p.891. Our present notion "$[oe$" of a half-line is slightly different (it is tailored for the structure $\langle Mn;\, Bw\rangle$, while the previous one was tailored for $\mathfrak{G}_{\mathfrak{M}}$), but the basic idea is the same.

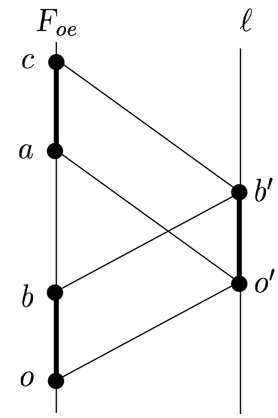
Definition 6.6.31 (The ordered field $\mathfrak{F}_{oe}$)
Assume $\langle Mn; Bw\rangle \models$ **opag**. Let $o, e \in Mn$ with $o \neq e$. We define an "ordered field" $\mathfrak{F}_{oe}$ corresponding to o and e as follows. Our o and e represent 0 and 1, respectively. Let

$$F_{oe} :\stackrel{\text{def}}{=} \{\, a \in Mn \;:\; coll(o, e, a)\,\},$$

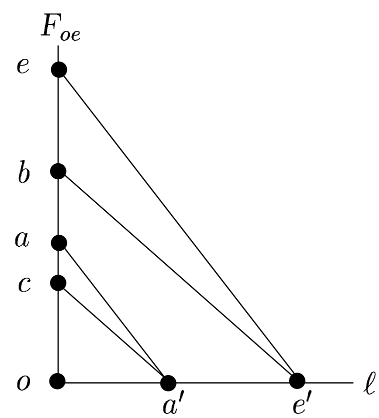
i.e. F_{oe} is the line determined by o and e. We will first define addition $+_{oe}$ as a ternary relation $+_{oe} \subseteq F_{oe} \times F_{oe} \times F_{oe}$ and later (in Prop.6.6.32) we will see that it is a function $+_{oe} : F_{oe} \times F_{oe} \longrightarrow F_{oe}$. We will define multiplication $\cdot_{oe} \subseteq F_{oe} \times F_{oe} \times F_{oe}$ in an analogous style. Further we will define "ordering" $\leq_{oe} \subseteq F_{oe} \times F_{oe}$.

Let $a, b, c \in \ell$.

$$+_{oe}(a, b, c)$$
$$\stackrel{\text{def}}{\iff}$$
$$(\exists \ell \in lines)\Big(o \notin \ell \;\wedge\; \ell \parallel F_{oe} \;\wedge$$
$$(\exists o', b' \in \ell)(\,\langle o, o'\rangle \parallel \langle b, b'\rangle \;\wedge\; \langle o', a\rangle \parallel \langle b', c\rangle\,)\Big).$$



$$\cdot_{oe}(a, b, c)$$
$$\stackrel{\text{def}}{\iff}$$
$$(\exists \ell \in L)\Big(\ell \cap F_{oe} = \{o\} \;\wedge$$
$$(\exists a', e' \in \ell)(\langle e, e'\rangle \parallel \langle a, a'\rangle \;\wedge\; \langle b, e'\rangle \parallel \langle c, a'\rangle)\Big).$$



$$(\forall d \in F_{oe})(o \leq_{oe} d \;\stackrel{\text{def}}{\iff}\; d \in [oe\,), \quad \text{and}$$
$$a \leq_{oe} b \;\stackrel{\text{def}}{\iff}\; (\exists d \in F_{oe})\,(\,a + d = b \;\wedge\; o \leq_{oe} d\,).$$

We define the algebraic structure $\mathfrak{F}_{oe}$ as

$$\mathfrak{F}_{oe} :\stackrel{\text{def}}{=} \langle F_{oe};\, +_{oe}, \cdot_{oe}, \leq_{oe} \rangle.$$

$\mathfrak{F}_{oe}$ is an ordered field by Prop.6.6.32 below.

$\lhd$

PROPOSITION 6.6.32 *Assume* $\langle Mn;\, Bw \rangle \models$ **opag**. *Assume* $o, e \in Mn$ *with* $o \neq e$. *Then* $\mathfrak{F}_{oe}$ *is an ordered field.*

On the proof: A proof can be recovered from Goldblatt [108, pp. 23-27, 71, 114] and Hilbert [134, §24]. ∎

Item (i) of the following proposition says that the ordered field $\mathfrak{F}_{oe}$ does not depend on the particular choice of o and e. I.e. if we choose o, e differently we obtain an ordered field isomorphic to $\mathfrak{F}_{oe}$. In item (ii) we state that there is an isomorphism beween the ordered fields $\mathfrak{F}_{oe}$ and $\mathfrak{F}_{o'e'}$ such that it is (uniformly) first-order definable over the structure $\langle Mn;\, Bw, o, e, o', e' \rangle$.

PROPOSITION 6.6.33 *Assume* $\langle Mn;\, Bw \rangle \models$ **opag**. *Assume* $o, e, o', e' \in Mn$ *are such that* $o \neq e$ *and* $o' \neq e'$. *Then (i)–(iii) below hold.*

(i) $\mathfrak{F}_{oe} \cong \mathfrak{F}_{o'e'}$.

(ii) *There is an isomorphism* $f_{oe}^{o'e'} : \mathfrak{F}_{oe} \longrightarrow \mathfrak{F}_{o'e'}$ *which is first-order definable over the structure* $\langle Mn;\, \mathrm{Bw}, o, e, o', e' \rangle$ *and the first-order definition of this isomorphism* $f_{oe}^{o'e'}$ *does not depend on the particular choice of* o, e, o', e'*; i.e.*

(iii) *the definition of the relation* $f_{oe}^{o'e'}$ *is uniform over the class*

$$\{\, \langle Mn;\, Bw, o, e, o', e' \rangle \;:\; \langle Mn;\, Bw \rangle \models \mathbf{opag},\ o, e, o', e' \in Mn,\ o \neq e,\ o' \neq e' \,\}$$

of models; where we note that $f_{oe}^{o'e'} \subseteq Mn \times Mn$.

Outline of proof: Assume the assumptions. Let $f_{oe}^{o'e'} \subseteq F_{oe} \times F_{o'e'}$ be defined as follows. Let $\langle a, a' \rangle \in F_{oe} \times F_{o'e'}$. Before reading the formula below the reader is advised to consult Figure 321. Then

$$\langle a, a' \rangle \in f_{oe}^{o'e'}$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$\Big([\, (\, o = o' \ \wedge\ \neg coll(o, e, e')\,) \ \rightarrow\ (\, (\textsc{i}) \text{ below hold}\,)\,] \ \wedge$$
$$[\, (\, o = o' \ \wedge\ coll(o, e, e')\,) \ \rightarrow\ (\, (\textsc{ii}) \text{ below hold}\,)\,] \ \wedge$$
$$[\, o \neq o' \ \rightarrow\ (\, (\textsc{iii}) \text{ below hold}\,)\,] \Big),$$

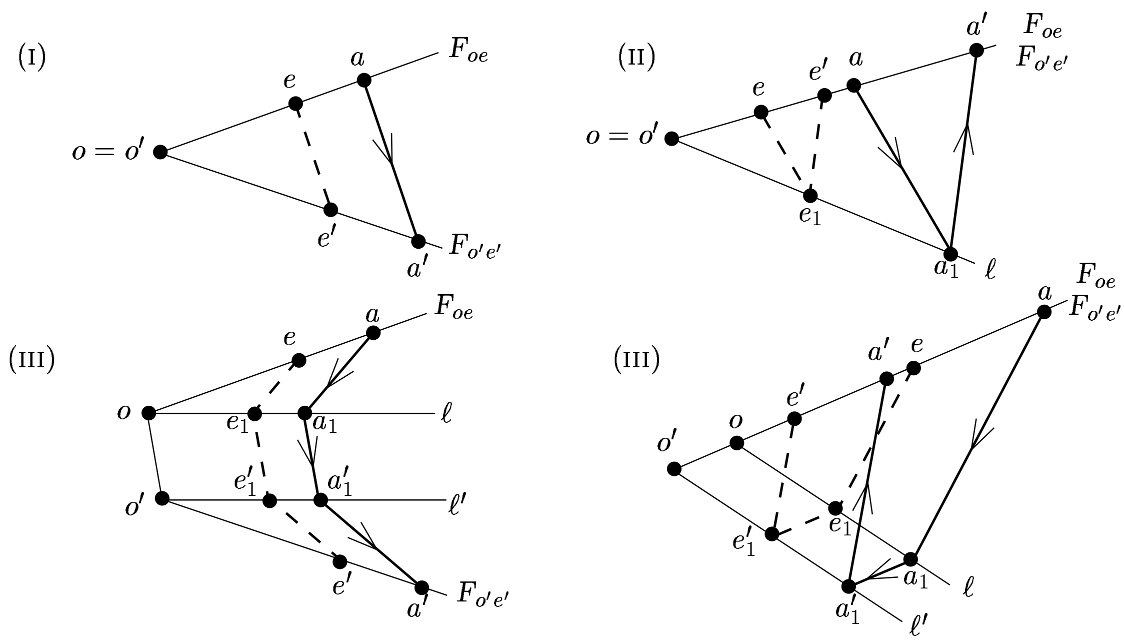


Figure 321: (I) is the easy case when $o = o'$ and o, e, e' are not collinear, (II) is somewhat more complicated because there o, e, e' are collinear, etc.

see Figure 321.

(I) $\langle e, e' \rangle \parallel \langle a, a' \rangle$.

(II) $(\exists \ell \in \mathit{lines})(\exists e_1, a_1 \in \ell)(\ell \cap F_{oe} = \{o\} \ \wedge \ \langle e, e_1 \rangle \parallel \langle a, a_1 \rangle \ \wedge \ \langle e_1, e' \rangle \parallel \langle a_1, a' \rangle\,)$.

(III) $(\exists$ distinct $\ell, \ell' \in \mathit{lines})(\exists e_1, a_1 \in \ell)(\exists e_1', a_1' \in \ell')$
$(\ell \cap F_{oe} = \{o\} \ \wedge \ \ell' \cap F_{o'e'} = \{o'\} \ \wedge \ \ell \parallel \ell' \ \wedge$
$\langle e, e_1 \rangle \parallel \langle a, a_1 \rangle \ \wedge \ \langle o, o' \rangle \parallel \langle e_1, e_1' \rangle \parallel \langle a_1, a_1' \rangle \ \wedge \ \langle e_1', e' \rangle \parallel \langle a_1', a' \rangle)$.

Then $f_{oe}^{o'e'}$ is an isomorphism between $\mathfrak{F}_{oe}$ and $\mathfrak{F}_{o'e'}$. A proof of this can be recovered from Goldblatt [108, pp. 23-27, 71, 114] and Hilbert [134, §24].

The present definition of the isomorphism $f_{oe}^{o'e'}$ is somewhat complicated. Probably we would obtain a less complicated definition for this isomorphism if we first defined it for the special cases (I)[1046] and $\langle o, o' \rangle \parallel \langle e, e' \rangle \ \wedge \ \langle o, e \rangle \parallel \langle o', e' \rangle$, and then we would obtain an isomorphism for the general case as a composition of three isomorphisms defined for the special cases. ∎

Definition 6.6.34 (The ordered field $\mathfrak{F}$ corresponding to $\langle Mn; Bw \rangle$) Assume $\langle Mn; Bw \rangle \models$ **opag**. We define the ordered field $\mathfrak{F}$ explicitly (in the sense of §6.3.2) over $\langle Mn; Bw \rangle$ as follows. First, we define the new relation

$$R := \left\{ \langle a, o, e \rangle \in {}^{3}F \ : \ o \neq e, \ a \in F_{oe} \right\}.$$

Then we define the new auxiliary sort U to be R together with the projection functions pj_0, pj_1, pj_2. Then we define the equivalence relation $\equiv$ on U as follows. Let $\langle a, o, e \rangle, \langle a', o', e' \rangle \in U$. Then

$$\langle a, o, e \rangle \equiv \langle a', o', e' \rangle \ \stackrel{\text{def}}{\Longleftrightarrow} \ \langle a, a' \rangle \in f_{oe}^{o'e'},$$

where $f_{oe}^{o'e'} : \mathfrak{F}_{oe} \longrightarrow \mathfrak{F}_{o'e'}$ is the isomorphism which was defined in (the proof of) Prop 6.6.33. Of course one uses pj_0, pj_1, pj_2 in the formal definition of $\equiv$. We define the sort F to be $U/\equiv$ together with $\in \subseteq U \times F$.[1047] Now, we define $+, \cdot \ \subseteq {}^{3}F$ and $\leq \ \subseteq {}^{2}F$ as follows. Let $a, b, c \in F$. Then

$$+(a, b, c)$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$

[1046] i.e. for the case $o = o'$ and $\neg \mathit{coll}(o, e, e')$

[1047] We use the notation pj and $\in$ in the style of §6.3.2. If someone want to avoid this then he can use a notation like $+(\langle a_0, a_1, a_2 \rangle/\equiv, \ldots, \langle c_0, c_1, c_2 \rangle/\equiv) \ \stackrel{\text{def}}{\Longleftrightarrow} \ \exists a'[a' \equiv a$ etc.

$$(\exists a' \in a)(\exists b' \in b)(\exists c' \in c)$$
$$\Big(pj_1(a') = pj_1(b') = pj_1(c') \ \wedge \ pj_2(a') = pj_2(b') = pj_2(c') \wedge$$
$$pj_0(a') \ +_{pj_1(a')pj_2(a')} \ pj_0(b') \ = \ pj_0(c')\Big),$$

$$\cdot(a, b, c)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists a' \in a)(\exists b' \in b)(\exists c' \in c)$$
$$\Big(pj_1(a') = pj_1(b') = pj_1(c') \ \wedge \ pj_2(a') = pj_2(b') = pj_2(c') \wedge$$
$$pj_0(a') \ \cdot_{pj_1(a')pj_2(a')} \ pj_0(b') \ = \ pj_0(c')\Big),$$

$$a \le b$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists a' \in a)(\exists b' \in b)\Big(pj_1(a') = pj_1(b') \ \wedge \ pj_2(a') = pj_2(b') \wedge$$
$$pj_0(a') \ \le_{pj_1(a')pj_2(a')} \ pj_0(b')\Big).$$

Let
$$\mathfrak{F} :\overset{\text{def}}{=} \langle F; +, \cdot, \le \rangle.$$
$\mathfrak{F}$ is first-order defined over $\langle Mn; Bw\rangle$. $\mathfrak{F}$ is an ordered field by Prop.6.6.35 below.

We will often use the elements of F in the form $\langle a, o, e\rangle/\equiv$ where $o \neq e$ and $a \in F_{oe}$.

$\lhd$

PROPOSITION 6.6.35 *Assume* $\langle Mn; Bw\rangle \models$ **opag**. *Let* $\mathfrak{F} = \langle F, \ldots\rangle$ *be the "ordered field" corresponding to* $\langle Mn; Bw\rangle$ *defined in Def.6.6.34. Assume* $o, e \in Mn$. *Let* $\mathfrak{F}_{oe} = \langle F_{oe}; \ldots\rangle$ *be the ordered field corresponding to* o, e *defined in Def.6.6.31. Let* $f_{oe} : F_{oe} \longrightarrow F$ *be defined by* $a \mapsto \langle a, o, e\rangle/\equiv$.

Then f_{oe} *is an isomorphism between* $\mathfrak{F}_{oe}$ *and* $\mathfrak{F}$.

On the proof: The proposition can be proved by Prop.6.6.33. ∎

Assume $\langle Mn; Bw\rangle \models$ **opag**. We will use $n+1$ tuples $\langle o, e_0, e_1, \ldots, e_{n-1}\rangle$ where $\{o, e_0, \ldots, e_{n-1}\}$ is an $n+1$ element independent subset of Mn to identify potential coordinate systems. We will think of o as the origin and $e_0, \ldots, e_{n-1}$ as the unit vectors. We will define a coordinatization for such $n+1$ tuples in Def.6.6.37 below. In Def.6.6.37 we will use the following notation.

Notation 6.6.36 Assume $\langle Mn; Bw\rangle$ is a geometry. Let $a, b \in Mn$ and $H \subseteq Mn$. Then

$$\langle a, b\rangle \parallel Plane'(H) \overset{\text{def}}{\iff} (\exists c, d \in Plane'(H))\, \langle a, b\rangle \parallel \langle c, d\rangle.$$

◁

Definition 6.6.37 (coordinatization)
Assume $\langle Mn;\, Bw\rangle \models$ **opag**. Recall that for every $o, e \in Mn$ with $o \neq e$ the ordered field $\mathfrak{F}_{oe} = \langle F_{oe};\, \ldots\rangle$ was defined in Def.6.6.31. Let $\mathfrak{F} = \langle F;\, \ldots\rangle$ be the ordered field corresponding to $\langle Mn;\, Bw\rangle$ defined in Def.6.6.34. Let $\langle o, e_0, \ldots, e_{n-1}\rangle \in {}^{n+1}Mn$ be such that $\{o, e_0, e_1, \ldots, e_{n-1}\}$ is an $n+1$ element independent subset of Mn. We define the coordinatization

$$Co_{\langle o,e_0,\ldots,e_{n-1}\rangle} : Mn \longrightarrow {}^{n}F$$

as follows. Let $a \in Mn$. For every $i \in n$, let $a_i \in F_{oe_i}$ be such that if $a \notin F_{oe_i}$ then $\langle a, a_i\rangle \parallel Plane'(\{o, e_0, \ldots, e_{n-1}\} \setminus \{e_i\})$, otherwise $a_i = a$, see Figure 322. Such a_i's

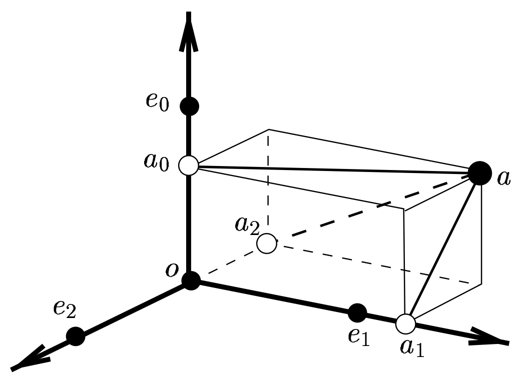


Figure 322:

exist and are unique.

We define

$$Co_{\langle o,e_0,\ldots,e_{n-1}\rangle}(a) \overset{\text{def}}{:=} \langle f_{oe_0}(a_0), \ldots, f_{oe_{n-1}}(a_{n-1})\rangle,$$

where $f_{oe_0}, \ldots, f_{oe_{n-1}}$ are as defined in Prop.6.6.35 (p.1050).

◁

PROPOSITION 6.6.38 *Assume* $\langle Mn; Bw\rangle \models$ **opag**. *Assume* $\langle o, e_0, \ldots, e_{n-1}\rangle \in {}^{n+1}Mn$ *is such that* $\{o, e_0, \ldots, e_{n-1}\}$ *is an* $n+1$ *element independent subset of* Mn. *Let* $\mathfrak{F} = \langle F; \ldots\rangle$ *be the ordered field corresponding to* $\langle Mn; Bw\rangle$ *defined in Def.6.6.34.*

Then $Co_{\langle o,e_0,\ldots,e_{n-1}\rangle}$ *is an isomorphism between* $\langle Mn; Bw\rangle$ *and* $\langle {}^nF, \mathsf{Betw}\rangle$.

On the proof: A proof can be recovered from Goldblatt [108, pp. 23-27, 71, 114] and Hilbert [134, §24]. ∎

Item 6.6.39 (Summary of some notation)
Let us return to $\mathsf{Ge}(\mathbf{Pax})$. Our definitions of *lines*, $\|$ make sense for the geometries in $\mathsf{Ge}(\mathbf{Pax})$, too. Now, we have strongly related triples of notions $L, Col, \|_{\mathfrak{G}}$ and *lines*, *coll*, $\|$. The differences between these two are rather small. The reason for the differences is that by the construction of $\mathfrak{G}_{\mathfrak{M}}$ some lines may be missing from L (in some sense).[1048] Assume $\mathbf{Pax} + \mathbf{Ax(diswind)}$. (Recall that L, Col, and $\|_{\mathfrak{G}}$ belong together, while *lines*, *coll* and $\|$ belong together.) Now, $L \subseteq lines$, $Col \subseteq coll$ and $\|_{\mathfrak{G}} \subseteq \|$. Further Col and $\|_{\mathfrak{G}}$ are the natural restrictions (of *coll* and $\|$) to the "world of L". If we assume $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ })$ in addition then L, Col, $\|_{\mathfrak{G}}$ coincide, respectively, with *lines*, *coll*, $\|$.

◁

6.6.3 Continuation of duality theory

Let us recall from p.1036 that our purpose with §6.6.2 was to prepare ourselves to the definition of our functor $\mathcal{M}$.

In Def.6.6.41 below we define the functor $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^{+}) \longrightarrow \mathsf{Mod}(\emptyset)$. In this definition we will use facts and propositions stated in §6.6.2 for ordered Pappian affine geometries (i.e. for **opag**) and notation introduced in §6.6.2. Therefore we include Prop.6.6.40 below. Intuitively, the proposition says that the windows of ($\mathbf{Pax}$+$\mathbf{Ax(Bw)}$)-geometries are ordered Pappian affine geometries.

[1048] The reason for this is that L was obtained from coordinate axes (and traces of photons) only. If we had defined L such that a set of events is in L if some inertial observer thinks that it is a Euclidean line then we would have obtained all of *lines* as elements of L. In other words L corresponds to inertial coordinate axes (and traces of photons), while *lines* corresponds to Euclidean lines. I.e. $\ell \in L$ if some inertial m thinks it is a coordinate axis (or is a trace of a photon), while $\ell \in lines$ if some inertial m thinks it is a Euclidean line.

PROPOSITION 6.6.40 *Assume* $\mathfrak{G} = \langle Mn, \ldots\rangle \in \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. *Assume* $o \in Mn$. *Let* Mn_o *be the "window of* o*", i.e.* $Mn_o :\stackrel{\text{def}}{=} \{ e \in Mn : e \sim o \}$.[1049] *Then*

$$\langle Mn_o;\, Bw \restriction Mn_o\rangle \models \mathbf{opag}.$$

Outline of proof: Let $\mathfrak{G} = \langle Mn, \ldots\rangle \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. Then $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}} = \langle Mn_{\mathfrak{M}}, \ldots\rangle$ for some $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. Let this $\mathfrak{M}$ be fixed. Let $o \in Mn_{\mathfrak{M}}$ and $(Mn_{\mathfrak{M}})_o = \{ e \in Mn_{\mathfrak{M}} : o \sim e\}$. To prove the proposition it is enough to prove

$$(*) \qquad \langle (Mn_{\mathfrak{M}})_o;\, Bw_{\mathfrak{M}} \restriction (Mn_{\mathfrak{M}})_o\rangle \models \mathbf{opag}.$$

Let $m \in Obs$ be such that $o \in Rng(w_m)$. Then by Thm.4.3.13 (p.482), w_m is an isomorphism between $\langle {}^nF;\, \mathsf{Betw}\rangle$ and $\langle (Mn_{\mathfrak{M}})_o;\, Bw_{\mathfrak{M}} \restriction (Mn_{\mathfrak{M}})_o\rangle$. Cf. Prop.6.2.79 (p.884). But then $(*)$ above holds. ∎

<u>*Intuitive idea for the definition of the functor* $\mathcal{M}$: *geometries* $\longrightarrow$ *frame models.*</u>

Assume we are given a geometry $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$. We want to define (by using first-order logic only) an observational model $\mathcal{M}(\mathfrak{G})$ over this geometry $\mathfrak{G}$. Moreover, we would like to choose $\mathcal{M}(\mathfrak{G})$ such that its geometry $\mathcal{G}(\mathcal{M}(\mathfrak{G}))$ should be <u>as close to the original</u> $\mathfrak{G}$ as possible (cf. potential theorem schemas (A)–(I) for duality on pp. 1009–1012). (In a sense, one could say, that using the functor $\mathcal{M}$ we would like to recover from $\mathbf{I}\mathfrak{G}$ [but using only the "legitimate geometrical" structure of $\mathfrak{G}$] that "long forgotten" observational model whose geometric counterpart $\mathfrak{G}$ is.) Cf. here the relevant motivational parts of the introduction (pp. 774-778) to the present chapter. What do we need in order to find an observational $\mathfrak{M}$ inside our geometry $\mathfrak{G}$? Surely we need to find a field $\mathfrak{F}^{\mathfrak{M}}$ in $\mathfrak{G}$, but <u>that</u> is no problem as we saw in §6.6.2 ("Coordinatization ..."). This is a good start, but what <u>else</u> do we need to find in $\mathfrak{G}$? Certainly we will need to <u>find observers</u> in $\mathfrak{G}$. But what is an observer? We can identify an observer m with his coordinatization[1050] $w_m : {}^nF \longrightarrow Mn$ of (a part of Mn). What is w_m? It is a coordinatization of (a part of) Mn [1051] by ${}^nF$. For simplicity, in this intuitive remark we fix $n = 3$. We can represent such a coordinatization $w_m : {}^nF \longrightarrow Mn$, by a choice of w_m's origin $o \in Mn$ and by w_m's three unit-vectors 1_t, 1_x, 1_y. More precisely, we are thinking of the w_m-images of the origin, and of the unit-vectors as they appear in Mn. Let us notice that in geometry, i.e. in Mn, vectors are easily represented by pairs of points. Actually, $w_m(\bar{0}), w_m(1_t), w_m(1_x), w_m(1_y)$ are nothing but 4 elements $o, e_t, e_x, e_y \in Mn$ of our

[1049] Recall that $\sim$ is a binary relation of connectedness on Mn defined in Def.6.2.12 (p.818).
[1050] or world-view function
[1051] Eventually, we will need a coordinatization of a part of $\mathcal{P}(B)$ instead of Mn but that change will be easy to make, hence we postpone worrying about it.

geometry satisfying some conditions.[1052] So the idea naturally comes to one's mind to try to represent (or code or define) observers as four-tuples $\langle o, \dots, e_y \rangle$ of points (in Mn) satisfying certain conditions.

To make this idea work, we still have to figure out how to reconstruct the whole of the coordinatization w_m from the origin o and the unit vectors $e_t, \dots, e_y$, but having access to the whole geometry $\mathfrak{G}_{\mathfrak{M}}$, one can believe that, one way or another, at least some w_m can be reconstructed from $\langle o, \dots, e_y \rangle$. So, our plan is to code (or represent[1053]) observers (found in $\mathfrak{G}$) by tuples $\langle o, \dots, e_y \rangle \in {}^4Mn$ satisfying some conditions. It is natural to identify photons with photon-like lines i.e. elements of L^{Ph}. It is also natural to choose $B = Ib = Obs \cup Ph$. At this point we already have a grasp on what the $\mathfrak{F}^{\mathfrak{M}}$, $B^{\mathfrak{M}}$, $Obs^{\mathfrak{M}}$, $Ph^{\mathfrak{M}}$ parts of our model $\mathcal{M}(\mathfrak{G}) = \mathfrak{M} = \langle B, \dots, \mathfrak{F}, G, \in, W \rangle$ will be. It is, again, natural to choose $G = \mathsf{Eucl}(\mathfrak{F})$. Hence the only remaining part of $\mathfrak{M}$ which we still have to define over $\mathfrak{G}$ is $W^{\mathfrak{M}}$ which in turn is equivalent to defining w_m for each $m \in Obs$. However, by knowing m's unit vectors[1054] and having the geometric tools of $\mathfrak{G}$ (e.g. $g, lines, \|$)[1055] at our hand it is only a matter of patience to work out a definition for w_m. E.g. for $\lambda \in F$, $w_m(\langle \lambda, 0, 0 \rangle) \in Mn$ is on the line determined by o, e_t and its g-distance from o is $|\lambda \cdot g(o, e_t)|$. There are only two such points in Mn, and it is easy to figure out (by using e.g. Bw) which one to choose. We leave the details of defining W to the formal definition below. Now, we are ready for the formal (first-order) definition of $\mathcal{M}(\mathfrak{G})$ over $\mathfrak{G}$, which comes below.

The definition given below becomes simpler and more intuitive if condition (e) is omitted. The so obtained simpler definition still works but less "spectacularly". What we mean by this is explained in footnote 1056.

Definition 6.6.41 (the functor $\mathcal{M}$)
We define $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\emptyset)$ as follows. Let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$. Then we define

$$\mathcal{M}(\mathfrak{G}) = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, \mathsf{Eucl}(\mathfrak{F});\ \in, W \rangle$$

as follows:

1. $Obs :\overset{\text{def}}{=} \{ \langle o, e_0, \dots, e_{n-1} \rangle \in {}^{n+1}Mn \ : \ $(a)–(f) below hold$\}$, see Figure 323.

   (a) $\{o, e_0, \dots, e_{n-1}\}$ is an $n+1$ element independent subset of Mn.

   (b) $o \prec e_0$.

[1052] e.g. $o \prec e_t$, $o \neq e_x$ and $\langle o, e_t \rangle \perp_r \langle o, e_x \rangle$ etc.
[1053] or identify
[1054] i.e. knowing $w_m(\bar{0}), w_m(1_t), \dots, w_m(1_y)$
[1055] $L \subseteq lines$, cf. Item 6.6.39 on p.1052.

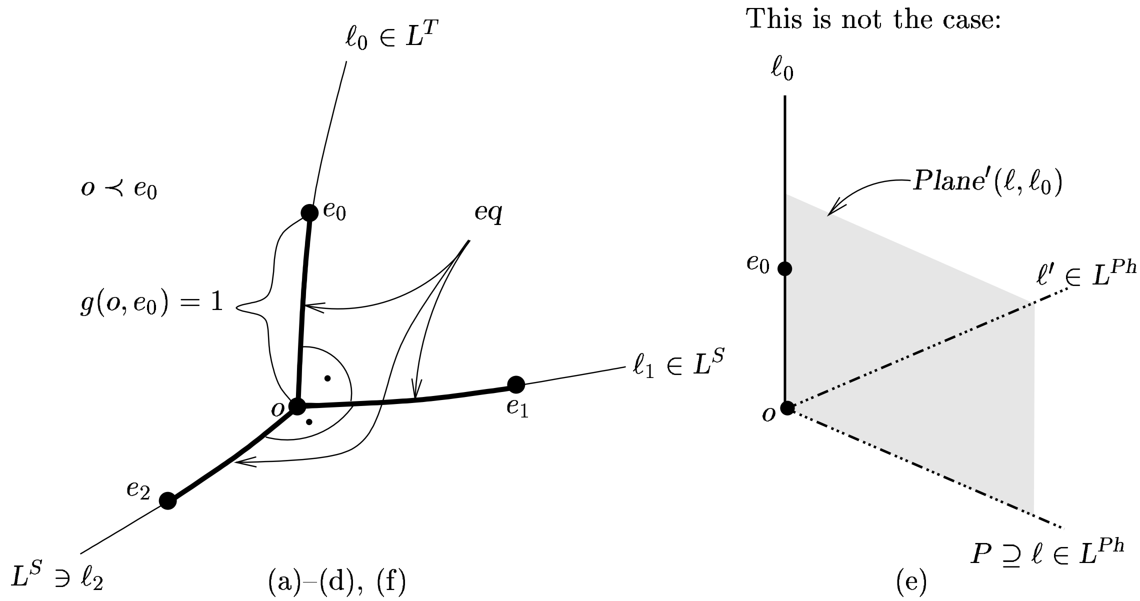


Figure 323: Illustration for the definition of *Obs*.

(c) $\langle o, e_0 \rangle \; eq \; \langle o, e_i \rangle$, for all $i \in n$.

(d) $(\exists \ell_0 \in L^T)(\exists \ell_1, \ldots, \ell_{n-1} \in L^S)$

$$\Big((\forall i \in n)\, o, e_i \in \ell_i \;\wedge\; (\forall \text{ distinct } i, j \in n)\, \ell_i \perp_r \ell_j\Big).$$

<u>*Convention:*</u> To each choice of $\langle o, \ldots, e_{n-1} \rangle$ we will use $\ell_0, \ell_1, \ldots, \ell_{n-1}$ as fixed by (d) above.

(e) $P := Plane'(\{o, e_1, \ldots, e_{n-1}\})$ is space-like in the following sense:

$$(\forall \ell, \ell' \in L^{Ph}) \Big([o \in \ell \subseteq P \;\wedge\; o \in \ell' \subseteq Plane'(\ell, \ell_0)] \quad \rightarrow \quad \ell = \ell' \Big),$$

see the right-hand side of Figure 323. In $\mathbf{Bax}^-$ geometries, intuitively, this means that if P contains the trace of a photon then the speed of this photon is infinite.[1056] Without assuming $\mathbf{Bax}^-$, condition (e) corresponds to axiom $\mathbf{Ax}(\infty ph)$ on p.1028 as part of the theory $\mathbf{Pax}^+$ (Def.6.6.10).

(f) $g(o, e_0) = 1$.

2. $Ph :\stackrel{\text{def}}{=} L^{Ph}$.

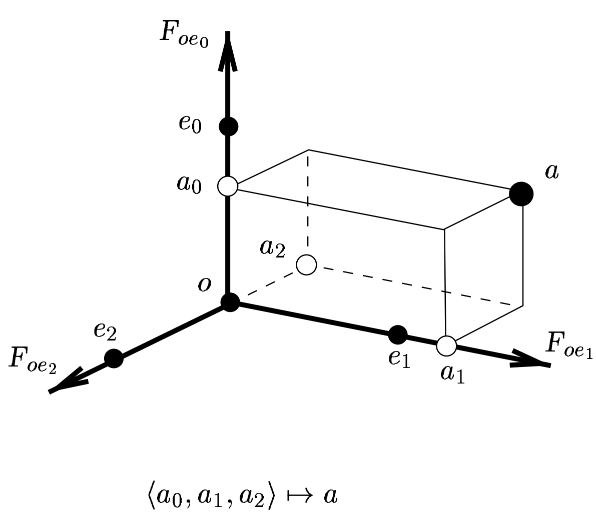
[1056] Item (e) is required only in order to make the following statement true: If $\mathfrak{G}$ is a $\mathbf{Bax}^{-\oplus}$ geometry then $\mathcal{M}(\mathfrak{G})$ is a $\mathbf{Bax}^{-\oplus}$ model, assuming $\mathbf{Pax}^+$ of course.

3. $B :\stackrel{\text{def}}{=} Ib :\stackrel{\text{def}}{=} Obs \cup Ph$.

4. Definition of the world-view relation W: First for every $m \in Obs$ we define the coordinatization function $w^0_m : {}^nF \rightarrowtail Mn$ as follows.[1057] Let $m = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs$. (Notice that, by (c), $o, e_0, \ldots, e_{n-1}$ are pairwise connected, i.e. $\sim$-related.) We use the notation F_{oe} introduced in Def.6.6.31, i.e. F_{oe} is the line determined by the points o and e. First, by using parallel lines[1058], we obtain a coordinatization mapping

$$F_{oe_0} \times F_{oe_1} \times \ldots \times F_{oe_{n-1}} \rightarrowtail Mn,$$

as depicted in the left-hand side of Fig.324. Next, for every $i \in n$, we identify F_{oe_i} with F_{oe_0} as depicted in the right-hand side of Fig.324, using lines parallel with $F_{e_ie_0}$. By these identifications and the above coordinatization, we obtain



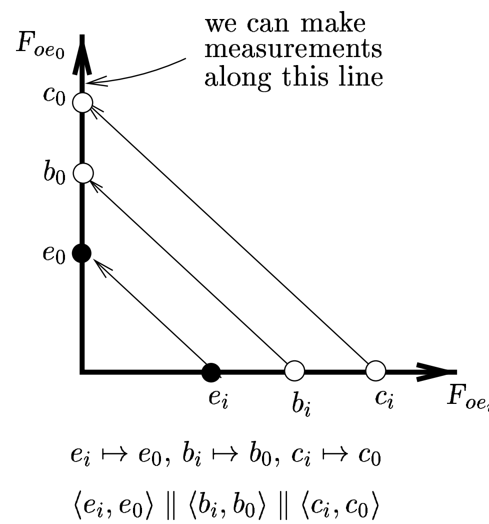


Figure 324: In the left-hand side of the picture we assume that $n = 3$.

a coordinatization

$${}^nF_{oe_0} \rightarrowtail Mn.$$

[1057]The problem which we will have to circumnavigate is that by g we can make reliable measurements only on the line determined by o, e_0 (since we assumed **Ax(eqtime)** but not **Ax(eqm)**). I.e. by g we can suitably measure the o, e_0 distance, while by the same g we cannot suitably measure the o, e_1 distance. This is why we will use parallel lines, cf. the right-hand side of Fig.324.

[1058]Here we use *lines* and $\parallel$ both definable in $\mathfrak{G}$, cf. Item 6.6.39 (p.1052).

We identify F by F_{oe_0} using g, the natural way, i.e. 0 and 1 get identified with o and e_0, repectively, and $x \in F$ gets identified with $a \in F_{oe_0}$ such that $g(o,a) = |x|$ and $(Bw(a,o,e_0) \Leftrightarrow x < 0)$. (This identification can be done because by the assumption $\mathbf{Pax}^+$ we can make reliable measurements along F_{oe_0} by g.) In this way, from the above coordinatization ${}^nF_{oe_0} \rightarrowtail Mn$ we obtain the coordinatization

$$w_m^0 : {}^nF \rightarrowtail Mn.$$

In the next step, from w_m^0 we define the real world-view function w_m (whose range is a subset of $\mathcal{P}(B)$). To this end we "represent" Mn as part of $\mathcal{P}(B)$ i.e. we define a mapping $f : Mn \longrightarrow \mathcal{P}(B)$ the natural way. Let $e \in Mn$. Then we say that a photon $\ell \in Ph$ is present in event e iff $e \in \ell$, and an observer $\langle o', e'_0, \ldots, e'_{n-1}\rangle \in Obs$ is present in event e iff $e \in F_{o'e'_0}$. Let

$$f : Mn \longrightarrow \mathcal{P}(B)$$

be defined by

$$f(e) \stackrel{\text{def}}{:=} \{\, b \in B \ : \ b \text{ is present in } e \,\}, \quad \text{for all } e \in Mn.$$

Let $w_m \stackrel{\text{def}}{:=} w_m^0 \circ f$. The world-view relation W is defined from the w_m's the obvious way, i.e.

$$W \stackrel{\text{def}}{:=} \{\, \langle m,p,b\rangle \in Obs \times {}^nF \times B \ : \ b \in w_m(p) \,\}\,.$$

Thus, all ingredients of $\mathcal{M}(\mathfrak{G})$ are defined except for the ordered field $\mathfrak{F}$. Now we turn to defining $\mathfrak{F}$.

5. *Definition of $\mathfrak{F}$:* To define the ordered field $\mathfrak{F}$ from the geometry $\mathfrak{G}$ it is enough to define multiplication on F (from $\mathfrak{G}$), since $\mathbf{F_1} = \langle F; 0, 1, +, \le\rangle$ is contained in $\mathfrak{G}$. Now we turn to doing this.

   First let us notice that there is an original ordered field $\mathfrak{F}^{\mathfrak{M}}$ behind $\mathfrak{G}$, since $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$, for some $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^+)$. Let such an $\mathfrak{M}$ be fixed. Let

$$\mathbf{F}_1^{\mathfrak{M}} \stackrel{\text{def}}{:=} \langle \mathbf{F}^{\mathfrak{M}}; 0^{\mathfrak{M}}, 1^{\mathfrak{M}}, +^{\mathfrak{M}}, \le^{\mathfrak{M}}\rangle.$$

   Now, $\mathbf{F_1} \cong \mathbf{F}_1^{\mathfrak{M}}$ by $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Of course we are not allowed to use $\mathfrak{F}^{\mathfrak{M}}$ when we are defining something from $\mathfrak{G}$, since $\mathfrak{F}^{\mathfrak{M}}$ is not explicitly included in $\mathfrak{G}$. (We use $\mathfrak{F}^{\mathfrak{M}}$ only for didactical [i.e. explanatory] purposes.) Now, we start defining multiplication over $\mathfrak{G}$. Assume $o, e \in Mn$, $o \equiv^T e$ and $g(o,e) = 1$. Such o, e exist by **Ax(eqtime)** (and by $\mathbf{AxE_{01}}$ + $(\forall m, k)(\forall 0 < i \in n)\, tr_m(k) \neq \bar{x}_i$) or by **Ax(eqm)** (and $\mathbf{AxE_{01}}$). Let $Mn_o :\stackrel{\text{def}}{=} \{\, a \,:\, o \sim a \,\}$. Then $\langle Mn_o;\, Bw \restriction Mn_o\rangle \models \mathbf{opag}$ by Prop.6.6.40. Let $\mathfrak{F}_{oe} = \langle F_{oe};\, +_{oe}, \cdot_{oe}, \leq_{oe}\rangle$ be the ordered field corresponding to o, e defined in Def.6.6.31. By Prop.6.6.32, $\mathfrak{F}_{oe}$ is indeed an ordered field (and is isomorphic to $\mathfrak{F}^{\mathfrak{M}}$). Let $g_{oe} : F_{oe} \longrightarrow F$ be defined as follows: Let $a \in F_{oe}$. Then

$$g_{oe}(a) :\stackrel{\text{def}}{=} \begin{cases} g(o,a) & \text{if } a \in [oe \\ -g(o,a) & \text{otherwise.} \end{cases}$$

Clearly, $g_{oe}(o) = 0$ and $g_{oe}(e) = 1$ by our choice of o, e. We note that $g_{oe} : F_{oe} \longrightarrow F$ is an isomorphism between $\langle F_{oe};\, o, e, +_{oe}, \leq_{oe}\rangle$ and $\mathbf{F_1}$. Now we use these g_{oe}'s to copy the multiplications $\cdot_{oe}$ on F_{oe}'s to obtain multiplication $\cdot$ on F. We define multiplication $\cdot \subseteq F \times F \times F$ as follows. Let $x, y, z \in F$

$$\cdot(x,y,z)$$
$$\stackrel{\text{def}}{\iff}$$
$$(\exists o, e \in Mn)\Big[\; o \equiv^T e \;\wedge\; g(o,e) = 1 \;\wedge\; g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)\Big].$$

By this, multiplication $\cdot$ is defined on F. By the above the structure $\mathfrak{F} := \langle F;\, +, \cdot, \leq\rangle$ is defined. We will prove as Claim 6.6.43 that $\mathfrak{F}$ is an ordered field isomorphic to $\mathfrak{F}^{\mathfrak{M}}$.

By items 1–5 above, the frame model $\mathcal{M}(\mathfrak{G})$ is defined.

END OF DEF. OF THE FUNCTOR $\mathcal{M}$.

$\lhd$

Remark 6.6.42 We note that, if $n > 2$, $\mathcal{M}$ is defined on $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqtime)})$ and $\mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)})$, by Proposition 6.6.11 (p.1029).

$\lhd$

Claim 6.6.43 below serves to prove correctness of Def.6.6.41 above.

Claim 6.6.43 Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^{+})$. Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^{+})$ be such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Let the structure $\mathfrak{F}$ be defined as in item 5 of Def.6.6.41 above. Then $\mathfrak{F}$ is an ordered field isomorphic to $\mathfrak{F}^{\mathfrak{M}}$.

Outline of proof: Let $\mathfrak{G}, \mathfrak{M}, \mathfrak{F}$ be as in the claim. Without loss of generality we can assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$, because the functor $\mathcal{M}$ was defined in such a style that it associates isomorphic models to isomorphic structures. Assume $o, e \in Mn$ are such that $o \equiv^T e$ and $g(o, e) = 1$. Let $Mn_o := \{ a \in Mn : a \sim o \}$. Let $m \in Obs$ be such that $o, e \in w_m[\bar{t}]$. Exists. Then

$$w_m : \langle {}^nF; \mathsf{Betw} \rangle \rightarrowtail\!\!\!\twoheadrightarrow \langle Mn_o; Mn_o \restriction Bw \rangle$$

is an isomorphism by Thm.4.3.13 (p.482). Let $o' := w_m^{-1}(o)$ and $e' := w_m^{-1}(e)$. Clearly $o', e' \in \bar{t}$. Let $\mathfrak{F}_{o'e'} = \langle F_{o'e'}; \ldots \rangle$ and $\mathfrak{F}_{oe} = \langle F_{oe}; \ldots \rangle$ be the ordered fields corresponding to o', e' and o, e, respectively defined in Def.6.6.31 (p.1046). Then $F_{o'e'} = \bar{t}$ and $|e'_t - o'_t| = 1$. The latter holds by

$$(*) \qquad \begin{array}{l} g(o,e) = 1 \quad \text{and} \quad \mathbf{AxE_{01}} + \\ \Big((\, \mathbf{Ax(eqtime)} \wedge (\forall m, k)(\forall 0 < i \in n) tr_m(k) \neq \bar{x}_i \,) \vee \mathbf{Ax(eqm)} \Big). \end{array}$$

Without loss of generality we may assume that $e'_t - o'_t = 1$. Let $g_{oe} : F_{oe} \longrightarrow F^{\mathfrak{M}}$ be defined as on p.1058. Now, $(w_m \restriction \bar{t}) \circ g_{oe} : F_{o'e'} \longrightarrow F^{\mathfrak{M}}$ and $(w_m \restriction \bar{t}) \circ g_{oe} : p \mapsto p_t - o'_t$ by $(*)$. Thus, $(w_m \restriction \bar{t}) \circ g_{oe} : \mathfrak{F}_{o'e'} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{F}^{\mathfrak{M}}$ is an isomorphism. By this and by noticing that $w_m \restriction \bar{t} : \mathfrak{F}_{o'e'} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{F}_{oe}$ is an isomorphism, we conclude that

$$g_{oe} : \mathfrak{F}_{oe} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{F}^{\mathfrak{M}}$$

is an isomorphism. By this it can be checked that the multiplication defined on $F^{\mathfrak{M}}$ on p.1058 coincides with the multiplication of $\mathfrak{F}^{\mathfrak{M}}$. Hence $\mathfrak{F}$ and $\mathfrak{F}^{\mathfrak{M}}$ are isomorphic. (Actually, by our assumption that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$ $\mathfrak{F}$ and $\mathfrak{F}^{\mathfrak{M}}$ coincide.) ∎

Next we state that the functor $\mathcal{M}$ constructed so far is of the kind we need for our duality theory outlined on pp.1007-1008, cf. Fig.310 (p.1007).

PROPOSITION 6.6.44 $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ *and* $\mathcal{M}$ *is a first-order definable meta-function. Hence* $\mathcal{M}[\mathsf{Ge}(\mathbf{Pax}^+)] \subseteq \mathsf{Mod}(\mathbf{Pax}^+)$ *is first-order definable over* $\mathsf{Ge}(\mathbf{Pax}^+)$.

Outline of proof: First-order definability of $\mathcal{M}$ comes immediately from the definition of $\mathcal{M}$ (by using Remark 6.3.36 on p.980). To prove $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^+) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^+)$ let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$. Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^+)$ be such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Without loss of generality we can assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$. The visibility relation $\overset{\odot}{\to}$ is an equivalence relation when restricted to $Obs^{\mathfrak{M}}$ by Thm.4.3.13. Let $O \subseteq Obs^{\mathfrak{M}}$ be a set of representatives for the equivalence relation $\overset{\odot}{\to}$. Recall that for every

$k \in Obs^{\mathfrak{M}}$ $\quad \mathfrak{G}_k = \langle {}^nF, \ldots \rangle$ is the observer-dependent geometry defined in Def.6.2.76 (p.880). Then similarly to item 3b of Prop.6.2.79 (p.889) the $\perp_r$-free reduct of $\mathfrak{G}$ is a photon-glued disjoint union of the family

$$\langle \perp_r\text{-free reduct of } \mathfrak{G}_k \ : \ k \in O \rangle.$$

Further $Bw_k = \mathsf{Betw}$ and $L_k \subseteq \mathsf{Eucl}$ by Thm.4.3.13 for every $k \in Obs^{\mathfrak{M}}$. Thus $\mathfrak{G}$ is a photon-glued disjoint union of the familiar ${}^nF$-geometries. By this, it can be checked that $\mathcal{M}(\mathfrak{G}) \models \mathbf{Pax} + \mathbf{Ax(Bw)} + \mathbf{AxE_{01}} + \mathbf{Ax}(\infty ph)$. Thus it remains to prove that $\mathcal{M}(\mathfrak{G}) \models (\mathbf{Ax(eqtime)} + (\forall m,k)(\forall 0 < i \in n) tr_m(k) \neq \bar{x}_i)$ or $\mathcal{M}(\mathfrak{G}) \models \mathbf{Ax(eqm)}$. By $\mathfrak{M} \models \mathbf{Pax^+}$, we have $\mathfrak{M} \models (\mathbf{Ax(eqtime)} + (\forall m,k)(\forall 0 < i \in n) tr_m(k) \neq \bar{x}_i)$ or $\mathfrak{M} \models \mathbf{Ax(eqm)}$. For the case $\mathfrak{M} \models (\mathbf{Ax(eqtime)} + \ldots)$ checking $\mathcal{M}(\mathfrak{G}) \models (\mathbf{Ax(eqtime)} + \ldots)$ is easy and is left to the reader. (Hint: $L^T \cap L^S = \emptyset$ and $L^T \cap L^{Ph} = \emptyset$ hold in this case.)

Assume $\mathfrak{M} \models \mathbf{Ax(eqm)}$. We will prove that $\mathcal{M}(\mathfrak{G}) \models \mathbf{Ax(eqm)}$. Let $g^* : Mn \times Mn \overset{\circ}{\longrightarrow} F$ be the partial function defined as follows. Let $e, e_1 \in Mn$ and $\lambda \in F$. Then

$$g^*(e, e_1) = \lambda$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs^{\mathfrak{M}})(\exists i \in n)(\exists p, q \in \bar{x}_i)(w_m(p) = e \ \wedge \ w_m(q) = e_1 \ \wedge \ |p - q| = \lambda).$$

By $\mathbf{Ax(eqm)}$, g^* is well defined. By $\mathbf{Ax(eqm)} + \mathbf{AxE_{01}}$, it is easy to check that g and g^* agree on time-like separated pairs of points. For every $m \in Obs^{\mathcal{M}(\mathfrak{G})}$ let $w_m^0 : {}^nF \longrightarrow Mn$ be defined as on p.1056 in Def.6.6.41. If we prove

$$(*) \qquad (\forall m \in Obs^{\mathcal{M}(\mathfrak{G})})(\forall i \in n)(\forall p, q \in \bar{x}_i) \, |p - q| = g^*(w_m^0(p), w_m^0(q))$$

then $\mathcal{M}(\mathfrak{G}) \models \mathbf{Ax(eqm)}$ will hold (by the definition of W on p.1057). Thus it is enough to prove $(*)$ above. For every $o, e \in Mn$ with $o \neq e$ and $o \sim e$ let $F_{oe} = \{a \in Mn \ : \ coll(a, o, e)\}$; and for every $o, e, o', e' \in Mn$ with $o \neq e$, $o' \neq e'$, $o \sim e$ and $o' \sim e'$ let $f_{oe}^{o'e'} : F_{oe} \longrightarrow F_{o'e'}$ be defined as in the proof of Prop.6.6.33 on p.1047. Now items 1 and 2 below hold because of the following. It is easy to check that items 1,2 hold when eq is replaced by eq_0 in them. By this, by $f_{oe}^{o'e'} \circ f_{o'e'}^{o''e''} = f_{oe}^{o''e''}$ and since eq is defined to be the transitive closure of eq_0 we have that 1 and 2 below hold. (In proving this, $L^T \cap L^{Ph} = \emptyset$ is used too).

1. $\langle a, b \rangle \ eq \ \langle c, d \rangle \ \Rightarrow \ g^*(a, b) = g^*(c, d)$.

2. $(\forall o, e, o', e' \in Mn)\Big((o \neq e \ \wedge \ o' \neq e' \ \wedge \ \langle o, e \rangle \ eq \ \langle o', e' \rangle) \ \rightarrow$
$$(\forall a, b \in F_{oe}) \, \langle a, b \rangle \ eq \ \langle f_{oe}^{o'e'}(a), f_{oe}^{o'e'}(b) \rangle\Big).$$

Now we turn to proving (*) above. Let $m \in Obs^{\mathcal{M}(\mathfrak{G})}$, $i \in n$, $p, q \in \bar{x}_i$. Then $m = \langle o, e_0, \ldots, e_{n-1}\rangle$ for some $o, e_0, \ldots e_{n-1} \in Mn$ satisfying (a)–(f) on p.1054. By 2 above and by $\langle o, e_0\rangle\ eq\ \langle o, e_i\rangle$, we have that $\langle w^0_m(p), w^0_m(q)\rangle\ eq\ \langle f^{oe_0}_{oe_i}(w^0_m(p)), f^{oe_0}_{oe_i}(w^0_m(q))\rangle$. Hence, by 1 above, $g^*(w^0_m(p), w^0_m(q)) = g^*(f^{oe_0}_{oe_i}(w^0_m(p), f^{oe_0}_{oe_i}(w^0_m(q)))$. By the definition of $\mathcal{M}(\mathfrak{G})$, [1059] we have $|p - q| = g^*(f^{oe_0}_{oe_i}(w^0_m(p)), f^{oe_0}_{oe_i}(w^0_m(q)))$. Thus $|p - q| = g^*(w^0_m(p), w^0_m(q))$ and this proves the proposition. ∎

The following theorem implies that the sentences in our frame language can be translated to sentences in the language of our relativistic geometries (in a meaning preserving way), assuming $\mathbf{Pax}^+$. More intuitively, whatever can be said in the language of the ("observational") frame models can be said in the "theoretical terminology" of relativistic geometries, too. (Cf. Thm.6.6.16 on p.1033.)

THEOREM 6.6.45 *There is a "natural" translation mapping*

$$T_{\mathcal{M}} : Fm(\mathsf{Mod}(\mathbf{Pax}^+)) \longrightarrow Fm(\mathsf{Ge}(\mathbf{Pax}^+))$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{Mod}(\mathbf{Pax}^+))$ with all its free variables belonging to sort F, $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$ and evaluation $\bar{a}$ of $\bar{x}$ (in F of course)

$$\mathcal{M}(\mathfrak{G}) \models \varphi[\bar{a}] \quad \Leftrightarrow \quad \mathfrak{G} \models T_{\mathcal{M}}(\varphi)[\bar{a}].$$

Proof: The theorem follows by Prop.6.6.44 and by Prop.6.4.4 (p.985). ∎

The following theorem says that for our $(\mathcal{G}, \mathcal{M})$-duality, theorem schemas (A)–(H), hold under some conditions.

THEOREM 6.6.46 *For the choice of $\mathcal{M}$ given in Def.6.6.41 above the conclusions of Theorems 6.6.12 (p.1030) and 6.6.17 (p.1034) hold. E.g. $\mathcal{G}$ and $\mathcal{M}$ are first-order definable meta-functions and*

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th),$$

assuming Th satisfies condition $(\star)$ in Thm.6.6.12 and **Ax(diswind)**. *Further, theorem schemas (A)–(H) hold, etc.*

[1059] and by noticing that $f^{oe_0}_{oe_i}(w^0_m(p))$, $f^{oe_0}_{oe_i}(w^0_m(q)) \in F_{oe_0}$, $\langle w^0_m(p), f^{oe_0}_{oe_i}(w^0_m(p))\rangle \parallel \langle e_i, e_0\rangle$, $\langle w^0_m(q), f^{oe_0}_{oe_i}(w^0_m(q))\rangle \parallel \langle e_i, e_0\rangle$

Outline of proof:

Case of Thm.6.6.12:
Let Th be as in Thm.6.6.12. Assume $n > 2$. Clearly, $Th \models \mathbf{Pax}^+$ (by Thm.4.3.24). Let $\mathfrak{M} \in \mathsf{Mod}(Th)$. Let $h_{Obs} : Obs^{\mathfrak{M}} \longrightarrow Obs^{(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})}$ be defined by $h_{Obs} : m \mapsto \langle w_m(\bar{0}), w_m(1_0), \ldots, w_m(1_{n-1})\rangle$ and $h_{Ph} : Ph^{\mathfrak{M}} \longrightarrow Ph^{(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})}$ be defined by $h_{Ph} : ph \mapsto \{e \in Mn_{\mathfrak{M}} : ph \in e\}$. By Remark 6.2.66 (ii) (p.867), Propositions 6.2.88 (p.895) and 6.2.92 (p.901) one can check that

$$\langle h_{Obs} \cup h_{Ph}, \mathrm{Id} \restriction F, \mathrm{Id} \restriction G\rangle : \mathfrak{M} \rightarrowtail\!\!\!\twoheadrightarrow (\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$$

is an isomorphism. Thus

$$(*) \qquad (\forall \mathfrak{M} \in \mathsf{Mod}(Th))(\mathcal{G}\circ\mathcal{M})(\mathfrak{M}) \cong \mathfrak{M}.$$

Let $\mathfrak{G} \in \mathsf{Ge}(Th)$. Then $\mathfrak{G} \cong \mathcal{G}(\mathfrak{M})$ for some $\mathfrak{M} \in \mathsf{Mod}(Th)$. Let this $\mathfrak{M}$ be fixed. Then $(\mathcal{G}\circ\mathcal{M})(\mathfrak{M}) \cong \mathfrak{M}$ by $(*)$ above. Hence, $(\mathcal{M}\circ\mathcal{G})(\mathcal{G}(\mathfrak{M})) \cong \mathcal{G}(\mathfrak{M})$. Thus, $(\mathcal{M}\circ\mathcal{G})(\mathfrak{G}) \cong \mathfrak{G}$. By the above, item (ii) of Thm.6.6.12 is proved. By $(*)$ above, and by the fact that $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism we conclude that $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$. Further, $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ holds by the definition of $\mathcal{G}$. First-order definability of $\mathcal{M}$ comes from Prop.6.6.44 while first-order definability of $\mathcal{G}$ comes from Thm.6.3.22 (p.961). By this Thm.6.6.12 is proved.

Case of Thm.6.6.17: For any $\mathfrak{G} \in \mathsf{Ge}(\emptyset)$ let $\mathfrak{G}^*$ be the geometry obtained from $\mathfrak{G}$ by omitting $\mathcal{T}$ and replacing g with $g \restriction \{\langle a, b\rangle \in Mn \times Mn : a \equiv^T b\}$. It can be checked that for any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$, $\mathfrak{G}^* \cong (\mathcal{M}\circ\mathcal{G})(\mathfrak{G}^*)$. Further, for any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$, $\mathcal{M}(\mathfrak{G}) \cong \mathcal{M}(\mathfrak{G}^*)$. Therefore, for any $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax}^+)$, $\mathcal{M}(\mathfrak{G}) \cong (\mathcal{G}\circ\mathcal{M})(\mathcal{M}(\mathfrak{G}))$. This proves item (i) of the theorem. Item (ii) follows from item (i) by the fact that $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism. For the proof of item (iii) cf. the proof for the case of Thm.6.6.12 above. Item (iv) follows by the proof of Prop.6.6.44 and by the proof of item (i). ∎

The next proposition says that for certain choices of Th, if $\mathfrak{G}$ is a Th-geometry then $\mathcal{M}(\mathfrak{G})$ is a Th-model. More intuitively, our duality theory works for these choices of Th.

PROPOSITION 6.6.47

$$\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th) \quad \textit{and} \quad \mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th),$$ [1060]

assuming

$$Th := Th_1 + \mathbf{Pax}^+,$$

[1060] The $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ part is easy by the definition of $\mathsf{Ge}(Th)$, so the emphasis is on the $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$ part.

where $Th_1 \in \{\emptyset,\ \mathbf{Bax}^{-\oplus},\ \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^{-} + \mathbf{Ax}(\sqrt{\ }) + Th_2,\ \mathbf{Flxbasax} + Th_2,\ \mathbf{Newbasax} + Th_2,\ \mathbf{Basax} + Th_2,\ \mathbf{Basax} + \mathbf{Ax}(\omega)^0 + Th_2\}$, *where* $Th_2 \subseteq \{\mathbf{Ax}(Triv),\ \mathbf{Ax}(Triv_t)^{-},\ \mathbf{Ax}(\|)\}$.

Further, for these choices of Th and for $\mathcal{M}$ *defined in Def.6.6.41 conclusions (i)–(iii) of Thm.6.6.17 (p.1034) hold when* $\mathbf{Pax}^{+}$ *is replaced by Th in them.*

On the proof: We will give a proof for the case $Th_1 = \mathbf{Bax}^{-\oplus}$ and $n > 2$. The proofs for the remaining cases can be obtained by Remark 6.2.66 (ii) (p.867), Propositions 6.2.88 (p.895) and 6.2.92 (p.901), and are left to the reader.

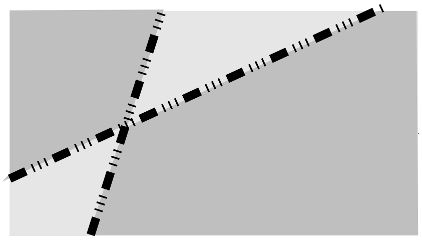
Assume $n > 2$. Let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Pax}^{+})$. Then $\mathcal{M}(\mathfrak{G}) \in \mathsf{Ge}(\mathbf{Pax}^{+})$ by Prop.6.6.44. Thus to prove $\mathcal{M}(\mathfrak{G}) \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Pax}^{+})$ it remains to prove $(*)$ below.

$(*)$ In the world-view of any observer $m \in Obs^{\mathcal{M}(\mathfrak{G})}$ for any point p and for any direction d the following holds. There is exactly one photon trace forwards in direction d passing through p and the "speed of this photon trace" is not ∞; and for all speeds slower than the speed of this photon trace there is an observer moving in direction d with this speed and passing through point p.

Throughout the proof we tacitly use Prop.6.2.79 (p.884). Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{-\oplus} + \mathbf{Pax}^{+})$ be such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Without loss of generality we may assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$.[1061] Let $m \in Obs^{\mathcal{M}}(\mathfrak{G})$. Then $m = \langle o, e_0, \dots, e_{n-1}\rangle$ for some $o, e_0, \dots, e_{n-1} \in Mn$ satisfying (a)–(f) on p.1054. Let $\ell_0 \in L^T$ be such that $o, e_0 \in \ell_0$. Let P be defined as in item (e) on p.1055. Intuitively P is the space part of observer m. We claim that there are no photon-like lines in P. To prove this claim, assume that there is a photon like line in P. Then, by Thm.4.3.17 (p.488), there is $\ell \in L^{Ph}$ such that $o \in \ell \subseteq P$. Let this ℓ be fixed. Then by item (e) on p.1055 there is exactly one photon-like line in the plane determined by ℓ and ℓ_0 passing through o. ℓ_0 is the life-line of some observer $k \in Obs^{\mathfrak{M}}$, i.e. $\ell_0 = \{e \in Mn : k \in e\}$. Let this k be fixed. Then, since $\mathfrak{M} \models \mathbf{Bax}^{-}$, and since there is only one photon-like line in the plane determined by ℓ and ℓ_0 passing through o we conclude that for k the photon whose life-line is ℓ moves with infinite speed. This contradicts "$\oplus$", i.e. contradicts $\mathbf{Bax}^{-\oplus}$. Thus there are no photon-like lines in P.

Now, we turn to proving $(*)$ above for m and for $p = \bar{0}$. Let P' be a 2-dimensional plane that contains ℓ_0. Since $\mathfrak{M} \models \mathbf{Bax}^{-\oplus}$ and the life-line of $k \in Obs^{\mathfrak{M}}$ is ℓ_0 there are exactly two photon-like lines in P' passing through o. These two photon-like lines divide the plane P' into two regions as illustrated below.

[1061]This is so since $\mathcal{M}$ preserve the property of being isomorphic as we already noted.



Let ℓ_P be the intersection of P' and P. Neither one of the two photon-like lines coincides with ℓ_P since in P there are no photon-like lines. We will prove that ℓ_P and ℓ_0 are in different regions. Assume that ℓ_P and ℓ_0 are in the same region. See the left-hand side of Figure 325. Then, since $\mathfrak{M} \models \mathbf{Bax}^-$ and the life-line of k is

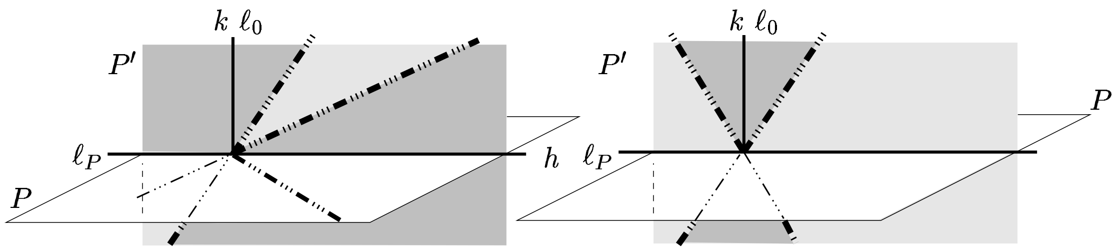


Figure 325:

ℓ_0, we conclude that k sees an observer h on ℓ_P, i.e. ℓ_P is the life-line of observer $h \in Obs^{\mathfrak{M}}$. Since through any point and in any direction h sees a photon and h's life-line ℓ_P is contained in P we conclude that there is a photon-like line in P. This leads to a contradiction since we proved that there are no photon-like lines in P. Thus, ℓ_0 and ℓ_P are in different regions, cf. the right-hand side of Figure 325. Then any line in the same region as ℓ_0 passing through o is time-like. This can be proved by using the world-view of observer k. But then it can be seen that any line in the same region as ℓ_0 passing through o is a "life-line" of an observer in the model $\mathfrak{M}^{\mathcal{M}(\mathfrak{G})}$, too.[1062] Thus we proved that $(*)$ above holds for m and for $p = \bar{0}$. Since, by Thm.4.3.17 (p.488), straight lines parallel to traces of photons are traces of photons again and since any line parallel to a time-like line is a time-like line by **Ax4**, we conclude that $(*)$ above holds for arbitrary p and not only for $\bar{0}$. ∎

QUESTION 6.6.48 *Does Proposition 6.6.47 above generalize from* $Th_1 = \mathbf{Bax}^{-\oplus}$ *to* $Th_1 = \mathbf{Bax}^-$*?*

◁

[1062] All observers of $\mathfrak{M}$ show up in $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$ in a modified form.

The next proposition says that the operator $\mathcal{G} \circ \mathcal{M}$ makes our models more "puritan" in some sense.

PROPOSITION 6.6.49 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax}^+)$. *Then*

$$(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \models \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit.$$

We omit the easy **proof**. ∎

It might be interesting to notice that by the above proposition some of the conditions of the categoricity theorem (Thm.3.8.7 on p.299) become true in $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$.

megnézni hogy ez kérdés-e?

Question for future research 6.6.50 It would be interesting to see for which reduct of $\mathfrak{G}_{\mathfrak{M}}$ does the above outlined duality theory still go through. We note that in §6.6.4 we will have an analogous duality theory for the $(g, \mathcal{T})$-free reduct of our geometries.

◁

We close the present sub-section with Remark 6.6.51 below. Further theorems about $(\mathcal{G}, \mathcal{M})$-duality will be stated in §6.6.5 (p.1078) and §6.6.6 (p.1084).

The following remark shows how to remove the condition **Ax(eqtime)** (or **Ax(eqm)**) from our duality theory $(\mathcal{G}, \mathcal{M})$, i.e. how to reconstruct $\mathfrak{M}$ (at least a version of $\mathfrak{M}$) from the geometry $\mathfrak{G}_{\mathfrak{M}}$ even if **Ax(eqtime)** is not assumed.

Remark 6.6.51 On a possible *more* general function $\mathcal{M}^+$: Geometries $\longrightarrow$ Models (not requiring the whole of $\mathbf{Pax}^+$ to be assumed before the definition):

(A) Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)})$. Let $o, e \in Mn$ with $o \neq e$ and $o \sim e$. Let $\mathfrak{F}_{oe} = \langle F_{oe}, \ldots \rangle$ be the ordered field corresponding to o, e as defined Def.6.6.31. An element a of F_{oe} is called <u>*positive*</u> iff $o \leq_{oe} a$ and $o \neq a$, as one would expect. Consider the possible properties (i), (ii) below.

(i) $(\forall$ positive $a, b \in F_{oe})\ [a \neq b \ \Rightarrow\ g(o,a) \neq g(o,b)]$.

Let $g_{oe} : F_{oe} \longrightarrow F$ be defined by

$$g_{oe}(a) :\stackrel{\text{def}}{=} \begin{cases} g(o,a) & \text{if } a \text{ is positive} \\ -g(o,a) & \text{otherwise.}^{1063} \end{cases}$$

(ii) $g_{oe} : \langle F_{oe};\ o, e, +_{oe}, \leq_{oe} \rangle \longrightarrow \mathbf{F_1}$ is an isomorphism.

[1063]This g_{oe} is the same as the g_{oe} on p.1058.

If (i) and (ii) hold for $o, e \in Mn$ with $o \neq e$ and $o \sim e$ then we say that g is nice on F_{oe}.

Question for future research: Do we need (ii) or is (i) enough? That is, is (i) $\Rightarrow$ (ii) true in *some* sense?

Def. of $\mathcal{M}^+(\mathfrak{G})$: We distinguish two cases.

Case (I): Assume $\mathfrak{G}$ is such that g is nice on *some* F_{oe}. Then we define multiplication "$\cdot$" on F as follows.

$$\cdot(x, y, z)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists o, e \in Mn)\Big[o \neq e \ \wedge \ o \sim e \ \wedge \ (g \text{ is nice on } F_{oe}) \ \wedge \\ g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)\Big].$$

Then we construct $\mathcal{M}^+(\mathfrak{G})$ the same way as $\mathcal{M}(\mathfrak{G})$ was constructed, except that we do not require item (f) to hold in the definition of Obs, i.e.

$$Obs := \big\{\langle o, e_0, \ldots, e_{n-1}\rangle \in {}^{n+1}Mn \ : \ \text{(a)–(e) hold on p.1054}\big\}$$

and $\mathfrak{F} := \langle F, +, \cdot, \leq\rangle$, where $\cdot$ is defined above. At the end of the remark we will prove that

(♣) $\mathfrak{F}$ is an ordered field.

The rest of the ingredients of $\mathcal{M}^+(\mathfrak{G})$ are defined exactly as those of $\mathcal{M}(\mathfrak{G})$.

Case (II): Assume that for any $o, e \in Mn$ with $o \neq e$ and $o \sim e$, g is *not nice* on F_{oe}. Then we throw g away and use an arbitrary $o, e \in Mn$ with $o \neq e$ and $o \sim e$ and an arbitrary isomorphism[1064] $i : \langle F_{oe}; o, e, +_{oe}, \leq_{oe}\rangle \longrightarrow \mathbf{F_1}$ to copy the multiplication $\cdot_{oe}$ of $\mathfrak{F}_{oe}$ to F obtaining an ordered field $\mathfrak{F}$. The rest of $\mathcal{M}^+(\mathfrak{G})$ is defined as in Case (I).

We note that in Case (II) $\mathcal{M}^+(\mathfrak{G})$ is not first-order definable over $\mathfrak{G}$ in general while in Case (I) $\mathcal{M}^+(\mathfrak{G})$ is first-order definable over $\mathfrak{G}$.

Now, we conjecture that the theorems stated for $\mathcal{M}$ go through for $\mathcal{M}^+$ with very little change (and the same conditions). Further we guess that some simple theorems like $(\mathcal{G}\circ\mathcal{M}^+)^2(\mathfrak{M}) \cong (\mathcal{G}\circ\mathcal{M}^+)(\mathfrak{M})$ will be true, for $\mathfrak{M} \models \mathbf{Pax}+\mathbf{Ax(Bw)}$.

(B) Item (A) above suggests the following possibility for improving/generalizing our $(\mathcal{G}, \mathcal{M})$-duality theory. First, one formulates an axiom in our frame language which implies about $\mathfrak{M}$ that in $\mathfrak{G}_{\mathfrak{M}}$ g is nice on some F_{oe}, assuming $\mathbf{Pax}$. Let us notice

[1064] It can be proved that $\langle F_{oe}; o, e, +_{oe}, \leq_{oe}\rangle$ is isomorphic with $\mathbf{F_1}$.

that there exist very mild choices for such an axiom, e.g. **Ax(mild)** below is such. We note that **Ax(mild)** is much weaker than **Ax(eqtime)** $\vee$ **Ax(eqm)**, assuming e.g. $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ })$ and $n > 2$.

Ax(mild) $(\exists m \in Obs)(\exists i \in n)\,[(\forall ph \in Ph) tr_m(ph) \neq \bar{x}_i \ \wedge\ (\forall p, q \in \bar{x}_i)(\forall k \in Obs)$ (the distance between events $w_m(p)$ and $w_m(q)$ as measured by k is not smaller than the distance between these two events as measured by m, i.e. if k sees both $w_m(p)$ and $w_m(q)$ on the same coordinate axis then the distance between $w_m(p)$ and $w_m(q)$ as measured by k is not smaller than $|p - q|$)].

Then, one can obtain a duality theory (between frame models and geometries) in which one uses the milder **Ax(mild)** in place of **Ax(eqtime)**. I.e. one defines a first-order definable meta-function $\mathcal{M}^* : \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)} + \mathbf{Ax(mild)}) \longrightarrow \mathsf{FM}$ exactly as $\mathcal{M}^+$ was defined in item (A) for Case (I).

<u>*Proof of (♣):*</u> Now we turn to proving that $\mathfrak{F}$ defined in Case (I) is an ordered field. Let $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(Bw)})$ be such that g is nice on some F_{oe} and let "$\cdot$" and $\mathfrak{F}$ be defined as in Case (I) above. Then there is $\mathfrak{M} \models \mathbf{Pax} + \mathbf{Ax(Bw)}$ such that $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$. Let this $\mathfrak{M}$ be fixed. Without loss of generality we may assume that $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$. Hence $\mathbf{F_1} = \mathbf{F_1^{\mathfrak{M}}}$. To avoid ambiguity we will denote the multiplication of the ordered field $\mathfrak{F}^{\mathfrak{M}}$ by "$*$" (instead of the usual "$\cdot$"). To prove that $\mathfrak{F}$ defined in Case (I) above is an ordered field it is enough to prove that $\cdot$ and $*$ coincide, i.e.

$$(\star) \qquad (\forall x, y, z \in F)\,(\,\cdot(x, y, z) \ \Leftrightarrow\ x * y = z\,).$$

Let $o, e \in Mn$ be fixed such that $o \neq e$ and $o \sim e$. Observer m is called <u>*good for F_{oe}*</u> iff m sees F_{oe} on a coordinate axis (i.e. $w_m[\bar{x}_i] = F_{oe}$ for some $i \in n$) and the distance between o and e as measured by m is 1 (i.e. $|w_m^{-1}(e) - w_m^{-1}(o)| = 1$). For every observer m which sees F_{oe} on a coordinate axis we define a function $g^m : F_{oe} \longrightarrow F$ as follows. Intuitively, $g^m(a)$ will be the signed distance between o and a as measured by m. Let $m \in Obs$ be such that m sees F_{oe} on a coordinate axis. Let $a \in F_{oe}$. Then

$$g^m(a) :\stackrel{\text{def}}{=} \begin{cases} \ \ |w_m^{-1}(a) - w_m^{-1}(o)| & \text{if } a \text{ is positive} \\ -|w_m^{-1}(a) - w_m^{-1}(o)| & \text{otherwise.} \end{cases}$$

By Thm.4.3.13 (p.482), it is easy to see that

$(\star\star)$ $g^m : \langle F_{oe};\, o, +_{oe}, \leq_{oe}\rangle \rightarrowtail\!\!\!\!\rightarrow \langle F;\, 0, +, \leq\rangle$ is an isomorphism and if m is good for F_{oe} then

$$g^m : \mathfrak{F}_{oe} \rightarrowtail\!\!\!\!\rightarrow \mathfrak{F}^{\mathfrak{M}}$$

is an isomorphism.

Claim **6.6.52** Assume that g is nice on F_{oe}. Then for every $x, y, z \in F$ there is an observer m such that m is good for F_{oe}, $(g^m)^{-1}(x) = g_{oe}^{-1}(x)$, $(g^m)^{-1}(y) = g_{oe}^{-1}(y)$, and $(g^m)^{-1}(z) = g_{oe}^{-1}(z)$.

Proof: Assume, g is nice on F_{oe}. To prove the claim it is enough to prove that for every $a, b, c \in F_{oe}$ there is an observer m such that m is good for F_{oe} and $g^m(a) = g_{oe}(a)$, $g^m(b) = g_{oe}(b)$, $g^m(c) = g_{oe}(c)$. Let $a, b, c \in F_{oe}$. For every $f \in F_{oe}$ by $-_{oe}f$ we denote the inverse of f taken in the group $\langle F_{oe}; o, +_{oe}\rangle$. Since for every $f \in F_{oe}$, $g^m(-_{oe}f) = -g^m(f)$ and $g_{oe}(-_{oe}f) = -g_{oe}(f)$ without loss of generality we may assume that a, b, c are non-negative, i.e. that $o \leq_{oe} a$ etc. Let

$$d := e +_{oe} a +_{oe} b +_{oe} c.$$

Let $m \in Obs$ be such that m sees F_{oe} on a coordinate axis and the distance between o and d as measured by m is $g(o, d)$, formally $|w_m^{-1}(d) - w_m^{-1}(o)| = g(o, d)$. Such an m exists by the definition of g. Hence,

$$g^m(d) = g_{oe}(d).$$

By $(\star\star)$,

$$g^m(d) = g^m(e) + g^m(a) + g^m(b) + g^m(c),$$

and $g^m(e)$, $g^m(a)$, $g^m(b)$, $g^m(c)$ are non-negative. Further, (since g_{oe} is nice on F_{oe}) we have,

$$g_{oe}(d) = g_{oe}(e) + g_{oe}(a) + g_{oe}(b) + g_{oe}(c),$$

and $g_{oe}(e)$, $g_{oe}(a)$, $g_{oe}(b)$, $g_{oe}(c)$ are non-negative. Further,

$$g_{oe}(e) \leq g^m(e), \quad g_{oe}(a) \leq g^m(a), \quad g_{oe}(b) \leq g^m(b), \quad g_{oe}(c) \leq g^m(c)$$

by the definitions of g, g_{oe}, g^m (i.e. by the fact that for every positive $f \in F_{oe}$ $g^m(f)$ is the distance between o and f as measured by m while $g_{oe}(f)$ is the minimum of the distances between o and f measured by observers who see F_{oe} on a coordinate axis). Therefore, $|w_m^{-1}(e) - w_m^{-1}(o)| =: g^m(e) = g_{oe}(e) = 1$, $g^m(a) = g_{oe}(a)$, etc., i.e. observer m has the desired properties.

(QED Claim 6.6.52)

Now, we turn to proving $(\star)$ above. Let $x, y, z \in F$.

Proof of direction "$\Rightarrow$": Assume $\cdot(x, y, z)$. Then there are $o, e \in Mn$ such that $o \neq e$, $o \sim e$, g is nice on F_{oe} and $g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)$. Let such o, e be fixed. Then, by Claim 6.6.52, there is an observer m such that m is good for F_{oe}, $(g^m)^{-1}(x) = g_{oe}^{-1}(x)$, $(g^m)^{-1}(y) = g_{oe}^{-1}(y)$, and $(g^m)^{-1}(z) = g_{oe}^{-1}(z)$. Let this m be fixed. Now, $(g^m)^{-1}(z) = (g^m)^{-1}(x) \cdot_{oe} (g^m)^{-1}(y)$. Thus, by the second part of ($\star\star$), $z = x * y$.

Proof of direction "⇐": Assume $z = x * y$. Let $o, e \in Mn$ be such that g is nice on F_{oe} (and, of course, $o \neq e$, $o \sim e$). If $m \in Obs$ is good for F_{oe} then by $z = x * y$ and ($\star\star$), we have $(g^m)^{-1}(z) = (g^m)^{-1}(x) \cdot_{oe} (g^m)^{-1}(y)$. By Claim 6.6.52 there is m such that m is good for o, e, $(g^m)^{-1}(x) = g_{oe}^{-1}(x)$, $(g^m)^{-1}(y) = g_{oe}^{-1}(y)$, and $(g^m)^{-1}(z) = g_{oe}^{-1}(z)$. Therefore $g_{oe}^{-1}(z) = g_{oe}^{-1}(x) \cdot_{oe} g_{oe}^{-1}(y)$. Hence $\cdot(x, y, z)$ and this completes the proof of (♣).

◁

6.6.4 Duality theory for the $(g, \mathcal{T})$-free reducts of our geometries

Motivation for looking at reducts of our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$ is given in the introduction of §6.7 ("Interdefinability ...") pp. 1134–1135 and on p.1124. A further motivation for the physicist might be that depending on which aspect of the physical world we want to concentrate on we will "see" different reducts[1065] of our $\mathfrak{G}_{\mathfrak{M}}$.

The main message of our $(\mathcal{G}, \mathcal{M})$-duality is that we can reconstruct the original observational model $\mathfrak{M}$ from the streamlined, more abstract geometry $\mathfrak{G}_{\mathfrak{M}}$ associated to it (under some conditions of course). So, we *do not loose information* if we move from the "detail-rich" world $\mathfrak{M}$ to the geometry abstracted from it. The question naturally comes up: How much of $\mathfrak{G}_{\mathfrak{M}}$ is needed for this reconstruction? In other words, from which reducts of $\mathfrak{G}_{\mathfrak{M}}$ is our "original world" $\mathfrak{M}$ reconstructible? Of course, if we take a too small reduct e.g. $\langle Mn, L; \in\rangle$ then we will not be able to reconstruct $\mathfrak{M}$ from this reduct. Below we will see that if we omit g and $\mathcal{T}$ from $\mathfrak{G}_{\mathfrak{M}}$ then $\mathfrak{M}$ remains reconstructible from this weaker geometry $\mathfrak{G}^0_{\mathfrak{M}} = \langle Mn, \dots, eq\rangle$, under some conditions.[1066] We will do more than just reconstructing $\mathfrak{M}$ from $\mathfrak{G}^0_{\mathfrak{M}}$, namely we will elaborate a duality theory (analogous to our original one) between $\mathsf{Mod}(Th)$ and our weaker geometries.[1067]

[1065] e.g. we may want to concentrate on the so called conformal structure (i.e. the light-cones) of space-time, or we may want to concentrate on orthogonality, or on the metric g etc.

[1066] A price we will have to pay for omitting g is that we will have to add **Ax6** to our assumptions.

[1067] We leave it, partially, to the reader to decide exactly which other reducts of $\mathfrak{G}_{\mathfrak{M}}$ are strong enough such that $\mathfrak{M}$ is recoverable from them. In other words: which reducts of $\mathfrak{G}_{\mathfrak{M}}$ are strong

In more detail: In the present sub-section we will see that even if we omit g from our geometries we can still develop a duality theory between geometries and models. As a contrast, later (in §6.6.10) we will see that we cannot omit much more from our geometries without loosing the possibility for building a (similarly strong) duality theory.

The present duality theory will be more symmetric than the previous one $(\mathcal{M}, \mathcal{G})$, namely in the new duality the geometries will be axiomatically defined just as the frame models are, cf. the text below Thm.6.6.17 on p.1036.

At the same time, we note that at least from a certain point of view, the new duality will involve loosing (or forgetting) a bit more "information" than in the case of $(\mathcal{M}, \mathcal{G})$. Namely, under some assumptions,

$$\mathfrak{M} \models \mathbf{Ax(eqtime)} \quad \Rightarrow \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \models \mathbf{Ax(eqtime)}.$$

I.e. $\mathcal{G} \circ \mathcal{M}$ "preserves" **Ax(eqtime)**. This property will be lost in the case of the new duality. (This can be sometimes be an advantage and some other times a disadvantage).

Definition 6.6.53

(i) For every frame model $\mathfrak{M}$, $\mathfrak{G}^0_{\mathfrak{M}}$ is defined to be the $(g, \mathcal{T})$-free reduct of $\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \ldots \rangle$, i.e.

$$\mathfrak{G}^0_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq \rangle.$$

(ii) For any set Th of formulas in our frame language the corresponding class $\mathsf{Ge}^0(Th)$ of geometries is defined as follows.

$$\mathsf{Ge}^0(Th) :\stackrel{\text{def}}{=} \{\, \mathfrak{G} \ :\ (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\, \mathfrak{G}^0_{\mathfrak{M}} \cong \mathfrak{G} \,\}.$$

(iii) GEO is defined to be the class of all structures of the <u>similarity type of $\mathsf{Ge}^0(\emptyset)$</u> in which the axiom of extensionality holds for the incidence relation $\in$ ($\in \subseteq Mn \times L$). Because of this, without loss of generality we may assume that our incidence relation is the real set theoretic $\in$. Actually throughout we will assume this.

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enough to support a duality theory analogous to $(\mathcal{G}, \mathcal{M})$-duality and the one below. Cf. also item 6.6.50 (p.1065). In §6.6.10 and §6.7 we will obtain some partial information in this direction.

(iv) For any set TH of formulas in the language of GEO

$$\mathsf{Mog}(TH) :\stackrel{\text{def}}{=} \{\mathfrak{G} \in \mathsf{GEO} \;:\; \mathfrak{G} \models TH\}.$$ [1068]

We introduce axioms $\mathbf{L_1}$ and $\mathbf{L_2}$ in the language of GEO. We use the abbreviation *coll* introduced in item 6.2.12 and the new sort *lines* which is first-order defined from *coll* (and Mn) on p.1037. Axioms $\mathbf{L_1}$, $\mathbf{L_2}$ below state that L-lines are also *lines*-lines, and that any point is the intersection of two photon-like lines.

$\mathbf{L_1}$ $L \subseteq \mathit{lines}$.

(This is one of the places where we heavily use the assumption in Def.6.6.53(iii), i.e. that the geometric incidence relation is the set theoretic $\in$. Of course the axiom could be formulated without relying on this assumption, but then it would become longer.)

$\mathbf{L_2}$ $(\forall a \in Mn)(\exists \ell, \ell' \in L^{Ph})\, \ell \cap \ell' = \{a\}$.

Recall that **opag** is the axiom system for ordered Pappian affine geometries defined on p.1044 in Def.6.6.27.

Definition 6.6.54 $\quad \mathbf{lopag} :\stackrel{\text{def}}{=} \mathbf{opag} + \mathbf{L_1} + \mathbf{L_2}$.

◁

In the following definition we define the functors $\mathcal{G}o$ and $\mathcal{M}o$ connecting the two worlds $\mathsf{Mod}(\ldots)$ and $\mathsf{Mog}(\mathbf{lopag})$; according to the pattern

$$\mathsf{Mod}(\ldots) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad \mathsf{Mog}(\mathbf{lopag})$$

and more generally

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad \mathsf{Mog}(TH),$$

where Th and TH are in two different languages.

Much of the intuitive idea for the definition of $\mathcal{M}$ on p.1053 applies to the definition of $\mathcal{M}o$ given below.

[1068] Since TH is a theory and $\mathsf{Mog}(TH)$ consists of the models of that theory we could have used the notation $\mathsf{Mod}(TH)$ in place of $\mathsf{Mog}(TH)$. However we wanted to emphasize that the language of our present TH is the geometric language of GEO. Therefore the models of TH will be geometries. To emphasize this we use the notation $\mathsf{Mog}(TH)$ to remind the reader that the language is now that of geometries.

Definition 6.6.55 (functors $\mathcal{G}o$ and $\mathcal{M}o$)

(i) We define the functor $\mathcal{G}o : \mathsf{FM} \longrightarrow \mathsf{GEO}$ to be the function $\mathfrak{M} \mapsto \mathfrak{G}^0_{\mathfrak{M}}$.

(ii) We define the functor $\mathcal{M}o : \mathsf{Mog}(\mathbf{lopag}) \longrightarrow \mathsf{FM}$ as follows. Let $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$. Then the model

$\mathcal{M}o(\mathfrak{G}) = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, \mathsf{Eucl}(\mathfrak{F});\ \in, W \rangle$ is defined as follows.

$Obs :\overset{\text{def}}{=} \{ \langle o, e_0, \ldots, e_{n-1} \rangle \in {}^{n+1}Mn \ :\ \text{(a)–(e) on p.1054 hold} \}$.

If $Obs = \emptyset$, then $\mathcal{M}o(\mathfrak{G})$ is defined to be the empty model, otherwise the rest of the ingredients of $\mathcal{M}o(\mathfrak{G})$ are defined as follows.

$Ph :\overset{\text{def}}{=} L^{Ph}$.

$B :\overset{\text{def}}{=} Ib \ :\overset{\text{def}}{=} Obs \cup Ph$.

$\mathfrak{F} = \langle F; \ldots \rangle$ is the ordered field corresponding to $\langle Mn;\ Bw \rangle$ defined in Def.6.6.34 (p.1049).

For every $\langle o, e_0, \ldots, e_{n-1} \rangle \in Obs$ the coordinatization

$$Co_{\langle o, e_0, \ldots, e_{n-1} \rangle} : Mn \longrightarrow {}^nF$$

is defined in Def.6.6.37 (p.1051). By Prop.6.6.38, we have that these coordinatizations are bijections. For every $m = \langle o, e_0, \ldots, e_{n-1} \rangle \in Obs$, we define

$$w^0_m :\overset{\text{def}}{=} Co^{-1}_{\langle o, e_0, \ldots, e_{n-1} \rangle}.$$

Now the world-view relation W is defined from the functions w^0_m's exactly as in Def.6.6.41. Let $m \in Obs$ and $p \in {}^nF$. Then

$w_m(p) :\overset{\text{def}}{=}$
$\{ \ell \in Ph \ :\ w^0_m(p) \in \ell \} \cup \{ \langle o, e_0, \ldots, e_{n-1} \rangle \in Obs \ :\ coll(o, e_0, w^0_m(p)) \}$.[1069]

W is defined from the w_m's the obvious way, i.e.

$$W :\overset{\text{def}}{=} \{ \langle m, p, b \rangle \in Obs \times {}^nF \times B \ :\ b \in w_m(p) \}.$$

◁

Now we introduce the axiom system **Wax** in our frame language which will nicely "match" with the geometrical axiom system **lopag**. **Ax**(Ph) below is one of the axioms of **Wax**.

[1069] For a more intuitive (but longer) formula defining w_m cf. the definition of $\mathcal{M}$, p.1057.

Ax(Ph) $(\forall m \in Obs)(\forall p \in {}^nF)(\exists ph_1, ph_2 \in Ph)\ tr_m(ph_1) \cap tr_m(ph_2) = \{p\}$.

Intuitively, each observer at any point p sees at least two photons, and these two photons do not meet at any point different from p.

Definition 6.6.56 $\mathbf{Wax} :\stackrel{\text{def}}{=} \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax6}, \mathbf{Ax(Bw)}, \mathbf{Ax}(Ph)\}$. $\lhd$

We note that the following "weak" axiom systems are stronger than **Wax**. $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6}$, $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(Bw)} + \mathbf{Ax6}$, $\mathbf{Pax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Ph) + \mathbf{Ax6}$, $\mathbf{Pax} + \mathbf{Ax(Bw)} + \mathbf{Ax}(Ph) + \mathbf{Ax6}$; and if $n > 2$ $\mathbf{Bax}^-(n) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6}$, $\mathbf{Bax}^-(n) + \mathbf{Ax(Bw)} + \mathbf{Ax6}$, $\mathbf{Bax}(n) + \mathbf{Ax6}$.

Item (ii) of the following theorem is of the pattern of theorem-schemas (G), (H) on p.1011 way above. (Cf. Thm.6.6.17 for a similar theorem.) The whole theorem is of the pattern

$$\mathsf{Mod}(\mathbf{Wax}) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad \mathsf{Mog}(\mathbf{lopag}).$$

THEOREM 6.6.57

(i)
$$\mathcal{G}o : \mathsf{Mod}(\mathbf{Wax}) \longrightarrow \mathsf{Mog}(\mathbf{lopag}), \qquad \mathcal{M}o : \mathsf{Mog}(\mathbf{lopag}) \longrightarrow \mathsf{Mod}(\mathbf{Wax}),$$

and $\mathcal{M}o$ is a first-order <u>*definable*</u> *meta-function.*

(ii) *Both $\mathcal{G}o \circ \mathcal{M}o$ and $\mathcal{M}o \circ \mathcal{G}o$ have fixed-point property in the sense that for any* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax})$ *and* $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$

$$(\mathcal{G}o \circ \mathcal{M}o)^2(\mathfrak{M}) \cong (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \quad \textit{and} \quad (\mathcal{M}o \circ \mathcal{G}o)^2(\mathfrak{G}) \cong (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}).$$

(iii) *For any* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax})$ *and* $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$

$$\mathcal{G}o(\mathfrak{M}) \longleftarrow\!\!\prec (\mathcal{M}o \circ \mathcal{G}o)(\mathcal{G}o(\mathfrak{M})) \quad \textit{and} \quad \mathcal{M}o(\mathfrak{G}) \succ\!\!\longrightarrow (\mathcal{G}o \circ \mathcal{M}o)(\mathcal{M}o(\mathfrak{G})).$$

We omit the **proof**, but cf. the proof of Thm.6.6.46. ∎

Galois connections will be introduced on p.1080, §6.6.5. Motivated by the above theorem we conjecture that there is a Galois connection between $Rng(\mathcal{G}o)$ and

$Rng(\mathcal{M}o)$,[1070] cf. Thm.6.6.70 (p.1083). Actually, this Galois connection can be regarded as an adjoint situation (to be introduced on p.1091) too according to the following pattern

$$Rng(\mathcal{M}o) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad Rng(\mathcal{G}o),$$

cf. Conjecture 6.6.81 (p.1093). Further, we conjecture that between $Rng(\mathcal{G}o \circ \mathcal{M}o)$ and $Rng(\mathcal{M}o \circ \mathcal{G}o)$ the same connection turns out to be an equivalence of categories (cf. p.1094) of the pattern

$$Rng(\mathcal{G}o \circ \mathcal{M}o) \quad \underset{\mathcal{M}o}{\overset{\mathcal{G}o}{\rightleftarrows}} \quad Rng(\mathcal{M}o \circ \mathcal{G}o),$$

cf. Conjecture 6.6.84 (p.1094).

Conjecture 6.6.58 *We conjecture that*

$$\mathsf{Mod}(Th) \equiv_{\Delta} \mathsf{Mog}(TH),$$

for certain natural choices of Th and TH. We note that these choices of Th we have in mind contain the axiom $(\forall m)(\forall h \in Exp)(\exists k)\mathsf{f}_{mk} = h$.[1071]

Filling out the details and including the proof is left to the interested reader. Hint: Use the construction in the proof of Thm.6.6.13 (p.1031) omitting of course any references to those parts to the geometry which do not exist in the present case like e.g. g.

◁

Further theorems in this line will be stated in the next two sub-sections.

The following theorem says that the sentences in our frame language can be translated to sentences in the language of our relativistic geometries (not involving the function g and the topology $\mathcal{T}$) in a meaning preserving way, assuming **lopag** on both sides. (Cf. Thm.6.6.45 for a similar theorem.)

[1070] To show that this is a Galois connection one has to define appropriate pre-orderings on the classes $Rng(\mathcal{G}o)$ and $Rng(\mathcal{M}o)$.

[1071] Intuitively, this means that there are arbitrarily large as well as arbitrarily small animals, cf. Remark 4.2.1 on p.458.

THEOREM 6.6.59 *There is a "natural" translation mapping*

$$T_{\mathcal{M}o} : Fm(\mathsf{FM}) \longrightarrow Fm(\mathsf{GEO})$$

such that for every $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$ *and sentence* $\varphi \in Fm(\mathsf{FM})$

$$\mathcal{M}o(\mathfrak{G}) \models \varphi \quad \Leftrightarrow \quad \mathfrak{G} \models T_{\mathcal{M}o}(\varphi).$$

Proof: The theorem follows by item (i) of Thm.6.6.57 and by Prop.6.4.4 (p.985). ∎

The next proposition says that the operators $\mathcal{G}o \circ \mathcal{M}o$ and $\mathcal{M}o \circ \mathcal{G}o$ make our models and geometries "smooth" in some sense. (Cf. Prop.6.6.49 for a similar proposition.) We already know, by Thm.6.6.57, that for any $\mathfrak{M} \models \mathbf{Wax}$ $(\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \models \mathbf{Wax}$. Item (i) of the proposition states that besides **Wax** some further axioms become true when $\mathcal{G}o \circ \mathcal{M}o$ is applied to $\mathfrak{M}$. A similar remark applies to **lopag** and item (ii) below.

PROPOSITION 6.6.60

(i) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax})$. *Then*

$$(\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \;\models\; \mathbf{Ax(ext)} \;+\; \mathbf{Ax}\heartsuit \;+\; (\forall m,k)(\mathsf{f}_{mk} \in Aftr) \;+$$

$$+\; \mathbf{Ax(}\infty ph\mathbf{)} \;+\; (\forall m)(\forall h \in Exp)(\exists k)\mathsf{f}_{mk} = h.$$

(ii) *Assume* $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag})$. *Then*

$$(\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}) \models \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}},$$

where axioms $\mathbf{L_3}, \ldots, \mathbf{L_{10}}$ *are introduced below the present proposition.*

Moreover;

(iii)

$$Rng(\mathcal{M}o) \;\models\; \mathbf{Ax(ext)} \;+\; \mathbf{Ax}\heartsuit \;+\; (\forall m,k)(\mathsf{f}_{mk} \in Aftr) \;+\; \mathbf{Ax(}\infty ph\mathbf{)} \;\textit{and}$$
$$Rng(\mathcal{G}o) \;\models\; \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}},$$

where axioms $\mathbf{L_3}, \ldots, \mathbf{L_{10}}$ *are introduced below.*

Itt meg lehetne említeni a hangya elefántot!

We omit the **proof**. ∎

Now we turn to introducing axioms $\mathbf{L_3}, \ldots, \mathbf{L_{10}}$ in the language of GEO. These axioms are motivated by item (ii) of the above proposition and/or by contemplating the idea that they are very natural (it is hard to imagine a reasonable geometry in which one of them would fail).

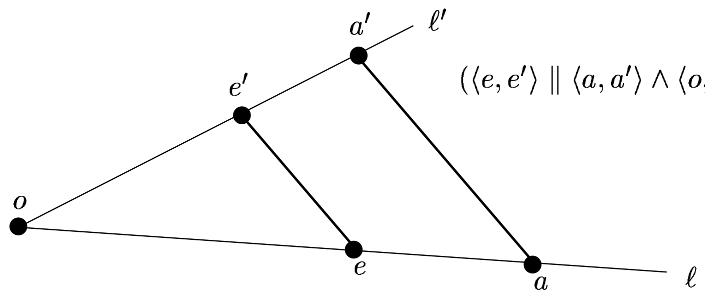
L₃ $([\,a \prec b \;\wedge\; (\,Bw(a,b,c) \;\vee\; Bw(a,c,b)\,)\,] \;\rightarrow\; a \prec c\,) \;\wedge$
$([\,a \prec b \;\wedge\; (\,Bw(c,a,b) \;\vee\; Bw(a,c,b)\,)\,] \;\rightarrow\; c \prec b\,).$

Intuitively, Bw and $\prec$ are both kinds of orderings. The axiom says that these two are "in harmony". In particular if we know Bw on a line ℓ, and two points of ℓ are $\prec$-related then this fact induces a $\prec$-connection between any two other points of ℓ.

L₄ Intuitively, eq is (very) symmetric, formally:
$$\langle a,b\rangle\ eq\ \langle c,d\rangle \quad\rightarrow\quad (\,\langle c,d\rangle\ eq\ \langle a,b\rangle \;\wedge\; \langle b,a\rangle\ eq\ \langle c,d\rangle \;\wedge\; \langle a,a\rangle\ eq\ \langle c,c\rangle\,).$$

L₅ eq is transitive, i.e.
$$(\,\langle a,b\rangle\ eq\ \langle c,d\rangle \;\wedge\; \langle c,d\rangle\ eq\ \langle e,f\rangle\,) \quad\rightarrow\quad \langle a,b\rangle\ eq\ \langle e,f\rangle.$$

L₆ (For the intuitive meaning of this axiom see Fig.326.)
$$(\forall \ell,\ell' \in L)(\forall o,e,e',a,a' \in Mn)\Big([\,\ell \cap \ell' = \{o\} \;\wedge\; e,a \in \ell \;\wedge\; e',a' \in \ell' \;\wedge$$
$$\langle e,e'\rangle \parallel \langle a,a'\rangle \;\wedge\; \langle o,e\rangle\ eq\ \langle o,e'\rangle\,] \quad\rightarrow\quad \langle o,a\rangle\ eq\ \langle o,a'\rangle\Big).$$



$$(\langle e,e'\rangle \parallel \langle a,a'\rangle \wedge \langle o,e\rangle\ eq\ \langle o,e'\rangle) \;\rightarrow\; \langle o,a\rangle\ eq\ \langle o,a'\rangle$$

Figure 326: Axiom **L₆**.

L₇ (For the intuitive meaning of this axiom see Fig.327.)
$$(\forall \ell \in L^T \cup L^S)(\forall a,b,c,d,e,f \in Mn)\,[\,(\,a,b,c,d \in \ell \;\wedge$$
$$\langle a,b\rangle \parallel \langle e,f\rangle \parallel \langle c,d\rangle \;\wedge\; \langle a,e\rangle \parallel \langle b,f\rangle \;\wedge\; \langle c,e\rangle \parallel \langle d,f\rangle\,) \;\rightarrow\; \langle a,b\rangle\ eq\ \langle c,d\rangle\,].$$

L₈ $\perp_r$ is symmetric, i.e.
$$(\forall \ell,\ell' \in L)\,(\ell \perp_r \ell' \;\rightarrow\; \ell' \perp_r \ell).$$

L₉ $\perp_r$ is closed under parallelism, i.e.
$$(\forall \ell,\ell_1,\ell_2 \in L)\,[\,(\,\ell \perp_r \ell_1 \;\wedge\; \ell_1 \parallel \ell_2\,) \quad\rightarrow\quad \ell \perp_r \ell_2\,].$$

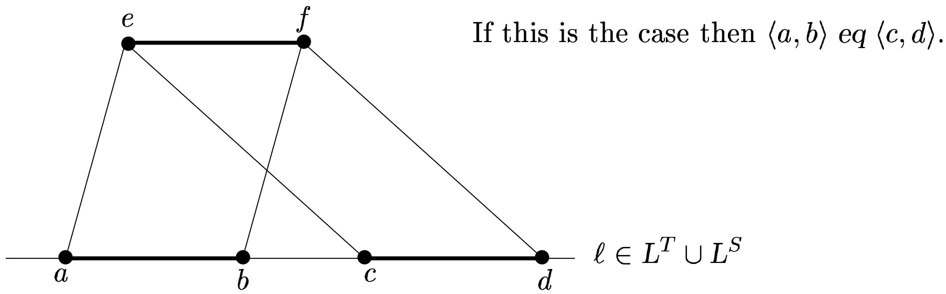


Figure 327: Axiom $\mathbf{L_7}$.

$\mathbf{L_{10}}$ $\perp_r$ is closed under taking limits, i.e. $\perp_r$ satisfies item (ii) on p.792. This property can be formulated in the language of GEO as follows. (See Fig.328.)

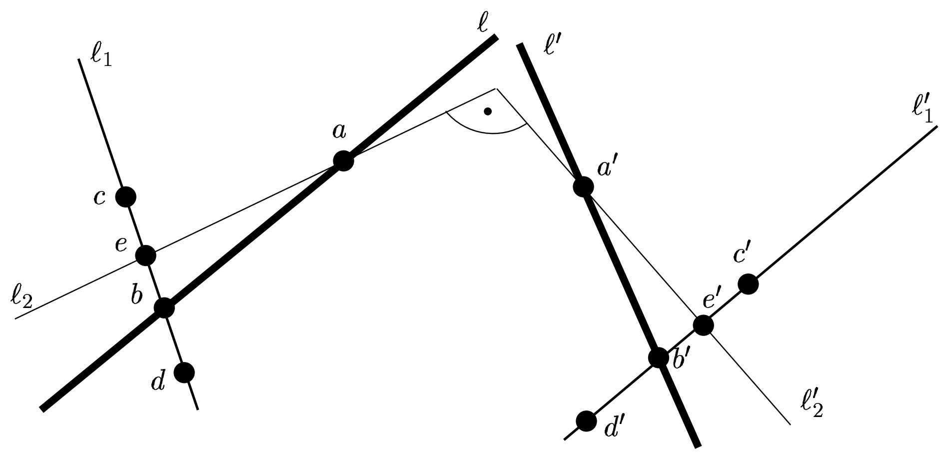


Figure 328: Illustration for axiom $\mathbf{L_{10}}$.

$$(\forall \ell, \ell' \in L)\Big((\exists \text{ distinct } a, b \in \ell)(\exists \text{ distinct } a', b' \in \ell')(\exists \ell_1, \ell'_1 \in L)$$
$$\Big[\, \ell \cap \ell_1 = \{b\} \;\wedge\; \ell' \cap \ell'_1 = \{b'\} \;\wedge\; (\forall c, d \in \ell_1)(\forall c', d' \in \ell'_1)$$
$$[\,(\, Bw(c,b,d) \;\wedge\; Bw(c',b',d')\,) \;\rightarrow\; (\exists e, e' \in Mn)(\exists \ell_2, \ell'_2 \in L)$$
$$(Bw(c,e,d) \;\wedge\; Bw(c',e',d') \;\wedge\; e, a \in \ell_2 \;\wedge\; e', a' \in \ell'_2 \;\wedge\; \ell_2 \perp_r \ell'_2\,)\,]\Big] \;\rightarrow$$
$$\ell \perp_r \ell'\Big),$$ cf. Fig.328.

6.6.5 Galois connections

In this sub-section we will see that $(\mathcal{G}, \mathcal{M})$ and $(\mathcal{G}o, \mathcal{M}o)$ form "Galois connections". In Def.6.6.62 below, we will recall from the literature the notion of a Galois connection cf. e.g. Adámek-Herrlich-Strecker [2, item 6.26(4), p.81]. We will compare Galois connections with adjoint functors and with further related concepts in the mathematical literature in item entitled "Connections between adjoint situations, Galois connections, ..." on p.1096 at the end of §6.6.6. Cf. also Remark 6.6.4 (pp. 1014–1027) as motivation for studying Galois connections.

Remark 6.6.61
(Motivations for Galois connections [for the physicist reader])

(I) Galois connection is a simplified form of adjoint situation (from category theory)[1072] which in turn is regarded as one of the most important[1073] conceptual tools of category theory. (To understand adjoint situations well, the first step is to understand Galois connections [as special adjoint situations].) Galois connections are obtained from adjointness by considering the simple kinds of categories called pre-orderings (where between any two objects there is at most one morphism); for these kinds of categories etc. cf. the subtitle "Connections between adjoint situations, Galois connections, ..." on p.1096.

Galois connection is a generalization of <u>isomorphism</u>. The idea is that isomorphism is very useful but it is a too <u>rigid</u> concept (and therefore it occurs rarely). So let us make isomorphism a little bit <u>more flexible</u> such that it would retain most of its useful properties[1074] but would become more flexible (more often applicable). The result i.e. the <u>flexible version of isomorphism</u> is called Galois connection (in the case when it connects pre-orderings). The definition is given in Def.6.6.62 below. In the general case (of categories) the name of "flexible isomorphisms" is adjoint situations or adjoint pairs of functors. To see a glimpse of the idea let us recall that an isomorphism from $\langle P, \leq \rangle$ onto $\langle Q, \leq \rangle$ is a homomorphism f such that there is a backward homomorphism g

$$\langle P, \leq \rangle \qquad \underset{g}{\overset{f}{\rightleftarrows}} \qquad \langle Q, \leq \rangle$$

[1072] Cf. §6.6.6 (p.1084) for category theory.
[1073] Cf. e.g. Adámek-Herrlich-Strecker [2], p.283 first sentence (Chap.18, Adjoint functors). There they write: "Perhaps the most successful concept of category theory is that of adjoint functor. Adjoint functors occur frequently in <u>many branches of mathematics</u> ... surprising range of applications." Cf. also (†) on p.1096 for importance of adjointness in physics.
[1074] e.g. we can transfer "constructions" from one side to the other

with $(f \circ g)(x) = x$ and $(g \circ f)(y) = y$. For easier formulation (of what comes) we replace homomorphism with dual-homomorphism (i.e. order reversing map). Now, to make the concept less rigid, we replace the condition $(f \circ g)(x) = x$ with the weaker one $(f \circ g)(x) \geq x$ and similarly for $g \circ f$. The result is summarized in Fact 6.6.63 below, but cf. also ($\star$) on p.1097 which might be a more suggestive (equivalent) definition of "flexible isomorphism". Then Fact 6.6.65 indicates that the resulting notion of "flexible isomorphisms" (i.e. Galois connections) retains many of the useful properties of isomorphisms.[1075] But to convince the reader that the so obtained notion of "flexible isomorphisms" really does the job it is supposed to do, one has to go through the literature of Galois connections and adjoint functors for which a few references and hints are collected on pp. 1014–1027, pp. 1084–1107; but perhaps pp. 1096–1105 is convincing in itself.

(II) Galois connections can serve as a unified theory of the research-branches mentioned on pp. 1096–1105 ranging from Boolean algebras with operators, residuated-residual pairs, conjugates of operators, linear logic, Lambek calculus, relation algebras, closure operators, geometry, vector spaces, C^*-algebras, but cf. also Janelidze [142] for more daring applications via Galois theories (which are of course strongly tied up[1076] with Galois connections).

In particular, studying Galois connections can serve as an abstract, *unified study of duality theories* or adjoint situations, which in turn, according to Adámek et al. [2], Lawvere [160] and others[1077] pervade much of mathematics and modern mathematical physics. We hope, recalling the patterns:

$$\langle P, \leq \rangle \quad \underset{g}{\overset{f}{\rightleftarrows}} \quad \langle Q, \leq \rangle \qquad\qquad \text{Galois connection}$$

[1075] The same idea in different words: A homomorphism f is called an isomorphism iff it admits a two-sided inverse g ($g \circ f = \mathrm{Id}$ and $f \circ g = \mathrm{Id}$). Now, in order to be a flexible isomorphism it is enough to admit a quasi-inverse as sketched in footnote 1093 on p.1097.

[1076] A Galois theory is always a (special) Galois connection, cf. items (I), (V) of Remark 6.6.4 (pp. 1014, 1027)

[1077] A sample of the references claiming and illustrating with examples that duality theories, i.e. adjoint situations ***are very broadly applicable*** (and applied) throughout mathematics and also in mathematical physics is Lawvere [160, 162, 161], Arbib-Manes [33, 32], Manes [184], Guitart [117], Mac Lane [168], Goldblatt [107], Handbook of Categorical Algebra [50], Barr-Wells [40, §1.9, p. 50–63], Freyd-Scedrov [89], Adámek et al. [2], [3], Varadarajan [270], Lawvere-Schanuel [163], Nel [202], Pelletier-Rosický [211], Dimov-Tholen [74], Janelidze [142], Davey-Priestley [68]. These references give examples ranging from algebraic geometry, compact Galois groups, geometry and analysis, sheaves of continuous maps, metric spaces, tensor algebra, Banach spaces and spaces of generalized Lipschitz functions, computability & automata & linear systems. (Cf. the works of Arbib, Manes, Guitart for the latter four topics.) Cf. also (†) on p.1096.

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th) \qquad \text{duality theory}$$[1078]

gives a hint for the above idea (of Galois connections serving as a unified, abstract study of dualities).

(III) Whenever we are given two sets or classes say K, L and a binary relation $R \subseteq K \times L$ between them then R induces a natural Galois connection between $\mathcal{P}(K)$ and $\mathcal{P}(L)$ as follows. For $X \subseteq K$, $f_R(X) = \{\, y \in L : (\forall x \in X)\, x\ R\ y \,\}$. So $f_R : \mathcal{P}(K) \longrightarrow \mathcal{P}(L)$ is order reversing. $g_R : \mathcal{P}(L) \longrightarrow \mathcal{P}(K)$ is defined analogously. Cf. item (IV) of Remark 6.6.4 (p.1026) which is the $(\mathsf{Mod}, \mathsf{Th})$-Galois connection induced by the relation $\models$. Cf. also p.453.

(IV) Cf. Remark 6.6.4, pp. 1014–1027.

END OF MOTIVATION FOR GALOIS CONNECTIONS.

$\lhd$

Definition 6.6.62 (Galois connection)
Let $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ be pre-ordered classes and

$$f : P \longrightarrow Q \quad \text{and} \quad g : Q \longrightarrow P.$$

The pair (f, g) is called a *Galois connection* between $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ iff for all $p \in P$ and $q \in Q$

$$p \leq g(q) \quad \Leftrightarrow \quad q \leq f(p).$$

$\lhd$

The following fact states a (known) equivalent reformulation of the definition of Galois connections.

FACT 6.6.63 *Assume $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ are pre-ordered clases and that $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$. Then the pair (f, g) is a Galois connection between $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ iff (a) and (b) below hold.*

- **(a)** *f and g are both order-reversing, i.e. if $p \leq p' \in P$ then $f(p) \geq f(p')$, and if $q \leq q' \in Q$ then $g(q) \geq g(q')$.*
- **(b)** *$f \circ g$ and $g \circ f$ are both monotonous, i.e.*
  $p \leq (f \circ g)(p)$ for all $p \in P$ and $q \leq (g \circ f)(q)$ for all $q \in Q$. ∎

[1078] We have not yet defined a structure like "$\leq$" on $\mathsf{Mod}(Th)$, $\mathsf{Ge}(Th)$ but that will come later (and is kind of implicit already in schemas (A)–(I) on pp.1009–1012).

Notation 6.6.64 Assume that $\langle P, \leq \rangle$ is a pre-ordered class. Then the binary relation $\simeq$ on P is defined as

$$p \simeq p' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (p \leq p' \ \wedge \ p' \leq p).$$

We note that $\simeq$ is an equivalence relation.

$\triangleleft$

Fact 6.6.65 below is known from algebra. Items (i)–(iii) of this fact say that if (f, g) is a Galois connection then both $f \circ g$ and $g \circ f$ are closure operators up to the equivalence relation $\simeq$ (cf. the notion of a closure operator up to isomorphism on p.1013.) Further, item (iv) says that the closed "up to $\simeq$" elements of $f \circ g$ are the elements of the range of g ("up to $\simeq$"). Similarly for $g \circ f$.

FACT 6.6.65 *Assume $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ are pre-ordered classes and (f, g) is a Galois connection between them. Then for all $p \in P$ and $q \in Q$, (i)–(iv) below hold.*

(i) $p \leq (f \circ g)(p)$ *and* $q \leq (g \circ f)(q)$.

(ii) *Both $f \circ g$ and $g \circ f$ have fixed-point property in the sense* $(f \circ g)^2(p) \simeq (f \circ g)(p)$ *and* $(g \circ f)^2(q) \simeq (g \circ f)(q)$.

(iii) *If $p \leq p' \in P$ and $q \leq q' \in Q$ then* $(f \circ g)(p) \leq (f \circ g)(p')$ *and* $(g \circ f)(q) \leq (g \circ f)(q')$.

(iv) $(g \circ f)(f(p)) \simeq f(p)$ *and* $(f \circ g)(g(q)) \simeq g(q)$. ∎

For the motivation of the following definition cf. Propositions 6.6.49 (p.1065) and 6.6.60 (p.1075).

Definition 6.6.66

$$\begin{aligned}
\mathbf{Pax}^{++} &\stackrel{\text{def}}{:=} \mathbf{Pax}^{+} + \mathbf{Ax(eqm)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit, \\
\mathbf{Wax}^{+} &\stackrel{\text{def}}{:=} \mathbf{Wax} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\infty ph) + (\forall m, k)(\mathsf{f}_{mk} \in Aftr) \\
\mathbf{lopag}^{+} &\stackrel{\text{def}}{:=} \mathbf{lopag} + \mathbf{L_3} + \mathbf{L_4} + \mathbf{L_5} + \mathbf{L_6} + \mathbf{L_7} + \mathbf{L_8} + \mathbf{L_9} + \mathbf{L_{10}}.
\end{aligned}$$

$\triangleleft$

Remark 6.6.67 We note that item (iii) of Prop.6.6.60 (p.1075) states, by Thm.6.6.57 (p.1073), that

$$Rng(\mathcal{M}o) \models \mathbf{Wax}^{+} \qquad \text{and} \qquad Rng(\mathcal{G}o) \models \mathbf{lopag}^{+}.$$

$\triangleleft$

ezekre jobb, természetesebb definiciót adni!

We will prove that $(\mathcal{G}o, \mathcal{M}o)$ forms a Galois connection between the classes $\mathsf{Mod}(\mathbf{Wax}^+)$ and $\mathsf{Mog}(\mathbf{lopag}^+)$ for a certain choice of pre-orderings $\leq_{\mathcal{M}o}$ and $\leq_{\mathcal{G}o}$ of these two classes. (I.e. $\leq_{\mathcal{M}o}$ is a pre-ordering of $\mathsf{Mod}(\mathbf{Wax}^+)$, and similarly for $\leq_{\mathcal{G}o}$ and $\mathsf{Mog}(\mathbf{lopag}^+)$). We will prove an analogous statement about $(\mathcal{G}, \mathcal{M})$ and $\mathsf{Mod}(\mathbf{Pax}^{++})$, $\mathsf{Ge}(\mathbf{Pax}^{++})$.

Definition 6.6.68 ($\leq_{\mathcal{M}o}$, $\leq_{\mathcal{G}o}$, $\leq_{\mathcal{M}}, \leq_{\mathcal{G}}$)

(i) We define $\leq_{\mathcal{M}o}$ to be the smallest transitive binary relation on $\mathsf{Mod}(\mathbf{Wax}^+)$ for which 1 and 2 below hold.

1. $\mathfrak{M} \leq_{\mathcal{M}o} (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M})$, and
2. $\mathfrak{M} \cong \mathfrak{N} \;\Rightarrow\; \mathfrak{M} \leq_{\mathcal{M}o} \mathfrak{N}$, for all $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Wax}^+)$.

(ii) We define $\leq_{\mathcal{G}o}$ to be the smallest transitive binary relation on $\mathsf{Mog}(\mathbf{lopag}^+)$ for which 1 and 2 below hold.

1. $\mathfrak{G} \leq_{\mathcal{G}o} (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G})$, and
2. $\mathfrak{G} \cong \mathfrak{H} \;\Rightarrow\; \mathfrak{G} \leq_{\mathcal{G}o} \mathfrak{H}$, for all $\mathfrak{G}, \mathfrak{H} \in \mathsf{Mog}(\mathbf{lopag}^+)$.

(iii) We define $\leq_{\mathcal{M}}$ to be the smallest transitive binary relation on $\mathsf{Mod}(\mathbf{Pax}^{++})$ for which 1 and 2 below hold.

1. $\mathfrak{M} \leq_{\mathcal{M}} (\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$, and
2. $\mathfrak{M} \cong \mathfrak{N} \;\Rightarrow\; \mathfrak{M} \leq_{\mathcal{M}} \mathfrak{N}$, for all $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Pax}^{++})$.

(iv) We define $\leq_{\mathcal{G}}$ to be the smallest transitive binary relation on $\mathsf{Ge}(\mathbf{Pax}^{++})$ for which 1 and 2 below hold.

1. $\mathfrak{G} \leq_{\mathcal{G}} (\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$, and
2. $\mathfrak{G} \cong \mathfrak{H} \;\Rightarrow\; \mathfrak{G} \leq_{\mathcal{G}} \mathfrak{H}$, for all $\mathfrak{G}, \mathfrak{H} \in \mathsf{Ge}(\mathbf{Pax}^{++})$.

$\lhd$

Next we state some simple properties of the pre-orderings $\leq_{\mathcal{M}o}$ etc.

PROPOSITION 6.6.69

(i) *Let* $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Wax}^+)$. *Then*

$$(\mathfrak{M} \leq_{\mathcal{M}o} \mathfrak{N} \;\wedge\; \mathfrak{N} \leq_{\mathcal{M}o} \mathfrak{M}) \quad\Rightarrow\quad \mathfrak{M} \cong \mathfrak{N}, \quad \textit{and}$$
$$\mathfrak{M} \leq_{\mathcal{M}o} \mathfrak{N} \quad\Rightarrow\quad \mathfrak{M} \rightarrowtail \mathfrak{N}.$$

(ii) *Let* $\mathfrak{G}, \mathfrak{H} \in \mathsf{Mog}(\mathbf{lopag}^+)$. *Then*

$$\begin{aligned}(\mathfrak{G} \leq_{\mathcal{G}o} \mathfrak{H} \ \wedge \ \mathfrak{H} \leq_{\mathcal{G}o} \mathfrak{G}) \quad &\Rightarrow \quad \mathfrak{G} \cong \mathfrak{H}, \quad \textit{and} \\ \mathfrak{G} \leq_{\mathcal{G}o} \mathfrak{H} \quad &\Rightarrow \quad \mathfrak{G} \longleftarrow\!\!\prec \mathfrak{H}.\end{aligned}$$

(iii) *Let* $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Pax}^{++})$. *Then*

$$\begin{aligned}(\mathfrak{M} \leq_{\mathcal{M}} \mathfrak{N} \ \wedge \ \mathfrak{N} \leq_{\mathcal{M}} \mathfrak{M}) \quad &\Rightarrow \quad \mathfrak{M} \cong \mathfrak{N}, \quad \textit{and} \\ \mathfrak{M} \leq_{\mathcal{M}} \mathfrak{N} \quad &\Rightarrow \quad \mathfrak{M} \succ\!\!\longrightarrow \mathfrak{N}.\end{aligned}$$

(iv) *Let* $\mathfrak{G}, \mathfrak{H} \in \mathsf{Ge}(\mathbf{Pax}^{++})$. *Then*

$$\begin{aligned}(\mathfrak{G} \leq_{\mathcal{G}} \mathfrak{H} \ \wedge \ \mathfrak{H} \leq_{\mathcal{G}} \mathfrak{G}) \quad &\Rightarrow \quad \mathfrak{G} \cong \mathfrak{H}, \quad \textit{and} \\ \mathfrak{G} \leq_{\mathcal{G}} \mathfrak{H} \quad &\Rightarrow \quad \mathfrak{G} \longleftarrow\!\!\prec \mathfrak{H}.\end{aligned}$$

We omit the **proof**. ∎

THEOREM 6.6.70

(i)

$$\begin{aligned}\mathcal{G}o \ &: \ \mathsf{Mod}(\mathbf{Wax}^+) \longrightarrow \mathsf{Mog}(\mathbf{lopag}^+) \quad \textit{and} \\ \mathcal{M}o \ &: \ \mathsf{Mog}(\mathbf{lopag}^+) \longrightarrow \mathsf{Mod}(\mathbf{Wax}^+).\end{aligned}$$

(ii) $(\mathcal{G}o, \mathcal{M}o)$ *is a Galois connection between* $\langle \mathsf{Mod}(\mathbf{Wax}^+), \ \leq_{\mathcal{M}o} \rangle$ *and* $\langle \mathsf{Mog}(\mathbf{lopag}^+), \ \leq_{\mathcal{G}o} \rangle$.

We omit the **proof**. ∎

We suggest that the reader compare Theorem 6.6.70 with the intuitive text on p.1073 below Thm.6.6.57 together with Remark 6.6.67 (p.1081).

The following corollary is of the pattern of theorem schemas (A), (B), (E)–(H) and it is a corollary of Theorem 6.6.70, Fact 6.6.65, and Prop.6.6.69.

COROLLARY 6.6.71
For any $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Wax}^+)$ and $\mathfrak{G} \in \mathsf{Mog}(\mathbf{lopag}^+)$, (i)–(iii) below hold.

(i)

$$\mathfrak{M} \succ\!\!\longrightarrow (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \quad \text{and} \quad \mathfrak{G} \longleftarrow\!\!\prec (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}).$$

(ii) The members of the range of $\mathcal{G}o$ are fixed-points of $\mathcal{M}o \circ \mathcal{G}o$ and the members of the range of $\mathcal{M}o$ are fixed-points of $\mathcal{G}o \circ \mathcal{M}o$, i.e.

$$(\mathcal{M}o \circ \mathcal{G}o)(\mathcal{G}o(\mathfrak{M})) \cong \mathcal{G}o(\mathfrak{M}) \quad \text{and} \quad (\mathcal{G}o \circ \mathcal{M}o)(\mathcal{M}o(\mathfrak{G})) \cong \mathcal{M}o(\mathfrak{G}).$$

(iii) Both $\mathcal{G}o \circ \mathcal{M}o$ and $\mathcal{M}o \circ \mathcal{G}o$ have fixed-point property in the sense

$$(\mathcal{G}o \circ \mathcal{M}o)^2(\mathfrak{M}) \cong (\mathcal{G}o \circ \mathcal{M}o)(\mathfrak{M}) \quad \text{and} \quad (\mathcal{M}o \circ \mathcal{G}o)^2(\mathfrak{G}) \cong (\mathcal{M}o \circ \mathcal{G}o)(\mathfrak{G}).$$

∎

THEOREM 6.6.72

(i) $\mathcal{M} : \mathsf{Ge}(\mathbf{Pax}^{++}) \longrightarrow \mathsf{Mod}(\mathbf{Pax}^{++})$ (*and* $\mathcal{G} : \mathsf{Mod}(\mathbf{Pax}^{++}) \longrightarrow \mathsf{Ge}(\mathbf{Pax}^{++})$).

(ii) $(\mathcal{G}, \mathcal{M})$ *is a Galois connection between* $\langle \mathsf{Mod}(\mathbf{Pax}^{++}), \leq_{\mathcal{G}} \rangle$ *and* $\langle \mathsf{Ge}(\mathbf{Pax}^{++}), \leq_{\mathcal{M}} \rangle$.

Proof: The theorem follows by Thm.6.6.17 (p.1034) and Fact 6.6.63. ∎

At this point we could formulate a corollary of Thm.6.6.72 which would be analogous with Corollary 6.6.71 of Thm.6.6.70. This corollary of Thm.6.6.72 basically coincides with our Thm.6.6.17 formulated on p.1034.

6.6.6 Adjoint functors, categories

Motivation for adjoint functors for the physicist reader is found in Remark 6.6.61 (p.1078). Cf. also p.1096. For adjoint situations in physics cf. e.g. Lawvere-Schanuel [163, pp. 5–6, pp. 76–77]; but see also the references in footnote 1077, p.1079.[1079]

[1079] Category theory has been becoming increasingly popular and often used in physics recently, cf. e.g. Baez-Dolan [36], Crane [63], Freed [87], Andai [6], Kassel [153], Baez [37]. Cf. also Lawvere-Schanuel [163].

The subject matter of this sub-section is strongly connected to Remark 6.6.4 (p.1014) entitled "Galois theories, Galois connections, duality theories all over mathematics ...."

In this sub-section we will see that $(\mathcal{M},\mathcal{G})$ and $(\mathcal{M}o,\mathcal{G}o)$ are "adjoint pairs of functors" in the category theoretic sense, under certain conditions.

We use the notion of a *category* in the usual category theoretic sense, cf. e.g. Mac Lane [168]. Assume $\mathbb{C}$ is a category. Then $\mathsf{Ob}\,\mathbb{C}$ and $\mathsf{Mor}\,\mathbb{C}$ denote the classes of *objects* and *morphisms* of $\mathbb{C}$, respectively. $f : A \longrightarrow B$ means that f is a morphism with *domain* $A \in \mathsf{Ob}\,\mathbb{C}$ and *codomain* $B \in \mathsf{Ob}\,\mathbb{C}$. For any $A, B \in \mathsf{Ob}\,\mathbb{C}$,

$$\hom(A,B) :\stackrel{\text{def}}{=} \{\, f \in \mathsf{Mor}\,\mathbb{C} \;:\; (f : A \longrightarrow B) \,\}\,.$$

Further, *composition* $\circ$ is a partial binary operation on $\mathsf{Mor}\,\mathbb{C}$, and if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ then $f \circ g : A \longrightarrow C$. We use the notion of a *functor* in the usual sense, i.e. a functor is a map from a category to a category which takes objects to objects, morphisms to morphisms, preserves domains and codomains, identities[1080] and composition $\circ$. If $\mathbb{C}$ and $\mathbb{D}$ are categories and $\mathcal{D}$ is a functor from $\mathbb{C}$ to $\mathbb{D}$, then we will write $\mathcal{D} : \mathbb{C} \longrightarrow \mathbb{D}$.

Definition 6.6.73 (strong embedding)
Terminology from model theory: Let $f : \mathfrak{A} \rightarrowtail \mathfrak{B}$ be an embedding of model $\mathfrak{A}$ into model $\mathfrak{B}$. By the *f-image* $f[\mathfrak{A}]$ of $\mathfrak{A}$ we understand the unique (weak) submodel of $\mathfrak{B}$ such that f is an isomorphism between $\mathfrak{A}$ and $f[\mathfrak{A}]$.

Now, $f : \mathfrak{A} \rightarrowtail \mathfrak{B}$ is called a *strong embedding* iff it is an embedding and the f-image $f[\mathfrak{A}]$ of $\mathfrak{A}$ is a strong submodel of $\mathfrak{B}$.

$\lhd$

Definition 6.6.74 (categories $\mathbb{M}\mathsf{od}(Th)$, $\mathbb{G}\mathsf{e}(Th)$, $\mathbb{M}\mathsf{og}(TH)$)
Let Th be a set of formulas in our frame language.

(i) $\mathsf{Mod}(Th)$ forms a category $\mathbb{M}\mathsf{od}(Th)$ the following way. The class of objects of $\mathbb{M}\mathsf{od}(Th)$ is $\mathsf{Mod}(Th)$ and the morphisms are those embeddings

$$f : \mathfrak{M}_0 \rightarrowtail \mathfrak{M}_1$$

[1080] A morphism $f : A \longrightarrow A$ is called an *identity* if for every morphism g with domain A, $f \circ g = g$ and for every morphism g' with codomain A, $g' \circ f = g'$.

which are surjective on the sets of photons (i.e. $f[Ph_0] = Ph_1$), unless $\mathfrak{M}_0$ is the empty model.[1081] More precisely, the morphisms of the category $\mathbb{M}\mathrm{od}(Th)$ are triples of the form $\langle \mathfrak{M}, f, \mathfrak{N} \rangle$, where $f : \mathfrak{M} \rightarrowtail \mathfrak{N}$ is such that $f[Ph^{\mathfrak{M}}] = Ph^{\mathfrak{N}}$ or $\mathfrak{M}$ is the empty model. The reason why we need triples instead of f in itself is that when looking at a morphism we have to be able to tell what its domain and codomain are. For simplicity, if there is no danger of confusion we will use f as a morphism instead of the triple $\langle \mathfrak{M}, f, \mathfrak{N} \rangle$. We hope context will help. The composition $\circ$ is the usual one.[1082]

(ii) $\mathsf{Ge}(Th)$ forms a category $\mathbb{G}\mathrm{e}(Th)$ in the following way. The class of objects of $\mathbb{G}\mathrm{e}(Th)$ is $\mathsf{Ge}(Th)$ and the morphisms are those embeddings

$$h : \mathfrak{G}_0 \rightarrowtail \mathfrak{G}_1$$

which are (i) strong embeddings on the $\langle Mn;\, Bw \rangle$ reducts and are (ii) surjective on the sets of photon-like lines (i.e. $h[L_0^{Ph}] = L_1^{Ph}$), unless $\mathfrak{G}_0$ is the empty model. (The composition $\circ$ is the usual one.)

(iii) For any set TH of formulas in the language of GEO, $\mathsf{Mog}(TH)$ forms a category $\mathbb{M}\mathrm{og}(TH)$ in a completely analogous way with item (ii), i.e. the class of objects of $\mathbb{M}\mathrm{og}(TH)$ is $\mathsf{Mog}(TH)$ and the morphisms are those embeddings

$$h : \mathfrak{G}_0 \rightarrowtail \mathfrak{G}_1$$

which are (i) strong embeddings on the $\langle Mn;\, Bw \rangle$ reducts and are (ii) surjective on the sets of photon-like lines (i.e. $h[L_0^{Ph}] = L_1^{Ph}$), unless $\mathfrak{G}_0$ is the empty model.

◁

Definition 6.6.75 $\quad$ $\mathbf{Pax}^+_+ \stackrel{\text{def}}{=} \mathbf{Pax}^+ + \mathbf{Ax(diswind)}$.

◁

[1081] Surjectiveness on the sets of photons is required only because eventually we want $\mathcal{M}$ to be a functor between $\mathbb{G}\mathrm{e}(Th)$ and $\mathbb{M}\mathrm{od}(Th)$. It is not quite obvious to see why this purpose (functoriality of $\mathcal{M}$) makes us to need the surjectiveness condition. Hint: this is connected to condition (e) on p.1055. If we omitted item (e) on p.1055 from the definition of $\mathcal{M}$, then we could define morphisms of $\mathbb{M}\mathrm{od}(Th)$ to be the embeddings. The reader is invited to elaborate an alternative version to our $(\mathcal{M}, \mathcal{G})$-duality by omitting condition (e) from the definition of $\mathcal{M}$ and then dropping the present surjectiveness condition w.r.t. Ph.

[1082] I.e. $\langle \mathfrak{M}, f, \mathfrak{N} \rangle \circ \langle \mathfrak{M}_1, g, \mathfrak{N}_1 \rangle = \langle \mathfrak{M}, f \circ g, \mathfrak{N}_1 \rangle$ if $\mathfrak{N} = \mathfrak{M}_1$ and is *undefined* otherwise.

The functions $\mathcal{M}$, $\mathcal{G}$, $\mathcal{M}o$, $\mathcal{G}o$ are defined on the objects of the categories $\mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+)$, $\mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+)$, $\mathbb{M}\mathrm{og}(\mathbf{lopag})$, $\mathbb{M}\mathrm{od}(\mathbf{Wax})$, respectively. In the following definition we extend these functions to the morphisms. In this way we obtain functors

$$\begin{array}{ll} \boldsymbol{\mathcal{M}} : \mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+), & \boldsymbol{\mathcal{G}} : \mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+) \longrightarrow \mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+), \\ \boldsymbol{\mathcal{M}o} : \mathbb{M}\mathrm{og}(\mathbf{lopag}) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax}), & \boldsymbol{\mathcal{G}o} : \mathbb{M}\mathrm{od}(\mathbf{Wax}) \longrightarrow \mathbb{M}\mathrm{og}(\mathbf{lopag}). \end{array}$$

Definition 6.6.76 (the functors $\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{M}o}, \boldsymbol{\mathcal{G}o}$)
To define a functor, one has to define what it does with the objects and what it does with the morphisms (of the category in question). On the objects $\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{M}o}, \boldsymbol{\mathcal{G}o}$ agree with $\mathcal{M}, \mathcal{G}, \mathcal{M}o, \mathcal{G}o$, respectively. It remains to define our functors on the morphisms.

$\boldsymbol{\mathcal{M}}$. For every morphism $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ of $\mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+)$ we will define the morphism $\boldsymbol{\mathcal{M}}(h) : \boldsymbol{\mathcal{M}}(\mathfrak{G}_0) \longrightarrow \boldsymbol{\mathcal{M}}(\mathfrak{G}_1)$ of $\mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+)$, see the left-hand side of Figure 329. Since the definition looks somewhat "longish" we note that in it we will do the "natural thing" (following the structure of the definition of $\mathcal{M}$). Let $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ be a morphisms of $\mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+)$, i.e. $\mathfrak{G}_0 = \langle Mn_0, \ldots \rangle$, $\mathfrak{G}_1 = \langle Mn_1, \ldots \rangle \in \mathsf{Ge}(\mathbf{Pax}_+^+)$ and h is an embedding satisfying the conditions in the definition of the category $\mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+)$, i.e. in Def.6.6.74(ii). Then h is a tuple $\langle h_M, h_F, h_L \rangle$ with $h_M : Mn_0 \rightarrowtail Mn_1$, $h_F : F_0 \rightarrowtail F_1$ and $h_L : L_0 \rightarrowtail L_1$. Further, $\boldsymbol{\mathcal{M}}(\mathfrak{G}_0) = \langle B_0, \ldots \rangle$, $\boldsymbol{\mathcal{M}}(\mathfrak{G}_1) = \langle B_1, \ldots \rangle \in \mathsf{Mod}(\mathbf{Pax}_+^+)$ by Prop.6.6.44 (p.1059). Then $\boldsymbol{\mathcal{M}}(h) := \langle \boldsymbol{\mathcal{M}}(h)_B, \boldsymbol{\mathcal{M}}(h)_F, \boldsymbol{\mathcal{M}}(h)_G \rangle$, where $\boldsymbol{\mathcal{M}}(h)_B : B_0 \longrightarrow B_1$, $\boldsymbol{\mathcal{M}}(h)_F : F_0 \longrightarrow F_1$ and $\boldsymbol{\mathcal{M}}(h)_G : G_0 \longrightarrow G_1$ are defined as follows. To define $\boldsymbol{\mathcal{M}}(h)_B$ let $b \in B_0$. Then either $b = \langle o, e_0, \ldots, e_{n-1} \rangle \in Obs_0 \subseteq {}^{n+1}Mn_0$, for some $o, \ldots, e_{n-1}$ or $b \in Ph_0 = L_0^{Ph}$. Now,

$$\boldsymbol{\mathcal{M}}(h)_B(b) :\stackrel{\text{def}}{=} \begin{cases} \langle h_M(o), h_M(e_0), \ldots, h_M(e_{n-1}) \rangle & \text{if } b = \langle o, e_0, \ldots, e_{n-1} \rangle \in Obs_0 \\ h_L(b) & \text{if } b \in Ph_0. \end{cases}$$

$\boldsymbol{\mathcal{M}}(h)_B$ takes observers to observers and photons to photons. $\boldsymbol{\mathcal{M}}(h)_F$ is defined to be h_F and $\boldsymbol{\mathcal{M}}(h)_G$ is naturally induced by $\boldsymbol{\mathcal{M}}(h)_F$, i.e. $\boldsymbol{\mathcal{M}}(h)_G : \mathsf{Eucl}(\mathfrak{F}_0) \longrightarrow \mathsf{Eucl}(\mathfrak{F}_1)$ is defined by $\ell \mapsto \widetilde{\boldsymbol{\mathcal{M}}(h)_F}[\ell]$.

We will prove as Claim 6.6.77(i) that $\boldsymbol{\mathcal{M}}(h) : \boldsymbol{\mathcal{M}}(\mathfrak{G}_0) \longrightarrow \boldsymbol{\mathcal{M}}(\mathfrak{G}_1)$ is indeed a morphism of $\mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+)$, moreover that $\boldsymbol{\mathcal{M}} : \mathbb{G}\mathrm{e}(\mathbf{Pax}_+^+) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+)$ is a functor.

$\boldsymbol{\mathcal{G}}$. For every morphism $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ of $\mathbb{M}\mathrm{od}(\mathbf{Pax}_+^+)$ we will define the morphism $\boldsymbol{\mathcal{G}}(f) : \boldsymbol{\mathcal{G}}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}}(\mathfrak{M}_1)$, see the right-hand side of Figure 329. Let

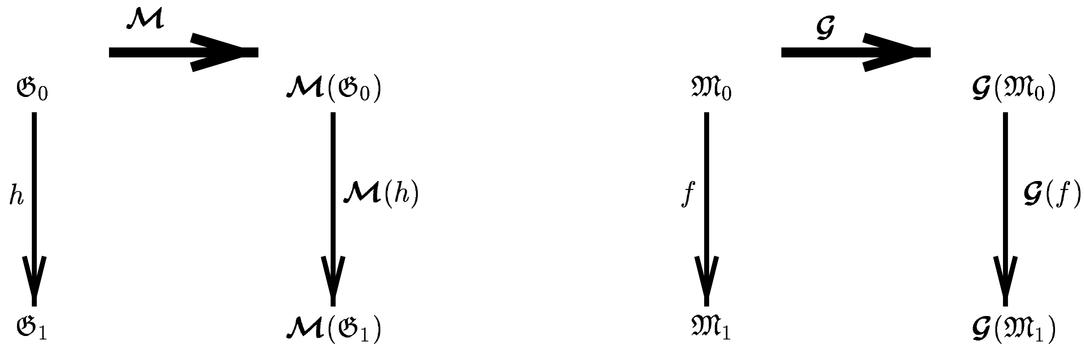


Figure 329:

$f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ be a morphism of $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$, i.e. $\mathfrak{M}_0 = \langle B_0, \ldots\rangle$, $\mathfrak{M}_1 = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Pax}^+_+)$ and f is an embedding satisfying the conditions in the definition of the category $\mathbb{M}\mathsf{od}(\mathbf{Pax}^+)$, i.e. in Def.6.6.74(i). Then f is a tuple $\langle f_B, f_F, f_G\rangle$ with $f_B : B_0 \rightarrowtail B_1$, $f_F : F_0 \rightarrowtail F_1$ and $f_G : G_0 \rightarrowtail G_1$. Further, $\mathcal{G}(\mathfrak{M}_0) = \langle Mn_0, \ldots\rangle$, $\mathcal{G}(\mathfrak{M}_1) = \langle Mn_1, \ldots\rangle \in \mathsf{Ge}(\mathbf{Pax}^+_+)$. Then $\mathcal{G}(f) := \langle \mathcal{G}(f)_M, \mathcal{G}(f)_F, \mathcal{G}(f)_L\rangle$, where $\mathcal{G}(f)_M \subseteq Mn_0 \times Mn_1$, $\mathcal{G}(f)_F : F_0 \longrightarrow F_1$ and $\mathcal{G}(f)_L \subseteq L_0 \times L_1$ are defined as follows. Let $\langle e_0, e_1\rangle \in Mn_0 \times Mn_1$ and $\langle \ell_0, \ell_1\rangle \in L_0 \times L_1$. Then

$$\langle e_0, e_1\rangle \in \mathcal{G}(f)_M$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs_0)(\exists p \in {}^nF_0)\Big(w_m(p) = e_0 \ \wedge \ w_{f_B(m)}(\widetilde{f_F}(p)) = e_1\Big).$$

Further

$$\langle \ell_0, \ell_1\rangle \in \mathcal{G}(f)_L$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs_0)\Big((\exists i \in n)\,(\,\ell_0 = w_m[\bar{x}_i] \ \wedge \ \ell_1 = w_{f_B(m)}[\bar{x}_i]\,)^{1083} \vee$$
$$(\exists ph \in Ph)\,(\,\ell_0 = w_m[tr_m(ph)] \ \wedge \ \ell_1 = w_{f_B(m)}[tr_{f_B(m)}(f_B(ph))]\,)\Big).$$

$\mathcal{G}(f)_F$ is defined to be f_F.

[1083]The first $\bar{x}_i$ is the i-th coordinate axis in ${}^nF_0$ while the second $\bar{x}_i$ is the i-th coordinate axis in ${}^nF_1$.

We will prove as Claim 6.6.77(ii) that $\mathcal{G}(f) : \mathcal{G}(\mathfrak{M}_0) \longrightarrow \mathcal{G}(\mathfrak{M}_1)$ is indeed a morphism of $\mathbb{G}\mathrm{e}(\mathbf{Pax}^+_+)$, moreover that $\mathcal{G} : \mathbb{M}\mathrm{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathrm{e}(\mathbf{Pax}^+_+)$ is a functor.

$\mathcal{M}\boldsymbol{o}$. For every morphism $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$ we will define the morphism $\mathcal{M}\boldsymbol{o}(h) : \mathcal{M}\boldsymbol{o}(\mathfrak{G}_0) \longrightarrow \mathcal{M}\boldsymbol{o}(\mathfrak{G}_1)$ of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$. Let $h : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_1$ be a morphism of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$, i.e. $\mathfrak{G}_0 = \langle Mn_0, \ldots\rangle$, $\mathfrak{G}_1 = \langle Mn_1, \ldots\rangle \in \mathsf{Mog}(\mathbf{lopag})$ and h is an embedding satisfying the conditions in the definition of the category $\mathbb{M}\mathrm{og}(\mathbf{lopag})$, i.e. in Def.6.6.74(iii). Then h is a pair $\langle h_M, h_L\rangle$ with $h_M : Mn_0 \rightarrowtail Mn_1$ and $h_L : L_0 \rightarrowtail L_1$. Further, $\mathcal{M}\boldsymbol{o}(\mathfrak{G}_0) = \langle B_0, \ldots\rangle$, $\mathcal{M}\boldsymbol{o}(\mathfrak{G}_1) = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Wax})$ by Thm.6.6.57. Then $\mathcal{M}\boldsymbol{o}(h) := \langle \mathcal{M}\boldsymbol{o}(h)_B, \mathcal{M}\boldsymbol{o}(h)_F, \mathcal{M}\boldsymbol{o}(h)_G\rangle$ where $\mathcal{M}\boldsymbol{o}(h)_B : B_0 \longrightarrow B_1$, $\mathcal{M}\boldsymbol{o}(h)_F \subseteq F_0 \times F_1$ and $\mathcal{M}\boldsymbol{o}(h)_G \subseteq G_0 \times G_1$ are defined as follows. $\mathcal{M}\boldsymbol{o}(h)_B$ is defined analogously to the case of $\mathcal{M}$, i.e. as follows. Let $b \in B_0$. Then either $b = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs_0 \subseteq {}^{n+1}Mn_0$, for some $o, \ldots, e_{n-1}$ or $b \in Ph_0 = L_0^{Ph}$. Now,

$$\mathcal{M}(h)_B(b) :\stackrel{\text{def}}{=} \begin{cases} \langle h_M(o), h_M(e_0), \ldots, h_M(e_{n-1})\rangle & \text{if } b = \langle o, e_0, \ldots, e_{n-1}\rangle \in Obs_0 \\ h_L(b) & \text{if } b \in Ph_0. \end{cases}$$

To define $\mathcal{M}\boldsymbol{o}(h)_F$ let $\langle p, q\rangle \in F_0 \times F_1$. In the definition below, we will use $F_0, F_1, \mathfrak{F}_0, \mathfrak{F}_1$ which were introduced in Definitions 6.6.34 (p.1049) and 6.6.55 (p.1072). Then,

$$\langle p, q\rangle \in \mathcal{M}\boldsymbol{o}(h)_F$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(\exists p' \in p)(\exists q' \in q)\Big(pj_i(q') = h_M(pj_i(p')), \quad \text{for all } i \in 3\Big).$$

$\mathcal{M}\boldsymbol{o}(h)_G$ is induced by $\mathcal{M}\boldsymbol{o}(h)_F$ the natural way, cf. the definition of $\mathcal{M}(h)_G$ in item $\mathcal{M}$. above.

We will prove as Claim 6.6.77(iii) that $\mathcal{M}\boldsymbol{o}(h) : \mathcal{M}\boldsymbol{o}(\mathfrak{G}_0) \longrightarrow \mathcal{M}\boldsymbol{o}(\mathfrak{G}_1)$ is indeed a morphism of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$, moreover that $\mathcal{M}\boldsymbol{o} : \mathbb{M}\mathrm{og}(\mathbf{lopag}) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax})$ is a functor.

$\mathcal{G}\boldsymbol{o}$. For every morphism $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$ we will define the morphism $\mathcal{G}\boldsymbol{o}(f) : \mathcal{G}\boldsymbol{o}(\mathfrak{M}_0) \longrightarrow \mathcal{G}\boldsymbol{o}(\mathfrak{M}_1)$ of $\mathbb{M}\mathrm{og}(\mathbf{lopag})$. Let $f : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_1$ be a morphism of $\mathbb{M}\mathrm{od}(\mathbf{Wax})$, i.e. $\mathfrak{M}_0 = \langle B_0, \ldots\rangle$, $\mathfrak{M}_1 = \langle B_1, \ldots\rangle \in \mathsf{Mod}(\mathbf{Wax})$ and f is an embedding satisfying the conditions in the definition of the category

$\mathbb{M}\mathsf{od}(\mathbf{Wax})$, i.e. in Def.6.6.74(i). Further, $\boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) = \langle Mn_0, \ldots\rangle$, $\boldsymbol{\mathcal{G}o}(\mathfrak{M}_1) = \langle Mn_1, \ldots\rangle \in \mathsf{Mog}(\mathbf{lopag})$ by Thm.6.6.57. We define the morphism

$$\boldsymbol{\mathcal{G}o}(f) : \boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}o}(\mathfrak{M}_1)$$

of $\mathbb{M}\mathsf{og}(\mathbf{lopag})$ to be $\langle \boldsymbol{\mathcal{G}}(f)_M, \boldsymbol{\mathcal{G}}(f)_L\rangle$, where $\boldsymbol{\mathcal{G}}(f)_M$ and $\boldsymbol{\mathcal{G}}(f)_L$ are defined as in item $\boldsymbol{\mathcal{G}}$. above.

We will prove as Claim 6.6.77(iv) that $\boldsymbol{\mathcal{G}o}(f) : \boldsymbol{\mathcal{G}o}(\mathfrak{M}_0) \longrightarrow \boldsymbol{\mathcal{G}o}(\mathfrak{M}_1)$ is indeed a morphisms, moreover that $\boldsymbol{\mathcal{G}o} : \mathbb{M}\mathsf{od}(\mathbf{Wax}) \longrightarrow \mathbb{M}\mathsf{og}(\mathbf{lopag})$ is a functor.

◁

Claim 6.6.77 below serve to prove correctness of Def.6.6.76 above.

Claim 6.6.77

(i) $\boldsymbol{\mathcal{M}} : \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+)$ is a functor.

(ii) $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$ is a functor.

(iii) $\boldsymbol{\mathcal{M}o} : \mathbb{M}\mathsf{og}(\mathbf{lopag}) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Wax})$ is a functor.

(iv) $\boldsymbol{\mathcal{G}o} : \mathbb{M}\mathsf{od}(\mathbf{Wax}) \longrightarrow \mathbb{M}\mathsf{og}(\mathbf{lopag})$ is a functor.

The **proof** is available from Judit Madarász. ∎

Next, we recall the notion of adjoint pair of functors from category theory e.g. from Mac Lane [168]. For this, first we introduce the notion of a reflection and coreflection in Def.6.6.78 below. We will use the notion of a subcategory in the usual way, cf. e.g. Mac Lane [168].

Definition 6.6.78 (reflection, coreflection) Let $\mathbb{C}$ and $\mathbb{D}$ be two categories.

(i) Assume $\mathbb{D}$ is a subcategory of $\mathbb{C}$. Let $A \in \mathsf{Ob}\,\mathbb{C}$.

(a) $B \in \mathsf{Ob}\,\mathbb{D}$ is called the *reflection* of A in $\mathbb{D}$ iff B is the "nearest" object to A in $\mathbb{D}$, i.e. iff there is a morphism $f : A \longrightarrow B$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(A, B'))(\exists! g \in \hom(B, B'))\, f \circ g = f',$$

see the left top picture in Figure 330.

(b) $B \in \mathsf{Ob}\,\mathbb{D}$ is called a *coreflection* of A in $\mathbb{D}$ iff there is a morphism $f : B \longrightarrow A$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(B', A))(\exists! g \in \hom(B', B))\, g \circ f = f',$$

see the right top figure in Figure 330.

(ii) Assume $\mathcal{C} : \mathbb{D} \longrightarrow \mathbb{C}$ is a functor. Let $A \in \mathsf{Ob}\,\mathbb{C}$.

(a) $B \in \mathsf{Ob}\,\mathbb{D}$ is called a *reflection* of A in $\mathbb{D}$ iff B is the nearest object to A in $\mathbb{D}$, i.e. there is a morphism $f : A \longrightarrow \mathcal{C}(B)$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(A, \mathcal{C}(B')))(\exists! g \in \hom(B, B'))\, f \circ \mathcal{C}(g) = f',$$

see the left bottom picture in Figure 330.
The morphism $f : A \longrightarrow \mathcal{C}(B)$ above is called the *$\mathcal{C}$-reflection arrow*[1084] *of the object A*.

(b) $B \in \mathsf{Ob}\,\mathbb{D}$ is called a *coreflection* of A in $\mathbb{D}$ iff there is a morphism $f : \mathcal{C}(B) \longrightarrow A$ which is the shortest one in the following sense:

$$(\forall B' \in \mathsf{Ob}\,\mathbb{D})(\forall f' \in \hom(\mathcal{C}(B'), A))(\exists! g \in \hom(B', B))\, \mathcal{C}(g) \circ f = f',$$

see the right bottom picture in Figure 330.
The morphism $f : \mathcal{C}(B) \longrightarrow A$ above is called the *$\mathcal{C}$-coreflection arrow of the object A*. ◁

Definition 6.6.79 (adjoint situation)[1085]
Let $\mathbb{C}$ and $\mathbb{D}$ be two categories and let

$$(\star) \qquad \mathcal{C} : \mathbb{D} \longrightarrow \mathbb{C} \quad \text{and} \quad \mathcal{D} : \mathbb{C} \longrightarrow \mathbb{D}$$

be two functors. Then $(\mathcal{C}, \mathcal{D})$ is called an *adjoint pair* iff for every $A \in \mathsf{Ob}\,\mathbb{C}$, $\mathcal{D}(A)$ is the reflection of A in $\mathbb{D}$ and for every $B \in \mathsf{Ob}\,\mathbb{D}$, $\mathcal{C}(B)$ is the coreflection of B in $\mathbb{C}$, cf. Figure 331.

Further, we say that $(\star)$ above is an *adjoint situation* iff $(\mathcal{C}, \mathcal{D})$ is an adjoint pair of functors.

◁

[1084] We could call this f intuitively $\mathbb{D}$-reflection arrow.

[1085] We refer to e.g. Mac Lane [168] for the "official" definition of adjointness. Cf. also Adámek [1, p. 138–148, (sub-section 3F)], and Adámek-Herrlich-Strecker [2, pp. 283-300] where a large number of mathematical applications/examples of adjointness and what we call here duality theories is given.

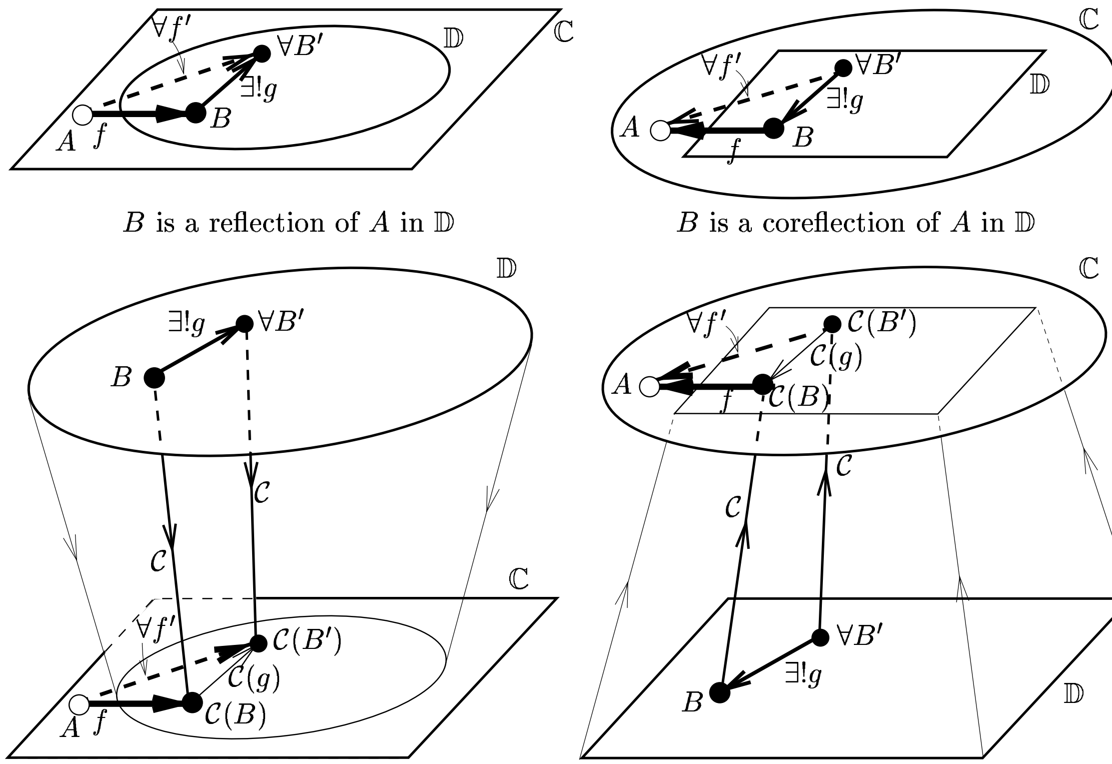


Figure 330: Reflection and coreflection.

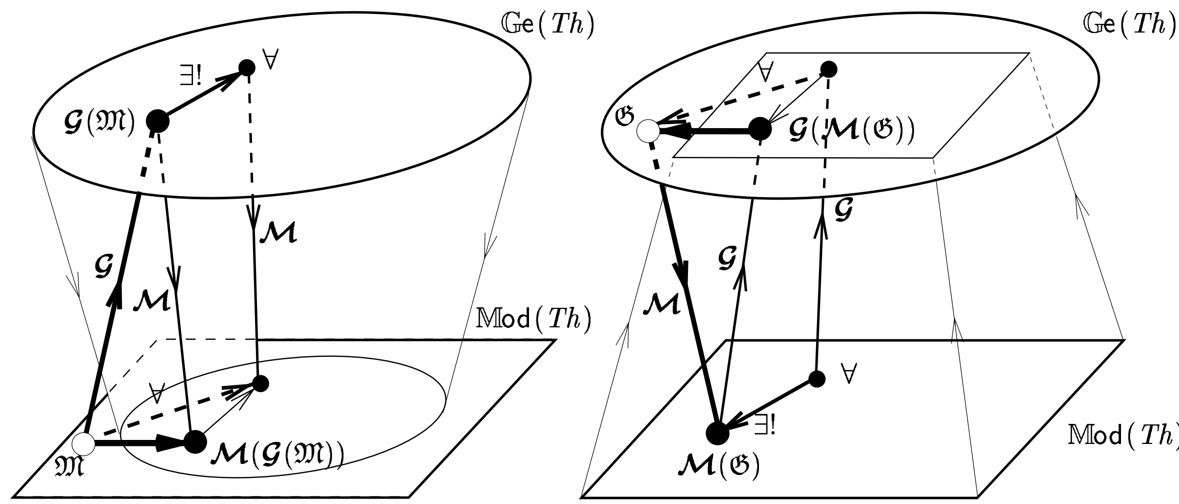


Figure 331: $(\mathcal{M}, \mathcal{G})$ is an adjoint pair of functors, under certain conditions.

Definition 6.6.80 $\quad \mathbf{Pax}^{++}_{+} \overset{\text{def}}{=} \mathbf{Pax}^{++} + \mathbf{Ax(diswind)}$.

◁

For the following conjectures recall that $\mathcal{M}$, $\mathcal{G}$, $\mathcal{M}o$, $\mathcal{G}o$ are functors by Claim 6.6.77 (p.1090).

Conjecture 6.6.81 *We strongly conjecture that (i) and (ii) below hold.*

(i) $\mathcal{M} : \mathbb{G}\mathrm{e}(\mathbf{Pax}^{++}_{+}) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Pax}^{++}_{+})$ *and* $\mathcal{G} : \mathbb{M}\mathrm{od}(\mathbf{Pax}^{++}_{+}) \longrightarrow \mathbb{G}\mathrm{e}(\mathbf{Pax}^{++}_{+})$ *is an adjoint situation,*[1086] *cf. Figure 331.*

(ii) $\mathcal{M}o : \mathbb{M}\mathrm{og}(\mathbf{lopag}^{+}) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax}^{+})$ *and*
$\mathcal{G}o : \mathbb{M}\mathrm{od}(\mathbf{Wax}^{+}) \longrightarrow \mathbb{M}\mathrm{og}(\mathbf{lopag}^{+})$ *is an adjoint situation.*

◁

Let $f : A \longrightarrow B$ be a morphism of the category $\mathbb{C}$. We call f an *isomorphism* (of $\mathbb{C}$) if

$$(\exists g \in \hom(B, A))(f \circ g \text{ and } g \circ f \text{ are identity morphisms}),$$

cf. footnote 1080 on p.1085 for identity morphisms.

[1086] In accordance with our Convention 6.6.2 (p.1008) here we are talking about the restrictions of $\mathcal{M}$ and $\mathcal{G}$ to $\mathbb{G}\mathrm{e}(\mathbf{Pax}^{++}_{+})$ and $\mathbb{M}\mathrm{od}(\mathbf{Pax}^{++}_{+})$. We will use this convention throughout the present section.

Definition 6.6.82 (equivalence of categories)[1087]
The categories $\mathbb{C}$ and $\mathbb{D}$ are called _equivalent_ iff there is an adjoint pair of functors

$$\mathcal{C} : \mathbb{D} \longrightarrow \mathbb{C} \quad \text{and} \quad \mathcal{D} : \mathbb{C} \longrightarrow \mathbb{D}$$

such that the following holds. For every object A of $\mathbb{C}$ the $\mathcal{C}$-reflection arrow is an isomorphism and the same holds for the $\mathcal{D}$-coreflection arrows of objects $B \in \mathsf{Ob}\,\mathbb{D}$. In such situations the pair $(\mathcal{C}, \mathcal{D})$ of functors is called an _equivalence of categories_ ($\mathbb{C}$ and $\mathbb{D}$).[1088]

$\lhd$

Conjecture 6.6.83 *We strongly conjecture that* $\mathbb{M}\mathsf{od}(Th)$ *and* $\mathbb{G}\mathsf{e}(Th)$ *are equivalent categories, and* $(\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}})$ *is an equivalence between these two categories, assuming* $n > 2$ *and* $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\|)^{-} + \mathbf{Ax(eqtime)} + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$.

$\lhd$

In connection with the above conjecture cf. Thm.6.6.13 (p.1031) saying that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent, assuming the assumptions of the above conjecture. Thm.6.6.13 already implies isomorphism, hence equivalence, between categories $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ if we choose elementary embeddings as morphisms, cf. p.1005.

Before stating our next conjecture we note the following. Consider the functor $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathsf{od}(\mathbf{Pax}^+_+) \longrightarrow \mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$. Then $\boldsymbol{\mathcal{G}}$ is surjective in the sense that $Rng(\boldsymbol{\mathcal{G}})$ is $\mathbb{G}\mathsf{e}(\mathbf{Pax}^+_+)$ up to isomorphism. This holds for any Th with $Th \models \mathbf{Pax}^+_+$ in place of $\mathbf{Pax}^+_+$.

Conjecture 6.6.84 *We strongly conjecture that (i) and (ii) below hold.*

- **(i)** *Consider the functors* $\boldsymbol{\mathcal{M}} : \mathbb{G}\mathsf{e}(\mathbf{Pax}^{++}_+) \longrightarrow \mathbb{M}\mathsf{od}(\mathbf{Pax}^{++}_+)$ *and* $\boldsymbol{\mathcal{G}} : \mathbb{M}\mathsf{od}(\mathbf{Pax}^{++}_+) \longrightarrow \mathbb{G}\mathsf{e}(\mathbf{Pax}^{++}_+)$. *Then* $Rng(\boldsymbol{\mathcal{M}})$ *is a category and* $Rng(\boldsymbol{\mathcal{M}})$ *and* $\mathbb{G}\mathsf{e}(\mathbf{Pax}^{++}_+)$ *are equivalent categories, and* $(\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{G}} \restriction Rng(\boldsymbol{\mathcal{M}}))$ *is an equivalence between these two categories.*

[1087] We refer to e.g. Mac Lane [168] or Adámek et al [2, p.26, Def.3.33] for the "official" definition of equivalence of categories. Officially a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is an equivalence iff it is a bijection on every $\hom(A, B)$, i.e. $F : \hom_{\mathbb{C}}(A, B) \rightarrowtail\!\!\!\twoheadrightarrow \hom_{\mathbb{D}}(F(A), F(B))$, and it is surjective with respect to isomorphisms.

[1088] An adjoint situation $(\mathcal{C}, \mathcal{D})$ could be called a _Galois-adjoint situation_ iff $Rng(\mathcal{C})$ and $Rng(\mathcal{D})$ are categories and $(\mathcal{C}, \mathcal{D})$ is an equivalence between categories $Rng(\mathcal{C})$ and $Rng(\mathcal{D})$. The so obtained notion could be considered as a special kind of adjoint situations and at the same time as a generalization of Galois connections.

(ii) *Consider the functors* $\boldsymbol{\mathcal{M}o} : \mathbb{M}\mathrm{og}(\mathbf{lopag}^+) \longrightarrow \mathbb{M}\mathrm{od}(\mathbf{Wax}^+)$ *and* $\boldsymbol{\mathcal{G}o} : \mathbb{M}\mathrm{od}(\mathbf{Wax}^+) \longrightarrow \mathbb{M}\mathrm{og}(\mathbf{lopag}^+)$. *Then* $Rng(\boldsymbol{\mathcal{M}o})$ *and* $Rng(\boldsymbol{\mathcal{G}o})$ *are equivalent categories and* $(\boldsymbol{\mathcal{M}o} \restriction Rng(\boldsymbol{\mathcal{G}o}), \boldsymbol{\mathcal{G}o} \restriction Rng(\boldsymbol{\mathcal{M}o}))$ *is an equivalence between these two categories.*

$\lhd$

Items (i) of Conjectures 6.6.81 and 6.6.84 together say that $(\mathcal{M}, \mathcal{G})$ is a Galois-adjoint situation in the sense of footnote 1088, assuming $\mathbf{Pax}_+^{++}$; while items (ii) of the same conjectures say that $(\mathcal{M}o, \mathcal{G}o)$ is a Galois-adjoint situation, assuming $\mathbf{Wax}^+$ and $\mathbf{lopag}^+$. Cf. also the intuitive text on p.1073 above Conjecture 6.6.58 together with Remark 6.6.67 and compare them with Conjectures 6.6.81, 6.6.84.

The (syntax, semantics)-duality described in Remark 6.6.4 item (III) (pp. 1020–1026) is actually an adjoint functor pair between two categories. The functors are "syntax" and "semantics" (or equivalently Th and Mod). This motivates the following

Exercise 6.6.85 Many of the duality theories introduced or outlined before introducing categories, i.e. before §6.6.6, extend to adjoint pairs of functors between two categories.

(i) An important example is the

$$\{\langle Fm(Th), Th\rangle \ :\ Th \text{ is a set of formulas}\} \underset{\mathsf{Th}}{\overset{\mathsf{Mod}}{\rightleftarrows}} \{\mathsf{K} \ :\ \mathsf{K} \text{ is a class of similar models}\}$$

duality.[1089] The first step is to turn the left-hand side and the right-hand side into categories by defining the morphisms between the indicated objects. Then one defines what the functors Mod and Th do with these morphisms. I.e. if Tr is a morphism between theories, then we have to define what $\mathsf{Mod}(Tr)$ is.

(ii) There are further examples between pp. 1003–1084. We invite the reader to select a few of these, then turn the left-hand side into a category, then same with right-hand side, and then turn the connections into functors.

$\lhd$

Notation 6.6.86 Let A be a set and let $\tau(x)$ be a term with input variable x, defined for $x \in A$. Recall that then $f := \langle \tau(x) \ :\ x \in A\rangle$ denotes a function $f : A \longrightarrow Rng(f)$, cf. p.27 where we used *expr* in place of τ.

[1089]Note that "*Th* is a set of formulas" is equivalent with saying that *Th* is a theory (by definition).

We will use the intuitive notation $\tau(-)$ for denoting this function f. I.e.

$$\tau(-) :\stackrel{\text{def}}{=} \langle \tau(x) \ : \ x \in A \rangle.$$

This notation is somewhat under-specified since A, i.e. the domain of $\tau(-)$, is not explicitly indicated. This intuitive notation $\tau(-)$ comes from category theory. Cf. also the notational convention $g(-, y, z)$ above Def.4.3.35 (partial derivative) on p.518 (in §4.3). That convention is the same as the present one (with some extra parameters added).

◁

Motivation for studying equivalences of categories, adjoint situations, etc:
If two categories $\mathbb{C}$ and $\mathbb{D}$ are equivalent then one can utilize this the following way. Assume we have a problem in the world of $\mathbb{C}$ (and assume that it is easier to think about such problems in $\mathbb{D}$). Then we may transform our problem from $\mathbb{C}$ to $\mathbb{D}$, solve the problem in $\mathbb{D}$ and then transform the result back into $\mathbb{C}$. Indeed this often happens e.g. in Stone duality between Boolean algebras and certain topological spaces.[1090] Similarly this kind of application often happens in algebraic logic (cf. [30] and §6.6.7). Of course the problems in question have to satisfy some conditions, e.g. they have to be isomorphism invariant. (In the case of adjoint situations not every problem can be translated to "back and forth", because the functors satisfy fewer conditions. However there are adjoint situations which have a fixed-point property like our theorem-schemas (G), (H) (p.1011). Then there is a category theoretic equivalence between the subcategories of "closed objects" (or equivalently fixed-points) and using these one can transform problems back and forth [such are e.g. the spectacular applications of Galois theories[1091]].) Cf. the introduction to §6.6 (p.1003) and Fig.309 (p.1003) in connection with the above ideas.

(†) The usefulness of adjoint situations in theoretical physics is emphasized e.g. in Baez [37] (e.g. on p.3 therein). But cf. also footnote 1079 on p.1084 in this connection. Cf. also p.3 lines 8–10 in Baez [37] for "the relation between category theory and quantum theory ... so important in topological quantum field theory".

Connections between adjoint situations, Galois connections, and other duality theories:
Before getting started, we note that Remark 6.6.61 (p.1078) is also about our present subject.

[1090] Cf. the intuitive text on p.1019 above item (III).
[1091] both for fields and for cylindric algebras

Assume that in our category $\mathbb{C}$ there is at most one morphism between any two objects, i.e. assume $|\hom(A,B)| \leq 1$ is valid in $\mathbb{C}$. Then $\mathbb{C}$ becomes a pre-ordering. (Hint: We use $A \leq B$ to denote $\hom(A,B) \neq \emptyset$.) Assume the same for category $\mathbb{D}$. Then functors $\mathcal{C} : \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathcal{D} : \mathbb{D} \longrightarrow \mathbb{C}$ become order preserving mappings between pre-orderings $\mathbb{C}$ and $\mathbb{D}$. Then it is a natural question to ask which pairs (f,g) of order preserving mappings between pre-orderings P, Q are actually adjoint situations. Translating the definition of adjoint situations way above (from the language of categories to that of pre-orderings) gives us a natural answer to this question. Assume for simplicity that our pre-orderings are actually partial orderings (posets for short). Then (f,g) forms an adjoint pair iff $(\star)$ below holds.

$$(\star) \qquad \begin{array}{l} f(p) = \inf\{\, q \in Q \ :\ p \leq g(q)\,\} \\ g(q) = \sup\{\, p \in P \ :\ q \geq f(p)\,\}. \end{array}$$

Actually, we note that $(\star)$ works for characterizing adjointness even in the more general case of pre-orderings, too. More precisely, if we want $(\star)$ to work for pre-orderings too, then it is enough to replace "$f(p) = \inf\{\ldots\}$" with "$f(p)$ is a smallest element[1092] of the set $\{\, q \in Q \ :\ p \leq g(q)\,\}$" and similarly for "$g(q) = \sup\{\ldots\}$".

Summing up, $(\star)$ is the order-theoretic counterpart of adjointness. The paper Andréka et al. [15] discusses and investigates equivalent versions and applications of (order-theoretic) adjointness of the form $(\star)$ above. In that paper $(\star)$ shows up in the fourth line beginning with "If (f,g) is such a pair, then $f(p) = \ldots$". (This is the second, equivalent definition they give for order-theoretic adjointness.) They call an (order-theoretic) adjoint pair (f,g) satisfying $(\star)$ a *residuated-residual pair*. Among others they show that residuated-residual pairs are equivalent with Galois connections. They discuss the connections with Galois theory, too. Residuatedness plays an extremely important role in many branches of algebra, in sophisticated duality theories, and in Algebraic Logic. One of the slogans in a large part of Algebraic Logic says that all extra Boolean operators in Algebraic Logic are residuated.[1093] Cf. e.g. Jónsson-Tarski [148], Jónsson [147], Jónsson-Tsinakis [149], Thompson [257, p.340] and Jipsen-Jónsson-Rafter [144] and the references in the latter. Actually, Birkhoff in his famous Lattice Theory book [47] introduces relation algebras as "residuated Boolean lattices" (where we note that relation algebras are one of the main themes

[1092] In pre-orderings, x is a smallest element of H iff $x \in H$ and $(\forall y \in H)\, x \leq y$.

[1093] An operator f on a Boolean algebra, or more generally a function f : pre-order $\longrightarrow$ pre-order is called *residuated* iff it is part of a residuated-residual pair (f,g). Then g is called the *residual* of f. (We could call g the "right residual" of f and f the "left-residual" of g, but we do not do this e.g. because it would cause confusion with the slashes to be discussed soon [the slashes are called left and right residuals of $\circ$].)
It is sometimes useful to think of the residual g of f as a kind of *quasi-inverse* (w.r.t. the pre-ordering $\leq$) of f. Hence f is residuated iff it is quasi-invertible w.r.t. the pre-order in question.

in the literature of Tarskian Algebraic Logic). In passing we note that the *residual* g of f is very strongly related what is called the *conjugate* of f in a large part of abstract algebra, cf. e.g. Jónsson [147, pp. 129-130], Thompson [257, p.340] and Henkin-Monk-Tarski [129, Part I, p.175]. If our posets are Boolean algebras then for any mapping g its dual[1094] g^∂ is also defined. Now, if (f, g) are residuated then g^∂ is exactly the conjugate of f. I.e. the conjugate of f is the dual g^∂ of the residual g of f. Therefore, conjugates of mappings are extremely close to residuals of mappings, e.g. in the case of Boolean algebras the two concepts are term-definable from each other.[1095] (More generally, the mathematical idea of a "conjugate" in general is strongly related to the idea of a residual pair, i.e. of an adjoint situation.) In the literature of Algebraic Logic and in that of Sub-structural Logics (e.g. Lambek calculus, linear logic etc.) the residuals of any fixed "central" binary operation, say $\circ$, are denoted by the slashes $/, \backslash$ while the conjugates of the same central operation are denoted by the triangles $\triangleleft, \triangleright$, cf. Andréka-Mikulás [26], Jipsen-Jónsson-Rafter [144], Marx-Venema [189], van Benthem [266, pp. 194, 195, 230, 231], [268, p.246] and Bahls-Cole-Galatos-Jipsen-Tsinakis [38][1096].

additivitás nem kell lábj.-ben?

The paper Andréka et al. [15] discusses further important applications and variants of adjointness of the form $(\star)$ above. About this subject cf. also our next sub-section on Algebraic Logic. The present subject (importance of Galois connections etc.) is continued in a broader perspective in the item below (... importance "omnipresence" ... duality theories).

On the importance, "omnipresence" and literature of duality theories:

(1) We did not have space to pay due credit to the importance of duality theories which range from the subjects we already mentioned to geometry, analysis, algebraic geometry, computability and other fields. What we described in our coordinatization sub-section §6.6.2 (p.1037) is easily developed to several duality theories (an obvious one acts between synthetic geometries and fields). This kind of duality is usually called *coordinatization* and the idea goes back to von Staudt [273] 1857 (where it was elaborated for projective spaces). Building on top of such coordinatizations further, useful kinds of duality theories were elaborated. Many of these act between

[1094] $g^\partial(x) := -g(-x)$

[1095] Though the residual of f is its quasi-inverse, the conjugate of f is not a quasi-inverse (but the dual of a quasi-inverse). E.g. if c is a complemented $(c(-c(x)) = -c(x))$ closure operator on a Boolean algebra (cf. Fig.313) then c is its own conjugate, cf. Henkin-Monk-Tarski [129, Part I, p.175], while the residual c^∂ of c is the interior operator $c^\partial(x) = -c(-x)$ naturally corresponding to c.

[1096] In this paper, though the residuals "/", "\" are defined, the lattice we are working in is not required even to be distributive.

a class of lattices on the left-hand side and a class like e.g. vector spaces[1097] on the other. The general pattern is:

$$\text{a class of lattices} \quad \underset{\mathcal{L}a}{\overset{\mathcal{V}e}{\rightleftarrows}} \quad \text{a class like that of vector spaces.}$$

In many of these examples the functor $\mathcal{L}a$ associates something like the subalgebra lattice $\langle \mathrm{Sub}(\mathbf{V}), \cap, \vee \rangle$ to the vector-space (-like structure) $\mathbf{V}$ coming from the right-hand side.

An example is the following. Let CMA-lattices denote *complemented modular algebraic lattices* . Then, there is a duality theory between

(CMA-lattices satisfying some extra conditions)
and
(Vector spaces over division rings),[1098]

cf. Grätzer [111, Thm.15, p.208].[1099] Cf. e.g. Grätzer [111] for more information on the above. This research direction proved to be rather useful and fruitful, e.g. von Neumann[1100] and his followers obtained strong and useful duality theories of this spirit, in addition to Grätzer [111], Czédli [65] and von Neumann [272], cf. also e.g. Andréka-Givant-Németi [13, p.17], Varadarajan [270], Freese [88], Urquhart [260], Kurucz [156] and the references in the last two works. These dualities are based on some version of coordinatization in the sense outlined way above. Lipshitz [165], and Urquhart [260] generalized von Neumann's duality (and/or coordinatization) to different kinds of lattices, and Urquhart [260] in §3 (entitled "duality theories") elaborated further useful kinds of duality theories e.g. in Theorems 3.1–3.3 therein. These are in turn related to the algebraic logic dualities e.g. in Hansoul [123], Jónsson-Tarski [148], Madarász [171], Jónsson-Gehrke [99], Goldblatt [109], the Algebraic Logic special issue Németi-Sain [207] to mention only a few, but cf. also subtitle "Connections ... duality theories" p.1096 herein. Urquhart's above mentioned duality theory was further generalized in the algebraic logic works Andréka-Givant-Németi [13, pp. 16–20] and Kurucz [156, pp. 22–27].

Von Neumann also developed a coordinatization with rings for *orthocomplemented modular lattices.* This can be developed into a duality theory[1101] (analogous

[1097] or one of their generalizations or variants (e.g. R-modules for certain kinds of R)
[1098] The division ring part can be eliminated (i.e. replaced by fields) by adding an extra condition on the lattice side.
[1099] A special case of this duality theory was announced by O. Veblen and W. H. Young in 1910, a full proof without gaps was published by von Neumann in 1936, and a generalization was given by Frink in 1946.
[1100] Cf. e.g. Grätzer [111], von Neumann [272] and Czédli [65, Theorems VI.36–37, p.138].
[1101] roughly, between Hilbert spaces and orthocomplemented modular lattices

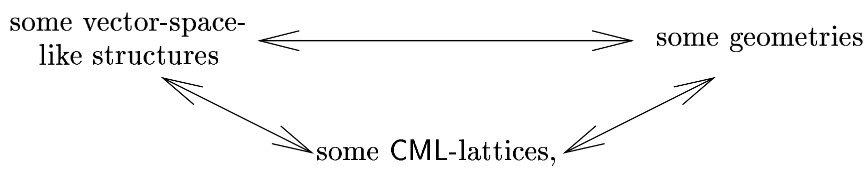
to the above ones); generalizations and improvements of this duality have been a central theme in the literature of mathematical physics especially in connection with quantum mechanics, quantum field theory[1102], cf. e.g. Foulis [86], Varadarajan [270].

Using the idea of coordinatization mentioned way above, one can obtain the following (with geometries in place of vector spaces). There is a duality theory

$$\mathsf{CMA}\text{-lattices} \quad \underset{\mathcal{L}a'}{\overset{\mathcal{G}e}{\rightleftarrows}} \quad \text{projective geometries}$$

where the functor $\mathcal{L}a'$ associates the <u>subspace lattice</u> $\mathcal{L}a'(\mathbb{G}) := \langle \mathrm{Sub}(\mathbb{G}), \cap, \vee \rangle$ to any geometry $\mathbb{G}$ coming from the right-hand side. Cf. Jónsson [146, Thm.5.5] and e.g. Czédli [65, Thm's VII.4, VII.18, VII.23 (p.152, p.166)].

Putting the above dualities together, with some extra work (and under extra conditions) we can have 3-way dualities



cf. also Shafarevich [238, item 10.VII]. For further developments we refer to e.g. Varadarajan [270], Andréka-Givant-Németi [13], Czédli [65], McKenzie et al. [192, Thm.4.88 on p.216, p.89, (Exercise 2 on p.216)], footnote 1077 on p.1079, item (4) on p.1104 herein.

A further duality theory between orthocomplemented, weakly modular lattices and Baer *-semigroups, relevant to physics, is in Foulis [86].[1103]

(2) *Duality between certain topological spaces and commutative C^*-algebras:*

A topological space $\mathbf{X} = \langle X, \mathcal{O} \rangle$ is <u>*locally compact*</u> if each point $p \in X$ has a neighborhood U such that U is compact, i.e.

$$(\exists U \subseteq X)(\exists A \in \mathcal{O})(p \in A \subseteq U \ \text{ and } \ U \text{ is compact}^{1104} \text{ in } \mathbf{X}).$$

Let $\mathbb{L}\mathsf{Top}$ denote the category of locally compact topological spaces. Roughly, C^*-algebras are vector spaces $\mathbf{V}$ over the field $\mathbf{C}$ of complex numbers such that $\mathbf{V}$ is

[1102] a unification of special relativity theory and quantum mechanics

[1103] This can be regarded as a generalization of von Neumann's coordinatization for orthocomplemented modular lattices. We note that practically all of the so called coordinatization theorems (including the one in §6.6.2 herein) can be regarded as <u>duality</u> theories. We note this to emphasize the unifying power of a "theory of duality theories".

[1104] i.e. $(\mathbf{X} \restriction U)$ is compact; in more detail $(\exists \mathcal{H} \subseteq \mathcal{O})[\bigcup \mathcal{H} \supseteq U \ \Rightarrow \ (\exists \text{ finite } \mathcal{H}_0 \subseteq \mathcal{H}) \bigcup \mathcal{H}_0 \supseteq U]$. Cf. also footnote 1008 on p.1018.

enriched with a binary operation "·",[1105] an antilinear involution * (unary) and a norm $\|-\| : V \longrightarrow \mathfrak{R}$.[1106] Then if $\langle \mathbf{V}, \cdot, ^*, \|-\|, \mathfrak{R} \rangle$ satisfies certain extra axioms then it is called a C^*-algebra. We do not recall the notion of a C^*-algebra in more detail but we note that they play an important role in physics (cf. e.g. Rédei [218, p.62, Chapter 6 (von Neumann Lattices)], [219]). Cf. also Reed-Simon [220, Vol.IV], Pelletier-Rosický [211], and Bratteli-Robinson [51].

Now, there is a duality

$$\mathbb{L}\mathsf{Top} \quad \underset{G}{\overset{F}{\rightleftarrows}} \quad \text{"commutative } C^*\text{-algebras"}$$

satisfying certain useful properties. E.g. F, G are functors (w.r.t. the natural morphisms), closed sets (of topologies) correspond to closed ideals (of $\langle \mathbf{V}, \ldots, \mathfrak{R} \rangle$), open sets (of $\mathbf{X}$) correspond to quotient algebras (of $\langle \mathbf{V}, \ldots, \mathfrak{R} \rangle$), etc.

(3) *Example of an important duality theory of which it is not obvious how to reformulate it as an adjointness:*[1107]

According to our philosophy, Laplace transformation is a duality theory.[1108] This duality theory is used in analysis and in particular in solving linear differential equations.

Roughly, the two "worlds" being connected is the world of (certain) *real functions* and the world of (certain) *complex functions*.

$^+$R denotes the positive half-line of the set of real numbers R. Similarly let $^+\mathbf{C}$ denote the positive half-plane of the complex plane[1109] with $\mathbf{C}$ the field of complex numbers. Then, roughly, our duality is of the form

$$^{+\mathrm{R}}\mathrm{R} \quad \underset{\mathcal{I}}{\overset{\circ \, \mathcal{L}}{\underset{\circ}{\rightleftarrows}}} \quad ^{+\mathbf{C}}\mathbf{C}$$

where $\mathcal{L}$ and $\mathcal{I}$ are partial functions such that $(\mathcal{L} \circ \mathcal{I})^2(f) = (\mathcal{L} \circ \mathcal{I})(f)$ whenever $(\mathcal{L} \circ \mathcal{I})(f)$ exists. $\mathcal{L}$ is called *Laplace transform* and $\mathcal{I}$ is the inverse transform.[1110]

[1105] Actually, $\langle \mathbf{V}, \cdot \rangle$ is an algebra over the field $\mathbf{C}$, in the sense of classical algebra, cf. e.g. Shafarevich [238, Chap.8, Example 3].

[1106] For the notation $\|-\|$ cf. item 6.6.86 on p.1095.

[1107] It may be an adjointness, it is not trivial to our minds how to bring it to an adjoint form in a short time. (We did not have time to try seriously.)

[1108] Cf. second motivation for duality theories at the beginning of §6.6 p.1004, p.1019 (above item III), p.1096 ("Motivation for ... equivalence of categories ..."), p.777 item (ii) in §6.1.

[1109] $^+\mathbf{C} = \{ a + ib \, : \, a \in {}^+\mathrm{R}, b \in \mathrm{R} \}$ where $i = \sqrt{-1}$.

[1110] We hope it will cause no confusion that in item (II) of Remark 6.6.4 (p.1015) $\mathcal{L}$ denoted the functor going from topological spaces to lattices.

It simplifies discussion if on the left-hand side we take the broader "world" ${}^{+\mathrm{R}}\mathbf{C}$. Then we have two (infinite dimensional) vector spaces over the field $\mathbf{C}$ and

$$\begin{aligned} \mathcal{L} &: {}^{+\mathrm{R}}\mathbf{C} \xrightarrow{\circ} {}^{+\mathbf{C}}\mathbf{C}, \\ \mathcal{I} &: {}^{+\mathbf{C}}\mathbf{C} \xrightarrow{\circ} {}^{+\mathrm{R}}\mathbf{C} \end{aligned}$$

are two partial vector space homomorphisms[1111] between them.[1112] Actually, $Dom(\mathcal{L}) \subseteq {}^{+\mathrm{R}}\mathbf{C}$ is a sub-vector-space of ${}^{+\mathrm{R}}\mathbf{C}$, hence

$${}^{+\mathrm{R}}\mathbf{C} \supseteq Dom(\mathcal{L}) \xrightarrow{\mathcal{L}} {}^{+\mathbf{C}}\mathbf{C}$$

is a _vector space homomorphism_. Moreover, $Dom(\mathcal{I}) \subseteq {}^{+\mathbf{C}}\mathbf{C}$ is also a sub-vector-space of ${}^{+\mathbf{C}}\mathbf{C}$, hence we have two vector space homomorphisms

$${}^{+\mathrm{R}}\mathbf{C} \supseteq Dom(\mathcal{L}) \quad \underset{\mathcal{I}}{\overset{\mathcal{L}}{\rightleftarrows}} \quad Dom(\mathcal{I}) \subseteq {}^{+\mathbf{C}}\mathbf{C}$$

between two vector spaces $Dom(\mathcal{L})$ and $Dom(\mathcal{I})$ satisfying our theorem schema (G) for duality theories (on p.1011) saying that $(\forall f \in Dom(\mathcal{L} \circ \mathcal{I}))(\mathcal{L} \circ \mathcal{I})(f)$ is a fixed point of $(\mathcal{L} \circ \mathcal{I})$. A similar statement can be made about the other side and $(\mathcal{I} \circ \mathcal{L})$, cf. theorem schema (H), p.1012. Further, $(\mathcal{L} \circ \mathcal{I})(f)$ differs from f only on a set of measure zero. We hope that what we have said so far indicates that what we are discussing here fits into the pattern of what we called duality theories at the beginning of §6.6, pp. 1003–1012 (ending with theorem schemas (A–H) for duality theories). Therefore we will also refer to the Laplace transform as the $(\mathcal{L}, \mathcal{I})$-duality. We do not describe $Dom(\mathcal{L})$ in detail but we note that all functions $f : {}^{+}\mathrm{R} \longrightarrow \mathrm{R}$ which are piecewise continuous _and_ do not grow too fast[1113] are in $Dom(\mathcal{L})$.

Why is the $(\mathcal{L}, \mathcal{I})$-duality useful? The answer is that certain problems formulated about elements $f, f_1, \ldots \in Dom(\mathcal{L})$ of the left-hand side world become much simpler when ($\mathcal{L}$-translated, i.e.) reformulated about their images $\mathcal{L}(f), \mathcal{L}(f_1), \ldots$ in the world ${}^{+\mathbf{C}}\mathbf{C}$.

To illustrate this, we note that if f' denotes the derivative of f, then $\mathcal{L}(f')$ can be obtained from $\mathcal{L}(f)$ by taking the simple function mapping $p \in {}^{+}\mathbf{C}$ to

[1111] The traditional expression is _linear operator_ for homomorphisms between infinite dimensional vector spaces.

[1112] Recall from p.42, that the vector space ${}^{+\mathrm{R}}\mathbf{C} := \langle {}^{+\mathrm{R}}\mathrm{C}; +, (\lambda \cdot -)\rangle_{\lambda \in \mathrm{C}}$ is a group $\langle {}^{+\mathrm{R}}\mathrm{C}; +\rangle$ enriched with the unary operations $(\lambda \cdot -)$ called scalar products, for each $\lambda \in \mathrm{C}$. Similarly for the other vector space ${}^{+\mathbf{C}}\mathbf{C}$. The operations preserved by $\mathcal{L}$ and $\mathcal{I}$ are the just indicated vector space operations.

[1113] i.e. $(\exists \lambda, \eta \in {}^{+}\mathrm{R})(\forall x \in {}^{+}\mathrm{R})\, |f(x)| \le \lambda \cdot e^{\eta \cdot x}$.

$p \cdot \mathcal{L}(f)(p) - f(0)$, i.e. $\mathcal{L}(f') = \langle p \cdot (\mathcal{L}(f)(p)) - f(0) \,:\, p \in {}^{+}\mathbf{C}\rangle$. Remark: Since $\mathcal{L}(f)$ is an element of a vector space over $\mathbf{C}$ and $p, f(0) \in \mathbf{C}$, we can write

$$\mathcal{L}(f') = \mathrm{Id} \cdot \mathcal{L}(f) - \langle f(0) : p \in {}^{+}\mathbf{C}\rangle$$

(where note that $\mathrm{Id} = \mathrm{Id} \restriction {}^{+}\mathbf{C}$ and the constant function $\langle f(0) \,:\, p \in {}^{+}\mathbf{C}\rangle$ are elements of our vector space ${}^{+\mathbf{C}}\mathbf{C}$).[1114]

As a further nice property of $\mathcal{L}$ we mention that it takes convolution to products (note that ${}^{+\mathbf{C}}\mathbf{C}$ is not only a vector space but also an algebra[1115]), where: The convolution of $f, g \in Dom(\mathcal{L})$ is defined as

$$(f \star g)(x) \stackrel{\text{def}}{=} \int_0^\infty f(t) \cdot g(x-t) dt$$

and what we said is

$$\mathcal{L}(f \star g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$$

where "$\cdot$" is the usual product in the algebra ${}^{+\mathbf{C}}\mathbf{C}$.[1116]

We do not recall more detail, but we hope that what we sketched above makes it imaginable that the $(\mathcal{L}, \mathcal{I})$-duality can be helpful in solving e.g. some linear differential equations. Actually, this duality is widely used e.g. in electrical engineering, and in various branches of physics.

A strongly related, but different, duality theory is called *Fourier transformation*. For some applications of the latter cf. e.g. Shafarevich [238, §5, Example 8]. For the definition and discussion of Fourier transformation cf. Kirillov [154, §2.8 "Duality and Fourier transformations"]; cf. also e.g. Pour-El & Richards [215, p.109] and/or Reed & Simon [220, Vol.II]. The definition of Laplace transform[1117] can be found

[1114]Recall that in the intuitive introduction of duality theories (cf. p.1003) we had a world on the left-hand side of the "bridge" and one on the right. The above observation about $\mathcal{L}(f')$ points in the direction that, in our present duality theory, the world on the left-hand side is $\langle {}^{+\mathrm{R}}\mathrm{R}$ with differentiation etc. as "structure" $\rangle$ while the world on the right-hand side is $\langle {}^{+\mathbf{C}}\mathbf{C}$ with algebra as "structure"$\rangle$. I.e. this duality translates (a part of) analysis to algebra (and vice versa).

[1115]In more detail ${}^{+\mathbf{C}}\mathbf{C}$ is a ring moreover the ring product "$\cdot$" is suitable for being the algebra product "$\cdot$", i.e. $(f \cdot g)(x) = f(x) \cdot g(x)$.

[1116]Cf. footnote 1115.

[1117]In passing we note that, roughly speaking, the definition of the Laplace transform $\mathcal{L}(f)$ is a relatively simple (perhaps improper) integral: $\mathcal{L}(f)(p) := \int_0^\infty e^{-p \cdot t} \cdot f(t)\, dt$, for any $p \in \mathbf{C}$. Here e is the usual constant (i.e. the base of natural logarithms).

e.g. in Concise Lexicon of Mathematics [84].[1118] A categorified version of the Fourier transformation can be found in Baez [37, §6.1, pp. 52-54)].[1119]

(4) Further examples, applications, explanations and motivation for duality theories i.e. adjointness can be found in the following references. Most of the expository works on categories emphasize that adjoint situations (hence duality theories) are extremely important for (almost) the whole of mathematics and that besides this they turn out to be a successful vehicle for unifying and deepening mathematical thought.[1120] Cf. Lawvere [160, 162, 161], Arbib-Manes [33, 32], Manes [184], Guitart [117], Mac Lane [168], Goldblatt [107], Handbook of Categorical Algebra [50], Barr-Wells [40, §1.9, p. 50–63], Freyd-Scedrov [89], Adámek et al. [2], [3], Varadarajan [270], Lawvere-Schanuel [163], Nel [202], Pelletier-Rosický [211], Dimov-Tholen [74], Janelidze [142], Davey-Priestley [68], Marx [187, Fig.1.2 (p.12) and §2.2 (... "duality theory")] and Mikulás [195, §1.3 "Bridge between..." (p.18)]. Several examples for application of duality theories and similar algebraic ideas in physics can be found in Shafarevich [238] cf. e.g. Example 2 in §21 or Example 8 in §5, or the parts on groups of symmetry and laws of nature, or on elementary particles and group representations in §18 item E, or the Galois theory of linear differential equations in §18 item B.

The study of duality theories is an active, fruitful and steadily growing branch of mathematics and mathematical physics nowadays. To illustrate this we mention only (i-v) below. (i) The duality between not-necessarily normal Boolean algebras with operators (non-normal BAO's for short) on the one side and partial Kripke-frames on the other was discovered only recently[1121], cf. Madarász [171]. This duality extends to a duality for non-normal modal logics with modalities of higher ranks. (Cf. e.g. Marx-Venema [189] or Blackburn et at. [48] for the latter.) (ii) The results in the very recent Hirsch-Hodkinson book [135] contains new developments on dualities under the name "representation theorems". (iii) The recent duality paper Goldblatt [109]. (iv) Makkai's duality for ultra-categories and first-order-logic theories [179]. (v) As a

[1118] We note the following connection between the Fourier transform and the Laplace transform. For $f \in {}^{+\mathrm{R}}\mathrm{R} \cup {}^{\mathrm{R}}\mathrm{R}$ we let $f^0 = (f \restriction {}^{+}\mathrm{R}) \cup \langle 0 : 0 > x \in \mathrm{R}\rangle$; i.e.

$$f^0(x) = \begin{cases} 0 & \text{if } x < 0 \\ f(x) & \text{else.} \end{cases}$$

Assume the Fourier transform $F(f^0)$ exists. Then $\mathcal{L}(f)$ exists and $\mathcal{L}(f)(x) = F(f^0)(i \cdot x)$, for $x \in {}^{+}\mathrm{R}$ where $i = \sqrt{-1}$.

[1119] The notions of "categorification" and "categorified version" are introduced in works of Baez and Baez-Dolan in connection with applying category theory in physics.

[1120] Typical examples for this are e.g. Lawvere [160], Mac Lane [168] (but almost all the remaining references say this with differences only in emphasis).

[1121] but already receives applications e.g. in connection with partial correctness of programs further illustration that duality theories are dynamically evolving with applications in physics, we include here a small sample of further references: Stinespring [241], Sankaran [234], Joyal-Street [150], Schauenburg [235], Gootman-Lazar [110].

As we indicated in Remark 6.6.61 item (II), footnote 1077 on p.1079 the application areas range from geometry, analysis, algebra, through to sheaves, computability, logic and other things.

6.6.7 Algebraic Logic as a duality theory, in analogy with the ones in the present work

There is a methodological connection here with algebraic logic (for the latter cf. e.g. Andréka-Németi-Sain [30]), as follows.

In algebraic logic, a logical system $\mathcal{L}$ is a tuple $\mathcal{L} = \langle Fm, \ldots, \vdash \rangle$ which, in some sense, is close to a certain intuitive conception of logic. Then a function Alg is defined which to each logic $\mathcal{L}$ associates a class $\mathsf{Alg}(\mathcal{L})$ of algebras. The idea is that $\mathsf{Alg}(\mathcal{L})$ is a mathematically more streamlined object than $\mathcal{L}$, while $\mathcal{L}$ is closer to a certain intuition. Therefore it is worthwhile to develop a so called duality theory "Logical systems"$\rightleftarrows$"Classes of algebras" which enables us to "translate" problems and results in both directions cf. Andréka-Németi-Sain [30].

For discussing the case of our present theory, let $\mathcal{G}$ and $\mathcal{M}$ be the functions as defined above. Then our frame models $\mathfrak{M}$ are in analogy with logical systems $\mathcal{L}$, $\mathfrak{M} \stackrel{\mathcal{G}}{\longmapsto} \mathfrak{G}_{\mathfrak{M}}$ is in analogy with the function $\mathcal{L} \mapsto \mathsf{Alg}(\mathcal{L})$ and $\mathcal{M}$ is in analogy with the construction of a logical system from a class of algebras (which we did not recall from Algebraic Logic). Indeed, as in the case of algebraic logic, $\mathfrak{M}$ is also close to a certain intuitive picture of bodies, motion, observation etc, while $\mathfrak{G}_{\mathfrak{M}}$ is a mathematically more streamlined object. (Just as our geometries $\mathfrak{G}_{\mathfrak{M}}$ ($\mathfrak{M} \in \mathsf{Mod}(Th)$) form a category the natural way, the same applies to the $\mathsf{Alg}(\mathcal{L})$'s [for $\mathcal{L} \in Logics$]. I.e. the $\mathsf{Alg}(\mathcal{L})$'s form a category.) In this connection cf. the observational/theoretical distinction in the introduction to the present chapter, e.g. p.774.

To pursue the analogy, for many logics, $\mathsf{Alg}(\mathcal{L})$ is a class of cylindric algebras (e.g. this is the case for classical first-order logic). It is customary to investigate "reducts" of $\mathsf{Alg}(\mathcal{L})$ e.g. a certain reduct of $\mathsf{Alg}(\mathcal{L})$ is a class of Boolean algebras, while another is a class of distributive lattices. The experience is that investigating these reducts helps us in understanding the behavior of $\mathsf{Alg}(\mathcal{L})$ and even the original object $\mathcal{L}$ itself. In analogous manner, in relativity theory it seems to be interesting to investigate reducts of $\mathfrak{G}_{\mathfrak{M}}$ one $\mathfrak{G}^1_{\mathfrak{M}} = \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, \mathcal{T}\rangle$ of which is obtained by omitting g and eq while another one $\mathfrak{G}^2_{\mathfrak{M}} = \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r\rangle$ is obtained by omitting (or forgetting) g, $\mathcal{T}$ and eq.

A point to make here is the observation that *none* of the two worlds (that $\mathcal{L}$ and that of $\mathsf{Alg}(\mathcal{L})$) is better than the other. The useful and illuminating thing is that we can move between the two (without making one superior to the other). Similar observation applies here to $\mathfrak{M}$ and $\mathfrak{G}_{\mathfrak{M}}$, the important thing is that we can reconstruct one from the other (i.e. move between them) without thinking that one is superior and the other should be forgotten forever.

Applications of duality theories to definability theory (as used in the present work) are e.g. in Madarász [173], [170], [169], Madarász-Sayed [178], Hoogland [138].

Remark 6.6.87 (On representation theorems, Field's book "Science without numbers" [85]:) Duality theories usually involve special kinds of results called *representation theorems.* E.g. Stone duality theory[1122] (between Boolean algebras and certain topological spaces), implies that every "abstract" Boolean algebra is *representable* as a concrete Boolean algebra of sets (with real intersection etc. as its operations).[1123]

A more complete version of our duality theory between relativistic models $\mathsf{Mod}(Th)$ and geometries $\mathsf{Ge}(Th)$ will also involve such kinds of representation theorems, among other things.

We are pointing this out here e.g. because Field's book [85] suggests using *representation theorems* for studying the logical (and philosophical) connections between so called "purely" physical theories on the one hand *and* mathematical theories on the other hand. (To be more precise instead of "mathematical theories" we should have said something like "mathematized physical theories".) In this connection we note that statements like Facts 6.6.21, 6.6.25, 6.6.28 (pp. 1041–1044) stating that certain "synthetic geometries" are representable as "analytic geometries", i.e. $\langle Points, Lines;\ \in, \dots\rangle$ type geometries satisfying certain axioms are representable over some field (or division ring) are also called representation theorems, cf. Field [85] and Tarski's school of logical approach to geometry, cf. [237].

[1122] Cf. pp. 1015, 1019 for Stone duality theory.

[1123] Recall that on p.787 we distinguished concrete classes of structures like Boolean set algebras and abstract classes of structures like the axiomatic class of Boolean algebras. Representation theorems can often be interpreted as saying that an abstract, axiomatically defined class can be "represented" by a certain concrete class, i.e. "Abstract class"$=\mathbf{I}$"Concrete class". Cf. e.g. Németi [206, Remark 2 (finitization)] for more on these ideas (abstract class, concrete class, representation theorems).

The connections between our *duality theories*, *representation theorems*, *adjoint functors* and the subject of the logical connections between physical and mathematical theories will be further discussed in a later work related to the present one. But we emphasize already here the following: Duality theories, adjoint situations, representation theorems are different words for the same thing. One uses different words for putting the emphasis on different aspects of the (same) subject.[1124] Galois theories are special versions of the above things[1125],where groups of symmetries connected with hierarchy of levels of ontology are emphasized.[1126] Galois connections represent a more abstract unifying view of all the above (and are very strongly related to the subjects listed on pp. 1096–1105, e.g "flexible isomorphisms", "quasi-inverse", cf. also Remark 6.6.4).

$\lhd$

6.6.8 On potential laws of nature, characterizing our symmetry axioms

Let us turn to separating out the law-like formulas from $Fm(\mathfrak{M})$, i.e. to distinguish the potential laws (of nature) from the "potential factual statements" in the language $Fm(\mathfrak{M})$ of our observational models $\mathfrak{M}$. (We discussed this goal in the introduction to Chapter 6, i.e. in §6.1 pp. 777–778).[1127] We will do all this relative to some (arbitrary but fixed) theory $Th \subseteq Fm(\mathfrak{M})$. We suggest that before reading this sub-section the reader re-reads pp. 777–778 (beginning with the title "Potential laws ...") in the introduction of the present chapter.

Throughout this sub-section we assume $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$. This is actually true by Theorems 6.3.22, 6.3.23 (p.961), under some conditions on Th. We could work with $\mathsf{Ge}'(Th)$ or $\mathsf{Ge}''(Th)$ and then we would need much weaker conditions

[1124]Baez [37], too, treats duality theories, representation theorems and adjointness as belonging together. He also writes about these concepts being important for physics.

[1125]duality theories etc.

[1126]It might be of interest to compare the mathematically oriented Galois theories mentioned herein (e.g. that of fields and of cylindric algebras) with the physics oriented considerations on groups of symmetries and levels of (physical) ontology.

[1127]The distinction between potential laws and potential contingent (i.e. accidental or factual) statements is not an absolute one. Anyway, here we are outlining only the first steps in the development of a model-theoretic or logical theory of the law-like/fact-like distinction. Cf. lawlike generalization on p.423 of the philosophical dictionary [34].

on Th (practically nothing) cf. e.g. Thm.6.3.24 (p.962). We leave generalizations in this direction to the interested reader.

Definition 6.6.88 $\mathsf{Mod}(Th)^+$ denotes the definitional expansion

$$\mathsf{Mod}(Th)^+ := \{ \langle \mathfrak{M},\ \mathfrak{G}_{\mathfrak{M}},\ \text{auxiliaries} \rangle \ :\ \mathfrak{M} \in \mathsf{Mod}(Th) \}$$ [1128]

of $\mathsf{Mod}(Th)$ without taking reducts, where for "auxiliaries" cf. p.964 under the name "auxiliary relations".

◁

Notation: Throughout the present sub-section $Fm(\mathfrak{M})$ denotes our frame language (for relativity), $Fm(\mathfrak{G}_{\mathfrak{M}})$ denotes the language of observer independent geometries, and $Fm(\mathfrak{M}^+)$ denotes the language of $\mathsf{Mod}(Th)^+$ defined in Def.6.6.88 above.

◁

By Thm.6.3.27 (second translation theorem) p.965 there is a translation mapping

$$Tr : Fm(\mathfrak{M}^+) \longrightarrow Fm(\mathfrak{M})$$

and formulas $code_i(x, \bar{x})$ such that

$$(\star) \qquad \mathsf{Mod}(Th)^+ \models code_i(x, \bar{x}) \ \to\ [\, \psi(x, \bar{z}) \ \leftrightarrow\ Tr(\psi)(\bar{x}, \bar{z}) \,]$$ [1129]

as indicated in Thm.6.3.27, p.965. Intuitively, $(\star)$ means that $Tr(\psi)$ is a meaning-preserving translation of ψ (to the narrower language of $\mathfrak{M}$) while $code_i(x, \bar{x})$ tells us how the free variables of ψ are represented in $Tr(\psi)$.

Our intuition is the following. From our observation-oriented model $\mathfrak{M}$ we defined a theoretical super-structure[1130] $\mathfrak{G}_{\mathfrak{M}}$ built up from more theoretical concepts (than the parts of $\mathfrak{M}$). Now, if a formula $\varphi(\bar{x}) \in Fm(\mathfrak{M})$ can be expressed in the language $Fm(\mathfrak{G}_{\mathfrak{M}})$ of this theoretical structure $\mathfrak{G}_{\mathfrak{M}}$ the chances are better for $\varphi(\bar{x})$ being a potential law (as compared to the case when $\varphi(\bar{x})$ is not expressible in the [theoretical] language of $\mathfrak{G}_{\mathfrak{M}}$). The idea is that if a statement $\varphi(x)$ can be formulated using (and involving) <u>theoretical concepts only</u> then, in some sense, it will be

[1128] We note that the common part of the vocabularies of $\mathfrak{M}$ and $\mathfrak{G}_{\mathfrak{M}}$ consists of the sort symbol F and relation/function symbols $+, \cdot, \leq$. Therefore in $\mathsf{Mod}(Th)^+$ we have only one copy of these things.

[1129] More generally, ψ may have more than one variables of sort not available in $\mathfrak{M}$. Let these be x, y. Then $\mathsf{Mod}(Th)^+ \models [\, code_i(x, \bar{x}) \ \wedge\ code_j(y, \bar{y}) \,] \ \to\ [\, \psi(x, y, \bar{z}) \ \leftrightarrow\ Tr(\psi)(\bar{x}, \bar{y}, \bar{z}) \,]$. Similarly for $\{x, y, u, \ldots\}$ in place of $\{x, y\}$. See Thm.6.3.27 for a general formulation.

[1130] Cf. Friedman [90] § VI.3 (p.236) under the title "Theoretical Structure and Theoretical Unification".

like what one would intuitivelly call a "theoretical statement" like "all bodies fall" or "electrons repel each others". Such "theoretical statements" have a better chance for being potential laws of nature than non-theoretical statements like e.g. "there are 3 apples in my basket" or "observer k sees 3 inertial bodies on life-line ℓ", or "there are 3247 ants in the cellar of our neighbor lady".

Definition 6.6.89 Let Th be fixed, $\varphi(\bar{x}) \in Fm(\mathfrak{M})$. We call $\varphi(\bar{x})$ a *Th-potential law* (of nature) iff there is a formula $\varphi'(\bar{x'}) \in Fm(\mathfrak{G}_{\mathfrak{M}})$ in the language of $\mathfrak{G}_{\mathfrak{M}}$ such that

$$\mathsf{Mod}(Th)^+ \models code(\bar{x}, \bar{x'}) \rightarrow [\varphi(\bar{x}) \leftrightarrow Tr(\varphi')(\bar{x'})].$$

Here $code(\bar{x}, \bar{x'})$ abbreviates the statement expressible by the formulas $code_i$ that those variables in $\bar{x}$ which do not belong to the sorts of $\mathfrak{G}_{\mathfrak{M}}$ are replaced by their codes in $\bar{x'}$ (while the common variables are left unchanged).

◁

Intuitively, the above definition utilizes the fact that $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{M}$, hence the ingredients (relation symbols, sorts etc) of $\mathfrak{G}_{\mathfrak{M}}$ can be regarded as abbreviations or defined terms in the language of $\mathfrak{M}$. In other words, the language $Fm(\mathfrak{G}_{\mathfrak{M}})$ of $\mathfrak{G}_{\mathfrak{M}}$ can be regarded as a (perhaps complicatedly defined) sublanguage of $Fm(\mathfrak{M})$. Now, a statement $\varphi(\bar{x})$ (about the entities $\bar{x}$) is called potential law iff it can be expressed in the sublanguage of $Fm(\mathfrak{M})$ corresponding to $\mathfrak{G}_{\mathfrak{M}}$.

Certainly, those formulas $\varphi(\bar{x}) \in Fm(\mathfrak{M})$ which can be expressed in the sublanguage[1131] $Fm(\mathfrak{G}_{\mathfrak{M}})$ built up from the theoretical concepts constituting the vocabulary of $\mathfrak{G}_{\mathfrak{M}}$ are more "theoretical", in some sense, than the rest of the formulas in $Fm(\mathfrak{M})$. Our above definition of potential laws expresses our faith that the more theoretical statements are more likely to turn out to be potential laws (than the less theoretical ones).

If instead of Th-potential law we write simply potential law then we assume that Th is implicitly understood, or that Th is one of the general theories for which we proved that $\mathsf{Ge}(Th)$ is definable over $\mathsf{Mod}(Th)$.

We note that our law-like/fact-like distinction could be based on our $(\mathcal{G}, \mathcal{M})$-duality theory

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th),$$

elaborated in §6.6, as follows. Roughly $\varphi(\bar{x})$ will be regarded a potential law if its truth-value does not change when passing from $\mathfrak{M}$ to $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$. I.e. potential

[1131] $Fm(\mathfrak{G}_{\mathfrak{M}})$ is the language of the geometry $\mathfrak{G}_{\mathfrak{M}}$. We can regard it as a sublanguage of $Fm(\mathfrak{M})$ only because $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{M}$.

laws are those formulas which are not sensitive to the change between $\mathfrak{M}$ and the model $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$ "recovered" from the geometry $\mathcal{G}(\mathfrak{M})$ associated to $\mathfrak{M}$. This duality theory based version might be more suitable for further refinement.

Assume *Th* is strong enough for ensuring the existence of a "canonical" partial homomorphism

$$f : \mathfrak{M} \overset{\circ}{\longrightarrow} (\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$$

for each $\mathfrak{M} \in \mathsf{Mod}(Th)$. Further assume f is defined on $Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}}$. What we understand by f being canonical is explained in Def.6.6.78 (reflection ...) p.1090 and in Fig.330 (p.1092), i.e. by f being canonical we mean that f is a kind of $\mathcal{M}$-reflection "arrow", cf. Fig.310 (p.1007) together with Figures 330 (p.1092), 331 (p.1093).[1132] In some sense, the homomorphism $f : \mathfrak{M} \overset{\circ}{\longrightarrow} (\dots)$ intends to "illustrate" how $(\mathcal{G} \circ \mathcal{M})$ modifies the original $\mathfrak{M}$, i.e. where can we find the original elements of $\mathfrak{M}$ in the modified model $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$.

Assume $\varphi(\bar{x}) \in Fm(\mathfrak{M})$ is such that f is defined over all the possible values of $\bar{x}$ in $\mathfrak{M}$.[1133] Consider $(\star\star)$ below.

For all evaluations $\bar{a}$ of $\bar{x}$ in $\mathfrak{M}$ such that[1134] $Rng(\bar{a}) \subseteq Dom(f)$,

$(\star\star)$ $$\mathfrak{M} \models \varphi[\bar{a}] \quad \Longleftrightarrow \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \models \varphi[\bar{a} \circ f].$$ [1135]

Now, we could call $\varphi(x) \in Fm(\mathfrak{M})$ a *Th*-potential law (in the $(\mathcal{G}, \mathcal{M})$-sense) if $(\star\star)$ holds for all $\mathfrak{M} \in \mathsf{Mod}(Th)$. Let us notice that this new, $(\mathcal{G}, \mathcal{M})$-oriented definition of *Th*-potential laws is more-or less equivalent with our first, "translation mapping (i.e. *Tr*)"-oriented version. We leave the task of comparing the mathematical content of the two versions to the interested reader.

* * *

Now, we can use our definition of potential laws for formulating Einstein's SPR as saying that no inertial observers $m, k \in Obs$ are distinguishable by *Th*-potential laws. I.e. for any *Th*-potential law $\varphi(m)$ we claim

$$(\forall m, k \in Obs)\,[\,\varphi(m) \leftrightarrow \varphi(k)\,].$$

[1132] Cf. the construction of $h_{Obs} \cup h_{Ph} : (Obs \cup Ph)^{\mathfrak{M}} \longrightarrow B^{(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})}$ in the outline of proof for Thm.6.6.46 above statement $(*)$ on p.1062, for the existence of f.

[1133] A more formal version of this condition on f and $\bar{x}$ is the $Rng(\bar{a}) \subseteq Dom(f)$ part in $(\star\star)$ below.

[1134] Here we think of an evaluation $\bar{a}$ as a function from the set of variables into the universe of $\mathfrak{M}$.

[1135] Since $Rng(\bar{a}) \subseteq Dom(f)$, $\bar{a} \circ f$ is an evaluation again.

We leave comparing the above formalized version of Einstein's SPR with both §6.2.8 ("Characterizing ... $\mathfrak{G}_{\mathfrak{M}}$") and our symmetry axioms (choosing appropriate versions of *Th*) as a future research task. Also we leave pursuing further the potential law/potential fact distinction as a future task. Here we wanted to indicate that having our theoretical super-structure $\mathfrak{G}_{\mathfrak{M}}$ definable over the more observationally oriented $\mathfrak{M}$ gives us a handle on beginning to classify the formulas in $Fm(\mathfrak{M})$ according to the more law-like/more fact-like distinction.

6.6.9 Geometric dualities, definability, Gödel incompleteness

The present section is related to the subject matter of §3.8 ("Making **Basax** complete...", pp.294-346), to the "relativity and Gödel incompleteness papers Andréka-Madarász-Németi [16], [17], and to the "Accelerated observers" materials, e.g. the Accelerated Observers Chapter in Andréka-Madarász-Németi-Sági-Sain [24], and [127], [23].

Notation 6.6.90 For any axiom system *Axi*, we write $\mathsf{T}(Axi)$ for the <u>*theory generated by*</u> *Axi*. I.e.

$$\mathsf{T}(Axi) :\stackrel{\text{def}}{=} \mathsf{Th}(\mathsf{Mod}(Axi)).$$

◁

Let $\mathfrak{G}^*_{\mathfrak{M}}$ be defined exactly as $\mathfrak{G}_{\mathfrak{M}}$ was defined in Def.6.2.2 (p.787) with the following changes.

$$L :\stackrel{\text{def}}{=} L^T \cup L^{Ph} \cup L^S \cup \text{ "life-lines of inertial bodies"; i.e.}$$
$$L :\stackrel{\text{def}}{=} L^T \cup L^{Ph} \cup L^S \cup \{\{e \in Mn : b \in e\} : b \in Ib\}.$$

Now,

$$\mathfrak{G}^*_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L, L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq, g, \mathcal{T}\rangle.$$

I.e. $\mathfrak{G}^*_{\mathfrak{M}}$ is obtained from $\mathfrak{G}_{\mathfrak{M}}$ by including the <u>life-lines of inertial bodies</u> as extra lines. This is in perfect harmony with our **Ax3** (p.48) (or even $\mathbf{Ax3_0}$) which say that the life-lines of inertial bodies are straight lines.

Instead of $\mathfrak{G}_{\mathfrak{M}}$, we could have investigated $\mathfrak{G}^*_{\mathfrak{M}}$ in the present chapter (Chap.6), the changes would be inessential. Actually, the reader is invited to elaborate a version of the present chapter based on $\mathfrak{G}^*_{\mathfrak{M}}$ in place of $\mathfrak{G}_{\mathfrak{M}}$. The only reason why we chose $\mathfrak{G}_{\mathfrak{M}}$ as a basis of the present chapter (instead of $\mathfrak{G}^*_{\mathfrak{M}}$) was to make it shorter. However, the nature of the present sub-section (§6.6.9) is such that $\mathfrak{G}^*_{\mathfrak{M}}$ is more suitable as a basis for it than $\mathfrak{G}_{\mathfrak{M}}$. So we will concentrate on $\mathfrak{G}^*_{\mathfrak{M}}$ instead of $\mathfrak{G}_{\mathfrak{M}}$ in the present sub-section. Since the differences are small, to avoid complicated, heavy notation, we will simply pretend in the present sub-section that $\mathfrak{G}_{\mathfrak{M}} := \mathfrak{G}^*_{\mathfrak{M}}$ (i.e. that $\mathfrak{G}_{\mathfrak{M}}$ denotes $\mathfrak{G}^*_{\mathfrak{M}}$) and that all the results, definitions etc. of the present chapter are about $\mathfrak{G}^*_{\mathfrak{M}}$.

CONVENTION 6.6.91 In the present sub-section (§6.6.9) we will pretend that $\mathfrak{G}_{\mathfrak{M}} := \mathfrak{G}^*_{\mathfrak{M}}$, hence in particular, that the life-lines of inertial bodies are lines in $\mathfrak{G}_{\mathfrak{M}}$. This convention is valid only inside this sub-section, after the end of this sub-section $\mathfrak{G}_{\mathfrak{M}}$ will retain its original definition. Whenever the present convention would lead to inconsistencies, we leave it to context and the reader's common sense to eliminate these inconsistencies.

We note that besides the present §6.6.9 in later investigations too (especially after Chapter 7) $\mathfrak{G}^*_{\mathfrak{M}}$ will be superior to $\mathfrak{G}_{\mathfrak{M}}$. Hence the definition of $\mathfrak{G}^*_{\mathfrak{M}}$ lives after the present sub-section. (The role of $\mathfrak{G}_{\mathfrak{M}}$ will be to keep discussions shorter. So, after §6.6.9, if a discussion is shorter for $\mathfrak{G}_{\mathfrak{M}}$ than for $\mathfrak{G}^*_{\mathfrak{M}}$ and it is obvious how to generalize it to $\mathfrak{G}^*_{\mathfrak{M}}$, then we will use $\mathfrak{G}_{\mathfrak{M}}$ instead of the more proper $\mathfrak{G}^*_{\mathfrak{M}}$.)

⋆ ⋆ ⋆

The purpose of this sub-section is threefold:

(i) We saw, e.g. in Thm.6.6.13 (p.1031), that the "world" of observation oriented models, the $\mathfrak{M}$'s, and the world of observer-independent geometries, the $\mathfrak{G}$'s, are definitionally equivalent (under some assumptions). From [16, 17], and/or from the relevant part of the present work we know that Gödel's incompleteness theorems do apply to many of the $\mathfrak{M}$'s.[1136] In brief, the limitative theorems[1137] of metamathematics do apply to the "world" of the $\mathfrak{M}$'s.

[1136] Hence e.g. T(**Basax**) is undecidable, moreover T(**Basax** + some extra axioms) is hereditarily undecidable, it admits a formulation Con(**Basax** + extra) of its own consistency etc. The techniques of proving this (formulatizability of own consistency) ensure that the Liar Paradox expressing "this sentence is not provable from (**Basax** + extra)" can be formulated in "**Basax** + extra", which in turn leads to strong hereditary incompleteness results. If someone wants to make this theory complete, then he will probably try by adding the Liar Paradox to (**Basax** + extra) as a new axiom. But this spectacularly fails, because then there will be a new incarnation of the "Liar" saying "this sentence is not provable from (**Basax** + extra + "Liar formulated for (**Basax** + extra)"). Etc.

[1137] See e.g. Bell-Machover [44, Chapter 7, "Logic-limitative results"] or Chaitin [58].

At the same time, one vaguely remembers from logic courses, that Gödel's incompleteness theorems have a tendency of not being applicable to geometric structures and in this respect geometries have a tendency of behaving similarly to real-closed fields (or $\mathfrak{R}$ itself) in that they usually do not satisfy the conditions of Gödel's incompleteness theorems (hence, these theorems do not apply to these structures).[1138] Cf. e.g. Goldbatt [108, p.169 lines 11-10 bottom up] where it is stated that the theory of Minkowskian geometry over $\mathfrak{R}$ is decidable. In particular, there are natural frame-theories $Th \supseteq$ **Specrel**, such that Gödel's incompleteness theorems apply to Th but do not apply to $\mathsf{Ge}(Th)$ or to $\mathcal{M}[\mathsf{Ge}(Th)] = (\mathcal{M} \circ \mathcal{G})[\mathsf{Mod}(Th)]$. All these lead to the following question: How is it possible that two "worlds" are equivalent and Gödel's theorems apply to one of them but not to the other? Similarly, we could ask, why does the $(\mathcal{G}, \mathcal{M})$-duality not "import" Gödel incompleteness properties from the side (or "world") of the $\mathfrak{M}$'s to the side (or "world") of the geometries, the $\mathfrak{G}_{\mathfrak{M}}$'s.[1140] (Below we will see that the answer is in the conditions of our theorems, and that the just outlined "tension"[1141] can lead to interesting observations.)

(ii) Can we extend our $(\mathcal{G}, \mathcal{M})$-duality to handling non-inertial bodies (or at least non-inertial observers) well? (I.e. can we extend our duality such that non-inertial bodies or observers would not necessarily disappear from $(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$?)

(iii) We will briefly ask ourselves whether the life-lines of some non-inertial bodies are definable in $\mathsf{Ge}(Th)$, for nice enough choices of Th.

Before going on, we note that the above three issues (i)-(iii) are interconnected as follows: If all non-inertial bodies of $\mathfrak{M}$ would reappear in $\mathcal{G} \circ \mathcal{M}(\mathfrak{M})$ [1142] then

[1138] In passing we note that if our field $\mathfrak{F}$ is strange enough (i.e. is far from being a real-closed field) then we can loose decidability of e.g. $\mathsf{Th}(Mink(4, \mathfrak{F}))$. Cf. [17]. But this is not too relevant to our present concerns, so we do not discuss this and we pretend that $\mathsf{Th}(\mathfrak{F})$ is always decidable. Although in the typical well behaved cases Gödel's theorems do not apply to $\mathfrak{G}_{\mathfrak{M}}$ whenever $\mathfrak{F}$ is a real-closed field,[1139] we note that there are exotic exceptions. E.g. we conjecture that either for the geometry $\mathfrak{G}_{\mathfrak{M}}$ constructed in the proof of Thm.6.2.24 (p.830) Gödel's incompleteness theorems do apply, or one can construct an analogous $\mathfrak{G}_{\mathfrak{M}}$ for which Gödel's incompleteness theorems apply. We leave solving this conjecture as a future exercise for the reader.

[1139] E.g. in Minkowskian geometries this is always so (i.e. [$\mathfrak{F}$ is real-closed] $\Rightarrow$ [Gödel's incompleteness theorems do not apply to $Mink(\mathfrak{F})$]), cf. e.g. Goldblatt [108, p.169] for this.

[1140] Of course, there are structures in $\mathsf{Ge}(Th)$ to which the conditions of Gödel's theorems do apply, but they are the exceptional ones, in some sense (from the physical point of view they are somewhat strange); while on the $\mathsf{Mod}(Th)$ side it is much more typical, frequent (and natural) to have these conditions satisfied, cf. Andréka-Madarász-Németi [16],[17] (e.g. having a periodically moving body is sufficient).

[1141] By tension we mean something which looks like a contradiction (but is not one).

[1142] This would be a positive answer to (ii).

probably all non-inertial bodies of $\mathfrak{M}$ would be (at least parametrically) definable in $\mathcal{G}(\mathfrak{M})$. (This would answer item (iii).) But, if this would be the case, then applicability of Gödel's incompleteness theorems for $\mathfrak{M}$ would probably be inherited by $\mathcal{G}(\mathfrak{M})$,[1143] because non-inertial bodies of $\mathfrak{M}$ played an essential role in applying these theorems to $\mathfrak{M}$ in [16], [17]. So items (i)-(iii) are interconnected.

A perspective on items (i)-(iii): In connection with item (i), in Statement ($\star$) below, we will see that $(\mathcal{M} \circ \mathcal{G})$ tends to streamline our models, it tends to make our originally complicated, "untidy" $\mathfrak{M}$ into a "streamlined", "tidy", and smooth variant $(\mathcal{M} \circ \mathcal{G})(\mathfrak{M})$ of the original $\mathfrak{M}$. As a byproduct, it may happen that $\mathfrak{M}$ satisfies the conditions of Gödel's incompleteness theorems but $(\mathcal{M} \circ \mathcal{G})(\mathfrak{M})$ does not.

Now, in items (ii), (iii) we ask ourselves: Is this good for us or is this bad for us? Roughly, the answer will be the following. At the present level of investigations this is not bad at all. However, in later generalizations toward general relativity, e.g. in the theory of accelerated observers[1144] this might create inconveniencies (which we will have to be careful to avoid).

Let us turn to discussing (some of) the questions (i)-(iii) above.

In §6.6.3 we had a proposition saying, roughly, that the operator $\mathcal{M} \circ \mathcal{G}$ makes our possibly complicated and "inhomogeneous"[1145] models $\mathfrak{M}$ (which might contain random features) symmetric, "tidy" and "smooth", e.g.

$$(\star) \qquad (\mathcal{M} \circ \mathcal{G})(\mathfrak{M}) \models \mathbf{Ax}\heartsuit + \mathbf{Ax(ext)}.$$ [1146]

In the "Gödel incompleteness" papers Andréka-Madarász-Németi [16], [17] related to the present work[1147], we saw that, roughly, such "smooth" models usually have a decidable theory to which Gödel's incompleteness theorems do *not* apply (assuming

[1143] at least in most of the cases (i.e. when non-inertial bodies were responsible for "incompleteness")

[1144] Cf. e.g. [16], [17], [23], [24, Chap. "Accelerated Observers"], [127], [196].

[1145] We mean here that on some (but not all) life-lines there may be many indistinguishable observers in a random manner, and that there may be many non inertial bodies with complicated life-lines in one part of $\mathfrak{M}$ but not in the another etc.

[1146] In passing we note that many other duality theories tend to do this "streamlining" of their objects. E.g. in the case of Galois connections (pp. 1078–1081) if $p \in P$ then $g(f(p))$ is the "closure" of p and usually has more symmetry properties than p. A similar remark applies to the $(\mathsf{Mod}, \mathsf{Th})$-duality on p.1026 where to the possibly "untidy" or "random" $\Sigma \subseteq Fm$, the streamlined $\mathsf{Th}(\mathsf{Mod}(\Sigma))$ is associated (which is closed under "$\models$").

[1147] cf. also the Gödel incompleteness chapter of this work (§7)

$\mathfrak{F}$ is a real-closed field).[1148]

Though ($\star$) can be viewed as a positive result, in a certain other sense it will turn out to be a *limitative* one, cf. e.g. Thm.6.6.95, Conj.6.6.97.

Independently of this, we saw in §§ 6.6.3, 6.6.4 that the function

$$\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

is a *first-order definable meta-function* (assuming Th is strong enough), i.e. that $\mathcal{M}(\mathfrak{G})$ is uniformly first-order definable over $\mathfrak{G}$. Moreover $\mathsf{Mod}(Th)$ is definable over $\mathsf{Ge}(Th)$ if Th is strong enough, cf. Theorems 6.6.12, 6.6.13 and Prop.6.6.44.

First-order definability of $\mathcal{M}(\mathfrak{G})$ over $\mathfrak{G}$ includes the claim that (intuitively speaking) *every observer* m of $\mathcal{M}(\mathfrak{G})$ *is* first-order *definable* from $\mathfrak{G}$ by using $n+1$ parameters. Namely, each m is definable by using (as *parameters*) $n+1$ points $o, e_0, \ldots, e_{n-1}$ satisfying (a)–(f) on p.1054. (This kind of definability is called *parametrical definability* in standard mathematical logic, cf. §6.3 [pp. 950, 935].)

Summing up, every observer of $\mathcal{M}(\mathfrak{G})$ is parametrically definable in $\mathfrak{G}$. Moreover

($\star\star$) every body of $\mathcal{M}(\mathfrak{G})$ is parametrically definable in $\mathfrak{G}$.

All the bodies in $\mathcal{M}(\mathfrak{G})$ are *inertial*. But in our relativity theories like e.g. (**Basax**+$\mathbf{Ax}(\omega)^\sharp$) *non-inertial* bodies also play some important role, cf. e.g. the formalization of the Twin Paradox in §2 (p.38 and Figure 5 on p.38) and the continuation of this work on accelerated observers [23], the accelerated observers chapter in [24], [127], and the related discussions in the present work.

Therefore, as we already said, the following question naturally comes up: *Can* we *define* (by first-order means) *strongly non-inertial bodies*[1149] from $\mathfrak{G}$? Further, can we extend our duality theory

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th), \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

by possibly strengthening Th (and improving the definition of $\mathcal{M}$) such that it would "handle" strongly non-inertial bodies too? We will see that the answer is no, at least if we want to keep our geometries $\mathfrak{G}_{\mathfrak{M}}$ at least remotely similar to the geometries considered in the literature, e.g. if we want to stick with the three sorts

[1148] As we already mentioned in connection with geometries (in footnote 1138), there might be exceptional models $\mathfrak{M}$ which are "smooth" in the above sense with $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ and still have an undecidable theory. Cf. Andréka et al. [16, Thm.9(iii)].

[1149] Cf. Def.6.6.96 for strongly non-inertial bodies.

Points, *Lines* and *Quantities* (i.e. F) only.[1150] On the other hand, we will indicate in Remark 6.6.92 that a positive answer is possible in the framework of first-order logic on the expense of making our structures richer than "geometries". Since accelerated observers with constant acceleration will play an important role later in generalizing our theory,[1151] we note the following. (Life-lines of) accelerated bodies with constant acceleration are parametrically definable in most of our geometries $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Pax})$.

Remark 6.6.92 We note that to recover strongly non-inertial bodies from $\mathfrak{G}_{\mathfrak{M}}$ we will need to add, among others, an extra sort representing, roughly, a possibly nonstandard model of Peano's Arithmetic as it was done in the development of nonstandard temporal logics and nonstandard dynamic logic cf. e.g. Sain [231], Andréka-Goranko et al. [14] and the references therein. We plan to do such things in a later work related to the present one. Such developments will also represent connections with nonstandard analysis.[1152] We note that in this approach we will add the following extra sorts to $\mathfrak{G}$. (i) A sort usually denoted as I which represents functions from the sort $\mathbf{F_1}$ into itself. I.e. $I \subseteq {}^F F$. (ii) Further, a binary operation $\mathsf{value} : I \times F \longrightarrow F$ such that for $f \in I$, $\mathsf{value}(f, x) \in F$ is considered to be the value "$f(x)$" for $x \in F$. (iii) A unary relation $N \subseteq F$ which plays the role of the positive integer elements of F, e.g. $0, 1 \in N$ and N is closed under $+, \cdot$ of $\mathbf{F_1}$, moreover $\langle N, 0, 1, +, \cdot \rangle$ is a model of Peano's Arithmetic. (iv) We will postulate the comprehension axiom-schema for I saying that all functions $f : F \longrightarrow F$ which are definable in the language of the so expanded model $\mathfrak{G}$ appear as elements of I. I.e. all first-order definable[1153] functions $f : F \longrightarrow F$ show up in I, roughly $f \in I$. The purpose of all this machinery is to enable us to express in first-order logic (i.e. in the first-order language of the so expanded $\mathfrak{G}$) the things which we want to express in order to develop our theory of, say, accelerated observers (and/or motion in general). This approach will be described in a later continuation of the present work.[1154]

$\lhd$

Notation 6.6.93 $\mathbf{Mink}(n, rc)$ denotes the class

$$\mathbf{I}\{\mathit{Mink}(n, \mathfrak{F}) \ : \ \mathfrak{F} \text{ is a real-closed field}^{1155}\}$$

[1150]or anything in the spirit of $\langle$*Points*, *Lines*, *Planes*, *Quantities*$\rangle$-like arrangement to which e.g. our definition of $\mathfrak{G}^*_{\mathfrak{M}}$ does conform

[1151]in the direction of general relativity theory

[1152]This would mean a connection between the presently discussed kind of "logic-based relativity" and nonstandard analysis.

[1153]We mean definable in the many-sorted structure $\langle \mathfrak{G}, I, \mathsf{value}, \text{etc.} \rangle$.

[1154]We note that at this point we did not explain why and how adding such extra sorts including an extra arithmetical sort will help. The reader does not have to see why this will work, all this will be elaborated in a later work. (But consulting Sain [231], Andréka-Goranko-et al [14], Montague [198], Gallin [96] may give useful hints.)

of all n-dimensional Minkowskian geometries over real-closed fields.

$\lhd$

Items 6.6.95, 6.6.97 below can be interpreted as saying that *not all important aspects of* (special) *relativity can be recovered from the geometries* $\mathfrak{G}_{\mathfrak{M}}$ (or from Minkowskian geometry).

A body $b \in B$ is called periodically moving, or periodical for short, if there is $m \in Obs$ such that $tr_m(b)$ can be interpreted as a function $tr_m(b) : \bar{t} \longrightarrow {}^{n-1}F$ and this function is periodical. See Figure 332. For simplicity we will use the following simpler definition.

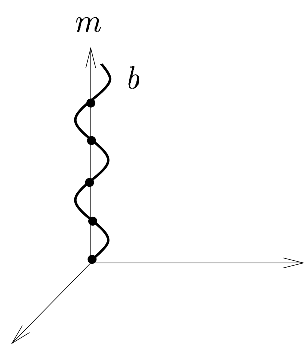


Figure 332: b is a periodically moving body in m's world-view.

Definition 6.6.94 Let $\mathfrak{M}$ be fixed. Body b is called *periodical* iff there is $m \in Obs$ such that letting $H := \bar{t} \cap tr_m(b)$ the set $H \subseteq F$ is discrete[1156] and cofinal in $\mathfrak{F}$, and for any two neighboring pairs $a, b, a', b' \in H$ we have $|b - a| = |b' - a'|$ (where a and b are neighbors[1157] and the same holds for a', b').

$\lhd$

Intuitively, the following theorem says that life-lines of periodical bodies are *not definable* in our geometries like $\mathsf{Ge}(\mathbf{Bax})$. Recall that $\mathbf{Ax(rc)}$ is the usual axiom system for real-closed fields defined on p.301 in §3.8.

[1155] Cf. p.301 for the notion of real-closed fields.

[1156] We use the language of $\mathfrak{F}$. H is *discrete* if any point in H has a successor and a predecessor in H unless it is an endpoint of H.

[1157] I.e., $[a < b$ and $(\nexists c \in H)\ a < c < b]$.

THEOREM 6.6.95

(i) *Let $n > 1$ and consider the class* $\mathbf{Mink}(n, rc)$ *of Minkowskian geometries. Then, there exists* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp)$ *such that* $\mathfrak{G}_\mathfrak{M} = \langle Mn, F, \dots \rangle \in \mathbf{Mink}(n, rc)$ *and for no periodical body b of $\mathfrak{M}$ is the life-line* $\{ e \in Mn : b \in e \}$ *of b definable parametrically in the geometry* $\mathfrak{G}_\mathfrak{M}$.[1158]

(ii) *Statement (i) remains true if we replace* $\mathbf{Mink}(n, rc)$ *by any one of our distinguished classes* $\mathsf{Ge}(Th)$ *of geometries. (Here Th ranges over our hierarchy* $\mathbf{Bax}^-, \mathbf{Bax}, \dots, \mathbf{Basax}$*).*

Outline of proof: In Andréka-Madarász-Németi [16], [17] as well as in the "decidability ... Gödel incompleteness" part of this work we will see that if we add the existence of a periodical body as an extra axiom (this extra axiom is denoted by $\boldsymbol{\nu}$ there) to any one of our distinguished theories Th, then the so obtained $(Th + \boldsymbol{\nu})$ becomes essentially undecidable as a theory, it satisfies the conditions of Gödel's incompleteness theorems, hence the conclusions of Gödel's incompleteness theorems (both of them) apply to the theory $(Th + \boldsymbol{\nu})$.[1159]

Therefore, if a periodical body was parametrically definable in $\mathsf{Mod}(Th)$ then this would render $\mathsf{Th}(\mathsf{Mod}(Th))$ essentially undecidable etc. (The parameters [in our notion of definability] cause no problem in this argument because we can use quantifiers in our language to make the parameters "disappear" when translating number theoretic formulas to formulas in the language of $\mathsf{Mod}(Th)$. This technique [for getting rid of the parameters] was used e.g. in Németi [205]).

Having seen that $\mathsf{Mod}(Th)$ would become essentially undecidable if formula $\boldsymbol{\nu}$ was added to it, one can push the same argument through to show that $\mathsf{Ge}(Th)$ would become hereditarily *undecidable if* $\boldsymbol{\nu}$ was *expressible* in the language of $\mathsf{Ge}(Th)$. Since we know that $\mathsf{Ge}(Th + (\mathfrak{F}$ is a real-closed field$))$ can be extended to a *decidable* consistent theory, cf. the "Making **Basax** complete ... " section, i.e. §3.8 pp.294-346, we conclude that $\boldsymbol{\nu}$ cannot be expressible in the first-order language of $\mathsf{Ge}(Th)$. But

[1158] I.e. for no finite number of parameters $p_1, \dots, p_k$ from $\mathfrak{G}_\mathfrak{M}$ (i.e. from $Uv(\mathfrak{G}_\mathfrak{M}) = Mn \cup F \cup L$) is the life-line of any periodical body of $\mathfrak{M}$ (first-order) definable in $\mathfrak{G}_\mathfrak{M}$ by using $p_1, \dots, p_k$ as parameters. That is, let $\bar{p} = \langle p_1, \dots, p_k \rangle$. Then *no* first-order formula $\varphi(x, \bar{p})$ in the language of $\mathfrak{G}_\mathfrak{M}$ defines the trace $\{ e \in Mn : b \in e \}$ of a periodical body b of $\mathfrak{M}$.

[1159] This can be seen by interpreting Robinson's arithmetic denoted by R in Monk [197, Def.14.9, p.247] in the theory $(Th + \boldsymbol{\nu})$. Note that this version R of arithmetic is much weaker than Peano's arithmetic, in particular, it involves no induction axiom schema. Hereditary undecidability etc. of R is in Thm.16.1, p.280 of Monk [197]. For more detail on $(Th + \boldsymbol{\nu})$ cf. Andréka-Madarász-Németi [17].

this implies that *no* periodical body can be parametrically definable[1160] in $\mathsf{Ge}(Th)$. This finishes the proof. ∎

Definition 6.6.96 Let $\mathfrak{M}$ be fixed. A body b is called *strongly non-inertial* iff there is an observer m such that $tr_m(b) \cap \bar{t}$ is a nonempty set and is gapy in the following sense:

$$(*) \quad (\forall p \in tr_m(b) \cap \bar{t})(\exists q, r \in \bar{t})(p_t < q_t < r_t \ \wedge \ q \notin tr_m(b) \ \wedge \ r \in tr_m(b)).$$ [1161]

◁

If a body b is periodical (in the sense of Def.6.6.94) then it is strongly non-inertial.

Intuitively, the next conjecture says that the life-lines of strongly non-inertial bodies are not definable in our geometries like e.g. $\mathsf{Ge}(\mathbf{Bax})$.

Conjecture 6.6.97 *Theorem 6.6.95 remains true for strongly non inertial bodies in place of periodical ones.*

Possible idea of proof: We choose a model $\mathfrak{M} \in \mathsf{Mod}(Th)$ such that $\mathfrak{F}^{\mathfrak{M}}$ is a real-closed field and such that $\mathfrak{G}_{\mathfrak{M}}$ is the Minkowskian geometry over $\mathfrak{F}^{\mathfrak{M}}$, up to isomorphism. Then $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{F}^{\mathfrak{M}}$. Assume that the life-line of a strongly non-inertial body b of $\mathfrak{M}$ is parametrically definable over $\mathfrak{G}_{\mathfrak{M}}$. Then $(*)$ above holds for some $m \in Obs$. Let this m be fixed. Then the intersection $\{ e \in Mn : b, m \in e \}$ of the life-lines of b and m is parametrically definable over $\mathfrak{G}_{\mathfrak{M}}$ by a formula $\varphi(x, \bar{p})$ with parameters $\bar{p}$. Since $\mathfrak{G}_{\mathfrak{M}}$ is definable over $\mathfrak{F}^{\mathfrak{M}}$, there is a definitional expansion $\mathfrak{G}^+_{\mathfrak{M}}$ of $\mathfrak{F}^{\mathfrak{M}}$ such that $\mathfrak{G}_{\mathfrak{M}}$ is a reduct of $\mathfrak{G}^+_{\mathfrak{M}}$. Now, by Thm.6.3.27 (p.965) there is a translation mapping $Tr : Fm(\mathfrak{G}^+_{\mathfrak{M}}) \longrightarrow Fm(\mathfrak{F}^{\mathfrak{M}})$ such that the conclusion of Thm.6.3.27 holds for this Tr. Now, if we apply this Tr to our formula $\varphi(x, \bar{p})$ then we obtain a new formula which defines a relation on $\mathfrak{F}^{\mathfrak{M}}$ parametrically. We conjecture that from this, one can obtain a further formula which defines a subset H of $\mathfrak{F}^{\mathfrak{M}}$ parametrically which is gapy in the sense of Def.6.6.96 immediately above. Now, to such a gapy H one can apply the proof of Lemma 6.2.28 (p.834) to derive a contradiction. Namely, by the proof of Lemma 6.2.28 it follows that H is not parametrically definable over $\mathfrak{F}^{\mathfrak{M}}$. (The proof of Lemma 6.2.28 goes through for the present case if one uses arbitrary polynomials in the proof and not only polynomials

[1160]Here, we mean uniform definability for the whole class $\mathsf{Ge}(Th)$. However one can refine the present argument to prove that there is a geometry $\mathfrak{G} \in \mathsf{Ge}(Th)$ in which no such body is parametrically definable.

[1161]Cf. Def.6.2.27 (p.834) and note that though the two definitions are similar they are not the same.

with rational coefficients and if one uses "gapy" in the sense of Def.6.6.96 above and not in the sense of Def.6.2.27).

$\lhd$

Remark 6.6.98 (On Gödel's logic proofs, relativity proof, and Escher:) Some of Escher's pictures can be associated both to Gödel's incompleteness proof (logic) and to his rotating universe construction for general relativity. So these two seemingly distant creations of Gödel seem to be more closely related than is usually acknowledged in the literature. But cf. Dawson [73, pp. 176–177] for a positive exception (where the "two Gödel's" are connected). See Figure 333. For Gödel's rotating universe see Figure 355 on p.1208.

$\lhd$

Items 6.6.95, 6.6.97 above seem to say that our duality theory

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th), \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

cannot be easily extended to a duality theory consisting of some $\mathcal{G}^+$ and $\mathcal{M}^+$ which would satisfactorily handle periodically moving (or strongly non-inertial) bodies present in the models $\mathfrak{M} \in \mathsf{Mod}(Th)$. Or in other words, the duality theory based on $\mathcal{G}$ and $\mathcal{M}$ abstracts from strongly non-inertial bodies (and therefore also from strongly non-inertial observers (!)), and this feature seems to be unavoidable in view of items 6.6.95, 6.6.97. More precisely, this seems to be so unless we expand our geometries in the "nonstandard dynamic logic" style mentioned/promised in Remark 6.6.92 way above.

Let us return to answering items/questions (i)-(iii) on p.1112 close to the beginning of this sub-section. The above discussion, theorem, etc. answer items (ii), (iii)[1162].

To answer (i), let us assume some nice, strong frame-theory[1163] e.g. $Th^+ \stackrel{\text{def}}{:=}$ $\mathbf{Basax}+\mathbf{Ax}(\omega)+\mathbf{Ax}(Triv)+\mathbf{Ax}(\|)+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(rc)}+\mathbf{Ax(eqm)}+\mathbf{Ax(eqtime)}$.

Now, we are looking at $\mathsf{Mod}(Th^+)$ and at $\mathcal{G}^*[\mathsf{Mod}(Th^+)] = \{\mathfrak{G}^*_{\mathfrak{M}} : \mathfrak{M} \models Th^+\}$ where $\mathcal{G}^* : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ with $\mathcal{G}^*(\mathfrak{M}) \stackrel{\text{def}}{:=} \mathfrak{G}^*_{\mathfrak{M}}$ for all $\mathfrak{M}$. According to the proofs in [16, 17], there are many models $\mathfrak{M} \models Th^+$ satisfying the conditions of Gödel's incompleteness theorems. At the same time, $\mathfrak{G}^*_{\mathfrak{M}}$ fails to satisfy the conditions of Gödel's theorems for many[1164] choices of the above $\mathfrak{M}$. The reason

[1162] at least to some extent

[1163] The purpose of assuming such a theory is to avoid being side-tracked by some, more-or-less, inessential detail.

[1164] We are inclined to write "for most choices".

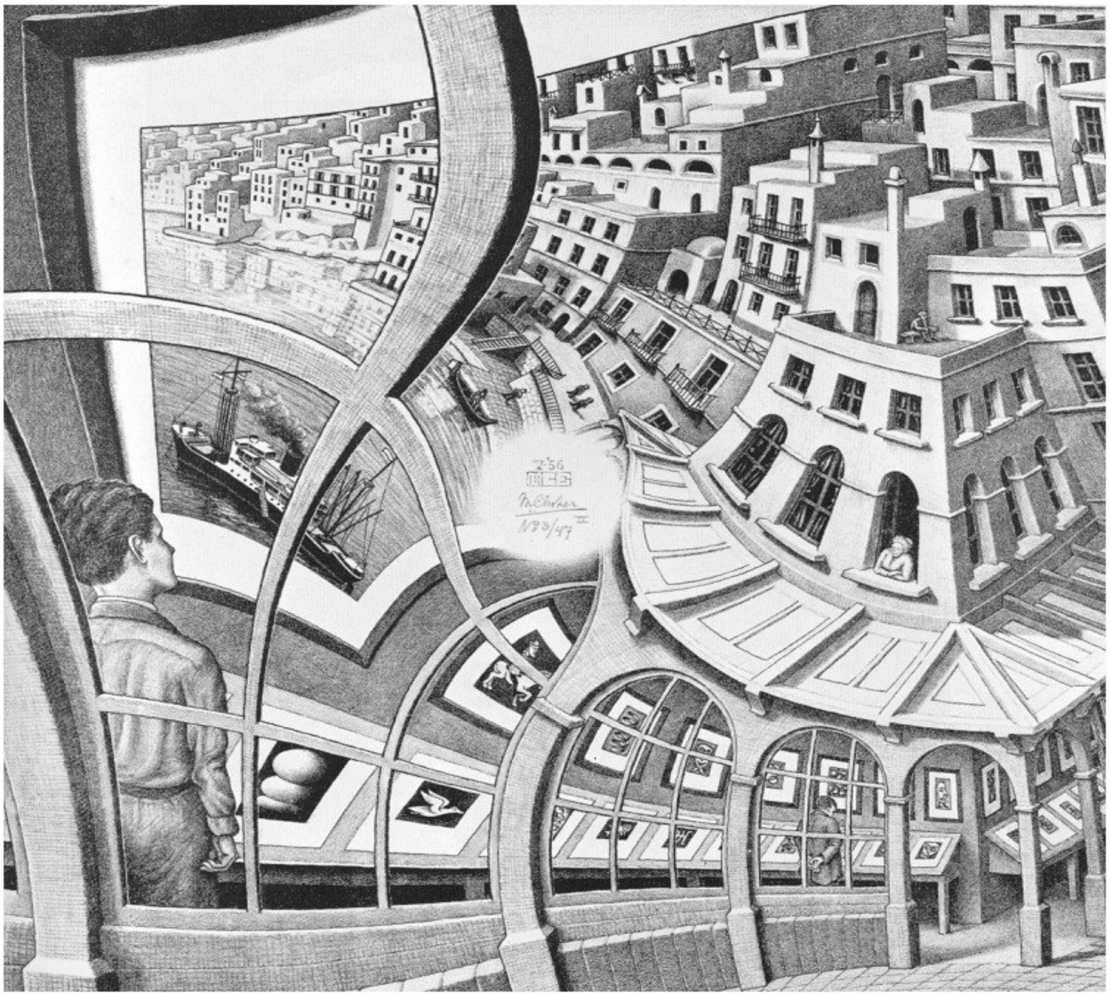


Figure 333: Print Gallery, by M.C. Escher. Cf. Fig.334 for the “logic” of this picture and for its connections with Gödel’s proof. A key idea in Gödel’s proof is self reference: “this sentence is not provable” (a variant of the well-known Liar paradox).

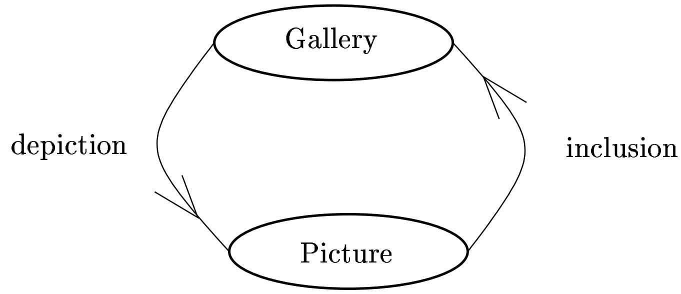


Figure 334: A collapsed version of Fig.333 (i.e. of Escher's Print Gallery).

for this is item ($\star$) on p.1114 together with the fact that in Thm.10 of [16] we used the presence of periodically moving bodies to prove the conditions of Gödel's theorems (for models satisfying Th^+). But the functor $\mathcal{G}^*$ removes (or forgets) the traces of such bodies. Hence the "periodical body method" in [16],[17] is no longer applicable to the structure $\mathfrak{G}^*_{\mathfrak{M}}$.[1165] Recall that here we pretend that the $(\mathcal{G}, \mathcal{M})$-duality is really some $(\mathcal{G}^*, \mathcal{M}^*)$-duality where $\mathcal{G}^*$ corresponds to $\mathfrak{G}^*_{\mathfrak{M}}$ defined on p.1111 (beginning of §6.6.9) and $\mathcal{M}^*$ matches $\mathcal{G}^*$ the same way and spirit as $\mathcal{M}$ matched $\mathcal{G}$. In summary, we can say that the apparent paradox in (i) is caused by the following. It is true that $\mathcal{M} \circ \mathcal{G}(\mathfrak{M})$ is almost the same as $\mathfrak{M}$ (hence almost all properties of $\mathfrak{M}$ should probably hold for $\mathcal{M} \circ \mathcal{G}(\mathfrak{M})$), but it is exactly that remaining little difference between $\mathfrak{M}$ and $\mathcal{M} \circ \mathcal{G}(\mathfrak{M})$ which really matters in the Gödel incompleteness issue. Namely, $(\mathcal{M} \circ \mathcal{G})$ preserves all nice properties but it forgets the non-inertial bodies. And it are exactly these bodies which are used in the proof in [16], [17].[1166]

So this is why our $(\mathcal{G}, \mathcal{M})$-duality or $(\mathcal{G}^*, \mathcal{M}^*)$-duality does not preserve the Gödel incompleteness properties of the structures involved.[1167] One still can ask

[1165] There are other "Gödel incompleteness methods" in [16], but they are less important from the physical point of view. (And even most of these are "killed" by the $\mathcal{M} \circ \mathcal{G}$-transition, with the exception of one or two.) Anyway, these alternative methods from [16] are excluded now by our choice of Th^+.

[1166] There were other incompleteness methods in [16], [17], but that is, so to speak, beside the point here, for various reasons.

[1167] There are also similar minor effects, e.g. $\mathcal{M} \circ \mathcal{G}$ makes **Ax(ext)** true which, by [16], eliminates further possibilities of applicability of Gödel's theorems, but to save space we do not discuss these here.

why the definitional equivalence theorem[1168]

$$\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$$

does not export Gödel incompleteness properties (e.g. hereditary undecidability) from $\mathsf{Mod}(Th)$ to $\mathsf{Ge}(Th)$. The answer is simple: The condition of the just quoted theorem (Thm.6.6.13) on Th excludes the kinds of applicability of Gödel's incompleteness theorems even to $\mathsf{Mod}(Th)$ which we used in e.g. [16]. Indeed, it is indicated in [16],[17] that **Ax♡**, **Ax(ext)**, **Ax($\sqrt{\ }$)**, **Ax(diswind)**, **Ax(eqtime)**[1169] are all axioms working <u>against</u> satisfiability of the conditions of Gödel's theorems. E.g. **Ax♡** excludes periodic (hence non-inertial) bodies.

Question for future research 6.6.99 Elaborate the present chapter (Chapter 6) for $\mathfrak{G}^*_{\mathfrak{M}}$ in place of $\mathfrak{G}_{\mathfrak{M}}$. Note that this implies (among many other things) defining two functors $\mathcal{G}^*$, $\mathcal{M}^*$ such that they form a duality theory analogous to the present $(\mathcal{G}, \mathcal{M})$-duality etc.

Our next two sub-sections (6.6.10, 6.6.11) are related to section 6.7 which in turn, is concerned with streamlining our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$ (among others), as was promised in the introduction.

6.6.10 Further connections between relativistic models and geometries

Let us return to the question, formulated at the beginning of this section of whether we can reconstruct $\mathfrak{M}$ from $\mathfrak{G}_{\mathfrak{M}}$ or from a reduct of $\mathfrak{G}_{\mathfrak{M}}$. In the duality theory developed in §§ 6.6.1–6.6.6 above we saw that $\mathfrak{M}$ can be reconstructed from $\mathfrak{G}_{\mathfrak{M}}$ (under some conditions on $\mathfrak{M}$). Below, we will look at the *same question* somewhat differently. We will look at *reduct* geometries $\mathfrak{G}^i_{\mathfrak{M}}$ and we will prove things which might be interpreted as saying that $\mathfrak{M}$ cannot be reconstructed from $\mathfrak{G}^i_{\mathfrak{M}}$. In *this form* these sound like negative results. However, in the form we will state them they will sound like positive results. Roughly speaking, assume we introduced the notation $\mathsf{Ge}^i(Th) = \mathbf{I}\{\mathfrak{G}^i_{\mathfrak{M}} : \mathfrak{M} \models Th\}$. Then for certain choices of Th_1 and Th_2 we will state that

$$(\star) \qquad \mathsf{Ge}^i(Th_1) = \mathsf{Ge}^i(Th_2);$$

[1168]Thm.6.6.13, p.1031

[1169]The condition of Thm.6.6.13 requires all these axioms to be provable from Th.

(for certain choices of i). This might be interpreted as a representation result stating that every geometry in $\mathsf{Ge}^i(Th_1)$ is *representable* as a geometry of some Th_2-model (and vice-versa). Theorems of style ($\star$) above can be read of from Fig.282 (p.863).

Intuitively, from a *relativity theoretic* point of view these results (of form ($\star$)) can be used the following way. Consider certain kinds of thought-experiments the *characteristic feature* of which is that they can be formulated in the language of $\mathfrak{G}^i_{\mathfrak{M}}$. Then a result of the type ($\star$) above can be interpreted by saying that the relativity theories Th_1 and Th_2 cannot be distinguished by thought-experiments of "type $\mathfrak{G}^i$". A result of this kind might be of interest e.g. when Th_1 is Reichenbachian version of relativity like **Reich**(**Basax**) and Th_2 is something more "classical" like **Basax**, cf. e.g. Theorems 6.6.107–6.6.110.

Below we will define several reducts $\mathfrak{G}^0_{\mathfrak{M}}$–$\mathfrak{G}^5_{\mathfrak{M}}$ of our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$. The physical motivation for looking at such reducts is given at the beginning of §6.6.4 on p.1069. The main idea is that at different times one may want to concentrate at different aspects of the world, and later one might want to compare the results and/or experiences so obtained. Concrete works on physics are listed in the preface of Schutz [236] which indeed concentrate on different aspects of the world e.g. on $\perp_r$, or on, $\prec$, or g. Some relatively significant physical conclusions (of the coming investigation of $\mathfrak{G}^0_{\mathfrak{M}}, \dots, \mathfrak{G}^i_{\mathfrak{M}}$) are summarized on p.1147 at the end of item (2) of §6.7.1.

Let us look at the geometry

$$\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq, g, \mathcal{T} \rangle.$$

Recall that $\mathfrak{G}^0_{\mathfrak{M}}$ is obtained from $\mathfrak{G}_{\mathfrak{M}}$ by forgetting g and $\mathcal{T}$ (hence also the universe $\mathbf{F_1}$), but keeping all the rest, i.e.

$$\mathfrak{G}^0_{\mathfrak{M}} = \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, eq \rangle,$$

cf. Def.6.6.53. Let

$$\mathfrak{G}^1_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r, \mathcal{T} \rangle$$

be obtained from $\mathfrak{G}_{\mathfrak{M}}$ by forgetting eq and g. Let

$$\mathfrak{G}^2_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r \rangle$$

be obtained from $\mathfrak{G}^1_{\mathfrak{M}}$ by forgetting the topology $\mathcal{T}$. Let

$$\mathfrak{G}^3_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L^R;\ L^T, L^{Ph}, \in, \prec, Bw, \mathcal{T} \rangle$$

be obtained from $\mathfrak{G}^1_{\mathfrak{M}}$ by forgetting L^S and $\perp_r$ and by replacing the universe L with the universe L^R, where $L^R = L^T \cup L^{Ph}$ as defined on p.800. Let

$$\mathfrak{G}^4_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L^R;\ L^T, L^{Ph}, \in, \prec, Bw\rangle$$

be obtained from $\mathfrak{G}^3_{\mathfrak{M}}$ by forgetting the topology $\mathcal{T}$. Let

$$\mathfrak{G}^5_{\mathfrak{M}} :\stackrel{\text{def}}{=} \mathfrak{G}^R_{\mathfrak{M}} = \langle Mn, \mathbf{F_1}, L^R;\ L^T, L^{Ph}, \in, \prec, Bw, g^R, \mathcal{T}^R\rangle$$

be the Reichenbachian version of the geometry $\mathfrak{G}_{\mathfrak{M}}$ defined on p.799.

Now, as we already said above, for various theories Th_1, Th_2 of relativity theory (like $\mathbf{Bax}^-$, $\mathbf{Reich}(\mathbf{Bax})$, etc.) the question whether

$(\star\star)$ $\qquad (\forall \mathfrak{M} \in \mathsf{Mod}(Th_1))[\mathfrak{G}^i_{\mathfrak{M}} \cong \mathfrak{G}^i_{\mathfrak{N}},$ for some $\mathfrak{N} \in \mathsf{Mod}(Th_2)]$ with $i \in 6$

is true, makes sense, and seems interesting for various choices of i and Th_1, Th_2.

Definition 6.6.100 Let Th be a set of formulas in our frame language. Let $i \in 6$. Then we define

$$\mathsf{Ge}^i(Th) :\stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{G}^i_{\mathfrak{M}} \ :\ \mathfrak{M} \models Th\}.$$

$\lhd$

In the style $(\star)$ above, $(\star\star)$ means

$$\mathsf{Ge}^i(Th_1) \subseteq \mathsf{Ge}^i(Th_2).$$

Next we state theorems of style $(\star)$ and $(\star\star)$ above. The next ten theorems say that if we restrict attention to certain reducts of our geometries then the geometries associated to different choices of Th will coincide. The first four of these theorems say that for certain choices of Th the $\mathfrak{G}^0_{\mathfrak{M}}$-geometries of Th coincide with the $\mathfrak{G}^0_{\mathfrak{M}}$-geometries of the symmetric versions of Th, i.e. $\mathsf{Ge}^0(Th) = \mathsf{Ge}^0(Th + \text{some symmetry axioms})$.

In connection with the theorems below recall that, by Thm.6.2.98 (p.910), in models of $\mathbf{Flxbasax}^{\oplus}$ + $\mathbf{Ax}(\sqrt{\ })$ + $\mathbf{Ax(}Triv_t\mathbf{)}^-$ almost all the symmetry axioms are equivalent with one another assuming some auxiliary axioms and $n > 2$, in particular $\mathbf{Ax(syt_0)}$ is equivalent with any one of $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle 1}$+$\mathbf{Ax(eqtime)}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\square 2}$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}$, $\mathbf{Ax(}\omega\mathbf{)}^0$, $\mathbf{Ax(}\omega\mathbf{)}^{00}$, $\mathbf{Ax(}\omega\mathbf{)}^\sharp$, $\mathbf{Ax(}\omega\mathbf{)}^{\sharp\sharp}$.

THEOREM 6.6.101 *For any $Th \in \{$ **Basax**, **Newbasax**, $\mathbf{Flxbasax}^{\oplus}$ $\}$ (i) and (ii) below hold.*

(i) $\mathsf{Ge}^0(Th + \mathbf{Ax}(\sqrt{\ })) = \mathsf{Ge}^0(Th + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)})$.

(ii) *Assume,* $\mathfrak{M} \in \mathsf{Mod}(Th + \mathbf{Ax}(\sqrt{\ }))$. *Then*

$$\mathfrak{G}^0_{\mathfrak{M}} = \mathfrak{G}^0_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(Th + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)}).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.102 *Assume* $n > 2$. *Then for any* $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}^{\oplus}\}$ *and for any* $\mathbf{Ax} \in \{\mathbf{Ax}(\omega)^0, \mathbf{Ax}(\omega)^{00}, \mathbf{Ax(syt_0)}, \mathbf{Ax(symm)}, \mathbf{Ax(speedtime)}, \mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)}, \mathbf{Ax\triangle 2}, \mathbf{Ax\square 2}\}$ *(i) and (ii) below hold.*

(i) $\mathsf{Ge}^0(Th + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ })) = \mathsf{Ge}^0(Th + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax})$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(Th + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }))$. *Then*

$$\mathfrak{G}^0_{\mathfrak{M}} = \mathfrak{G}^0_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(Th + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.103 *Assume* $n > 2$. *Then (i) and (ii) below hold.*

(i) $\mathsf{Ge}^0(\mathbf{Basax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ })) = \mathsf{Ge}^0(\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ }))$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ }))$. *Then*

$$\mathfrak{G}^0_{\mathfrak{M}} \cong \mathfrak{G}^0_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ })).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.104 *Assume* $n > 2$. *Then for any*

$$Th \in \{\mathbf{Reich}(\mathbf{Basax}), \mathbf{Reich}(\mathbf{Newbasax}), \mathbf{Reich}(\mathbf{Flxbasax})^{\oplus}\} \quad \text{and}$$
$$\mathbf{Ax} \in \{\mathbf{R}(\mathbf{Ax\ syt_0}), \mathbf{R}(\mathbf{Ax\ eqsp}), \mathbf{R}^+(\mathbf{Ax\ eqsp}), \mathbf{R}(\mathbf{sym})\}$$

(i) and (ii) below hold.

(i) $\mathsf{Ge}^0(Th + \mathbf{Ax}(Triv)) = \mathsf{Ge}^0(Th + \mathbf{Ax}(Triv) + \mathbf{Ax})$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(Th + \mathbf{Ax}(Triv))$. *Then*

$$\mathfrak{G}^0_{\mathfrak{M}} = \mathfrak{G}^0_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(Th + \mathbf{Ax}(Triv) + \mathbf{Ax}).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.105

(i) $\mathsf{Ge}^1(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ })) = \mathsf{Ge}^1(\mathbf{Newbasax} + \mathbf{Ax}(\sqrt{\ }))$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }))$. *Then*

$$\mathfrak{G}^1_{\mathfrak{M}} = \mathfrak{G}^1_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax} + \mathbf{Ax}(\sqrt{\ })).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.106

(i) $\mathsf{Ge}^2(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ })) = \mathsf{Ge}^2(\mathbf{Newbasax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)})$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }))$. *Then*

$$\mathfrak{G}^2_{\mathfrak{M}} = \mathfrak{G}^2_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)}).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.107

(i) $\mathsf{Ge}^3(\mathbf{Reich}(\mathbf{Bax})^{\oplus}) = \mathsf{Ge}^3(\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }))$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}(\mathbf{Bax})^{\oplus})$. *Then*

$$\mathfrak{G}^3_{\mathfrak{M}} = \mathfrak{G}^3_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ })).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.108

(i) $\mathsf{Ge}^4(\mathbf{Reich}(\mathbf{Bax})^{\oplus}) = \mathsf{Ge}^4(\mathbf{Newbasax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)})$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}(\mathbf{Bax})^{\oplus})$. *Then*

$$\mathfrak{G}^4_{\mathfrak{M}} = \mathfrak{G}^4_{\mathfrak{N}}, \text{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)}).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.109

(i) $\mathsf{Ge}^4(\mathbf{Reich}(\mathbf{Bax})^{\oplus} + \mathbf{Ax(\uparrow\uparrow)}) = \mathsf{Ge}^4(\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ }))$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}(\mathbf{Bax})^{\oplus} + \mathbf{Ax(\uparrow\uparrow)})$. *Then*

$$\mathfrak{G}^4_{\mathfrak{M}} \cong \mathfrak{G}^4_{\mathfrak{N}}, \textit{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ })).$$

The **proof** is available from Judit Madarász. ∎

THEOREM 6.6.110
For any $\mathbf{Ax} \in \{\, \mathbf{R}(\mathbf{Ax\ syt_0}),\ \mathbf{R}(\mathbf{Ax\ eqsp}),\ \mathbf{R}^+(\mathbf{Ax\ eqsp}),\ \mathbf{R}(\mathbf{sym}) \,\}$ *(i) and (ii) below hold.*

(i) $\mathsf{Ge}^5(\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax(\uparrow\uparrow)} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}) = \mathsf{Ge}^5(\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ }))$.

(ii) *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax(\uparrow\uparrow)} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax})$. *Then*

$$\mathfrak{G}^5_{\mathfrak{M}} \cong \mathfrak{G}^5_{\mathfrak{N}}, \textit{ for some } \mathfrak{N} \in \mathsf{Mod}(\mathbf{BaCo} + \mathbf{Ax}(\sqrt{\ })).$$

The **proof** is available from Judit Madarász. ∎

We plan to go into more detail about questions like $(\star\star)$ in a later work related to the present one.

In passing, we note that a variant[1170] of $(\star\star)$ can be formulated in terms of (i) accelerated observers and in terms of (ii) general relativity where the import of $(\star\star)$ would be replacing the principle of locality[1171] with embeddability of finite neighborhoods of certain events into spec. rel. space-time, at the expense of using only a reduct of our geometries involved. We leave the further discussion of these ideas to a later work related to the present one (in the meanwhile we refer to Dávid [70]).

[1170] Using only partial isomorphisms, i.e. of the form $\mathfrak{G}^i_{\mathfrak{M}} \restriction H \cong \mathfrak{G}^i_{\mathfrak{N}} \restriction H'$, for certain $H \subseteq Mn_{\mathfrak{M}}$ and H'. This form of $(\star\star)$ says that $\mathfrak{G}^i_{\mathfrak{M}} \restriction H \rightarrowtail \mathfrak{G}^i_{\mathfrak{N}}$ i.e. $\mathfrak{G}^i_{\mathfrak{M}} \restriction H$ is embeddable into $\mathfrak{G}^i_{\mathfrak{N}}$.
[1171] Using the terminology of nonstandard analysis: infinitely small neighborhoods (of gen. rel. space-time) are embeddable into spec. rel. space-time.

6.6.11 Some properties of our relativistic geometries

In this sub-section we will see that for certain choices of Th, the Th geometries restricted to hyper-planes are either Euclidean or Minkowskian or so called Robb geometries, cf. e.g. Goldblatt [108] for analogous results as well as for the definition of Robb geometries.[1172]

Definition 6.6.111 Assume $\mathfrak{F}$ is Euclidean and $n \geq 1$. By the *Euclidean geometry* over $\mathfrak{F}$ we understand the usual geometric structure

$$Euclgeom(\mathfrak{F}) :\stackrel{\text{def}}{=} Euclgeom(n, \mathfrak{F}) :\stackrel{\text{def}}{=} \langle {}^nF, \mathbf{F_1}, \mathsf{Eucl}(n, \mathbf{F});\ \in, \mathsf{Betw}, \perp, eq, g, \mathcal{T}\rangle,$$

where $g : {}^nF \times {}^nF \longrightarrow F$ is defined by $g : (p, q) \mapsto |p - q|$, $eq \subseteq {}^2({}^nF) \times {}^2({}^nF)$ is defined as

$$\langle p, q\rangle\ eq\ \langle r, s\rangle \stackrel{\text{def}}{\iff} |p - q| = |r - s|,$$

and $\mathcal{T}$ is the usual Euclidean topology on ${}^nF$. Further, $Euclgeom^0(\mathfrak{F})$ is defined to be the $(g, \mathcal{T})$-free reduct of $Euclgeom(\mathfrak{F})$, i.e.

$$Euclgeom^0(\mathfrak{F}) :\stackrel{\text{def}}{=} Euclgeom^0(n, \mathfrak{F}) :\stackrel{\text{def}}{=} \langle {}^nF, \mathsf{Eucl}(n, \mathbf{F});\ \in, \mathsf{Betw}, \perp, eq\rangle.$$

◁

Let $\mathfrak{M}$ be a frame model. Then we define $\mathfrak{G}^6_{\mathfrak{M}}$ to be $(L^T, L^{Ph}, L^S, \prec)$-free reduct of of $\mathfrak{G}_{\mathfrak{M}}$, i.e.

$$\mathfrak{G}^6_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ \in, Bw, \perp, eq, g, \mathcal{T}\rangle.$$

Let

$$\mathfrak{G}^7_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, L;\ \in, Bw, \perp, eq\rangle.$$ [1173]

be the $(g, \mathcal{T})$-free reduct of $\mathfrak{G}^6_{\mathfrak{M}}$. The classes $\mathsf{Ge}^6(Th)$ and $\mathsf{Ge}^7(Th)$ of geometries are defined as in Def.6.6.100, for any Th.

Definition 6.6.112 Assume $\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \rangle$ is an n-dimensional geometry.

(i) Let $H \subseteq Mn$. H is called a *hyper-plane* iff $(\forall a, b \in H) a \sim b$ and H is an n element independent subset of Mn (in the sense of Def.6.6.18).

[1172] They are called Robb threefolds, fourfolds there.

[1173] $\mathfrak{G}^7_{\mathfrak{M}}$ is the same as the Goldblatt-Tarski reduct $GT_{\mathfrak{M}}$ of $\mathfrak{G}_{\mathfrak{M}}$ introduced on p.923.

(ii) A hyper-plane H is called a <u>*space-like hyper-plane*</u> iff $(\forall \ell \in L)(\ell \subseteq H \ \rightarrow \ell \in L^S)$.

(iii) A hyper-plane H is called a <u>*time-like hyper-plane*</u> iff H contains a time-like line, i.e. $(\exists \ell \in L^T)\ell \subseteq H$.

(iv) A hyper-plane H is called a <u>*Robb hyper-plane*</u> iff H contains a photon-like line, and H is not a time-like hyper-plane.

$\lhd$

The next two theorems say that $\mathbf{Bax}^{\oplus}$ geometries restricted to space-like hyper-planes are Euclidean geometries, under certain assumptions. For stating these theorems we need the notion of a new kind of restriction of our geometries to a subset of their points, introduced below.

Definition 6.6.113 Assume $\mathfrak{G} = \langle Mn, \ldots \rangle$ is an observer independent geometry, and $N \subseteq Mn$. Then $\mathfrak{G} \restriction N$ is defined as in Def.6.2.77 (p.882). Further, $\mathfrak{G} \restriction^* N$ is defined to be the geometry which is obtained from $\mathfrak{G} \restriction N$ by replacing the universe $L \restriction N$ of lines with $L^* :\stackrel{\text{def}}{=} \{\, \ell \in L \restriction N \ : \ |\ell| > 1 \,\}$ and by replacing $L^T \restriction N$, $L^{Ph} \restriction N$, $L^S \restriction N$, $\perp_N$ by $(L^T \restriction N) \cap L^*$, $(L^{Ph} \restriction N) \cap L^*$, $(L^S \restriction N) \cap L^*$, $\perp_N \cap (L^* \times L^*)$ respectively.

$\lhd$

THEOREM 6.6.114

$\mathbf{I}\big\{\, Euclgeom^0(n-1, \mathfrak{F}) \ : \ \mathfrak{F} \textit{ is Euclidean} \,\big\} =$
$\qquad = \mathbf{I}\,\{\, \mathfrak{G}^7_{\mathfrak{M}} \restriction^* H \ : \ \mathfrak{M} \in \mathsf{Mod}(Th(n)), \ H \textit{ is a space-like hyper-plane of } \mathfrak{G}_{\mathfrak{M}} \,\}$,
i.e. these two classes of geometries coincide, assuming

$$Th = \mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(||)^- + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\sqrt{\ }).$$

On the proof: For the case $n = 2$ the proof is easy and is left to the reader. Assume $n > 2$. By the proof of Prop.6.2.92 (and the proof of Thm.6.2.10) it is enough to prove the theorem in place of Th (in the formulation of the theorem) for

$$Th' := \mathbf{Newbasax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(||)^- + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\sqrt{\ }).$$

But the $\mathfrak{G}^7_{\mathfrak{M}}$ reducts of Th' geometries are photon-glued disjoint unions of Minkowskian geometries by Thm.6.2.75 (p.879). The remaining part of the proof is left to the reader. ∎

THEOREM 6.6.115 *Assume that Th satisfies the assumptions of Thm.6.6.114 above. Assume $n > 2$. Then*

$$\mathbf{I}\{\, Euclgeom(n-1,\mathfrak{F}) \;:\; \mathfrak{F} \text{ is Euclidean}\,\} = \mathbf{I}\{\, \mathfrak{G}^6_{\mathfrak{M}} \restriction^* H \;:\; \mathfrak{M} \in \mathsf{Mod}(Th(n) + \mathbf{Ax(eqspace)}),\; H \text{ is a space-like hyper-plane of } \mathfrak{G}_{\mathfrak{M}}\,\},$$

i.e. these two classes of geometries coincide.

On the proof: Similarly to the case of Thm.6.6.114 it is enough to prove the theorem for

$$\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\|)^- + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqm)}$$

in place of *Th*. ∎

Our next theorem says that for certain choices of *Th*, the n-dimensional *Th* geometries when restricted to time-like hyper-planes coincide with the $(n-1)$-dimensional *Th* geometries.

THEOREM 6.6.116 *Assume $n \geq 3$ and $Th \in$* {**BaCo**, **Basax** + $\mathbf{Ax}(\omega)^\sharp$, **Basax**, **Newbasax**, **Flxbasax**, **Bax**, **NewtK**, **Relnoph**, **Reich**(**Basax**), **Reich**(**Newbasax**), **Reich**(**Flxbasax**), **Reich**(**Bax**), $\mathbf{Bax}^-$, **Pax**}.

(i) *Assume, $\mathfrak{G} \in \mathsf{Ge}(Th(n))$ and H is a time-like hyper-plane of $\mathfrak{G}$. Then*

$$\mathfrak{G} \restriction^* H \in \mathsf{Ge}(Th(n-1)).$$

(ii) *Assume $n \geq 4$. Then*

$$\mathsf{Ge}(Th(n-1) + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ })) = \{\mathfrak{G} \restriction^* H \;:\; \mathfrak{G} \in \mathsf{Ge}(Th(n) + \mathbf{Ax}(\sqrt{\ })),\; H \text{ is a time-like hyper-plane of } \mathfrak{G}\},$$

i.e. these two classes of geometries coincide.

We omit the **proof**. ∎

We note that the assumption $\mathbf{Ax}(\sqrt{\ })$ is needed in item (ii) of the above theorem, e.g. $\mathbf{Basax}(2) \not\models \mathbf{Ax}(\sqrt{\ })$ while $\mathbf{Basax}(3) \models \mathbf{Ax}(\sqrt{\ })$, cf. Thm.3.6.17 (p.274).

The following is a corollary of Theorems 6.6.116, 6.2.59 (p.861), 6.2.64 (p.866).

COROLLARY 6.6.117 Assume $n \geq 3$. Then (i) and (ii) below hold.

(i) Assume, $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax}(n) + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow))$ and that H is a time-like hyper-plane. Then $\mathfrak{G} \restriction^* H$ is a Minkowskian geometry up to isomorphism, i.e.

$$\mathfrak{G} \restriction^* H \;\cong\; Mink(n-1, \mathfrak{F}),$$

for some Euclidean $\mathfrak{F}$.

(ii) Assume, $\mathfrak{G} \in \mathsf{Ge}^2(\mathbf{Bax}^\oplus + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\uparrow\uparrow))$ and that H is a time-like hyper-plane of $\mathfrak{G}$. Then $\mathfrak{G} \restriction^* H$ is a $(eq, g, \mathcal{T})$-free reduct of a Minkowskian geometry up to isomorphism. ∎

Remark 6.6.118 Assume $n \geq 3$. Let Th be as in Thm.6.6.114. Assume H is a Robb hyper-plane of $\mathfrak{G}$. Then $\mathfrak{G} \restriction^* H$ is a Robb geometry in the sense of Goldblatt [108].

◁

Future research task 6.6.119 Consider the classes $\mathsf{Ge}^i(Th)$ $(i \in 8)$ for our distinguished theories $\mathbf{Bax}^-(n), \ldots, (\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp)(n)$, and $n > 1$. This gives us several classes of geometries.

It would be nice to find axiomatizations (in first-order logic) of the classes $\mathsf{Ge}^i(Th)$ for various choices of i, of Th (and of $n > 1$). Some of these axiomatizations will probably be like axiomatizations obtained by Tarski and his followers cf. e.g. Schwabhäuser-Szmielew-Tarski [237], and Goldblatt [108]. Fig.282 (p.863) and §6.2.9 (p.923) are relevant here.

◁

Question for future research 6.6.120 For which Th's is $\mathsf{Ge}(Th)$ an elementary class? (Here we mean Th to be one of the theories discussed in this work.)

◁

Conjecture 6.6.121 *We conjecture that* $\mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$ *is an elementary class.*

◁

In connection with the above question we note the following. Let Th be fixed. It is easy to see that $\mathsf{Ge}(Th)$ is closed under ultraproducts, since $\mathsf{Mod}(Th)$ is closed under ultraproducts, and the function $\mathcal{G} : \mathfrak{M} \mapsto \mathfrak{G}_{\mathfrak{M}}$, defined on p.1007, commutes over ultraproducts. So to prove that $\mathsf{Ge}(Th)$ is an elementary class it remains to prove that $\mathsf{Ge}(Th)$ is closed under elementary equivalence (actually, being closed under taking ultraroots[1174] is sufficient). We conjecture, if for Th the duality theory, described in §§ 6.6.1, 6.6.3, 6.6.4 works, that is if $\mathcal{M} : \mathsf{Ge}(Th) \to \mathsf{Mod}(Th)$ (cf. Def.6.6.41 on p.1054 and Prop.6.6.47 on p.1062) then $\mathsf{Ge}(Th)$ is closed under elementary equivalence.

[1174]Ultraroots are the "reverse" of ultraproducts.

6.7 Interdefinability questions; on the choice of our geometrical vocabulary (or language $L, L^T, \ldots, g, \mathcal{T}$)

Our $\mathfrak{G}_{\mathfrak{M}}$ has a large number of components. As we have promised in the introduction (§6.1), in the present section we explore how $\mathfrak{G}_{\mathfrak{M}}$ can be streamlined such that it will consist only of a few components and each remaining component will either be definable in terms of these or turn out to be superfluous. Our criteria here are that (i) the theory of the streamlined geometry be simple and perspicuous and (ii) the streamlined geometry be a familiar mathematical structure.[1175] This streamlining will begin with §6.7.2, thus the impatient reader can go directly to §6.7.2. In other words: In this section we will investigate how the various ingredients (i.e. non-logical symbols) of our geometries in $\mathsf{Ge}(Th)$ are *definable from each other.* Among others, this amounts to asking ourselves whether one or another ingredient is superfluous (in presence of the others).

As we said above, the main purpose of the present section is streamlining $\mathfrak{G}_{\mathfrak{M}}$. However, this will be obvious only in the second part of this section (i.e. in §§ 6.7.2, 6.7.3, 6.7.4). Namely, if $\mathfrak{G}_{\mathfrak{M}}$ can be streamlined, if several of its ingredients turn out to be superfluous then the question naturally comes up: Why did we introduce these superfluous ingredients and why do we still keep them around if they are superfluous? To prepare ourselves for answering these kinds of questions, in the first part of the present section we will investigate interdefinability properties of the ingredients of $\mathfrak{G}_{\mathfrak{M}}$. To the above formulated question of why we introduced so many parts of $\mathfrak{G}_{\mathfrak{M}}$ despite of its reducibility (streamlineability) to a few parts only will turn out to be threefold:

(i) It is true that $\mathfrak{G}_{\mathfrak{M}}$ is definable over e.g. its streamlined reduct $\langle Mn, \mathbf{F_1}; g^{\prec}\rangle$ to be introduced on p.1170, but for this we need to assume some axioms in Th. For other streamlinings of $\mathfrak{G}_{\mathfrak{M}}$ we need some other axioms, cf. e.g. Theorems 6.7.20 (p.1157), 6.7.30 p.1164, 6.7.37 (p.1167), 6.7.39 (p.1168), 6.7.47 (p.1172) and Corollaries 6.7.38 (p.1167) 6.7.40 (p.1168). One of the reasons why we do not throw away the ingredients which turn out to be superfluous (e.g. definable over $\langle Mn, \mathbf{F_1}; g^{\prec}\rangle$) in the just quoted theorems is that we are not sure that we want to assume all these conditions on Th throughout our future research activities.

[1175] These two criteria were kept in mind by Tarski and his followers while building up algebraic logic. Cf. §6.6.7.

(ii) It is useful to have, roughly, two definitionally equivalent versions of the structure $\mathfrak{G}_{\mathfrak{M}}$ we want to use. Namely, a streamlined version and a "rich" version. We use the streamlined version when we want to prove some properties of $\mathfrak{G}_{\mathfrak{M}}$ (or properties of its theory). On the other hand, we use the rich version of $\mathfrak{G}_{\mathfrak{M}}$ when we want to apply $\mathfrak{G}_{\mathfrak{M}}$ to some purpose. (The more ingredients of $\mathfrak{G}_{\mathfrak{M}}$ are available, the more likely it is that some of them will be applicable to the purpose in question.)

(iii) Our third reason for keeping $\mathfrak{G}_{\mathfrak{M}}$ rich is summarized in item (8) on p.852, and at the points (of this work) to which we refer from that item (e.g. on p.1069).

* * *

By definability we mean explicit definability in first-order logic without parameters in the sense of §6.3. Our terminology in the present section differs slightly from that of §6.3 [1176] : e.g. if we say that *Bw* is definable from *Col* in $\mathsf{Ge}(Th)$, then this means that there exists a formula β, in which the only non-logical symbol is *Col*, such that

$$\mathsf{Ge}(Th) \models (\forall a, b, c \in Mn)[\beta(a, b, c) \ \leftrightarrow \ Bw(a, b, c)].$$

In the present section the orthogonality symbol $\perp$ denotes both Euclidean and relativistic orthogonality in an ambiguous way, but context will help. Further let us also recall that for a set *Th* of formulas in our frame language we defined

$$\mathsf{Ge}(Th) := \mathbf{I}\{\mathfrak{G}_{\mathfrak{M}} \ : \ \mathfrak{M} \in \mathsf{Mod}(Th)\}.$$

Throughout we will distinguish three cases (when studying geometries similar to $\mathsf{Ge}(\emptyset)$). These are the following:

(i) *Euclidean case:* By a Euclidean geometry we understand an isomorphic copy of the usual geometric structure

$$Euclgeom(\mathfrak{F}) := \langle {}^nF, \mathbf{F_1}, \mathsf{Eucl}(n, \mathbf{F}); \in, \mathsf{Betw}, \perp, eq, g, \mathcal{T}\rangle$$

over an arbitrary Euclidean field $\mathfrak{F}$ defined in Def.6.6.111 on p.1129.

(ii) *Minkowskian case:* By a Minkowskian geometry we understand an isomorphic copy of the Minkowskian geometric structure

$$Mink(\mathfrak{F}) = \langle {}^nF, \mathbf{F_1}, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, \prec_\mu, Bw_\mu, \perp_\mu, eq_\mu, g_\mu, \mathcal{T}_\mu\rangle$$

[1176]The present terminology remains consistent with that of §6.3. The difference is that it will be more specialized to certain purposes of the present section.

constructed in Def.6.2.58 on p.859 from an arbitrary Euclidean field $\mathfrak{F}$.

(iii) *General case:* By the general case we understand investigations of the classes of the form $\mathsf{Ge}(Th)$, where Th ranges over our distinguished theories **Pax**, $\mathbf{Bax}^-$, ..., $\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$.

It is important to recall that the Minkowskian case is a special part of our general case, moreover "Minkowskian geometries" $= \mathsf{Ge}(\mathbf{Basax}+\mathbf{Ax}(\omega)^\sharp+\mathbf{Ax}(\uparrow\uparrow))$, assuming $n > 2$, cf. Thm.6.2.59 (p.861). As a contrast, the Euclidean case is not a part of the general case, unless we omit everything except L, Bw and $\mathcal{T}$ (cf. item 3 on p.1147). However, if we restrict the geometries in $\mathsf{Ge}(Th)$ to space-like hyperplanes then the $(\prec, L^T, L^{Ph}, L^S)$-free reducts of our geometries will turn out to be Euclidean geometries, under some assumptions on Th, cf. Thm.6.6.115 (p.1131).

6.7.1 On Col, Bw, $\perp$, eq, g

In what follows we will use L and Col interchangeably since we have seen that in most cases they are definitionally equivalent[1177], cf. Theorems 6.5.3 (p.993), 6.5.5 (p.996).[1178] For the definition of Col we refer to pp. 992, 996 in §6.5. In this sub-section we will concentrate on the sublanguage $\langle L, Bw, \perp, eq, g\rangle$ or equivalently $\langle Col, Bw, \perp, eq, g\rangle$. For completeness we note that this sublanguage is the language of the geometric model $\langle Mn, \mathbf{F_1}; Col, Bw, \perp, eq, g\rangle$ (if we disregard the language of $\mathbf{F_1}$). The reason for concentrating first on this sublanguage is, that this sublanguage makes sense in all three of the Euclidean, the Minkowskian, and in the present more general (i.e. $\mathsf{Ge}(Th)$-style) case, i.e. in all three cases (i)–(iii) discussed above. We should have included the topology $\mathcal{T}$ into this sub-language,[1179] but to save space we will discuss $\mathcal{T}$ only very briefly and tangentially, e.g. on p.1158 (cf. also the discussion in $(\star\star\star)$ of Remark 6.2.8 on p.809). In passing we note that $\mathcal{T}$ is definable over $\langle Mn, \mathbf{F_1}; g\rangle$.

(1) On (definability from) *Col*

First let us consider the question, whether from the simplest reduct $\langle Points, Lines; \in\rangle$ or equivalently $\langle Points; Col\rangle$ of our geometries any of the re-

[1177] i.e. they are definable from each other (they are interdefinable)

[1178] E.g. L and Col are definitionally equivalent in $\mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(diswind)})$.

[1179] because $\mathcal{T}$ too makes sense in all three cases. Actually, $\langle Mn, \mathbf{F_1}; Col, Bw, \perp, eq, g, \mathcal{T}\rangle$ is the maximal reduct of our geometries making sense in all three cases.

maining ingredients $Bw, \perp, eq, g$ is definable. Since Bw is the simplest (in some sense) of these extra ingredients, first we ask ourselves if Bw is definable from $\langle Points, Lines; \in\rangle$.

Our first theorem says that in $\mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$ Bw is indeed definable from $\langle Points, Lines; \in\rangle$. (Recall that **Pax** is weaker than $\mathbf{Bax}^-$.)

THEOREM 6.7.1 *Betweenness (Bw) is first-order definable from the collinearity relation (Col) in* $\mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$, *i.e. there is a first-order formula* $\beta(x, y, z)$ *in the language of Col explicitly defining Bw.*

Outline of proof: The idea of the proof is depicted in Fig.335. First we define *coll* from *Col*, by using Fig.344 (p.1162). (We note that $Col \subseteq coll$, while $Col \supseteq coll$ does not necessarily hold since *coll* was defined by *Bw* and *Col* was defined by *L*.) Then in $\langle Mn; coll\rangle$ we define the new sort *lines* together with the incidence relation $\in$ as they were defined on p.1037. Then we define the ternary relation H on Mn as follows, cf. Figure 335. (Intuitively, $H(a, b, c)$ means that c is on the half-line with origin a and containing b.) Let $a, b, c \in Mn$. Then

$$H(a,b,c)$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists \ell, \ell' \in lines)[\, a, b, c \in \ell \ \ \wedge \ \ \{a\} = \ell \cap \ell' \ \ \wedge$$
$$(\exists d \in \ell)(\exists b', d' \in \ell')(\langle b, b'\rangle \parallel \langle d, d'\rangle \ \wedge \ \langle d, b'\rangle \parallel \langle c, d'\rangle)\,],$$

see Figure 335. Finally Bw is defined as follows.

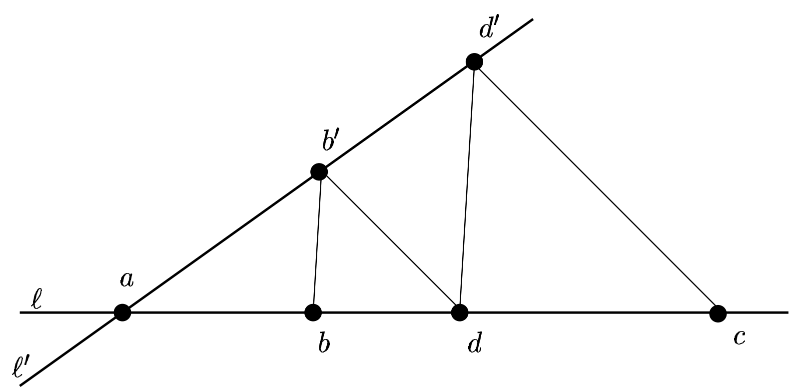


Figure 335: Illustration for the proof of Thm.6.7.1. c is on the half-line $\vec{\ell}_{ab}$.

$$Bw(a,b,c) \overset{\text{def}}{\Longleftrightarrow} [\, a \neq b \neq c \neq a \ \ \wedge \ \ coll(a,b,c) \ \ \wedge \ \ \neg H(b,c,a)\,]. \ \blacksquare$$

For completeness we note that condition $\mathbf{Ax}(\sqrt{\ })$ in Thm.6.7.1 is needed: Bw is not definable from $\langle Col, \perp, eq\rangle$ e.g. in $\mathsf{Ge}(\mathbf{Basax}(2))$ and in $\mathsf{Ge}(\mathbf{Flxbasax}(n))$, cf. Thm.6.7.13 on p.1143. (We note that Bw is not definable from $\langle Col, \perp, eq\rangle$ in $\mathfrak{G}_{\mathfrak{M}^+}$, where the counterexample $\mathfrak{M}^+ \in \mathsf{Mod}(\mathbf{Basax}(2))$ was constructed in the proof of Thm.2.7.3 on p.111.)

Question for future research 6.7.2

(i) Does Thm.6.7.1 remain true if the assumption **Ax(diswind)** is omitted?

(ii) If the answer to (i) turns out to be "NO" then we ask the following. Does Thm.6.7.1 remain true if the assumption **Ax(diswind)** is omitted and the assumption **Pax** is replaced by $\mathbf{Bax}^-$ or $\mathbf{Bax}^{-\oplus}$ or $\mathbf{Reich}(\mathbf{Bax})^{\oplus}$?

◁

COROLLARY 6.7.3 [1180] The relation Bw of betweenness is uniformly first-order definable from $\langle Points, Lines; \in\rangle$ both in the Euclidean and in the Minkowskian cases.[1181] Actually, the explicit definition $\beta(x, y, z)$ mentioned in Thm.6.7.1 works here, too.

The **proof** is available from Judit Madarász. ∎

THEOREM 6.7.4 *Assume* $\mathfrak{F} = \langle \mathbf{F}; \leq\rangle$ *is an ordered field and* $n \geq 2$. *Let* $\mathfrak{A}(n, \mathfrak{F}) = \langle {}^nF;\ Col, Bw\rangle$ *be the usual n-dimensional Cartesian geometry over* $\mathfrak{F}$, *i.e.* Bw *and* Col *are* Betw *and* $coll_{\mathbf{F}}$, *respectively (for the latter cf. p.1040). Then*

$$Bw \text{ is definable by a first-order formula from } Col \text{ in } \mathfrak{A}(n,\mathfrak{F}).$$
$$\Updownarrow$$
$$\leq \text{ is definable by a first-order formula from } \mathbf{F} = \langle F;\ 0, 1, +, \cdot\rangle \text{ in } \mathfrak{F}.$$

The **proof** will be given on p.1139.

The Minkowskian geometry $Mink_{nonE}(n, \mathfrak{F})$ over an arbitrary ordered field $\mathfrak{F}$ will be defined on p.1160 in Def.6.7.25. The above theorem implies that

$$Bw \text{ is definable by a first-order formula from } Col \text{ in } Mink_{nonE}(n,\mathfrak{F}).$$
$$\Updownarrow$$
$$\leq \text{ is definable by a first-order formula from } \mathbf{F} = \langle F;\ 0, 1, +, \cdot\rangle \text{ in } \mathfrak{F}.$$

[1180] This is basically Theorem 1 of Royden [227]. Cf. also Lenz [164].

[1181] The relation of betweenness was denoted by Betw in the Euclidean case and it was denoted by Bw_μ in the Minkowskian case.

We will give two proofs for Thm.6.7.4. The first proof will be based on Lemma 6.7.5 below, while the second one will use the "coordinatization procedure" recalled from the literature in §6.6.2. We will give the proofs after Lemma 6.7.5 below. We note that Thm.6.7.4 does not generalize to $n = 1$ because of the following. For every ordered field $\mathfrak{F}$, every permutation of the universe F of $\mathfrak{F}$ preserve the structure $\langle F;\ Col\rangle$, but obviously there are permutations of F which do not preserve Bw, e.g. each transposition is such. So, by the above argument, for every $\mathfrak{F}$ we have that Bw is not definable from $\langle F, Col\rangle$; and clearly there is an ordered field $\mathfrak{F}$ in which $\leq$ is definable from the field reduct $\mathbf{F}$ of $\mathfrak{F}$, e.g. each Euclidean $\mathfrak{F}$ is such.

LEMMA 6.7.5 (Definability and ultraproducts) *Let $\mathfrak{M} = \langle \mathfrak{M}_0;\ R\rangle$ be a (possibly many-sorted) model of first-order logic,*[1182] *where R is a distinguished relation of $\mathfrak{M}$. Then (i) and (ii) below hold.*

(i) *If R is not (first-order) definable*[1183] *from $\mathfrak{M}_0$, then*

($\star$) *there is an ultrapower*[1184] *${}^{\mathrm{I}}\mathfrak{M}/U = \langle {}^{\mathrm{I}}\mathfrak{M}_0/U;\ {}^{\mathrm{I}}R/U\rangle$ and an automorphism h of ${}^{\mathrm{I}}\mathfrak{M}_0/U$ such that h is not an automorphism of ${}^{\mathrm{I}}\mathfrak{M}/U$ (i.e. $\widetilde{h}[{}^{\mathrm{I}}R/U] \neq {}^{\mathrm{I}}R/U$).*

(ii) *R is definable from $\mathfrak{M}_0$ iff statement ($\star$) above fails, i.e. iff for all ultrafilters U, every automorphism of ${}^{\mathrm{I}}\mathfrak{M}_0/U$ is an automorphism of ${}^{\mathrm{I}}\mathfrak{M}/U$, too.*

On the proof: The proof is based on the Keisler-Shelah isomorphic ultrapowers theorem as stated e.g. in Chang-Keisler [59] and on Beth's definability property. (We note that the Keisler-Shelah theorem is used two times in the proof.) The proof is available from Judit Madarász. ∎

Proof of Thm.6.7.4: As we already said, we will give two proofs for this theorem.
First proof: Let $n \geq 2$. For every ordered field $\mathfrak{F}$ let

$$\mathfrak{A}(n,\mathfrak{F}) :\stackrel{\text{def}}{=} \langle {}^{n}F;\ Col, Bw\rangle \qquad \text{and} \qquad \mathfrak{B}(n,\mathfrak{F}) :\stackrel{\text{def}}{=} \langle {}^{n}F;\ Col\rangle.$$

Claim **6.7.6** For every ultrafilter U on a set I (i.e., $U \subseteq \mathcal{P}(\mathrm{I})$) we have

$${}^{\mathrm{I}}\mathfrak{A}(n,\mathfrak{F})/U \cong \mathfrak{A}(n, {}^{\mathrm{I}}\mathfrak{F}/U) \qquad \text{and} \qquad {}^{\mathrm{I}}\mathfrak{B}(n,\mathfrak{F})/U \cong \mathfrak{B}(n, {}^{\mathrm{I}}\mathfrak{F}/U).$$

[1182] The notation $\mathfrak{M} = \langle \mathfrak{M}_0;\ R\rangle$ means that $\mathfrak{M}_0$ is the R-free reduct of $\mathfrak{M}$.
[1183] We mean explicit definability by a single first-order formula.
[1184] If $\mathfrak{M}$ is a model then ${}^{\mathrm{I}}\mathfrak{M}/U$ denotes the ultrapower of $\mathfrak{M}$, where U is an ultrafilter over the index set I, i.e. $U \subseteq \mathcal{P}(I)$, cf. e.g. Chang-Keisler [59] or Enderton [82].

Claim 6.7.6 follows from Lemma 6.7.27 way below (p.1160).

QED (Claim 6.7.6)

Claim **6.7.7** For every ordered field $\mathfrak{F} = \langle \mathbf{F}; \leq \rangle$ we have

$$\begin{aligned} Aut(\mathfrak{A}(n,\mathfrak{F})) &= \{A \circ \widetilde{\varphi} \,:\, A \in Aftr(n,\mathbf{F}) \;\wedge\; \varphi \in Aut(\mathfrak{F})\}, \text{ and} \\ Aut(\mathfrak{B}(n,\mathfrak{F})) &= \{A \circ \widetilde{\varphi} \,:\, A \in Aftr(n,\mathbf{F}) \;\wedge\; \varphi \in Aut(\mathbf{F})\}. \end{aligned}$$

Claim 6.7.7 follows from Lemma 3.1.6 on p.163.

QED (Claim 6.7.7)

Let $\mathfrak{F}$ be an ordered field. Then:

Bw is definable from *Col* in $\mathfrak{A}(n,\mathfrak{F})$.

$\Updownarrow$ (by Lemma 6.7.5 and Claim 6.7.6)

For every ultrapower ${}^{\mathrm{I}}\mathfrak{F}/U$ of $\mathfrak{F}$ every automorphism of $\mathfrak{B}(n,{}^{\mathrm{I}}\mathfrak{F}/U)$ is an automorphism of $\mathfrak{A}(n,{}^{\mathrm{I}}\mathfrak{F}/U)$.

$\Updownarrow$ (by Claim 6.7.7)

For every ultrapower ${}^{\mathrm{I}}\mathfrak{F}/U$ of $\mathfrak{F}$ every automorphism of the field reduct ${}^{\mathrm{I}}\mathbf{F}/U$ of ${}^{\mathrm{I}}\mathfrak{F}/U$ is order preserving, that is $Aut({}^{\mathrm{I}}\mathbf{F}/U) = Aut({}^{\mathrm{I}}\mathfrak{F}/U)$.

$\Updownarrow$ (by Lemma 6.7.5)

$\leq$ is definable from $\mathbf{F}$ in $\mathfrak{F}$.

By the above Thm.6.7.4 is proved.

Second proof: This proof is based on Thm.6.6.29 (p.1045) and Prop.6.6.38 (p.1052). The details are left to the reader. ∎

Recall that Thm.6.7.1 says that *Bw* is uniformly first-order definable from *Col* assuming $\mathsf{Ge}(\mathbf{Pax}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(diswind)})$. Thm.6.7.8 below says that the assumption $\mathbf{Ax}(\sqrt{\ })$ becomes superfluous if in Thm.6.7.1 we replace **Pax** by the stronger **Newbasax** and uniform definability by weaker one-by-one definability.

THEOREM 6.7.8 *Assume $n > 2$ and $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$. Then betweenness Bw is first-order definable from the relation Col of collinearity in $\mathfrak{G}$. I.e. in $\mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$ Bw is one-by-one definable[1185] from Col, if $n > 2$.*

[1185] Cf. p.951 for the notion of one-by-one definability.

On the proof: Thm.6.7.8 can be considered as a kind of corollary of Thm.6.7.4 (p.1138) and Thm.6.7.10 below (since *coll* is definable from *Col* by the proof of Thm.6.7.1, cf. also Fig.344 on p.1162). ∎

In connection with Thm.6.7.8 we include the following question.

QUESTION 6.7.9 *Assume* $n > 2$. *Consider the class* Ge(**Newbasax** + **Ax(diswind)**) *of geometries. Is Bw uniformly first-order definable by a single formula from Col in this class?*[1186]

In connection with the above question see Item 6.7.44 on p.1169.

THEOREM 6.7.10 *Assume,* $n > 2$ *and* $\mathfrak{M} \models$ **Newbasax**. *Let* $\mathfrak{F}^{\mathfrak{M}} = \langle \mathbf{F}^{\mathfrak{M}}; \leq \rangle$ *be the ordered field corresponding to* $\mathfrak{M}$. *Then* $\leq$ *is first-order definable from* $\mathbf{F}^{\mathfrak{M}}$.

On the proof: The proof goes by contradiction. Assume $n > 2$. Let $\mathfrak{M}$ be a model of **Newbasax** such that $\leq$ is not first-order definable from $\mathbf{F}^{\mathfrak{M}}$. Then, by Lemma 6.7.5 above (p.1139), there is an ultrafilter U such that the ultrapower $\mathfrak{F}_U = \langle \mathbf{F}_U; \leq \rangle$ of $\mathfrak{F}^{\mathfrak{M}} = \langle \mathbf{F}^{\mathfrak{M}}; \leq \rangle$ according to ultrafilter U has the following property: $\mathbf{F}_U$ has an automorphism φ which is not order preserving, i.e. φ is not an automorphism of $\mathfrak{F}_U$. Let such U, φ be fixed. Let $\mathfrak{M}_U$ be obtained by taking the ultrapower of $\mathfrak{M}$ according to ultrafilter U. Clearly $\mathfrak{M}_U \models$ **Newbasax** and the ordered field corresponding to $\mathfrak{M}_U$ is $\mathfrak{F}_U$. Now one can use the (non order preserving) automorphism φ and the model $\mathfrak{M}_U$ to construct a model $\mathfrak{M}_U^+$ of **Newbasax** such that in $\mathfrak{M}_U^+$ FTL observers exist. But this contradicts to Thm.3.4.2 on p.204 which says that **Newbasax** does not allow FTL observers.

Construction of $\mathfrak{M}_U^+$ *from* $\mathfrak{M}_U = \langle (B; Obs, Ph, Ib), \mathfrak{F}_U, \mathsf{Eucl}(n, \mathbf{F}_U); \in, W \rangle$:
Let $m_0 \in Obs$ be arbitrary, but fixed. Let

$$H :\stackrel{\text{def}}{=} \{\ell \in (\mathsf{Eucl}(n, \mathfrak{F}_U) \setminus \mathsf{SlowEucl}) \ : \ \widetilde{\varphi}[\ell] \in \mathsf{SlowEucl}\}.$$

We note that $H \neq \emptyset$ by Lemma 6.6.6 (p.1028) since φ is not order preserving. Let

$$Obs^+ :\stackrel{\text{def}}{=} Obs \cup H.$$

Let $k \in Obs^+$. We define w_k^+ as follows:
Case 1: $k \in Obs$ and $\neg(m_0 \stackrel{\odot}{\to} k)$. Then $w_k^+ :\stackrel{\text{def}}{=} w_k$.
Case 2: $k \in Obs$ and $m_0 \stackrel{\odot}{\to} k$. Then

$$(\forall p \in {}^nF)\ w_k^+(p) :\stackrel{\text{def}}{=} w_k(p) \cup \{\ell \in H \ : \ p \in \mathsf{f}_{m_0k}[\ell]\}.$$

[1186] To avoid misunderstandings we note that, as we have already said, by definability we mean uniform first-order definability, cf. §6.3.

Case 3: $k \in H$. Then $\widetilde{\varphi}[k] \in \mathsf{SlowEucl}$, hence $\widetilde{\varphi}[k] = tr_{m_0}(m)$, for some $m \in Obs$ by **Ax5**. Let such an m be fixed. Now,

$$\begin{aligned} \mathsf{f}^+_{m_0 k} &\stackrel{\text{def}}{:=} \widetilde{\varphi} \circ \mathsf{f}_{m_0 m}, \text{ and} \\ w^+_k &\stackrel{\text{def}}{:=} (\mathsf{f}^+_{m_0 k})^{-1} \circ w^+_{m_0}; \end{aligned}$$

where $w^+_{m_0}$ is defined in Case 2. Further $B^+ \stackrel{\text{def}}{:=} B \cup H$, $Ib^+ \stackrel{\text{def}}{:=} Ib \cup H$, and W^+ is defined from w^+_k's the obvious way. Now,

$$\mathfrak{M}^+_U \stackrel{\text{def}}{:=} \langle (B^+;\ Obs^+, Ph, Ib^+), \mathfrak{F}_U, \mathsf{Eucl}(n, \mathbf{F}_U);\ \in, W^+ \rangle.$$

One has to check that $\mathfrak{M}^+_U \models$ **Newbasax**. Clearly, there are FTL observers in $\mathfrak{M}^+_U$, e.g. each observer in $Obs^+ \setminus Obs$ moves FTL for observer m_0. ∎

QUESTION 6.7.11 *Does Thm.6.7.10 above generalize from* **Newbasax** *to* $\mathbf{Bax}^{\oplus}$*? More concretely: Assume* $n > 2$ *and* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(diswind)})$*. Let* $\mathfrak{F}^{\mathfrak{M}} = \langle \mathbf{F}^{\mathfrak{M}}; \le \rangle$ *be the ordered field corresponding to* $\mathfrak{M}$*. Is* $\le$ *definable from* $\mathbf{F}^{\mathfrak{M}}$*?*

If the answer to the above question turned out to be "YES" then Bw would be definable from Col in every $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(diswind)})$. We note that Bw is definable from Col in $\mathsf{Ge}(\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$ by Thm.6.7.1 (p.1137), and for every $n > 1$ there is $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax} + \mathbf{Ax(diswind)})$ such that Bw is not definable from Col in $\mathfrak{G}$ by Thm.6.7.13 below.

The following is a corollary of Thm.6.7.10 above.

COROLLARY 6.7.12 Assume $n > 2$. Let $\mathfrak{M} \models$ **Newbasax**. Let $\mathfrak{F}^{\mathfrak{M}} = \langle \mathbf{F}^{\mathfrak{M}}; \le \rangle$ be the ordered field corresponding to $\mathfrak{M}$. Then $Aut(\mathfrak{F}^{\mathfrak{M}}) = Aut(\mathbf{F}^{\mathfrak{M}})$, that is each automorphism of $\mathbf{F}^{\mathfrak{M}}$ is order preserving. ∎

For completeness we note the following. The geometry $Mink_{nonE}(n, \mathfrak{F})$ will be defined on p.1160 in Def.6.7.25. Assume $n > 2$. Now, Thm.6.7.10 implies that not every ordered field $\mathfrak{F}$ can be the reduct of a **Newbasax** model. Hence, if we start out from an arbitrary field $\mathfrak{F}$ and construct the Minkowski style geometry $Mink_{nonE}(n, \mathfrak{F})$, then $Mink_{nonE}(n, \mathfrak{F}) \notin \mathsf{Ge}(\mathbf{Newbasax})$ may happen. These observations are relevant to Question 3.6.19 and Thm.3.6.17 on p.274. We leave it as an exercise to experiment with searching for models $\mathfrak{M}$ such that $\mathfrak{G}_{\mathfrak{M}} = Mink_{nonE}(n, \mathfrak{F})$. What are the properties of $\mathfrak{M}$ if $\mathfrak{F}$ is not Euclidean? (Hint: $\mathfrak{M}$ may be strange, it may even not exist.)

THEOREM 6.7.13 *For every $n \geq 2$ there is $\mathfrak{G} \in$ Ge(**Flxbasax**(n) + **Ax(diswind)**) such that Bw is not definable from $\langle Col, \perp, eq\rangle$ in $\mathfrak{G}$. Moreover Bw is not definable from the rest of $\mathfrak{G}$.*

Outline of proof: Let $n \geq 2$. Let $\mathfrak{F} = \langle \mathbf{F}; \leq\rangle$ be an ordered field such that there is an automorphism φ of **F** which is not order preserving. Let such a φ be fixed.

Now, we construct a model $\mathfrak{N}$ of **Flxbasax** + $c = \infty$ + **Ax(ext)** over $\mathfrak{F}$ such that all the f_{mk}'s are affine and we include all possible observers into this model with all possible choices of unit-vectors. (Therefore observers with their clocks running backwards are also included.) Now, φ induces an automorphism of the *Bw*-free reduct of $\mathfrak{G}_{\mathfrak{N}}$ which does not preserve $Bw_{\mathfrak{N}}$. Hence *Bw* is not definable from the *Bw*-free reduct of $\mathfrak{G}_{\mathfrak{N}}$ in $\mathfrak{G}_{\mathfrak{N}}$. ∎

Remark 6.7.14 What we write in item (1) (of §6.7) about definability of *Bw* from *Col* (e.g. in geometries of **Pax** + **Ax(diswind)** + **Ax**($\sqrt{\ }$)) can be considered as a (modest) generalization of Theorem 1 of Royden [227]. Cf. also Lenz [164].

◁

We note that in all three cases ((i) the Euclidean case, (ii) the Minkowskian case, and (iii) the general case) neither $\perp$ nor *eq* is definable from $\langle Points; Col\rangle$. Similarly g is not definable either. As a contrast we will see on p.1151 that $\mathbf{F_1}$ is definable over $\langle Points; Col\rangle$ under some assumptions.

Intuitive summary of item (1): Under some reasonable conditions *Bw* and $\mathbf{F_1}$ are definable over the structure $\langle Points; Col\rangle$, while no one of the rest $(\perp, eq, g)$ of the list $Col, Bw, \perp, eq, g$ addressed in the title of §6.7.1 is definable from *Col*.

(2) On *Col*, *Bw*, $\perp$, *eq*

The sublanguage $\langle Col, Bw, \perp, eq\rangle$ or equivalently $\langle L, Bw, \perp, eq\rangle$ was introduced and used already by Hilbert, Tarski, and their followers (as we have already mentioned, Tarski used *Col* in place of *L*).[1187] Actually, we called $\langle Mn; Col, Bw, \perp, eq\rangle$ the Goldblatt-Tarski reduct of our geometry on p.923. Hilbert and Tarski did not include $\perp$ into the basic vocabulary, because *in the Euclidean case*, $\perp$ is *definable* from *eq*.[1188] We recall from the literature that this can be seen as follows: First

[1187] As we indicated before we treat *Col* and *L* as equivalent concepts, hence we use them interchangeably. When we use *Col* then $\perp$ is defined between pairs of points, i.e. it is a 4-ary relation on the set of points.

[1188] In passing we note, that perhaps the simplest first-order language for Euclidean geometry is that of the structure $\langle Points; eq\rangle$. Namely, *Col*, *Bw*, and $\perp$ are first-order definable from $\langle Points; eq\rangle$ hence Tarski's axiom system can be written up as a theory about these very simple structures. (On the other hand, *eq* is not definable in $\langle Points; Col, Bw\rangle$ and in $\langle Points, Lines; \in, Bw\rangle$.)

one defines *Col* (in terms of *eq*) as it is illustrated in Figure 336 (cf. e.g. Tarski-Givant [254]), and then using *Col* and *eq* one defines $\perp$ as it is shown in Figure 337. We leave it as an exercise to the reader to show that the definition in Fig.337 does not work in the case of Minkowskian geometries.

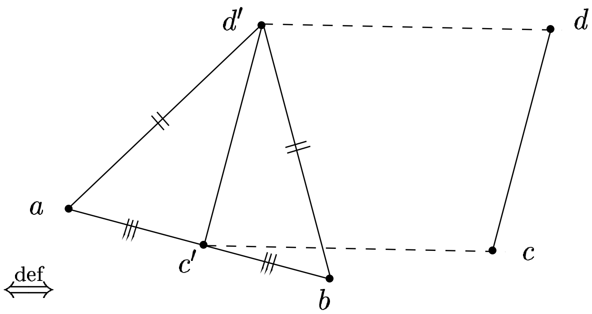
$$Col(a,b,c) \stackrel{\text{def}}{\Longleftrightarrow} (\forall x)\ [eq(a,b,a,x)\ \wedge\ eq(c,b,c,x))\ \Rightarrow\ x = b]$$

Figure 336: Definition of *Col* from *eq* in Euclidean geometry.

For completeness we note that $\perp$ is definable from *eq* in the Minkowskian case for $n > 2$,[1189] and this generalizes to $\mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)^-)$ for $n > 2$.[1190] So why do we not throw $\perp$ away? The answer is threefold: (i) $\perp$ is not definable from *Col*, *Bw* and *eq* in the class $\mathsf{Ge}(\emptyset)$ of _all_ our geometries moreover it is not definable from *Col*, *Bw* and *eq* even in the smaller class $\mathsf{Ge}(\mathbf{Bax})$. (ii) We want to keep compatibility with Goldblatt [108] and there $\perp$ is a basic symbol. (iii) We want to consider reducts of $\mathfrak{G}_{\mathfrak{M}}$ without *Bw* and *eq* in which $\perp$ is still available.[1191] The relation $\perp$ is _not_ definable from $\langle Mn, L; \in, Bw\rangle$ uniformly even in $\mathsf{Ge}(\mathbf{BaCo} + \mathbf{Ax(rc)})$. The same applies for the Euclidean case. So, we include $\perp$ into our language.

(Let us recall that in this case $\perp$ is a 4-ary relation on points.) It is not hard to see that *Col* is definable from $\langle Points; \perp\rangle$ in the Euclidean case, Minkowskian case and some of our general cases,[1192] e.g. in the case of $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- +$

[1189] We conjecture that this generalizes to $n = 2$, too.
[1190] This holds by Corollary 6.7.38 on p.1167.
[1191] One of the points in looking at reducts $\langle Mn, L; \in, \perp\rangle$ is that they are compatible with the structures in Goldblatt [108, §2.3, p.36].
[1192] A possible definition of *Col* from $\perp$ is the following: $Col(a,b,c) \stackrel{\text{def}}{\Longleftrightarrow} (\forall d)(\langle a,b\rangle \perp \langle a,d\rangle \leftrightarrow$



$$\langle a,b\rangle \perp \langle c,d\rangle \stackrel{\text{def}}{\Longleftrightarrow}$$

$$(\exists c',d')\,[\,\overline{cc'} \parallel \overline{dd'} \;\wedge\; \overline{cd} \parallel \overline{c'd'} \;\wedge\; Col(a,c',b) \;\wedge\; eq(a,c',b,c') \;\wedge\; eq(a,d',b,d')\,]$$

Figure 337: Definition of $\perp$ from *Col* and *eq* in Euclidean geometry.

Ax(diswind) + **Ax**($\sqrt{\ }$)) for $n > 2$ (the latter holds by Thm.6.2.71 on p.877).

COROLLARY 6.7.15 *Col* and $\perp$ are definable from *eq* in Ge(**Basax** + **Ax(**$Triv_t$**)**$^-$ + **Ax**($\sqrt{\ }$)), as well as in Euclidean and in Minkowskian geometry; assuming $n > 2$.[1193]

As a contrast we include the following proposition.

PROPOSITION 6.7.16 *The relation Col of collinearity is* <u>*not*</u> *definable from the 4-ary relation* $\perp$ *on points in*

$$\mathsf{Ge}(\mathbf{Reich}(\mathbf{Basax}) + \mathbf{R}(\mathbf{sym}) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\parallel)).$$

The **proof** is available from Judit Madarász. ∎

We conjecture that in Ge(**Bax**) *Col* is not definable from $\perp$. Further, definability of *Bw* from *Col* was discussed in item (1) above. Assume **Bax**$^\oplus$ + **Ax(**$Triv_t$**)**$^-$ + **Ax(diswind)** + **Ax**($\sqrt{\ }$) and $n > 2$. Since <u>now</u> *Col* is definable from $\perp$; and *Bw* is definable from *Col* (cf. item 1), we conclude that *Bw* is definable from $\perp$. This is so in the Euclidean geometry, too. A direct definition of *Bw* from $\perp$ and *Col* for the Euclidean case will be shown in Figure 338 below.

The question which remains to be discussed (in item 2) is whether *eq* is definable from $\perp$ and *Col*. We turn to this question now.

$\langle a,c\rangle \perp \langle a,d\rangle$).

[1193] It would be nice to know what happens if n=2.

Bw :

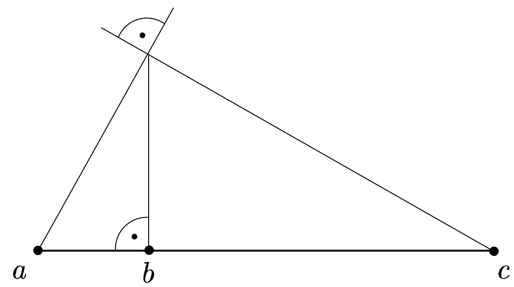
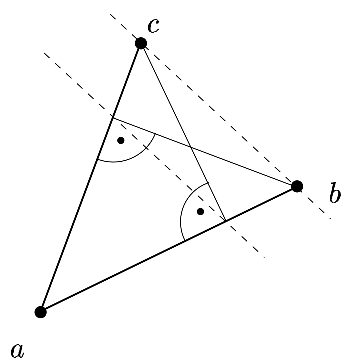


Figure 338: Definition of Bw from L and $\perp$ in Euclidean geometry.

<u>*Euclidean case:*</u> eq is definable from $\langle Points; \perp\rangle$. Instead of prooving this we include Figure 339.

eq :



for $eq(a, b, a, c)$.

The rest uses only parallel lines.

Figure 339: Definition of eq from L and $\perp$ in Euclidean geometry.

<u>*Minkowskian case:*</u> eq is definable from $\langle Points; \perp\rangle$ for $n > 2$, and we strongly conjecture that this holds for $n = 2$, too.

<u>*The general case:*</u> eq is definable from $\langle Mn, L; \in, \perp\rangle$ even in $\mathsf{Ge}(\mathbf{Basax}+\mathbf{Ax(}Triv_t\mathbf{)}^{-}+\mathbf{Ax}(\sqrt{\ }))$ by Corollary 6.7.41. However eq is not definable from $\langle Mn, L; \in, Bw, \perp\rangle$ in $\mathsf{Ge}(\mathbf{Bax}^{\oplus})$.

We will return to definability <u>from</u> $\perp$ and eq at the end of §6.7.2 in items 6.7.38, 6.7.41. We will see that almost everything is definable from any one of these two under some assumptions.

In the summary below, we will indicate that there are considerable physical consequences of the above investigations: E.g. we obtain information on how one should choose the basic concepts of our theoretical model(s) of the physical world and what the consequences of such a choice are.

Summing up (of items 1 and 2): We need to keep eq in our language because it is not definable from $\langle L, Bw, \perp\rangle$ in our general case $\mathsf{Ge}(\mathbf{Bax}^{\oplus})$.[1194] On the other hand, we keep Bw because it is not definable from $\langle Col, \perp, eq\rangle$ e.g. in $\mathsf{Ge}(\mathbf{Basax}(2))$ and in $\mathsf{Ge}(\mathbf{Flxbasax})$ (cf. Thm.6.7.13). A further reason for keeping Bw is that it is a more "primitive/elementary" concept than eq or $\perp$ (in some sense cf. e.g. Goldblatt [108]), hence at some point, we might want to consider the reduct $\langle Points, Lines; \in, Bw\rangle$ without $\perp$. In this connection, we recall that $\perp$ is not definable from $\langle Points, Lines; \in, Bw\rangle$ in practically all non-trivial cases, e.g. in the Euclidean case or in the Minkowskian case.[1195] To see this consider e.g. a usual geometry over the real field $\mathfrak{R}$ and a linear transformation which does not preserve $\perp$. These considerations lead up to the subject of the following item.

(3) The reduct $\langle Mn, L; \in, Bw\rangle$

The point in looking at the reduct

$$G^E_{\mathfrak{M}} = \langle Mn, L; \in, Bw\rangle$$

is that at this level of abstraction Euclidean geometry and relativistic geometries do not get separated. More precisely assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ }))$. Then $G^E_{\mathfrak{M}}$ is a reduct of the Euclidean geometry over the field $\mathfrak{F}^{\mathfrak{M}}$ with perhaps some lines missing. This can be proved using Thm.4.3.13 on p.482.

Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t)^-)$. Then $G^E_{\mathfrak{M}}$ is a reduct of the Euclidean geometry over the field $\mathfrak{F}^{\mathfrak{M}}$. Actually we may even include the topology, and have

$$G^{ET}_{\mathfrak{M}} = \langle Mn, L; \in, Bw, \mathcal{T}\rangle$$

a Euclidean structure (over the field $\mathfrak{F}^{\mathfrak{M}}$) if we assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp})$.

[1194] A similar observation applies to g as we will see in item (5) way below. Further, no one of eq or g is definable from the other in $\mathsf{Ge}(Th)$ for some of our distinguished choices of Th. This is one of the reasons why we keep both eq and g in our language.

[1195] As we indicated at the beginning of the present item (item 2) this is part of the reason why we keep $\perp$ in our language.

Motivated by these observations, we feel that one could call $G^{ET}_{\mathfrak{M}}$ the <u>*Euclidean reduct*</u> of the relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$, for any model $\mathfrak{M}$.

$G^{ET}_{\mathfrak{M}}$ is maximal among the Euclidean reducts in the sense that if we add any one (e.g. $\perp$ or eq) of the ingredients of $\mathfrak{G}_{\mathfrak{M}}$ missing from $G^{ET}_{\mathfrak{M}}$ to $G^{ET}_{\mathfrak{M}}$ then what we get will no more be representable as an isomorphic copy of a Euclidean geometry, assuming $\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp}$.

Remark 6.7.17 (On the affine reduct or part of $\mathfrak{G}_{\mathfrak{M}}$)
We could call $G^{E}_{\mathfrak{M}}$ the <u>*affine part*</u> (or reduct) of our relativistic geometry $\mathfrak{G}_{\mathfrak{M}}$. The reason for this is that $G^{E}_{\mathfrak{M}}$ consists exactly of those parts of $\mathfrak{G}_{\mathfrak{M}}$ which are preserved under affine transformations, under some assumptions[1196] on $\mathfrak{M}$. This might sound a little sloppy because affine transformations act of ${}^{n}F$ while the universe of $\mathfrak{G}_{\mathfrak{M}}$ is Mn. However what we said can be made precise by saying that $(\forall m \in Obs)$ [the image of $G^{E}_{\mathfrak{M}}$ under w_m^{-1} is preserved under all affine transformations of ${}^{n}\mathbf{F}$] while the other parts (like eq, $\perp$, L^{Ph} or g) of $\mathfrak{G}_{\mathfrak{M}}$ do not have this property.

For completeness, we note that under reasonably mild assumptions[1197] on $\mathfrak{M}$, $G^{E}_{\mathfrak{M}}$ satisfies the usual definition of an <u>*affine geometry*</u>[1198]. For more in this direction we refer to Coxeter [62] Chapter 13 beginning with p.191 (cf. also pp.175–176 for connections with relativity). For affine geometry and the claim that $G^{E}_{\mathfrak{M}}$ satisfies its axioms we also refer to Schwabhäuser et al. [237] II.§7 (Allgemeine affine Geometrie) pp.413-447 where the axioms are on p.415 cf. also item 7.63 on p.447 for the n-dimensional case. Cf. also Szczerba-Tarski [245] axioms A1–A6, E on the third page of the paper.

When we say that $G^{E}_{\mathfrak{M}}$ satisfies the axioms of affine geometry, we mean only that it satisfies the axioms of **lopag** without $\mathbf{L_2}$, i.e. $\mathbf{lopag} \setminus \{\mathbf{L_2}\}$, introduced on p.1071 (Def.6.6.54). The acronym $\mathbf{lopag} \setminus \{\mathbf{L_2}\}$ abbreviates ordered Pappian affine geometry with distinguished lines. The models of $\mathbf{lopag} \setminus \{\mathbf{L_2}\}$ can be considered as the abstract, axiomatic versions of the affine reduct $G^{E}_{\mathfrak{M}}$ (with some conditions[1199] on $\mathfrak{M}$ as we already indicated).

We hope that the above discussion clarifies in what sense (and why) we could call $G^{E}_{\mathfrak{M}}$ the affine part of our geometry $\mathfrak{G}_{\mathfrak{M}}$.

$\triangleleft$

[1196] e.g. $\mathbf{Bax}^{\oplus}$, $\mathbf{Ax}(Triv_t)$, **Ax6**
[1197] e.g. **Ax1**–**Ax3**, **Ax6**, **Ax(Bw)**
[1198] Whose study goes back to Euler but became intensive starting with Klein's Erlangen program.
[1199] e.g. **Ax1**–**Ax3**, **Ax6**, **Ax(Bw)**

(4) On circles or spheres

Before turning to richer languages, we note that having *eq* around is nice because it enables us to speak about circles or spheres.[1200] We note that for $n > 2$ in $\mathbf{Basax} + \mathbf{Ax}(Triv_t)^-$ in terms of *eq* a sphere looks like as in Figure 340 when intersected with $\mathsf{Plane}(\bar{t}, \bar{x})$. So far we talked about circles based on *eq*. Let us call them *eq*-circles. Similarly we can consider circles based on g. Let us call these second kind of circles g-circles.[1201] We use circles in 2-dimensional models and spheres in $n > 2$ dimensional ones. We note that the set of neighborhoods $T_0 :\stackrel{\text{def}}{=} \{\, S(e,\varepsilon) \,:\, e \in Mn,\ \varepsilon \in {}^+F \,\}$ defined on p.797 coincides with the set of g-circles (in any $\mathfrak{G} \in \mathsf{Ge}(\emptyset)$).

(i) A g-circle in $\mathbf{Basax}(2) + \mathbf{Ax}(\omega)^\sharp$ looks like as in Figure 340 (where the lines of our sheet of paper represent the lines in $\mathfrak{G}_{\mathfrak{M}}$).

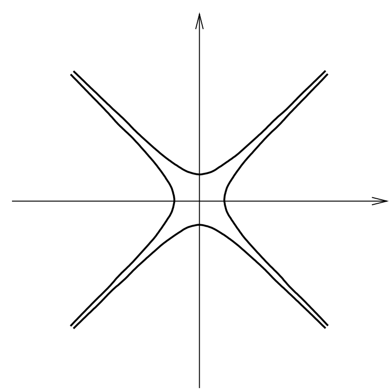


Figure 340: A g-circle in $\mathbf{Basax}+\mathbf{Ax}(\omega)^\sharp$. An *eq*-circle in **Basax** may look like this. Cf. also Fig.29 on p.88.

(ii) However, a g-circle in $\mathbf{Basax}(2)$ may look like as any one of those in Figure 341.

(iii) A g-circle in $\mathbf{Bax}(2)$ may even look like as in Figure 342.

[1200] For completeness we note that circles were already touched upon in Chapter 2 (cf. p.89).

[1201] By a g-sphere we understand a maximal set of such points of Mn whose g-distance is the same (constant) from a given point. Similarly for g-circles and for *eq* in place of g.

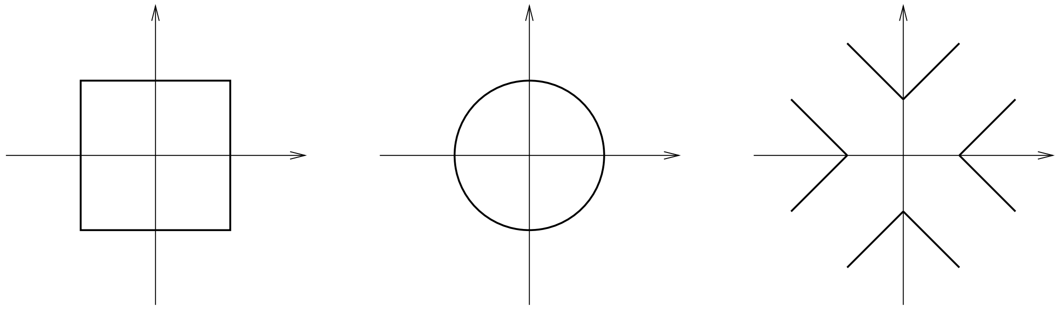


Figure 341: A g-circle in **Basax** may look like any of these. No one of these can be an eq-circle of **Basax**, cf. also Fig.29 on p.88.

(iv) If $n > 2$, a g-sphere as well as an eq-sphere in $\mathbf{Bax}^{\oplus}$ + **Ax(eqspace)** + **Ax(eqtime)** + $\mathbf{Ax}(\mathit{Triv}_t)^{-}$ may look like as in Figure 343. We note that the hyperboloid part is necessary, and the horizontal part is an (almost) arbitrary surface. Under these axioms the sides of the sphere always form a hyperboloid, while the top may be an arbitrarily complicated surface. The bottom surface is the reflection of the top one w.r.t. the origin. This g-sphere is typical of $\mathbf{Bax}^{\oplus}$ + "auxiliaries". If we throw **Ax(eqtime)** away then the top and bottom surfaces of the g-sphere may be replaced by clouds of points. If we throw $\mathbf{Ax}(\mathit{Triv}_t)^{-}$ away then the sides of the g-sphere may become "gapy".

A possible way of visualizing a relativistic geometry say $\mathfrak{G}_{\mathfrak{M}}$ (or equivalently the model $\mathfrak{M}$) is to draw a g-sphere or g-circle as in Figures 340–343. More precisely if we do not assume any "symmetry" property on $\mathfrak{M}$ then this picture will represent the model or geometry from the point of view of a certain observer. However assuming the axioms listed in item (iv) together with $\mathbf{Ax}(\uparrow\uparrow)$ ensure that such a drawing contains information about the world-views of all other observers too, hence about the whole model $\mathfrak{M}$ (or geometry), assuming **Ax♡** and **Ax(ext)** of course. Cf. Figure 29 on p.88 for more information in this direction.

(5) On g

Let us turn to definability of g over the geometry $\langle Mn; Col, Bw, \perp, eq\rangle$. First let us notice that g has a codomain[1202] $\mathbf{F_1}$, i.e. $g : Mn \times Mn \overset{\circ}{\longrightarrow} \mathbf{F_1}$, where we recall

[1202]For the notion of codomain cf. p.1085.

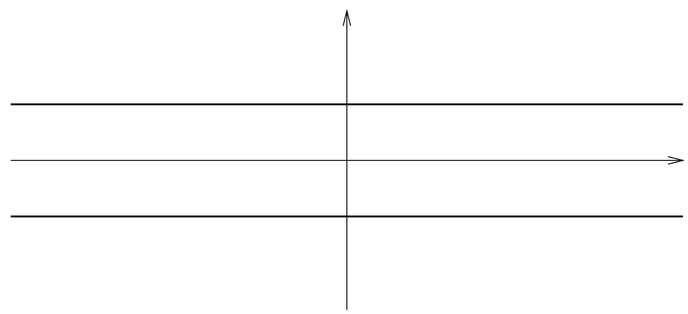


Figure 342: A g-circle in **Bax** may even look like this.

that $\mathbf{F_1} = \langle F;\, 0, 1, +, \leq \rangle$. Therefore, defining g requires defining $\mathbf{F_1}$ too, because $\mathbf{F_1}$ is not available in the geometry $\langle Mn;\ Col, Bw, \perp, eq \rangle$ from which we are supposed to define the *metric-geometry* $\langle Mn, \mathbf{F_1};\ Col, Bw, \perp, eq, g \rangle$.[1203] In passing we note that in $\mathsf{Ge}(\mathbf{Pax}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax6})$ the structure $\mathbf{F_1}$ is definable over each one of $\langle Mn;\ Bw \rangle$ and $\langle Mn;\ Col \rangle$ by Propositions 6.6.40 (p.1053) and 6.6.38 (p.1052) and Thm.6.7.1.

First, let us consider the *reduct* when the codomain of g is the ordered group

$$\mathbf{F_0} :\stackrel{\text{def}}{=} \langle F;\, 0, +, \leq \rangle$$

(instead of $\mathbf{F_1} = \langle F;\, 0, 1, +, \leq \rangle$).

PROPOSITION 6.7.18 *In* $\mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp)$, $\mathbf{F_0}$ *and*

$$g : Mn \times Mn \stackrel{\circ}{\longrightarrow} \mathbf{F_0}$$

are uniformly first-order definable in the language of $\langle Mn;\ Bw, eq \rangle$. *I.e. in the* $\langle Mn;\ Bw, eq \rangle$*-reduct of* $\mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp)$, *the structure* $\mathbf{F_0}$ *and*

$$g : Mn \times Mn \stackrel{\circ}{\longrightarrow} F$$

are uniformly first-order definable.

[1203] To be precise, the reason for this (i.e. for our saying that defining g requires defining $\mathbf{F_1}$ too) is a "subjective" decision: Namely, at this point we decide to identify g with $\langle Mn, \mathbf{F_1};\ g \rangle$, because to be able to use g we usually need its domain and codomain Mn and $\mathbf{F_1}$.

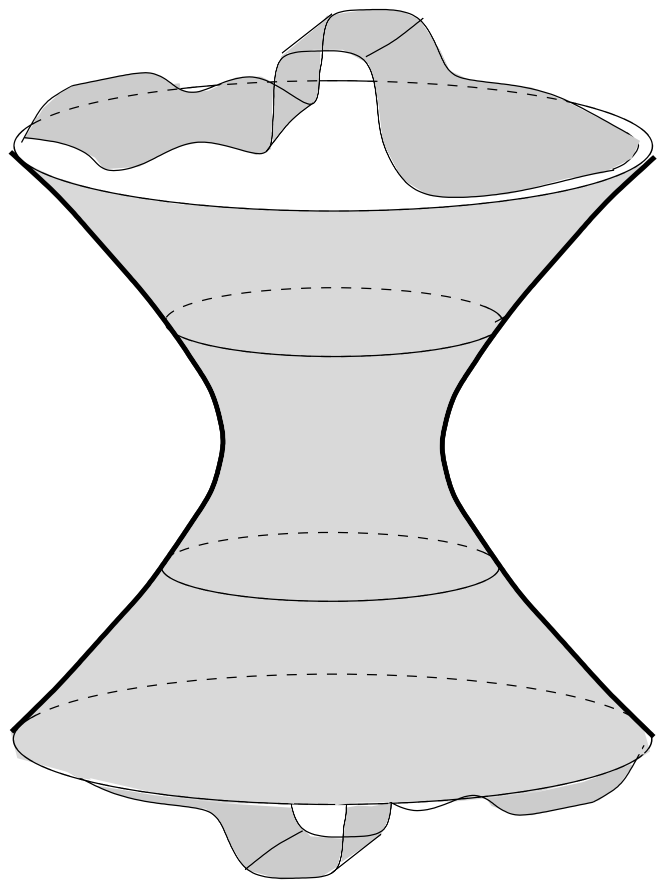


Figure 343: A g-sphere or an eq-sphere in $\mathbf{Bax}^{\oplus}(3)$+**Ax(eqspace)**+**Ax(eqtime)**+ $\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^{-}$.

On the proof: Assume the hypotheses of the proposition. The proof for the case $n = 2$ is available from Judit Madarász. Assume $n > 2$. First we define the relation

$$R :\stackrel{\text{def}}{=} \left\{ \langle a, o, e \rangle \in Mn \times Mn \times Mn \;:\; o \not\equiv^{Ph} e,\; coll(a, o, e) \right\}.$$

Then we define the auxiliary sort U to be R together with pj_0, pj_1, pj_2. The equivalence relation $\equiv$ on U is defined as follows.

$$\langle a, o, e \rangle \equiv \langle a', o', e' \rangle \;\stackrel{\text{def}}{\iff}\; \Big((a \in [oe \;\leftrightarrow\; a' \in [oe'\,) \;\wedge\; \langle a, o \rangle \; eq \; \langle a', o' \rangle \Big).$$

(Of course one uses pj_0, pj_1, pj_2 in the formal definition of $\equiv$.) F is defined to be $U/\!\equiv$ together with $\in\; \subseteq U \times U/\!\equiv$. For every $o, e \in Mn$, F_{oe}, $+_{oe} \subseteq F_{oe} \times F_{oe} \times F_{oe}$, and $\leq_{oe} \subseteq F_{oe} \times F_{oe}$ are defined as in Def.6.6.31 (p.1046). Now, we define the addition $+ \subseteq F \times F \times F$ and the ordering $\leq\; \subseteq F \times F$ as follows. Let $a, b, c \in F$. Then

$$\begin{gathered} +(a, b, c) \\ \stackrel{\text{def}}{\iff} \\ (\exists a' \in a)(\exists b' \in b)(\exists c' \in c) \\ \Big(pj_1(a') = pj_1(b') = pj_1(c') \;\wedge\; pj_2(a') = pj_2(b') = pj_2(c') \;\wedge \\ pj_0(a') \;+_{pj_1(a')pj_2(a')}\; pj_0(b') \;=\; pj_0(c') \Big), \end{gathered}$$

$$\begin{gathered} a \leq b \\ \stackrel{\text{def}}{\iff} \\ (\exists a' \in a)(\exists b' \in b)\Big(pj_1(a') = pj_1(b') \;\wedge\; pj_2(a') = pj_2(b') \;\wedge \\ pj_0(a') \;\leq_{pj_1(a')pj_2(a')}\; pj_0(b') \Big). \end{gathered}$$

Further the constant 0 is defined by

$$x = 0 \quad \stackrel{\text{def}}{\iff} \quad x + x = x.$$

By the above $\mathbf{F_0} = \langle F;\, 0, +, \leq \rangle$ is defined. Finally we define $g : Mn \times Mn \stackrel{\circ}{\longrightarrow} \mathbf{F_0}$ as follows. Let $a, b \in Mn$ and $x \in F$. Then

$$\begin{gathered} g(a, b) = x \\ \stackrel{\text{def}}{\iff} \\ \Big((a \equiv^{Ph} b \;\wedge\; x = 0) \;\vee\; (x > 0 \;\wedge\; (\exists x' \in x)(pj_0(x') = a \;\wedge\; pj_1(x') = b) \Big). \end{gathered}$$

It can be checked that this is a correct explicit definition of a partial function g.

We leave it to the reader to check that the above outlined explicit definitions of $\mathbf{F_0}$ and g have the desired properties. Hint: Thm.6.2.60, saying that that the $\prec$-free reducts of $(\mathbf{Basax}+\mathbf{Ax}(\omega)^\sharp)$-geometries are the $\prec$-free reducts of Minkowskian geometries up to isomorphism, helps in checking this. ∎

Throughout the remaining part of the present item (item 5) we assume $\mathbf{Basax}+\mathbf{Ax}(\omega)^\sharp$. If in our enriched geometries $\mathfrak{G}_{\mathfrak{M}}$, only $\mathbf{F}_0$ was present as an extra sort, then we could avoid including g and $\mathbf{F}_0$ into $\mathfrak{G}_{\mathfrak{M}}$ by arguing that they are definable from the simpler, more elegant one-sorted geometry $\langle Mn; Bw, eq\rangle$, cf. Prop.6.7.18 above. However, into $\mathfrak{G}_{\mathfrak{M}}$ we included the richer structure $\mathbf{F_1}$ as the codomain of g. Our definability statement in Prop.6.7.18 does <u>not</u> extend from $\mathbf{F}_0$ to $\mathbf{F_1}$. In other words while the expanded "metric" geometry $\langle Mn, \mathbf{F_0}; Bw, eq, g\rangle$ <u>is</u> definable from its one-sorted reduct $\langle Mn; Bw, eq\rangle$ the richer expanded geometry $\langle Mn, \mathbf{F_1}; Col, Bw, \perp, eq, g\rangle$ is <u>not</u> definable from its one-sorted reduct $\langle Mn; Col, Bw, \perp, eq\rangle$. Moreover, g is not definable even from the g-free reduct of $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp)$. The *intuitive* reason for definability of $\mathbf{F_0}$ and undefinability of $\mathbf{F_1}$ (in our geometries $\langle Mn; Bw, \dots, eq\rangle$) is the following:

We can easily express geometrically statements like $g(a,b) = 0$, $g(a,b) = g(b,c) + g(d,e)$, and $g(a,b) \leq g(b,c)$ by using Bw and eq only, cf. the proof of Prop.6.7.18 above. This leads to definability of the ordered group $\mathbf{F}_0$. At the same time we <u>cannot</u> express the property $g(a,b) = 1$ of points a, b (using only Bw and eq). This can be seen for $n = 2$ by looking at the simplest 2-dimensional Minkowskian geometry $\langle \mathrm{R} \times \mathrm{R}, Col, \dots, eq\rangle$ over $\mathfrak{R}$ and considering its automorphism h defined as follows:

$$(\forall x, y \in \mathrm{R})\ h(x,y) = (2x, 2y).$$

Now, there are points p, q here, such that

$$(*) \qquad g(p,q) = 1 \text{ but } g(hp, hq) \neq 1.$$

Since h is an automorphism, this proves that the *property* $g(p,q) = 1$ of a pair of points p, q is <u>not</u> definable in this reduct of Minkowskian geometry. One can push this argument further to show that in the 1-free reducts of Minkowskian geometries the binary relation defined by $g(p,q) = 1$ is not definable.[1204]

[1204] Let $un := \{\langle p,q\rangle \in Mn \times Mn \ :\ g(p,q) = 1\}$. Then the binary relation un on points is not definable in the g-free reduct of $\mathfrak{G}_{\mathfrak{M}}$. Moreover, in some intuitive sense, it is this undefinability of un which is the real reason for undefinability of g. E.g. $\langle g, \mathbf{F_1}\rangle$ is definable from the "simple", one-sorted geometry $\langle Mn; Bw, eq, un\rangle$. (Recall that $\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp$ is assumed here.) Here "un" is an acronym for "unit distance".

We call transformations like h *expansions.* Now, expansions are automorphisms of the $Col, \ldots, eq$ part but they are typically *not* extendable to automorphisms of $\mathfrak{G}_{\mathfrak{M}}$ because of $(*)$.

Summing up (of item 5):
We were discussing definability of g over $\langle Mn; Col, Bw, \perp, eq\rangle$. If

$$g : Mn \times Mn \overset{\circ}{\longrightarrow} \mathbf{F_0}$$

was the case then g would be definable (even from $\langle Mn; Bw, eq\rangle$) under some assumptions on $\mathfrak{M}$. But since in our g there is a distinguished constant "1" i.e. since we identify[1205] g with the structure $\langle Mn, \mathbf{F_1}; g\rangle$ our g is not definable even over the g-free reducts $\langle Mn, L; \ldots, eq, \mathcal{T}\rangle$ of our geometries. Moreover $\langle g, \mathbf{F_1}\rangle$ is not definable from the rest of the vocabulary in any Minkowskian geometry $\mathfrak{G}$. This is a quite strong form of undefinability. Therefore we include g in our language. This completes the discussion of $\langle L, Bw, \perp, eq, g\rangle$. As an afterthought, in this connection we also state the following (which was already mentioned informally).

PROPOSITION 6.7.19 *Let* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp})$. *Then (i) and (ii) below hold.*

- **(i)** $\langle g, \mathbf{F_1}\rangle$ *is not definable over the* $(g, \mathbf{F_1})$*-free reduct of* $\mathfrak{G}$.
- **(ii)** g *is not definable over the* g*-free reduct* $\langle Mn, \mathbf{F_1}; Col, \ldots, eq, \mathcal{T}\rangle$ *of* $\mathfrak{G}$. *Note that* $\mathbf{F_1}$ *is present in the reduct in which* g *is not definable.*

On the proof: A proof can be obtained by using expansions the same way as we used them around $(*)$ above. ∎

We guess that the above proposition extends to $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)}$, $n > 2$.

In the present sub-section we did not address the question of which parts of $\mathfrak{G}_{\mathfrak{M}}$ are definable from the pseudo-metric $\langle g, \mathbf{F_1}\rangle$. However, this question will be addressed in §6.7.2 below, cf. e.g. items 6.7.38–6.7.40.

We will return to g and to recoverability of things like L^T, L^{Ph}, L^S from g in §6.8 devoted to geodesics. Geodesics play an important role in generalizations towards accelerated observers and eventually towards general relativity, hence they deserve a special section.

The summary (of item 5) way above might be considered as a logic-based explanation of the following experience in physics. In a majority of the physics books

[1205] cf. footnote 1203 on p.1151

on relativity (both special and general) the mathematical model of the world is a structure of the form $\langle Mn, g\rangle$. This can be caused by the fact that as we saw in the summary, the $\langle g, \mathbf{F_1}\rangle$ part of $\mathfrak{G}_{\mathfrak{M}}$ is not definable from the rest therefore one has to include it into our mathematical model if we do not want to loose information. (We also saw that under strong enough conditions[1206] we can define $\mathfrak{G}_{\mathfrak{M}}$ over $\langle Mn, \mathbf{F_1};\, g\rangle$.)

6.7.2 On $\prec, Col^T, Col^{Ph}, Col^S, \equiv^T, \equiv^{Ph}$

For the definition of Col^T, Col^{Ph}, Col^S we refer to p.998 in §6.5. Before plunging into the subject matter of the present sub-section seriously we point out the following. No part of the vocabulary studied in §6.7.1 is made superfluous by the "*Col*-free" part of the vocabulary of the present sub-section §6.7.2. Namely, at the end of §6.7.2 we will see that no part of the vocabulary $Col, Bw, \perp, eq, g$ discussed in §6.7.1 is definable from the *Col*-free part $\prec, \equiv^T, \equiv^{Ph}, \equiv^S$ of the vocabulary discussed in §6.7.2 in all of our distinguished classes $\mathsf{Ge}(Th)$,[1207] cf. Thm.6.7.42 on p.1168.

(I) On the "causality" pre-ordering $\prec$

Next we turn to discussing the status of the causality pre-ordering $\prec$ (of Mn). It is not definable from the rest of the vocabulary of $\mathsf{Ge}(Th)$ e.g. in $\mathsf{Ge}(\mathbf{Basax}+\mathbf{Ax(\uparrow\uparrow)})$.

On a connection with the literature: In *some* of our models $\prec$ might behave quite differently from the behavior of Robb's relation called "after"; the latter is described in e.g. Goldblatt [108, Appendix B, p.170]. Let $\mathfrak{G} \in \mathsf{Ge}(\emptyset)$. We define $\succ^r \subseteq Mn \times Mn$ as follows.

$$b \succ^r a \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad a \neq b \ \wedge\ (\forall c \in Mn)(b \prec c \ \rightarrow\ a \prec c).$$

In Minkowskian geometries our $\succ^r$ is the same as Robb's after. However our $\succ^r$ is defined for more general classes $\mathsf{Ge}(Th)$, where it can behave quite differently from the behavior of Robb's after in Minkowskian geometries. To mention one of the

[1206] These conditions are quite restrictive, therefore many authors e.g. Friedman [90] and ourselves do not utilize this possibility of restricting the model to $\langle Mn, \mathbf{F_1};\, g\rangle$. Also, as we mentioned keeping the other parts gives us the possibility of "abstraction" i.e. concentrating on aspects of the world.

[1207] By the *Col*-free part here we mean the (Col^T, Col^{Ph}, Col^S)-free part. We have to restrict attention to this *Col*-free part because from $\langle Col^T, Col^{Ph}, Col^S\rangle$ one can trivially define *Col* (then from *Col* we obtain *Bw*, under some mild assumptions, cf. Thm.6.7.1).

differences, for some $\mathfrak{M} \models$ **Basax** the relation $\succ^r$ is symmetric. However, if we assume **Basax**+**Ax(↑↑)**, our $\succ^r$ behaves the same way as Robb's "after" does.[1208] E.g. it is a partial ordering. (In the other direction our $\prec$ is definable from Robb's "after" in Minkowskian geometries, of course.)

The following theorem is a generalization of the Alexandrov-Zeeman theorem which was proved for standard Minkowskian geometry over $\mathfrak{R}$, cf. e.g. Goldblatt [108, Appendix B] or Alexandrov [4, 5] or Zeeman [276]. In the rest of this sub-section $\perp$ is a 4-ary relation on the set of points (and is relativistic).

THEOREM 6.7.20 *Assume $n > 2$. Then (i)–(iii) below hold.*

(i) *Col^T, Col^{Ph}, Bw are definable from $\langle Mn; \prec\rangle$ in $\mathsf{Ge}(Th)$, assuming*

$$Th \models \mathbf{Reich}(\mathbf{Bax})^{\oplus} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\mathbf{diswind}).$$

(ii) *$Col, Col^T, Col^{Ph}, Col^S, Bw, \perp$ are definable from $\langle Mn; \prec\rangle$ in $\mathsf{Ge}(Th)$, assuming*

$$Th \models (\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\mathbf{diswind})).$$

(iii) *$eq, Col, Col^T, Col^{Ph}, Col^S, Bw, \perp$ are definable from $\langle Mn; \prec\rangle$ in $\mathsf{Ge}(Th)$, assuming*

$$Th \models (\mathbf{Newbasax} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\mathbf{diswind})).$$

On the proof: Before reading the proof cf. Fig.282 (p.863). A proof can be obtained by the proof of Alexandrov-Zeeman theorem in Goldblatt [108] and by Theorems 6.2.71 (p.877), 6.2.74 (p.878), 6.6.109 (p.1128).

∎

The condition **Newbasax** is needed in the above theorem since if we replace in (iii) **Newbasax** with **Flxbasax**$^{\oplus}$ then *eq* will not be definable from $\prec$, moreover it will be undefinable even from the *eq*-free reduct of $\mathsf{Ge}(Th)$. Hence, in particular in $\mathsf{Ge}(\mathbf{Flxbasax}^{\oplus})$, *eq* is not definable from the rest of the vocabulary of $\mathsf{Ge}(Th)$.

In connection with item (i) of Thm.6.7.20, i.e. in connection with putting the emphasis on $\prec$, Col^T and Col^{Ph} we refer to e.g. Busemann [56, 55] (to be precise we note that instead of Col^T Busemann [56, 55] uses time-like geodesics in the same sense as we do in §6.8).

[1208] The definition of Robb's after generalizes in a very natural (and non-problematic) way to $\mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\uparrow\uparrow))$. According to Robb's after, $b \succ^r a$ iff b is on or in the future directed light-cone of a; i.e. if a "can send a signal" to b without using FTL particles.

As a contrast to Thm.6.7.20 we note that $g, \mathbf{F_1}$ is *not definable* from $\prec$ even in Minkowskian geometries, cf. the end of §6.7.1 on p.1155. (Actually the reason for this is undefinability of $\mathbf{F_1}$, i.e. the constant 1, cf. Prop.6.7.18 on p.1151.)

In connection with Thm.6.7.20 see Theorems 6.7.35, 6.7.36 (p.1166).

QUESTION 6.7.21 *Does item (i) of Thm.6.7.20 generalize from* $\mathbf{Reich}(\mathbf{Bax})^{\oplus}$ *to* $\mathbf{Bax}^{-\oplus}$*?*

◁

We note that if $Th = \mathbf{Basax}^{\oplus} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\uparrow\uparrow)$ and $n > 2$ then all parts of our geometry $Col, \ldots, \mathcal{T}$ are definable from $\langle Mn; \prec\rangle$ in $\mathsf{Ge}(Th)$ with the exception of $\langle \mathbf{F_1}, g\rangle$.[1209] Even $\langle \mathbf{F_0}, g\rangle$ is definable. (Note that the topology $\mathcal{T}$ is definable, too.) This means that everything in our geometry $\mathfrak{G}_{\mathfrak{M}}$ is recoverable from the simple structure $\langle Mn; \prec\rangle$ with the only exception of the "size of a hydrogen atom"[1210] (i.e. with the exception of the units of measurement). But this is quite natural since we cannot expect the causality pre-ordering $\prec$ to contain information like the "size of a hydrogen atom".
We will further refine these observations in §6.7.3. Namely, we will add to $\langle Mn; \prec\rangle$ the restriction $g^{\prec}$ of g to "$\prec$" obtaining the streamlined time-like-metric structure $\langle Mn, \mathbf{F_1}; g^{\prec}\rangle$ which in turn will prove to be satisfying both from the point of view of mathematical elegance (streamlined-ness) and expressive power.

Remark 6.7.22 (On causality) Although, following the literature, we call the pre-ordering $\prec$ "causality pre-ordering", we do not claim that the way we introduced and discussed $\prec$ would represent a well justified and well understood theory of causality. (Perhaps "possible future" would be a better name for $\prec$, but we decided to follow the majority of the literature.) One of the reasons why we mention this is that we feel that elaborating a carefully developed, well understood, sufficiently "subtle" (or deep) theory of causality which would be also well founded from the point of view of mathematical logic, would be highly desirable. (So we do not want to make the impression that we already have such a theory.)

◁

[1209]By $\langle \mathbf{F_1}, g\rangle$ (as well as by $\langle g, \mathbf{F_1}\rangle$ or $\langle Mn, g, \mathbf{F_1}\rangle$) we mean the many-sorted structure $\langle Mn, \mathbf{F_1}; g\rangle$. (Of these $\langle Mn, g, \mathbf{F_1}\rangle$ corresponds to the category theoretical spirit, cf. p.1086.)
[1210]Cf. §2.8, p.139 for an intuitive explanation connecting the size of a hydrogen atom with the units of measurement (i.e. with the constant 1 of $\mathbf{F_1}$). Cf. also the intuitive explanation on the role of the constant $1 \in \mathbf{F_1}$ in our geometry $\mathfrak{G}_{\mathfrak{M}}$ on p.850.

(II) On $\boldsymbol{Col^T}$, $\boldsymbol{Col^{Ph}}$ and $\boldsymbol{Col^S}$

In the present item we discuss the status of Col^T, Col^{Ph}, Col^S.

Actually in Thm.6.7.20 above we have already started discussing this. No one of Col^T, Col^{Ph}, Col^S is definable from Col in all cases. On the other hand each one of them is definable from Col and $\perp$ under assuming $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$ and $n > 2$ (cf. Thm.6.2.115 on p.926).

Consider the incidence geometry $\mathbf{G}^{Ph}_{\mathfrak{M}} := \langle Mn;\ Col^{Ph}\rangle$. Assume that $\mathfrak{M}$ is the standard, Minkowski model over some real-closed field $\mathfrak{F}$, $n > 2$. Then in $\mathbf{G}^{Ph}_{\mathfrak{M}}$, $Col, Col^T, Col^S, \perp, Bw, eq, \mathcal{T}$ and g_0 are definable, where g_0 is obtained from g by forgetting the constant 1 from the codomain $\mathbf{F_1}$ of g. This means that the reduct of $\mathfrak{G}_{\mathfrak{M}}$ without $\prec$ and 1 is definable from $\mathbf{G}^{Ph}_{\mathfrak{M}}$. (The same applies to the other two incidence geometries $\mathbf{G}^{T}_{\mathfrak{M}}$ and $\mathbf{G}^{S}_{\mathfrak{M}}$.) The proof is based on the Alexandrov-Zeeman Theorem in Appendix B of Goldblatt [108].

Next, we consider some generalizations of the above mentioned results.

Let $Mink(\mathfrak{F})$ be the Minkowskian geometry over an ordered field $\mathfrak{F}$ defined in Def.6.2.58 (p.859). We will use $Col_\mu, Col^T_\mu, Col^{Ph}_\mu, Col^S_\mu$ instead of $L_\mu, L^T_\mu, L^{Ph}_\mu, L^S_\mu$ of $Mink(\mathfrak{F})$ in the present item. Earlier we used L and Col interchangeably, in this material. In the present item Col is more convenient for our purposes than L. This is why we use Col here. Also $\perp_\mu$ is a 4-ary relation on the set of points. We let $Mink(n)$ to be the class of n-dimensional Minkowskian geometries.

$$\mathbf{Mink}(n) :\stackrel{\text{def}}{=} \mathbf{I}\{\, Mink(n, \mathfrak{F}) \ :\ \ \mathfrak{F} \text{ is Euclidean}\,\}\,.$$

THEOREM 6.7.23 [1211] *Assume $n > 2$, and $\mathfrak{F}$ is Euclidean. Then $Col_\mu, Col^T_\mu, Col^S_\mu, Bw_\mu, \perp_\mu, eq_\mu$ are uniformly definable from $\langle {}^nF;\ Col^{Ph}_\mu\rangle$ in $\mathbf{Mink}(n)$.*

On the proof: The proof is based on the proof of Alexandrov-Zeeman theorem in Goldblatt [108]. ∎

In connection with Thm.6.7.23 above see Thm.6.7.33 on p.1165.

THEOREM 6.7.24 *Let $n = 2$ and $\mathfrak{F} = \mathfrak{R}$ the ordered field of reals. Consider the photon geometry $\langle {}^2F;\ Col^{Ph}_\mu\rangle$. Now, Col_μ is not definable over $\langle {}^2F;\ Col^{Ph}_\mu\rangle$.*

[1211] We guess that this theorem was probably known.

Proof: We note that the idea is to construct an automorphism of $\langle {}^2F;\ Col^{Ph}_\mu\rangle$ which does not preserve Col_μ. The proof can be found in [16], in more detail the proof of the statement in [16] saying that the axiom system Specrel(2) is independent proves our Thm.6.7.24. ∎

We note, that the weaker statement saying that Col^T_μ is not definable from Col^{Ph}_μ in $Mink(2,\mathfrak{R})$ is proved e.g. in Goldblatt [108].

A large portion of Minkowskian geometry can be defined over arbitrary ordered fields $\mathfrak{F}$ (assuming $\mathbf{Ax}(\sqrt{\ })$ is not necessary). Therefore we define the following.

Definition 6.7.25
$Mink_{nonE}(\mathfrak{F}) :\stackrel{\text{def}}{=} Mink_{nonE}(n,\mathfrak{F}) :\stackrel{\text{def}}{=}$
$$\langle {}^nF, \mathbf{F_1};\ Col_\mu, Col^T_\mu, Col^{Ph}_\mu, Col^S_\mu, \prec_\mu, Bw_\mu, \perp_\mu, eq_\mu, g^2_\mu\rangle$$
is defined completely analogously with the definition of $Mink(n,\mathfrak{F})$ in Def.6.2.58 (p.859). Let us notice, that the only essential difference is that we had to replace g_μ with g^2_μ since the definition of g_μ was the only part where we used square roots.[1212] Another difference is that we use now Col^T_μ etc. instead of L^T_μ etc: As we said, in the present item Col is more convenient for our purposes than L. Also $\perp_\mu$ is 4-ary. (To keep the definition short we omitted the topology, but this was not essential since it is definable from g^2_μ or Bw_μ or eq. [1213])

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Next, we consider Alexandrov-Zeeman-style theorems about Minkowskian geometries over arbitrary (possibly non-Euclidean) ordered fields $\mathfrak{F}$, cf. e.g. Goldblatt [108, Appendix B].

THEOREM 6.7.26
Let n and $\mathfrak{F}$ be arbitrary. Then $Col_\mu, Col^{Ph}_\mu, Col^S_\mu, Bw_\mu, \perp_\mu, eq_\mu$ are (first-order) definable from $\langle {}^nF;\ Col^T_\mu\rangle$ in the geometry $Mink_{nonE}(n,\mathfrak{F})$.

For the proof of Thm.6.7.26 we will need Lemma 6.7.27 below.

LEMMA 6.7.27 *Let n and $\mathfrak{F}$ be arbitrary. Then for every ultrafilter $U \subseteq \mathcal{P}(\mathrm{I})$ (I is an arbitrary set) we have*

$$^{\mathrm{I}}Mink_{nonE}(n,\mathfrak{F})/U \cong Mink_{nonE}(n, {}^{\mathrm{I}}\mathfrak{F}/U)$$

[1212] More precisely, in the definition of eq_μ we used g_μ, but g_μ can be replaced by g^2_μ in the definition of eq_μ.
[1213] We could have included the topology $\mathcal{T}_\mu$ into $Mink_{nonE}(n,\mathfrak{F})$, and then the definability theorems extend to definability of $\mathcal{T}_\mu$ too (in some sense).

We omit the **proof**. ∎

On the idea of proof of Thm.6.7.26:
Definition of Col, Col^{Ph}, Col^S:
Below we will use L, L^T, L^{Ph}, L^S instead of $Col, Col^T, Col^{Ph}, Col^S$ because the intuitive idea of the proof is easier to see with the L's.
Case 1: $n > 2$. First one defines P the set of planes from L^T. Then one defines L from P. Then one defines a partial orthogonality $\perp_p \subseteq L^T \times L$ as follows. Let $\ell \in L^T$ and $\ell_1 \in L$. Then

$$\ell \perp_p \ell_1 \overset{\text{def}}{\iff} \Big(\ell \cap \ell_1 \neq \emptyset \ \wedge\ (\exists b, c \in \ell_1)[b \neq c \ \wedge\ (\forall a \in \ell)(\overline{ab} \in L^T \ \Leftrightarrow\ \overline{ac} \in L^T)]\Big).$$ [1214]

Now,

$$\ell \in L^S \overset{\text{def}}{\iff} (\exists \ell_1 \in L^T)\ \ell_1 \perp_p \ell.$$

Now, $L^{Ph} :\overset{\text{def}}{=} L \setminus (L^T \cup L^S)$.

Case 2: $n = 2$. The proof for this case is analogous with the proof that **Basax** $\Rightarrow$ (f_{mk} preserves L), see Figure 344.

$Col(a, b, c) \overset{\text{def}}{\iff} \Big($the L^T-lines and L^T-triangles indicated in Figure 344 exist (and the triangles are similar etc)$\Big)$. Since we are in $n = 2$, the relation of parallelism is definable in the geometry $\langle {}^nF, L^T; \in\rangle$. The definitions of L^{Ph}, and L^S are the same as they were in Case 1.

Proof that Bw is definable from $\langle {}^nF;\ Col, Col^T\rangle$:

By Lemmas 6.7.5(ii) (p.1139) and Lemma 6.7.27 (p.1160), it is enough to prove that for every $\mathfrak{F}$, each automorphism of $\langle {}^nF;\ Col, Col^T\rangle$ preserves Bw. To see this let $\mathfrak{F} = \langle \mathbf{F}; \leq\rangle$ be arbitrary, and let h be an automorphism of $\langle {}^nF;\ Col, Col^T\rangle$. Since h is an automorphism of $\langle {}^nF;\ Col\rangle$ we conclude that $h = \widetilde{\varphi} \circ A$ for some $\varphi \in Aut(\mathbf{F})$ and $A \in Aftr$ by Lemma 3.1.6 on p.163. Let this φ and A be fixed. Since h preserves Col^T too, we conclude that $\varphi \in Aut(\mathfrak{F})$, by Lemma 6.6.6 on p.1028. But now we have that h preserves Bw since both φ and A preserve Bw.

Proof that eq and $\perp$ are definable from $\langle {}^nF;\ Col, Col^T, Col^{Ph}\rangle$:
The proof of this will be similar to the proof given for definability of Bw. By

[1214] An alternative definition is the following: $((\ell \cap \ell_1 = \{o\}$, for some $o \in {}^nF) \ \wedge\ (\forall a \in \ell)(\forall b, c \in \ell_1)$ ["$\langle o, b\rangle$ and $\langle o, c\rangle$ are equidistant" $\Rightarrow$ $(\overline{ab} \in L^T \ \Leftrightarrow\ \overline{ac} \in L^T)$]); where the statement "$\langle o, b\rangle$ and $\langle o, c\rangle$ are equidistant" can be formalized as follows: There is a parallelogram which one diagonal is segment $\langle b, c\rangle$ and the intersection of the diagonals of this parallelogram is $\{o\}$.

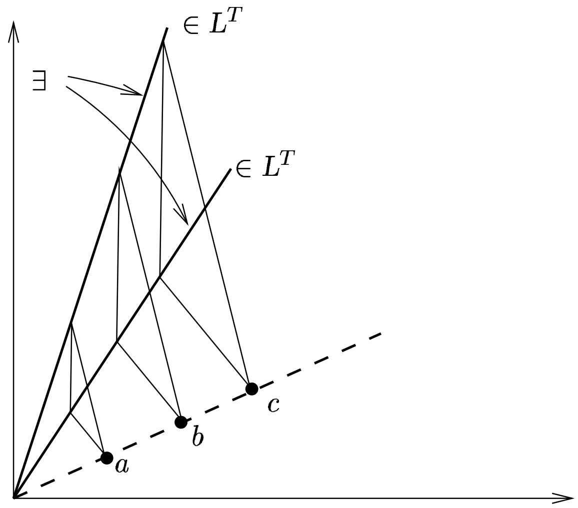

Figure 344: Illustration for the proof of Thm.6.7.26.

Lemmas 6.7.5 (p.1139), 6.7.27 (p.1160) it is enough to prove that for every $\mathfrak{F}$, every automorphism of $\langle {}^nF;\ Col, Col^T, Col^{Ph}\rangle$ preserves $\perp$ and eq. To see this let h be an automorphism of $\langle {}^nF;\ Col, Col^T, Col^{Ph}\rangle$. Now in the proof for Bw we have seen that $h = \widetilde{\varphi}\circ A$, for some $\varphi \in Aut(\mathfrak{F})$ and $A \in Aftr(n, \mathbf{F})$. Let this φ and A be fixed. Now, $\widetilde{\varphi}$ is an automorphism of the structure $\langle {}^nF;\ Col, Col^T, Col^{Ph}\rangle$ and it is not hard to check that it preserves $\perp$ and eq since $\perp$ and eq were defined by equations in the language of $\mathbf{F}$. So we have that A is an automorphism of $\langle {}^nF;\ Col, Col^T, Col^{Ph}\rangle$, and it remains to prove that A preserves $\perp$ and eq. To prove this let $\mathfrak{F}_* = \langle \mathbf{F}_*;\ \leq\rangle$ be the real closure of $\mathfrak{F} = \langle \mathbf{F};\ \leq\rangle$, and let $A_* \in Aftr(n, \mathbf{F}_*)$ such that $A_* \restriction {}^nF = A$. Now, by Lemma 3.4.5 on p.205, we have that A_* preserves Col_*^{Ph} of $Mink_{nonE}(n, \mathfrak{F}_*)$, and it is not hard to check that A_* preserves Col_*^T of $Mink_{nonE}(n, \mathfrak{F}_*)$, too. So we have that

(392) $\quad A_*$ preserves the reduct $\langle {}^nF_*;\ Col_*^T, Col_*^{Ph}, Col_*\rangle$ of $Mink_{nonE}(n, \mathfrak{F}_*)$.

Now, $\perp_*$ and eq_* of $Mink_{nonE}(n, \mathfrak{F}_*)$ can be defined from the structure $\langle {}^nF_*;\ Col_*^T, Col_*^{Ph}, Col_*\rangle$ by Alexandrov-Zeeman theorem since $\mathfrak{F}_*$ is a real-closed field. Therefore, by (392), A_* preserves $\perp_*$ and eq_*, and this implies that A preserves $\perp$ and eq since $A_* \restriction {}^nF = A$. ∎

THEOREM 6.7.28 *Let $\mathfrak{F}$ and n be arbitrary. Consider the Minkowskian geometry $Mink_{nonE}(n,\mathfrak{F})$ over $\mathfrak{F}$. Then, with the exception of g^2_μ, $\prec_\mu$, and 1 the whole of $Mink_{nonE}(n,\mathfrak{F})$ is (first-order) definable from $\langle {}^nF;\ Col^S_\mu\rangle$.*

On the proof: The case of $n = 2$ is completely analogous with the proof of Thm.6.7.26.

<u>*Assume $n > 2$.*</u>

Below we will use L, L^T, L^{Ph}, L^S instead of $Col, Col^T, Col^{Ph}, Col^S$ because the intuitive idea of the proof is easier to see with the L's. First we define the set P of planes from L^S (any two intersecting L^S-line determines a plane). Then we define L from P (any intersection of two different planes is a line or has fewer than two elements).

The only task which remains is to separate out L^{Ph} from $L \setminus L^S$. For this we classify the planes into 3 categories (i) space-like planes (all their lines are in L^S), (ii) time-like planes (they contain many non L^S-lines through every point in them), and (iii) Robb planes, where $H \in P$ is a Robb plane iff

$$(\forall p \in H)\ [(\exists! \ell \in (L \setminus L^S))\ p \in \ell \subseteq H].$$ [1215]

Now,

$$\ell \in L^{Ph} \overset{\text{def}}{\Longleftrightarrow} (\ell \notin L^S \text{ and } \ell \subseteq H, \text{ for some Robb plane } H);$$

and $L^T :\overset{\text{def}}{=} L \setminus (L^S \cup L^{Ph})$. Next, one checks that this definition of L^T, L^{Ph}, L from L^S is "correct" and that we really did not use $\mathbf{Ax}(\sqrt{\ })$ in showing that it works.

$Bw, \perp, eq$ are definable from $\langle {}^nF;\ Col^T\rangle$ by Thm.6.7.26 on p.1160. ∎

Future research task 6.7.29 Let

$$\mathbf{Mink}_{nonE}(n) :\overset{\text{def}}{=} \mathbf{I}\,\{\, Mink_{nonE}(n,\mathfrak{F})\ :\ \mathfrak{F} \text{ is an ordered field}\,\}\,.$$

Which ones of the above proved definability results (e.g. Theorems 6.7.26 and 6.7.28) carry over to uniform definability in the class $\mathbf{Mink}_{nonE}(n)$? E.g. are $Bw_\mu, \perp_\mu$ and eq_μ <u>uniformly</u> definable from Col^T_μ in the class $\mathbf{Mink}_{nonE}(n)$?

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[1215] The present definition of time-like planes etc. is slightly different from our "official" definition of time-like hyper-planes etc. on p.1129 which is in force throughout of the present work except for the duration of the present proof.

The next two theorems are a corollaries of the proofs of Theorems 6.7.23, 6.7.26, 6.7.28. These theorems discuss among others the simple geometries $\langle Mn;\ L^{Ph}\rangle$. We note that our structure $\langle Mn;\ \equiv^{Ph}\rangle$ is denoted in Friedman [90, p.164, line 11] by the similar notation $\langle M, \lambda\rangle$. In passing we note that the reduct $\langle Mn;\ L^{Ph}\rangle$ of our geometries already forms [an important] geometry. In the general relativistic case this simple geometry can behave in a very non-Euclidean way, e.g. two L^{Ph} lines may meet exactly in two points etc. In such a general relativistic context the "simple" geometry $\langle Mn;\ L^{Ph}\rangle$ is strongly related to what is called a conformal structure [of general relativistic space-time] in Ehlers-Pirani-Schild [78]. Studying only the $\langle Mn;\ L^{Ph}\rangle$ geometry (of general relativity) in itself can lead to interesting insights. We also note that the other geometry $\langle Mn;\ L^{T}\rangle$ is called a projective structure of general relativity in the same work [78].

elmagyarázni hogy hogyan alakult ki a szögtartóságból ez a fajta relativisztikus konformális fogalom relativisztikus ortogonalitásra való koncenrálással.)

THEOREM 6.7.30 *Statements (i)–(iii) below hold for any Th satisfying condition (⋆) way below.*

(i) $eq, Col, Col^T, Col^S, Bw, \perp$ *<u>are definable</u> from* $\langle Mn;\ Col^{Ph}\rangle$ *in* $\mathsf{Ge}(Th)$*, assuming* $n > 2$.

(ii) $eq, Col, Col^{Ph}, Col^S, Bw, \perp$ *<u>are definable</u> from* $\langle Mn;\ Col^T\rangle$ *in* $\mathsf{Ge}(Th)$.

(iii) $eq, Col, Col^T, Col^{Ph}, Bw, \perp$ *<u>are definable</u> from* $\langle Mn;\ Col^S\rangle$ *in* $\mathsf{Ge}(Th)$.

(⋆) $Th \models (\mathbf{Newbasax} + \mathbf{Ax(}Triv_t\mathbf{)}^- + \mathbf{Ax(}\sqrt{\ }\mathbf{)} + \mathbf{Ax(diswind)})$.

Proof: Before reading the proof cf. Fig.282 (p.863). The proof is based on the proofs of Theorems 6.7.23, 6.7.26, 6.7.28. Cf. also Thm.6.2.74 on p.878. ∎

THEOREM 6.7.31 *Statements (i)–(iii) below hold for any Th satisfying condition (⋆) way below.*

(i) $Col, Col^T, Col^S, Bw, \perp$ *<u>are definable</u> from* $\langle Mn;\ Col^{Ph}\rangle$ *in* $\mathsf{Ge}(Th)$*, assuming* $n > 2$.

(ii) $Col, Col^{Ph}, Col^S, Bw, \perp$ *<u>are definable</u> from* $\langle Mn;\ Col^T\rangle$ *in* $\mathsf{Ge}(Th)$.

(iii) $Col, Col^T, Col^{Ph}, Bw, \perp$ *<u>are definable</u> from* $\langle Mn;\ Col^S\rangle$ *in* $\mathsf{Ge}(Th)$.

(⋆) $Th \models (\mathbf{Bax}^{\oplus} + \mathbf{Ax(}Triv_t\mathbf{)}^- + \mathbf{Ax(}\sqrt{\ }\mathbf{)} + \mathbf{Ax(diswind)})$.

Proof: Before reading the proof cf. Fig.282 (p.863). The theorem follows from Thm.6.7.30 above and Thm.6.6.105 on p.1127. ∎

THEOREM 6.7.32 *Assume $n > 2$. Then Col^T and Bw are definable from $\langle Mn;\ Col^{Ph}\rangle$ in* $\mathsf{Ge}(\mathbf{Reich}(\mathbf{Bax})^{\oplus} + \mathbf{Ax(diswind)})$.

Proof: The theorem follows by Thm.6.7.30 above and Thm 6.6.107 (p.1127). ∎

For completeness, we note that $\mathbf{Ax}(\sqrt{\ })$ is "necessary" in Thm.6.7.23 and in items (i) of Theorems 6.7.30, 6.7.31 in the following sense:

There is an ordered field $\mathfrak{F}$ such that in the Minkowskian geometry $Mink_{nonE}(3,\mathfrak{F})$ Col^T_μ is <u>not</u> definable from $\langle {}^3F, Col^{Ph}_\mu\rangle$ because this geometry has an automorphism h for which $(\exists \ell \in L^T_\mu)\ h[\ell] \notin L^T_\mu$. (As a contrast *Col* is definable in the same structure by Thm.6.7.34.) Of course, this geometry cannot be completed to a model of **Basax** (because of our theorem that $\mathbf{Basax}(3) \models \mathbf{Ax}(\sqrt{\ })$). In other words: The Alexandrov-Zeeman theorem does <u>not</u> generalize to usual 4 (or 3) dimensional Minkowskian geometries $Mink_{nonE}(\mathfrak{F})$ over arbitrary ordered fields $\mathfrak{F}$ (from $Mink(\mathfrak{R})$). To such a generalization we need to assume that $\mathfrak{F}$ is Euclidean. In connection with this we state Thm.6.7.33.

THEOREM 6.7.33

(i) *Thm.6.7.23 (p.1159) generalizes to the Minkowskian geometry $Mink_{nonE}(n,\mathfrak{F})$ (for $n > 2$) over an ordered field $\mathfrak{F} = \langle \mathbf{F};\ \leq\rangle$ iff the ordering $\leq$ is definable (by a first-order formula) over $\mathbf{F}$. More concretely:*

(ii) *Let $n > 2$, and let $\mathfrak{F}$ be arbitrary. Consider the Minkowskian geometry $Mink_{nonE}(n,\mathfrak{F})$. Then (a) and (b) below hold.*

(a)

$$\Big(Col^T_\mu \text{ or } Col^S_\mu \text{ or } Bw_\mu \text{ is definable from } \langle {}^nF;\ Col^{Ph}_\mu\rangle \text{ in } Mink_{nonE}(n,\mathfrak{F})\Big)$$
$$\Downarrow$$
$$\Big(\leq \text{ is definable from } \mathbf{F} \text{ in } \mathfrak{F} = \langle \mathbf{F};\ \leq\rangle\Big).$$

(b)

$$\Big(\langle Col_\mu, Col^T_\mu, Col^S_\mu, Bw_\mu, \perp_\mu, eq_\mu\rangle \text{ is definable from } \langle {}^nF;\ Col^{Ph}_\mu\rangle \text{ in } Mink_{nonE}(n,\mathfrak{F})\Big)$$
$$\Updownarrow$$
$$\Big(\leq \text{ is definable from } \mathbf{F} \text{ in } \mathfrak{F} = \langle \mathbf{F};\ \leq\rangle\Big).$$

The **proof** is based on Lemma 6.7.5 on p.1139, and it is available from Judit Madarász. ∎

THEOREM 6.7.34 *Let $n > 2$ and let $\mathfrak{F}$ be arbitrary. Then Col_μ is definable from $\langle {}^nF;\ Col_\mu^{Ph} \rangle$ in $Mink_{nonE}(n, \mathfrak{F})$.*

The **proof** is available from Judit Madarász. ∎

THEOREM 6.7.35 *Let $n > 2$ and let $\mathfrak{F}$ be arbitrary. Consider the geometry $Mink_{nonE}(n, \mathfrak{F})$. Then with the exception of g_μ^2 and 1, the whole of $Mink_{nonE}(n, \mathfrak{F})$ is definable from $\langle {}^nF;\ \prec_\mu \rangle$ as well as from $\langle {}^nF;\ \succ_\mu^r \rangle$, where $\succ_\mu^r$ denotes Robb's "after" and is defined as:*

$$q \succ_\mu^r p \quad \overset{\text{def}}{\Longleftrightarrow} \quad p_t < q_t \ \wedge\ (\exists \ell \in \mathsf{SlowEucl} \cup \mathsf{PhtEucl})\, p, q \in \ell,$$

for any $p, q \in {}^nF$.

The **proof** is available from Judit Madarász. We note that, as we already said in item (II), $\succ_\mu^r$ can be defined from $\prec_\mu$ as follows: $b \succ_\mu^r a \ \Leftrightarrow\ [\, b \neq a \ \wedge\ (\forall c)(b \prec_\mu c \ \rightarrow\ a \prec_\mu c)\,]$.

∎

The following theorem says that Theorems 6.7.35, 6.7.34 above do not generalize to $n = 2$ even if we assume that $\mathfrak{F} = \mathfrak{R}$ the ordered field of reals.

THEOREM 6.7.36 *Col_μ is not definable from either $\langle {}^2\mathrm{R};\ \prec_\mu \rangle$ or $\langle {}^2\mathrm{R};\ Col_\mu^{Ph}, \prec_\mu \rangle$.*

Proof: The proof of Thm.6.7.24 on p.1159 goes through for the present case, too. ∎

(III) On $\equiv^T$, $\equiv^{Ph}$, $\equiv^S$

In this item we concentrate on $\mathfrak{G}_{\mathfrak{M}}^{\equiv}$ instead of $\mathfrak{G}_{\mathfrak{M}}$. (We mentioned that we will usually identify the two.) The connection between g and L will be discussed beginning with Def.6.8.2 (Geodesics, p.1179) way below, therefore we do not go into that here.[1216] To define e.g. L^T from L and g we do need $\equiv^T$. (In the standard literature g is defined in such a way that $\equiv^T$ is recoverable from g, cf. Remark 6.2.45 on p.849.) Similarly for L^{Ph} and $\equiv^{Ph}$ etc. This is the reason why we included $\equiv^T$ etc. in our language. Although one can define $\equiv^T$ from L^T, we wanted a simple device like $\equiv^T$ which would help us to define L^T directly from the pseudo-metric g. Similar considerations apply to $\equiv^S$ (we omit them). So, in some sense we consider $\equiv^T$ as an additional "part" of g, or in other words an extra datum for using g. Thus our "pseudo-metric" could be considered the tuple $\langle g, \equiv^T, \equiv^{Ph}, \equiv^S \rangle$ or equivalently the structure $\langle Mn, \mathbf{F_1};\ g, \equiv^T, \equiv^{Ph}, \equiv^S \rangle$.

[1216] In our present setting L^T etc. can be recovered from $g, \equiv^T, \equiv^{Ph}$ etc. by the Alexandrov-Zeeman and Latzer type theorems above and below. The purpose of geodesics goes beyond these concerns and is connected with e.g. accelerated observers and further generalizations.

In the following theorem, we briefly discuss what parts of our geometry $\mathfrak{G}_{\mathfrak{M}}$ can be obtained from $\equiv^{Ph}$. We could call this theorem an Alexandrov-Zeeman type theorem, cf. Theorems 6.7.30–6.7.32 (p.1164) and the text above Thm.6.7.26. The theorem is also a generalization of a result of Latzer [159].

THEOREM 6.7.37 *Assume $n > 2$. Then (i)–(iii) below hold.*

(i) *Let Th be as in ($\star$) of Thm.6.7.30 (p.1164).*
Then $eq, Col, Col^T, Col^{Ph}, Col^S, \mathrm{Bw}, \perp$ are definable from $\langle Mn; \equiv^{Ph}\rangle$ in $\mathsf{Ge}(Th)$.

(ii) *Let Th be as in ($\star$) of Thm.6.7.31 (p.1164).*
Then $Col, Col^T, Col^{Ph}, Col^S, \mathrm{Bw}, \perp$ are all definable from $\langle Mn; \equiv^{Ph}\rangle$ in $\mathsf{Ge}(Th)$.

(iii) *Col^T, Col^{Ph}, Bw are definable from $\langle Mn; \equiv^{Ph}\rangle$ in $\mathsf{Ge}(\mathbf{Reich}(\mathbf{Bax})^{\oplus} + \mathbf{Ax(diswind)})$.*

Proof: The theorem follows by Theorems 6.7.30–6.7.32 (p.1164) and by the fact that Col^{Ph} is definable from $\equiv^{Ph}$ in $\mathsf{Ge}(\mathbf{Reich}(\mathbf{Bax})^{\oplus} + \mathbf{Ax(diswind)})$ as follows.

$$Col^{Ph}(a,b,c) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad a \equiv^{Ph} b \equiv^{Ph} c \equiv^{Ph} a. \quad \blacksquare$$

The next results are here because they are corollaries of our above theorems. Namely, using our above results we can show properties of $g, eq, \perp$ which are of interest in themselves. In more detail, below we briefly indicate that under some assumptions, almost the whole of the geometry $\mathfrak{G}_{\mathfrak{M}}$ is recoverable from (relativistic) distance as measured by either eq or g. The non-recoverable part is $\prec$ (which cannot be recovered since it involves "direction of time" or "direction of causality", which is usually asymmetric and is not "coded" in g very roughly because $g(a,b) = g(b,a)$).

COROLLARY 6.7.38 Let Th be as in Thm.6.7.31, $n > 2$. Then in $\mathsf{Ge}(Th)$, all parts of our geometry are definable from eq with the exception of $\prec, g, \mathbf{F_1}$.

On the proof: We use Thm.6.7.37(ii). From eq first we define $\sim$ as follows.

$$e \sim e_1 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists e_2)(eq(e,e_2,e,e_2) \ \wedge \ eq(e_1,e_2,e_1,e_2)).$$

Then we define $\equiv^{Ph}$ as follows.

$$e \equiv^{Ph} e_1 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad [\, e \sim e_1 \ \wedge \ (e = e_1 \ \vee \ \neg eq(e,e_1,e,e_1))\,].$$

Now we apply Thm.6.7.37(ii). ∎

The next theorem says that all of $\mathfrak{G}_{\mathfrak{M}}$ is <u>definable from g</u>, with the exception of $\prec$, under certain conditions.

THEOREM 6.7.39 *All parts of* $\mathsf{Ge}(Th)$ *are <u>definable</u> over its reduct* $\langle Mn, \mathbf{F_1}; g\rangle$ *with the only exception of* $\prec$*, assuming Th is as in Thm.6.7.30 and* $n > 2$.

On the proof: Assume Th. Then we can define $\equiv^{Ph}$ from $\langle g, \mathbf{F_0}\rangle$ as follows.[1217]

$$e \equiv^{Ph} e_1 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad g(e, e_1) = 0.$$

Now we apply Thm.6.7.37. ∎

Actually, in the above proof we did not use the whole of $\mathbf{F_1}$ but only $\mathbf{F_0}$.

COROLLARY 6.7.40 All parts of $\mathsf{Ge}(Th)$ are <u>definable</u> over its reduct $\langle Mn, \mathbf{F_0}; g\rangle$ with the only exception of $\prec$ and 1 (of $\mathbf{F_1}$), assuming the conditions of Thm.6.7.39 above. ∎

Items 6.7.38–6.7.40 above generalize (and formalize) the "general wisdom" from relativity theory saying that "everything is recoverable from relativistic distance" (or somewhat sloppily, from the "pseudo-metric") with the exception of $\prec$ since $\prec$ is not symmetric. Of course, if we want, we can modify g such that even $\prec$ will be recoverable from the new, non-symmetric pseudo-metric $g^{\prec}$. We will further explore this possibility in §6.7.3.

COROLLARY 6.7.41 Let Th be as in Thm.6.7.30, $n > 2$. Then in $\mathsf{Ge}(Th)$, all parts of our geometry are definable from the 4-ary relation $\perp$ on points with the exception of $\prec, g, \mathbf{F_1}$.

On the proof: First, we define $\equiv^{Ph}$ as follows.

$$a \equiv^{Ph} b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (a = b \ \vee \ \langle a, b\rangle \perp \langle a, b\rangle).$$

Now, we apply Thm.6.7.37. ∎

THEOREM 6.7.42
In $\mathsf{Ge}(\mathbf{Flxbasax}(n))$*, <u>no one</u> of* $Col, Bw, \perp, eq, g, Col^T, Col^{Ph}, Col^S$ *is <u>definable</u> from the <u>rest</u> of our geometry (i.e. from* $\prec, \equiv^T, \equiv^{Ph}, \equiv^S, \mathbf{F_1}$*).*

[1217] Let us notice that if e and e_1 are not photon-like separated then either $g(e, e_1)$ is undefined or $g(e, e_1) > 0$.

On the proof: We leave it to the reader to modify the proof of Thm.6.7.13 in order to obtain a proof for the present theorem. (A different possible approach is the following. Let $n > 2$ and $\mathfrak{M} \models$ **NewtK**. Let $m \in Obs$ be fixed and S be the space of m. Pretend that $S \subseteq Mn$. Take a permutation of Mn which leaves all points outside of S fixed. This permutation will be an automorphism of $\langle Mn, \mathbf{F_1}; \prec, \equiv^T, \equiv^{Ph}, \equiv^S\rangle$. If we choose this permutation suitably than it will not preserve any one of $Col, Bw, \perp, eq, g, Col^T, Col^{Ph}, Col^S$.) ∎

The following should be known from field theory.

Exercise level question 6.7.43

(i) Is there an elementary class K of ordered fields such that for each $\mathfrak{F} \in \mathsf{K}$ the ordering $\leq$ of $\mathfrak{F}$ is definable by a first-order formula from $0, 1, +, \cdot$, but there is no formula uniformly defining $\leq$ from $0, 1, +, \cdot$ in K?

(ii) Is there an ordered field $\mathfrak{F} = \langle \mathbf{F}; \leq\rangle$ such that $\leq$ is definable from $\mathbf{F}$, and such that there is an ordering $\leq'$, different from $\leq$, on F such that $\mathfrak{F}' := \langle \mathbf{F}, \leq'\rangle$ is an ordered field too?

◁

Item 6.7.44 The answer to Question 6.7.9 (p.1141) will turn out to be "YES" (by Thm.6.7.8 on p.1140) if **(i) or (ii)** below hold .

(i) The answer to question 6.7.43(ii) turns out to be "NO".

(ii) Conjecture 6.6.121 (p.1132) is true and the answer to question 6.7.43(i) turns out to be "NO".

◁

6.7.3 The streamlined, partial metric $g^{\prec}$

Recall that the Reichenbachian relativistic geometry[1218] $\mathfrak{G}^R_{\mathfrak{M}} = \langle Mn, \dots, g^R, \mathcal{T}^R\rangle$ associated to $\mathfrak{M}$ is defined in item (VI) of Def.6.2.2 on p.799 and is motivated by

[1218]Reichenbachian relativistic geometry is a short name for Reichenbachian version of the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$.

§4.5. $\mathsf{Ge}^R(Th)$ is the class of Reichenbachian relativistic geometries associated to Th, i.e.

$$\mathsf{Ge}^R(Th) :\stackrel{\text{def}}{=} \mathbf{I}\{ \mathfrak{G}^R_{\mathfrak{M}} \;:\; \mathfrak{M} \in \mathsf{Mod}(Th) \}.$$ [1219]

Definition 6.7.45 Assume $\mathfrak{G}$ is a relativistic (or a Reichenbachian relativistic) geometry.

(i) The reflexive hull $\preceq := \prec \cup \mathrm{Id}$ of $\prec$ is defined as follows:

$$a \preceq b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad [\, a \prec b \;\text{ or }\; a = b\,], \quad a, b \in Mn.$$

(ii) The *time-like-metric*[1220] $g^{\prec}$ is defined to be $g \restriction (\preceq)$, i.e.

$$g^{\prec} \stackrel{\text{def}}{=} \{\, \langle a, b, \lambda\rangle \in Mn \times Mn \times F \;:\; a \preceq b \;\text{ and }\; g(a, b) = \lambda \,\}.$$ [1221]

(iii) $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ is called the *time-like-metric reduct* of $\mathfrak{G}$. For "time-like-metric reduct" we will also use the expressions "time-like-metric geometry", "time-like-metric structure", and "time-like-metric relativistic geometry".

◁

We will see that under some assumptions on $\mathfrak{M}$, $g^{\prec}$ satisfies certain very nice and familiar looking axioms, e.g. is more "streamlined" than g is, from the mathematical point of view, cf. p.1172. Therefore we will often refer to $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ as the streamlined partial metric reduct of $\mathfrak{G}_{\mathfrak{M}}$. Beginning with p.1172 we will see that in many regards $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ is the most streamlined reduct of $\mathfrak{G}_{\mathfrak{M}}$ and at the same time it seems to be rather suitable (to serve as a stepping-stone) for generalizations in the direction of general relativity.

The next theorem says that the Reichenbachian geometry $\mathfrak{G}^R_{\mathfrak{M}}$ is definable from its streamlined, time-like-metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$, under mild assumptions on $\mathfrak{M}$. The second theorem (Thm.6.7.47) says the same for the full geometry $\mathfrak{G}_{\mathfrak{M}}$, under some stronger conditions on $\mathfrak{M}$.

$\mathsf{Ge}^{\prec}(Th)$ definicióját a fejezetke végéről előredobni, és ebben a tételben fölhasználni

THEOREM 6.7.46

[1219] We note that $\mathsf{Ge}^R(Th)$ coincides with $\mathsf{Ge}^5(Th)$, where $\mathsf{Ge}^5(Th)$ was defined on p.1125.
[1220] "Time-like-metric" is the same as "streamlined partial metric".
[1221] I.e. $g^{\prec}(a, b) = g(a, b)$ if $a \preceq b$ else is undefined.

(i) $\mathsf{Ge}^R(Th)$ *is* <u>*definable*</u> *from its streamlined, simple reduct* $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ *more precisely from its reduct of language* $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$, *assuming* $n > 2$ *and* $Th \models \mathbf{Bax}^{-\oplus} + \mathbf{Ax(TwP)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$.

(ii) *Statement (i) above remains true if the assumption* $\mathbf{Ax(TwP)}$ *is replaced by any one of* $\mathbf{R}(\mathbf{Ax\ syt_0}) + \mathbf{Ax}(\mathit{Triv})$, $\mathbf{Bax} + \mathbf{Ax(syt_0)}$.

(iii) *Statements (i) and (ii) above remain true if we omit the assumption* $n > 2$ *and assume instead* $\mathbf{Ax}(\uparrow\uparrow_\mathbf{0})$ *as a substitute.*

Idea of proof:

<u>*Case of (i):*</u> Assume the assumptions. Then $Th \models \mathbf{Ax(eqtime)}$ by Prop.6.8.25 on p.1201 and there are no FTL observers by Thm.4.3.24 on p.497. By these (and by the assumptions, of course), one can check that the following definitions work.

$$\begin{aligned} Col^T(a,b,c) &\overset{\text{def}}{\iff} \Big(g^+(a,b) = g^+(a,c) + g^+(c,b) \ \vee \\ &\qquad g^+(a,c) = g^+(a,b) + g^+(b,c) \ \vee \\ &\qquad g^+(b,c) = g^+(b,a) + g^+(a,c)\Big), \quad \text{where} \\ g^+(a,b) = \lambda &\overset{\text{def}}{\iff} g^{\prec}(a,b) = \lambda \ \vee \ g^{\prec}(b,a) = \lambda. \end{aligned}$$

Bw is definable from Col^T by the <u>proof</u> of Thm.6.7.1 (p.1137) and Fig.344 on p.1162.[1222]

$$\begin{aligned} a \equiv^T b &\overset{\text{def}}{\iff} (\exists c \in Mn)\ Col^T(a,b,c). \\ a \equiv^{Ph} b &\overset{\text{def}}{\iff} a = b \ \vee\ \Big(a \not\equiv^T b \ \wedge\ (\exists c \in Mn)[c \neq b \ \wedge\ c \sim b \ \wedge \\ &\qquad\qquad (\forall d \in Mn)(Bw(b,d,c) \ \to\ a \equiv^T d)]\Big). \\ Col^{Ph}(a,b,c) &\overset{\text{def}}{\iff} a \equiv^{Ph} b \equiv^{Ph} c \equiv^{Ph} a. \\ a \prec b &\overset{\text{def}}{\iff} a \neq b \ \ \wedge\ (\exists \lambda \in F)\, g^{\prec}(a,b,\lambda). \\ g^R(a,b,\lambda) &\overset{\text{def}}{\iff} g^{\prec}(a,b,\lambda) \ \vee\ g^{\prec}(b,a,\lambda) \ \vee (a \equiv^{Ph} b \ \wedge\ \lambda = 0). \end{aligned}$$

$\mathcal{T}^R$ is defined by g^R.

<u>*Case of (ii):*</u> Item (ii) follows by item (i), Thm.4.7.15 (p.622) and Thm.4.2.9 (p.461).

<u>*Case of (iii):*</u> Item (iii) follows by the proof of item (i) and Prop.6.2.32 on p.840. ∎

[1222] To avoid misunderstandings we note that this is Bw for all lines and not only for e.g. L^T or $L^T \cup L^{Ph}$.

THEOREM 6.7.47 *$\mathsf{Ge}(Th)$ is definable from $\langle Mn, \mathbf{F_1};\ g^{\prec}\rangle$ i.e. from its reduct of language $\langle Mn, \mathbf{F_1};\ g^{\prec}\rangle$, assuming $n > 2$ and $Th \models$* **Newbasax** + $\mathbf{Ax}(\omega)^{\sharp\sharp}$ + $\mathbf{Ax}(\sqrt{\ })$ + **Ax(diswind)**.

Idea of proof: Assume the assumptions. By Thm.6.2.60 (p.862) and by Examples 6.2.69 (p.875), the $\prec$-free reducts of members of $\mathsf{Ge}(Th)$ are disjoint unions of $\prec$-free reducts of Minkowskian geometries. Using this fact together with Thm.6.7.46 and the theorems in §6.7.2 one can complete the proof. ∎

Axiomatics of $g^{\prec}$

Under some mild assumptions on $\mathfrak{M}$,[1223] the following simple axioms $\mathbf{G_1}$–$\mathbf{G_4}$ hold in the time-like-metric reduct $\langle Mn, \mathbf{F_1};\ g\rangle$ of $\mathfrak{G}_{\mathfrak{M}}$.

$\mathbf{G_1}$ The domain $\preccurlyeq\ :=\ Dom(g^{\prec})$ is a reflexive partial ordering.

$\mathbf{G_2}$ $g^{\prec}(x,y) \geq 0$ if it is defined.

$\mathbf{G_3}$ $g^{\prec}(x,y) = 0 \quad \Leftrightarrow \quad x = y$.

$\mathbf{G_4}$ $g^{\prec}(x,y) + g^{\prec}(y,z) \leq g^{\prec}(x,z)$ if $x \preccurlyeq y \preccurlyeq z$.

We define the axiom system **busg** as follows.

$$\mathbf{busg} :\stackrel{\text{def}}{=} \mathbf{G_1} + \mathbf{G_2} + \mathbf{G_3} + \mathbf{G_4}.$$

It is interesting to compare **busg** with the usual[1224] axiomatizations of metric spaces (we feel that **busg** is closer to the usual axiomatizations of metrics[1225] than e.g. the axioms which could describe g).

The above axiomatization **busg** is not unrelated to the one given in Busemann [56, p.7]. Unlike Busemann, however, we regard the topology on $\langle Mn, \mathbf{F_1};\ g^{\prec}\rangle$ to be defined from the partial metric $g^{\prec}$ (or from $\prec$) in the style of either Def.6.2.31(ii) (p.838) or of Def.6.2.2(VI) (p.800), i.e. in the style of our defining the Reichenbachian topology $\mathcal{T}^R$ from the Reichenbachian partial metric g^R.[1226] I.e. for $e \in Mn$ and $\varepsilon \in {}^+F$ we let

$$S^{\prec}(e,\varepsilon) :\stackrel{\text{def}}{=} \{\, e_1 \in Mn\ :\ 0 < g^{\prec}(e,e_1) < \varepsilon \,\}.$$

[1223] e.g. $\mathbf{Bax}^{-\oplus}$, **Ax(TwP)**, $\mathbf{Ax}(\sqrt{\ })$, $\mathbf{Ax}(\uparrow\uparrow_0)$ are sufficient

[1224] non-relativistic

[1225] both in complexity and in spirit

[1226] The difference between g^R and $g^{\prec}$ seems to be minor but is not negligible. Else: We note that instead of $g^{\prec}$ we could use $\prec$ for defining the topology in the style of Fig.279, p.839. Cf. Def.6.2.31 (ii), p.838.

Now, our topology $\mathcal{T}^{\prec}$ is the one generated by the subbase

$$\{\, S^{\prec}(e,\varepsilon) \;:\; e \in Mn,\;\; \varepsilon \in {}^{+}F \,\}.$$

When the topology $\mathcal{T}^{\prec}$ is present, we add to **busg** the extra axiom

$\mathbf{G_5}$ $\langle Mn, \mathcal{T}^{\prec}\rangle$ is a Hausdorff (i.e. T_2) space[1227] and $g^{\prec} : Mn \times Mn \overset{\circ}{\longrightarrow} \mathbf{F_0}$ is continuous.

It is shown in Busemann [56] that the topological structure

$$\langle Mn, \mathbf{F_1};\, g^{\prec}, \mathcal{T}^{\prec}\rangle$$

has desirable properties from the point of view of mathematical elegance, and at the same time admits a relatively natural generalization in the direction of general relativity theory (cf. e.g. Busemann [56, p.7, axioms T_1–T_4]).

The generalization in the "local" direction of **busg** tailored for general relativity theory states only that first we are given a Hausdorff topology $\mathcal{T}^{\prec}$ and then for any point $e \in Mn$ there is a neighborhood U_e of e such that a partial ordering $\preccurlyeq_e$ and a partial function $g_e^{\prec}$ are defined on U_e. Then the axioms of **busg** are stated only for the little structures $\langle U_e, \mathbf{F_1};\, \preccurlyeq_e, g_e^{\prec}\rangle$, $e \in Mn$.[1228] In addition to these axioms one has to add some consistency axioms for the case when U_e and $U_{e'}$ overlap. These consistency axioms are rather simple and natural, we do not recall them, they can be found in Busemann [56, p.7] axiom T_4. The so obtained local version of **busg** is completely consistent with (and is applicable to) general relativity theory, cf. Busemann [56] for more information on this. Summing up, the general relativistic versions of the time-like-metric structures $\langle Mn, \mathbf{F_1};\, g^{\prec}, \mathcal{T}^{\prec}\rangle$ look like $\langle Mn, \mathbf{F_1};\, \mathcal{T}^{\prec}, \preccurlyeq_e, g_e^{\prec}\rangle_{e\in Mn}$ (cf. the definition of ${}^{n}\mathbf{F}_1$ on p.42 for the $\langle \ldots, g_e^{\prec}\rangle_{e\in Mn}$ notation). Further, the class of these structures is axiomatized by the list of axioms just quoted from Busemann [56, p.7] (ending with T_4).

In connection with the general relativistic (i.e. localised) structures $\langle Mn, \mathbf{F_1};\, \mathcal{T}^{\prec}, \preccurlyeq_e, g_e^{\prec}\rangle_{e\in Mn}$ we note that although we included the topology $\mathcal{T}^{\prec}$ into the structure, it is definable from the rest $GG := \langle Mn, \mathbf{F_1};\, \preccurlyeq_e, g_e^{\prec}\rangle_{e\in Mn}$. Therefore

[1227] For Hausdorff spaces cf. footnote 1009 on p.1018.

[1228] It would be sufficient to write $\langle U_e, \mathbf{F_1};\, g_e^{\prec}\rangle$, $e \in Mn$ for these structures, since $\preccurlyeq_e$ is obviously definable from $g_e^{\prec}$.

one can define GG without $\mathcal{T}^{\prec}$ and then later one can define $\mathcal{T}^{\prec}$ from GG. Namely, assume $e \in Mn$ and $\varepsilon \in {}^{+}F$. Then

$$S^{\prec}(e,\varepsilon) :\stackrel{\text{def}}{=} \{\, e_1 \in Mn \;:\; 0 < g_e^{\prec}(e,e_1) < \varepsilon \,\}$$

is an open set, and it is an element of the subbase of $\mathcal{T}^{\prec}$ we want to define. Now, we postulate that

$$\{\, S^{\prec}(e,\varepsilon) \;:\; e \in Mn,\ \varepsilon \in {}^{+}F \,\}$$

is a subbase of our topology $\mathcal{T}^{\prec}$. We note this only as a possibility; we do not explore the general relativistic time-like-metric structures GG, in this section any further.

Remark 6.7.48 In the language of time-like-metric structures $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ we could define a kind of collinearity relation $coll^{\prec}$ the following way and could enrich the axiom-system **busg** by adding natural conditions on this collinearity: First we define

$$Bw^{\prec}(a,b,c) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad g^{\prec}(a,c) = g^{\prec}(a,b) + g^{\prec}(b,c).$$

Then we define $coll^{\prec}$ from $Bw^{\prec}$ basically the same way as $coll$ was defined from Bw on p.818.

◁

It would be interesting to know how many further axioms we need to add to **busg** in order to ensure that the partial metric structure $\langle Mn, \mathbf{F_0};\, g^{\prec}\rangle$ comes from a model of one of our relativity theories $\mathsf{Mod}(Th)$. Looking into this might be a nice future research task.

Lehet, hgoy Def.1.1.1 (VI)-ban a Reichenbachi metrikat erdemes kicserelni az uj $g^{\prec}$-re, mert (1) $g^{\prec}$ szebben viselkedik es (2) $\equiv^{Ph}$ arany-lag konnyen defho $g^{\prec}$-bol.

Since the time-like-metric reduct of $\mathfrak{G}_{\mathfrak{M}}$ is an important one we introduce the following distinguished class of geometries. Let Th be a set of frame formulas. Then

$$\mathsf{Ge}^{\prec}(Th) :\stackrel{\text{def}}{=} \mathbf{I}\{\, \langle Mn, \mathbf{F_1};\, g^{\prec}\rangle \;:\; \langle Mn, \mathbf{F_1};\, g^{\prec}\rangle \text{ is the time-like-metric reduct of } \mathfrak{G}_{\mathfrak{M}} \text{ for some } \mathfrak{M} \models Th\}.$$

Recall from p.1174 that the topology $\mathcal{T}^{\prec}$ is definable in $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ therefore we can use $\mathsf{Ge}^{\prec}(Th)$ as if its definition were

$$\mathsf{Ge}^{\prec}(Th) = \mathbf{I}\{\langle Mn, \mathbf{F_1};\, g^{\prec}, \mathcal{T}^{\prec}\rangle \;:\; \ldots \text{ the usual conditions on } \mathcal{T}^{\prec} \;\; ^{1229}\,\}.$$

[1229]e.g. the definition of $\mathcal{T}^{\prec}$ on p.1174 is suitable for this

6.7.4 Relativistic incidence geometries

Let $\mathfrak{G}^{inc}_{\mathfrak{M}}$ be the reduct

$$\mathfrak{G}^{inc}_{\mathfrak{M}} \stackrel{\text{def}}{:=} \langle Mn, L;\ L^T, L^{Ph}, L^S, \in \rangle$$

of $\mathfrak{G}_{\mathfrak{M}}$. We call $\mathfrak{G}^{inc}_{\mathfrak{M}}$ the *incidence geometry* associated to $\mathfrak{M}$.

$\mathsf{Ge}^{inc}(Th) \stackrel{\text{def}}{:=} \mathbf{I}\{\, \mathfrak{G}^{inc}_{\mathfrak{M}} : \mathfrak{M} \models Th \,\}$ is the class of relativistic incidence geometries associated to Th.

It is attractive to discuss relativistic incidence geometries, since they look "pure and clean" in their language and since they look so similar to incidence geometries $\langle Points, Lines;\ \in \rangle$ known from Euclidean geometry, projective geometry etc. Despite of this apparent "purity", we know that

($\star$) all parts of our geometries $\mathsf{Ge}(Th)$ are definable from $\mathsf{Ge}^{inc}(Th)$, with the exception of $\prec, g, \mathbf{F_1}$, assuming $n > 2$ and $Th \models$ **Newbasax** $+$ $\mathbf{Ax}(Triv_t)^-$ $+$ $\mathbf{Ax}(\sqrt{\ })$ $+$ **Ax(diswind)**, cf. Theorems 6.7.30, 6.7.31

I.e. almost all parts of $\mathfrak{G}_{\mathfrak{M}}$ are definable from the "pure and nice" $\mathfrak{G}^{inc}_{\mathfrak{M}}$, assuming some conditions. This implies two things:

(1) We can base our study of relativistic geometry on the "nice and pure" incidence geometries $\mathfrak{G}^{inc}_{\mathfrak{M}}$ (under some assumptions) if we want to. (The prize is that we loose $\prec, g, \mathbf{F_1}$ [but we can use eq in place of g in many situations].) Perhaps it would be a useful future activity to rewrite the present chapter (Chapter 6) with first concentrating on the incidence geometries $\mathsf{Ge}^{inc}(Th)$, and later introducing the parts like g etc. not definable over $\mathsf{Ge}^{inc}(Th)$ when they are needed. Then one could compare the two versions of "Chapter Geometry" and discuss the advantages of both.

(2) In the present work we do not need to discuss the "attractive" geometries $\mathsf{Ge}^{inc}(Th)$ since in definitionally equivalent forms they were already discussed: cf. e.g. Theorems 6.7.30, 6.7.31, p.1164 *and* the duality theory $(\mathcal{G}o, \mathcal{M}o)$ in §6.6.4 (pp. 1069–1078). Our reason for referring to the $(\mathcal{G}o, \mathcal{M}o)$-duality is that on the geometry side it uses ingredients definable over the incidence geometries $\mathsf{Ge}^{inc}(Th)$, with the exception of $\prec$. It does not seem hard to adapt the $(\mathcal{G}o, \mathcal{M}o)$-duality to the $\prec$-free reduct. Of course, in this generalization, one has to adjust the assumptions on the relativity theories Th.

It might be a useful future research task to generalize our $(\mathcal{G}o, \mathcal{M}o)$-duality to (i) the $\prec$-free reducts of our geometries and (ii) to $\mathsf{Ge}^{inc}(Th)$ in place of $\mathsf{Ge}^{0}(Th)$. This would yield a duality of the pattern

$$\mathsf{Mod}(Th) \;\overset{\longrightarrow}{\longleftarrow}\; \mathsf{Ge}^{inc}(Th)$$

with some assumptions on Th. Of course, one should try to make as few and weak assumptions on Th as possible.

In this connection we note that our $(\mathcal{G}, \mathcal{M})$-duality is of the pattern

$$\mathsf{Mod}(Th) \;\overset{\longrightarrow}{\longleftarrow}\; \mathsf{Ge}(Th)$$

while the $(\mathcal{G}o, \mathcal{M}o)$-duality yields the pattern

$$\mathsf{Mod}(Th) \;\overset{\longrightarrow}{\longleftarrow}\; \mathsf{Ge}^{0}(Th)$$

(with appropriate assumptions on Th in both cases, of course).[1230] The new duality would be of the pattern

$$\mathsf{Mod}(Th) \;\overset{\longrightarrow}{\longleftarrow}\; \mathsf{Ge}^{inc}(Th).$$

Here we do not discuss the just outlined “incidence geometries only” direction further.

[1230]The $(\mathcal{G}o, \mathcal{M}o)$-duality does more than this, since it also yields a pattern

$$\mathsf{Mod}(Th) \;\overset{\longrightarrow}{\longleftarrow}\; \mathsf{Mog}(TH),$$

where Th and TH are in two <u>different languages</u>.

6.8 Geodesics

In the present section we discuss geodesics which, among other things, will help us to understand the connections between g and L. In later work, in moving in the direction of general relativity, geodesics will play an important role (they do so already in the case of accelerated observers even in "flat" space-time).[1231] In moving towards general relativity geodesics will *replace* L as possible life-lines of inertial bodies. (They will play other important roles, e.g. they can be used for recognizing curvature of space-time). At the same time, studying geodesics may be considered as a continuation of §6.7 discussing recoverability of various parts (or reducts) of our relativistic geometries from each other. Geodesics can be regarded as an attempt to recover the lines of our geometry, basically, from g, in a style different from the Alexandrov-Zeeman style proofs in AMN [18, §6.7.2].[1232] For completeness we note that by Corollary 6.7.15 in AMN [18], p.1145, the present author proved that L and $\perp$ are first-order logic definable from *eq* as well as from g under some reasonable assumptions on $\mathfrak{M}$ (e.g. $(\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)})$ is sufficient for this).

Though we will not prove this, by using geodesics one can recover from $g, \equiv^T, \equiv^{Ph}, \equiv^S$ [1233] the potential life-lines of inertial bodies even when the axiom **Det** [1234] is not assumed (but certain conditions are still needed, of course). Roughly speaking, in generalizations of our geometries in the direction of general relativity (cf. e.g. the geometries GG on p.1173 in §6.7.3), geodesics will remain suitable for representing life-lines of inertial bodies. Further, time-like geodesics will be the possible life-lines of inertial observers, photon-like geodesics will be the life-lines of photons, while space-like geodesics can be regarded as potential life-lines of hypothetical

[1231] Cf. e.g. [24], [19], [23]. For completeness we note that sometimes geodesics are used in special relativity, too, cf. e.g. Friedman [90, pp.125-126, 128ff].

[1232] To be able to use g we will need its codomain $\mathbf{F_0}$, too. To make our life easier we will also use $\equiv^T$, $\equiv^{Ph}$, $\equiv^S$ but with sufficient (coding) effort these data could be recovered from g, where g is understood together with its domain Mn and codomain $\mathbf{F_0}$. We will not discuss here how, under sufficient conditions $\equiv^T$, $\equiv^{Ph}$ are recoverable from $\langle Mn, \mathbf{F_0}; g\rangle$. Cf. Remark 6.2.45 on p.849. Cf. also the first 15 lines of (III) on p.1166. On p.1150 we used $\mathbf{F_1}$ as the codomain of g. The reason for the difference is that here we think of g slightly differently than we did there. So this is not an inconsistency, but simply a change in perspective. The choice of perspective depends on for what purposes we want to use g. (Once we identify it with $\langle Mn, \mathbf{F_0}; g\rangle$ and once with $\langle Mn, \mathbf{F_1}; g\rangle$.) For completeness we note that $\equiv^T, \equiv^{Ph}, \equiv^S$ are definable from g (more precisely, from $\langle Mn, \mathbf{F_0}; g\rangle$) if $n > 2$ and some conditions hold, cf. items 6.7.38-6.7.39 (p.1167) in AMN [18].

[1233] and $\mathbf{F_0}$, Mn of course

[1234] Cf. §6.5, p.992 for **Det** (**Det** says that "points determine lines").

FTL particles called tachyons in the literature (assuming such things exist); all this is understood under sufficient conditions. Already in the world-view of an accelerated observer[1235], say m, it will be convenient to say that for m the life-lines of inertial bodies are geodesics [determined by $g, \mathbf{F_0}, \equiv^T, \equiv^{Ph}$] because in the world-view $w_m : {}^nF \longrightarrow Mn$ of m the Euclidean lines of ${}^n\mathbf{F}$ do not necessarily correspond to inertial bodies (if m is really accelerated).[1236]

To make a long story short, the present section on geodesics intends to prepare the road for generalizations (in the direction of general relativity). For further motivation we refer to Figure 355 on p.1208, to Figure 281 on p.855 and to Figure 308 on p.1002. For further motivation we refer to Figure 355 (p.1208), Figure 281 (p.855) and Figure 308 (p.1002).

Remark 6.8.1 We note that we could have based our theory of geodesics entirely on the streamlined, time-like metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$ of $\mathfrak{G}_{\mathfrak{M}}$. This would have advantages (i) from the point of view of aesthetics and (ii) from the point of view of generalizability towards general relativity (as the latter is illustrated in Busemann [56]). To save space we use below a "bigger" reduct. We leave it as a future research task to elaborate a version of the present section (§6.8 "Geodesics") based entirely on the streamlined, time-like metric reduct $\langle Mn, \mathbf{F_1};\, g^{\prec}\rangle$.

$\triangleleft$

We base our definition of geodesics in $\mathfrak{G}_{\mathfrak{M}}$ (Def.6.8.2) below on the definition of geodesics in e.g. Busemann [55], [56], cf. also Busemann-Beem [**?**]. Part of the relevant mathematical literature uses the same kind of definition while another part uses a definition (of geodesics) which goes e.g. via using derivatives[1237]. (Within this, they distinguish "affine geodesics" and "metric geodesics" which distinction is nicely illuminated e.g. in Friedman [90, pp.349,357].)[1238] Busemann's version is simpler (as far as we have a metric around). One might think that a large part of the literature uses the derivatives oriented version because that is needed for general relativity. However, this is not the case since Busemann [56] shows that

[1235] Cf. e.g. [23] and the relevant parts of this work.

[1236] A more important point will be that in general relativity the life-lines of inertial bodies do not satisfy the axiom **Det**, i.e. different geodesics can meet in several points. This is true in the approximation of general relativity built on "special relativity" + "accelerated observers" + "Newtonian approximations" in Rindler [224, §7.7, e.g. item (7.28) on p.124].

[1237] Cf. e.g. d'Inverno [75, pp.75, 83, 99] or Misner-Thorne-Wheeler [196], or Hawking-Ellis [126], or Hicks [132, pp.19,27].

[1238] In Friedman [90, p,357] it is explained that the above "metric-affine" distinction behaves differently in non-relativistic geometries and in relativistic ones (this might perhaps be related to our Corollary 6.8.21).

general relativity can be based on his simple definitions.[1239] So, here we stick with Busemann's simple definition (especially because in the introduction to AMN [18] we adopted a policy to keep things as simple as possible, postponing the introduction of more complicated ideas to the point where they become useful/needed). A further motivation for adopting Busemann's definition of geodesics is that Busemann [55] is an ambitious mathematics (modern geometry) book whose main subject matter is the study of geodesics.

The definition of geodesics (Def.6.8.2) below is not intended to be a first-order logic definition over (a reduct of) the structure $\mathfrak{G}$. This causes no harm to our first-order logic oriented philosophy (for building up physical theories). We will return to discussing this briefly in Remark 6.8.3 below the definition.

Definition 6.8.2 (Geodesics) Assume $\mathfrak{G}$ is a relativistic geometry.

1. Throughout $\mathbf{F_0} = \langle F; 0, +, \leq \rangle$ is the ordered group reduct of the sort $\mathbf{F_1}$ of $\mathfrak{G}$.

2. The *pseudo-metric reduct* M of $\mathfrak{G}$ is defined as follows.

$$M :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_0};\ g, \equiv^T, \equiv^{Ph}, \equiv^S \rangle.$$

   In the definition of geodesics of the geometry $\mathfrak{G}$ we will use only its pseudo-metric reduct. If we wanted to concentrate on the time-like geodesics, then it would be sufficient to use the streamlined, time-like-metric reduct $\langle Mn, \mathbf{F_1};\ g^{\prec} \rangle$ discussed in §6.7.3 (p.1170).

3. Let $\ell \subseteq Mn$. Then ℓ is called a *photon-like geodesic* iff

$$(\forall a, b \in \ell)\ a \equiv^{Ph} b.$$

   Any photon-like geodesic is also called a *photon-like quasi geodesic*, and a *photon-like Archimedean geodesic*.

4. By an *interval of* $\mathbf{F_0}$ we mean an open interval

$$(x, y) := \{\, z \in F\ :\ x < z < y \,\},$$

   where $x, y \in F \cup \{-\infty, \infty\}$, and $x < y$.[1240]

[1239] It seems a more likely explanation that the derivatives-oriented version is suitable for discussing the metric geodesic affine geodesic distinction and that it can be used on a level of abstraction where we throw g and eq away (i.e. we don't have a metric) e.g. in differential topological approaches to relativity.

[1240] In this section $-\infty \neq \infty$ deviating from our convention on p.534 of AMN [18]. As usual, $-\infty < x < \infty$, for any $x \in F$.

5. Let $\ell \subseteq Mn$. By a *parameterization* of ℓ we understand a function h mapping an interval of $\mathbf{F_0}$ onto ℓ, such that h is locally distance preserving, i.e. for any $z \in Dom(h)$ there is $\varepsilon \in {}^+F$ such that, letting $D := (z - \varepsilon, z + \varepsilon)$, $(*)$ below holds.[1241]

   $(*)$ $h \restriction D$ is distance preserving, i.e.
   $$(\forall x, y \in D)\, g(h(x), h(y)) = |x - y|.$$

   If ℓ admits such a parametrization, then we call it a *parametrizable curve*.

6. Let $\ell \subseteq Mn$. ℓ is called a *time-like quasi geodesic* iff there is a parameterization h of ℓ such that for every $z \in Dom(h)$ there is $\varepsilon \in {}^+F$ such that, for $D := (z - \varepsilon, z + \varepsilon)$, $(**)$ below holds.

   $$(**) \qquad (\forall x, y \in D)\, h(x) \equiv^T h(y).$$

7. A time-like quasi geodesic ℓ is called a *short time-like geodesic* iff there is a parameterization h of ℓ such that, for $D := Dom(h)$, $(*)$ and $(**)$ above hold.

8. Let ℓ, h be as in item 5 above. Intuitively, ℓ is a *space-like quasi geodesic* if it is a union of "intervals" $h[D]$ each one of which consists of events 1/2-simultaneous for some observer, cf. Figure 345. Formally:

   ℓ is called a *space-like quasi geodesic* iff there is a parameterization h of ℓ such that for any $z \in Dom(h)$ there is $\varepsilon \in {}^+F$ such that, for $D := (z - \varepsilon, z + \varepsilon)$, $(* * *)$ below holds. Intuitively, the second part of $(* * *)$ says that there is an observer who thinks that all the events in $h[D]$ are 1/2-simultaneous, cf. Figure 345.

   $$(* * *) \quad \begin{array}{l} (\forall x, y \in D)\, h(x) \equiv^S h(y) \text{ and} \\ \text{there are a short time-like geodesic } \ell' \text{ and } a \in \ell' \\ \text{such that } (\forall x \in D)(\exists c, d \in \ell') \\ [c \neq d \ \wedge \ g(a, c) = g(a, d) \ \wedge \ c \equiv^{Ph} h(x) \equiv^{Ph} d\,], \end{array}$$ [1242]

   see Figure 345.

[1241] Note that such a parameterization h : "interval of $\mathbf{F_0}$" $\longrightarrow \ell$ is always continuous w.r.t. the natural topology on $\mathbf{F_0}$ and the topology induced by g on ℓ. I.e. condition $(*)$ (postulated for every D as above) implies this kind of continuity. This continuity is slightly weaker than continuity w.r.t. the topology $\mathcal{T}$ of $\mathfrak{G}$; the latter amounts to viewing h as h : "interval of $\mathbf{F_0}$" $\longrightarrow Mn$.

[1242] We note for "general relativitists" that if we make the above condition local by requiring $\ell' \cap D \neq \emptyset$ then the condition will get only stronger which means that our theorems will get weaker, i.e. omitting this locality condition makes our theorems stronger.

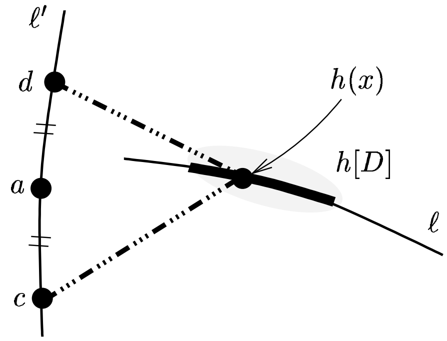


Figure 345: Illustration for $(***)$.

9. Let $\ell \subseteq Mn$. ℓ is called a <u>*quasi geodesic*</u> iff it is a time-like or a photon-like or a space-like quasi geodesic.

10. A quasi geodesic ℓ is called a <u>*time-like geodesic*</u> iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $\varepsilon \in {}^{+}F$ such that for any $z \in (x, y)$, letting $D := (z - \varepsilon, z + \varepsilon)$, $(*)$ and $(**)$ above hold.

11. A quasi geodesic ℓ is called a <u>*space-like geodesic*</u> iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $\varepsilon \in {}^{+}F$ such that for any $z \in (x, y)$, letting $D := (z - \varepsilon, z + \varepsilon)$, $(*)$ and $(***)$ above hold.

12. A quasi geodesic ℓ is called a <u>*geodesic*</u> iff it is a time-like or a photon-like or a space-like geodesic.

13. A geodesic ℓ is called a <u>*time-like Archimedean geodesic*</u> iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $k \in \omega$ and intervals $I_0, \ldots, I_k$ of $\mathbf{F_0}$ such that

    $(x, y) \subseteq I_0 \cup \ldots \cup I_k \quad \wedge \quad (\forall i \in k)\, I_i \cap I_{i+1} \neq \emptyset \quad \wedge$
    $(\forall i \in (k+1))\, [\, (*) \text{ and } (**) \text{ above hold for } D := I_i\,]$.

14. A geodesic ℓ is called a <u>*space-like Archimedean geodesic*</u> iff there is a parameterization h of ℓ such that for every $x, y \in Dom(h)$ with $x < y$ there is $k \in \omega$ and intervals $I_0, \ldots, I_k$ of $\mathbf{F_0}$ such that

    $(x, y) \subseteq I_0 \cup \ldots \cup I_k \quad \wedge \quad (\forall i \in k)\, I_i \cap I_{i+1} \neq \emptyset \quad \wedge$
    $(\forall i \in (k+1))\, [\, (*) \text{ and } (***) \text{ above hold for } D := I_i\,]$.

15. A geodesic ℓ is called an *Archimedean geodesic* iff it is a time-like or a photon-like or a space-like Archimedean geodesic.

16. A space-like geodesic ℓ is called a *short space-like geodesic* iff there is a parameterization h of ℓ such that, for $D := Dom(h)$, $(*)$ and $(***)$ above hold.

17. A geodesic ℓ is called a *short geodesic* iff it is a photon-like geodesic or it is a time-like short geodesic or it is a space-like short geodesic.

18. A geodesic ℓ is called a *strong geodesic* iff it is either photon-like or there is a parameterization h of ℓ which is continuous w.r.t. the natural topology on $\mathbf{F_0}$ and the relativistic topology $\mathcal{T}$ of $\mathfrak{G}$,[1243] and h satisfies the conditions in the definition of geodesics (in items 10–12 above).[1244] We define the *strong* versions of our special kinds of geodesics defined in items 6–17 completely analogously, i.e. by recuiring that the parameterization h occuring in their definitions is continuous w.r.t. the relativistic topology $\mathcal{T}$ of $\mathfrak{G}$.

19. A geodesic ℓ is called a *maximal geodesic* iff

$$(\forall \text{ geodesic } \ell')[\ell' \supseteq \ell \;\rightarrow\; \ell' = \ell].$$

The definition of "maximality" remains completely analogous for special kinds of geodesics in place of just geodesics. (E.g. a maximal strong space-like quasi geodesic is a strong space-like quasi geodesic which is maximal among the strong space-like quasi geodesics.)

20. A geodesic ℓ is called a *divisible geodesic* iff

$$(\forall a,b \in \ell)\Big(g(a,b) \text{ is defined } \;\rightarrow\; (\exists \kappa \in {}^+F)(\forall \varrho \in {}^+F)(\exists c \in \ell)$$

$$\Big[\frac{g(a,c)}{g(c,b)} = \varrho \quad \wedge \quad g(a,c) + g(c,b) < \kappa\Big]\Big).$$

21. Let $e \in Mn$ and $\varepsilon \in {}^+F$. Let us recall that the ε-neighborhood of e is defined as

$$S(e,\varepsilon) := \{\, e_1 \in Mn \;:\; g(e,e_1) < \varepsilon \,\}.$$ [1245]

[1243] i.e. h is continuous from an interval of $\mathbf{F_0}$ into the topology $\langle Mn; \mathcal{T}\rangle$

[1244] Assume ℓ is a strong geodesic with parameterization h. Then h is a "local" homeomorphism in the sense that $(\forall x \in Dom(h))(\exists \varepsilon \in {}^+F)[\, h \restriction (x-\varepsilon, x+\varepsilon)$ is a homeomorphism w.r.t. the relativistic topology $\mathcal{T}$ of $\mathfrak{G}\,]$. Cf. the notion of a parameterized curve in Hicks [132, p.10] and curves in Kurusa [157]. In passing we note that in general continuity w.r.t. $\langle Mn; \mathcal{T}\rangle$ is stronger than continuity w.r.t. the topology on ℓ induced by g as discussed in footnote 1241 (p.1180). Hence, in general, there are fewer strong geodesics than geodesics.

22. Let $\ell \subseteq Mn$. Then ℓ is called a *weak geodesic* iff

$$(\forall e \in \ell)(\exists \varepsilon \in {}^{+}F)\,[\, g \restriction (\ell \cap S(e,\varepsilon)) \text{ is } \underline{\textit{additive}}\,],$$

where *additivity* means that, letting $D := \ell \cap S(e,\varepsilon)$,

$$(\forall a,b \in D)\ (g(a,b) \text{ is defined})\ \wedge$$
$$(\forall a,b,c \in D)\ [\, g(a,b), g(b,c) \le g(a,c) \ \rightarrow\ g(a,c) = g(a,b) + g(b,c)\,].$$

A quasi geodesic which is also a weak geodesic will be called *locally additive*.[1246]

23. Let $\ell \subseteq Mn$. ℓ is called *additive* iff $g \restriction \ell$ is additive.

24. A weak geodesic ℓ is called a *continuous weak geodesic* iff there is a continuous function h mapping an interval of $\mathbf{F_0}$ onto ℓ.

$\lhd$

We will see in Thm.6.8.20 (p.1197) that the second part of condition $(***)$ on space-like quasi geodesics and geodesics in the above definition is needed, cf. Figure 351 (p.1198).

Remark 6.8.3 (Connections with first-order logic definability) We also note that our definition of geodesics over $\langle Mn, \mathbf{F_0};\ g, \ldots, \equiv^S \rangle$ is not a first-order logic definition in the sense of §6.3. To save space, here we do not address the question of how and under what price[1247] could we turn the definition of geodesics into a first-order logic one. We only note that if we include the geodesics together with their parameterization into $\mathfrak{G}$ obtaining a structure $\langle \mathfrak{G}, \textit{geodesics}, \textit{pameterizations} \ldots \rangle$ as extra sorts[1248], then things can be arranged so that the class of so expanded structures does admit a first-order logic axiomatization. We note that by the above we

[1245] In the case of Minkowskian geometry our notation $S(e,\varepsilon)$ might be ambiguous since it both denotes the relativistic "ε-sphere" and the Euclidean "ε-sphere". Throughout the present section by $S(e,\varepsilon)$ we always mean the relativistic sphere.

[1246] Therefore being locally additive is a property of geodesics and in some situations there may be fewer locally additive geodesics than geodesics.

[1247] we mean under what extra conditions and what modification of the definition of $\mathfrak{G}_{\mathfrak{M}}$

[1248] Actually, it is enough to incude *parameterizations* of geodesics as an extra sort Geod together with an extra inter-sort operation $\mathsf{valueof} : \mathsf{Geod} \times F \rightarrow Mn$. Intuitively, for $h \in \mathsf{Geod}$, $e = \mathsf{valueof}(h,x)$ means that $e = h(x)$, i.e. e is the value of parameterization h at value $x \in F$. Actually, $\mathsf{valueof}$ is a partial function only since we do not want to require $Dom(h) = F$. The details are analogous with the style of Nonstandard Dynamic Logic, cf. e.g. Sain [231], Andréka, Goranko et al. [14].

do not mean that the $\mathfrak{G}$-reduct of $\langle \mathfrak{G}, geodesics, \ldots\rangle$ would determine the rest of the structure (e.g. the sort geodesics) uniquely. We only mean to say that in the expanded structure $\langle \mathfrak{G}, geodesics, \ldots\rangle$ the geodesics would behave well enough for our working with them in accordance with our intuition and for basing our relativity theoretic ideas on them. (This is very much like the difference between standard models of higher-order logic and Henkin-style nonstandard models of that logic. Our expanded structures $\langle \mathfrak{G}, geodesics, \ldots\rangle$ are very much like Henkin-style nonstandard models.)

In passing we note that if we assume enough axioms of special relativity on $\mathfrak{M}$, then geodesics become first-order logic definable over $\langle Mn;\ g, \equiv^T, \equiv^{Ph}, \equiv^S\rangle$, but the greatest value of geodesics is in their use in general relativity where these axioms are not assumed. Hence we do not discuss this direction here. $\lhd$

In passing we note that for the purposes of accelerated observers and general relativity (to come in a later work) "quasi geodesic", "short geodesic" and "geodesic" are "local" concepts while "maximal geodesic" seems to be more on the "global" side. Further, in general relativity the emphasis is on time-like and photon-like geodesics, cf. e.g. Busemann [55, 56] or Ehlers-Pirani-Schild [78].

In the present section we will concentrate on quasi geodesics, geodesics, Archimedean geodesics, and the maximal versions of these geodesics. By our definition,

$$\ell \text{ is an Archimedean geodesic} \quad\Rightarrow\quad \ell \text{ is a geodesic} \quad\Rightarrow\quad \ell \text{ is a quasi geodesic.}$$

The analogous statements hold, respectively, for time-like, space-like, and photon-like versions of Archimedean geodesics, geodesics and quasi geodesics, e.g. (ℓ is an Archimedean time-like geodesic) $\Rightarrow$ (ℓ is a time-like geodesic) $\Rightarrow$ (ℓ is a time-like quasi geodesic). In some of the cases these implications hold in the other direction, too. In connection with this we include Propositions 6.8.4 and 6.8.6 below.

PROPOSITION 6.8.4 *Assume $M = \langle Mn, \mathbf{F_0}; \ldots\rangle$ is the pseudo-metric reduct of a relativistic geometry. Assume that $\mathbf{F_0}$ is isomorphic with the ordered additive group reduct of the ordered field $\mathfrak{R}$ of reals. Let $\ell \subseteq Mn$. Then*

$$\ell \textit{ is an Archimedean geodesic} \quad\Leftrightarrow\quad \ell \textit{ is a geodesic} \quad\Leftrightarrow\quad \ell \textit{ is a quasi geodesic.}$$

The analogous statements hold, respectively, for space-like, time-like and photon-like versions of Archimedean geodesics, geodesics and quasi geodesics.

We omit the easy **proof**. ∎

For stating our next proposition we need the following definition.

Definition 6.8.5 An ordered group $\langle G; 0, +, \leq\rangle$ is said to be _Archimedean_ iff for any $a, b \in G$

$$(\forall i \in \omega) ia < b \quad \Rightarrow \quad a = 0.$$[1249]

$\triangleleft$

We note that an ordered field $\mathfrak{F}$ is Archimedean iff its ordered additive group reduct $\mathbf{F_0}$ is Archimedean in the sense of the above definition.

PROPOSITION 6.8.6 *Assume $M = \langle Mn, \mathbf{F_0}; \ldots\rangle$ is the pseudo-metric reduct of a relativistic geometry. Assume that $\mathbf{F_0}$ is Archimedean. Let $\ell \subseteq Mn$. Then*

$$\ell \textit{ is an Archimedean geodesic} \quad \Leftrightarrow \quad \ell \textit{ is a geodesic.}$$

The analogous statements hold, respectively, for space-like, time-like and photon-like versions of Archimedean geodesics and geodesics.

We omit the easy **proof**. ∎

Let us consider how the notion of geodesics helps us to recover the "truly geometric parts" L^T, L^{Ph} etc. from g and $\equiv^T, \equiv^{Ph}$.

Let us recall that the geometry $\mathfrak{G}_{\mathfrak{M}}$ associated to a model $\mathfrak{M}$ looks like

$$\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \mathbf{F_1}, L; L^T, L^{Ph}, L^S, \in, \prec, \equiv^T, \equiv^{Ph}, \equiv^S, Bw, \perp_r, eq, g, \mathcal{T}\rangle.$$[1250]

Among others, below we compare lines (i.e. elements of L) with geodesics.[1251] We have time-like, photon-like and space-like lines and the same applies to geodesics. This gives us, roughly, 4 kind of comparisons, lines with geodesics in general, and then special lines with special geodesics.

Recall that we call the elements of L lines of $\mathfrak{G}_{\mathfrak{M}}$. Above we defined the geodesics of $\mathfrak{G}_{\mathfrak{M}}$, but they are not necessarily the same as lines of $\mathfrak{G}_{\mathfrak{M}}$. We will elaborate this subject in the following discussion of the theorems which will come soon. We will see that, under some assumptions on Th, all elements of L turn out to be geodesics, i.e.

$$L \subseteq \text{(geodesics)},$$

[1249] Here ia is understood in the sense $3a = a + a + a$.
[1250] As we already said, we identify $\mathfrak{G}_{\mathfrak{M}}$ with its expansion $\mathfrak{G}^{\equiv}_{\mathfrak{M}} = \langle \mathfrak{G}_{\mathfrak{M}}; \equiv^T, \equiv^{Ph}, \equiv^S\rangle$.
[1251] We mean to compare lines of $\mathfrak{G}$ with geodesics of the same $\mathfrak{G}$.

in $\mathsf{Ge}(Th)$ of course (cf. Prop.6.8.7).[1252] Under stronger assumptions, L coincides with the set of maximal geodesics, i.e.

$$L \;=\; \text{(maximal geodesics)}$$

(Corollary 6.8.33, p.1204). Under somewhat milder assumptions, we will already have

$$L^T \;=\; \text{(maximal time-like geodesics)}$$

(Thm.6.8.24, p.1200 and Corollary 6.8.27, p.1202).

$$L^{Ph} \;=\; \text{(maximal photon-like geodesics)},$$

under some (reasonably mild) assumptions on Th (item (v) of Prop.6.8.8). The conditions in the above quoted theorems are quite strong, hence we will address the issue wether they are really needed. We will do this in the form of conjectures, open problems, etc.

(We will also see that the various kinds of geodesics (e.g. "maximal geodesics") introduced in Def.6.8.2 are needed for forming a clear picture of the subject of this section.)

The following proposition says that lines (i.e. elements of L) are geodesics under certain assumptions.

PROPOSITION 6.8.7 *Assume* $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqm)}$. *Then the elements of L are geodesics.*

We omit the easy **proof**. ∎

The following proposition is a more detailed version of Prop.6.8.7 above. Among others, it says that the elements of L^T, L^{Ph}, L^S are geodesics under certain assumptions.

PROPOSITION 6.8.8

(i) *Assume* $\mathbf{Ax1}$, $\mathbf{Ax2}$, $\mathbf{Ax3_0}$, $\mathbf{Ax4}$, $\mathbf{Ax6_{00}}$, $\mathbf{AxE_{01}}$ *and* $\mathbf{Ax(eqm)}$. *Then*

$$L^T \subseteq \textit{(time-like Archimedean geodesics)},^{1253} \quad \textit{and}$$
$$L^T \subseteq \textit{(short time-like geodesics)}.$$

[1252] As a contrast, we will have no theorem saying the reverse of this, i.e. that under some assumptions on Th, say, $L \supseteq$ (maximal geodesics) without claiming equality, i.e. without claiming $L =$ (maximal geodesics).

(ii) *Assume* $\mathbf{Bax}^{-\oplus}+\mathbf{Ax(eqm)}$, *<u>or</u>* $n>2$ *and* $\mathbf{Bax}^{-\oplus}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(eqtime)}$. *Then*

$$L^T \;\subseteq\; (\textit{maximal locally additive time-like geodesics}).$$

(iii) $L^{Ph} \;\subseteq\; (\textit{photon-like geodesics}) \;=\; (\textit{photon-like Archimedean geodesics}).$

(iv) *Assume* $\mathbf{Bax}^{-\oplus}+\mathbf{Ax(diswind)}$. *Then*

$$L^{Ph} \;\subseteq\; (\textit{maximal photon-like geodesics}).$$

(v) *Assume* $\mathbf{Reich(Bax)}^{\oplus}+\mathbf{Ax(diswind)}$. *Then*

$$L^{Ph} \;=\; (\textit{maximal photon-like geodesics}).$$

(vi) *Assume* $\mathbf{Bax}^{-\oplus}+\mathbf{Ax(eqm)}$, *<u>or</u>* $n>2$ *and* $\mathbf{Bax}^{-\oplus}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(eqspace)}$. *Then*

$$L^S \;\subseteq\; (\textit{space-like Archimedean geodesics}).$$

(vii) *Assume* $n=2$ *and* $\mathbf{Bax}^{-\oplus}+\mathbf{Ax(eqm)}$. *Then*

$$L^S \;\subseteq\; (\textit{maximal locally additive space-like geodesics}).$$

Outline of proof: The proofs of items (i), (iii) and (iv) are easy and are left to the reader.

<u>Proof of (ii):</u> Assume the assumptions. Let $\ell \in L^T$. It is easy to see that ℓ is a locally additive time-like geodesic. We will prove that ℓ is a maximal one among these geodesics. Let $e \in Mn$ such that $e \notin \ell$. Then there are $a,b \in \ell$ such that $a \neq b$ and $a \equiv^{Ph} e \equiv^{Ph} b$, see the left-hand side of Fig.346. g is not additive on $\{a,b,e\}$, since $g(a,e)=g(b,e)=0$ and $g(a,b)>0$. Further, for any $\varepsilon \in {}^+F$, $a,b,e \in S(e,\varepsilon)$. Thus, g is not locally additive on $\ell \cup \{e\}$. Therefore ℓ is a maximal locally additive time-like geodesic.

<u>Proof of (v):</u> Item (v) follows by item (iv) and by the fact that in $\mathbf{Reich(Bax)}^{\oplus}$ geometries

$$a \equiv^{Ph} b \equiv^{Ph} c \equiv^{Ph} a \quad\to\quad coll(a,b,c),$$

[1253]Note that (Archimedean geodesics) $\subseteq$ (geodesics) and the same holds for time-like ones etc. Therefore the conclusions of the present proposition remain true if the adjective Archimedean is omitted. Later we will not return to indicating the consequences of this observations explicitly.

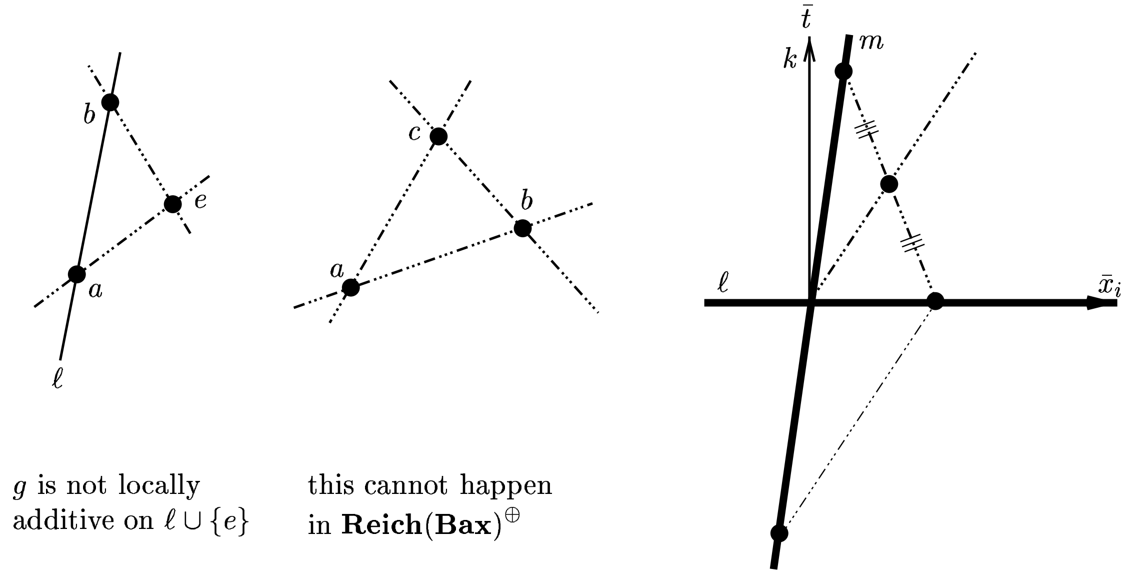


Figure 346: Illustration for the proof of Prop.6.8.8.

see the middle of Fig.346. This fact holds by Thm.6.6.107 (p.1127).

<u>*Proof of (vi):*</u> We claim that for every $\mathfrak{M} \models \mathbf{Bax}^{-\oplus}$ and $\ell \in L^S_{\mathfrak{M}}$ there is an observer m such that m sees that all the events on ℓ are 1/2-simultaneous. The proof of this claim is depicted in the right hand-side of Fig.346. Using this claim and item (i) of our proposition one can easily prove item (vi).

<u>*Proof of item (vii):*</u> The proof of item (vii) is analogous to that of (ii). ∎

Remark 6.8.9 (Discussing some of the conditions of Prop.6.8.8)

(i) In item (iv) of Prop.6.8.8 the condition **Ax(diswind)** cannot be omitted. Moreover, for every $n > 2$, there is $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Newbasax}+\mathbf{Ax(eqm)})$ such that there is $\ell \in L^{Ph}$ which is not a maximal photon-like geodesic. Hint for the idea of a proof is illustrated in the left-hand side of Figure 347. In the figure $\ell \in L^{Ph}$, $e \in Mn$, ℓ and e are in different windows and $(\forall a \in \ell)(\exists \ell' \in L^{Ph}) a, e \in \ell'$. Thus, $\ell \cup \{e\}$ is a photon-like geodesic. Hence, ℓ is not a maximal photon-like geodesic. Further, item (iv) of the above proposition does not generalize to $\mathbf{Bax}^{-}+\mathbf{Ax(diswind)}$ because of the following. If $n > 2$, in models of **NewtK** the photon-like lines are not maximal geodesics.

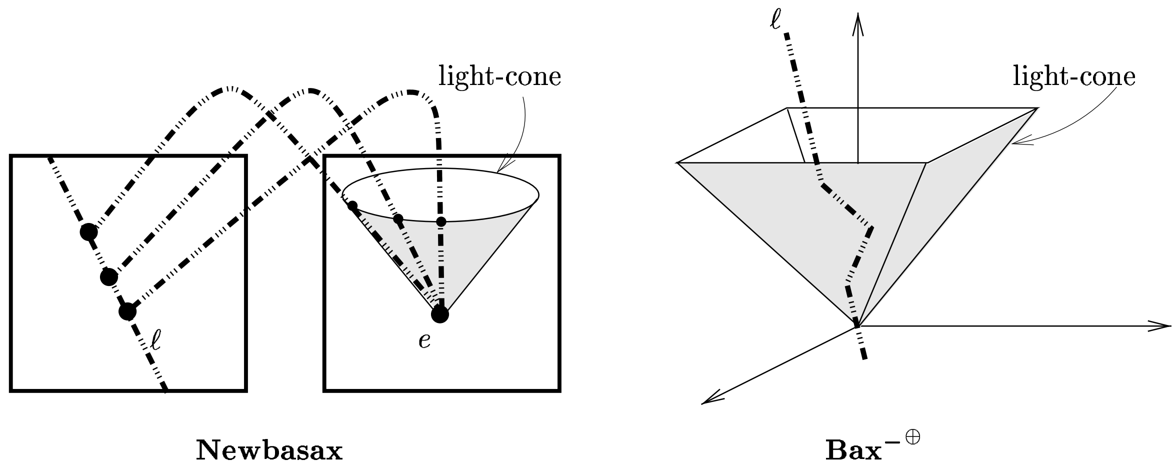


Figure 347: Illustration for Remark 6.8.9. On the right-hand side ℓ is a maximal photon-like geodesic. (Here ℓ is on the surface of the light-cone.) On the left-hand side, $\ell \cup \{e\}$ is a photon-like geodesic.

(ii) Item (v) of Prop.6.8.8 does not generalize form $\mathbf{Reich}(\mathbf{Bax})^{\oplus}$ to $\mathbf{Bax}^{-\oplus}$. The idea of a proof is illustrated in the right-hand side of Figure 347. In the figure ℓ is a maximal photon-like geodesic. We note that in $\mathbf{Reich}(\mathbf{Bax})$ the right-hand side of Figure 347 is excluded by the characterization theorem of $\mathbf{Reich}(Th)$-models in AMN [18, §4.5]. This is the theorem which states that every model of $\mathbf{Reich}(Th)$ can be obtained from some model of Th by "relativizing" with an artificial simultaneity (under some conditions on Th).

$\lhd$

In connection with Propositions 6.8.7 and 6.8.8 above we ask the following.

QUESTION 6.8.10 *Assume $n > 2$, $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(eqtime)})$ and that $\mathbf{F_0}$ is isomorphic with the ordered additive group reduct of $\mathfrak{R}$.*

(i) *Are the members of L^T maximal time-like geodesics?*

(ii) *Is there a time-like geodesic ℓ such that ℓ has a non-injective parameterization?*

$\lhd$

In connection with the above question we note that if we assume that $\mathbf{F_0}$ is non-Archimedean (instead of $\mathbf{F_0} \cong \langle \mathrm{R}; 0, +, \leq \rangle$), then the answer to (i) is "no", while the answer to (ii) is "yes", cf. Theorem 6.8.16 (p.1193), the proof of this theorem and Fig.348 (p.1194).

Intuitively, item (ii) of Question 6.8.10 is equivalent with the following question. Does there exist a geodesic *time-travel*, i.e. can the life-line of a time-traveler who meets his younger himself be a geodesic?

The following theorem says that (i) in Minkowskian geometries the maximal Archimedean geodesics are exactly the lines, (ii) in Minkowskian geometries over Archimedean fields the maximal geodesics are exactly the lines, and (iii) in the Minkowskian geometry over the field $\mathfrak{R}$ of reals the maximal quasi geodesics are exactly the lines.

THEOREM 6.8.11 *Assume $\mathfrak{F}$ is Euclidean and $n \geq 2$. Then in the Minkowskian geometry $Mink(n, \mathfrak{F})$ over $\mathfrak{F}$ (i)–(iii) below hold.*

(i)

$$\begin{aligned} L &= (\textit{maximal Archimedean geodesics}), \\ L^T &= (\textit{maximal time-like Archimedean geodesics}), \\ L^{Ph} &= (\textit{maximal photon-like Archimedean geodesics}) \\ &= (\textit{maximal photon-like geodesics}) \\ &= (\textit{maximal photon-like quasi geodesics}), \quad \textit{and} \\ L^S &= (\textit{maximal space-like Archimedean geodesics}). \end{aligned}$$

(ii) *Assume $\mathfrak{F}$ is Archimedean. Then*

$$\begin{aligned} L &= (\textit{maximal geodesics}), \\ L^T &= (\textit{maximal time-like geodesics}), \\ L^S &= (\textit{maximal space-like geodesics}). \end{aligned}$$

(iii) *Assume $\mathfrak{F} = \mathfrak{R}$. Then*

$$\begin{aligned} L &= (\textit{maximal quasi geodesics}), \\ L^T &= (\textit{maximal time-like quasi geodesics}), \quad \textit{and} \\ L^S &= (\textit{maximal space-like quasi geodesics}). \end{aligned}$$

Proof: Let $\mathfrak{F}$ be Euclidean and $n \geq 2$. By Propositions 6.8.4, 6.8.6 (p.1185), (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Hence, to prove the theorem, it is enough to prove (i) for $Mink(n, \mathfrak{F})$. By

Thm.6.2.59 (p.861), the eq-free reduct of $Mink(n,\mathfrak{F})$ is isomorphic with the eq-free reduct of $\mathfrak{G}_{\mathfrak{M}}$ for some $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(\uparrow\uparrow)})$. Let such an $\mathfrak{M}$ be fixed. To prove (i) for $Mink(n,\mathfrak{F})$ it is enough to prove it for $\mathfrak{G}_{\mathfrak{M}}$. So we will prove (i) for $\mathfrak{G}_{\mathfrak{M}}$.

Claim **6.8.12** Let $a, b \in Mn$ and $m \in Obs$.

(i) Assume that a, b are in m's life-line, i.e. $a, b \in w_m[\bar{t}\,]$. Then the time elapsed between events a and b for observer m is $g(a,b)$.

(ii) Assume that a, b are simultaneous for m and $a \equiv^S b$. Then the (spatial) distance between a and b for m is $g(a,b)$.

Proof: By item 4(e)ii of Prop.6.2.79 (p.889) the irreflexive parts of the relations $\equiv^T$, $\equiv^{Ph}$, $\equiv^S$ are pairwise disjoint. Further, $\mathbf{Ax(eqtime)}$ holds in $\mathfrak{M}$ (by $\mathbf{Ax(symm)} = \mathbf{Ax(symm_0)} + \mathbf{Ax(eqtime)}$). By these we conclude that item (i) of the claim holds. By Thm.2.8.11 (p.133), $\mathbf{Ax(eqspace)}$ holds in $\mathfrak{M}$. Therefore, we conclude that item (ii) of the claim holds.

(QED Claim 6.8.12)

Claim **6.8.13** Assume that ℓ is a short time-like geodesic of $\mathfrak{G}_{\mathfrak{M}}$. Then

$$(\exists \ell' \in L^T)\, \ell \subseteq \ell'.$$

Proof: Assume that ℓ is a short time-like geodesic. Then there is a parameterization h of ℓ such that, for $D := Dom(h)$, $(*)$ and $(**)$ on p.1180 hold. Therefore h is additive on ℓ and $(\forall a, b \in \ell)\, a \equiv^T b$. To prove the claim it is enough to prove that $(\forall a, b, c \in \ell)(\exists \ell' \in L^T)\, a, b, c \in \ell'$. By Thm.2.8.18 (p.140), the twin paradox $\mathbf{Ax(TwP)}$ holds in $\mathfrak{M}$. Let $a, b, c \in \ell$. Since $a \equiv^T b \equiv^T c \equiv^T a$, there are observers m_a, m_b, m_c such that b, c are on the life-line of m_a, a, c are on the life-line of m_b, and a, b are on the life-line of m_c. Let such $m_a, m_b, m_c \in Obs$ be fixed. If the life-lines of two observers from $\{m_a, m_b, m_c\}$ coincide then for this life-line $\ell' \in L^T$ we will have that $a, b, c \in \ell'$. Assume, the life-lines of m_a, m_b, m_c are pairwise distinct. We will derive a contradiction. It can be checked (even in $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(\uparrow\uparrow_0)}$ and if $n > 2$ in $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ })$) that there is $e \in \{a, b, c\}$ such that observer m_e thinks that event e is temporally between the other two events. But then by $\mathbf{Ax(TwP)}$ and Claim 6.8.12(i), we have that

$$\begin{aligned} g(a,b) &> g(a,c) + g(c,b) \quad \text{or} \\ g(a,c) &> g(a,b) + g(b,c) \quad \text{or} \\ g(b,c) &> g(b,a) + g(a,c). \end{aligned}$$

This contradicts the fact that g is additive on ℓ.

(QED Claim 6.8.13)

Claim 6.8.14 Assume that ℓ is a time-like Archimedean geodesic of $\mathfrak{G}_{\mathfrak{M}}$. Then

$$(\exists \ell' \in L^T)\, \ell \subseteq \ell'.$$

Proof: Assume that ℓ is a time-like Archimedean geodesic of $\mathfrak{G}_{\mathfrak{M}}$. Then there is a parameterization h of ℓ such that h satisfies the conditions in item 13 of Def.6.8.2 on p.1181. Let such an h be fixed. To prove the claim it is enough to prove that for every $x, y \in Dom(h)$ with $x < y$ there is $\ell' \in L^T$ such that $h[(x, y)] \subseteq \ell'$. Let $x, y \in Dom(h)$ be such that $x < y$. By our choice of h, there are $k \in \omega$ and intervals[1254] $I_0, \ldots, I_k$ of $\mathbf{F_0}$ such that $(x, y) \subseteq I_0 \cup \ldots \cup I_k$, $(\forall i \in k) I_i \cap I_{i+1} \neq \emptyset$ and $(\forall i \in (k+1))\, [\, h[I_i]$ is a short time-like geodesic$\,]$. Therefore, by Claim 6.8.13, we conclude that there is $\ell' \in L^T$ such that $h[(x, y)] \subseteq \ell'$.

(QED Claim 6.8.14).

Claim **6.8.15** Assume that ℓ is a space-like Archimedean geodesic of $\mathfrak{G}_{\mathfrak{M}}$. Then

$$(\exists \ell' \in L^S)\, \ell \subseteq \ell'.$$

Proof: Assume, ℓ is a space-like Archimedean geodesic of $\mathfrak{G}_{\mathfrak{M}}$. Then there is a parameterization h of ℓ such that h satisfies the conditions in item 14 of Def.6.8.2 on p.1181. Let such an h be fixed. To prove the claim it is enough to prove that for every $x, y \in Dom(h)$ with $x < y$ there is $\ell' \in L^S$ such that $h[(x, y)] \subseteq \ell'$. Let $x, y \in Dom(h)$ be such that $x < y$. By our choice of h, there are $k \in \omega$ and intervals $I_0, \ldots, I_k$ of $\mathbf{F_0}$ such that

$$(\dagger) \qquad (x, y) \subseteq I_0 \cup \ldots \cup I_k \quad \wedge \quad (\forall i \in k)\, I_i \cap I_{i+1} \neq \emptyset,$$

and

$$(\forall i \in (k+1))\, [\, (*) \text{ and } (***) \text{ on p.1180 hold for } D := I_i\,].$$

Thus, to prove that there is $\ell' \in L^S$ such that $h[(x, y)] \subseteq \ell'$, by $(\dagger)$ it is enough to prove that for every $i \in (k+1)$ there is $\ell' \in L^S$ such that $h[I_i] \subseteq \ell'$. Let $i \in (k+1)$. Since $(***)$ on p.1180 holds for $D := I_i$ and, by Claims 6.8.12(i) and 6.8.13, there is an observer m, such that

$(\ddagger)$ m thinks that all the events in $h[I_i]$ are simultaneous and $(\forall a, b \in h[I_i])\, a \equiv^S b$.

Further, since $(*)$ on p.1180 holds for $D := I_i$, we conclude that

$$g \text{ is additive on } h[I_i].$$

[1254]By definition, these intervals are open, cf. Def.6.8.2(4).

By this, by (‡), by item (ii) of Claim 6.8.12, and by the fact that the triangle inequality holds in Euclidean geometry, we conclude that m sees that any three events in $h[I_i]$ are collinear (and $\equiv^S$-related). Thus, there is $\ell' \in L^S$ such that $h[I_i] \subseteq \ell'$.

(QED Claim 6.8.15).

It can be easily checked that in $\mathfrak{G}_{\mathfrak{M}}$,

$$\begin{aligned} L^T &\subseteq \text{(time-like Archimedean geodesics)}, \quad \text{and} \\ L^S &\subseteq \text{(space-like Archimedean geodesics)}. \end{aligned}$$

Therefore, by Claims 6.8.14 and 6.8.15, in $\mathfrak{G}_{\mathfrak{M}}$

$$\begin{aligned} L^T &= \text{(maximal time-like Archimedean geodesics)}, \quad \text{and} \\ L^S &= \text{(maximal space-like Archimedean geodesics)}. \end{aligned}$$

Further,

$$\begin{aligned} L^{Ph} &= \text{(maximal photon-like Archimedean geodesics)} = \text{etc.} \quad \text{and} \\ L &= \text{(maximal Archimedean geodesics)}. \quad \blacksquare \end{aligned}$$

Among others, the following theorem says that the condition that $\mathfrak{F}$ is Archimedean cannot be omitted from item (ii) of Thm.6.8.11 above.

THEOREM 6.8.16 *Assume $\mathfrak{F}$ is a non-Archimedean and Euclidean ordered field. Then (i)–(iv) below hold.*

(i) *For any $n \geq 2$ in $Mink(n, \mathfrak{F})$*

$$\begin{aligned} L^T &\cap \textit{(maximal time-like geodesics)} = \emptyset, \qquad \underline{\textit{but}} \\ L^T &\subseteq \textit{(maximal locally additive time-like geodesics)}, \qquad \textit{while} \\ L^T &\not\supseteq \textit{(maximal locally additive time-like geodesics)}. \end{aligned}$$

(ii) *For any $n \geq 2$ in $Mink(n, \mathfrak{F})$*

$$L^S \cap \textit{(maximal space-like geodesics)} = \emptyset,$$

i.e. if $\ell \in L^S$ then ℓ is not a maximal space-like geodesic.

(iii) *In $Mink(2, \mathfrak{F})$*

$$\begin{aligned} L^S &\subseteq \textit{(maximal locally additive space-like geodesics)}, \qquad \textit{while} \\ L^S &\not\supseteq \textit{(maximal locally additive space-like geodesics)}. \end{aligned}$$

(iv) *As a contrast with item (iii), for any $n > 2$ in $Mink(n, \mathfrak{F})$*

$$L^S \cap (\textit{maximal locally additive space-like geodesics}) = \emptyset \quad \textit{and}$$
$$L^S \cap (\textit{maximal additive space-like geodesics}) = \emptyset.$$

Outline of proof: Assume $\mathfrak{F}$ is non-Archimedean and Euclidean. Idea of proof for

$$L^T \cap (\text{maximal time-like geodesics}) = \emptyset \quad \text{in } Mink(\mathfrak{F})$$

is depicted in Figure 348. In the figure $\ell \in L^T$ and $\ell \cup \ell'$ is a time-like geodesic. Hence, ℓ is not a maximal time-like geodesic. By Prop.6.8.8(ii) on p.1186 (and by Thm.6.2.59 on p.861), in $Mink(\mathfrak{F})$

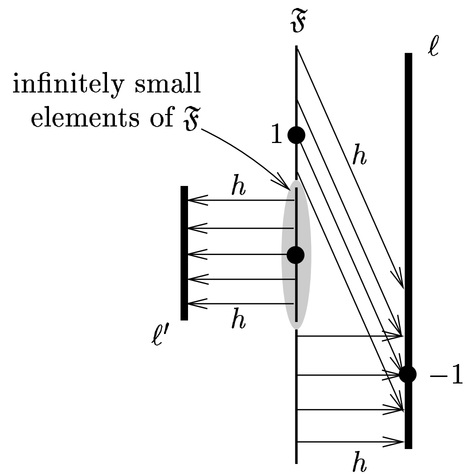


Figure 348: $\ell \in L^T$ and $\ell \cup \ell'$ is a time-like geodesic.

$$L^T \subseteq (\text{maximal locally additive time-like geodesics}).$$

Idea for the proof of

$$L^T \not\supseteq (\text{maximal locally additive time-like geodesics}) \quad \text{in } Mink(\mathfrak{F})$$

is depicted in Figure 349. In the figure ℓ is a maximal locally additive time-like geodesic. This holds by the proof of item (ii) of Prop.6.8.8. Clearly, ℓ (in Fig.349) is not a time-like line. By these item (i) of our theorem is proved. Proofs for items (ii) and (iii) can be obtained by the proof of item (i). (The proofs of (ii) and (iii) are left to the reader.)

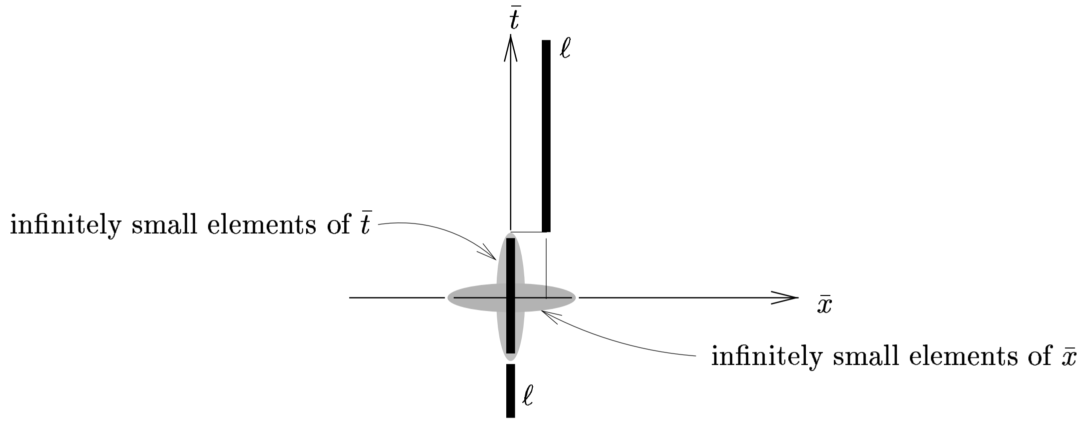


Figure 349: ℓ is a maximal locally additive time-like geodesic in the Minkowskian geometry over a non-Archimedean $\mathfrak{F}$.

To prove item (iv) let $\ell \in L^S$. Consider a Robb plane[1255] that contains ℓ. Let ℓ' be constructed as in Figure 348 but such that ℓ' is contained in the Robb plane, see Figure 350. Then, in Figure 350, $\ell \cup \ell'$ is an additive space-like geodesic, cf. hint for the proof of Thm.6.8.20 on p.1198. Hence, ℓ is not a maximal locally additive space-like geodesic and is not a maximal additive space-like geodesic. ∎

The geodesic $\ell \cup \ell'$ represented in Fig.348 is not a divisible geodesic.[1256] This motivates Question 6.8.17 below. We did not have time to check whether the geodesic in Fig.349 is divisible or not. (Németi guesses that it might perhaps be divisible after all.) We note that the geodesic in Fig.350 is a divisible one.

QUESTION 6.8.17 *Assume that $\mathfrak{F}$ is non-Archimedean and Euclidean. Is the following true in Mink($\mathfrak{F}$)?*

$$L^T \ \cap \ (\textit{maximal divisible time-like geodesics}) \ = \ \emptyset?$$

[1255]cf. e.g. Goldblatt [108] or p.1163 in the present work for the notion of a Robb plane. If $n > 3$ then we can talk about ***Robb hyper-planes*** (cf. p.804 in AMN [18]) which in Goldblatt [108] are called Robb threefolds (if $n = 4$). However, there still exist ***Robb planes***, too, which are (two-dimensional) and planes with the Robb property. In the above proof of Thm.6.8.20 it is important that we talk about Robb planes and not about Robb hyper-planes.

[1256]Hint: Without loss of generality we may assume that $n = 3$. We choose p on ℓ and q on ℓ' such that $p_t = q_t$. Without loss of generality we may assume $p = \bar{0}$ and $q = 1_x$. Then $g_\mu(p,q) = 1$. Assume $s \in \ell \cup \ell'$ is such that $g_\mu(p,s) = g_\mu(q,s)$. If $\ell \cup \ell'$ is a divisible geodesic then such an s exists. We will derive a contradiction. Without loss of generality we may assume $s = (x, 1, 0)$. Now a straightforward computation gives a contradiction.

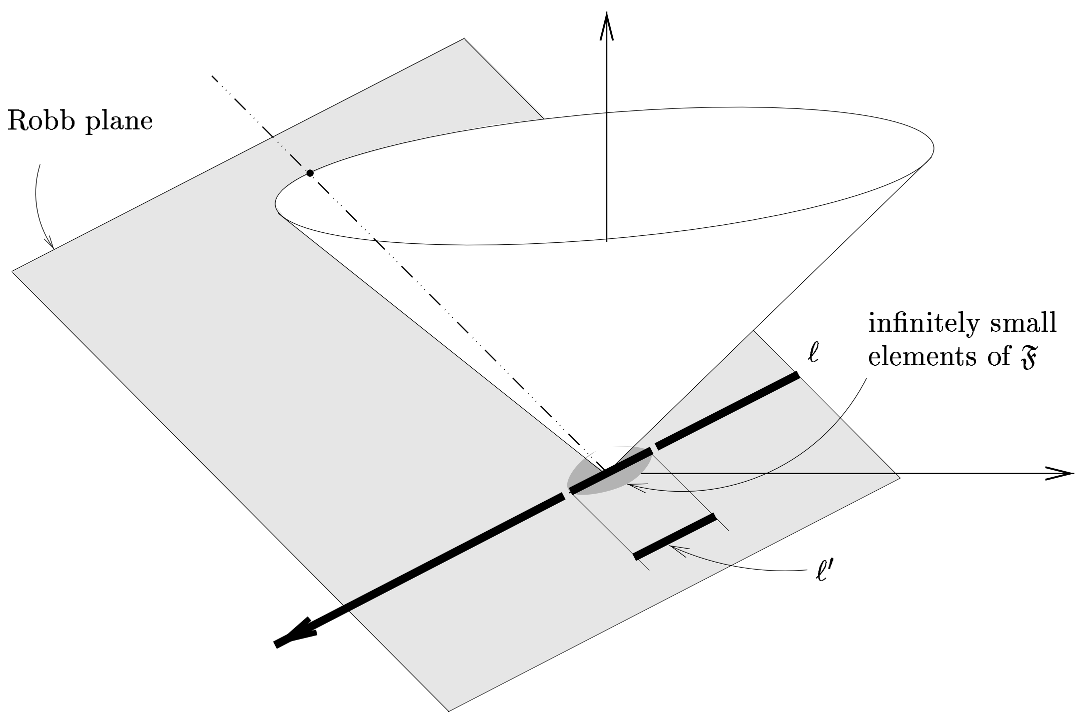


Figure 350: $\ell \cup \ell'$ is an additive space-like geodesic in the Minkowskian geometry over a non-Archimedean $\mathfrak{F}$.

$\triangleleft$

In connection with the proof of items (i) and (ii) of Thm.6.8.16 above we ask the following.

QUESTION 6.8.18 *Assume $\mathfrak{F}$ is as in Thm.6.8.16 above and $n \geq 2$. Are there maximal space-like geodesics or maximal time-like geodesics in $Mink(n, \mathfrak{F})$?*

$\triangleleft$

Among others, the following theorem says that in item (iii) of Thm.6.8.11 (p.1190) above the ordered field $\mathfrak{R}$ of reals cannot be replaced by any Euclidean ordered field $\mathfrak{F}$ which is not isomorphic with $\mathfrak{R}$.

THEOREM 6.8.19 *Assume $\mathfrak{F}$ is Euclidean and $\mathfrak{F}$ is not isomorphic with $\mathfrak{R}$. Then items (i)–(iv) in Thm.6.8.16 hold if we replace "geodesics" with "quasi geodesics" in them.*

We omit the **proof**. ∎

The following theorem says that the second part of condition $(***)$ (on p.1180) is needed in the definition of space-like quasi geodesics, geodesics and Archimedean geodesics. In other words, if we omit condition $(***)$ from the definition of geodesics, then they *do not "work"* in relativistic geometries, e.g. in Minkowskian space-times. Although they do work in Euclidean geometry and more generally in Riemannian geometries. This further implies that if we use the definition of geodesics as given e.g. in the book "Geometry of Geodesics" (Busemann [55]), then they do not work in relativistic geometries ($n > 2$), e.g. in Minkowskian geometry.[1257]

THEOREM 6.8.20 *Assume $n \geq 3$. Then in the Minkowskian geometry $Mink(n, \mathfrak{R})$ there is a "curve" $\ell \subseteq {}^n\mathrm{R}$ such that $(\forall p, q \in \ell)\, p \equiv^S q$,*

$$(\forall e \in \ell)(\exists \varepsilon \in {}^+F)\,[\,\underline{no}\ \text{three distinct points of}\ \ell \cap S(e, \varepsilon)\ \underline{\text{are collinear}}\,],$$

and there is a homeomorphism $h : \mathfrak{R} \rightarrowtail\!\!\!\twoheadrightarrow \ell$ which is differentiable infinitely many times and is distance preserving in the sense that

$$(\forall x, y \in \mathrm{R})\, |x - y| = g_\mu(h(x), h(y)),$$

[1257]This entails nothing negative about Busemann [55], since it does not deal with relativistic geometries. Caution is needed with the word "Minkowskian geometry", since here (cf. also Goldblatt [108], Schutz [236]) we use it for certain relativistic geometries while e.g. in Busemann [55, §17] it is used for other kinds of (non-relativistic) spaces.

see Figure 351. Moreover this function h is a homeomorphism w.r.t. (the usual topology on $\mathfrak{R}$ and) any one of the following topologies on ℓ: the topology induced by g_μ, the relativistic topology $\mathcal{T}_\mu$ of Mink($\mathfrak{R}$) and the Euclidean topology on ${}^n\mathrm{R}$. *Actually these topologies coincide on ℓ.*

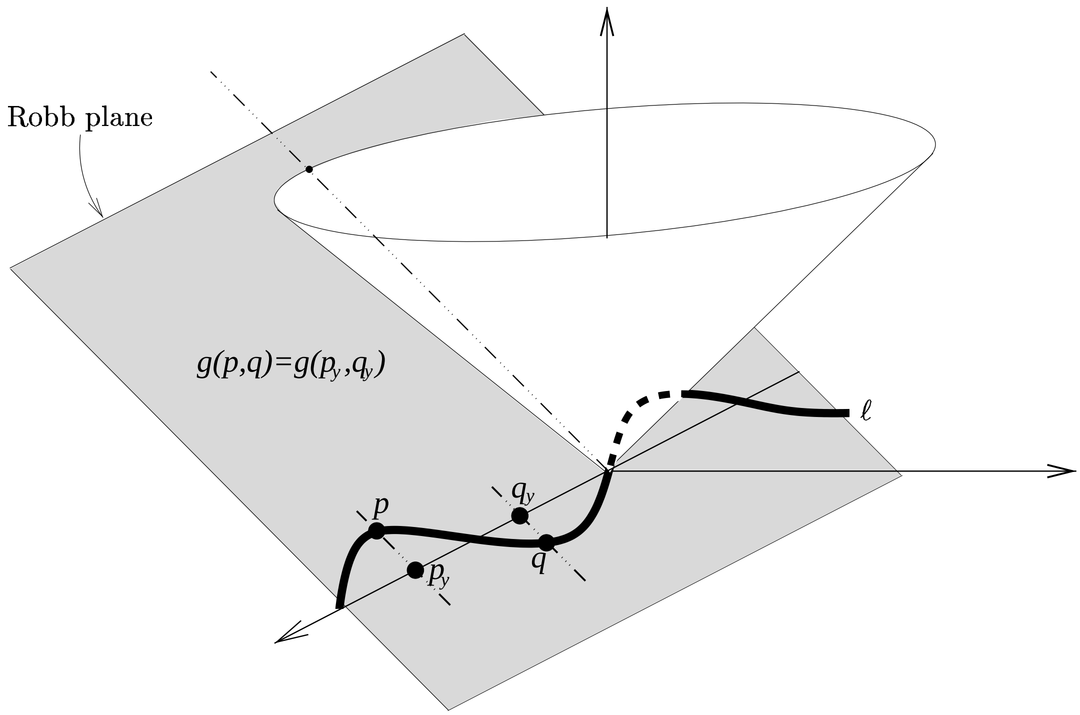


Figure 351: Condition $(***)$ is needed in the definition of space-like geodesics.

On the proof: Hint: Assume $n \geq 3$. The Robb planes[1258] have the following "exotic" property in *Mink*($\mathfrak{R}$) (in connection with the metric g_μ and geodesics). Let P be a Robb plane containing the $\bar{y}$ axis. Then the relativistic distance $g_\mu(p,q)$ between points $p, q \in P$ coincides with the absolute value of the difference between the y-coordinates p_y and q_y of p and q, respectively. Cf. Figure 351. Therefore the metric g_μ is additive on the whole Robb plane. Actually this idea works in many of our relativistic geometries, e.g. in the case of $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqspace)} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$ they do. ∎

[1258] cf. e.g. Goldblatt [108] or p.1163 in the present work for the notion of a Robb plane.

COROLLARY 6.8.21 Assume $n \geq 3$ and consider $\mathfrak{G} \stackrel{\text{def}}{=} Mink(n, \mathfrak{R})$ as in Thm.6.8.20 above. Then

(i) If we omit condition $(***)$ from the definition of geodesics, then there are geodesics in $\mathfrak{G}$ which are not straight lines. Further,

(ii) there exist many Robb planes[1259] in $\mathfrak{G}$, and

(iii) almost[1260] every curve in every Robb plane counts as a geodesic if we omit condition $(***)$ from the definition of geodesics.

Discussion of Thm.6.8.20 and Corollary 6.8.21. The condition $(***)$ is not present in the usual definition of geodesics. Items 6.8.20, 6.8.21 say that this condition is needed in relativistic geometries if we want to discuss space-like geodesics, too.

The definition of *usual geodesics* is obtained from Def.6.8.2 by replacing all occurrences of condition $(***)$ with $(\forall x, y \in D) x \equiv^S y$.

What we obtain this way is more or less the usual definition of geodesics (cf. Busemann [55]) adapted to the relativistic situation where we have $\equiv^T, \equiv^{Ph}, \equiv^S$.[1261] Now, what items 6.8.20, 6.8.21 say is that even in the most classical, most standard form of special relativity, i.e. in Minkowskian space-time with $n > 2$, usual geodesics (as defined above) do not "work". (They do not behave as we wanted them to behave when defining them.)

COROLLARY 6.8.22 Let $n > 2$ and consider $\mathfrak{G} \stackrel{\text{def}}{=} Mink(n, \mathfrak{R})$. Then there are *usual geodesics* ℓ in $\mathfrak{G}$ which are not straight lines, moreover ℓ can be chosen to be continuous and differentiable such that $(\forall p \in \ell)(\forall \varepsilon \in {}^+F)$ *the ε-neighbourhood of p in ℓ is not straight.* Moreover, this ℓ is an *Archimedean*, short, usual geodesic, cf. Def.6.8.2 items 14, 17. Further, it is a *maximal* geodesic, and a *strong, divisible* geodesic. Through any two distinct space-like separated points of $\mathfrak{G}$ there are continuum many such usual geodesics.

Proof. The proof goes by inspecting Figure 351 (and the proof of Thm.6.8.20) and by checking all the items quoted from Def.6.8.2. ∎

[1259] each photon line is contained in a Robb plane which is unique iff $n = 3$. So, if $n > 3$, then the Robb plane in question is not unique.

[1260] Instead of defining precisely which curves in the Robb plane we mean, we give only an intuitive description: Let ℓ be a "continuous, differentiable" connected curve in the Robb plane as illustrated in Figure 351. Assume $(\forall p, q \in \ell) p \equiv^S q$. Assume further that ℓ is a homeomorphic image of some connected interval of $\mathbf{F_0}$. Then ℓ counts as a geodesic (without $(***)$).

[1261] i.e., so to speak, adapted from Riemannian geometries to pseudo-Riemannian ones; or in other words, adapted to so called "indefinite metrics".

COROLLARY 6.8.23 Let $n > 2$. A statement analogous to items 6.8.20-6.8.22 applies to our geometries in $\mathsf{Ge}(\mathbf{Bax}^{\oplus} + ax(eqspace) + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }))$.

Proof. The proof goes by checking that already under the axioms $\mathbf{Bax}^{\oplus} + \ldots + \mathbf{Ax}(\sqrt{\ })$ listed above, the Robb plane exhibits the strange properties illustrated in Figure 351. ∎

Items 6.8.20-6.8.23 show that condition $(***)$ is really needed and is not easily replaced with something "more traditional". Further, they indicate that the (simplest) usual notion of geodesics[1262] does not work in relativistic situations for *space-like geodesics*. This might be connected to the historical fact that *in general relativity* much less attention is paid to space-like geodesics than to time-like or photon-like ones. E.g. the basic book Hawking-Ellis [126] does not even mention space-like geodesics.[1263] A further indication of this[1264] is that in the world-famous basic book of relativity Misner-Thorne-Wheeler [196] the statement of Exercise 13.6 on p.324 (discussing space-like geodesics) seems to be either false or not very carefully formulated. (We mean this of course wrt. the definitions given in that book.[1265]) Further, as far as we know, this (about the book) has not yet been pointed out in the literature. With this we stop discussing items 6.8.20-**??** (and return to discussing our notion of geodesics in our relativity theories).

Next, we generalize our earlier positive results from the concrete case of Minkowskian geometries to a broader class of our observer independent geometries of the "axiomatic form" $\mathsf{Ge}(Th)$.

Recall that $\mathbf{Ax(TwP)}$ is the twin paradox defined in Def.4.2.6 on p.460 of AMN [18]. Among others, the next theorem says that, assuming $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)}$, $n > 2$ and that $\mathfrak{F}$ is Archimedean, the maximal time-like geodesics are exactly the time-like lines.

THEOREM 6.8.24 *Assume* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)} + \mathbf{Ax(\uparrow\uparrow_0)})$, *or* $n > 2$ *and* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)})$. *Then (i)–(iii) below hold.*

(i) $L^T = $ (*maximal time-like Archimedean geodesics*).

[1262] Cf. the definition of usual geodesics above.

[1263] This in turn might be motivated by the famous quotation for Eddington [56, p.22] "Assuming that a material particle cannot travel faster than light ... we ourselves are limited by material bodies and have direct experience of time-like intervals."

[1264] i.e. that relativity theorists seem to pay little attention to space-like geodesics

[1265] But it seems to remain false for any usual definition of geodesics known to the present author.

(ii) *Assume that* $\mathbf{F_0}$ *is an Archimedean ordered group. Then*

$$L^T = (\textit{maximal time-like geodesics}).$$

(iii) *Assume that* $\mathbf{F_0}$ *is isomorphic with the ordered additive group reduct of the field* $\mathfrak{R}$ *of reals. Then*

$$L^T = (\textit{maximal time-like quasi geodesics}).$$

Proof: The proof is obtained by pushing through the time-like part of the proof of Thm.6.8.11 under the present more general conditions. For this generalization of the proof one uses Proposition 6.8.25 below. ∎

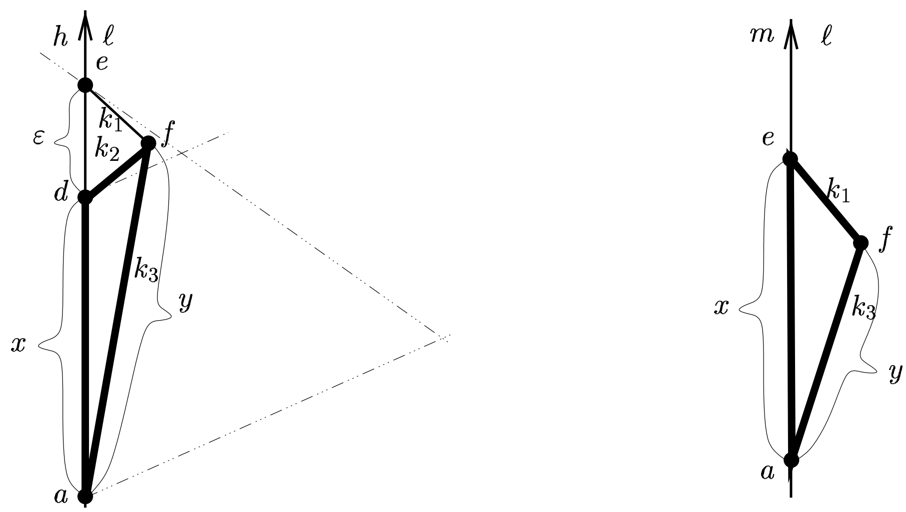
PROPOSITION 6.8.25 $\mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax(TwP)} \to \mathbf{Ax(eqtime)}$.

Proof: The proof goes by contradiction. Assume $\mathfrak{M} \models \mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(TwP)}$ and that $\mathfrak{M} \not\models \mathbf{Ax(eqtime)}$. Then there are $m, h \in Obs$ with common life-line ℓ, i.e. $w_m[\bar{t}] = w_h[\bar{t}] := \ell$, and $e, a \in \ell$ such that the time elapsed between e and a for m is x, while the time elapsed between e and a for h is $x+\varepsilon$, for some positive x and ε. Let such $m, h, a, e, \ell, x, \varepsilon$ be fixed. Let $d \in \ell$ be such that h thinks that the time elapsed between e and d is ε and that the time elapsed between d and a is x, cf. the left-hand side of Figure 352. Let $k_1, k_2, k_3 \in Obs$ be slower than light as seen by h and let they be as depicted in the left-hand side of Figure 352.[1266] Let $f \in w_{k_1}[\bar{t}] \cap w_{k_2}[\bar{t}] \cap w_{k_3}[\bar{t}]$. Let the time elapsed between f and a for k_3 be y ($y \in {}^+F$). Since $\mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ })$ is assumed and k_1, k_2, k_3 are slower than light as seen by h, it can be checked that k_3 thinks that d is happened temporally between a and f, cf. the left-hand side of Fig.352. By $\mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ })$ and Prop.6.6.5(i) (p.1028), m thinks that f happened temporally between e and a (since h thinks this is so). Now, applying $\mathbf{Ax(TwP)}$ for k_3, k_2, h and m, k_1, k_3, respectively, we get that $y > x$ and $x > y$. This is a contradiction. ∎

The following is a corollary of Thm.6.8.24 herein, and Thm.4.2.9 (p.461) of AMN [18].

COROLLARY 6.8.26 Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^\oplus + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\uparrow\uparrow_0))$, <u>or</u> $n > 2$ and $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^\oplus + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\sqrt{\ }))$. Then items (i)–(iii) in Thm.6.8.24 hold for $\mathfrak{G}$. ∎

[1266] I.e. there is an event f such that $f \notin \ell$, h thinks that f is happened temporally between e and d, $\ell \cap w_{k_1}[\bar{t}] = \{e\}$, $\ell \cap w_{k_2}[\bar{t}] = \{d\}$, $\ell \cap w_{k_3}[\bar{t}] = \{a\}$, and $w_{k_1}[\bar{t}] \cap w_{k_2}[\bar{t}] \cap w_{k_3}[\bar{t}] = \{f\}$.



world-view of h world-view of m

Figure 352: Illustration for the proof of Prop.6.8.25

The following item, for $n > 2$ is a corollary of Thm.6.8.24 herein, and Thm.4.7.15 (p.622) of AMN [18], while for $n = 2$ it is a corollary of the proof of Thm.6.8.24 herein, and Theorems 4.7.15 (p.622), 4.7.9 (p.617) of AMN [18].

COROLLARY 6.8.27 Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{R}(\mathbf{Ax\ syt_0}))$. Then items (i)–(iii) in Thm.6.8.24 hold for $\mathfrak{G}$. ∎

The following theorem says, among others, that assuming $n > 2$, $\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqspace)} + \mathbf{Ax(TwP)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}$ and that $\mathfrak{F}$ is Archimedean, the maximal geodesics are exactly the lines.

THEOREM 6.8.28 *Assume* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqspace)} + \mathbf{Ax(TwP)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax(diswind)})$, *<u>or</u>* $n > 2$ *and* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(eqspace)} + \mathbf{Ax(TwP)} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Then (i)–(iii) below hold for* $\mathfrak{G}$.

(i)

$$L \quad = \quad (\mathit{maximal\ Archimedean\ geodesics}),$$

$$\begin{aligned}
L^T &= (\textit{maximal time-like Archimedean geodesics}),\\
L^{Ph} &= (\textit{maximal photon-like Archimedean geodesics})\\
&= (\textit{maximal photon-like geodesics})\\
&= (\textit{maximal photon-like quasi geodesics}), \quad \textit{and}\\
L^S &= (\textit{maximal space-like Archimedean geodesics}).
\end{aligned}$$

(ii) *Assume* $\mathbf{F_0}$ *is Archimedean. Then*

$$\begin{aligned}
L &= (\textit{maximal geodesics}),\\
L^T &= (\textit{maximal time-like geodesics}),\\
L^S &= (\textit{maximal space-like geodesics}).
\end{aligned}$$

(iii) *Assume that* $\mathbf{F_0}$ *is isomorphic with the ordered additive group reduct of* $\mathfrak{R}$. *Then*

$$\begin{aligned}
L &= (\textit{maximal quasi geodesics}),\\
L^T &= (\textit{maximal time-like quasi geodesics}), \quad \textit{and}\\
L^S &= (\textit{maximal space-like quasi geodesics}).
\end{aligned}$$

Proof: The theorem follows by Thm.6.8.24 and by the proof of Thm.6.8.11. I.e. the proof is obtained by pushing through the proof of Thm.6.8.11 under the present more general conditions. ∎

QUESTION 6.8.29 *Does the "space-like part" of Thm.6.8.28 remain true if we replace* **Ax(TwP)** *with* **Ax(eqtime)** *in the assumptions of Thm.6.8.28?*

◁

The following is a corollary of Thm.6.8.28 herein, and Thm.4.2.9 (p.461) in AMN [18].

COROLLARY 6.8.30 Thm.6.8.28 remains true if **Ax(TwP)** is replaced by $\mathbf{Ax(syt_0)}$ in it. ∎

In the corollary above $\mathbf{Bax}^{\oplus}$+**Ax(eqspace)**+$\mathbf{Ax(syt_0)}$+$\mathbf{Ax}(\sqrt{\ })$ was assumed. In connection with this we include the following conjecture. Roughly, it says that **Ax(eqspace)** + $\mathbf{Ax(syt_0)}$ blurs the distinction between **Bax** and **Flxbasax** if $n > 2$.

Conjecture 6.8.31 *Assume $n > 2$. Then (i) and (ii) below hold.*

(i) $\mathbf{Bax}+\mathbf{Ax(eqspace)}+\mathbf{Ax(syt_0)}+\mathbf{Ax}(\sqrt{\ })\ \models\ (\forall m,k)\,(m \stackrel{\odot}{\to} k\ \to\ c_m = c_k)$.

(ii) $\mathbf{Bax}+\mathbf{Ax(eqspace)}+\mathbf{Ax(syt_0)}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax6}\ \models\ \mathbf{Flxbasax}$.

◁

We base our conjecture above on Thm.4.2.4 (p.458) of AMN [18] and on Thm.4.7.11 (p.619) of AMN [18].

QUESTION 6.8.32 *Is the above conjecture true if we replace* $\mathbf{Ax(syt_0)}$ *with* $\mathbf{Ax(TwP)}$*?*

◁

Recall that $\mathbf{Ax}(\omega)^0$ is a very weak symmetry principle introduced in Def.6.2.37 (p.844) and that $\mathbf{Ax}(\omega)^{00}$ is weaker than $\mathbf{Ax}(\omega)^0$. The following is a corollary of Thm.6.8.28, Thm.6.2.98 (p.910) herein, and Thm.4.2.9 (p.461) of AMN [18]

COROLLARY 6.8.33

(i) Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Flxbasax}^{\oplus}+\mathbf{Ax}(\omega)^0+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax}(\uparrow\uparrow_0)+\mathbf{Ax(diswind)})$, <u>or</u> $n > 2$ and $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Flxbasax}^{\oplus}+\mathbf{Ax}(\omega)^{00}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(diswind)})$. Then items (i)–(iii) in Thm.6.8.28 hold for $\mathfrak{G}$.

(ii) The statement in item (i) remains true if we replace $\mathbf{Ax}(\omega)^0$ with any one of $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle1}+\mathbf{Ax(eqtime)}$, $\mathbf{Ax\triangle2}$, $\mathbf{Ax\Box1}+\mathbf{Ax(eqtime)}$, $\mathbf{Ax\Box2}$.

(iii) The statement in item (i) remains true if we replace $\mathbf{Ax}(\omega)^{00}$ with any one of $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}+\mathbf{Ax}(\mathit{Triv}_t)^-$, $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle1}+\mathbf{Ax(eqtime)}$, $\mathbf{Ax\triangle2}$, $\mathbf{Ax\Box1}+\mathbf{Ax(eqtime)}$, $\mathbf{Ax\Box2}$. ∎

Remark 6.8.34 We note that items 6.8.11 (p.1190), 6.8.28 (p.1202), 6.8.30 (p.1203), 6.8.33 (p.1204) above remain true if we replace "geodesics" with "strong geodesics" in them.

◁

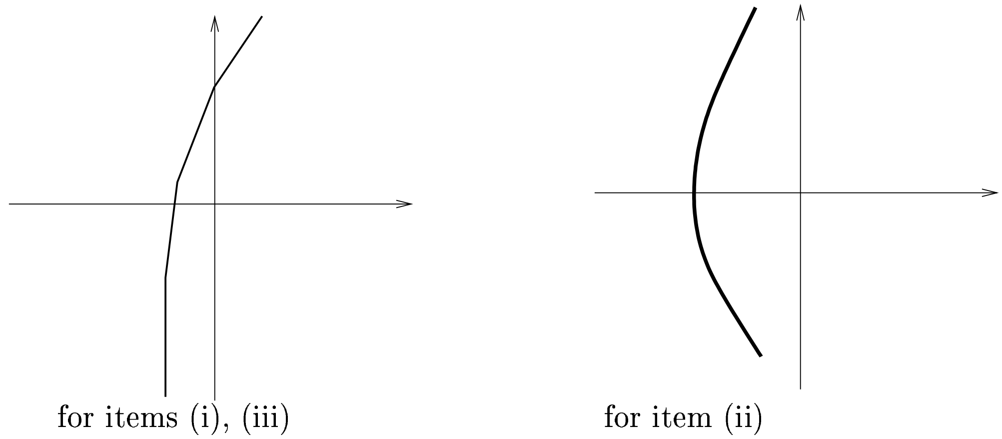


Figure 353: Illustration for the idea of a possible proof for Conjecture 6.8.35.

By Corollary 6.8.33 above, assuming **Basax** + $\mathbf{Ax(syt_0)}$ + $\mathbf{Ax}(\sqrt{\ })$ + $\mathbf{Ax(\uparrow\uparrow)}$ and that $\mathfrak{F}$ is Archimedean,

$$\text{(maximal time-like geodesics)} \;=\; L^T.$$

This does not remain so if we omit the assumption $\mathbf{Ax(syt_0)}$, since then $Rng(g) = \emptyset$ can happen. Further, in our next item we conjecture that the maximal time-like geodesics are not necessarily members of the set L of lines even if we assume **Basax** and that $\mathfrak{F} = \mathfrak{R}$.

Conjecture 6.8.35 *Let* $Th := \mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(\uparrow\uparrow)}$. *Then (i)-(iv) below hold.*

(i) *Assume* $n = 2$. *Then there is* $\mathfrak{M} \in \mathsf{Mod}(Th + \mathbf{Ax(}Triv\mathbf{)} + \mathbf{Ax(eqm)})$ *with* $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ *and with the topology* $\mathcal{T}_{\mathfrak{M}}$ *Euclidean such that*

$$(\exists \text{ maximal time-like geodesic } \ell \text{ of } \mathfrak{G}_{\mathfrak{M}})\ \ell \notin L_{\mathfrak{M}}.$$

Moreover this ℓ *is not contained in any* $L_{\mathfrak{M}}$*-line.*

(ii) *Assume* $n = 2$. *Then there is* $\mathfrak{M} \in \mathsf{Mod}(Th + \mathbf{Ax(eqm)})$ *with* $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ *and with the topology* $\mathcal{T}_{\mathfrak{M}}$ *Euclidean, and there is a maximal geodesic* ℓ *of* $\mathfrak{G}_{\mathfrak{M}}$ *such that no 3 distinct points of* ℓ *are* $L_{\mathfrak{M}}$*-collinear.*

(iii) *Assume* $n > 2$. *Then there is* $\mathfrak{M} \in \mathsf{Mod}(Th + \mathbf{Ax(}Triv_t\mathbf{)}^-)$ *with* $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ *and with the topology* $\mathcal{T}_{\mathfrak{M}}$ *Euclidean such that there is a maximal time-like geodesic* ℓ *of* $\mathfrak{G}_{\mathfrak{M}}$ *such that* $\ell \notin L_{\mathfrak{M}}$. *Moreover this* ℓ *is not contained in any* $L_{\mathfrak{M}}$*-line.*