(i) w_m is an injection.

(ii) Assume $m \overset{\odot}{\to} k$. Then f_{mk} is a bijective collineation by Thm.4.3.11 (p.481).

(iii) $(Rng(w_m) \cap Rng(w_k) = \emptyset$ or $Rng(w_m) = Rng(w_k))$ and $(m \overset{\odot}{\to} k \quad \Leftrightarrow \quad Rng(w_m) = Rng(w_k))$. This holds by Thm.4.3.11 and **Ax4**.

(iv) Assume $\mathbf{Ax}(\sqrt{\ })$ and $m \overset{\odot}{\to} k$. Then f_{mk} is betweenness preserving by Fact 4.7.7 (p.617) and Remark 3.6.7 (p.268).

(v) There are no photons at rest by $\mathbf{AxE_{01}}$.

(vi) Assume **Ax(diswind)** and $Rng(w_m) \cap Rng(w_k) = \emptyset$. Assume ph is a photon such that $(\exists e \in Rng(w_m))\, ph \in e$. Then $(\forall e \in Rng(w_k))\, ph \notin e$.

(vii) Assume $\mathbf{Ax}(\sqrt{\ }) + (\mathsf{c}_m(d) < \infty)$ and $(n > 2$ or $\mathbf{Ax}(\uparrow\uparrow_0))$. Then there are no FTL observers by items 4.3.24 (p.497) and 6.2.32 (p.840). ∎

In connection with the following remark recall that **Pax** is weaker than $\mathbf{Bax}^-$, cf. p.482 in §4.3.

Remark 6.2.80 The following items remain true if the assumption $\mathbf{Bax}^-$ is replaced by **Pax** in them: Remark 6.2.13 (p.819), Prop.6.2.14 (p.819), Prop.6.2.16 (p.820), Prop.6.2.36 (p.843), Thm.6.2.44(ii) (p.847), and almost the whole of Prop.6.2.79, i.e. Prop.6.2.79 with the exception of items 1h, 4c, 4d, 4e. The new proofs (i.e. for **Pax**) can be obtained from the old ones (i.e. for $\mathbf{Bax}^-$) by replacing Thm.4.3.11 (p.481) with Thm.4.3.13 (p.482) in them.

◁

Remark 6.2.81 (On Figure 290) Figure 289 shows that our geometries $\mathfrak{G}_{\mathfrak{M}}$ can be viewed as being glued together from "windows" which in turn can be regarded as world-views of individual observers. There is a (deliberate) analogy here with the so called Penrose diagrams from general relativity. (We will not explain Penrose diagrams here but certain properties are "visible" without explanation.) Figure 290 on p.888 represents a Penrose diagram (of a general relativistic space-time geometry) from Hawking-Ellis [126]. It is visible on Figure 290 that this geometry, too, consists of regions like our windows on Figure 289. (Cf. e.g. regions I, II. III on the diagram.) Roughly, each of these regions can be regarded as the window of some observer (just like in our Fig.289). Of course, besides the similarities there are some dissimilarities which we do not discuss here. We note that the fact that our geometries are glued together from windows is intended to make transitions towards general relativity easier (in later continuations of the present work). In passing we note that Figure 290 is the Penrose diagram of a rotating black hole which contains closed time-like geodesics ("time travel"). Therefore it is related to Figure 355 on p.1208 which also contains closed time-like geodesics (among other exotic and exciting features). Figure 290 is in "Penrose-diagram form" while Figure 355 is in a more usual space-time diagram form.

$\triangleleft$

6.2.6 Proof of Theorems 6.2.22 and 6.2.23 which say that eq is first-order definable in $\mathfrak{M}$

The reader may safely skip the present sub-section. However Propositions 6.2.83 (p.892), 6.2.88 (p.895) and 6.2.92 (p.901) herein might be interesting.

This sub-section is devoted to the proof[841] of Theorems 6.2.22 and 6.2.23 which say that the relation eq of equidistance becomes first-order definable in the "observational world" $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, under some assumptions. In proving these theorems the key propositions will be Propositions 6.2.83, 6.2.88 and 6.2.92. The proof of the above mentioned theorems comes on p.906.

Definition 6.2.82 Let $\mathfrak{G}$ be an observer-independent geometry, $\ell \in L$ and $o, e \in \ell$ with $o \neq e$. The half-line $\vec{\ell}_{o,e}$ with origin o and containing e is defined as follows.

$$\vec{\ell}_{o,e} \stackrel{\text{def}}{:=} \{\, e_1 \in \ell \;:\; \neg Bw(e_1, o, e) \,\}.$$

$\triangleleft$

The Euclidean geometry $Euclgeom(\mathfrak{F})$ over $\mathfrak{F}$ will be introduced on p.1129. We will use it without recalling it. Assume $\mathbf{Bax}^{-} + \mathbf{Ax}(\sqrt{\ })$. Then without proving them we will use the facts that our notion of a half-line satisfies the usual (geometric) properties of half-lines and that a pre-image of a half-line along a world-view is again a half-line (or the empty set) but now in $Euclgeom(\mathfrak{F})$ (i.e. the w_m^{-1}-image of a half-line is a Euclidean half-line or the empty set), cf. Fact 6.2.85 and items 1f, 2b of Prop.6.2.79.

[841] We will have a single proof which will prove both theorems.

Figures 291 and 294 (p.895) represent the two key steps (or the two key ideas) of the proof which is the subject matter of the present sub-section.

The proposition below is one of the key propositions in proving Theorem 6.2.22. For the intuitive meaning of the proposition see Figure 291.

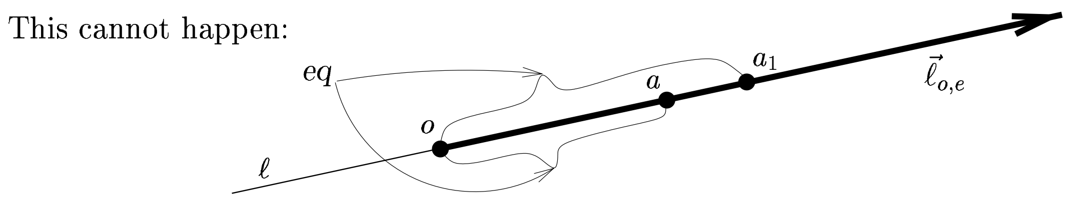

Figure 291: Illustration for Proposition 6.2.83.

PROPOSITION 6.2.83 *Assume* **Basax** + **Ax**($\sqrt{\ }$). *Let* $\ell \in L$ *and* $o, e \in \ell$ *with* $o \neq e$. *Then*

$$[\, a, a_1 \in \vec{\ell}_{o,e} \quad \wedge \quad \langle o, a\rangle \ eq \ \langle o, a_1\rangle \,] \quad \Rightarrow \quad a = a_1,$$

see Figure 291.[842] *In other words*

$$Bw(o, a, a_1) \quad \Rightarrow \quad \neg \ \langle o, a\rangle \ eq \ \langle o, a_1\rangle.^{843}$$

How the arrangement in Figure 291 can happen in systems weaker than **Basax** is illustrated in item 6.2.96 at the end of this sub-section.

Intuitively Proposition 6.2.83 above means that eq behaves well under some assumptions. In some sense this may be relevant to the question why relativity can be so nicely geometrized. Cf. also Propositions 6.2.92 and 6.2.96.

Proof of Prop.6.2.83: Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }))$. Consider the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$. Let us recall that $g^2_\mu : {}^nF \times {}^nF \longrightarrow F$ is the square of the Minkowski-distance defined in Def.2.9.1.

Claim **6.2.84** Let $a, b, c, d \in Mn$. Assume $\langle a, b\rangle \ eq \ \langle c, d\rangle$. Then, each observer thinks that the Minkowski-distances between a, b and between c, d coincide; formally: $(\forall m \in Obs)\, g^2_\mu(w_m^{-1}(a), w_m^{-1}(b)) = g^2_\mu(w_m^{-1}(c), w_m^{-1}(d))$.

[842] We conjecture that $\mathbf{Ax}(\sqrt{\ })$ might not be needed in Proposition 6.2.83.

[843] We note that, assuming $\mathbf{Bax}^-$, $[\,Bw(o, a, a_1) \ \wedge \ \langle o, a\rangle \ eq \ \langle o, a_1\rangle\,] \Rightarrow (\exists \ell \in L)\, o, a, a_1 \in \ell$.

Proof: We will show that the claim holds when eq is replaced by eq_0 in it. This will prove the claim since eq is defined to be the transitive closure of eq_0. Let $a, b, c, d \in Mn$ be such that $\langle a, b\rangle \; eq_0 \; \langle c, d\rangle$. Then, by the definition of eq_0, we have that $(\exists k \in Obs)(\exists i, j \in n)(\exists p, q \in \bar{x}_i)(\exists r, s \in \bar{x}_j)$

$$\Big(|p - q| = |r - s| \;\wedge\; w_k(p) = a \;\wedge\; w_k(q) = b \;\wedge\; w_k(r) = c \;\wedge\; w_k(s) = d \Big),$$

cf. Figure 292. Let such k, i, j, p, q, r, s be fixed. Now, let $m \in Obs$ be arbitrary but

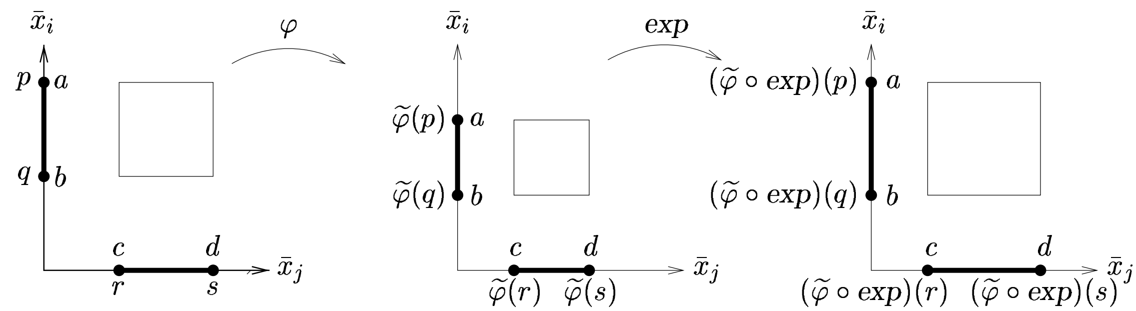

Figure 292: Illustration for the proof of Claim 6.2.84.

fixed. By Theorem 2.9.4(i), $\mathsf{f}_{km} = \widetilde{\varphi} \circ exp \circ poi$, for some $poi \in Poi$, $exp \in Exp$ and $\varphi \in Aut(\mathfrak{F})$. Let these be fixed. It can be easily checked that $(\widetilde{\varphi} \circ exp)(p), (\widetilde{\varphi} \circ exp)(q) \in \bar{x}_i$, $(\widetilde{\varphi} \circ exp)(r), (\widetilde{\varphi} \circ exp)(s) \in \bar{x}_j$ and

$$|(\widetilde{\varphi} \circ exp)(p) - (\widetilde{\varphi} \circ exp)(q)| = |(\widetilde{\varphi} \circ exp)(r) - (\widetilde{\varphi} \circ exp)(s)|,$$

cf. Figure 292. Thus

$$g^2_\mu((\widetilde{\varphi} \circ exp)(p), (\widetilde{\varphi} \circ exp)(q)) = g^2_\mu((\widetilde{\varphi} \circ exp)(r), (\widetilde{\varphi} \circ exp)(s)).$$

Since the Poincaré transformations preserve g^2_μ and $\mathsf{f}_{km} = \widetilde{\varphi} \circ exp \circ poi$, we have

$$g^2_\mu(\mathsf{f}_{km}(p), \mathsf{f}_{km}(q)) = g^2_\mu(\mathsf{f}_{km}(r), \mathsf{f}_{km}(s)).$$

But $\mathsf{f}_{km}(p), \mathsf{f}_{km}(q), \mathsf{f}_{km}(r), \mathsf{f}_{km}(s)$ are $w_m^{-1}(a), w_m^{-1}(b), w_m^{-1}(c), w_m^{-1}(d)$ respectively. So

$$g^2_\mu(w_m^{-1}(a), w_m^{-1}(b)) = g^2_\mu(w_m^{-1}(c), w_m^{-1}(d)),$$

and this proves the claim.

QED (Claim 6.2.84)

Now we turn to proving Proposition 6.2.83. Let $\ell \in L$ and $o, e \in \ell$ with $o \neq e$. Let $a, a_1 \in \vec{\ell}_{o,e}$ be such that $\langle o, a\rangle \; eq \; \langle o, a_1\rangle$. If $\ell \in L^{Ph}$ then, by the definition of eq, it can be checked that $o = a = a_1$, since there is no observer who sees a photon-like line on a coordinate axis. So, we may assume $\ell \in L^S \cup L^T$, i.e. ℓ is space-like or time-like. Then there is an observer m who sees ℓ on some coordinate axis, i.e. $\ell = w_m[\bar{x}_i]$, for some $i \in n$. Let such m and $\bar{x}_i$ be fixed. Clearly, $w_m^{-1}(o), w_m^{-1}(a), w_m^{-1}(a_1) \in w_m^{-1}[\vec{\ell}_{o,e}]$. Further, $w_m^{-1}[\vec{\ell}_{o,e}]$ is contained in $\bar{x}_i$ and is a Euclidean half-line, i.e. letting $\lambda := w_m^{-1}(o)_i \in F$,

$$(\star) \qquad w_m^{-1}[\vec{\ell}_{o,e}] = \{\, p \in \bar{x}_i \;:\; p_i \geq \lambda \,\} \quad \vee \quad w_m^{-1}[\vec{\ell}_{o,e}] = \{\, p \in \bar{x}_i \;:\; p_i \leq \lambda \,\},$$

see Figure 293, cf. Fact 6.2.85. By Claim 6.2.84,

Figure 293: $w_m^{-1}[\vec{\ell}_{o,e}]$ is contained in $\bar{x}_i$ and is a Euclidean half-line.

$$g_\mu^2(w_m^{-1}(o), w_m^{-1}(a)) = g_\mu^2(w_m^{-1}(o), w_m^{-1}(a_1)).$$

Therefore $w_m^{-1}(a) = w_m^{-1}(a_1)$. Hence $a = a_1$. ∎

FACT 6.2.85 *Assume* $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$. *Assume* $\vec{\ell}_{o,e}$ *is a half-line contained in the line* $\ell \in L$. *Assume observer* m *sees* ℓ *on the coordinate axis* $\bar{x}_i$ *(i.e.* $w_m[\bar{x}_i] = \ell$*). Then* m *will see the half-line* $\vec{\ell}_{o,e}$ *as illustrated in Figure 293, i.e.* $(\star)$ *above holds for some* λ.[844]

Proof: The fact follows by item 1f of Prop.6.2.79 (p.884). ∎

Definition 6.2.86 Let $\mathfrak{G}$ be an observer-independent geometry. We extend the definition of connectedness $\sim$ from points Mn to lines L the natural way, i.e. as follows. Let $\ell, \ell' \in L$. Then

$$\ell \sim \ell' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists e \in \ell)(\exists e' \in \ell')\, e \sim e'.$$

◁

[844]The assumption $\mathbf{Ax}(\sqrt{\ })$ is needed in Fact 6.2.85.

Remark 6.2.87 Assume $\mathbf{Bax}^-$. Let $\ell, \ell' \in L^T \cup L^S$. Then

$$\ell \sim \ell' \iff (\forall e \in \ell)(\forall e' \in \ell')\, e \sim e'.$$ [845]

◁

For the intuitive meaning of the proposition below see Figure 294.

PROPOSITION 6.2.88 *Assume $n > 2$ and $\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$. Let $\ell, \ell' \in L^T \cup L^S$ be such that $\ell \sim \ell'$. Further, let $a, b \in \ell'$ and $c, u \in \ell$ with $c \neq u$. Then there is $d \in \vec{\ell}_{c,u}$ such that $\langle a, b\rangle\ eq_2\ \langle c, d\rangle$. See Figure 294.*[846]

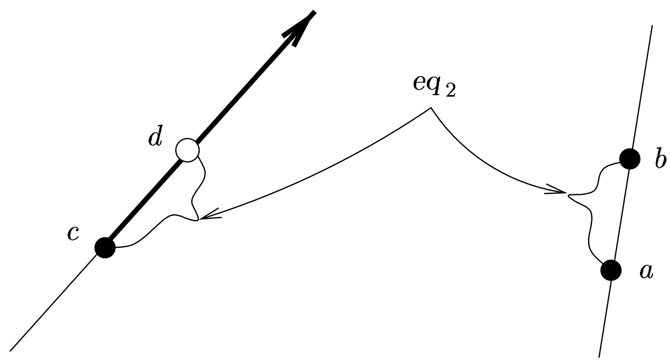

Figure 294: Illustration for Proposition 6.2.88: $(\forall a, b, c)(\exists d$ as in the figure$)$.

Proof: Assume $n > 2$ and $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$. Consider the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$. First we prove Lemma 6.2.89 below.

The intuitive idea of the proof in the present sub-section (i.e. the proof of Thm.'s 6.2.22, 6.2.23) is explained below Definition 6.2.91 on p.899. To implement that intuitive idea we will use Lemma 6.2.89 below.

Lemma **6.2.89** Let $\ell \in L^T \cup L^S$ and $\ell_1 \in L^T$ be such that $\ell \sim \ell_1$. Then there are $m, k \in \mathit{Obs}$ and $\ell_2 \in L^S$ such that (a)–(c) below hold, cf. Figure 295. Intuitively, (a) both m and k see ℓ_2 on $\bar{y}$-axis, (b) m sees ℓ_1 on $\bar{t}$-axis and (c) k sees ℓ on $\bar{x}$-axis or on $\bar{t}$-axis; formally:

[845] This holds by item 2c of Prop.6.2.79 (p.885) and by Remark 6.2.13 (p.819).

[846] We note that the assumption $n > 2$ is needed in Proposition 6.2.88, cf. the first 8 lines of the proof of Thm.6.2.22 on p.906.

(a) $\ell_2 = w_m[\bar{y}] = w_k[\bar{y}]$.

(b) $\ell_1 = w_m[\bar{t}\,]$.

(c) $\ell = w_k[\bar{x}]$ or $\ell = w_k[\bar{t}\,]$.

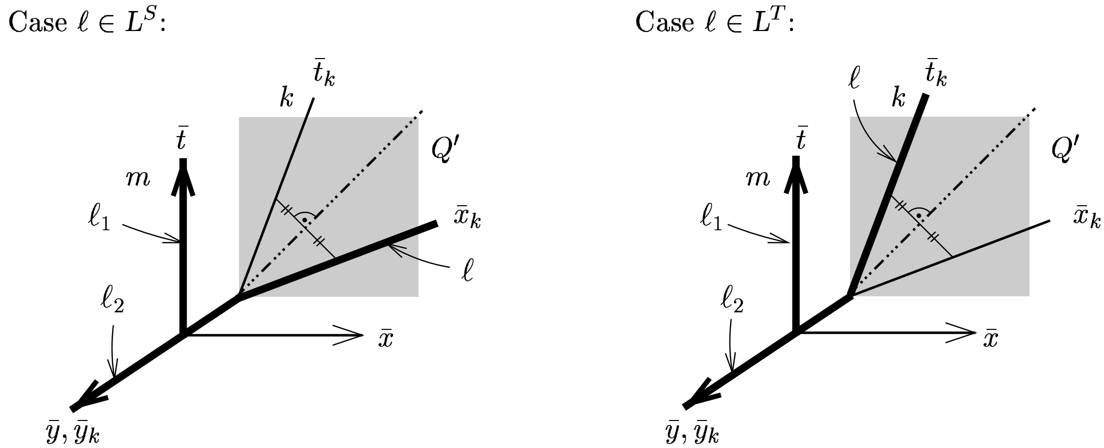

Figure 295: Illustration for Lemma 6.2.89; we are in the world-view of m.

Proof: Recall that $\mathfrak{M}$ was fixed at the beginning of the proof of Prop.6.2.88. Let $\mathfrak{N}$ be a model of **Newbasax** obtained from $\mathfrak{M}$ by changing the units of measurement for time, formally: Assume $\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W\rangle$. Recall that c_m is the speed of light for observer m. Define $W^{\mathfrak{N}} \subseteq Obs \times {}^nF \times B$ by

$$W^{\mathfrak{N}}(m, p_0, \ldots, p_{n-1}, h) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad W(m, 1/\sqrt{c_m} \cdot p_0, p_1, \ldots, p_{n-1}, h).$$

Now,

$$\mathfrak{N} :\stackrel{\text{def}}{=} \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W^{\mathfrak{N}}\rangle.$$

By the definition of $\mathfrak{N}$ it can be checked that

$\mathfrak{N} \models$ **Newbasax** $+$ **Ax(**$Triv_t$**)**$^-$ $+$ **Ax**$(\sqrt{\ })$,

$(\forall m \in Obs)(\forall i \in n)\ w_m^{\mathfrak{M}}[\bar{x}_i] = w_m^{\mathfrak{N}}[\bar{x}_i]$, and

$\langle Mn_{\mathfrak{M}};\ L^T_{\mathfrak{M}}, L^S_{\mathfrak{M}}, \in_{\mathfrak{M}}\rangle = \langle Mn_{\mathfrak{N}};\ L^T_{\mathfrak{N}}, L^S_{\mathfrak{N}}, \in_{\mathfrak{N}}\rangle$, i.e. the reducts involving only Mn, L^T, L^S of the geometries associated to $\mathfrak{M}$ and $\mathfrak{N}$ coincide.

By these, it is enough to prove the lemma for $\mathfrak{N}$ in place of $\mathfrak{M}$. To prove the lemma for $\mathfrak{N}$, let $\ell \in L^T \cup L^S$ and $\ell_1 \in L^T$ be such that $\ell \sim \ell_1$. (Throughout of the proof of the lemma we are in $\mathfrak{N}$ [and not in $\mathfrak{M}$].) Then there is an observer m' whose life-line is ℓ_1, i.e. $w_{m'}[\bar{t}] = \ell_1$. Let such an m' be fixed. Let $\ell^E := w_{m'}^{-1}[\ell]$. Since $\ell \sim \ell_1$, we have that $\ell^E \in \mathsf{Eucl}$ holds by items 6.2.13, 6.2.48, and 4.3.11. We will work in the world-view of observer m', represented in Figure 296, for a while. Throughout this part of the proof the reader is advised to consult Figure 296. We are in the

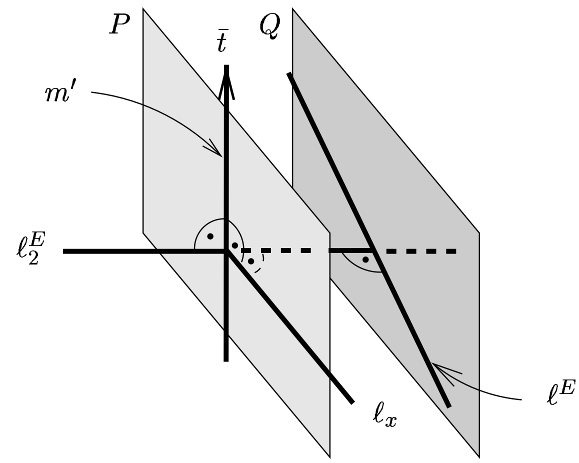

Figure 296: We are in Euclidean geometry; in the world-view of m'.

Euclidean geometry over an Euclidean field. Let $\ell_2^E \in \mathsf{Eucl}$ be such that

$$\ell_2^E \perp_e \bar{t}, \quad \ell_2^E \perp_e \ell^E, \quad \ell_2^E \cap \bar{t} \neq \emptyset \quad \text{and} \quad \ell_2^E \cap \ell^E \neq \emptyset.$$

Since $n > 2$ such an ℓ_2^E exists by Euclidean geometry. In passing we note that if $\bar{t} \cap \ell^E = \emptyset$ then ℓ_2^E is uniquely determined, while if $n > 3$ and $\bar{t} \cap \ell^E \neq \emptyset$ then it is not uniquely determined. Let $P, Q \in \mathsf{Planes}(n, \mathbf{F})$ be such that

$$\bar{t} \subseteq P, \quad \ell^E \subseteq Q, \quad P \parallel Q \quad \text{and} \quad \ell_2^E \perp_e P.$$

Such P, Q exist by Euclidean geometry. Let $m \in Obs$ be a brother of m' such that the world-view of m looks like as in Figure 295, i.e. $(*)$ below holds for m. The existence of such an m can be proved by $\mathbf{Ax}(\mathit{Triv}_t)^-$, and is proved in more detail below $(*)$.

$(*)$ $\quad w_m[\bar{t}] = w_{m'}[\bar{t}] = \ell_1, \quad w_m[\mathsf{Plane}(\bar{t}, \bar{x})] = w_{m'}[P] \quad$ and
$\quad w_m[\bar{y}] = w_{m'}[\ell_2^E]$.

In more detail, there is an observer m satisfying $(*)$ because of the following: Let $\ell_x \in \mathsf{Eucl}$ be such that

$$\ell_x \subseteq P, \quad \bar{t} \cap \ell_x \cap \ell_2^E \neq \emptyset, \quad \ell_x \perp_e \bar{t}.$$

Then, since $\ell_2^E \perp_e P$ and $\ell_x \subseteq P$, we have $\ell_2^E \perp_e \ell_x$. Thus, $\bar{t}, \ell_x, \ell_2^E$ are pairwise $\perp_e$-orthogonal. Therefore there is a Triv_t transformation f that takes $\bar{t}, \bar{x}, \bar{y}$ to $\bar{t}, \ell_x, \ell_2^E$ respectively. Let such an f be fixed. Clearly, f takes $\mathsf{Plane}(\bar{t}, \bar{x})$ to P. By $\mathbf{Ax}(\mathit{Triv}_t)^-$, there is an observer m such that $\mathsf{f}_{mm'}[\bar{t}] = \mathsf{f}[\bar{t}] = \bar{t}$, $\mathsf{f}_{mm'}[\bar{x}] = \mathsf{f}[\bar{x}] = \ell_x$ and $\mathsf{f}_{mm'}[\bar{y}] = \mathsf{f}[\bar{y}] = \ell_2^E$. Clearly, for this choice of m $(*)$ holds.

Throughout the remaining part of the proof of the lemma the reader is advised to consult Figure 295 (p.896). Let

$$\ell_2 :\stackrel{\text{def}}{=} w_m[\bar{y}] \in L^S.$$

(We note that in the world-view of m' [i.e. in Figure 296] ℓ_2 appears as ℓ_2^E, i.e. m' sees ℓ_2 on ℓ_2^E.)

Thus, to complete the proof of the lemma it is sufficient to find an observer k such that $\ell_2 = w_k[\bar{y}]$ and $(\ell = w_k[\bar{x}]$ or $\ell = w_k[\bar{t}])$. Let $Q' := \mathsf{f}_{m'm}[Q]$. Then $Q' \parallel \mathsf{Plane}(\bar{t}, \bar{x})$ by $P \parallel Q$ and $(*)$. Further, $w_m^{-1}[\ell] \subseteq Q'$, $w_m^{-1}[\ell] \in \mathsf{Eucl}$ and $\ell \cap \ell_2 \neq \emptyset$. We distinguish two cases.

Case 1: Assume $\ell \in L^S$. Then, by our no FTL theorem Thm.3.4.2 (or Thm.4.3.24), $\mathit{ang}^2(w_m^{-1}[\ell]) > 1$. Let $\bar{t}_k$ be the mirror image of $w_m^{-1}[\ell]$ w.r.t. a photon-line lying in plane Q' and passing through $w_m^{-1}(\ell \cap \ell_2)$, see the left-hand side of Figure 295 (p.896). Then $\mathit{ang}^2(\bar{t}_k) < 1$. Let $k' \in \mathit{Obs}$ be such that $\mathit{tr}_m(k') = \bar{t}_k$. Such a k' exists by **Ax5**. We claim that, since we are in **Newbasax**, in the world-view[847] of k' the time axis, ℓ, ℓ_2 appear as pairwise $\perp_e$-orthogonal; formally $\bar{t}$, $w_{k'}^{-1}[\ell], w_{k'}^{-1}[\ell_2]$ are pairwise $\perp_e$-orthogonal, cf. Figure 295 on p.896. One proves this claim by using properties of **Basax** models and noticing that **Basax** models and **Newbasax** models are very close to each other, cf. Thm.3.3.12. One of the just mentioned properties of **Basax** models is that every world-view transformation is a composition of a PT transformation and a field automorphism, cf. Proposition 3.6.5 (cf. also Thm.2.9.4(i)). Here we may ignore the field automorphism because it does not affect orthogonality. We advise the reader to prove this claim first for the special case $Q' = \mathsf{Plane}(\bar{t}, \bar{x})$, see Figure 295 on p.896.

Now, by $\mathbf{Ax}(\mathit{Triv}_t)^-$ there is a brother k of k' such that k sees ℓ on $\bar{x}$-axis and sees ℓ_2 on $\bar{y}$-axis; formally: $\ell = w_k[\bar{x}]$ and $\ell_2 = w_k[\bar{y}]$.

[847]When we speak about the world-view of an observer say m then we can mean a structure whose universe is ${}^nF$ or equivalently a structure whose universe is $\mathit{Rng}(w_m) \subseteq \mathit{Mn}$. Since these two are isomorphic it does not matter which one we choose. If the choice mattered for some reason then we leave it to the reader to make the appropriate choice based on context.

Case 2: Assume $\ell \in L^T$. See the right-hand side of Figure 295. Then, by our no FTL theorem, $ang^2(w_m^{-1}[\ell]) < 1$. Let $\bar{x}_k$ be the mirror image of $w_m^{-1}[\ell]$ w.r.t. a photon-line lying in plane Q' and passing through $w_m^{-1}(\ell \cap \ell_2)$. Then $ang^2(\bar{x}_k) > 1$. Now, similarly to the proof for Case 1, one can prove that there is $k \in Obs$ such that k sees ℓ on $\bar{t}$-axis and ℓ_2 on $\bar{y}$-axis, i.e. $\ell = w_k[\bar{t}]$ and $\ell_2 = w_k[\bar{y}]$ (and $w_m[\bar{x}_k] = w_k[\bar{x}]$).

QED(Lemma 6.2.89)

Claim **6.2.90** Let $\ell \in L^T \cup L^S$ and $\ell_1 \in L^T$ be such that $\ell \sim \ell_1$. Then (i) and (ii) below hold.

(i) Assume $a, b \in \ell$. Then there are $e, f \in \ell_1$ such that $\langle a, b\rangle\ eq_1\ \langle e, f\rangle$.

(ii) Let $e, f \in \ell_1$ be arbitrary and $c, u \in \ell$ with $c \neq u$. Then there is $d \in \vec{\ell}_{c,u}$ such that $\langle e, f\rangle\ eq_1\ \langle c, d\rangle$.

Before proving the claim we need a definition.

We will use the following definition outside the scope of the present proof, too. I.e. the definition applies to arbitrary $\mathfrak{M}$ and m with w_m injective.

Definition 6.2.91 Assume $m \in Obs$ and $e, e_1, e_2, e_3 \in Mn$. Assume the world-view function w_m is injective (as usual). Then observer m is called a *witness to $\langle e, e_1\rangle\ eq_0\ \langle e_2, e_3\rangle$* iff there are coordinate-axes $\bar{x}_i$ and $\bar{x}_j$ $(i, j \in \omega)$ such that m sees e, e_1 on $\bar{x}_i$-axis, sees e_2, e_3 on $\bar{x}_j$-axis, and sees that the distances between e, e_1 and between e_2, e_3 coincide; formally: $e, e_1 \in w_m[\bar{x}_i]$, $e_2, e_3 \in w_m[\bar{x}_j]$ and $|w_m^{-1}(e) - w_m^{-1}(e_1)| = |w_m^{-1}(e_2) - w_m^{-1}(e_3)|$. Cf. the definition of eq_0 on p.793. See Figure 297.

$\triangleleft$

The idea of the proof in the present sub-section[848] is the following. Assume $\langle a, b\rangle\ eq\ \langle c, d\rangle$. Then there is a finite (but possibly long) chain of eq_0-witnesses establishing this eq-connection. Then we want to replace that original (possibly long) chain with a short one. The short chain establishes $\langle a, b\rangle\ eq_2\ \langle c, d\rangle$. This way one proves $eq \subseteq eq_2$.

Proof of Claim 6.2.90: Recall that $\mathfrak{M}$ was fixed at the beginning of the proof of Prop.6.2.88. Let $\ell \in L^T \cup L^S$ and $\ell_1 \in L^T$ be such that $\ell \sim \ell_1$. Then there are $m, k \in Obs$ and $\ell_2 \in L^S$ such that (a)–(c) of Lemma 6.2.89 hold for $m, k, \ell, \ell_1, \ell_2$, see Figure 295 (p.896). Let such m, k, ℓ_2 be fixed.

To prove (i) assume $a, b \in \ell$. Recall (from (a)–(c) of 6.2.89) that k sees ℓ either on $\bar{x}$-axis or on $\bar{t}$-axis and sees ℓ_2 on $\bar{y}$-axis. Let $e_0, f_0 \in \ell_2$ be such that k is a witness to

$$\langle a, b\rangle\ eq_0\ \langle e_0, f_0\rangle.$$

[848] i.e. the proof of Thm.'s 6.2.22, 6.2.23

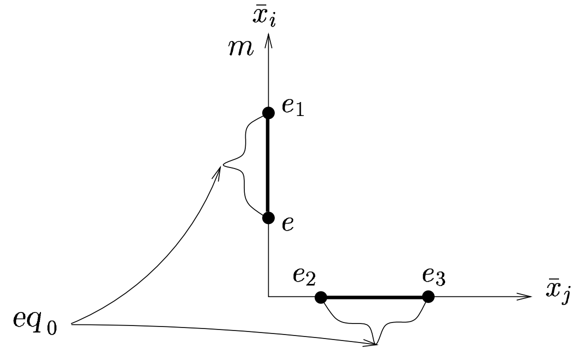

Figure 297: m is a witness to $\langle e, e_1\rangle\ eq_0\ \langle e_2, e_3\rangle$.

Recall (from (a)–(c) of 6.2.89) that m sees ℓ_2 on $\bar{y}$-axis and sees ℓ_1 on $\bar{t}$-axis. Let $e, f \in \ell_1$ be such that m is a witness to

$$\langle e_0, f_0\rangle\ eq_0\ \langle e, f\rangle.$$

Then, $\langle a, b\rangle\ eq_1\ \langle e, f\rangle$. This proves item (i).

To prove (ii) assume $e, f \in \ell_1$ and $c, u \in \ell$ with $c \neq u$. Let $c_0, d_0 \in \ell_2$ be such that m is a witness to

$$\langle e, f\rangle\ eq_0\ \langle c_0, d_0\rangle.$$

Let $d \in \vec{\ell}_{c,u}$ be such that k is a witness to

$$\langle c_0, d_0\rangle\ eq_0\ \langle c, d\rangle.$$ [849]

Thus, $\langle e, f\rangle\ eq_1\ \langle c, d\rangle$. This proves item (ii).

QED (Claim 6.2.90)

To prove Proposition 6.2.88 let $\ell, \ell' \in L^T \cup L^S$ be such that $\ell \sim \ell'$. Further let $a, b \in \ell'$ and $c, u \in \ell$ with $c \neq u$. Now, let $\ell_1 \in L^T$ be such that $\ell \sim \ell_1$. Clearly, $\ell' \sim \ell_1$.[850] Applying item (i) of Claim 6.2.90 for ℓ' and ℓ_1 we get that there are $e, f \in \ell_1$ such that

$$\langle a, b\rangle\ eq_1\ \langle e, f\rangle.$$

Let such e, f be fixed. Applying item (ii) of Claim 6.2.90 we get that there is $d \in \vec{\ell}_{c,u}$ such that

$$\langle e, f\rangle\ eq_1\ \langle c, d\rangle.$$

[849] Such a d exists by Fact 6.2.85.

[850] This holds by Remark 6.2.87 (p.895).

For this choice of $d \in \vec{\ell}_{c,u}$ we have $\langle a,b\rangle\ eq_2\ \langle c,d\rangle$. This completes the proof. Therefore
Proposition 6.2.88 has been proved. ∎

The following proposition is a "generalization" of Proposition 6.2.83. For the intuitive meaning of the proposition see Figure 298.

PROPOSITION 6.2.92 *Assume $n > 2$ and* $\mathbf{Bax}^{\oplus} + \mathbf{Ax(\|)}^- + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$. *Let $\ell \in L$ and $o, e \in \ell$ with $o \neq e$. Then*

$$[\,a, a_1 \in \vec{\ell}_{o,e} \quad \wedge \quad \langle o,a\rangle\ eq\ \langle o,a_1\rangle\,] \quad \Rightarrow \quad a = a_1,$$

see Figure 298.[851] *In other words*

$$Bw(o,a,a_1) \quad \Rightarrow \quad \neg\ \langle o,a\rangle\ eq\ \langle o,a_1\rangle.$$

This cannot happen:

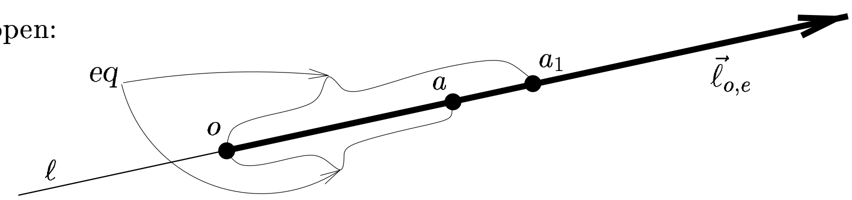

Figure 298: Illustration for Proposition 6.2.92.

Proof:

Intuitive idea of the proof: We start out with $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$. Then we transform $\mathfrak{M}$ to a **Newbasax** model $\mathfrak{N}$. We check that *eq* remains the same on space-like separated pairs of points (i.e. on such pairs of points *eq* of $\mathfrak{M}$ agrees with *eq* of $\mathfrak{N}$). We already know that in **Newbasax** models *eq* behaves well. Hence *eq* behaves well in $\mathfrak{N}$. From these we infer that *eq* behaves well on space-like separated pairs of points in $\mathfrak{M}$. It remains to check that *eq* behaves well on time-like separated pairs of points too in $\mathfrak{M}$. The idea of checking this is illustrated in Figure 300 on p.906.

Formal proof: For the formal proof first we need a definition and a lemma.

Definition : Assume $\mathfrak{M}$ is a frame model such that in $\mathfrak{M}$ the world-view transformations w_m are injections. Consider the observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$. We

[851] The assumption $n > 2$ is needed in Proposition 6.2.92, cf. Proposition 6.2.96(ii) (p.907).

define the spatial version eq^S of eq as follows. First we define $eq^S_0 \subseteq {}^4Mn$. Intuitively, we think of eq^S_0 as a binary relation between pairs of points. Let $a, b, c, d \in Mn$. Then intuitively $\langle a, b\rangle$ and $\langle c, d\rangle$ are eq^S_0-related iff there is an observer m such that for m (i)–(iv) below hold. (i) a and b are simultaneous, (ii) c and d are simultaneous, (iii) the distances between events a, b and between events c, d coincide and (iv) the lines connecting a, b and c, d intersect the time-axis (cf. Figure 299); formally:

$$\langle a, b\rangle \; eq^S_0 \; \langle c, d\rangle$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$(\exists m \in Obs)(\exists \ell, \ell' \in \mathsf{Eucl}) \; (\text{ (i)–(iv) below hold}).$$

See the left-hand side of Figure 299.

(i) $w_m^{-1}(a), w_m^{-1}(b) \in \ell \perp_e \bar{t}$.

(ii) $w_m^{-1}(c), w_m^{-1}(d) \in \ell' \perp_e \bar{t}$.

(iii) $|w_m^{-1}(a) - w_m^{-1}(b)| = |w_m^{-1}(c) - w_m^{-1}(d)|$.

(iv) $\ell \cap \bar{t} \neq \emptyset$ and $\ell' \cap \bar{t} \neq \emptyset$.

Now,
eq^S is defined to be the transitive closure of eq^S_0.

Let $m \in Obs$. Then observer m is called a <u>witness to $\langle a, b\rangle \; eq^S_0 \; \langle c, d\rangle$</u> iff there are $\ell, \ell' \in \mathsf{Eucl}$ such that for m, ℓ, ℓ' (i)–(iv) above hold, cf. the left-hand side of Figure 299.

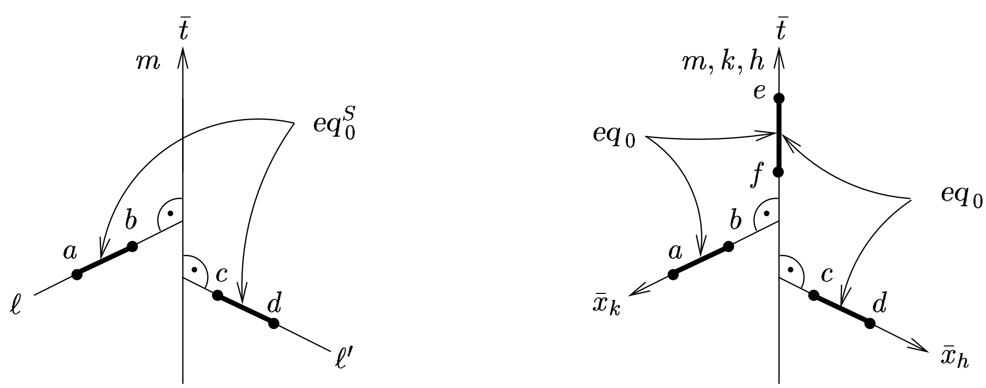

Figure 299: Illustration for eq^S_0 and the proof of Lemma 6.2.93.

The lemma below says that eq and eq^S coincide on space-like separated pairs of points, under certain conditions.

Lemma **6.2.93** Let $a, b, c, d \in Mn$. Then

$$\langle a, b\rangle \; eq^S \; \langle c, d\rangle \qquad \Longleftrightarrow \qquad \Big(a \equiv^S b \quad \wedge \quad c \equiv^S d \quad \wedge \quad \langle a, b\rangle \; eq \; \langle c, d\rangle\Big);$$

assuming $n > 2$ and $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^- + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ })$.

<u>*Proof:*</u>

<u>Proof of direction "$\Longrightarrow$":</u> Since eq^S is the transitive closure of eq_0^S and eq is transitive, it is sufficient to prove direction "$\Longrightarrow$" for eq_0^S in place of eq^S. Let $a, b, c, d \in Mn$ be such that $\langle a, b\rangle \; eq_0^S \; \langle c, d\rangle$. Let $m \in Obs$ be a witness to $\langle a, b\rangle \; eq_0^S \; \langle c, d\rangle$, see Figure 299. Let e, f be events on m's life-line such that for m the distances between events a, b, between events e, f and between events c, d coincide,[852] cf. Figure 299. By $\mathbf{Ax}(Triv_t)^-$, m has brothers k and h such that both k and h see events e, f on the $\bar{t}$-axis, k sees events a, b on the $\bar{x}$-axis and h sees events c, d on the $\bar{x}$-axis,[853] see the right-hand side of Figure 299. Let such $k, h \in Obs$ be fixed. Then $a \equiv^S b$ and $c \equiv^S d$. By $\mathbf{Ax}(\|)^-$, the world-view transformation between m and k is an isometry composed by an expansion. Thus, for observer k the distances between events a, b and between events e, f coincide,[854] since they coincide for m. Thus, k is a witness to

$$\langle a, b\rangle \; eq_0 \; \langle e, f\rangle,$$

for witness to eq_0 cf. Def.6.2.91 and Figure 297 on p.900. Similarly, by $\mathbf{Ax}(\|)^-$, for observer h the distances between events e, f and between events c, d coincide, since they coincide for m. Thus, h is a witness to

$$\langle e, f\rangle \; eq_0 \; \langle c, d\rangle.$$

$\langle a, b\rangle \; eq_0 \; \langle e, f\rangle$ and $\langle e, f\rangle \; eq_0 \; \langle c, d\rangle$ imply $\langle a, b\rangle \; eq \; \langle c, d\rangle$. This completes the proof of direction "$\Longrightarrow$" of Lemma 6.2.93.

<u>Proof of direction "$\Longleftarrow$":</u> Let $a, b, c, d \in Mn$ be such that $a \equiv^S b$, $c \equiv^S d$ and $\langle a, b\rangle \; eq \; \langle c, d\rangle$. We want to prove $\langle a, b\rangle \; eq^S \; \langle c, d\rangle$. If $a = b$ then $c = d$, by the definition of eq and by injectiveness of w_m's. Further $\langle a, a\rangle \; eq^S \; \langle c, c\rangle$ can be easily

[852]Formally: $e, f \in w_m[\bar{t}]$ and $|w_m^{-1}(a) - w_m^{-1}(b)| = |w_m^{-1}(e) - w_m^{-1}(f)| = |w_m^{-1}(c) - w_m^{-1}(d)|$.
[853]Formally: $e, f \in w_k[\bar{t}] = w_h[\bar{t}] = w_m[\bar{t}]$, $a, b \in w_k[\bar{x}]$ and $c, d \in w_h[\bar{x}]$.
[854]Formally: $|w_k^{-1}(a) - w_k^{-1}(b)| = |w_k^{-1}(e) - w_k^{-1}(f)|$.

checked. Thus, we can assume $a \neq b$. Since $\langle a, b\rangle \; eq \; \langle c, d\rangle$, there is a finite chain of pairs $\langle a^i, b^i\rangle$ with $a^i \neq b^i$ $(i \leq k \in \omega)$ such that

$$\langle a, b\rangle = \langle a^0, b^0\rangle \; eq_0 \; \langle a^1, b^1\rangle \; eq_0 \; \ldots \; eq_0 \langle a^k, b^k\rangle = \langle c, d\rangle.$$

Without loss of generality we may assume that our chain $\langle a^0, b^0\rangle \ldots \langle a^k, b^k\rangle$ is of minimal length (i.e. it cannot be replaced by a shorter similar chain).

Let $m^1, \ldots, m^k \in Obs$ be such that for every $0 < i \leq k$

$$m^i \quad \text{is a witness to} \quad \langle a^{i-1}, b^{i-1}\rangle \; eq_0 \; \langle a^i, b^i\rangle,$$

for witness to eq_0 cf. Def.6.2.91 and Figure 297 on p.900. For each pair $\langle a^i, b^i\rangle$ in the above chain we have either $a^i \equiv^T b^i$ or $a^i \equiv^S b^i$, briefly each pair is either time-like or is space-like.[855] Both $\langle a^0, b^0\rangle$ and $\langle a^k, b^k\rangle$ are space-like.

Assume that a pair $\langle a^i, b^i\rangle$ in the above chain is time-like. Then, we claim that its neighbors $\langle a^{i-1}, b^{i-1}\rangle$ and $\langle a^{i+1}, b^{i+1}\rangle$ (exist and) are space-like because of the following. Assume e.g. that $\langle a^{i+1}, b^{i+1}\rangle$ is time-like. Then the witness m^{i+1} will see both pairs $\langle a^i, b^i\rangle$ and $\langle a^{i+1}, b^{i+1}\rangle$ on the time-axis. Then witness m^i sees the pairs $\langle a^i, b^i\rangle$ and $\langle a^{i+1}, b^{i+1}\rangle$ on its time-axis too, further m^i too thinks that they are of the same length since m^i and m^{i+1} have the same life-line (and by Thm.4.3.11). But since m^i is a witness to $\langle a^{i-1}, b^{i-1}\rangle \; eq_0 \; \langle a^i, b^i\rangle$ it will be a witness to $\langle a^{i-1}, b^{i-1}\rangle \; eq_0 \langle a^{i+1}, b^{i+1}\rangle$. Therefore we could throw away $\langle a^i, b^i\rangle$ from our chain. A contradiction[856], proving our claim.

Next, we turn to proving that $\langle a^0, b^0\rangle$ and $\langle a^k, b^k\rangle$ are in eq^S, i.e. that there is an eq^S_0-chain connecting them. To see this we will replace the eq_0-chain between them by an eq^S_0-chain in a step-by-step fashion. Let $i < k$. If both $\langle a^i, b^i\rangle$ and $\langle a^{i+1}, b^{i+1}\rangle$ are space-like then they are in eq^S_0 (m^{i+1} is a witness to this). Therefore we may assume that one of them is time-like. For simplicity assume $\langle a^i, b^i\rangle$ is time-like. Then $i \neq 0$ and both $\langle a^{i-1}, b^{i-1}\rangle$ and $\langle a^{i+1}, b^{i+1}\rangle$ are space-like (and m^i, m^{i+1} have the same life-line). Now, we claim that $\langle a^{i-1}, b^{i-1}\rangle \; eq^S_0 \; \langle a^{i+1}, b^{i+1}\rangle$ holds. Actually both m^i and m^{i+1} are witnesses to this. In the last step we strongly used $\mathbf{Ax}(||)^-$. This completes the proof of the lemma.

QED (Lemma 6.2.93)

Now, we turn to proving Proposition 6.2.92. Assume $n > 2$. Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(||)^- + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$. Let $\mathfrak{N}$ be a model of **Newbasax** obtained from $\mathfrak{M}$ by changing the units of measurement for time, i.e. $\mathfrak{N}$ is obtained

[855] This holds by our no FTL theorem Thm.4.3.24, $a^i \neq b^i$ and the definition of eq_0.

[856] since we assumed minimality of our chain.

from $\mathfrak{M}$ exactly the same way as in the proof of item 6.2.89 on p.896. It can be checked that

$$\mathfrak{N} \models \mathbf{Newbasax} + \mathbf{Ax}(\|)^- + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }),$$

in particular $\mathfrak{N} \models \mathbf{Ax}(\|)^-$ holds by $\mathfrak{M} \models [\mathbf{Ax}(\|)^- + (tr_m(k) = \bar{t} \rightarrow c_m = c_k)]$, cf. the text below $\mathbf{Ax}(\|)^-$ on p.828.

Claim **6.2.94** $\langle Mn_{\mathfrak{M}}, L_{\mathfrak{M}};\ Bw_{\mathfrak{M}}, eq^S_{\mathfrak{M}}, \in_{\mathfrak{M}}\rangle = \langle Mn_{\mathfrak{N}}, L_{\mathfrak{N}};\ Bw_{\mathfrak{N}}, eq^S_{\mathfrak{N}}, \in_{\mathfrak{N}}\rangle$.

<u>*Proof:*</u> $\langle Mn_{\mathfrak{M}}, L_{\mathfrak{M}};\ Bw_{\mathfrak{M}}, (eq^S_0)_{\mathfrak{M}}, \in_{\mathfrak{M}}\rangle = \langle Mn_{\mathfrak{N}}, L_{\mathfrak{N}};\ Bw_{\mathfrak{N}}, (eq^S_0)_{\mathfrak{N}}, \in_{\mathfrak{N}}\rangle$ can be checked by the definitions of eq^S_0 and $\mathfrak{N}$. Since eq^S is defined to be the transitive closure of eq^S_0 we conclude that the claim holds.

QED (Claim 6.2.94)

By Proposition 6.2.83 (p.892) and by noticing that **Basax** models and **Newbasax** models are very close to each other (cf. Thm.3.3.12), the conclusion of the position holds for the **Newbasax** model $\mathfrak{N}$. Further, by Lemma 6.2.93 (p.903), the conclusion of the proposition holds for $\mathfrak{N}$ when eq is replaced by eq^S in the conclusion of the proposition. Thus, by Claim 6.2.94, we have the same for $\mathfrak{M}$; formally:

Claim **6.2.95** If $o, e \in \ell \in L_{\mathfrak{M}}$ with $o \neq e$ then

$$[\, a, a_1 \in \vec{\ell}_{o,e} \ \wedge\ \langle o, a\rangle\, eq^S \langle o, a_1\rangle\,] \quad \Rightarrow \quad a = a_1.$$

To prove the proposition assume $\ell \in L$, $o, e \in \ell$ with $o \neq e$. Let $a, a_1 \in \vec{\ell}_{o,e}$ be such that

$$\langle o, a\rangle\ eq\ \langle o, a_1\rangle.$$

We want to prove $a = a_1$. We distinguish three cases.

<u>*Case 1:*</u> Assume $\ell \in L^S$. Then $o \equiv^S a$ and $o \equiv^S a_1$. So, by Lemma 6.2.93, we have $\langle o, a\rangle\ eq^S\ \langle o, a_1\rangle$. Thus, by Claim 6.2.95, $a = a_1$.

<u>*Case 2:*</u> Assume $\ell \in L^{Ph}$. Then, by the definition of eq, it can be checked that $o = a = a_1$, since there is no observer who sees a photon-like line on a coordinate axis.

<u>*Case 3:*</u> Assume $\ell \in L^T$. Let m be an observer who sees ℓ on $\bar{t}$-axis, i.e. $w_m[\bar{t}\,] = \ell$. Let

$$\ell' :\stackrel{\text{def}}{=} w_m[\bar{x}] \in L^S$$

and $c, u \in \ell'$ with $c \neq u$. Let $d, d_1 \in \vec{\ell'}_{c,u}$ be such that m is a witness to both

$$\langle o, a\rangle\ eq_0\ \langle c, d\rangle \quad \text{and} \quad \langle o, a_1\rangle\ eq_0\ \langle c, d_1\rangle,$$

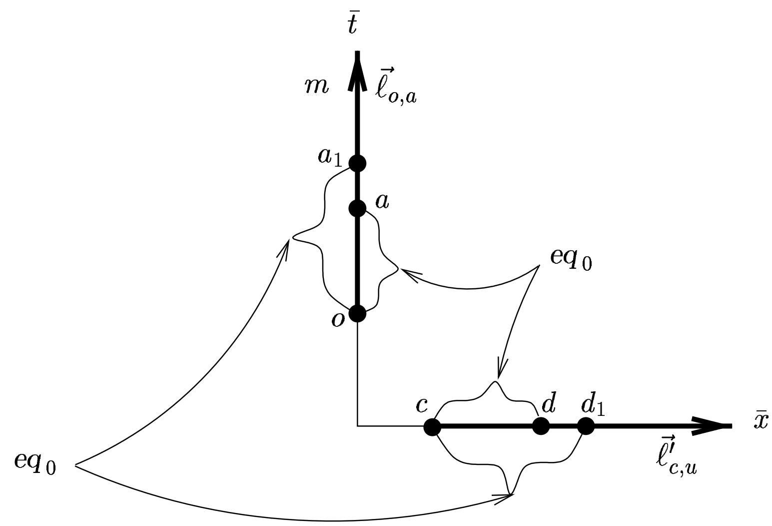

Figure 300: Illustration for the proof of Proposition 6.2.92.

see Figure 300. Then $a = a_1$ iff $d = d_1$, cf. Fact 6.2.85. Further, by $\langle o, a\rangle\ eq\ \langle o, a_1\rangle$, we have $\langle c, d\rangle\ eq\ \langle c, d_1\rangle$. By Case 1, we have $d = d_1$. Thus $a = a_1$. This completes the proof. Therefore
<u>*Proposition 6.2.92 has been proved.*</u> ∎

Proof of Theorems 6.2.22 and 6.2.23: The proof of Thm.6.2.22 for the case $n = 2$ is left to the reader, but we note the following. Assuming **Basax**(2), eq has the following property: Assume $\langle a, b\rangle\ eq\ \langle c, d\rangle$. Then there is an observer who sees $\langle a, b\rangle$ on a coordinate axis, $\langle c, d\rangle$ parallel to a coordinate axis and that the distances between a, b and between c, d coincide. Actually, the first eq_0-witness in the eq_0-chain establishing $\langle a, b\rangle\ eq\ \langle c, d\rangle$ will be such an observer, cf. Def.6.2.91 and the intuitive text below it. (Therefore, assuming **Basax**(2), each observer sees the pairs $\langle a, b\rangle$ and $\langle c, d\rangle$ on parallel lines or on Minkowski-orthogonal lines.)

Assume $n > 2$ and that

$$\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) \quad \text{or}$$
$$\mathfrak{M} \models \mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\|)^- + \mathbf{Ax}(\sqrt{\ }).$$

To prove the theorems we have to prove $eq = eq_2$ (because this is what the conclusion of the theorems say). $eq_2 \subseteq eq$ is obvious. Thus, we have to prove $eq \subseteq eq_2$. Let $a, b, c, d \in Mn$ be such that

$$\langle a, b\rangle\ eq\ \langle c, d\rangle.$$

Then, there are $\ell, \ell' \in L^T \cup L^S$ such that $\ell \sim \ell'$, $a, b \in \ell'$ and $c, d \in \ell$ (this follows from the definition of eq and from item 6.2.13). Let $u := d$ if $c \neq d$, otherwise let $u \in \ell$ with $u \neq c$ be arbitrary. Clearly, $d \in \vec{\ell}_{c,u}$. By Proposition 6.2.88, there is $d_1 \in \vec{\ell}_{c,u}$ such that

$$\langle a, b\rangle \; eq_2 \; \langle c, d_1\rangle,$$

see Figure 294 (p.895). Thus, it is sufficient to prove $d = d_1$. Since eq is transitive and reflexive (and since $eq_2 \subseteq eq$) we have

$$\langle c, d\rangle \; eq \; \langle c, d_1\rangle.$$

This and $d, d_1 \in \vec{\ell}_{c,u}$ imply $d = d_1$ by Propositions 6.2.83 and 6.2.92, see Figure 298 (p.901). This completes the proof of Thm.'s 6.2.22 and 6.2.23. Therefore
Theorems 6.2.22 and 6.2.23 have been proved. ∎

In connection with the following proposition cf. Thm.6.2.24 (p.830).

PROPOSITION 6.2.96

(i) *Proposition 6.2.83 (p.892) does not generalize from* **Basax** *to* $\mathbf{Bax}^{\oplus}$*, i.e. for any $n > 1$ there is a model $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\surd))$ in which the conclusion of Proposition 6.2.83 fails, i.e. the arrangement in Figure 291 (p.892) becomes possible.*

(ii) *Proposition 6.2.92 (p.901) does not generalize to $n = 2$, i.e. there is a model*

$$\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|)^{-} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\surd))$$

such that the conclusion of Proposition 6.2.92 fails in $\mathfrak{M}$, i.e. the arrangement in Figure 298 (p.901) becomes possible.

Idea of proof:

Case of (i): Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\surd))$ be with $m, k \in Obs$ such that m and k are brothers, m thinks that the speed of light is 1 while k thinks that the speed of light is 2^2 (i.e. $c_m = 1$ and $c_k = 2^2$), and m and k agree on coordinate axes (i.e. $\mathsf{f}_{mk}[\bar{x}_i] = \bar{x}_i$ for all $i \in n$). Now it is not hard to find events o, a, a_1 in $\mathfrak{G}_{\mathfrak{M}}$ as represented in Figure 291; actually one can choose these three events to be $w_m(\bar{0})$, $w_m(1_x)$ and $w_m(2 \cdot 1_x)$.

Case of (ii): Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\surd))$ be such that there are $m, k \in Obs$ with the following properties: k moves FTL relative to m, moreover $\mathsf{f}_{mk}[\bar{x}] = \bar{t}$ and $\mathsf{f}_{mk}[\bar{t}] = \bar{x}$. Further $c_m = 1$ while $c_k = 2^2$. Now it is not hard to check that in this model $\mathfrak{M}$ there are events o, a, a_1 validating the arrangement in Figure 298. Hint: both choices $\ell = w_m[\bar{x}]$ and $\ell = w_m[\bar{t}]$ work. ∎

6.2.7 Connections between Ax(eqm) and the rest of our axioms

In this sub-section we discuss some connections between the axiom **Ax(eqm)** of equi-measure introduced in §6.2.1 and some earlier introduced axioms. For simplicity, throughout the present sub-section we assume **Ax2**. This does not restrict generality since in all our theories introduced so far **Ax2** was assumed.[857] We consider **Ax(eqm)** as a stronger version of each one of the following axioms: **Ax(eqtime)**, **Ax(eqspace)**, **Ax(||)** (cf. §2.8 for these axioms). Namely, intuitively,

Ax(eqm) says that observers *agree on distances*,

Ax(eqtime) says that observers with common life-line *agree on time-like distances*,

Ax(eqspace) says that observers *agree on spatial distances*, and

Ax(||) says that observers with parallel life-lines *agree* (on time-like and space-like separatedness and) *on distances.*

Ax(eqm) is equivalent with **Ax(eqtime)+Ax(eqspace)**, under mild assumptions, cf. item 2b of Proposition 6.2.97 below. Interestingly, **Ax(eqm)** is equivalent with the weaker **Ax(eqspace)**, assuming $\mathbf{Bax}^{\oplus}$ and some auxiliary axioms, cf. items 5b, 5a of Prop.6.2.97. Further, assuming $\mathbf{Flxbasax}^{\oplus}(2)$ and some auxiliary axioms, **Ax(eqm)** turns out to be equivalent with both **Ax(||)** and **Ax(eqtime)**, cf. items 3b, 4a of Prop.6.2.97. Some of the connections between **Ax(eqm)**, **Ax(eqtime)**, **Ax(eqspace)** and **Ax(||)** are summarized in the following proposition. Recall that

$$\mathbf{Basax} \models \mathbf{Newbasax} \models \mathbf{Flxbasax}^{\oplus} \models \mathbf{Bax}^{\oplus} \models \mathbf{Bax} \models \mathbf{Bax}^{-}.$$

Below, the boxed notation $\boxed{\mathbf{Ax(eqm)}}$ means simply **Ax(eqm)**, the role of the box is only to call attention to the place where **Ax(eqm)** appears.

[857]For completeness we note that later we will have important theories in which **Ax2** will not be assumed. These theories will pave the road to generalizations toward general relativity.

PROPOSITION 6.2.97 *1–5 below hold.*

1. *(a)–(d) below hold.*

   (a) $\mathbf{Bax}^- \models \boxed{\mathbf{Ax(eqm)}} \to \mathbf{Ax(eqtime)}$.

   (b) $\mathbf{Bax}^- + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }) \models \boxed{\mathbf{Ax(eqm)}} \to \mathbf{Ax(eqspace)}$.

   (c) $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }) \models \boxed{\mathbf{Ax(eqm)}} \to \mathbf{Ax}(\|)$.

   (d) $\mathbf{Ax}(Triv)$ *cannot be omitted neither in item (b) nor in item (c) above.*

2. *Assume*
   $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) +$ (*there are no observers moving with infinite speed*)[858].
   Then (a)–(c) below hold.

   (a) $(\mathbf{Ax(eqtime)} + \mathbf{Ax(eqspace)}) \to \boxed{\mathbf{Ax(eqm)}}$.

   (b) *Assume* $\mathbf{Ax}(Triv)$. *Then*
   $(\mathbf{Ax(eqtime)} + \mathbf{Ax(eqspace)}) \leftrightarrow \boxed{\mathbf{Ax(eqm)}}$.

   (c) *The assumption* $\mathbf{Ax}(Triv)$ *cannot be omitted in item (b) above.*

3. *Assume* $\mathbf{Bax}^{\oplus}(2)$ + (*there are no observers moving with infinite speed*)[858].
   Then (a)–(c) below hold.

   (a) $\mathbf{Ax}(\|) \to \boxed{\mathbf{Ax(eqm)}}$.

   (b) *Assume* $\mathbf{Ax}(Triv)$. *Then* $\mathbf{Ax}(\|) \leftrightarrow \boxed{\mathbf{Ax(eqm)}}$.

   (c) *The assumption* $\mathbf{Ax}(Triv)$ *cannot be omitted in item (b) above.*

4. *(a)–(c) below hold.*

   (a) *Assume* $\mathbf{Flxbasax}^{\oplus}(2) + \mathbf{Ax}(Triv) +$ (*there are no observers moving with infinite speed*)[858]. *Then* $\mathbf{Ax(eqtime)} \leftrightarrow \boxed{\mathbf{Ax(eqm)}}$.

   (b) $\mathbf{Ax}(Triv)$ *cannot be omitted in item (a) above.*

   (c) *The statement in item (a) above does not generalize to* $n > 2$. *<u>Moreover:</u>* *For any* $n > 2$ *there is a model* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(n) + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\|) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }))$ *such that* $\mathfrak{M} \not\models \boxed{\mathbf{Ax(eqm)}}$.

---

[858]Formally: $(\forall m, k \in Obs)(\, m \overset{\odot}{\to} k \ \to\ v_m(k) \neq \infty\,)$.

5. *Assume $n > 2$ or that (there are no observers moving with infinite speed)*[858]. *Then (a) and (b) below hold.*

   (a) $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) \models \boxed{\mathbf{Ax(eqm)}} \leftrightarrow \mathbf{Ax(eqspace)}$.

   (b) $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\|)^- \models$
   $\boxed{\mathbf{Ax(eqm)}} \leftrightarrow \mathbf{Ax(eqspace)}$.

We omit the **proof**. ∎

In connection with Prop.6.2.97 above we note that, assuming $\mathbf{Bax}^{\oplus}(2)$, (there are no observers moving with infinite speed) ↔ (there are no FTL observers).

The following theorem is a stronger version of Theorems 2.8.17 (p.138), 3.9.11 (p.356) saying that, under certain conditions, the symmetry axioms in models of **Basax** are equivalent. Among others, the following theorem says that the axiom of equi-measure is equivalent with the symmetry axioms introduced in §§ 2.8, 3.9, assuming $n > 2$, $\mathbf{Flxbasax}^{\oplus}$ and some auxiliary axioms. Further it shows that in Thm.'s 2.8.17, 3.9.11 the auxiliary axiom $\mathbf{Ax}(\mathit{Triv}_t)$ can be replaced by its weaker version $\mathbf{Ax}(\mathit{Triv}_t)^-$.

THEOREM 6.2.98 *Assume $n > 2$. Let*

$$H \stackrel{\text{def}}{:=} \{\mathbf{Ax(syt_0)},\ \mathbf{Ax(speedtime)},\ \mathbf{Ax(eqspace)},\ \mathbf{Ax\triangle 2}\},$$

$$H_1 \stackrel{\text{def}}{:=} \{\boxed{\mathbf{Ax(eqm)}},\ \mathbf{Ax(symm)},\ \mathbf{Ax\triangle 1} + \mathbf{Ax(eqtime)},\ \mathbf{Ax\Box 2}\},$$

$$H_2 \stackrel{\text{def}}{:=} \{\mathbf{Ax\Box 1} + \mathbf{Ax(eqtime)}\}.$$

Then (i)–(vi) below hold.

(i) $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }) \models$
"all the axioms in H are equivalent with one another".

(ii) $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)^- \models$
"all the axioms in $H \cup H_1$ are equivalent with one another".

(iii) $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\uparrow\uparrow_0) \models$
"all the axioms in $H \cup H_1 \cup H_2$ are equivalent with one another".

(iv) $\mathbf{Flxbasax}^{\oplus}(2) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\uparrow\uparrow_0) \models$ *"all the axioms in $(H \cup H_1 \cup H_2) \setminus \{\mathbf{Ax(eqm)}, \mathbf{Ax(eqspace)}\}$ are equivalent with one another".*

(v) $\mathbf{Flxbasax}^{\oplus}(2) + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t)^{-} +$ (*there are no observers moving with infinite speed*) $\models$ *"all the axioms in* $(H \cup H_1) \setminus \{\mathbf{Ax(eqm)}, \mathbf{Ax(eqspace)}\}$ *are equivalent with one another".*

(vi) $\mathbf{Flxbasax}^{\oplus}(2) + \mathbf{Ax}(\sqrt{\ }) +$ (*there are no observers moving with infinite speed*) $\models$ *"all the axioms in* $H \setminus \{\mathbf{Ax(eqspace)}\}$ *are equivalent with one another".*

On the proof: A proof can be obtained by Theorems 2.8.17 (p.138), 3.9.11 (p.356), by item 5a of Prop.6.2.97, by footnote 354 (p.432), by noticing that each axiom in $H \cup H_1 \cup H_2$ implies **Ax(eqtime)**, and by noticing that $\mathbf{Ax(eqtime)} + \mathbf{Ax}(\mathit{Triv}_t)^{-}$ implies $\mathbf{Ax}(\mathit{Triv}_t)$ in models of $\mathbf{Flxbasax}^{\oplus}$. Cf. also [174]. ∎

As a contrast to Theorem 6.2.98 above we state Proposition 6.2.99 below. It says that

$$\mathbf{Basax}(2) + \boxed{\mathbf{Ax(eqm)}} + \text{“auxiliary axioms”}$$

does not imply any of the symmetry axioms, except for the axiom **Ax(eqspace)** of equi-space. In models of $\mathbf{Flxbasax}^{\oplus}$, for $n = 2$ too, **Ax(eqm)** and **Ax(eqspace)** are equivalent, assuming some auxiliary axioms, cf. item 5a of Prop.6.2.97.

PROPOSITION 6.2.99 *There is a model*

$$\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax}(2) + \boxed{\mathbf{Ax(eqm)}} + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ }))$$

such that in $\mathfrak{M}$ *neither one of the following symmetry axioms* $\mathbf{Ax(syt_0)}$*,* **Ax(speedtime)***,* **Ax(symm)***,* **Ax△1***,* **Ax△2***,* **Ax□1***,* **Ax□2** *holds.*

The **proof** is available from Judit Madarász. ∎

Roughly, the following theorem says that each one of the symmetry axioms implies **Ax(eqm)**, assuming $\mathbf{Flxbasax}^{\oplus}$.

THEOREM 6.2.100 *(i) and (ii) below hold.*

(i) $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Ax}(\omega)^{00} \rightarrow \boxed{\mathbf{Ax(eqm)}}$.

(ii) *The statement in (i) remains true if we replace* $\mathbf{Ax}(\omega)^{00}$ *with any one of* $\mathbf{Ax}(\omega)$*,* $\mathbf{Ax}(\omega)^{0}$*,* $\mathbf{Ax}(\omega)^{\sharp}$*,* $\mathbf{Ax}(\omega)^{\sharp\sharp}$*,* $\mathbf{Ax(syt_0)}$*,* **Ax(symm)***,* **Ax(speedtime)***,* **Ax△1** + **Ax(eqtime)***,* **Ax△2***,* **Ax□1** + **Ax(eqtime)***,* **Ax□2***,* **Ax(eqspace)***.*

The **proof** is available from Judit Madarász. ∎

By Thm.6.2.98, in models of $\mathbf{Flxbasax}^{\oplus}$ **Ax(eqm)** and **Ax(symm)** are equivalent, assuming $n > 2$ and some auxiliary axioms. In Thm.4.2.4 (p.458) and Prop.3.9.37 (p.386) we have seen that, assuming some auxiliary axioms, **Ax(symm)** blurs the distinction between **Bax** and **Flxbasax**. The next proposition says that this is not the case if we assume **Ax(eqm)** in place of **Ax(symm)**.

PROPOSITION 6.2.101 *For any* $n > 1$

$$\mathbf{Bax}^{\oplus} + \boxed{\mathbf{Ax(eqm)}} + \mathbf{Ax(\|)} + \mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax(}\sqrt{\ }\mathbf{)} + \mathbf{Ax6} \not\models \mathbf{Flxbasax}.$$

Let $\mathbf{Ax}_0 := \mathbf{Bax}^{\oplus} + \ldots + \mathbf{Ax6}$ which is the set of axioms on the left-hand side of $\not\models$ in Prop.6.2.101 (i.e. Prop.6.2.101 says that $\mathbf{Ax}_0 \not\models \mathbf{Flxbasax}$). $\mathbf{Ax}_0$ will play a distinguished role in our duality theories, moreover it already played a role in our earlier propositions about *eq* (cf. Prop.6.2.88 on p.895, Prop.6.2.92 on p.901, and Thm.6.2.23 on p.829). The observation that $\mathbf{Ax}_0$ is weak (e.g. $\mathbf{Ax}_0 \not\models \mathbf{Flxbasax}$) makes our theorems based on $\mathbf{Ax}_0$ (e.g. the just mentioned duality theories) stronger. In this direction we note that $\mathbf{Ax}_0$ is consistent with the existence of photons with arbitrary finite positive speeds. As a contrast $\mathbf{Ax}_0$ is strong enough to imply that "simultaneities" i.e. space-like hyper-planes[859] are Euclidean. In more detail, assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Ax}_0)$ and H is a space-like hyper-plane of $\mathfrak{G}$. Then the "subgeometry" $\mathfrak{G} \restriction H = \langle H, \mathbf{F_1}, \ldots \rangle$ [860] is Euclidean by Thm.6.6.115 (p.1131).[861]

[859] For the definition of space-like hyper-planes we refer to p.1130, Def.6.6.112.

[860] where $L, Bw, \ldots, \mathcal{T}$ are restricted to H the natural way

[861] In passing we note that general relativistic geometries usually fail to have this property (i.e. "pure" space is already curved). However, the simplified black hole geometry in Andréka et al. [23] enjoys this property at least for some (kinds of) space-like hyper-planes (more precisely space-like "geodesic hyper-surfaces"). Besides being Euclidean, these space-like hyper-surfaces are disjoint from each other, and their union covers the whole of space-time. The same applies to the simplified black hole geometry in Rindler [224] on p.124 given by equation (7.28). (The two simplified geometries, in [23] and in [224], are obtained via different trains of thought.) As a curiosity we note that one of the main features of the model constructed in Gödel's cosmological papers (and refined in Ozsváth-Schücking [209] for a finite universe) is that the whole of space-time of that model cannot be obtained as a disjoint union of space-like geodesic hyper-surfaces. Such a disjoint union of space-like geodesic hyper-surfaces could be regarded as a kind of "absolute" (even if artificial) temporal structure for the whole universe. In passing, universes with rotating black holes have the same "Gödelian" property. (When we write "universe" e.g. in connection with the works of Gödel, Ozsváth etc. we mean a mathematical structure which is in many respects similar to our $\mathfrak{G}$'s but in which geodesics discussed in §6.8 way below play a dominant role.) Cf. footnote 622 on p.775 for references etc. See Figure 355 on p.1208 for an intuitive picture of Gödel's (cosmological) model.

QUESTION 6.2.102 *In Thm.4.7.1 (p.606) we have seen that, assuming some auxiliary axioms,* **Ax(symm)** *blurs the distinction between* **Reich**(**Basax**) *and* **Basax**. *Is this true for* **Ax(eqm)** *in place of* **Ax(symm)**? *I.e. does*

$$\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \boxed{\mathbf{Ax(eqm)}} \models \mathbf{Basax}$$

hold?

◁

Further connections between the symmetry axioms and **Ax(eqm)** were discussed in §3.9.3 (in section Symmetry axioms).

Remark 6.2.103

1. In §3.9.3, we used the axiom **Ax(Gal)**, which is almost equivalent with **Ax(*Triv*)**. In the present chapter we will use only **Ax(*Triv*)**. All the theorems stated in §3.9.3 remain true if we replace **Ax(Gal)** with **Ax(*Triv*)**. We will use this fact without mentioning it.

2. **Ax(eqm)** is formulated in the present section slightly differently than its formulation in §3.9.3. All the theorems in §3.9.3 remain true if we use the form of **Ax(eqm)** stated in the present section. ◁

6.2.8 Characterizing our symmetry axioms by the automorphisms of the geometry $\mathfrak{G}_{\mathfrak{M}}$

In this sub-section we turn to characterizing our symmetry axioms (like $\mathbf{Ax}(\omega)$) in model theoretic or algebraic terms. More concretely, we will prove that, basically, our symmetry axioms are equivalent with the statement that

$$Aut(\mathfrak{G}_{\mathfrak{M}}) \;=\; \text{“World-view transformations of } \mathfrak{M}\text{”},$$ [862]

i.e. that the automorphisms of our observer-independent geometry coincide[863] with the world-view transformations between observers. We will prove this statement

[862] More precisely, we mean some kind of duals $\widehat{\mathsf{f}_{mk}}$ of the world-view transformations f_{mk} which will be defined soon. The point is that without loss of generality, one may identify the f_{mk}'s with their duals.

[863] in some sense

under mild assumptions[864] on $\mathfrak{M}$, cf. Thm.6.2.106 (p.916). Cf. also the introduction to the present chapter, p.778.[865]

We will also see that in our characterization (symmetry axiom $\Leftrightarrow$ $Aut(\mathfrak{G})$ = "World-view trf.'s") of the symmetry axioms, on the right-hand side both inclusions ($Aut(\mathfrak{G}) \subseteq$ "W..." and $Aut(\mathfrak{G}) \supseteq$ "W...") can fail if we make the assumptions weaker, cf. Theorems 6.2.109, 6.2.110, 6.2.111 (pp. 919–921).

Before elaborating the details, we note that the ideas in this sub-section, i.e. the idea of characterizing (instances of) Einstein's SPR (when applied to $\mathfrak{M}$) by looking at automorphisms of $\mathfrak{G}_{\mathfrak{M}}$, is not unrelated to the subject matter of our Remark 6.6.4 (p.1014, §6.6.1) on Galois theories.[866] Here Lemma 6.7.5 (p.1139) is also relevant.

Notation 6.2.104 Assume $\mathfrak{M}$ is a frame model. Assume that the world-view functions w_m are bijections, i.e. that for each $m \in Obs$, $w_m : {}^nF \rightarrowtail\!\!\twoheadrightarrow Mn$.

(i) Let $m, k \in Obs$. Intuitively $\widehat{\mathsf{f}_{mk}}$ will be the function on the observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$ induced by the world-view transformation f_{mk} the natural way, see Fig.301. For the formal definition of $\widehat{\mathsf{f}_{mk}}$ let us notice that

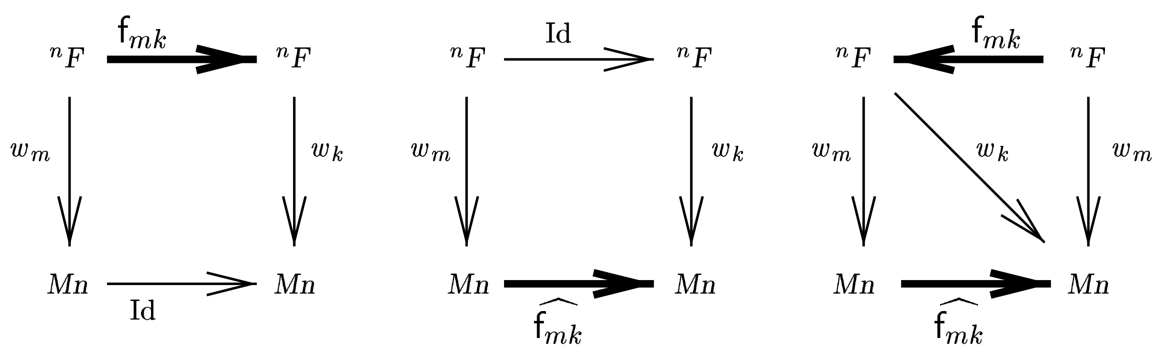

Figure 301: The diagrams above commute. (Illustration for $\widehat{\mathsf{f}_{mk}}$ and f_{mk}.)

[864]These assumptions are very mild in comparison with the assumptions needed in §3.9 for similar purposes (cf. **BaCo**$^-$ on p.347 and Thm.3.9.2).

[865]The goals of this sub-section were discussed in greater detail there, further they were put into broader perspective.

[866]The two subjects together are related to what is called symmetry breaking (and/or invariance principles or equivalence principles) in natural sciences, in particular in physics, cf. e.g. Gruber-Millman [114], Greene [113, pp. 122–123], or Darvas [67].

$w_m^{-1} \circ w_k : Mn \rightarrowtail\!\!\twoheadrightarrow Mn$ is a bijection. The function

$\widetilde{w_m^{-1} \circ w_k} : L \longrightarrow \mathcal{P}(Mn)$ is defined by $\widetilde{w_m^{-1} \circ w_k} : \ell \mapsto (w_m^{-1} \circ w_k)[\ell]$. Now,

$$\widehat{\mathsf{f}_{mk}} \stackrel{\text{def}}{:=} \langle\, w_m^{-1} \circ w_k,\ \mathrm{Id} \restriction F,\ \widetilde{w_m^{-1} \circ w_k} \,\rangle.$$

$\widehat{\mathsf{f}_{mk}}$ is a potential (three-sorted) automorphism[867] of $\mathfrak{G}_{\mathfrak{M}}$.

In some sense, $\widehat{\mathsf{f}_{mk}}$ contains the same "mathematical information" as f_{mk} does, cf. Fig.301. Therefore one can identify f_{mk} with $\widehat{\mathsf{f}_{mk}}$. In the present subsection, by <u>*world-view transformations*</u> we will always mean the $\widehat{\mathsf{f}_{mk}}$'s instead of the f_{mk}'s.

(ii) Let $f \in Aut(\mathfrak{G}_{\mathfrak{M}})$. Then, f is called <u>*nice*</u> if it is the identity function on the sort F.[868] Hence, the $\widehat{\mathsf{f}_{mk}}$'s are potential nice automorphisms.

(iii) $Aut^F(\mathfrak{G}_{\mathfrak{M}})$ denotes the set of nice automorphisms of $\mathfrak{G}_{\mathfrak{M}}$.

◁

Assume $\mathfrak{N}$ is a frame model. Recall that for every $m \in Obs^{\mathfrak{N}}$, $\mathfrak{G}_m$ is the observer-dependent geometry corresponding to m, cf. Def.6.2.76, p.880. Intuitively, $\mathfrak{G}_m$ is the world-view of observer m, if we abstract from bodies which are not photons or observers. Roughly, the following proposition says that $\widehat{\mathsf{f}_{mk}}$ is a nice automorphism (of $\mathfrak{G}_{\mathfrak{N}}$) iff the world-views of m and k coincide.

PROPOSITION 6.2.105 *Assume $\mathfrak{N}$ is a frame model such that for every $m \in Obs^{\mathfrak{N}}$, $w_m : {}^nF \rightarrowtail\!\!\twoheadrightarrow Mn$ is a bijection. Then, for every $m, k \in Obs$,*

$$\widehat{\mathsf{f}_{mk}} \in Aut^F(\mathfrak{G}_{\mathfrak{N}}) \qquad \Longleftrightarrow \qquad \mathfrak{G}_m = \mathfrak{G}_k.$$

Idea of proof: Assume the assumptions. Then for every $m \in Obs$, w_m induces an isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}}$ the natural way, cf. footnote 834, p.885 for details. The rest of the proof is depicted in Fig.302. ∎

Note that (i), (iv), (v) in the theorem below are statements of similar kind. The main message of the theorem below is that any one of (i), (iv), (v) is equivalent with (iii). I.e. the main message is the following:

symmetry axiom $\Leftrightarrow$ $Aut^F(\mathfrak{G}_{\mathfrak{M}}) =$ "World-view transformations of $\mathfrak{M}$".

[867]Cf. item (II) of Def.6.2.2 (p.798) for the notion of an isomorphism between geometries.
[868]I.e. f is nice if f leaves the elements of F fixed.

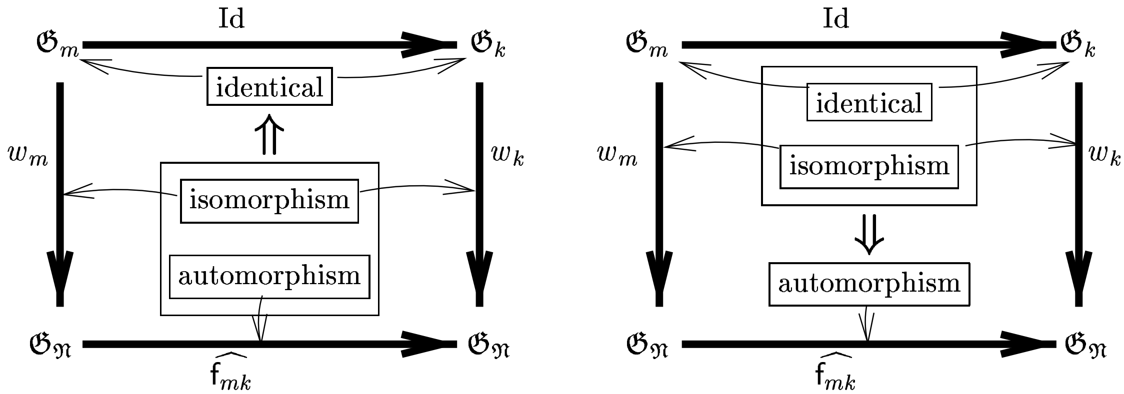

Figure 302: $\widehat{\mathsf{f}_{mk}} \in Aut^F(\mathfrak{G}_{\mathfrak{N}}) \quad \Longleftrightarrow \quad \mathfrak{G}_m = \mathfrak{G}_k.$

THEOREM 6.2.106 *Assume $n > 2$ and*

$$\mathfrak{M} \models \mathbf{Bax}^{-\oplus} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax(}\|\mathbf{)}.$$

Then (i)–(v) below are equivalent.

(i) $\mathfrak{M} \models \mathbf{Ax\square 1}$.

(ii) $(\forall m, k \in Obs)\, \widehat{\mathsf{f}_{mk}} \in Aut^F(\mathfrak{G}_{\mathfrak{M}})$.

(iii) $Aut^F(\mathfrak{G}_{\mathfrak{M}}) = \{\, \widehat{\mathsf{f}_{mk}} \ : \ m, k \in Obs \,\}$.

(iv) $\mathfrak{M} \models \mathbf{Ax(}\omega\mathbf{)}$.

(v) $\mathfrak{M} \models \mathbf{Flxspecrel}^{\oplus} + \mathbf{Ax\square 1}$.

Outline of proof: Assume $n > 2$, and that $\mathfrak{M}$ satisfies the assumptions. Then (i) $\Rightarrow$ (v), by Thm.5.2.17 on p.759 (or equivalently by Thm.4.3.18 on p.490). (v) $\Rightarrow$ (iv) by Thm.6.2.98 on p.910 (and by **Flxspecrel** $\models$ **Ax(symm)**). (iv) $\Rightarrow$ (i) by $\mathbf{Ax(}\omega\mathbf{)} := \mathbf{Ax\square 1} + \ldots$, cf. p.351. Therefore, (i), (iv), and (v) are equivalent.

(v) $\Rightarrow$ (ii) is not hard to check by using Prop.6.2.105 (and by **Ax□1** and the fact that the speed of light is the same for all observers). Cf. also the intuitive text above Prop.6.2.105. The details of this part of the proof are left to the reader.

Now, we turn to proving (ii) $\Rightarrow$ (i). Assume (ii). First, we prove that $\mathfrak{M} \models$ **Flxbasax**. To prove this it is enough to prove that $(\forall m, k \in Obs)(\forall d, d' \in$

directions) $\mathsf{c}_m(d) = \mathsf{c}_k(d')$. Let $m, k \in Obs$ and $d, d' \in$ directions. Let $k' \in Obs$ be a brother of k such that $\mathsf{f}_{kk'} \in Triv$ and $\mathsf{f}_{kk'}$ takes d to d'.[869] Then,

$$\mathsf{c}_k(d') = \mathsf{c}_{k'}(d).$$

By Prop.6.2.105 and (ii), $\mathfrak{G}_m = \mathfrak{G}_{k'}$. Hence,

$$\mathsf{c}_m(d) = \mathsf{c}_{k'}(d).$$

But then, $\mathsf{c}_m(d) = \mathsf{c}_k(d')$. Hence, $\mathfrak{M} \models$ **Flxbasax**. Next, we turn to proving $\mathfrak{M} \models \mathbf{Ax\Box 1}$. (ii) is still assumed. Let $m, k, m' \in Obs$. We want to prove that $\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$, for some k'. By Prop.6.2.105,

$$\mathfrak{G}_m = \mathfrak{G}_{m'}.$$

Let $h \in Obs$ be such that

$$tr_m(k) = tr_{m'}(h) \qquad \text{and} \qquad m \uparrow k \;\Leftrightarrow\; m' \uparrow h,$$

cf. Fig.303. Such a h exists by $\mathfrak{G}_m = \mathfrak{G}_{m'}$. Without loss of generality, by $\mathbf{Ax(}Triv\mathbf{)}$, we may assume that

$$\mathsf{f}_{km}(\bar{0}) = \mathsf{f}_{hm'}(\bar{0}),$$

see Fig.303.

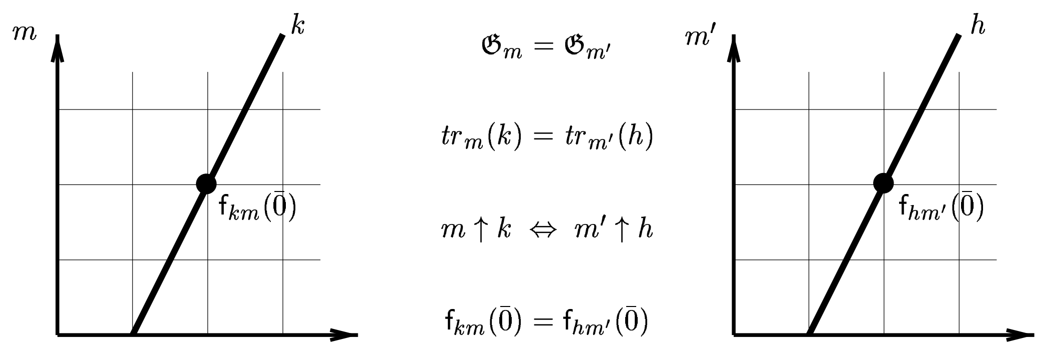

Figure 303: $h \in Obs$ is as depicted above.

[869]Such a k' exists by $\mathbf{Ax(}Triv\mathbf{)}$.

By Thm.4.3.24 (p.497), there are no FTL observers in $\mathfrak{M}$. Hence, by $\mathbf{Ax(\|)}+\mathbf{AxE_{01}}$,

$$(*)\qquad (\forall e, e_1 \in Mn)\,[\, e \equiv^T e_1 \quad \Rightarrow \quad (\text{the time elapsed between } e, e_1 \text{ is } g(e,e_1) \text{ for any observer who sees } e, e_1 \text{ on his life-line})\,].$$

By (∗) and the statements listed in the middle of Fig.303,

$$\mathsf{f}_{km} \restriction \bar{t} = \mathsf{f}_{hm'} \restriction \bar{t}.$$

Thus, $\mathsf{f}_{hm'} \circ \mathsf{f}_{mk}$ leaves the time-axis $\bar{t}$ pointwise fixed and leaves the set of "photon-lines"[870] fixed.[871] Hence, by the proof of Lemma 3.6.20 (p.275), $\mathsf{f}_{hm'} \circ \mathsf{f}_{mk}$ is a trivial transformation. Thus, $\mathsf{f}_{mk} = \mathsf{f}_{m'h} \circ f$, for some $f \in Triv$. Let this f be fixed. Let k' be a brother of h such that $\mathsf{f}_{hk'} = f$.[872] But then $\mathsf{f}_{mk} = \mathsf{f}_{m'h} \circ \mathsf{f}_{hk'} = \mathsf{f}_{m'k'}$, for this k'. So, $\mathfrak{M} \models \mathbf{Ax\Box 1}$, and this completes the proof of (ii) ⇒ (i).

So far we have seen that (i), (ii), (iv), (v) are equivalent. It remains to prove that (iii) is equivalent with these.

(iii) ⇒ (ii) is vacuously true. Now, to prove the theorem it is enough to prove that

$$(\text{v}) \qquad \Longrightarrow \qquad Aut^F(\mathfrak{G}_{\mathfrak{M}}) \subseteq \{\, \widehat{\mathsf{f}_{mk}} \;:\; m, k \in Obs \,\}.$$

Now, we turn to proving this. Assume (v). Then the f_{mk}'s are affine transformations by Prop.3.9.50 (p.392) and by the fact that $\mathbf{Bax}^- \models \mathbf{Ax(symm)} \to \mathbf{Ax(syt_0)}$. Let $f \in Aut^F(\mathfrak{G}_{\mathfrak{M}})$. We have to prove that $f = \widehat{\mathsf{f}_{mk}}$, for some $m, k \in Obs$. Let $m \in Obs$ be arbitrary but fixed. For simplicity assume that $n = 4$. Let $\ell_0, \ell_1, \ell_2, \ell_3 \in L$ be the coordinate axes of m in the observer-independent geometry, i.e. $\ell_i := w_m[\bar{x}_i]$, for $i < 4$. Let $o, e_0, e_1, e_2, e_3 \in Mn$ be, respectively, $w_m(\bar{0})$, $w_m(1_0)$, $w_m(1_1)$, $w_m(1_2)$, $w_m(1_3)$. Hence, by (∗) above and by the definition of $\mathfrak{G}_{\mathfrak{M}}$,

$\ell_0 \in L^T$, $\ell_1, \ell_2, \ell_3 \in L^S$, the ℓ_i's are pairwise $\perp_r$-orthogonal, $\langle o, e_0\rangle\; eq\; \langle o, e_i\rangle$ for all $i < 4$, $g(o, e_0) = 1$, and $o \prec e_0$.

The above statement holds when ℓ_i, o, e_i $(i < 4)$ are replaced, respectively, by $f(\ell_i)$, $f(o)$, $f(e_i)$ $(i < 4)$ in it. By this, Remark 6.2.66(ii) (p.867), (the proofs of) Propositions 6.2.88 (p.895) and 6.2.92 (p.901), (∗) above, and $\mathbf{Ax(}Triv\mathbf{)}$ one can prove that there is an observer k such that the coordinate axes of k are $f(\ell_i)$ $(i < 4)$ in the observer independent geometry, and $f(o) = w_k(\bar{0})$, $f(e_i) = w_k(1_i)$, for $i < 4$. Recall that the world-view tarnsformations are affine. Now, it is not hard to check that $\widehat{\mathsf{f}_{mk}} = f$ (the details are left to the reader). This completes the proof. ∎

---

[870]Here, "photon-lines":= $\{\, \ell \in \mathsf{Eucl} \;:\; ang^2(\ell) = c \,\}$, where c is the (square of the) speed of light in $\mathfrak{M}$. (Recall that $\mathfrak{M} \models$ **Flxbasax**.)

[871]The latter is so by **Flxbasax**.

[872]Such a k' exists by $\mathbf{Ax(}Triv\mathbf{)}$.

Conjecture 6.2.107 *We strongly conjecture that Thm.6.2.106 above remains true if we replace the assumption* $\mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv})$ *with* $\mathbf{Ax(eqtime)} + \mathbf{Ax}(\mathit{Triv}_t)$.

$\triangleleft$

Thm.6.2.108 below is a version of Thm.6.2.106 above. The main message of Thm.6.2.108 is that

<u>any</u> symmetry axiom $\Leftrightarrow$ $Aut^F(\mathfrak{G}_{\mathfrak{M}})$ = "World-view transformations of $\mathfrak{M}$".

Because of the "<u>any</u>" part (in the previous sentence) the assumptions of Thm.6.2.108 below are much stronger than those of Thm.6.2.106 above.

THEOREM 6.2.108 *Assume*

$$\mathfrak{M} \models \mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax(eqtime)} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }).$$

Then (i)–(iv) below are <u>equivalent</u>.

(i) $\mathfrak{M} \models \mathbf{Ax}(\omega)^0$.

(ii) $(\forall m, k \in Obs)\, \widehat{\mathsf{f}_{mk}} \in Aut^F(\mathfrak{G}_{\mathfrak{M}})$.

(iii) $Aut^F(\mathfrak{G}_{\mathfrak{M}}) = \{\, \widehat{\mathsf{f}_{mk}} \ :\ m, k \in Obs \,\}$.

(iv) $\mathfrak{M} \models \mathbf{Ax}$, *where* $\mathbf{Ax}$ *is <u>any</u> one of the following symmetry axioms* $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle 1}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\Box 1}$, $\mathbf{Ax\Box 2}$, $\mathbf{Ax}(\omega)^{\sharp}$.

On the proof: The proof is similar to the proof of Thm.6.2.106, and it is left to the reader. Among others, the proof uses Thm.6.2.98 (p.910), Thm.6.2.32 (p.840).

■

Since Theorems 6.2.106 and 6.2.108 above are rather important (they characterizes symmetry principles in terms of $Aut^F(\mathfrak{G}_{\mathfrak{M}})$) below we show that their conditions cannot be omitted. In other words the above connections between $Aut^F(\mathfrak{G}_{\mathfrak{M}})$, world-view transformations, and symmetry principles are not automatically true in all of our relativity theories.

THEOREM 6.2.109 *Assume* $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ })$. *Then*

weak symmetry axioms (e.g. $\mathbf{Ax\Box 1}$*)* $\not\Rightarrow$ $Aut^F(\mathfrak{G}_{\mathfrak{M}}) \subseteq \{\, \widehat{\mathsf{f}_{mk}} \ :\ m, k \in Obs \,\}$.

In more detail

(i) $\mathfrak{M} \models \mathbf{Ax\Box 1} + \mathbf{Ax\triangle 1} + \mathbf{Ax(symm_0)}$, *but*

(ii) $Aut^F(\mathfrak{G}_{\mathfrak{M}}) \not\subseteq \{\, \widehat{\mathsf{f}_{mk}} \;:\; m, k \in Obs\,\}$,

for some $\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(\sqrt{\ }).$

Outline of proof: Let $\mathfrak{N} := \mathfrak{M}^M_{\mathfrak{R}}$ be the Minkowski model over $\mathfrak{R}$ defined in Def.3.8.42 (p.331). The new model $\mathfrak{M}$ will be constructed from $\mathfrak{N}$ in such a way that for each $i \in \mathrm{Z}$ we add new observers whose meter rods are 2^i-times shorter than the meter-rods of the old observers and whose clocks are ticking 2^i-times faster than the clocks of the old observers. Formally,

$$\begin{aligned}
\mathfrak{M} &\stackrel{\text{def}}{=} \langle (B^{\mathfrak{M}};\ Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, Ib^{\mathfrak{M}}), \mathfrak{R}, \mathsf{Eucl}(\mathfrak{R});\ \in, W^{\mathfrak{M}} \rangle \ \text{ where}\\
Obs^{\mathfrak{M}} &\stackrel{\text{def}}{=} Obs^{\mathfrak{N}} \times \mathrm{Z},\\
Ph^{\mathfrak{M}} &\stackrel{\text{def}}{=} Ph^{\mathfrak{N}} \times \mathrm{Z},\\
B^{\mathfrak{M}} &\stackrel{\text{def}}{=} Ib^{\mathfrak{M}} \stackrel{\text{def}}{=} Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}}, \quad \text{and}
\end{aligned}$$

for any $\langle m, i\rangle \in Obs^{\mathfrak{M}}$, $p \in {}^n\mathrm{R}$, and $\langle b, j\rangle \in B^{\mathfrak{M}}$,

$$W^{\mathfrak{M}}(\langle m, i\rangle, p, \langle b, j\rangle) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad W^{\mathfrak{N}}(m, 2^i \cdot p, b).$$

Now, we claim that $\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax\Box 1} + \mathbf{Ax\triangle 1} + \mathbf{Ax(symm_0)}$. Proving this claim is left to the reader. (Hint: the "median observer" proof methods from §§ 3.8, 3.9 and [174] might help in proving this claim.)

Next, we turn to constructing a nice automorphism of $\mathfrak{G}_{\mathfrak{M}}$ which is not induced by any world-view transformation. Let $m \in Obs^{\mathfrak{M}}$ be fixed. Let $f : Mn \longrightarrow Mn$ be defined by $f : e \mapsto w_m(3 \cdot w_m^{-1}(e))$. Let $\tilde{f} : L \longrightarrow \mathcal{P}(Mn)$ be defined by $f : \ell \mapsto f[\ell]$. Now, it is not hard to see that $\langle f, \mathrm{Id} \restriction \mathrm{R}, \tilde{f}\rangle$ is a nice automorphism of $\mathfrak{G}_{\mathfrak{M}}$, but $\langle f, \mathrm{Id} \restriction \mathrm{R}, \tilde{f}\rangle \neq \widehat{\mathsf{f}_{mk}}$ for any m, k. ∎

It would be interesting to know if one could replace $\mathbf{Ax\Box 1}$ in Thm.6.2.109 above with a stronger symmetry axiom on the expense of weakening the basic axioms "**Basax** + …".[873]

---

[873]Under the present conditions, $\mathbf{Ax}(\omega) \Rightarrow Aut^F(\mathfrak{G}_{\mathfrak{M}}) \subseteq \{\widehat{\mathsf{f}_{mk}} : m, k \in Obs\}$. Even if we replace **Basax** by $\mathbf{Bax^-}$.

THEOREM 6.2.110 *Assume* **Basax** + **Ax(↑↑)** + **Ax(||)** + **Ax**($\sqrt{\ }$). *Then*

weak symmetry axioms (e.g. **Ax(syt)**) $\not\Rightarrow$ $Aut^F(\mathfrak{G}_{\mathfrak{M}}) \supseteq \{\widehat{\mathsf{f}_{mk}} : m, k \in Obs\}$.

In more detail,

(i) $\mathfrak{M} \models$ **Ax(syt)** + **Ax□2** + **Ax△2** + **Ax(speedtime)** + **Ax**$(\omega)^0$, *but*

(ii) $(\exists m, k \in Obs)\widehat{\mathsf{f}_{mk}} \notin Aut(\mathfrak{G}_{\mathfrak{M}})$,

for some $\mathfrak{M} \models$ **Basax** + **Ax(↑↑)** + **Ax(||)** + **Ax**($\sqrt{\ }$).

Idea of proof: The idea of proof is illustrated in Fig.304. Assume for simplicity that $n = 2$. We choose $\mathfrak{M} \models$ **Basax** + ... + **Ax(syt)** + ... such that there are $m, k \in Obs$ as depicted in Fig.304. For this choice of m and k, $\mathfrak{G}_m \neq \mathfrak{G}_k$. Hence, by Prop.6.2.105, $\widehat{\mathsf{f}_{mk}}$ is not an automorphism of $\mathfrak{G}_{\mathfrak{M}}$. ∎

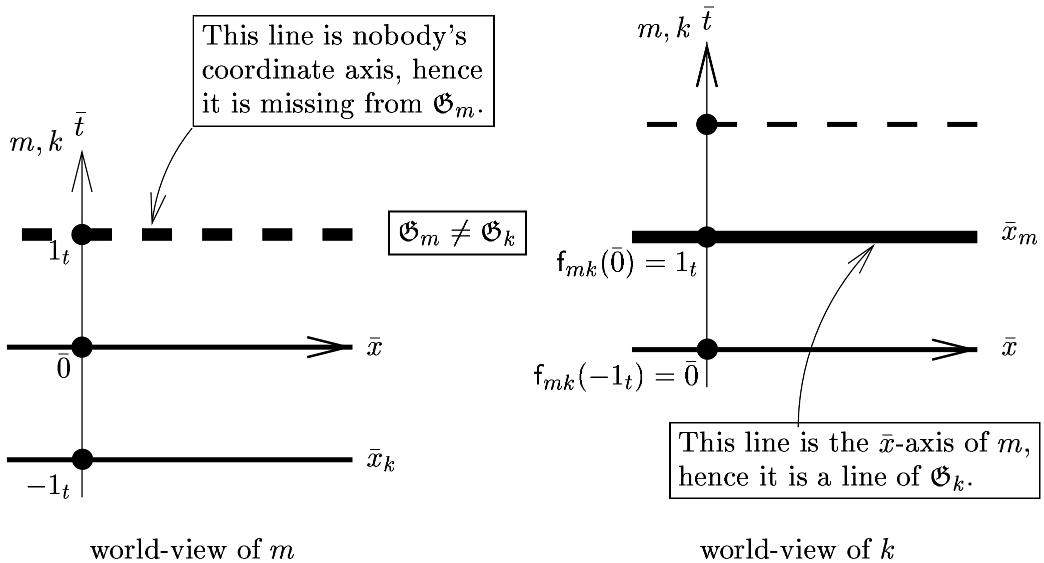

Figure 304: Idea of proof for Thm.6.2.110.

THEOREM 6.2.111 *Assume* **Pax** + **Ax(*Triv*)** + **Ax(||)** + **Ax**($\sqrt{\ }$). *Then*

strong symmetry axioms (e.g. **Ax(ω)**) $\not\Rightarrow$ $Aut^F(\mathfrak{G}_{\mathfrak{M}}) \supseteq \{\widehat{\mathsf{f}_{mk}} : m, k \in Obs\}$.

In more detail,

(i) $\mathfrak{M} \models \mathbf{Ax}(\omega) + \mathbf{Ax(symm)}$, *but*

(ii) $(\exists m, k \in Obs)\ \widehat{\mathsf{f}_{mk}} \notin Aut^F(\mathfrak{G}_{\mathfrak{M}})$,

for some $\mathfrak{M} \models \mathbf{Pax} + \mathbf{Ax}(Triv) + \mathbf{Ax}(\|) + \mathbf{Ax}(\sqrt{\ })$.

Outline of proof: Let $\mathfrak{M} \models$ **"BaCo"** [874] be such that $Ph^{\mathfrak{M}}$ is a one element set. Then $\mathfrak{G}_m \neq \mathfrak{G}_k$, for some $m, k \in Obs$. For this choice of m, k $\widehat{\mathsf{f}_{mk}} \notin Aut^F(\mathfrak{G}_{\mathfrak{M}})$, by Prop.6.2.105. But, $\mathfrak{M} \models \mathbf{Ax}(\omega) + \ldots$. ∎

Notice that by the above three results[875] if we weaken our basic axioms then symmetry does not imply

$$Aut^F(\mathfrak{G}_{\mathfrak{M}}) \ = \ \text{"World-view transformations of } \mathfrak{M}\text{"},$$

moreover symmetry does not imply neither $Aut^F(\mathfrak{G}_{\mathfrak{M}}) \subseteq$ "W…" nor $Aut^F(\mathfrak{G}_{\mathfrak{M}}) \supseteq$ "W…". The next theorem addresses the reverse direction:

$$\text{symmetry} \quad \Leftarrow \quad \text{automorphism properties.}$$

THEOREM 6.2.112 *Assume* $\mathbf{Basax} + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ })$. *Then*

$$\boxed{Aut^F(\mathfrak{G}_{\mathfrak{M}}) = \{\widehat{\mathsf{f}_{mk}} \ : \ m, k \in Obs\} \quad \not\Rightarrow \quad \mathbf{Ax}(\omega)^0.}$$

Outline of proof: Let $\mathfrak{N} := \mathfrak{M}^M_{\mathfrak{R}}$ be the Minkowski model over $\mathfrak{R}$ defined in Def.3.8.42 (p.331). The new model $\mathfrak{M}$ will be constructed from $\mathfrak{N}$ by including all the brothers of the observers in $\mathfrak{N}$ with all the possible units of measurement. Intuitively, $\mathfrak{M}$ will be the "ant and elephant" version of the Minkowski model, where cf. Remark 4.2.1 (p.458) for the "ant and elephant" version of relativity. Formally,

$$\begin{aligned}
\mathfrak{M} &\stackrel{\text{def}}{=} \langle (B^{\mathfrak{M}}; Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, Ib^{\mathfrak{M}}), \mathfrak{R}, \mathsf{Eucl}(\mathfrak{R}); \in, W^{\mathfrak{M}} \rangle \ \text{ where} \\
Obs^{\mathfrak{M}} &\stackrel{\text{def}}{=} Obs^{\mathfrak{N}} \times {}^{+}\mathrm{R}, \\
Ph^{\mathfrak{M}} &\stackrel{\text{def}}{=} Ph^{\mathfrak{N}} \times {}^{+}\mathrm{R}, \\
Ib^{\mathfrak{M}} &\stackrel{\text{def}}{=} B^{\mathfrak{M}} \stackrel{\text{def}}{=} Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}}, \quad \text{and}
\end{aligned}$$

for any $\langle m, \lambda \rangle \in Obs^{\mathfrak{M}}$, $p \in {}^{n}\mathrm{R}$, and $\langle b, \eta \rangle \in B^{\mathfrak{M}}$,

$$W^{\mathfrak{M}}(\langle m, \lambda \rangle, p, \langle b, \eta \rangle) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad W^{\mathfrak{N}}(m, \lambda \cdot p, b).$$

It is not hard to check that $\mathfrak{M}$ has the desired properties. ∎

[874] The notation $\mathfrak{M} \models$ "*Th*" was introduced on p.708.

[875] i.e. by Thm's 6.2.109, 6.2.110, 6.2.111

Problem 6.2.113 It would be interesting to know whether $\mathbf{Ax}(\omega)^0$ can be replaced with $\mathbf{Ax}\Box\mathbf{1}$ in Thm.6.2.112 above.

◁

Future research task 6.2.114 It would be interesting to see how the negative statements 6.2.109, 6.2.110, 6.2.111, 6.2.112 can be improved by e.g. modifying the assumptions on the background theory.

◁

6.2.9 Some reducts of our relativistic geometries; connections with the literature

The reader might feel that the geometric object $\mathfrak{G}_{\mathfrak{M}}$ defined in Definition 6.2.2 (pp. 787–798) seems to have too many components. However, we will concentrate on discussing *reducts* of $\mathfrak{G}_{\mathfrak{M}}$ instead of the full structure.

A very nicely streamlined reduct is called the time-like-metric reduct which will be introduced and discussed in §6.7.3 (p.1169). About that reduct we note that it is not only mathematically elegant, but also is most useful e.g. can be generalized smoothly such that it becomes a suitable framework for a possible formalization of the basics of general relativity, cf. Busemann [56]. All the same, below we start our discussion with a more "classical", more "Euclidean" reduct (of the incidence geometry kind).[876]

(1) Perhaps the most well known reduct of $\mathfrak{G}_{\mathfrak{M}}$ is

$$GT_{\mathfrak{M}} := \langle Mn, L; \in, Bw, \bot, eq\rangle$$

which we call the *Goldblatt-Tarski reduct* of $\mathfrak{G}_{\mathfrak{M}}$. This is a geometry of the form

$$\langle Points, Lines; \in, Bw, \bot, eq\rangle.$$

Tarski's axiomatic approach to Euclidean geometries over ordered fields $\mathfrak{F}$, basically, studies structures of this form:

$$\langle Points, Lines; \in, Bw, \bot, eq\rangle.$$

[876] Our excuse for starting with this reduct is that, stretching it a little bit, one could say that it was known already by the ancient Greeks.

More precisely, there, the first part $\langle Points, Lines; \in\rangle$ is coded up into a one sorted structure[877] $\langle Points, collinear(x, y, z)\rangle$. But as it will be discussed in §6.5 (p.991) below, this causes no essential difference. Actually, Tarski omitted $\perp$ because it is definable from Bw and eq, and Goldblatt in [108] did not include eq probably because it is definable from the rest of $GT_{\mathfrak{M}}$ in the cases of Minkowskian and Euclidean geometries. From now on we will ignore the fact that Tarski and Goldblatt omitted $\perp$ and eq, respectively.[878]

As we will recall, Hilbert, Tarski and their followers proved that for the Euclidean case the language of $\langle Points, Lines; \in, Bw, \perp, eq\rangle$ is expressive enough in the sense that all familiar concepts of classical geometry like e.g. circles can be defined in the *first-order language* of these structures $\langle Points, \dots, eq\rangle$.[879]

In order to continue Tarski's approach in the direction of geometries of special relativity, Goldblatt in [108] puts the emphasis on reducts

$$\begin{aligned} \mathfrak{G}_1 &= \langle Points, Lines; \in, Bw, \perp\rangle, \\ \mathfrak{G}_2 &= \langle Points, Lines; \in, \perp\rangle, \text{ and} \\ \mathfrak{G}_3 &= \langle Points, Lines; \in\rangle. \end{aligned}$$

Goldblatt in [108] calls $\mathfrak{G}_3$ an *incidence geometry* on p.18 or an incidence structure. (For more on incidence geometries cf. pp.1175–1176.) If $\mathfrak{G}_3$ satisfies certain axioms on p.19 of [108] then it is called an *affine plane* (which is the two-dimensional version of an affine geometry). If it satisfies other axioms (on p.74 of [108]) then it is called a *projective plane*. Further, Goldblatt in [108] calls $\mathfrak{G}_2$ a *metric plane* if it satisfies certain axioms collected on p.36. Finally, [108] calls $\mathfrak{G}_1$ an *ordered metric plane* if some further axioms (pp. 70–71) are satisfied by it (without $\perp$ [108] calls it an

[877]To be precise, Tarski uses $\langle Points; Bw\rangle$ to "code" $\langle Points; collinear\rangle$. Hence under very mild assumptions the geometries of form $\langle Points; Bw\rangle$ are definitionally equivalent with the geometries of form $\langle Points; collinear, Bw\rangle$. The latter version is used extensively in the literature.

[878]For some of our choices of $\mathfrak{M}$, eq is not definable from the rest of $GT_{\mathfrak{M}}$ and $\perp$ is not definable from the rest of $GT_{\mathfrak{M}}$ either. Cf. §6.7.

[879]A difference between Hilbert's and Tarski's approach to axiomatizing geometry is that Tarski insisted on using *purely first-order* logic, and to consider all the models (in the model theoretic sense) of his first-order axioms. (Hilbert used a second-order axiom besides first-order ones.) The approach to studying geometries over arbitrary Euclidean fields was started well before Hilbert's and Tarski's work. Referring to so many people would render our present discussion a little cumbersome. Therefore, instead of writing "Hilbert's, Tarski's, their precursor's and their follower's work" we will simply write Tarski's work or something similar. This is *only* for simplicity and by this we do not want to belittle the importance of Hilbert's, their precursor's and their follower's work. An incomplete list of references includes e.g. [62, 108, 131, 133, 134, 227, 237, 245, 246, 247, 251, 254]. We refer to Appendix ("Why first-order logic?") for more information, as well as for an explanation of why it is more useful to axiomatize something in first-order logic than in second-order logic.

ordered affine plane).[880] In passing we note, that Goldblatt [108] shows that if a 2-dimesional geometry $\mathfrak{G}_1$ satisfies certain axioms then it is either (i) Euclidean, or (ii) Minkowskian or (iii) a rather simple kind called Robb plane.

The reason why we recalled all this is that there seems to be a possibility for a unified axiomatic study of geometries coming from relativity theory (like our $GT_{\mathfrak{M}}$) and Euclidean geometry. Indeed Goldblatt presents such a unified treatment for geometries $\langle Points, Lines; \in, Bw, \perp\rangle$.

It would be interesting to study the *connections* between our geometries $GT_{\mathfrak{M}}$ ($\mathfrak{M} \in$ models of some of our distinguished theories like $\mathbf{Bax}^-$) and what Goldblatt [108] writes about the kinds of geometries recalled above. Actually, on this connection item (2) below taken together with Figures 282, 283 (pp. 863–864) gives some information; and further information will be given in §6.6.11 (p.1129).

(2) Let us look, briefly, at further reducts of $\mathfrak{G}_{\mathfrak{M}}$.

$$\begin{aligned}
GT^1_{\mathfrak{M}} &:= \langle GT_{\mathfrak{M}};\, L^T, L^{Ph}\rangle,\\
GT^2_{\mathfrak{M}} &:= \mathbf{G}_{\mathfrak{M}} = \langle GT^1_{\mathfrak{M}};\, \prec\rangle,\\
GT^0_{\mathfrak{M}} &:= (eq\text{-free } \textit{reduct } \langle Mn, L;\, \in, Bw, \perp\rangle \text{ of } GT_{\mathfrak{M}}),\\
GT^{-1}_{\mathfrak{M}} &:= (\perp\text{-free } \textit{reduct } \langle Mn, L;\, \in, Bw, eq\rangle \text{ of } GT_{\mathfrak{M}}),\\
GT^{-2}_{\mathfrak{M}} &:= \mathsf{G}_{\mathfrak{M}} = (Bw\text{-free } \textit{reduct } \langle Mn, L;\, \in, \perp\rangle \text{ of } GT^0_{\mathfrak{M}}).
\end{aligned}$$

If we assume $\mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\uparrow\uparrow)$ and $n > 2$ then the structures $\mathfrak{G}_{\mathfrak{M}}, GT_{\mathfrak{M}}, GT^i_{\mathfrak{M}}$ (for $-2 \le i \le 2$) are well known and well investigated. Under these axioms they become parts or variants of Minkowskian geometries (cf. Thm.6.2.59 on p.861) in which form their properties have been thoroughly studied cf. Goldblatt [108], Kostrikin-Manin [155], and other works on Minkowskian geometries. We can say more than this, namely for $n > 2$, the classes

$$\begin{aligned}
&\mathbf{I}\left\{\, GT_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) \,\right\},\\
&\mathbf{I}\left\{\, GT^i_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) \,\right\},\ \text{ for } -2 \le i \le 1\\
&\mathbf{I}\left\{\, GT^2_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\uparrow\uparrow) \,\right\}
\end{aligned}$$

coincide with the classes of isomorphic copies of the corresponding reducts of the Minkowskian geometries (cf. Thm.6.2.65, p.867). See Figures 282, 283. Further, the classes

[880] Goldblatt [108], investigated the more than two-dimensional versions of these geometries, too. Under the name fourfolds Goldblatt in [108] also investigates the 4-dimensional case. Since fourfolds have more sorts than "*Points*" and "*Lines*" we will return to this connection in §6.5 (after having studied how to create and eliminate new sorts in the framework of definitional equivalence [§6.3]). For brevity, we do not recall these in more detail.

$$\mathbf{I}\left\{ GT^i_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Bax}^{\oplus} + \mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6} \right\}, \text{ for } i \in \{0, -2\}$$

coincide with the classes of isomorphic copies of the corresponding reducts of the Minkowskian geometries (cf. Thm.6.2.64). Cf. also Theorems 6.2.71, 6.2.73, 6.2.74, 6.2.75, pp. 877–879. We obtain new structures if we consider e.g.

$$\begin{array}{lll} \mathbf{I}\{ GT_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Bax} \} & \text{or} & \mathbf{I}\{ GT_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Reich}(\mathbf{Bax}) \}, \\ \mathbf{I}\left\{ GT^i_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Bax} \right\} & \text{or} & \mathbf{I}\left\{ GT^i_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Reich}(\mathbf{Bax}) \right\}, \text{ for } -2 \le i \le 2, \\ \mathbf{I}\{ \mathfrak{G}_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Bax} \} & \text{or} & \mathbf{I}\{ \mathfrak{G}_{\mathfrak{M}} \,:\, \mathfrak{M} \models \mathbf{Reich}(\mathbf{Bax}) \}. \end{array}$$

Some of these structures are not isomorphic to any reduct of Minkowskian geometries. The mathematical properties of these geometries remain to be investigated. (There are some analogies with Tarski's fist-order axiomatization of geometry over ordered fields which could be utilized here.)

In the spirit of the above discussion, we could consider the Goldblatt-Tarski reduct $GT_{\mathfrak{M}} = \langle Mn, L;\, \in, Bw, \perp, eq\rangle$ as our core geometry. (As we mentioned all concepts needed for geometrical constructions [by straight-edge and compass] are expressible in $GT_{\mathfrak{M}}$.) However, we do not consider $GT_{\mathfrak{M}}$ to be our core geometry because from some other points of view other reducts will turn out to be important. Cf. e.g. §6.7.3 devoted to a rather nicely streamlined version of $\mathfrak{G}_{\mathfrak{M}}$, called time-like-metric reduct which can be nicely connected to general relativity as is shown e.g. in Busemann [56].

For better expressibility of some relativistic ideas, we include into this geometry L^T and L^{Ph} obtaining

$$GT^1_{\mathfrak{M}} := \langle Mn, L;\, L^T, L^{Ph}, \in, Bw, \perp, eq\rangle.$$

We consider $GT^1_{\mathfrak{M}}$ as a kind of core of our geometry $\mathfrak{G}_{\mathfrak{M}}$, because in this core we can both talk about the usual geometric concepts and also we can talk about time-like lines L^T, photon-like lines L^{Ph} and lines in general, too.[881]

If we assume some conditions on $\mathfrak{M}$, then instead of $GT^1_{\mathfrak{M}}$ it is enough to keep $GT_{\mathfrak{M}}$ because of the following.

THEOREM 6.2.115 *Assume $n > 2$. Assume $\mathfrak{M} \models (\mathbf{Bax}^{\oplus} + \mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. Then L^T, L^{Ph}, L^S, Bw are definable in first-order logic over $GT^{-2}_{\mathfrak{M}} = \mathsf{G}_{\mathfrak{M}} = \langle Mn, L;\, \in, \perp\rangle$, and therefore over $GT_{\mathfrak{M}}$, too.*

[881] We omitted L^S from the definition of $GT^1_{\mathfrak{M}}$ because under very mild conditions L^S is definable from the rest.

Idea of proof: Let $\ell \in L$. Then $\ell \in L^{Ph}$ iff $\ell \perp \ell$. Further $\ell \in L^T$ iff

$$(\forall \text{ 2-dimensional plane}^{882}\ P)$$
$$[\ell \subseteq P \text{ there is a photon line in } P \text{ intersecting } \ell \text{ in a single point}].$$

Of course, one has to prove that these definitions work. The details are available from Judit Madarász. Definability of Bw follows by Thm.6.7.1 (p.1137). ∎

Cf. Corollary 6.7.41 on p.1168 in connection with the above theorem.

We will discuss the remaining interdefinability connections between the basic relations $L^T, L^{Ph}, \ldots, Bw, \perp, eq, g$ of our language for geometries in the section "On the choice of our geometrical vocabulary (or language)" (pp. 1134–1169).

[882] Cf. footnote 798 on p.857 for the notion of a 2-dimensional plane.

6.3 Definability in many-sorted logic, defining new sorts

Historical remark:

The theory of definability as understood in the present work is a branch of mathematical logic (and its model theory) which goes back to Tarski's pioneering work [249]. Beginning with the just quoted paper of 1934 (and its precursor from 1931), Tarski did much to help the theory of definability to become a fully developed branch of mathematical logic which is worth of studying in its own right. Of the many works illustrating Tarski's concern for the theory of definability we mention only [129, PartI], Tarski-Givant [253], Tarski-Mostowski-Robinson [255] and Tarski [249, 250], cf. also Tarski [248] and [252, Volume 1, pp. 517-548] (which first appeared in 1931 and which already addresses the theory of definability).

In passing we note that the creation of the theory of cylindric algebras can be viewed as a by-product of Tarski's interest in developing and publicizing the theory of definitions (a cylindric algebra over a model can be viewed as the collection of all relations definable in that model).

Below, we try to summarize the theory of definability (allowing definitions of new sorts) in a style tailored for the needs of the present work *and* in a spirit consistent with Tarski's original ideas and views on the subject. Here the emphasis will be on defining new sorts (which is usually not addressed in classical logic books such as e.g. Chang-Keisler [59]).

The subject matter of the present sub-section is relevant to the definability issues discussed in the literature of relativity cf. e.g. Friedman [90, pp. 62–63, 65, 378 (index)]. In Reichenbach's book "Axiomatization of the Theory of Relativity" [223] already on the first page of the Introduction (p.3) he explains the difference between explicit and implicit definitions and emphasizes their importance. (He also traces this distinction (underlying definability theory) to Hilbert's works.) In passing we note that on p.5, Reichenbach [223] also explains in considerable detail why it is desirable to start out with observational concepts first when building up our theory (like we do in Chapters 1,2) and define theoretical concepts later over observational ones using definability theory (as we do in the present chapter). For the time being we do not discuss connections between definability theory and definability issues in relativity theory explicitly, but we plan to do so in a later work.[883]

[883]But we note that Reichenbach [223] makes it clear that he considers definability theory very important for relativity, and he also explains rather convincingly why he does so. This is also clear from the relativity works Friedman [90], or Grünbaum [115],[116], to mention only a few. Cf. also

For the physical importance of definability cf. the relevant parts of the introduction of this chapter. Further, we note the following. If in our language we allow using certain concepts and some other concept is definable from these, then this other concept *is* available in our language even if we do not include it (explicitly). So if we allow only such concepts which are definable from observational ones, then the effect will be the same as if we allowed only observational concepts. I.e. the physical principle of Occam's razor has been respected.

* * *

Let $\mathfrak{M} = \langle U_0, \ldots, U_j;\, R_1, \ldots, R_l \rangle$ be a many-sorted model with universes or sorts $U_0, \ldots, U_j$, and relations $R_1, \ldots, R_l \quad (j, l \in \omega)$.[884] Since functions are special relations we do not indicate them explicitly in the present discussion. We use the semicolon ";" to separate the sorts (or universes) from the relations of $\mathfrak{M}$.

When discussing many-sorted models, we always assume that they have *finitely many sorts* only.[885] The "big universe" $Uv(\mathfrak{M})$ of the model $\mathfrak{M}$ is the union of its universes (or sorts). Formally

$$Uv :\stackrel{\text{def}}{=} Uv(\mathfrak{M}) :\stackrel{\text{def}}{=} \bigcup \{\, U_i \;:\; U_i \text{ is a universe of } \mathfrak{M} \,\}\,.^{886}$$

In passing we note that although the sorts $U_0, \ldots, U_j$ of $\mathfrak{M}$ need not be disjoint, the following holds. To every many-sorted model $\mathfrak{M}$ there is an isomorphic copy $\mathfrak{M}'$ of $\mathfrak{M}$ such that the sorts $U'_0, \ldots, U'_j$ of $\mathfrak{M}'$ *are* mutually disjoint (i.e. $U'_0 \cap U'_1 = \emptyset$

Stein [240].

[884]The assumption that l is finite is irrelevant here in the sense that we will never make use of it (except when we state this explicitly). What we write in this section makes perfect sense if the reader replaces l with an arbitrary ordinal. As a contrast, we do use the assumption that $j \in \omega$.

[885]In some minor items there may be exceptions from this rule but then this will be clearly indicated.

[886]Although, in general, Uv is not a universe of $\mathfrak{M}$, we can *pretend* that it is a universe because there are only finitely many sorts. E.g. if we want to simulate the formula $(\exists x \in Uv)\, \psi(x)$ then we write $[\, (\exists x \in U_0)\, \psi(x) \;\vee\; (\exists x \in U_1)\, \psi(x) \;\vee\; \ldots \vee\; (\exists x \in U_j)\, \psi(x)\,]$. Then although the first formula $(\exists x \in Uv)\, \psi(x)$ does not belong to the language of $\mathfrak{M}$, the second formula "$[\, (\exists x \in U_0) \ldots]$" does belong to this language (assuming $(\exists x \in U_i)\, \psi(x)$ already belongs to the language) and the meaning of the second formula is the same as the intuitive meaning of the first one. If $(\exists x \in U_i)\, \psi(x)$ did still not belong to our many-sorted language then there is some extra routine work to do in translating this formula into our many-sorted language. This translation is explained in detail in the logic books which reduce many-sorted logic to one-sorted logic (cf. [43, 82, 197]). These books were quoted in §2 where we first encountered many-sorted logic. We also note that the quoted translation is straightforward. For more on why and how we can pretend that $Uv(\mathfrak{M})$ is a universe of $\mathfrak{M}$ we refer to the just quoted logic books.

etc.). Therefore we *are permitted to* pretend that the sorts (i.e. universes) of $\mathfrak{M}$ are disjoint from each other whenever we would need this.

By a <u>*reduct*</u> of a many-sorted model $\mathfrak{M}$ we understand a model $\mathfrak{M}^-$ obtained from $\mathfrak{M}$ by omitting some of the sorts and/or some of the relations of $\mathfrak{M}$. I.e. if

$$\mathfrak{M} = \langle U_0, \ldots, U_j;\ R_1, \ldots, R_l \rangle$$

then the reduct $\mathfrak{M}^-$ may be of the form

$$\langle U_0, \ldots, U_{j-1};\ R_1, \ldots, R_{l-1} \rangle$$

(assuming $R_1, \ldots, R_{l-1}$ do not involve the sort U_j).

A model $\mathfrak{M}^+$ is called an <u>*expansion*</u> of $\mathfrak{M}$ iff $\mathfrak{M}$ is a reduct of $\mathfrak{M}^+$. I.e. an expansion $\mathfrak{M}^+$ is obtained by adding new sorts and/or new relations to $\mathfrak{M}$. We will use the following abbreviation for denoting expansions:

$$\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new};\ \bar{R}^{new} \rangle$$

where U^{new} is the <u>*new sort*</u> and $\bar{R}^{new} = \langle R_1^{new}, \ldots, R_r^{new} \rangle$ is the sequence of <u>*new relations*</u>. Of course there may be more new sorts too, then we write

$$\mathfrak{M}^+ = \langle \mathfrak{M}, U_1^{new}, \ldots, U_\varrho^{new};\ \bar{R}^{new} \rangle.$$

However, we will concentrate on the case $\varrho = 1$ (for didactical reasons). Informally the general pattern is:

"New model" = ⟨"Old model", "New sorts"; "New relations/functions"⟩.

We will ask ourselves when $\mathfrak{M}^+$ will be *(first-order logic) definable* over[887] $\mathfrak{M}$. By *definable* we will *always* (throughout this work) mean first-order logic definable. If $\langle \mathfrak{M}, U^{new};\ \bar{R}^{new} \rangle$ is definable over $\mathfrak{M}$ then we will say that the new sort U^{new} together with $\bar{R}^{new}$ are *definable* in $\mathfrak{M}$. When defining a new sort U^{new} (in an "old" model $\mathfrak{M}$) we need the new relations $\bar{R}^{new}$ too because it is $\bar{R}^{new}$ which will specify the connections between the new sort U^{new} and the old sorts of $\mathfrak{M}$.

Although we will start out with discussing definability over a single model $\mathfrak{M}$, the really important part will be when we generalize this to definability (of an expanded class K^+) *over a class* K of models (which is first-order axiomatizable).

We will discuss *two kinds* of definability in many-sorted logic: <u>*implicit*</u> definability in §6.3.1 and <u>*explicit*</u> definability in §6.3.2.[888]

[887] "Definable over" is the same as "definable in".

[888] In passing, we note that in the *special* case of the most traditional one-sorted logic when *only* relations are defined (i.e. defining new sorts is not considered) the distinction between implicit and explicit definability is well investigated and is well understood cf. e.g. Chang-Keisler [59, p.90] or Hodges [136, pp.301-302].

Throughout model theory there is a *distinction* between symbols like *Obs* and objects like $Obs^{\mathfrak{M}}$ denoted by these symbols in a model $\mathfrak{M}$. This distinction between symbols and objects they denote is even more important in the theory of definitions than in other parts of logic. Therefore, in the next two items we clarify notions and notation connected to this distinction.

CONVENTION 6.3.1 By the *vocabulary* of a model $\mathfrak{M}$ we understand the system of sort-symbols, relation symbols and function symbols interpreted by $\mathfrak{M}$. Since function symbols are special relation symbols, we will restrict our attention to sort symbols and relation symbols. Assume e.g. that $\mathfrak{M}$ is of the form

$$\mathfrak{M} = \langle U_0^{\mathfrak{M}}, \dots, U_j^{\mathfrak{M}}; R_1^{\mathfrak{M}}, \dots, R_l^{\mathfrak{M}} \rangle \ ,$$

and assume that U_i is the sort *symbol* "denoting" $U_i^{\mathfrak{M}}$ and R_i is the relation *symbol* "denoting" $R_i^{\mathfrak{M}}$. Then the vocabulary of $\mathfrak{M}$ is

$$Voc(\mathfrak{M}) \stackrel{\text{def}}{=} \langle \{U_0, \dots, U_j\}, \{R_1, \dots, R_l\} \rangle .$$

Throughout we assume that a relation symbol R' contains the extra information which we call the *rank* of R'. This can be implemented by postulating that R' is an ordered pair $R' = \langle R'_0, R'_1 \rangle$ where R'_0 is the symbol we write on paper while R'_1 is the rank of R'. E.g. in the case of the usual model $\mathfrak{N} = \langle \omega, \leq, + \rangle$ the rank of "$\leq$" is 2 while that of "$+$" is 3. If there is more than one sort, then the rank of a relation is a sequence of sort symbols. So, a vocabulary is an ordered pair

$$Voc = \langle \text{"Sort symbols"}, \text{"Relation symbols"} \rangle$$

where "Sort symbols" and "Relation symbols" are two sets as discussed above subject to the condition that the sorts occurring in the ranks of the relation symbols all occur in the set of sort symbols. Now, a model $\mathfrak{M}$ of vocabulary *Voc* can be regarded as a pair $\mathfrak{M} = \langle \mathfrak{M}_0, \mathfrak{M}_1 \rangle$ of functions such that

$$\mathfrak{M}_0 : \text{"Sort symbols"} \longrightarrow \text{"Universes of } \mathfrak{M}\text{"}$$

and

$$\mathfrak{M}_1 : \text{"Relation symbols"} \longrightarrow \text{"Relations of } \mathfrak{M}\text{"},$$

with the restriction that $\mathfrak{M}_1$ is "rank-preserving" in a natural sense.
E.g. if $\mathfrak{M} = \langle U_0^{\mathfrak{M}}, \dots, U_j^{\mathfrak{M}}; R_1^{\mathfrak{M}}, \dots, R_l^{\mathfrak{M}} \rangle$, then

$$\mathfrak{M}_0 : \{U_i : i \leq j\} \longrightarrow \{U_i^{\mathfrak{M}} : i \leq j\}$$

$$\mathfrak{M}_1 : \{R_i : 0 < i \le l\} \longrightarrow \{R_i^{\mathfrak{M}} : 0 < i \le l\}.$$

I.e. to each sort symbol in $Voc(\mathfrak{M})$, $\mathfrak{M}$ associates a universe (i.e. a set) and to each relation symbol R' in $Voc(\mathfrak{M})$, $\mathfrak{M}$ associates a relation (of rank R'_1 as indicated way above).

We call two models $\mathfrak{M}$ and $\mathfrak{N}$ *similar* if they have the same vocabulary, i.e. if $Voc(\mathfrak{M}) = Voc(\mathfrak{N})$.

Let Voc', Voc be two vocabularies. We say that Voc' is a *sub-vocabulary* of Voc if the natural conditions $Voc'_0 \subseteq Voc_0$ and $Voc'_1 \subseteq Voc_1$ hold. Assume Voc' is a sub-vocabulary of $Voc(\mathfrak{M})$ for a model $\mathfrak{M}$. Then the *reduct* $\mathfrak{M} \upharpoonright Voc'$ of $\mathfrak{M}$ to the sub-vocabulary Voc' is defined as

$$\mathfrak{M} \upharpoonright Voc' \stackrel{\text{def}}{=} \langle \mathfrak{M}_0 \upharpoonright Voc'_0, \;\; \mathfrak{M}_1 \upharpoonright Voc'_1 \rangle.$$

◁

Remark 6.3.2 (On the intuitive content of Convention 6.3.1 above) On a very intuitive informal level, one can think of a model $\mathfrak{M}$ as a *function* associating objects to symbols. E.g. $\mathfrak{M}$ associates $U_i^{\mathfrak{M}}$ to the symbol U_i and $R_i^{\mathfrak{M}}$ to R_i. It is then a matter of notational convention that we write $U_i^{\mathfrak{M}}$ for the value $\mathfrak{M}(U_i)$ and $R_i^{\mathfrak{M}}$ for $\mathfrak{M}(R_i)$. Then the domain of the function $\mathfrak{M}$ is the collection of those symbols which $\mathfrak{M}$ can interpret. Hence, the *domain* of $\mathfrak{M}$ is the same thing as its vocabulary.

If the best way (from the intuitive point of view) of thinking about a model is regarding it as a function, then why did we formalize the notion of a model as a pair of functions (instead of a single function)? The answer is that *formally* it is easier to handle models as pairs of functions, but *intuitively* we think of models as functions, we think of vocabularies as domains of these functions and we consider two models similar if they have the same domain when they are regarded as functions.[889]

◁

CONVENTION 6.3.3 Throughout, by a class K of models we understand a class of similar models, i.e. we always assume $(\forall \mathfrak{M}, \mathfrak{N} \in \mathsf{K})\ Voc(\mathfrak{M}) = Voc(\mathfrak{N})$. For any class K of similar models, $Voc(\mathsf{K}) = Voc\mathsf{K}$ denotes the vocabulary of K, that is, the vocabulary of an arbitrary element of K.

A *reduct* K^- of K is obtained from K by omitting a part of the vocabulary of K, i.e. K^- is a reduct of K iff $Voc(\mathsf{K}^-) \subseteq Voc(\mathsf{K})$ and

$$\mathsf{K}^- = \left\{ \mathfrak{M} \upharpoonright Voc(\mathsf{K}^-) \ : \ \mathfrak{M} \in \mathsf{K} \right\}.$$

[889] We do not claim that it is always the case that the best way of thinking about models is regarding them as functions. What we claim is that in many situations, e.g. in definability theory, this is a rather good way. In other situations it might be better to visualize a model as a set of objects equipped with some relations and functions.

Expansion is the opposite of reduct. K^+ is an *expansion* of the class K iff K is a reduct of K^+, i.e. K^+ is an expansion of K iff $Voc(\mathsf{K}^+) \supseteq Voc(\mathsf{K})$ and

$$\mathsf{K} = \{\, \mathfrak{M} \restriction Voc(\mathsf{K}) \ : \ \mathfrak{M} \in \mathsf{K}^+ \,\}\,.$$

Note that forming expansions or reducts of a class K is somehow *uniform* over the members of K. E.g. we forget the *same* symbols (relation symbols or sort symbols) from all models $\mathfrak{M} \in \mathsf{K}$, when taking a reduct of K.

If Voc is a vocabulary with $Voc \subseteq Voc(\mathsf{K})$, then we use the following abbreviation:

$$\mathsf{K} \restriction Voc \stackrel{\text{def}}{=} \{\mathfrak{M} \restriction Voc \ : \ \mathfrak{M} \in \mathsf{K}\}.$$

Examples: $\mathsf{FM}^- = \{\, \mathfrak{F}^{\mathfrak{M}} \ : \ \mathfrak{M} \in \mathsf{FM} \,\}$ is a reduct of our class FM of frame models. Let $\mathsf{L} = \{\, \mathbf{F} \ : \ \mathbf{F} \text{ is a field} \,\}$. Then $\{\, \langle F; + \rangle \ : \ \langle F; +, \cdot, 0, 1 \rangle \in \mathsf{L} \,\}$ is a reduct of L.

Intuitively, we think of $Voc(\mathsf{K})$ as a set of symbols where each symbol contains information about its nature, i.e. about whether it is a sort symbol or a relation symbol of a certain rank. Therefore, we will write $Voc \cap Voc'$ for $\langle Voc_0 \cap Voc_0', Voc_1 \cap Voc_1' \rangle$, similarly for $Voc \cup Voc'$, for $Voc \subseteq Voc'$ etc.

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Before getting started, we emphasize that in order to define something over a model $\mathfrak{M}$ or over a class K of models, first of all we need *new symbols* R_i^{new}, U_i^{new} (with i in some index set) not occurring in the language of $\mathfrak{M}$ or of K. (The new symbols may be relation symbols like R_i^{new} or sort symbols U_i^{new} or both.) What we will define then (using definability theory) will be the meanings of the new symbols in $\mathfrak{M}^+$ or K^+. Most of the time we will not talk about the new *symbols* like R_i^{new} because we will identify them with the new relations like $(R_i^{new})^{\mathfrak{M}^+}$ which they denote in the expansion $\mathfrak{M}^+$ of the model $\mathfrak{M}$. Our reason for identifying the "symbol" with the "object" it denotes is to simplify the discussion. However, occasionally it will be useful to remember that an expansion $\mathfrak{M}^+ = \langle \mathfrak{M}, R \rangle$ of a model $\mathfrak{M}$ involves two new things not available in $\mathfrak{M}$, namely: a relation symbol and a relation denoted by this symbol (in $\mathfrak{M}^+$).

6.3.1 Implicit definability in many-sorted (first-order) logic

Let $\mathfrak{M}$ be a many-sorted model. Assume, $\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new};\, \bar{R}^{new}\rangle$ is an expansion of $\mathfrak{M}$. We say that $\mathfrak{M}^+$ is *definable implicitly up to isomorphism* over $\mathfrak{M}$ iff

> for any model
> $$\langle \mathfrak{M}, U';\, \bar{R}'\rangle \models \mathsf{Th}(\mathfrak{M}^+)$$
> (expanding $\mathfrak{M}$) there is an isomorphism
>
> ($\star$) $$h : \mathfrak{M}^+ \rightarrowtail\!\!\!\twoheadrightarrow \langle \mathfrak{M}, U';\, \bar{R}'\rangle$$
>
> such that h is the *identity* function on the sorts of $\mathfrak{M}$ (i.e. for each sort U_i of $\mathfrak{M}$ we have $h \restriction U_i = \mathrm{Id} \restriction U_i$).

$\mathfrak{M}^+$ is said to be *definable implicitly without taking reducts* over $\mathfrak{M}$ iff *in addition* to the above the isomorphism h mentioned above is *unique*.

We say that *U^{new}, $\bar{R}^{new}$ are definable implicitly* over $\mathfrak{M}$ iff $\langle \mathfrak{M}, U^{new};\, \bar{R}^{new}\rangle$ is definable implicitly without taking reducts over $\mathfrak{M}$. Informally we might say in such situations that the new sort U^{new} is definable implicitly in $\mathfrak{M}$ (but then $\bar{R}^{new}$ should be understood from the context, otherwise the definability claim is sort of under-specified).

In the above notion of definability, the *set of formulas defining U^{new}, $\bar{R}^{new}$ implicitly* over $\mathfrak{M}$ is $\mathsf{Th}(\mathfrak{M}^+)$. Hence, $\mathsf{Th}(\mathfrak{M}^+)$ is called an *implicit definition* of U^{new}, $\bar{R}^{new}$ over $\mathfrak{M}$ if ($\star$) above holds and the isomorphism h is unique. Further, for any set Δ of formulas in the language of $\mathfrak{M}^+$, Δ is called an *implicit definition* of $U^{new}, \bar{R}^{new}$ over $\mathfrak{M}$ iff ($\star$) above holds with Δ in place of $\mathsf{Th}(\mathfrak{M}^+)$ in such a way that h is unique.[890]

Remark 6.3.4 The reader might feel that the above notion of (implicit) definability without taking reducts (of $\mathfrak{M}^+$) is not strong enough and he might want to replace h with the identity function (requiring $U^{new} = U'$, $\bar{R}^{new} = \bar{R}'$). However, we claim that the above notion is "best possible" because (i) it is reasonable to assume that the first-order definition of $\mathfrak{M}^+$ (over $\mathfrak{M}$) is included in $\mathsf{Th}(\mathfrak{M}^+)$ and (ii) *any* isomorphic copy $\mathfrak{M}' = \langle \mathfrak{M}, U';\, \bar{R}'\rangle$ of $\mathfrak{M}^+$ will automatically validate $\mathsf{Th}(\mathfrak{M}^+)$ hence, in first-order logic we cannot define the new sort U^{new}, $\bar{R}^{new}$ more closely than up to (a

[890] The set Δ of formulas which we call an implicit definition is called a "rigidly relatively categorical" theory in Hodges [136, p.645]. If Δ is an implicit definition up to isomorphism only, then it is called a "relatively categorical" theory on p.638 of [136] (§12.5 therein).

unique) isomorphism.[891]

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$\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new}; \bar{R}^{new}\rangle$ is said to be *definable implicitly with parameters* over $\mathfrak{M}$ iff there are $s \in \omega$ and $\bar{p} \in {}^sUv(\mathfrak{M})$ such that the expansion $\langle \mathfrak{M}^+, \bar{p}\rangle$ is *definable implicitly without taking reducts* over the expansion $\langle \mathfrak{M}, \bar{p}\rangle$.[892]

* * *

Let us turn to definability over *classes of models.* Let K be a class of models with U^{new}, $\bar{R}^{new}$ in the language of K. For $\mathfrak{M} \in \mathsf{K}$ let $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ obtained by omitting (forgetting) U^{new}, $\bar{R}^{new}$. Let

$$\mathsf{K}^- := \{\, \mathfrak{M}^- \;:\; \mathfrak{M} \in \mathsf{K} \,\}.$$

We ask ourselves when K is definable over K^- or equivalently (but informally) when U^{new}, $\bar{R}^{new}$ are definable over K^-. We say that the class K of models is *definable implicitly without taking reducts* over K^- iff there is a set $\Delta \subseteq \mathsf{Th}(\mathsf{K})$ of formulas such that condition $(\star\star)$ below holds.

$(\star\star)$ For every $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(\Delta)$ similar to members of K and such that $\mathfrak{M}^- = \mathfrak{N}^- \in \mathsf{K}^-$, there is a unique isomorphism $h : \mathfrak{M} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{N}$ which is the identity on the universes of $\mathfrak{M}^-$.

If the isomorphism h is not necessarily unique then we say that K is *definable implicitly up to isomorphism* over K^-. Informally, we say that the *new sort* U^{new} and $\bar{R}^{new}$ are *definable implicitly* over K^- iff K as understood above is definable implicitly without taking reducts over K^-. When speaking about definability of U^{new}, $\bar{R}^{new}$ over K^-, it should be clear *from context* how K is obtained from the data K^- and U^{new}, $\bar{R}^{new}$. If $(\star\star)$ holds, then Δ in $(\star\star)$ is called an *implicit definition* of K over K^-.

We leave it to the reader to generalize the above definitions to the case when we have arbitrary sequences $\bar{U}^{new}$ and $\bar{R}^{new}$ of new sorts and new relations. However, herein we restrict our attention to the case when there are finitely many new symbols

[891] A possible way out of this would be if we required $\bar{R}^{new}$ to contain membership relations "$\in$" and projection functions pj_i (and then add some restrictions postulating e.g. that $\in$ and pj_i are the "real" set theoretic ones etc., cf. p.947 for the definition of the pj_i's). We will not do this because we feel that it would lead to too many complications without yielding enough benefits.

[892] We use "definable implicitly" and "implicitly definable" as synonyms. I.e. we are flexible about word order.

(i.e. both $\bar{U}^{new}$ and $\bar{R}^{new}$ are finite sequences of sorts and relations respectively). The classical notion of definability of new relations (without new sorts) is obtained as a special case of our general notion by choosing $\bar{U}^{new} = \emptyset$, i.e. $\bar{U}^{new}$ is the empty sequence.

Let K and L be two classes of models, i.e. L is not necessarily a reduct of K. We say that K *is definable implicitly over* L iff some expansion K^+ of K is definable implicitly without taking reducts over L. (In this case, L will be a reduct of K^+, of course.)[893] This means that statements (i) and (ii) below hold for some expansion K^+ of K:

(i) L is a reduct of K^+,

(ii) K^+ is definable implicitly over L without taking reducts. (Since here L is a reduct of K^+, our *earlier* definition of implicit definability without taking reducts on p.935 can be applied.)

We note that here we have to take seriously that our languages are finite, i.e. K^+ has only finitely many new symbols that do not occur in L.[894] In this case we say that Δ is an *implicit definition of* K *over* L if Δ is an implicit definition of K^+ over L. Thus an implicit definition of K over L may contain symbols not occurring in K.

We will apply the same convention for single models too, i.e. $\mathfrak{N}$ *is definable implicitly over* $\mathfrak{M}$ iff this holds for $\{\mathfrak{N}\}$ and $\{\mathfrak{M}\}$. We will sometime abbreviate "implicitly definable without taking reducts" by "*nr-implicitly definable*", where "nr" stands for "taking no reducts".

Example 6.3.5 The new sort ${}^nF$ together with the projection functions $pj_i : {}^nF \longrightarrow F$ $(i < n)$, cf. p.947, are definable nr-implicitly over the class FM of our frame models.

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Note that $(\star\star)$ above is a straightforward generalization of $(\star)$ on p.934. Therefore $\mathfrak{M}^+$ is definable nr-implicitly over $\mathfrak{M}$ iff the class $\{\mathfrak{M}^+\}$ is definable nr-implicitly over the class $\{\mathfrak{M}\}$.

[893] It would be more careful of us if we would call this new implicit definability (which permits taking reducts) weak implicit definability. This is so because when taking reducts then the uniqueness condition, cf. p.934, on isomorphisms may get lost.

[894] Cf. Examples 6.3.9 (2).

In situations like the one involving statement ($\star\star$) above, we also say that U^{new}, $\bar{R}^{new}$ are *uniformly* definable (implicitly) over K^-.[895] The set Δ of formulas is considered as a *uniform (implicit) definition* of U^{new}, $\bar{R}^{new}$ over K^-. Hence in the example above we can also say that ${}^nF$ etc. are uniformly definable over FM. We have not yet discussed *non-uniform* definability which is also called *"local"* or *"one-by-one"* definability: We will discuss this notion below Examples 6.3.9, on p.943.

Although we began this sub-section with discussing definability over a single model $\mathfrak{M}$, the main emphasis in this work will be on definability over a class K of models such that $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ i.e. such that K is axiomatizable in first-order logic.

We note that implicit definability without taking reducts of K over K^- is strictly stronger than implicit definability up to isomorphism. This remains so even if we assume that K and K^- are first-order axiomatizable classes of models. We leave the construction of a simple counterexample to the reader, but cf. Example 6.3.9(8) way below. For the connections between the various notions of definability we refer the reader to Figure 305 on p.979.

Remark 6.3.6 The following are intended to provide a kind of "intuitive" characterization of implicit definability without taking reducts of a class K of models over its reduct K^- (as was defined above).

(1) Assume K^- is a reduct of the class K (i.e. K^- is of the form $\{\mathfrak{M}^- : \mathfrak{M} \in \mathsf{K}\}$).

Then K is definable implicitly over K^- without taking reducts iff (i)–(ii) below hold.

  (i) $(\forall \mathfrak{M} \in \mathsf{K})\mathfrak{M}$ is definable nr-implicitly over its reduct $\mathfrak{M}^-$.

  (ii) There is a single set Δ of formulas such that for every $\mathfrak{M} \in \mathsf{K}$, Δ is an implicit definition of $\mathfrak{M}$ over $\mathfrak{M}^-$. In other words, not only each $\mathfrak{M}$ is nr-implicitly definable over $\mathfrak{M}^-$, but this defining can be done uniformly for the whole of K.

(2) Further, assume K is implicitly definable without taking reducts over its reduct K^-. Then the function

$$\mathsf{rd} \stackrel{\text{def}}{=} \{\langle \mathfrak{M}, \mathfrak{M}^- \rangle : \mathfrak{M} \in \mathsf{K}\}$$

[895] We will explain soon, beginning with item 11 of Examples 6.3.9 (p.940), what aspect of the above situation we are referring to with the adjective "uniform" here.

is a bijection up to isomorphism[896]

$$\mathsf{rd} : \mathsf{K} \longrightarrow\!\!\!\!\!\succ \mathsf{K}^-$$

such that each $\mathfrak{M} \in \mathsf{K}$ is definable nr-implicitly over $\mathsf{rd}(\mathfrak{M})$ and these definitions coincide for all choices of $\mathfrak{M}$.

$\lhd$

Remark 6.3.7 (properties of "general" definability of classes) Assume K is definable implicitly over L. Then (1)-(2) below hold.

(1) K and L agree on their common vocabulary, i.e.

$$\mathsf{K} \upharpoonright (Voc\mathsf{K} \cap Voc\mathsf{L}) = \mathsf{L} \upharpoonright (Voc\mathsf{K} \cap Voc\mathsf{L}).$$

(2) There is a surjective function[897] $f : \mathsf{L} \longrightarrow\!\!\!\!\!\succ \mathsf{K}$ such that for all $\mathfrak{M} \in \mathsf{L}$, $f(\mathfrak{M})$ is implicitly definable over $\mathfrak{M}$[898]; moreover the definition of $f(\mathfrak{M})$ over $\mathfrak{M}$ is the same (set of formulas) for all choices of $\mathfrak{M}$.

$\lhd$

Now we turn to giving examples.

Examples 6.3.8 (Traditional, one-sorted examples)

1. Let PA be the class of models of Peano's Arithmetic, cf. any logic book, e.g. Monk [197] or Chang-Keisler [59] for PA. The operation symbols of PA are $+, \cdot, 0, 1$. Consider the extra unary operation symbol "!" intended to denote the factorial. Let $\Delta_!$ be the set of the following two formulas

   $!(0) = 1$
   $\forall x[!(x+1) = (x+1)\cdot!(x)].$

   I.e. $\Delta_! = \{\ !(0) = 1,\ \forall x[!(x+1) = (x+1)\cdot!(x)]\ \}$. We claim that $\Delta_!$ is a (correct) implicit definition of "!" over PA. (The proof is not easy but is available in almost any logic book.) The point in the above example is that PA is an axiomatizable class and that $\Delta_!$ works over each member of PA. If we want an implicit definition over a single model instead of an axiomatizable class, that is easy:

[896] i.e. $\mathsf{rd}(\mathfrak{M}) = \mathsf{rd}(\mathfrak{N}) \ \Rightarrow\ \mathfrak{M} \cong \mathfrak{N}$. Roughly, something holds "up to isomorphism" iff it holds modulo identifying some of the isomorphic models.
[897] f is a function only up to isomorphism, cf. footnote 941 on p.971 for more detail.
[898] i.e. there is an implicit definitional expansion $\mathfrak{M}^+$ of $\mathfrak{M}$ with $f(\mathfrak{M})$ a reduct of $\mathfrak{M}^+$.

2. Consider the model $\langle \omega, 0, \mathsf{suc}, + \rangle$.[899] Let Δ_+ be the set of the following formulas:

$$x + y = y + x$$
$$0 + x = x$$
$$x + \mathsf{suc}(y) = \mathsf{suc}(x + y).$$

Now, Δ_+ defines $+$ implicitly over the model $\langle \omega, 0, \mathsf{suc} \rangle$. However, it is important to note that over the axiomatizable hull $\mathsf{Mod}(\mathsf{Th}(\langle \omega, 0, \mathsf{suc} \rangle))$ of this model, Δ_+ is not an implicit definition[900], and moreover addition is not nr-implicitly definable in $\mathsf{Mod}(\mathsf{Th}(\langle \omega, 0, \mathsf{suc} \rangle))$.

This shows that nr-implicit definability over a single model is much weaker than nr-implicit definability over an axiomatizable class of models. (Since primarily we are interested in theories, and theories correspond to axiomatizable classes, we are more interested in definability over axiomatizable classes than over single models.)

3. Let $E = \{2 \cdot n : n \in \omega\}$ be the set of even numbers. Then E as a *unary relation* is definable nr-implicitly over the model $\langle \omega, \mathsf{suc} \rangle$.

4. Let BA_0 be the class of Boolean algebras with "$\cap$", "$\cup$", $0, 1$ as basic operations. Now, $\{x \cap -x = 0,\ x \cup -x = 1\}$ is an implicit definition of complementation over BA_0. This implicit definition, however, can easily be rearranged into the form of an explicit definition as follows[901]:

$$-(x) = y \quad \Leftrightarrow \quad [x \cap y = 0 \ \wedge \ x \cup y = 1].$$

5. We recommend that the reader experiment with (i) defining the Boolean partial ordering "$\leq$" over BA_0, (ii) defining "$\cup$" over the basic operations "$\cap, -$" (and the same with the roles of "$\cup$" and "$\cap$" interchanged).

6. The model $\langle \omega \ \leq \rangle$ is implicitly definable over $\langle \omega, 0, \mathsf{suc} \rangle$, but it is not nr-implicitly definable because $\langle \omega, \leq \rangle$ is not an expansion of $\langle \omega, 0, \mathsf{suc} \rangle$. If $\mathfrak{M}^+ = \langle \mathfrak{M}; \bar{R}^{new} \rangle$, i.e. if $\mathfrak{M}^+$ does not contain new sorts, then $\mathfrak{M}^+$ is nr-implicitly definable over $\mathfrak{M}$ iff $\mathfrak{M}^+$ is implicitly definable over $\mathfrak{M}$. This is not necessarily true when $\mathfrak{M}^+$ contains new sorts, too.

$\lhd$

[899]Where $\mathsf{suc} : \omega \longrightarrow \omega$ is the usual successor function on ω, i.e. $\mathsf{suc}(n) = n + 1$ for all $n \in \omega$.

[900]i.e. it does not satisfy $(\star\star)$ way above

[901]We have not discussed explicit definitions yet, but they will be discussed soon (beginning with §6.3.2 on p.944).

Examples 6.3.9 (More advanced, many-sorted examples)

1. Let $\mathfrak{F}$ be an ordered field. Then the two-sorted model $\langle \mathfrak{F}, \mathcal{P}(F); \in \rangle$ is *not definable implicitly up to isomorphism* over $\mathfrak{F}$. Hence it is not nr-implicitly definable, either.

   Proof-idea: Assume $|F| = \omega$. Then $|\mathcal{P}(F)| > \omega$. But by the downward Löwenheim-Skolem theorem $\langle \mathfrak{F}, \mathcal{P}(F); \in \rangle$ has an elementary submodel with each sort countable.

2. Let $\bar{R}$ be any countable sequence of relations defined on the sorts $F, \mathcal{P}(F)$ in example 1 above. Then
$$\langle \mathfrak{F}, \mathcal{P}(F); \in, \bar{R} \rangle$$
   is *not definable implicitly up to isomorphism* over $\mathfrak{F}$.

   Hint: The reason remains the same as in example 1.

   This means that $\mathfrak{F}^+ \stackrel{\text{def}}{=} \langle \mathfrak{F}, \mathcal{P}(F); \in \rangle$ is *not implicitly definable* over $\mathfrak{F}$, either.

   However, there is an expansion $\mathfrak{F}^{++}$ of $\mathfrak{F}^+$ with uncountably many new relations such that $\mathfrak{F}^{++}$ is nr-implicitly definable over $\mathfrak{F}$. Indeed, let us take a new constant c_x for each element x of $F \cup \mathcal{P}(F)$. Then $\mathfrak{F}^{++} \stackrel{\text{def}}{=} \langle \mathfrak{F}, \mathcal{P}(F), \in, \langle c_x : x \in F \cup \mathcal{P}(F) \rangle \rangle$ is an nr-implicitly definable expansion of $\mathfrak{F}$. This shows the importance of allowing only finitely many relation symbols in our languages when defining implicit definability, cf. p.936.

3. Let $\mathbf{F}$ be a finite field. Then $\langle \mathbf{F}, \mathcal{P}(F); \in \rangle$ *is definable nr-implicitly* over $\mathbf{F}$. The same applies for any finite structure in place of $\mathbf{F}$.

   Notation: For any set H and cardinal κ we let $\mathcal{P}_\kappa(H)$ be the collection of those subsets of H whose cardinality is smaller than κ. In particular, $\mathcal{P}_\omega(H)$ denotes the set of finite subsets of H.

4. Let $\mathfrak{A}$ be a(n infinite) structure with universe A. Then $\langle \mathfrak{A}, \mathcal{P}_i(A); \in \rangle$ *is nr-implicitly definable* over $\mathfrak{A}$ for any $i \in \omega$.

5. Let $\mathfrak{A} = \langle \omega, \leq \rangle$ be the set of natural numbers with the usual ordering. Then the expansion $\langle \mathfrak{A}, \mathcal{P}_\omega(\omega); \in \rangle$ *is nr-implicitly definable* over $\mathfrak{A}$.

   Hint: An implicit definition is the following set of formulas:

$$\{\forall x_1 \dots x_n \in \omega \exists y \in \mathcal{P}_\omega(\omega) y = \{x_1, \dots, x_n\} : n \in \omega\} \cup$$
$$\{\forall y \in \mathcal{P}_\omega(\omega) \exists x \in \omega \forall z \in \omega (z \in y \to z \leq x)\} \cup$$

$\{\forall y, z \in \mathcal{P}_\omega(\omega)(y = z \leftrightarrow \forall x \in \omega(x \in y \leftrightarrow x \in z))\}$.

(In the above, $y = \{x_1, \ldots, x_n\}$ abbreviates any formula with the intended meaning.)

As a contrast, we include the following example.

6. Consider the expansion $\langle \omega, \mathcal{P}_\omega(\omega); \in\rangle$ of the "plain" structure $\langle\omega\rangle$. Then this structure (i.e. $\langle \omega, \mathcal{P}_\omega(\omega); \in\rangle$) *is not implicitly definable up to isomorphism* over $\langle\omega\rangle$.

   Hint: Take any countable elementary submodel $\mathfrak{B}$ of an ultrapower of $\langle \omega, \mathcal{P}_\omega(\omega); \in\rangle$ which contains a "nonstandard" element in $\mathcal{P}_\omega(\omega)$. Then the "$\omega$-part" of $\mathfrak{B}$ is isomorphic to $\langle\omega\rangle$, but $\mathfrak{B}$ is not isomorphic to $\langle \omega, \mathcal{P}_\omega(\omega); \in\rangle$.

7. $\langle \omega, \mathcal{P}_\omega(\omega); \mathsf{suc}, \in\rangle$ is implicitly definable over $\langle \omega, \mathsf{suc}\rangle$. We do not know whether it is nr-implicitly definable over $\langle \omega, \mathsf{suc}\rangle$ or not. (We conjecture that the answer is in the negative.)

8. $\langle \mathfrak{A}, U^{new}\rangle$ *is not implicitly definable up to isomorphism* over $\mathfrak{A}$, for any structure $\mathfrak{A}$ and infinite set U^{new}. Here U^{new} is a new sort, and there are no new relations. If $1 < |U^{new}| < \omega$, then U^{new} is implicitly definable and implicitly definable up to isomorphism, but not implicitly definable without taking reducts. If $|U^{new}| \leq 1$, then U^{new} is implicitly definable without taking reducts.

9. Let $\mathfrak{A}$ be any structure and let $\mathfrak{B}$ be any finite structure. Then $\langle \mathfrak{A}, \mathfrak{B}\rangle$ as a two-sorted structure is impicitly definable over $\mathfrak{A}$.

10. Let $\mathfrak{A}$ be a fixed structure. Consider

$$\mathsf{K} = \{\, \langle \mathfrak{A};\, U^{new}\rangle \;:\; |U^{new}| < \omega \,\}.$$

   Then K is *not nr-implicitly definable* over $\{\mathfrak{A}\}$ (not even up to isomorphism).

Understanding the examples below is *not* a prerequisite for understanding the rest of the present work. (They concern the distinction between uniform and non-uniform definability.)

11. For $k \in \omega$, let $\mathfrak{L}_k$ be the usual $k+1$ element linear ordering $\mathfrak{L}_k = \langle\{0, \ldots, k\}, <\rangle$ where "$<$" is the usual ordering of the natural numbers. Recall from set theory that $\aleph_k$ is the k'th infinite cardinal regarded as a special ordinal. E.g. $\aleph_0 = \omega$. Let

$$\mathsf{K} := \{\, \langle \aleph_k, \mathfrak{L}_k\rangle \;:\; k \in \omega \,\}$$

where $\langle U^{new}, \bar{R}^{new} \rangle = \mathfrak{U}_k$. I.e. K^- is obtained by forgetting the $\mathfrak{U}_k$-part. Then K is *not* uniformly nr-implicitly definable over K^- although for each $\mathfrak{M} \in \mathsf{K}$, we have that $\mathfrak{M}$ is nr-implicitly definable over $\mathfrak{M}^-$, i.e. $\mathfrak{U}_k$ is nr-implicitly definable over $\langle \aleph_k \rangle$.

12. The following is a generalization of item 11 above. Let $\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots$ $(k \in \omega)$ be *any* ω-sequence of elementarily equivalent one-sorted models.[902] Let $\mathfrak{U}_k$ be as in item 11 above.

$$\mathsf{K} := \{ \langle \mathfrak{A}_k, \mathfrak{U}_k \rangle \ : \ k \in \omega \} .$$

Then K is *not uniformly nr-implicitly definable* over $\mathsf{K}^- = \{ \mathfrak{A}_k \ : \ k \in \omega \}$ while every $\mathfrak{M} \in \mathsf{K}$ *is nr-implicitly definable* over $\mathfrak{M}^-$.

Hint: The key idea can be formulated with using $\mathfrak{A}_1$, $\mathfrak{A}_2$ only. The rest of the $\mathfrak{A}_k$'s serve only as decoration. So, one starts with $\mathfrak{A}_1 \equiv_{ee} \mathfrak{A}_2$ and[903] $|U_1| \neq |U_2|$ are finite. (Where U_i is the universe of $\mathfrak{U}_i$, similarly for A_i.) It is important to note that there are no inter-sort relations permitted here i.e. sort A_i is isolated from sort U_i. Next, one uses the following property of many-sorted logic. Assume $\mathfrak{A}, \mathfrak{B}$ are two structures of *disjoint languages*. Consider the new many-sorted structure $\langle \mathfrak{A}, \mathfrak{B} \rangle$. We claim that $\mathsf{Th}(\langle \mathfrak{A}, \mathfrak{B} \rangle) = \mathsf{Th}(\mathfrak{A}) \cup \mathsf{Th}(\mathfrak{B})$. The reason for this is the fact that an atomic formula xRy belongs to a many-sorted language *only* if x and y are of the *same* sort. Hence e.g. $(\exists x \in U_0)(\exists y \in U_1)\, x \neq y$ is not a (many-sorted) formula.

The present example does not work for "implicitly definable" in place of "implicitly definable without taking reducts".

Someone might think that the reason why the above counterexample works is that all elements of K^- are elementarily equivalent. Below we show that this is *not* the case.

13. Let the language of K^- consist of countably many constant symbols $c_0, \dots, c_i, \dots$ and just one sort. Let $\mathfrak{U}_k$ $(k \in \omega)$ be as in item 11 above.

$$\begin{aligned} \mathsf{K}^- &:= \{ \langle U, c_i \rangle_{i \in \omega} \ : \ \text{the set } \{ i \in \omega \ : \ c_i = c_0 \} \text{ is finite and} \\ &\qquad\qquad U \text{ is a set with } (\forall i \in \omega)\, c_i \in U \} . \\ \mathsf{K} &:= \{ \langle U, c_i ; \mathfrak{U}_k \rangle_{i \in \omega} \ : \ k = |\{ i \in \omega \ : \ c_i = c_0 \}| \text{ and } \langle U, c_i \rangle_{i \in \omega} \in \mathsf{K}^- \} . \end{aligned}$$

That is

[902] I.e. $(\forall k \in \omega)\, \mathsf{Th}(\mathfrak{A}_0) = \mathsf{Th}(\mathfrak{A}_k)$.

[903] Recall that $\equiv_{ee}$ denotes the binary relation of elementary equivalence defined between models.

$$\mathsf{K} \quad = \quad \{\, \langle \mathfrak{M}; \mathfrak{U}_k \rangle \ : \ \mathfrak{M} \in \mathsf{K}^- \text{ and } k = |\, \{\, i \in \omega \ : \ \text{in } \mathfrak{M} \text{ we have } c_i = c_0 \,\} \,| \,\}.$$

Now, K is *not uniformly nr-implicitly definable* over K^- *while* each concrete $\mathfrak{M} \in \mathsf{K}$ *is nr-implicitly definable* over $\mathfrak{M}^-$, *further*

$$(\forall \mathfrak{M}, \mathfrak{N} \in \mathsf{K})[\, \mathfrak{M}^- \equiv_{ee} \mathfrak{N}^- \quad \Rightarrow \quad \mathfrak{M} \equiv_{ee} \mathfrak{N} \,].$$

Idea for a proof:

Assume $\Delta = \mathsf{Th}(\mathsf{K})$ defines K implicitly over K^- (up to isomorphisms). Then by using ultraproducts one can show that there is $\mathfrak{N} = \langle U, c_i; \mathfrak{U}_2 \rangle_{i\in\omega} \in \mathsf{Mod}(\Delta)$ such that $(\forall i > 0)(c_i \neq c_0$ holds in $\mathfrak{N})$. But clearly for $\mathfrak{M} := \langle \mathfrak{N}^-; \mathfrak{U}_1 \rangle$ we have $\mathfrak{N}^- = \mathfrak{M}^- \in \mathsf{K}^-$ and $\mathfrak{M} \in \mathsf{K}$ hence by $\mathfrak{M} \not\cong \mathfrak{N}$ we conclude that Δ cannot be a definition of K.

$\lhd$

The above three examples were designed to illustrate the difference between uniform (nr-implicit) definability and *one-by-one (nr-implicit) definability* where by the latter we understand the case when each $\mathfrak{M} \in \mathsf{K}$ is definable over its reduct $\mathfrak{M}^-$ in K^- (but these definitions might be different for different choices of $\mathfrak{M}$);in more detail: Let K be a class of models with U^{new}, $\bar{R}^{new}$ in the language of K. For $\mathfrak{M} \in \mathsf{K}$ let $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ obtained by omitting (forgetting) U^{new}, $\bar{R}^{new}$. Let $\mathsf{K}^- := \{\, \mathfrak{M}^- \ : \ \mathfrak{M} \in \mathsf{K} \,\}$. Then we say that K is _one-by-one nr-implicitly definable_ over K^- iff each $\mathfrak{M} \in \mathsf{K}$ is nr-implicitly definable over its reduct $\mathfrak{M}^- \in \mathsf{K}^-$. Sometimes, informally we will use instead of one-by-one definability _“non-uniform”_ or _“local”_ definability as synonyms. We hope that the above three examples illustrate (the generally accepted opinion) that uniform definability is a more useful concept than one-by-one definability (when considering classes K of models) and is closer to what one would intuitively understand under definability.

For completeness, we refer the interested reader to the distinction between the *“local”* and the “usual” versions of explicit definability described in Andréka-Németi-Sain [30] Definitions 55–56 (Beth definability properties) therein. We also note that most standard textbooks concentrate on uniform definability only and they do not mention what we call here one-by-one definability. We too will concentrate on uniform definability and unless otherwise specified, by _definability_ we will always understand _uniform definability_.

Remark 6.3.10 A useful refinement of the notion of nr-implicit definability is *finite nr-implicit definability.* Assume K and K^- are as above statement $(\star\star)$ on p.935 (definition of nr-implicit definability). Assume, K is nr-implicitly definable over K^-.

Then K is said to be *finitely nr-implicitly definable* over K^- iff there is a finite set $\Delta_0 \subseteq \mathsf{Th}(\mathsf{K})$ of formulas such that Δ_0 defines K implicitly over K^-, i.e. ($\star\star$) holds for $\Delta = \Delta_0$. In most of our concrete examples and applications we will have *finite* nr-implicit definability, but for simplicity we will write just "definability".

To illustrate the importance of finite nr-implicit definability, consider the simple model $\langle \omega, \mathsf{suc} \rangle$. There are continuum many different implicit definitions (involving one new relation symbol R) over this model while there are only countably many finite implicit definitions (and we will see that there are only countably many explicit definitions over this model). (This example cannot be generalized from a single model like $\mathfrak{M} = \langle \omega, \mathsf{suc} \rangle$ to first-order-axiomatizable classes K of models, assuming there are only finitely many sorts).[904]

$\triangleleft$

6.3.2 Explicit definability in many-sorted (first-order) logic

So far we have discussed implicit definability which is a quite general notion of definability. Below we will turn to a special kind of implicit definability which we call *explicit definability*. Each explicit definition can be considered as an implicit definition. The other direction is not true however, there are implicit definitions which are not explicit definitions. (I.e. there is an implicit definition Δ which in its *given form* is not an explicit definition.) In definability theory, the connection between explicit and implicit definitions is an important subject. We will return to this subject at the end of the "definability" section (§6.3). In particular, we will state a generalization of Beth's theorem, saying that implicit definability is equivalent with explicit definability (even in our general framework where we allow definitions of new sorts, too [besides definitions of new relations], cf. Theorem 6.3.32 and Corollary 6.3.33 on p.977.

Explicit definability will turn out to be (i) a special case of implicit definability and (ii) a strong and useful concept e.g. in the following way. Assume $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ and that K^+ is an expansion of K which is explicitly definable

[904] The reason for this is the following. In the above reasoning we heavily used the fact that every element of $\langle \omega, \mathsf{suc} \rangle$ is definable "as a constant". (Therefore infinite implicit definitions can be given by listing the elements of R and the non-elements of R.) This does not remain true in $\mathsf{Mod}(\mathsf{Th}(\langle \omega, \mathsf{suc} \rangle))$.

over the class K of models. Then the theories $\mathsf{Th}(\mathsf{K})$ and $\mathsf{Th}(\mathsf{K}^+)$ as well as the languages of K and K^+ will be seen to be equivalent in a rather strong sense to be explained later, see Theorems 6.3.26 and 6.3.27 on p.962.

The key ingredients of explicit definability will be introduced in items (1)–(2.2) below. Then, on p.950, they will be combined into a description of what we mean by explicit definability. The generalization from definability over single models $\mathfrak{M}$ to definability over classes K of models will be given on p.950.

Notation: Assume $\mathfrak{M}$ is a many-sorted model and that ψ is a formula in the language of $\mathfrak{M}$ such that all the free variables of ψ belong to $x_0, \ldots, x_i, \ldots$. Assume $\bar{a} \in {}^{\omega}Uv(\mathfrak{M})$ and that the sort of a_i coincides with the sort of the variable x_i, for every $i \in \omega$. Then

$$\mathfrak{M} \models \psi[\bar{a}]$$

is the standard model theoretic notation for the statement that ψ is true in $\mathfrak{M}$ under the *evaluation* $\bar{a}$ of its free variables cf. e.g. Monk [197], Enderton [82], Chang-Keisler [59]. Sometimes we write $\mathfrak{M} \models \psi[a_1, \ldots, a_n]$ in which case it is understood that the free variables of ψ are among $x_1, \ldots, x_n$. The latter is often indicated by writing $\psi(x_1, \ldots, x_n)$ instead of ψ. I.e. if we write $\psi(x_1, \ldots, x_n)$ in place of ψ then this means that while talking about the formula ψ we want to indicate casually that the free variables of ψ are among $x_1, \ldots, x_n$.

The following is also a standard notation from logic. Assume τ is a term. Then $\psi(x/\tau)$ denotes the formula obtained from ψ by replacing all free occurrences of x by τ. Similarly for $\psi(x_1/\tau_1, \ldots, x_n/\tau_n)$. We could say that "$(x/\tau)$" is the "operator" of substituting τ for x.

If $\psi(x)$ is a formula and y is a variable (of the same sort as x), then $\psi(y)$ denotes $\psi(x/y)$; and similarly for a sequence $\bar{x}$ of variables.

Sometimes below we will write "definable" for "explicitly definable" to save space. Similarly, we write "definitional expansion" for "explicit definitional expansion". In general, we will tend to omit the adjective "explicit", because our primary interest will be explicit definability.

(1) Explicit definability of relations and functions in $\mathfrak{M}$.

Let $\mathfrak{M} = \langle U_0, \ldots, U_j;\ R_1, \ldots, R_l \rangle$ be a many-sorted model with universes or sorts $U_0, \ldots, U_j$, and relations $R_1, \ldots, R_l$. Let $R^{new} \subseteq U_{i_1} \times \ldots \times U_{i_m}$ be a (new) relation, with $i_1, \ldots, i_m \in (j+1)$. Now, R^{new} is called *(explicitly) definable* (as a relation) over $\mathfrak{M}$ iff there is a formula $\psi(x_{i_1}, \ldots, x_{i_m})$ in the language of $\mathfrak{M}$ such that

$$R^{new} = \{\, \langle a_{i_1}, \ldots, a_{i_m} \rangle \in U_{i_1} \times \ldots \times U_{i_m} \ :\ \mathfrak{M} \models \psi[a_{i_1}, \ldots, a_{i_m}] \,\}\,.$$

Such definable relations can be added to $\mathfrak{M}$ as new basic relations obtaining a(n explicit) *definitional expansion* of $\mathfrak{M}$ in the form

$$\mathfrak{M}^+ := \langle U_0, \ldots, U_j;\ R_1, \ldots, R_l, R^{new} \rangle.$$

To make $\mathfrak{M}^+$ "well defined" we have to add a *new relation symbol* to the language of $\mathfrak{M}$ denoting R^{new}. The formula $R^{new}(\bar{x}) \leftrightarrow \psi(\bar{x})$ is called an *(explicit) definition* of R^{new} (over $\mathfrak{M}$). Notice that $\Delta \stackrel{\text{def}}{=} \{R^{new}(\bar{x}) \leftrightarrow \psi(\bar{x})\}$ is also a(n implicit) definition of $\mathfrak{M}^+$ over $\mathfrak{M}$. We call Δ an *explicit definition of type (1)*. If Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then we say that *$\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by step (1)*. Note that if $\mathfrak{M}'$ is defined over $\mathfrak{M}$ by Δ, then $\mathfrak{M}'$ is $\mathfrak{M}^+$ above.

(2) Explicit definability of new sorts (i.e. universes) in $\mathfrak{M}$.

Defining a new sort explicitly (in $\mathfrak{M}$) takes a bit more care than defining a new relation. This is understandable, since now we want to define (or create) a *new universe* of entities (in terms of the old universes and old relations already available in $\mathfrak{M}$) while when defining a relation we defined only a new property of *already existing* entities (or of tuples of such entities) in $\mathfrak{M}$. If we define a new relation, then this amounts to defining a new property of already existing entities. I.e. we remain on the same *ontological level*. In contrast, if we define *new entities* which "did not exist" before, then we go up to a higher ontological level.[905]

If we want to define a new sort in $\mathfrak{M}$, first of all we need a new sort-symbol, say U^{new}, which does not yet occur in the language of $\mathfrak{M}$. If there is no danger of confusion then we will *identify* a sort-symbol like U^{new} with the universe, say $(U^{new})^{\mathfrak{M}^+}$, which it denotes in a model $\mathfrak{M}^+$.

An explicit definition of a new sort, say U^{new}, describes the elements of U^{new} as being constructed from "old" elements in a systematic, "tangible" and uniform way. More concretely, first we will introduce a few (basic constructions or) basic kinds of explicit definition and then "general" explicit definitions will be obtained by iterating these basic kinds. We will refer to the just mentioned basic kinds (of explicit definition) as basic steps of explicit definitions. Our basic steps for building up explicit definitions of new sorts are described in items (2.1), (2.2) below. Our choice of basic steps might look ad-hoc at first reading, but Theorem 6.3.32 at the end of this section will say that our selected few steps (i.e. examples of explicit definitions) cover (via iteration) all cases of implicit definitions (assuming there is a

[905] In connection with defining new sorts, for completeness, we also refer e.g. to the definition of the "new" many-sorted structure A^{eq} from the "old" structure A in Hodges [136, p.151] (cf. also pp. 148, 212, 213 therein). Cf. also the definition of relative categoricity in Hodges [136] p.638 together with p.638 line 3 bottom up to p.639 line 9.

sort with more than one elements). We will return to a more careful discussion of the present issue of choosing our basic steps in Remark 6.3.37.

(2.1) The first way of defining a new sort U^{new} in $\mathfrak{M}$ explicitly.

The simplest way of defining a new sort U^{new} in a model $\mathfrak{M} = \langle U_0, \ldots, U_j;\ R_1, \ldots, R_l \rangle$ is the following. Let $R \in \{R_1, \ldots, R_l\}$ be fixed. Assume R is an r-ary relation, i.e. $R \subseteq {}^r Uv(\mathfrak{M})$. We want to postulate that U^{new} coincides with R. So the first part of our definition of U^{new} is the postulate:

$$U^{new} :\stackrel{\text{def}}{=} R.$$

But, if we want to expand $\mathfrak{M}$ with U^{new} as a new sort obtaining something like

$$\mathfrak{M}' := \langle U_0, \ldots, U_j, U^{new};\ R_1, \ldots, R_l \rangle$$

then we *need* some new relations or functions *connecting* the new sort U^{new} to the old ones $U_0, \ldots, U_j$. In our present case (of $U^{new} = R$) we use the <u>*projection functions*</u> $pj_i : R \longrightarrow Uv(\mathfrak{M})$ with $i < r$. Formally,

$$pj_i(\langle a_0, \ldots, a_{r-1} \rangle) :\stackrel{\text{def}}{=} a_i.$$

To identify the domain of pj_i we should write something like pj_i^R, but for brevity we omit the superscript R. Now, the (explicit) *definitional expansion* of $\mathfrak{M}$ obtained by the choice $U^{new} := R$ is

$$\mathfrak{M}^+ := \langle U_0, \ldots, U_j, U^{new};\ R_1, \ldots, R_l, pj_0, \ldots, pj_{r-1} \rangle = \langle \mathfrak{M}, U^{new};\ pj_i \rangle_{i<r}.$$

We note that

$$\mathfrak{M}^+ = \langle U_0, \ldots, U_j, R;\ R_1, \ldots, R_l, pj_i^R \rangle_{i<r}.$$

If x is a variable, then $(\exists\,!x)\psi(x)$ denotes the formula expressing that there is exactly one value for which ψ holds, i.e. it denotes the formula $(\exists x)(\psi(x) \wedge (\forall z)[\psi(z) \rightarrow z = x])$. Let

$$\begin{aligned} \Delta \stackrel{\text{def}}{=} \ & \{(\exists\,!u \in U^{new})(pj_1(u, x_1) \wedge \ldots \wedge pj_r(u, x_r)) \leftrightarrow R(x_1, \ldots, x_r)\ , \\ & (\exists u \in U^{new})(pj_1(u, x_1) \wedge \ldots \wedge pj_r(u, x_r)) \rightarrow R(x_1, \ldots, x_r)\ , \\ & (\forall u \in U^{new})(\exists\,!x_i)pj_i(u, x_i) : 1 \le i \le r\}\ . \end{aligned}$$

Then Δ is an implicit definition of $\mathfrak{M}^+$ over $\mathfrak{M}$. We call Δ an <u>*explicit definition of type (2.1)*</u>. If Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then we say that <u>*$\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by Step (2.1)*</u>. Notice that if Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}^+$ above via an isomorphism which is identity on $\mathfrak{M}$.

Remark 6.3.11 This second form of $\mathfrak{M}^+$ might induce the (misleading) impression that $\mathfrak{M}^+$ contains nothing new: it consists of a rearranged version of the old parts of $\mathfrak{M}$. However, let us notice that as a first step we might define a new relation R^{new} in $\mathfrak{M}$ (in the style of item (1) above) obtaining

$$\mathfrak{M}^+ := \langle U_0, \ldots, U_j;\ R_1, \ldots R_l, R^{new}\rangle$$

and *then* we may define $U^{new} := R^{new}$ obtaining the *definitional expansion*

$$\mathfrak{M}^{++} := \langle U_0, \ldots, U_j, U^{new};\ R_1, \ldots, R^{new}, pj_i\rangle_{i<r}$$

of $\mathfrak{M}^+$. Now, we will *postulate* that a definitional expansion of a definitional expansion of $\mathfrak{M}$ is called a definitional expansion of $\mathfrak{M}$ again. Hence the above obtained $\mathfrak{M}^{++}$ is a definitional expansion of the original $\mathfrak{M}$. Using our abbreviation from p.930 we can write:

$$\langle \mathfrak{M}, U^{new};\ R^{new}, pj_i\rangle_{i<r} := \mathfrak{M}^{++}.$$

Now, *if* we do not want to have R^{new} as a relation, we can take the reduct

$$\mathfrak{M}^{++-} := \langle \mathfrak{M}, U^{new};\ pj_i\rangle_{i<r}$$

by forgetting R^{new} as a relation but not as a sort. We will call $\mathfrak{M}^{++-}$ a *generalized definitional expansion* of $\mathfrak{M}$ (cf. p.950).

$\lhd$

Example 6.3.12 Let $\mathbf{F} = \langle F, \ldots, \cdot\ \rangle$ be a field. We want to define the plane $F \times F$ over $\mathbf{F}$ as a new *sort* expanding $\mathbf{F}$. First we define the *relation* $R = F \times F$ by the formula $(x_0 = x_0\ \wedge\ x_1 = x_1)$. Clearly, in $\mathbf{F}$ this formula defines the relation $F \times F$. Then we expand $\mathbf{F}$ with this as a new relation obtaining

$$\mathbf{F}^+ = \langle F;\ +, \cdot\ , F \times F\rangle$$

where $F \times F$ is used as a relation interpreting the relation symbol $Rel_{F\times F}$. Now, in $\mathbf{F}^+$ we define the *new sort* $U^{new} := F \times F$ together with the projection functions as indicated above, obtaining the model

$$\mathbf{F}^{++} = \langle F, F \times F;\ +, \cdot\ , F \times F, pj_0, pj_1\rangle$$

where $pj_i : F \times F \longrightarrow F$. Now, we take a reduct of $\mathbf{F}^{++}$ by forgetting the relation symbol $Rel_{F\times F}$, but not the sort $F \times F$. We obtain

$$\mathbf{F}^{++-} = \langle F, F \times F;\ +, \cdot\ , pj_0, pj_1\rangle = \langle \mathbf{F}, F \times F;\ pj_0, pj_1\rangle.$$

Clearly this model $\mathbf{F}^{++-}$ is the expansion of the field $\mathbf{F}$ with the plane $F \times F$ as a new sort as we wanted.

The above example shows that the usual expansion of $\mathbf{F}$ with the plane as a *new sort*, is indeed a definitional expansion i.e. the plane as a new sort is *(first-order) definable explicitly* in $\mathbf{F}$.

$\triangleleft$

Similarly to the above example, ${}^nF$ is first-order definable (explicitly) as a *new sort* in any frame model $\mathfrak{M}$. Later we will introduce uniform explicit definability over a class K of models. Then we will see that ${}^nF$ as a new sort is uniformly (explicitly) definable over the class of all frame models. (In defining ${}^nF$ we use $pj_i : {}^nF \longrightarrow F$, $i \in n$, the same way as we did in the case of $\mathbf{F}^{++-}$.)

(2.2) The second way of defining a new sort U^{new} in $\mathfrak{M}$ explicitly.

To define a new sort U^{new} in a model $\mathfrak{M} = \langle U_0, \dots, U_j; R_1, \dots, R_l \rangle$ explicitly the second way, we begin by selecting an old sort $U := U_i$ and old relation $R := R_k$ ($i \le j$, $0 < k \le l$) in $\mathfrak{M}$. We proceed *only if* R happens to be an equivalence relation over U (i.e. if $R \subseteq U \times U$ etc.). We define the new sort to be the quotient set of R-equivalence classes[906]

$$U^{new} := U/R.$$

Again, similarly to the case of pj_i's in item (2.1) above, we need a new relation connecting the new sort U^{new} to the old ones. Now we choose the set theoretic membership relation

$$\in \; := \; \in_{U^{new}} \; := \; \in_{U,U^{new}} \; := \{ \langle a, a/R \rangle \; : \; a \in U \}$$

acting between U and U/R. Since $\in_{U^{new}} \subseteq U_i \times U^{new}$, this relation connects the new sort U^{new} with the old one U_i. Let us notice that from the notation $\in_{U,U^{new}}$ we may omit the first index obtaining the simpler notation $\in_{U^{new}}$ or we may omit both indices obtaining $\in$. The (explicit) *definitional expansion* of $\mathfrak{M}$ obtained by the choice $U^{new} = U_i/R_k$ is defined to be the model

$$\begin{aligned} \mathfrak{M}^+ &= \langle U_0, \dots, U_j, U^{new}; \; R_1, \dots R_l, \in_{U^{new}} \rangle \\ &= \langle U_0, \dots, U_i/R_k; \; R_1, \dots, R_l, \in \rangle \\ &= \langle \mathfrak{M}, U^{new}; \; \in_{U^{new}} \rangle \\ &= \langle \mathfrak{M}, U_i/R_k; \; \in_{U^{new}} \rangle. \end{aligned}$$

Let

[906] $U/R \stackrel{\text{def}}{=} \{a/R : a \in U\}$ where $a/R \stackrel{\text{def}}{=} \{b \in U : \langle a, b \rangle \in R\}$. I.e. U/R is the set of all "blocks" of R, and a/R is the "block" of R a is in.

$$\Delta \stackrel{\text{def}}{=} \{(\exists u \in U^{new})(\in(x,u) \land \in(y,u)) \leftrightarrow R(x,y),$$
$$[\in(x,u) \land \in(x,v)] \to u = v \quad \}.$$

Then Δ is an implicit definition of $\mathfrak{M}^+$ over $\mathfrak{M}$. We call Δ an *explicit definition of type (2.2)*. If Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then we say that *$\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by Step (2.2)*. Notice that if Δ is a definition of $\mathfrak{M}'$ over $\mathfrak{M}$, then $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}^+$ above via an isomorphism which is identity on $\mathfrak{M}$.

* * *

We are ready for defining our notion of explicit definability. We call a new *sort or relation (explicitly) definable in* $\mathfrak{M}$ iff it is definable by repeated applications of the steps described in items (1), (2.1), (2.2) above.

A model $\mathfrak{N}$ is called a *definitional expansion of* $\mathfrak{M}$ iff $\mathfrak{N}$ is obtained from $\mathfrak{M}$ by repeated applications of steps (1), (2.1), (2.2) above (involving finitely many steps only). An *explicit definition of $\mathfrak{N}$ over $\mathfrak{M}$* is the union of the explicit definitions of type (1), (2.1), (2.2) involved in a sequence leading from $\mathfrak{M}$ to $\mathfrak{N}$. We call Δ an *explicit definition* if Δ is an explicit definition of some definitional expansion.

A model $\mathfrak{N}$ is called a *generalized definitional expansion* of $\mathfrak{M}$ if (i), (ii) below hold.

(i) $\mathfrak{N}$ is a reduct of a definitional expansion, say $\mathfrak{M}^+$, of $\mathfrak{M}$.

(ii) $\mathfrak{N}$ is an expansion of $\mathfrak{M}$, i.e. $\mathfrak{M}$ is a reduct of $\mathfrak{N}$.

We call $\mathfrak{N}$ *(explicitly) definable in* $\mathfrak{M}$ iff item (i) above holds. If we want to indicate that we do not take a reduct while defining say $\mathfrak{M}^+$ from $\mathfrak{M}$ explicitly (i.e. that $\mathfrak{M}^+$ is obtainable by repeatedly applying steps (1), (2.1), (2.2) to $\mathfrak{M}$) then we say that $\mathfrak{M}^+$ is *explicitly definable* in $\mathfrak{M}$ *without taking reducts*. Sometimes we write *"definitional expansion without taking reducts"* to emphasize that we mean definitional expansion and not generalized definitional expansion.

We emphasize that a precise statement claiming that U^{new} is definable as a new sort should also mention the relations and/or functions (of $\mathfrak{N}$) connecting U^{new} to the original sorts of $\mathfrak{M}$. Examples for such "connecting relations" are pj_i and $\in_{U^{new}}$ discussed above.

We note that *explicit definability with parameters* is completely analogous with implicit definability with parameters cf. p.935.

Let us turn to (explicit) definability over a *class* K of models (instead of over a single model $\mathfrak{M}$). We say that K is a(n explicit) *definitional expansion* of its reduct K^- iff K can be obtained from K^- by (a finite sequence of) repeated (uniform) appications of the steps described in items (1), (2.1), (2.2) on pp.945–950. This is equivalent to saying that there is an explicit definition which defines K over K^- (as an implicit definition). In this case we also say that K *is (explicitly) definable over (or in)* K^- *without taking reducts*. We say that K is a *generalized definitional expansion* of K^- if K is an expansion of K^- and K is a reduct of a definitinal expansion of K^-. We say that K is *(explicitly) definable in* L if K is a reduct of a definitional expansion of L.

This is completely analogous with the case of implicit definability. *Uniform (explicit) definability* and *one-by-one (explicit) definability* are obtained from the notion of (explicit) definability for single models the same way as their counterparts were obtained in the case of implicit definability, cf. pp. 937, 943.

Finally, we introduce one more notion of definability which we will call *rigid definability.* We will use this in our examples to come. About the importance of this notion see Theorem 6.3.28 on p.969.

Assume $\mathfrak{M}^+ = \langle \mathfrak{M}, \bar{U}^{new}; \bar{R}^{new} \rangle$ is an expansion of $\mathfrak{M}$ (with new sorts and relations). We say that $\mathfrak{M}^+$ is *(explicitly) rigidly definable over* $\mathfrak{M}$ if $\mathfrak{M}^+$ is definable in $\mathfrak{M}$ and the identity is the only automorphism of $\mathfrak{M}^+$ which is the identity on $\mathfrak{M}$. Informally, we will say that the new sorts and relations $\bar{U}^{new}, \bar{R}^{new}$ are rigidly definable over $\mathfrak{M}$ if $\langle \mathfrak{M}; \bar{U}^{new}, \bar{R}^{new} \rangle$ is rigidly definable over $\mathfrak{M}$.

Further, K^+ is *rigidly definable over* K iff K^+ is a generalized definitional expansion of K and each $\mathfrak{M}^+ \in \mathsf{K}^+$ is rigid(ly definable) over its K-reduct.

In our opinion, rigid definability is "just as good" as definability without taking reducts. In other words, we feel that if $\bar{U}^{new}$ etc. are rigidly definable over K then $\bar{U}^{new}$ etc. are almost as well determined by K (or describable in K) as if they were definable without taking reducts. We note that rigid definability seems to be perhaps, our most important (or most central) version of definability[907] (cf. e.g. Theorem 6.3.27, Theorem 6.3.28 and Theorem 6.3.32).

CONVENTION 6.3.13 Assume K^+ is a definitional expansion of K. For $\mathfrak{M}^+ \in \mathsf{K}^+$ the reduct $\mathfrak{M}^+ \upharpoonright Voc\mathsf{K}$ may have more than one definitional expansions in K^+. (However these expansions are isomorphic.) Therefore K may have several different definitional expansions $\mathsf{K}^\oplus$ with the same set of defining formulas say Δ which defines K^+ from K. In such cases, of course we have $\mathbf{I}\mathsf{K}^\oplus = \mathbf{I}\mathsf{K}^+$. The largest such class is called a *maximal* definitional expansion of K. Since most of the time we will

[907] Our definition of K^+ being explicitly definable over K is strongly related to the notion of K^+ being "*coordinatisable over*" K as defined in Hodges [136, p.644], while K^+ is rigidly definable over K is strongly related to "*coordinatised over*" as defined in [136] (same page). We will return to discussing this connection in the sub-section beginning on p.976.

be interested in classes of models *closed under isomorphisms*, sometimes, but not always, we will concentrate on maximal definitional expansions. There are important exceptions to this[908], e.g. the class of two-sorted geometries[909] is not closed under isomorphisms and despite of this we will say that it is a definitional expansion of the class of one-sorted geometries (in Tarski's sense), under some conditions of course. $\triangleleft$

Remark 6.3.14 In Convention 6.3.13 above, and in the definition of definitional equivalence "$\equiv_\Delta$" (p.969) way below, we are "navigating around" two different trends both present in the present work (i.e. we are trying to make the consequences of these two trends "consistent" with each other). These are the following.

Trend 1. When discussing definability over $\mathfrak{M}$ or over K, what we are really interested in is definability over $\mathbf{I}\{\mathfrak{M}\}$ or $\mathbf{I}\mathsf{K}$. More generally in the present work, most of the time, we tend to concentrate our attention to isomorphism-closed classes $\mathsf{K} = \mathbf{I}\mathsf{K}$ of models, moreover we are inclined to identify isomorphic models.

A motivation behind Trend 1 (i.e. isomorphism invariance) is that when discussing definability over a structure like $\mathfrak{M}$, we want to regard $\mathfrak{M}$ as an abstract structure (and not a concrete structure).[910] Cf. also the note on p.786 on this and cf. Remark 6.2.4 on p.801.

Trend 2. For purely aesthetical reasons, some of our distinguished classes of models are not quite closed under isomorphisms. E.g. in the definition of our class FM of frame models we insisted that the relation $\in$ connecting ${}^nF$ and G should be the real set theoretical membership relation.[911] This aesthetics motivated decision is the only reason why $\mathsf{FM} \neq \mathbf{I}\mathsf{FM}$. Similarly in our two-sorted geometries of the kind $\langle Points, Lines; \in\rangle$ we insisted that $Lines \subseteq \mathcal{P}(Points)$ and "$\in$" is the real set theoretic one. This is the only reason why our two-sorted geometries are not closed under isomorphisms.

If only Trend 1 were present then we could simplify much of the presentation in this sub-section by discussing only isomorphism closed classes $\mathsf{K} = \mathbf{I}\mathsf{K}$, $\mathsf{K}^+ = \mathbf{I}\mathsf{K}^+$ etc. However, we cannot carry through this simplification because Trend 2 presents a "purely administrative" obstacle to it. We call this obstacle purely administrative

[908] i.e. to concentrating on maximal definitional expansions

[909] in the sense of $\langle Points, Lines; \in\rangle$, cf. p.991

[910] Recall that a structure is called abstract if it is defined only up to isomorphism. I.e. when discussing an abstract structure we want to abstract from knowing what its elements are. (Since our foundation is set theory the elements of a structure $\mathfrak{A}$ are sets whose elements are again sets etc. When regarding a structure as abstract, we want to disregard these details about the elements of the elements of our structure.)

[911] This is so if we understand the definition of FM in accordance with Convention 2.1.2 on p.35. (Otherwise FM can be understood in such a way that it becomes closed under isomorphisms.)

because the decision behind Trend 2 is purely aesthetical (everything would go through smoothly if we worked with IFM in place of FM). As a consequence we do the following: On the intuitive level we tend to follow the simplifications suggested by Trend 1. At the same time, on the formal level we take Trend 2 into account in order to make our results (and definitions) applicable to classes like FM or to two-sorted geometries even when we take the formal details fully into account. Therefore on the formal level, we try to make sure that our definitions make sense (and mean what they should) even when $\mathsf{K} \neq \mathbf{I}\mathsf{K}$. We suggest that the reader keep in mind the "intuitive level" (when we use only Trend 1 and replace FM with IFM etc.) and to treat the "formal level" as secondary, because this simplifies the picture *without* loosing any of the essential ideas.

$\lhd$

We close sub-section 6.3.2 with some examples. As an application, we also will apply the just defined notions to the geometries we defined earlier in this section.

Example 6.3.15 (Explicit definability of the rational numbers in the ring Z of integers.)
Let $\mathbf{Z} = \langle \mathrm{Z}; 0, 1, +, \cdot \rangle$ be the (usual) ring of integers. We will discuss how the set Q of rationals is definable explicitly as a *new sort* in $\mathbf{Z}$.[912] (Moreover with a little stretching of our terminology, we can say that the field $\mathbb{Q}$ of rationals is definable in $\mathbf{Z}$.) Here, the new *functions connecting* the new sort Q to the old one Z are (i) the ring-operations $+_{\mathrm{Q}}$ and $\cdot_{\mathrm{Q}}$ on the sort Q, and (ii) an injection $repr : \mathrm{Z} \rightarrowtail \mathrm{Q}$ representing the integers as rationals. The role of *repr* is to tell us which member of sort Z is considered to be equal with which member of the new sort Q. (Although the present "connecting-functions" do not coincide with our standard "explicit definability theoretical" ones pj_i and $\in$, we will see that they are first-order definable from the latter.)

Let us get started! We start out with $\mathbf{Z}$. First we define

$$R = \{ \langle a, b \rangle \ : \ a, b \in \mathrm{Z}, \ b \neq 0 \}$$

as a new relation, obtaining the expansion $\langle \mathbf{Z}; R \rangle$. Then we define the new sort U to be R with projections pj_0, pj_1 *and* for simplicity we forget R as a relation (but we keep it as a sort named U). This yields the definitional expansion

$$\mathbf{Z}^+ = \langle \mathrm{Z}, U; \ 0, 1, +, \cdot \ , pj_0, pj_1 \rangle = \langle \mathbf{Z}, U; \ pj_0, pj_1 \rangle$$

912 Although we promised, in §2, not to use the letter Q for other purposes than denoting the "quantity"-sort of our frame language, in the examples of the present sub-section we make an exception (since here there is no danger of creating a confusion).

where $pj_i : U \longrightarrow \mathrm{Z}$ are the usual. Next, we define the equivalence relation $\equiv$ on U as follows

$$\langle a, b\rangle \equiv \langle c, d\rangle \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad a \cdot d = b \cdot c.$$

Note, that it is this point where we need the operations pj_i, namely "$\langle a, b\rangle$" is not an expression of our first-order language, but we can simulate it by using the projections as follows. We define $\equiv$ by

$$x \equiv y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad pj_0(x) \cdot pj_1(y) = pj_1(x) \cdot pj_0(y),$$

where x, y are of sort Q. By using item (2.2) of our outline for definability, we define the *new sort* Q by $\mathrm{Q} := U/{\equiv}$ together with the usual membership relation $\in$ connecting sort U with sort Q.

Now, using the symbols $\in, pj_0, pj_1$ one can define the operations $+_{\mathrm{Q}}$, $\cdot_{\mathrm{Q}}$, *repr* as follows.

Assume $x \in \mathrm{Z}$ and $y \in \mathrm{Q}$. Then

$$repr(x) = y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists z \in y)\,[\,pj_0(z) = x \ \wedge \ pj_1(z) = 1].$$

Assume $x, y, z \in \mathrm{Q}$. Then

$$x \cdot_{\mathrm{Q}} y = z \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists x' \in x)(\exists y' \in y)(\exists z' \in z)$$
$$[\,pj_0(x') \cdot pj_0(y') = pj_0(z') \ \wedge \ pj_1(x') \cdot pj_1(y') = pj_1(z')].$$

The rest is easy, hence we omit it.

The above shows that the structure

$$\mathbf{Z}^{++} = \langle \mathbf{Z}, \mathrm{Q};\ +_{\mathrm{Q}}, \cdot_{\mathrm{Q}}, repr\rangle$$

is definable over $\mathbf{Z}^{+}$ hence it is also definable over $\mathbf{Z}$.

In passing, we note that the above definitional expansion makes sense and remains first-order if instead of $\mathbf{Z}$ we start out with an arbitrary ring, say $\mathfrak{A}$.

$\triangleleft$

Examples 6.3.16

1. Let $\mathbf{F}$ be a field. Consider the geometric expansion

$$\mathbf{G_F} := \langle \mathbf{F}, Points, Lines;\ pj_0, pj_1, \mathrm{E}\rangle$$

of $\mathbf{F}$ where $Points = F \times F$ and $pj_i : F \times F \longrightarrow F$ and $\mathrm{E} \subseteq Points \times Lines$ is the incidence relation (the usual way) and $Lines \subseteq \mathcal{P}(Points)$ is the set of lines in the Euclidean sense.

Then $\mathbf{G_F}$ *is rigidly definable* over $\mathbf{F}$. See the Hint in Example 2 below.

2. To each field $\mathbf{F}$ let $\mathbf{G_F}$ be associated as in item 1 above. Then

$$\mathsf{K}^+ := \{\, \mathbf{G_F} \;:\; \mathbf{F} \text{ is a field} \,\}$$

is rigidly definable (explicitly) over the class K of fields.[913]

<u>*Hint:*</u> First we define $\mathit{Points} = F \times F$ (with pj_i) as a new sort. Then we define

$$R = \{\, \langle p, q\rangle \in \mathit{Points} \times \mathit{Points} \;:\; p \neq q \,\},$$

as a new relation. Then we define the new auxiliary sort U to be R with the *new* projections $\overline{pj}_i : R \longrightarrow \mathit{Points}$ *and* we forget R as a relation (but we keep it as a sort named U). Then we define the equivalence relation $\equiv$ on U by saying

$$\langle p, q\rangle \equiv \langle r, s\rangle$$
$$\stackrel{\text{def}}{\Longleftrightarrow}$$
$$(p, q, r, s \text{ are collinear in the Euclidean sense}).$$

Then we define the new sort $\mathit{Lines} := U/\!\equiv$ together with $\in\; \subseteq U \times \mathit{Lines}$. From these data we define our final incidence relation $\mathrm{E} := \mathrm{E}_{\text{Points,Lines}}$ the usual way.[914]

$\lhd$

In the case of implicit definability we saw that uniform and one-by-one definability are wildly different. The example below is intended to demonstrate, for the case of explicit definability, the same kind of difference between uniform and one-by-one (explicit) definability. In this example we restricted ourselves to the most classical case: one sort only and the defined thing is a relation over the old sort. Besides providing explanation, this example was also designed to provide motivation for consistently sticking with the *uniform* versions of the kinds of definability we consider.

Example 6.3.17 Let $\underline{\omega} = \langle \omega;\, 0, 1, +, \cdot\, \rangle$ be the usual standard model of Arithmetic. Let us choose $R \subseteq \omega$ such that R is not explicitly definable even in higher-order logic over $\underline{\omega}$ (and even with parameters). Such an R exists.[915] Let

$$\mathsf{K} := \{\, \langle \underline{\omega};\, c, P\rangle \;:\; c \in \omega,\; P \subseteq \omega \text{ and } (c \in R \;\Rightarrow\; P = \{c\}) \text{ and } (c \notin R \;\Rightarrow\; P = \emptyset) \,\}.$$

[913] From now on we will tend to omit “explicitly” since we agreed that definability automatically means explicit definability.

[914] I.e. $p \,\mathrm{E}\, \ell \stackrel{\text{def}}{\Longleftrightarrow} (\exists x \in \ell)[\overline{pj}_0(x), \overline{pj}_1(x), p$ are collinear as computed in $\mathbf{F}\,]$.

[915] One can choose R to be so far from being computable that R is not even in the so called Analytical Hierarchy cf. [42].

Let K^- be the P-free reduct of K i.e.

$$\mathsf{K}^- := \{ \langle \underline{\omega}, c \rangle \;:\; c \in \omega \} .$$

<u>*Claim:*</u> Each member $\mathfrak{M} = \langle \underline{\omega};\, c, P \rangle$ of K is *explicitly* definable over its P-free reduct $\mathfrak{M}^- = \langle \underline{\omega}, c \rangle$. I.e. K is one-by-one explicitly definable over its reduct K^-.

We will see that K is very far from *being uniformly explicitly* definable over K^-. (Moreover K is far from being uniformly finitely implicitly definable.)

For $n \in \omega$, we denote the constant-term $\underbrace{1 + \ldots + 1}_{n\text{-times}}$ by $\bar{n}$. Assume P is uniformly explicitly definable over K^-. Then

$$\mathsf{K} \models [\, P(x) \leftrightarrow \psi(c, x) \,],$$

for some formula $\psi(x, y)$ in the language of $\underline{\omega}$.[916] Now, for any $n \in \omega$ we have the following:

$$\begin{array}{lllllll}
n \in R & \Rightarrow & [\,\mathsf{K} & \models & \bar{n} = c \;\rightarrow\; P(\bar{n}) & & \text{hence} \\
 & & \mathsf{K} & \models & \bar{n} = c \;\rightarrow\; \psi(c, \bar{n}) & & \text{hence} \\
 & & \mathsf{K}^- & \models & \bar{n} = c \;\rightarrow\; \psi(c, \bar{n}) & & \text{hence} \\
 & & \mathsf{K}^- & \models & \bar{n} = c \;\rightarrow\; \psi(\bar{n}, \bar{n}) & & \text{hence}^{917} \\
 & & \mathsf{K}^- & \models & \psi(\bar{n}, \bar{n}) & & \text{hence} \\
 & & \underline{\omega} & \models & \psi(\bar{n}, \bar{n}) \,]. & &
\end{array}$$

$$\begin{array}{lllllll}
n \notin R & \Rightarrow & [\,\mathsf{K} & \models & \bar{n} = c \;\rightarrow\; \neg P(\bar{n}) & & \text{moreover} \\
 & & \mathsf{K} & \models & \bar{n} = c \;\rightarrow\; P = \emptyset & & \text{hence} \\
 & & \mathsf{K}^- & \models & \bar{n} = c \;\rightarrow\; \neg\psi(c, \bar{n}) & & \text{hence} \\
 & & \mathsf{K}^- & \models & \bar{n} = c \;\rightarrow\; \neg\psi(\bar{n}, \bar{n}) & & \text{hence}^{917} \\
 & & \underline{\omega} & \models & \neg\psi(\bar{n}, \bar{n}) \,]. & &
\end{array}$$

But then $\psi(x, x)$ explicitly defines $R(x)$ in $\underline{\omega}$, which is a contradiction.

We have seen that while in K^- the new relation P is one-by-one explicitly definable (in other words locally explicitly definable), P is very far from being *uniformly* explicitly definable over the same K^-.

$\lhd$

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[916] This is so because $\psi(c, x)$ is in the language of K^-, which is the same as the language of $\underline{\omega}$ expanded with a constant symbol c.

[917] by $\mathsf{K} \not\models n \neq c$ (i.e. by $(\exists \mathfrak{M} \in \mathsf{K})\mathfrak{M} \models n = c$) and since under any evaluation of the variables (in a member of K) the value of the constant term $\bar{n}$ coincides with the element n of ω.

We hope that the above construction and proof explain why and how one-by-one definability is so much weaker than[918] uniform definability. We also hope that the above example illustrates why most authors simply identify uniform definability with definability.

Application: definability of the observer-independent geometries

Now we turn to the issue of definability of the observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$ over the ("observational") frame models $\mathfrak{M}$, which has already been discussed in §6.2.2 and Remark 6.2.8 (p.807). The propositions and the theorems below serve to illuminate parts of Remark 6.2.8.

The following proposition says, roughly, that the set of points Mn, and our various kinds of lines $L, \ldots, L^S$ are definable over the "observational" models $\mathfrak{M}$.

PROPOSITION 6.3.18 *For every frame model $\mathfrak{M}$ let $\mathfrak{M}^+ := \langle \mathfrak{M}, Mn; \in_{Mn} \rangle$ be the expansion of $\mathfrak{M}$ with the set of events $Mn := \bigcup \{\, Rng(w_m) \ : \ m \in Obs \,\}$ and the set theoretic membership relation $\in_{Mn} \subseteq B \times Mn$. Let*

$$\mathsf{FM}^+ := \left\{ \mathfrak{M}^+ \ : \ \mathfrak{M} \in \mathsf{FM} \right\}.$$

Then (i) and (ii) below hold.

(i) FM^+ *is <u>rigidly definable</u> over the class* FM *of frame models.*

(ii) *For every $\mathfrak{M}^+ \in \mathsf{FM}^+$ let $\mathfrak{M}^{++} := \langle \mathfrak{M}^+, L; L^T, L^{Ph}, L^S, \in_L \rangle$ be the expansion of $\mathfrak{M}^+$, where L^T, L^{Ph}, L^S, L are, respectively, the sets of time-like, photon-like, space-like, and all lines as defined in item 5 of Def.6.2.2(I); and $\in_L \subseteq Mn \times L$ is the membership (or equivalently the incidence) relation between points (elements of Mn) and lines. Then the class*

$$\mathsf{FM}^{++} := \left\{ \mathfrak{M}^{++} \ : \ \mathfrak{M}^+ \in \mathsf{FM}^+ \right\}$$

is <u>rigidly definable</u> over the class FM *of frame models.*

Proof:
<u>*Proof of (i):*</u> The new sort ${}^nF$ together with the projection functions are rigidly

[918] One-by-one definability is not only weaker than uniform definability, but also it is much *less satisfactory* from the point of view of re-capturing the intuitive idea of definability. In our opinion one-by-one definability does not capture the intuitive notion of definability while uniform definability does. (All the same, one-by-one definability is useful as a mathematical *auxiliary* concept.)

definable over FM, therefore we will pretend that ${}^nF$ is an old sort of FM. In defining FM^+ over FM up to unique isomorphism, first we define

$$R := \{ \langle m, p \rangle \in B \times {}^nF \ : \ m \in Obs \}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1 *and* we forget R as a relation (but we keep it as a sort named U). Then we define the equivalence relation $\equiv$ on U by saying

$$\langle m, p \rangle \equiv \langle k, q \rangle \ \stackrel{\text{def}}{\Longleftrightarrow} \ w_m(p) = w_k(q); \quad \text{formally:}$$
$$\langle m, p \rangle \equiv \langle k, q \rangle \ \stackrel{\text{def}}{\Longleftrightarrow} \ (\forall b \in B) \, [\, W(m, p, b) \leftrightarrow W(k, q, b) \,];$$

while if we want to get rid of the notation "$\langle m, p \rangle$" we can write the following. Let $a, d \in U$. Then

$$a \equiv d \ \stackrel{\text{def}}{\Longleftrightarrow} \ (\forall b \in B) \, [\, W(pj_0(a), pj_1(a), b) \leftrightarrow W(pj_0(d), pj_1(d), b) \,].$$

Then we define the new sort $Mn := U/\equiv$ together with $\in \subseteq U \times Mn$. From these data finally we define the "membership" relation $\mathrm{E}_{Mn} \subseteq B \times Mn$ as follows. Let $b \in B$ and $e \in Mn$. Then

$$b \ \mathrm{E}_{Mn} \ e \ \stackrel{\text{def}}{\Longleftrightarrow} \ (\exists a \in e) \ W(pj_0(a), pj_1(a), b).$$

So far we have defined Mn and E_{Mn}, hence all parts of (an isomorphic copy of) FM^+ have been defined (over FM). The "rigid-ness" (i.e. "uniqueness") part of definability stated in (i) comes from the fact that the axiom of extensionality holds for E_{Mn}, i.e.

$$(\forall e, e_1 \in Mn) \, [\, e = e_1 \ \leftrightarrow \ (\forall b \in B)(b \ \mathrm{E}_{Mn} \ e \leftrightarrow b \ \mathrm{E}_{Mn} \ e_1) \,].$$

Proof of (ii): By item (i) it is sufficient to prove that FM^{++} is rigidly definable over FM^+. In defining FM^{++} over FM^+ up to unique isomorphism, first we define

$$R := \{ \langle h, i \rangle \in B \times F \ : \ h \in Obs \cup Ph, \ i \in n, \ (h \notin Obs \ \Rightarrow \ i = 0) \, \}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1 *and* we forget R as a relation (but we keep it as a sort named U). Intuitively, the elements of U will code the lines.[919] We define a kind of incidence relation $E \subseteq Mn \times U$ as follows. Let $e \in Mn$ and $\ell \in U$. Then

[919]We code lines by elements of U according to the following intuition. Photon-like lines and time-like lines are coded by $\langle h, 0 \rangle$ where h is a photon or an observer (then $\langle h, 0 \rangle$ codes the life-line of h). Space-like lines are coded by an observer h and an axis $\bar{x}_i$ ($i \neq 0$) and the coded line is what h sees on the $\bar{x}_i$ axis i.e. it is $w_h[\bar{x}_i]$.

$$e \; E \; \ell$$
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$[\, pj_1(\ell) = 0 \;\wedge\; pj_0(\ell) \in_{Mn} e \,] \;\vee\; \textstyle\bigvee_{0<i\in n} [\, pj_1(\ell) = i \;\wedge\; (\exists q \in \bar{x}_i)\,(e = w_{pj_0(\ell)}(q))^{920}\,].$$

Then we define the equivalence relation $\equiv$ on U as follows. Let $\ell, \ell' \in U$. Then

$$\ell \equiv \ell' \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall e \in Mn)\,(e \; E \; \ell \;\leftrightarrow\; e \; E \; \ell').$$

We define the sort $L := U/\!\equiv$ together with the membership relation $\in \subseteq U \times L$. Now, the "membership" (or incidence) relation $\mathrm{E}_L \subseteq Mn \times L$ is defined as follows. Let $e \in Mn$ and $\ell \in L$. Then

$$e \; \mathrm{E}_L \; \ell \;\overset{\text{def}}{\Longleftrightarrow}\; (\exists \ell' \in \ell)\; e \; E \; \ell'.$$

Finally, the unary relations L^T, L^{Ph}, L^S on L are defined as

$$\begin{aligned} L^T &:= \{\, \ell \in L \,:\, (\exists \ell' \in \ell)\,(\, pj_0(\ell') \in Obs \;\wedge\; pj_1(\ell') = 0\,)\,\}, \\ L^{Ph} &:= \{\, \ell \in L \,:\, (\exists \ell' \in \ell)\,(\, pj_0(\ell') \in Ph \;\wedge\; pj_1(\ell') = 0\,)\,\}, \\ L^S &:= \{\, \ell \in L \,:\, (\exists \ell' \in \ell)\,(\, pj_0(\ell') \in Obs \;\wedge\; pj_1(\ell') > 0)\,\}. \end{aligned}$$

So far we have defined L, L^T, L^{Ph}, L^S and E_L, hence all parts of (an isomorphic copy of) FM^{++} have been defined (over FM^{+}). The "rigid-ness" (i.e. "uniqueness") part goes exactly as in the case of (i). ∎

Our next proposition says, roughly, that the topology part $\mathcal{T}$ (of our geometries) is definable over the "observational" models $\mathfrak{M}$.

PROPOSITION 6.3.19 *Let FM^{+} be as in Proposition 6.3.18 above. For every $\mathfrak{M}^{+} \in \mathsf{FM}^{+}$ let $\langle \mathfrak{M}^{+}, T_0; \in \rangle$ be the expansion of $\mathfrak{M}^{+}$ with the subbase*

$$T_0 = \big\{ S(e, \varepsilon) \;:\; e \in Mn,\; \varepsilon \in {}^{+}F \,\big\}$$

for the topology $\mathcal{T}$ (as defined in item 14 of Def.6.2.2(I)) and with the (standard) membership relation $\in \subseteq Mn \times T_0$. Then the class

$$\mathsf{FM}^{++} := \big\{\, \langle \mathfrak{M}^{+}, T_0; \in \rangle \;:\; \mathfrak{M}^{+} \in \mathsf{FM}^{+} \,\big\}$$

is rigidly definable over the class FM of frame models. Roughly, this means that the ("heart" of the) topology part of our geometries associated to FM is also definable over FM, but cf. the discussion in $(\star\star\star)$ of Remark 6.2.8 on p.809.

[920] We note that "$e = w_{pj_0(\ell)}(q)$" is a formula since the following is a formula.
$(\forall b \in B)\,[\, b \in e \;\leftrightarrow\; W(pj_0(\ell), q, b)\,]$.

Proof: Let $\mathsf{FM}^+, \mathsf{FM}^{++}$ be as above. By Prop.6.3.18(i) it is sufficient to prove that FM^{++} is rigidly definable over FM^+. In defining FM^{++} over FM^+ up to unique isomorphism, first we define the pseudo-metric $g : Mn \times Mn \overset{\circ}{\longrightarrow} F$ as a new relation as it was defined in item 13 of Def.6.2.2(I) (p.797). It can be checked that the just quoted definition of g can be translated to a first-order formula in the language of FM^+. Then we define

$$R := \{ \langle e, \varepsilon \rangle \ : \ e \in Mn, \ \varepsilon \in {}^+F \}$$

as a new relation. Then we define the new auxiliary sort U to be R together with pj_0, pj_1 *and* we forget R as a relation (but we keep it as a sort named U). Then we define the equivalence relation $\equiv$ on U by saying

$$\langle e, \varepsilon \rangle \equiv \langle e_1, \varepsilon_1 \rangle \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall e_2 \in Mn)\,(\, g(e, e_2) < \varepsilon \ \leftrightarrow \ g(e_1, e_2) < \varepsilon_1 \,).$$

(Of course one uses the projection functions pj_0, pj_1 to formalize the above definition of $\equiv$.)

Then, we define the new sort $T_0 := U/\equiv$ together with the membership relation $\in_{U,T_0} \subseteq U \times T_0$. Finally we define the "membership" relation $\mathrm{E} \subseteq Mn \times T_0$ as follows. Let $e \in Mn$, $A \in T_0$. Then

$$e \mathrel{\mathrm{E}} A \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\exists a \in U)\,[\, a \in_{U,T_0} A \ \wedge \ g(pj_0(a), e) < pj_1(a) \,].$$

So far we have defined T_0 and E, hence all parts of (an isomorphic copy of) FM^{++} have been defined (over FM^+). The "rigid-ness" (i.e. "uniqueness") part goes exactly as in the case of Prop.6.3.18(i). ∎

In connection with the following two propositions recall that alternative versions $\mathcal{T}'$ and $\mathcal{T}''$ of the topology part $\mathcal{T}$ of our geometries were defined in Def.6.2.31 (p.838). Further, T_0' and T_0'' are subbases for $\mathcal{T}'$ and $\mathcal{T}''$, respectively, as defined in Def.6.2.31.

PROPOSITION 6.3.20 *Proposition 6.3.19 remains true if we replace T_0 with T_0'' in it, where T_0'' was defined in Def.6.2.31(ii).*

We omit the easy **proof**.

Our next proposition says, roughly, that the topology part $\mathcal{T}'$ (of our geometries) is definable over the "observational" models $\mathfrak{M}$, assuming $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$.

PROPOSITION 6.3.21

(i) *For every frame model* $\mathfrak{M}$ *let* $\mathfrak{M}^+$ *be defined as in Prop.6.3.18, i.e.* $\mathfrak{M}^+ := \langle \mathfrak{M}, Mn; \in_{Mn} \rangle$. *Further, let* $\langle \mathfrak{M}^+, T_0'; \in \rangle$ *be the expansion of* $\mathfrak{M}^+$ *with the subbase* T_0' *for* $\mathcal{T}'$, *where* T_0' *is defined in Def.6.2.31(i); and with the membership relation* $\in \subseteq Mn \times T_0'$. *Then the class*

$$\mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\surd\,))^+ := \left\{ \langle \mathfrak{M}^+, T_0'; \in \rangle \ : \ \mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\surd\,)) \right\}$$

is rigidly definable over the class $\mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\surd\,))$.

(ii) *Since in* $\mathbf{Bax}^- + \mathbf{Ax}(\surd\,)$ T_0' *is a base for our topology* $\mathcal{T}'$, *for all practical purposes (i) "means" that the topology* $\mathcal{T}'$ *is definable over these models (cf.* $(\star\star\star)$ *on p.809).*

We omit the **proof**, but we note that a proof can be obtained using Propositions 6.2.16 (p.820), 6.2.79 (p.884), cf. also Thm.6.2.36 (p.843) and the proof of Thm.6.2.34 (p.840). ∎

Our next three theorems say, roughly, that our class $\mathsf{Ge}(Th)$ of relativistic geometries is definable over the corresponding class of observational models.[921]

THEOREM 6.3.22 *The class* $\mathsf{Ge}(Th)$ *is definable over the class* $\mathsf{Mod}(Th)$, *assuming that* $n > 2$ *and* Th *is a set of formulas in our frame language such that* $Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^- + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax(diswind)} + \mathbf{Ax}(\surd\,)$.
More precisely, instead of definability of the topology part $\mathcal{T}$ *we claim definability of only a subbase* T_0 *for* $\mathcal{T}$, *together with* $\in \subseteq Mn \times T_0$ *of course.*[922]

Proof: The theorem follows by Propositions 6.3.18 (p.957), 6.3.19 (p.959) and by Theorems 6.2.10 (p.813), 6.2.19 (p.823) and 6.2.23 (p.829). Cf. Remark 6.2.8 (p.807). ∎

The theorem below says that if in Thm.6.3.22 above **Basax** is assumed in place of $\mathbf{Bax}^{\oplus}$ then the assumptions $n > 2$, $\mathbf{Ax}(\|)^-$ and $\mathbf{Ax(diswind)}$ are not needed.

THEOREM 6.3.23 *The class* $\mathsf{Ge}(Th)$ *is definable over the class* $\mathsf{Mod}(Th)$, *assuming that* Th *is a set of formulas in our frame language such that* $Th \models \mathbf{Basax} + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\surd\,)$. *(More precisely instead of definability of* $\mathcal{T}$ *we claim definability of* T_0 *only.)*

[921] These three theorems were stated as Theorem 6.2.44 in the previous sub-section on p.847.
[922] Cf. the discussion of definability of $\mathcal{T}$ in $(\star\star\star)$ of Remark 6.2.8 on p.809.

Proof: The theorem follows by Propositions 6.3.18 (p.957), 6.3.19 (p.959) and by Theorems 6.2.10 (p.813), 6.2.22 (p.827). Cf. the proof of Thm.6.3.22 and Remark 6.2.8 (p.807). ∎

In connection with the next two theorems recall that $\mathsf{Ge}'(Th)$ and $\mathsf{Ge}''(Th)$ are alternative versions of $\mathsf{Ge}(Th)$ and are introduced in Definition 6.2.43 (p.846).

THEOREM 6.3.24

(i) $\mathsf{Ge}'(Th)$ *is* *definable* *over* $\mathsf{Mod}(Th)$, *for any set* Th *of formulas in our frame language. (More precisely instead of definability of* $\mathcal{T}$ *we claim definability of* T_0 *only.)*

(ii) $\mathsf{Ge}''(Th)$ *is* *definable* *over* $\mathsf{Mod}(Th)$,[923] *assuming that* Th *is a set of formulas in our frame language such that* $Th \models \mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$.

Proof: The theorem follows by Propositions 6.3.18, 6.3.19, 6.3.20, 6.3.21. Cf. the proof of Thm.6.3.22 and Remark 6.2.8 (p.807). ∎

6.3.3 Eliminability of defined concepts. Definitional equivalence of theories. An extension of Beth's theorem.

Notation 6.3.25 For a class K of (many-sorted, similar) models, $Fm(\mathsf{K})$ denotes the set of formulas of the language of K. Hence $\mathsf{Th}(\mathsf{K}) \subseteq Fm(\mathsf{K})$. Sometimes we refer to $Fm(\mathsf{K})$ as the language of K.[924]

◁

THEOREM 6.3.26 (First translation theorem) *Let* K *and* K^+ *be two classes of (many-sorted) models. Assume that* K^+ *is a generalized expansion of* K. *Then there is a "natural" translation mapping*

$$Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$$

[923] Cf. Prop.6.3.21(ii) in connection with definability of $\mathcal{T}'$ in $\mathsf{Ge}''(Th)$.

[924] According to our philosophy, $Fm(\mathsf{K})$ is the language, while the system of basic symbols (like relation symbols, sort symbols etc.) is the *vocabulary* of this language, cf. Convention 6.3.1 on p.931. We note this because some logic books use the word "language" for what we call the vocabulary (of a language or a model).

having the following property (called preservation of meaning):[925]

> *Assume* $\psi(\bar{x}) \in Fm(\mathsf{K}^+)$ *is such that all its free variables (indicated as* $\bar{x}$*) belong to "old"*[926] *sorts, i.e. to sorts of* K*. Then*

($\star$)

$$\mathsf{K}^+ \models [\,\psi(\bar{x}) \leftrightarrow Tr(\psi)(\bar{x})\,].$$

Further, for all $\psi \in Fm(\mathsf{K}^+)$

$$\mathsf{K}^+ \models \psi \;\Leftrightarrow\; \mathsf{K} \models Tr(\psi).$$

Moreover, Tr is very simple (transparent) from the computational point of view, e.g. it is Turing-computable in linear time.

Theorem 6.3.26 follows from the stronger Theorem 6.3.27 (and its proof) to be stated soon, so we do not prove it here.

Before stating the stronger version of Theorem 6.3.26, let us ask ourselves in what sense Tr in Thm.6.3.26 preserves the meanings of formulas. To answer this question, let us notice that the conclusion of Theorem 6.3.26 implies (i) and (ii) below.

(i) ψ and $Tr(\psi)$ have the same free variables $\bar{x}$, and in some intuitive sense they say the same thing about these variables $\bar{x}$.

(ii) Let $\mathfrak{M} \in \mathsf{K}^+$, $\mathfrak{M}^-$ be the reduct of $\mathfrak{M}$ in K and let $\bar{a}$ be a sequence of members of $Uv(\mathfrak{M}^-)$ matching the sorts of $\bar{x}$. In other words $\bar{a}$ is an evaluation of the variables $\bar{x}$. Then

$$\mathfrak{M} \models \psi[\bar{a}] \quad\Longleftrightarrow\quad \mathfrak{M}^- \models (Tr(\psi))[\bar{a}]\,;$$

cf. the notation on p.945. Intuitively, *whatever can be said about* some "old" elements $\bar{a}$ in a model $\mathfrak{M}$ in K^+, *it can be said* (about the same elements $\bar{a}$) ***already*** in the "old" model $\mathfrak{M}^-$ (in K). This will be generalized to "new" elements also (i.e. to arbitrary elements), in our next theorem.

[925] The existence of such a translation mapping Tr is often called in the literature "***uniform reduction property***", cf. Hodges [136, p.640]. A result of Pillay and Shelah is that for first order axiomatizable classes implicit definability without taking reducts implies the reduction property, cf. [214]. Cf. also Lemma 12.5.1 in Hodges [136, p.641].

[926] A symbol (e.g. a sort) is called old if it is available already in K (and not only in K^+).

Recall that K is a reduct of K^+. In some sense (i) and (ii) above mean that the poorer class K and the richer class K^+ of models are equivalent from the point of view of expressive power of language. So, the "language + theory" of K^+ is equivalent with the "language + theory" of K in means of expression. Therefore, on some level of abstraction, we may consider the languages of K and K^+ to be the *same* except that they[927] choose different "basic vocabularies" for representing this language. (In passing we note that a stronger form of this kind of *sameness* will appear in the form of definitional equivalence $\equiv_\Delta$, cf. beginning with p.969 (and the figure on p.975).)

Generalization of Theorem 6.3.26 to permitting free variables of new sorts to occur in ψ and $Tr(\psi)$

Let us turn to discussing the restriction in Theorem 6.3.26 which says (in statement ($\star$)) that the free variables of ψ belong to the sorts of K. The theorem does admit a generalization which is without this restriction on the free variables. This will be stated in Theorem 6.3.27 below. But then two things happen discussed in items (I), (II) below.

(I) Consider the process of defining K^+ over K as a sequence of steps (as described on p.951). Assume that a relation like pj_i or $\in_U$ connecting a new sort to an old one is introduced in one step and then is *forgotten* at the (last) reduct step. Then we call the relation (e.g. pj_i) in question an *auxiliary relation* of the definition of K^+ over K. Now, for the generalization of Theorem 6.3.26 we have in mind, we have to assume that all auxiliary relations (of the definition of K^+) remain definable in K^+. We will formulate this condition as "K^+ and K have a common (explicit) definitional expansion (without taking reducts)".

(II) The formulation of the theorem gets somewhat complicated. Intuitively, the generalized theorem says that all new objects[928] can be represented as equivalence classes of tuples of old objects, and then (using this representation) whatever can be said about elements of $Uv(\mathfrak{M})$ in an expanded model $\mathfrak{M} \in \mathsf{K}^+$ can be already said in the reduct $\mathfrak{M}^- \in \mathsf{K}$ of $\mathfrak{M}$. This intuitive statement is intended to serve as a generalization the text below item (ii) in the discussion of the intuitive meaning of Theorem 6.3.26 (presented immediately below Theorem 6.3.26).

Notation: $Var(U_i)$ denotes the (infinite) set of variables of sort U_i (where U_i is treated as a sort symbol or *equivalently* U_i is the name of one of the universes of the models in K^+).

[927] i.e. K and K^+

[928] By objects we mean elements of some sort.

THEOREM 6.3.27 (Second translation theorem) *Assume* K *is a reduct of* K^+ *and* K *and* K^+ *have a common definitional expansion (without taking reducts). This holds e.g. whenever* K^+ *is a definitional expansion of* K. *Assume* $U_1^{new}, \ldots, U_k^{new}$ *are the new sorts.*[929] *Then there is a translation mapping*

$$Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$$

for which the following hold. For each U_i^{new} *there is a formula* $code_i(x, \vec{x}) \in Fm(\mathsf{K}^+)$ *such that the following 1-2 hold.*

1. $x \in \mathrm{Var}(U_i^{new})$ *and* $\vec{x}$ *is a sequence of variables of old sorts.*

2. *(a)–(c) below hold.*

   (a) $\mathsf{K}^+ \models \forall x\, \exists \vec{x}\; code_i(x, \vec{x})$, [930]

   (b) $\mathsf{K}^+ \models [\, code_i(x, \vec{x}) \;\wedge\; code_i(y, \vec{x})\,] \;\rightarrow\; x = y$, *where* $y \in \mathrm{Var}(U_i^{new})$. [931]

   (c) *Our translation mapping*[932]

   $$Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$$

   satisfies the following stronger[933] *property of meaning preservation. Assume* $\psi(y, \bar{z}) \in Fm(\mathsf{K}^+)$ *is such that* $y \in \mathrm{Var}(U_i^{new})$ *and* $\bar{z}$ *is (a sequence of variables) of old sorts such that the variables in* $\bar{z}$ *are* distinct *from those occurring in* $\vec{y}$. *Then*

   $$\mathsf{K}^+ \;\models\; code_i(y, \vec{y}) \;\rightarrow\; [\, \psi(y, \bar{z}) \leftrightarrow (Tr(\psi))(\vec{y}, \bar{z})\,].$$

   Intuitively, whatever is said by ψ *about* y *and* $\bar{z}$*, the same is said by the translated formula* $Tr(\psi)$ *about the code* $\vec{y}$ *of* y *and* $\bar{z}$. *The case when* ψ *contains an arbitrary sequence, say* $\bar{y}$*, of variables of various new sorts is a straightforward generalization and is left to the reader.*

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[929] i.e. they are available in K^+ but not in K.

[930] Note that here "$\forall x$" means "$\forall x \in U_i^{new}$" automatically since we know that x is of sort U_i^{new} (as a variable symbol of the language of K^+).

[931] Note that items (a), (b) mean that $code_i$ represents an unambiguous coding of elements of U_i^{new} with equivalence classes of tuples of elements of old sorts, cf. (II) preceding the statement of the theorem and the text immediately below the theorem.

[932] fixed at the beginning of the formulation of the present theorem

[933] stronger than in Theorem 6.3.26

We note that the intuitive meaning of "$code_i(x,\bar{y})$" is "$\bar{y}$ codes x". Property (b) then says that "$\bar{y}$ codes only one element", property (a) says that "every new element has a code", and property (c) then tells us that "whatever can be said of a new element x in the new language, can be said of any of its codes $\bar{y}$ in the old language", cf. (II) before the statement of Theorem 6.3.27.

Proof:
(I) The case of step (2.1): Assume that K^+ is obtained from K by applying step (2.1) so that we defined $U^{new} \stackrel{\text{def}}{=} R$ where R is an old r-ary relation. For simplicity we assume $r = 2$ and $R \subseteq U_0 \times U_1$ where U_0, U_1 are old sorts. Then the new symbols (in K^+) are U^{new} and pj_0, pj_1. We want to represent objects (variables) of sort U^{new} with pairs of objects of ("old") sorts. To this end, we fix an injective function

$$rep : Var(U^{new}) \rightarrowtail Var(U_0) \times Var(U_1)$$

such that the values $rep(x)_i$ of rep are all distinct.[934] For simplicity, we will denote $rep(x)_i$ by x_i. We also assume that x_0, x_1 do not occur in the formulas to be translated.

Now, we define Tr by recursion as follows.

- $Tr((\exists x \in U^{new})\psi) := (\exists x_0 \in U_0, x_1 \in U_1)[R(x_0,x_1) \wedge Tr(\psi)]$; if $x \in Var(U^{new})$;
- $Tr((\exists y)\psi) := (\exists y) Tr(\psi)$; if y is a variable of old sort;
- $Tr(\neg\psi) := \neg Tr(\psi)$, $\quad Tr(\psi \wedge \varphi) := Tr(\psi) \wedge Tr(\varphi)$;
- $Tr(x = y) := (x_0 = y_0 \ \wedge \ x_1 = y_1)$, for any $x, y \in Var(U^{new})$;
- for any other *atomic* formula ψ, $Tr(\psi)$ is obtained from ψ by replacing each occurrence of $pj_i(x)$ with x_i (i.e. with $rep(x)_i$) in ψ for *every* variable $x \in Var(U^{new})$ and $i \in 2$; i.e. $Tr(\psi) := \psi(pj_i(x)/x_i)_{x \in Var(U^{new}), i<2}$.

We introduce the formula $code(x, x_0, x_1)$ (saying explicitly that the values of x_0, x_1 form really the code of the value of x) as follows:

$$code(x, x_0, x_1) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad [\, x_0 = pj_0(x) \ \wedge \ x_1 = pj_1(x) \ \wedge \ R(x_0, x_1)\,].$$

Now, it is not difficult to check that $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ is well defined, and (a)-(c) in the statement of Theorem 6.3.27 hold.

[934] $rep(x) = \langle rep(x)_0, rep(x)_1 \rangle$; and $rep(x)_i = rep(y)_j$ iff $\langle x, i \rangle = \langle y, j \rangle$.

(II) The case of step (2.2): Assume that K^+ is obtained from K by applying step (2.2) so that the only new symbols (in K^+) are $U^{new} = U/R$ and $\in$, where U is an (old) sort of K, and $R(x,y) \in \mathit{Fm}(\mathsf{K})$ where x, y are variables of sort U.

We fix an injective function

$$\mathit{rep} : \mathit{Var}(U^{new}) \rightarrowtail \mathit{Var}(U)$$

and we denote $\mathit{rep}(x)$ by $\underline{x}$. So $\underline{x} \in \mathit{Var}(U)$ if $x \in \mathit{Var}(U^{new})$. As before, we assume that the variables $\underline{x}$ do not occur in the formulas to be translated.

Now, we define Tr by recursion as follows.

- $\mathit{Tr}((\exists x \in U^{new})\psi) := (\exists \underline{x} \in U)\,\mathit{Tr}(\psi)$; if $x \in \mathit{Var}(U^{new})$;
- $\mathit{Tr}((\exists y)\psi) := (\exists y)\,\mathit{Tr}(\psi)$; if y is a variable of old sort;
- $\mathit{Tr}(\neg\psi) := \neg\,\mathit{Tr}(\psi)$, $\qquad \mathit{Tr}(\psi \wedge \varphi) := \mathit{Tr}(\psi) \wedge \mathit{Tr}(\varphi)$;
- $\mathit{Tr}(x = y) := R(\underline{x}, \underline{y})$, where $x, y \in \mathit{Var}(U^{new})$;
- $\mathit{Tr}(\in(z, \underline{x})) := R(z, \underline{x})$ and $\mathit{Tr}(\psi) := \psi$; for any other *atomic* formula ψ with no variables of new sort.

We introduce the formula $\mathit{code}(x, \underline{x})$ as follows:

$$\mathit{code}(x, \underline{x}) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \in(\underline{x}, x)\,.$$

Now, it is not difficult to check that $\mathit{Tr} : \mathit{Fm}(\mathsf{K}^+) \longrightarrow \mathit{Fm}(\mathsf{K})$ is well defined, and (a)-(c) in the statement of Theorem 6.3.27 hold.

(III) The case of explicit definability without taking reducts: If K^+ is obtained from K by step (1) then we have an obvious translation with all the good properties known from classical definability theory.[935]

By this we have covered all the steps (i.e. (1)–(2.2)) which might occur in an explicit definition. I.e. we defined *code*, *Tr* to all three kinds of “one-step” explicit definitions represented by items (1)–(2.2).

Assume now that K^+ is explicitly defined over K without taking reducts. Now, the definition of K^+ is a finite sequence of steps with each step using one of items (1), (2.1), (2.2). Hence by the above, we have a meaning preserving translation mapping Tr_k for the k'th step for each number

$$k < n := \text{“number of steps in the definition of } \mathsf{K}^+\text{”}.$$

[935] In the case of step (1), “*code*” is not needed because there are no new sorts involved. Hence (*if* we want to preserve uniformity of the steps) we can choose $\mathit{code}(x, y)$ to be $x = y$.

Besides Tr_k we also have a formula $code_k$ for each number k. Also for each Tr_k we have that (a)-(c) in the statement of Theorem 6.3.27 hold. But then we can take the composition $Tr := Tr_1 \circ Tr_2 \circ \ldots \circ Tr_n$ of these meaning preserving functions, and then the composition too will be meaning preserving if we also combine the formulas $code_1, \ldots, code_n$ into a single "big" formula $code$.

One can check that for the just defined Tr and $code$, (a)-(c) in the statement of Theorem 6.3.27 hold.

(IV) The general case: Assume now that K is a reduct of K^+ and that K^{++} is a common definitional expansion of K and K^+. By the previous case we have translation mappings $Tr_1 : Fm(\mathsf{K}^{++}) \longrightarrow Fm(\mathsf{K})$ and $Tr_2 : Fm(\mathsf{K}^{++}) \longrightarrow Fm(\mathsf{K}^+)$ together with appropriate $code_1, code_2$ which satisfy (a)-(c) in the statement of Theorem 6.3.27. Note that $Fm(\mathsf{K}^+) \subseteq Fm(\mathsf{K}^{++})$. Now we define

$$Tr \stackrel{\text{def}}{=} Tr_1 \restriction Fm(\mathsf{K}^+), \;\; code(x, \vec{x}) \stackrel{\text{def}}{=} Tr_2(code_1(x, \vec{x}))$$

whenever x is a variable of new sort in the language of K^+. One can check that Tr and $code$ as defined above satisfy (a)-(c). In more detail: Assume that U_i is a new sort of K^+, i.e. U_i is not a sort of K. Then U_i is a new sort of K^{++}, therefore there is $code_i^1(x, \bar{x}) \in Fm(\mathsf{K}^{++})$ which "matches" Tr_1. We cannot use $code_i^1$ in the interpretation from K^+ to K because $code_i^1$ may not be in the language of K^+. We will use Tr_2 to translate $code_i^1$ to the language of K^+ as follows. Since K^+ is an expansion of K, all the variables in $x, \bar{x}$ have sorts which occur in K^+. Thus by the properties of Tr_2 we have

$$\mathsf{K}^{++} \models code_i^1(x, \bar{x}) \leftrightarrow Tr_2(code_i^1(x, \bar{x})).$$

Let $code_i(x, \bar{x}) \stackrel{\text{def}}{=} Tr_2(code_i^1(x, \bar{x}))$. Then $code_i(x, \bar{x}) \in Fm(\mathsf{K}^+)$ and

$$\mathsf{K}^+ \models code_i(x, \bar{x}) \to [\psi(x, \bar{x}) \leftrightarrow Tr(\psi)(\bar{x}, \bar{z})]$$

because

$$\mathsf{K}^{++} \models code_i^1(x, \bar{x}) \to [\psi(x, \bar{z}) \leftrightarrow Tr_1(\psi)(\bar{x}, \bar{z})].$$

This finishes the proof. ∎

A special case of Theorem 6.3.27 above will be presented in §6.5, cf. items 6.5.5–6.5.6.

More is true than stated in Theorem 6.3.27, namely the existence of a translation mapping as in the theorem is actually sufficient for definability, as the following theorem states.

If Tr and $code_i$ satisfy the conclusion of Theorem 6.3.27, then we say that they *interpret* K^+ in K.[936]

[936]Cf. the definition of interpretations in Hodges [136, p.212, 221]. The existence of a

THEOREM 6.3.28 *Assume* K *is a reduct of* K^+. *Then (i) and (ii) below are equivalent and they imply (iii). If, in addition,* K^+ *is closed under taking ultraproducts, then (i)-(iii) below are equivalent.*

(i) K^+ *and* K *have a common definitional expansion.*

(ii) K^+ *is interpreted in* K *by some* Tr *and* $code_i$*, i.e. the conclusion of Theorem 6.3.27 is true: there are* Tr *and* $code_i$ *satisfying 1, 2 in Theorem 6.3.27.*

(iii) K^+ *is rigidly definable over* K.

A proof of this stronger version is in Andréka-Madarász-Németi [21].

Remark 6.3.29 (In connection with Theorems 6.3.26, 6.3.27.) These theorems state that the expressive powers of two languages $Fm(\mathsf{K}^+)$ and $Fm(\mathsf{K})$ coincide. However, the proofs of these theorems prove more. Namely there exists a *computable* translation mapping Tr acting between the two languages. Even more than this, Tr preserves the logical structure of the formulas i.e. in the sense of algebraic logic, Tr is a "linguistic homomorphism" i.e. grammar preserving mapping, cf. footnote 950 on p.985. (Whether one is interested in this extra property of being a "linguistic homomorphism" is related to a difference between the algebraic logic approach and the abstract model theoretic approach to defining the equivalence of logics [hence, in particular, to how one approaches characterizations of logics like the celebrated Lindström theorems].)

$\lhd$

Definitional equivalence

Until now we dealt with the case when one of the classes was an expansion of the other one. In this sub-section we turn to dealing with the case when our L is not necessarily an expansion of K.

tuple $Tr, code_i$ interpreting K^+ in K (as in Theorem 6.3.27) is strictly stronger than the *uniform reduction property* in [136, p.640]. Actually, the existence of $Tr, code_i$ is equivalent with K^+ being coordinatised over K in the sense of [136, p.644]. This equivalence is proved in [21].

Definition 6.3.30 Let K and L be two classes of models. We say that they are <u>*definitionally equivalent*</u>, in symbols $\mathsf{K} \equiv_\Delta \mathsf{L}$ iff they admit a common (explicit) definitional expansion M (without taking reducts).[937]

Further, $\mathfrak{M} \equiv_\Delta \mathfrak{N}$ abbreviates $\{\mathfrak{M}\} \equiv_\Delta \{\mathfrak{N}\}$. If $\mathfrak{M} \equiv_\Delta \mathfrak{N}$, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ are <u>*definitionally equivalent models*</u>.

$\triangleleft$

Cf. also in Hodges [136] under the name "*definitional equivalence*" pp. 60–61; cf. also Henkin-Monk-Tarski [129, Part I, e.g. p.56].

The relation $\equiv_\Delta$ defined above is symmetric and reflexive. For certain "administrative" reasons it is not transitive,but the counterexamples (to transitivity) are so artificial that we will not meet them (in this work). We could define $\equiv^*_\Delta$ to be the transitive closure of $\equiv_\Delta$ and then use $\equiv^*_\Delta$ as definitional equivalence. If this were a logic book we would do that. However, in the present work we will not need $\equiv^*_\Delta$, hence we do not discuss it, and we call $\equiv_\Delta$ definitional equivalence (though it is $\equiv^*_\Delta$ which is the really satisfactory notion of definitional equivalence.)

Discussion of the definition of $\equiv_\Delta$

(1) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then K and L agree on the common part of their vocabularies.[938] I.e.

$$\mathsf{K} \equiv_\Delta \mathsf{L} \qquad \Rightarrow \qquad \mathsf{K} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}) = \mathsf{L} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L}).$$

(2) For any definitional expansion K^+ of K we have $\mathsf{K} \equiv_\Delta \mathsf{K}^+$.

(3) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then L and K are definable over each other. Moreover their definitions over each other enjoy the following coherence property. Recall that every explicit definition is a special implicit definition. Now, we can choose the definitions of K and L[939] as sets of formulas $\Delta_\mathsf{K}, \Delta_\mathsf{L} \subseteq Fm(\mathsf{M})$ such that $\mathsf{M} \models \Delta_\mathsf{K}$ and $\mathsf{M} \models \Delta_\mathsf{L}$.[940] Here, M is the common definitional expansion of K and L mentioned in the definition of $\equiv_\Delta$. Hence, e.g.

$$\mathsf{Th}(\mathsf{K}) + \text{“the definition of } \mathsf{L} \text{ over } \mathsf{K}\text{”} + \text{“the definition of } \mathsf{K} \text{ over } \mathsf{L}\text{”}$$

[937] I.e. M is a definitional expansion (without taking reducts) of K and the same holds for L in place of K. Note that $\mathsf{Th}(\mathsf{M})$ can be regarded as an implicit definition of M over K, and the same for L in place of K.

[938] As a contrast, $\mathsf{K} \equiv^*_\Delta \mathsf{L}$ does not imply this, however as we said we will not need the generality of $\equiv^*_\Delta$ in this work.

[939] over each other.

[940] Actually, one may choose $\Delta_\mathsf{K} = \Delta_\mathsf{L} = \mathsf{Th}(\mathsf{M})$.

is a conservative extension of the theory $\mathsf{Th}(\mathsf{K})$. Formally $\mathsf{Th}(\mathsf{K}) + \Delta_{\mathsf{K}} + \Delta_{\mathsf{L}}$ is a conservative extension of $\mathsf{Th}(\mathsf{K})$ and the same holds with L in place of K.

(4) Let K, L be classes of (classical) one-sorted models. We say that K is *(classically) definable* over L if K is definable over L without taking reducts (in our sense, cf. p.951). (This means that we allow repeated applications of Step (1) only.) This notion of classical definability of one class over another is in agreement with the spirit of classical one-sorted model theory, cf. e.g. "definitional extension" in [136, p.60] or in [129, Part I, p.56]. Now, there exist classes of one-sorted models K, L such that they are classically definable over each other, their vocabularies are disjoint but $\mathsf{K} \not\equiv_\Delta \mathsf{L}$. Hence, definitional equivalence is stronger than mutual (explicit) definability, cf. Andréka-Madarász-Németi [22].

(5) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then there exists a kind of bijection[941] $f : \mathsf{K} \rightarrowtail\!\!\!\twoheadrightarrow \mathsf{L}$ such that $(\forall \mathfrak{M} \in \mathsf{K})$ $\mathfrak{M}$ and $f(\mathfrak{M})$ are definable over each other. Moreover, the uniform definitions $\Delta_{\mathsf{K}}, \Delta_{\mathsf{L}}$ in item (3) above are such that

$$(*) \qquad \Delta_{\mathsf{L}} \text{ defines } f(\mathfrak{M}) \text{ over } \mathfrak{M}, \quad \text{and } \Delta_{\mathsf{K}} \text{ defines } \mathfrak{M} \text{ over } f(\mathfrak{M})$$

for each $\mathfrak{M} \in \mathsf{K}$.

Further, $\mathfrak{M}$ and $f(\mathfrak{M})$ are reducts of a single model $\mathfrak{M}^+ \in \mathsf{M}$, hence M can be regarded as being a class

$$\{\, \langle \mathfrak{M}, f(\mathfrak{M}) \rangle \;:\; \mathfrak{M} \in \mathsf{K} \,\}$$

of pairs representing the "bijection" (up to isomorphism) f.

$(**)$ When we say that f is a bijection up to isomorphism, then we mean *all* properties of a bijection only up to isomorphism, i.e. f need not be a function instead

$$\langle \mathfrak{M}, \mathfrak{N} \rangle, \langle \mathfrak{M}, \mathfrak{N}_1 \rangle \in f \quad \Rightarrow \quad \mathfrak{N} \cong \mathfrak{N}_1\,,$$

similarly $\mathsf{K} \subseteq \mathbf{I}(Dom(f))$ is stated only instead of $\mathsf{K} = Dom(f)$ etc. In other words this means that f induces a bijection on the isomorphism equivalence classes of K and L, i.e. between $\mathsf{K}/\cong$ and $\mathsf{L}/\cong$. (It is natural to work with the elements of $\mathsf{K}/\cong$ and $\mathsf{L}/\cong$. The elements of $\mathsf{K}/\cong$ are called isomorphism types in [129, Part I, p.71, lines 8–10]).

[941] f is a bijection *only* up to isomorphism. I.e. $f(\mathfrak{M}) \cong f(\mathfrak{N}) \;\Rightarrow\; \mathfrak{M} \cong \mathfrak{N}$, *and* $(\forall \mathfrak{N} \in \mathsf{L})(\exists \mathfrak{M} \in \mathsf{K})\ \mathfrak{N} \cong f(\mathfrak{M})$, cf. $(**)$ for more on this. Our using properties *"up to isomorphism"* is based on the practice in model theory and universal algebra of identifying isomorphic structures in some (but not in all (!)) situations. If we assume $\mathbf{I}\mathsf{K} = \mathsf{K}$ and $\mathbf{I}\mathsf{L} = \mathsf{L}$ then f can be chosen to be a "real" bijection (and not only up to isomorphism).

(6) Assume $\mathsf{K} \equiv_\Delta \mathsf{L}$. Then the bijection $f : \mathsf{K} \rightarrowtail\!\!\!\twoheadrightarrow \mathsf{L}$ in (5) above has the following property. For all $\mathfrak{M} \in \mathsf{K}$, the automorphism group of $\mathfrak{M}$ is isomorphic to the automorphism group of $f(\mathfrak{M})$, in symbols

$$\langle Aut(\mathfrak{M}), \circ\rangle \cong \langle Aut(f(\mathfrak{M})), \circ\rangle.$$

This is so because of the following. Let $\mathfrak{M}^+ \in \mathsf{M}$ be such that $\mathfrak{M}^+$ is implicitly definable without taking reducts both over $\mathfrak{M}$ and over $f(\mathfrak{M})$. Since $\mathfrak{M}^+$ is implicitly definable without taking reducts over $\mathfrak{M}$, each automorphism of $\mathfrak{M}$ extends in a unique way to an automorphism of $\mathfrak{M}^+$, and this implies that the automorphism groups of $\mathfrak{M}$ and $\mathfrak{M}^+$ are isomorphic. We get the same for $f(\mathfrak{M})$ and $\mathfrak{M}^+$ completely analogously, and this proves that the automorphism groups of $\mathfrak{M}$ and $f(\mathfrak{M})$ are isomorphic.

(7) For more on definitional equivalence, its importance, and for motivation for the way we defined and use $\equiv_\Delta$ we refer to [129, pp. 56-57, Remark 0.1.6], [136, pp. 58-61].

$\lhd$

* * *

Two *theories* Th_1, Th_2 are called <u>*definitionally equivalent*</u> iff

$$\mathsf{Mod}(Th_1) \equiv_\Delta \mathsf{Mod}(Th_2).$$

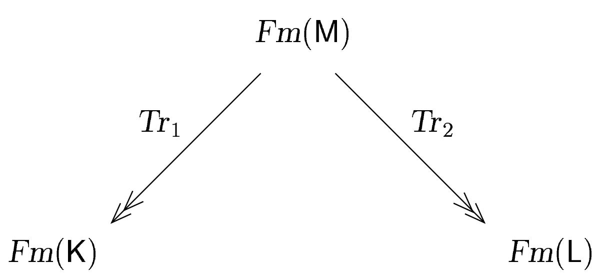
In view of the discussion of the two translation theorems (Theorems 6.3.26 and 6.3.27 way above), one can say that two definitionally equivalent theories can be regarded as being *essentially the same* theory and the difference between them is only that their *"syntactic decorations"* are different (i.e. they "choose" to represent their [essentially] common language with *different basic vocabularies*).

The same applies to classes of models K, L when $\mathsf{K} \equiv_\Delta \mathsf{L}$. As an example, choose K to be Boolean algebras with $\{\cap, -\}$ as their basic operations while choose L to be Boolean algebras with $\{\cup, -, 0, 1\}$ as basic operations. (Then $\mathsf{K} \equiv_\Delta \mathsf{L}$.) At a certain level of abstraction, K and L can be regarded as a collection of the *same* mathematical structures (namely, Boolean algebras) and the difference (between K and L) is only in the choice of their basic vocabularies (which is "$\cap, -$" in the one case while "$\cup, -, 0, 1$" in the other). Summing up: In some sense, definitionally equivalent theories $Th_1 \equiv_\Delta Th_2$ can be considered as just one theory with two *different linguistic representations.* The same applies to definitionally equivalent classes of models.

The above expounded train of thoughts works really smoothly if we restrict our attention to finite vocabularies (for our theories, and classes of models). However, it easily extends to infinite vocabularies if in the definition of "K^+ is definable over K" we permit infinitely many new symbols (sorts and relations) to be defined. The only reason for our not discussing this "infinitary" case is that in the present work we work with theories etc. of finite vocabularies (practically all the time).

Remark 6.3.31 (How and why can definitionally equivalent theories [and classes of models] be regarded as identical [as a corollary of Theorems 6.3.26, 6.3.27]?)
In addition to the text below, we also refer the reader to [129, p.56] and [136, pp.58–61] for explanations of why definitionally equivalent classes of models can be regarded as (in some sense) identical.

Let K and L be two definitionally equivalent classes of models (formally, $\mathsf{K} \equiv_\Delta \mathsf{L}$). Then, by the definition of $\equiv_\Delta$, there is a class M which is a definitional expansion (without taking reducts) of both K and L. We will argue below that this M establishes a very strong connection between K and L. (Cf. also item (5) in the discussion of the definition of $\equiv_\Delta$.) Our argument begins with the following: We can apply Theorem 6.3.26 to the pair M and K with M in place of K^+ in that theorem. The same applies to the pair M and L. By Theorem 6.3.26, then we have two translation mappings



both of which preserve meaning (in the sense of Theorem 6.3.26). Both of Tr_1 and Tr_2 are surjective. Intuitively, Tr_1 identifies K with M while Tr_2 identifies M with L. Hence K *gets identified with* L. (Perhaps the best way of thinking about this is that we identify both K and L with their common expansion M. As a by-product of this we identify K and L with each other, too.)

By surjectiveness of Tr_1 and Tr_2, whatever can be said in the language $Fm(\mathsf{K})$, the same can be said in $Fm(\mathsf{M})$ and hence (using Tr_2) the same can be said in the language $Fm(\mathsf{L})$ of L. Similarly, whatever can be said in $Fm(\mathsf{L})$ the same can be said in $Fm(\mathsf{K})$, too.

Now, if we want some more detail, let $\varphi(\bar{z}) \in Fm(\mathsf{K})$ with a sequence $\bar{z}$ of variables belonging to common sorts K and L. Then there are $\varphi'(\bar{z}) \in Fm(\mathsf{M})$, $\varphi''(\bar{z}) \in Fm(\mathsf{L})$ such that $Tr_1(\varphi') = \varphi$ and $Tr_2(\varphi') = \varphi''$. I.e.

$$\varphi(\bar{z}) \xleftarrow{\ Tr_1\ } \varphi'(\bar{z}) \xmapsto{\ Tr_2\ } \varphi''(\bar{z}).$$

Actually, we can choose $\varphi' = \varphi$ if we want to. Using Theorem 6.3.26 we can conclude

(361) $$\mathsf{M} \models \varphi(\bar{z}) \leftrightarrow \varphi''(\bar{z}).$$

I.e. the same things can be said about the common variables $\bar{z}$ in $Fm(\mathsf{K})$ and in $Fm(\mathsf{L})$. Hence the languages of K and L have the *same* expressive power.

On the basis of (361) above and what was said before (361), we can introduce two, more direct, translation mappings

$$Fm(\mathsf{K}) \underset{T_1}{\overset{T_2}{\rightleftarrows}} Fm(\mathsf{L})$$

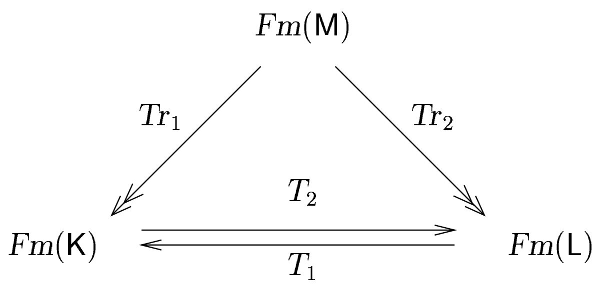
defined as follows. In defining T_1 and T_2 we can rely on the fact that

$$Fm(\mathsf{K}) \subseteq Fm(\mathsf{M}) = Dom(Tr_1)$$

and that $Tr_1 \restriction Fm(\mathsf{K}) = \mathrm{Id} \restriction Fm(\mathsf{K})$ which is the identity function. Hence we can choose

$$\begin{aligned} T_1 &:= Tr_1 \restriction Fm(\mathsf{L}) \quad \text{and} \\ T_2 &:= Tr_2 \restriction Fm(\mathsf{K}).^{942} \end{aligned}$$

[942] In passing, we also note that Tr_1 can be regarded as *injective in the sense* that if $\psi(\bar{z}), \gamma(\bar{z}) \in Fm(\mathsf{M})$ involve free variables of K only then $[\,Tr_1(\psi) = Tr_1(\gamma) \;\Rightarrow\; \mathsf{M} \models \psi(\bar{z}) \leftrightarrow \gamma(\bar{z})]$. Similarly for Tr_2 and L.



Assume $\varphi \in Fm(\mathsf{M})$ involves only common free variables of K and L. Then

$$\begin{aligned} \mathsf{M} &\models (T_2 Tr_1\varphi) \leftrightarrow Tr_2\varphi. \\ \mathsf{M} &\models (T_1 Tr_2\varphi) \leftrightarrow Tr_1\varphi. \end{aligned}$$

So in this "logical sense" the above diagram commutes.

For completeness, about the above diagram we also note the following commutativity property:

$$\begin{aligned} T_2 &\subseteq (Tr_1)^{-1} \circ Tr_2\,, \\ T_1 &\subseteq (Tr_2)^{-1} \circ Tr_1\,. \end{aligned}$$

Here we note that $(Tr_1)^{-1} \circ Tr_2$ is a binary relation but not necessarily a function.

Using Theorem 6.3.26, and (361) way above, one can check that for all $\varphi \in Fm(\mathsf{K})$ and for all $\psi \in Fm(\mathsf{L})$, if φ and ψ use only variables of common sorts (of K and L) then:

(362)
$$\begin{aligned} \mathsf{M} &\models \varphi(\bar{z}) \leftrightarrow (T_2\varphi)(\bar{z}), \\ \mathsf{M} &\models (T_1\psi)(\bar{z}) \leftrightarrow \psi(\bar{z}), \qquad \text{further} \end{aligned}$$

(363)
$$\begin{aligned} \mathsf{K} &\models \varphi(\bar{z}) \leftrightarrow (T_1T_2\varphi)(\bar{z}), \\ \mathsf{L} &\models \psi(\bar{z}) \leftrightarrow (T_2T_1\psi)(\bar{z}). \end{aligned}$$

These statements can be interpreted as saying that T_1 and T_2 are kind of *inverses* of each other and that they establish a kind of logical isomorphism between equivalence classes of formulas in $Fm(\mathsf{K})$ and $Fm(\mathsf{L})$ involving free variables of common sorts only. For completeness, we note that (362–363) can be generalized to formulas involving free variables of arbitrary sorts by using Theorem 6.3.27. For formulating this generalized version of (362–363) one needs to use the formulas "*code*" as they were used in Theorem 6.3.27. E.g. the first line of (363) becomes

$$\mathsf{K} \models code(x, \vec{x}) \rightarrow [\varphi(x, \bar{z}) \leftrightarrow (T_1 T_2 \varphi)(\vec{x}, \bar{z})],$$

where x belongs to a sort of K not in L, and $\bar{z}$ is a sequence of variables of common sorts of K and L. Here $code(x, \vec{x})$ is the formula we get from combining the corresponding formulas belonging to Tr_1 and Tr_2. We leave the details of generalizing (361–363) to treating free variables not in the common language to the interested reader.

(We note that the generalization of (363) above reminds us of the notion of equivalence between two categories, in the sense of category theory, cf. Definition 6.6.82 on p.1094.)

We hope, the above shows how and to what extent we consider two definitionally equivalent classes (and theories) as being essentially identical.

◁

A generalization of Beth's theorem[943]

Many-sorted definability theory with new sorts (i.e. the notion of implicit and explicit definition) is a generalization of one-sorted definability theory (without new elements) discussed in traditional logic books. This observation leads to several natural questions which we discuss here only tangentially. One of these is the question whether Beth's theorem (about the equivalence of the two notions of definability) generalizes to our present case.

THEOREM 6.3.32 *Assume* $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ *is a reduct of* K^+ *such that* K^+ *has only finitely many sorts. Assume that the language of* K^+ *is countable, and that* K *has a sort with more than one element. Then (i) and (ii) below are equivalent.*

(i) K^+ *is implicitly definable over* K *without taking reducts.*

(ii) K^+ *is a definitionally equivalent expansion of* K.

The **proof** uses Gaifman's theorem (cf. Hodges [136, Thm.12.5.8, p.645]), which is about one-sorted structures, together with ideas from Pillay & Shelah [214], and can be found in Andréka-Madarász-Németi [21]. ∎

[943]Acknowledgement: The results in this sub-section were obtained with help from Wilfrid Hodges.

COROLLARY 6.3.33 (Beth's theorem generalized to defining new sorts) Assume $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ is a reduct of K^+ such that K^+ has only finitely many sorts. Assume that the language of K^+ is countable, and that K has a sort with more than one element. Then (i) and (ii) below are equivalent.

(i) K^+ is implicitly definable over K.

(ii) K^+ is explicitly definable over K.

QUESTION 6.3.34 *Can Theorem 6.3.32 and Corollary 6.3.33 above be generalized for the case when infinitely many sorts are allowed? (First one has to generalize the definition of explicit definability. This can be done easily, e.g. we may allow iteration of steps (1), (2.1), (2.2) along an infinite ordinal, taking "unions" of ascending chains of expansions in the limit steps.)* $\lhd$

The above question seems to be more about logic than about relativity, so we do not discuss it here. We did not have time to think about it, but it seems to be an interesting question.

Connections with the literature

For investigations related to definability of new sorts as discussed in the present section (§6.3 herein) we refer to Hodges [136] Chapter 12, and within that chapter to §12.3 (pp.624-632), §12.5 (pp.638-652). E.g. p.638 last 3 lines – p.639 line 9 discusses generalizability of Beth's theorem, and similarly for p.645 line 6, p.649 lines 5-6. (We would also like to point out Exercises 13, 14 on p.649 of [136].) We also refer to Myers [200], Hodges-Hodkinson-Macpherson [137], Pillay-Shelah [214], Shelah [239]. In passing we note that our subject matter (i.e. definability of new sorts) is related to the directions in recent (one-sorted) model theory called "relative categoricity" or "categoricity over a predicate", and "theory of stability over a predicate".

Below we outline some connections between our notions and the ones used in a substantial part of the above quoted (one-sorted) literature. We will systematically refer to Hodges [136].

Assume $\mathsf{K}^+ = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$ and that K has finitely many sorts $U_0, \ldots, U_k$. Let $P = U_0 \cup \ldots \cup U_k$ be the union of these sorts regarded as a unary predicate. Then:

(1) "K^+ is implicitly definable up to isomorphism over K" is equivalent with "$\mathsf{Th}(\mathsf{K}^+)$ is relatively categorical over P".

(2) "K^+ is implicitly definable without taking reducts over K" is equivalent with "$\mathsf{Th}(\mathsf{K}^+)$ is rigidly relatively categorical over P".

(3) "K^+ is explicitly definable over K" is not equivalent with "$\mathsf{Th}(\mathsf{K}^+)$ is coordinatisable over P".

(4) "K^+ is a definitionally equivalent expansion of K" is equivalent with "$\mathsf{Th}(\mathsf{K}^+)$ is coordinatised over P".

In items (1)-(4) above, on the left hand side we have many-sorted notions, while on the right-hand side we have one-sorted notions (like relative categoricity). So it needs some explanation what we mean by claiming their equivalence. The answer is the following: First we translate our many-sorted notions to one-sorted ones (by treating the sorts as unary predicates of one-sorted logic) the usual, natural way, and then we claim that the so translated version of our many-sorted notion is equivalent with the other one-sorted notion quoted from Hodges [136]. E.g., the so elaborated version of item (1) looks like the following. "The one-sorted translation of (K^+ is implicitly definable up to isomorphism over K)" is equivalent with "(the one-sorted version of $\mathsf{Th}(\mathsf{K}^+)$) is relatively categorical over P". The point here is that relative categoricity is defined only for one-sorted logic in Hodges [136]. Therefore, to use it as a possible equivalent of (our many-sorted) "implicit definability up to isomorphism", first we have to translate everything to one-sorted logic, and then make the comparison. Indeed, items (1)-(4) are understood this way.

Connections between the various notions of definability

Figure 305 below shows the connections between the various notions introduced in this sub-section. It also indicates the above outlined connections with some notions used in the literature (relative categoricity, coordinatisability). The connections indicated are fairly easy to show, except for the following proposition (and, of course where Theorem 6.3.32 and Corollary 6.3.33 are indicated).

PROPOSITION 6.3.35 (Hodges [136]) *Assume the hypotheses of Theorem 6.3.32 (which are the same as the hypotheses used in Figure 305). Then "K^+ is implicitly definable over K up to isomorphism" does not imply "K^+ is implicitly definable over K".*

Proof. A 6-element counterexample proving this is given in Hodges [136, Example 2 on p.625]. There two structures are defined, **A** and **B**, with **A** a reduct of **B**. **B**

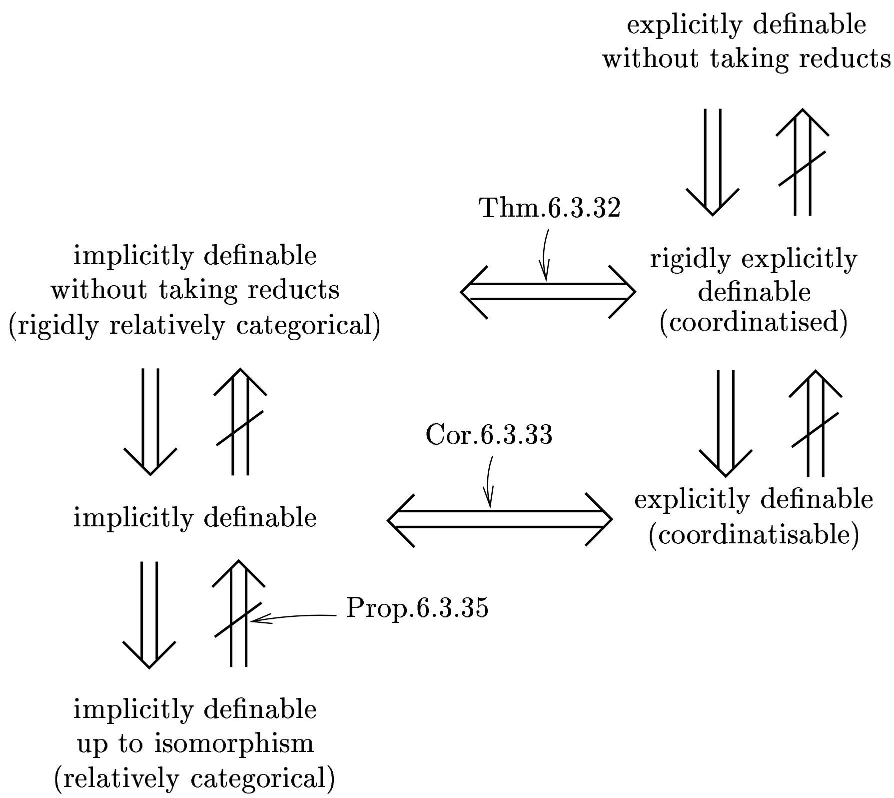


Figure 305: Connections between the various notions of definability. We assume that $\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$ is a reduct of K^+ such that K^+ has only finitely many new sorts. We also assume that the language of K^+ is countable, and that K has a sort with more than one element. On the figure we write "implicitly definable without taking reducts" for "K^+ is implicitly definable over K without taking reducts", and similarly for the other notions. For the implication "implicitly definable" to "implicitly definable up to isomorphisms" we need the extra assumption $\mathsf{K}^+ = \mathsf{Mod}(\mathsf{Th}(\mathsf{K}^+))$.

is implicitly definable up to isomorphism over $\mathbf{A}$ (this follows from the fact that $\mathbf{B}$ is finite). At the same time, $\mathbf{B}$ is not definable implicitly over $\mathbf{A}$, because $\mathbf{A}$ has an automorphism α of order 2 (i.e. $\alpha \circ \alpha = Id_A$) which cannot be extended to an automorphism β of $\mathbf{B}$ of order 2. Indeed, if $\mathbf{B}$ was implicitly definable over $\mathbf{A}$, then an expansion $\mathbf{B}^+$ of $\mathbf{B}$ would be implicitly definable over $\mathbf{A}$ without taking reducts. Hence the automorphism α would extend to an automorphism β of $\mathbf{B}^+$. Since the identity of $\mathbf{A}$ extends to a unique automorphism of $\mathbf{B}^+$, then $\beta \circ \beta = Id_{B^+}$ should hold. But then $\beta \restriction B$ would be an automorphism of $\mathbf{B}$ of order 2 and extending α. (Cf. Thm.12.5.7 in [136, p.644].) Since $\mathbf{A}$ and $\mathbf{B}$ are finite structures, we can take $\mathsf{K} = \mathbf{I}\{\mathbf{A}\}$ and $\mathsf{K}^+ = \mathbf{I}\{\mathbf{B}\}$, and then the hypotheses of Proposition 6.3.35 hold for K and K^+. This finishes the proof. ∎

On the choice of basic steps in explicit definability

Remark 6.3.36 (Forming disjoint union of two sorts) For didactical reasons we will refer to items (1)–(2.2) as steps (1)–(2.2) to emphasize their roles in constructing an explicit definition (for some new class K^+) in a step-by-step manner.

We could have included in this list of steps as step (2.3) the definition of a new sort as a disjoint union of two old sorts. This goes as follows:

Assume U_k, U_m are old sorts, i.e. sorts of $\mathfrak{M}$, while U^{new} is not a sort of $\mathfrak{M}$. Then, we can define the new sort as

$$U^{new} := U_k \,\dot{\cup}\, U_m$$

with two injections

$$i_1 : U_k \rightarrowtail U^{new} \qquad \text{and} \qquad i_2 : U_m \rightarrowtail U^{new}$$

such that U^{new} is the union of $Rng(i_1)$, $Rng(i_2)$ and $Rng(i_1) \cap Rng(i_2) = \emptyset$. Here $k = m$ is permitted. But even if $k = m$, i_1 and i_2 are different. Now the expanded model is

$$\mathfrak{M}^+ := \langle \mathfrak{M}, U^{new};\ i_1, i_2 \rangle.$$

We note that such an $\mathfrak{M}^+$ is always implicitly definable over $\mathfrak{M}$, further *all the nice properties* of *explicit definitions*[944] in items (1)–(2.2) hold for this new kind of explicit definition which from now on we will consider as step (2.3) of explicit definability.

All the same, we do not include step (2.3) into the list of permitted steps of building up an explicit definition. We have two reasons for this.

[944] As an example we mention that explicitly defined symbols can be eliminated from the language, cf. sub-section 6.3.3 on p.962.

(i) Step (2.3) can be reduced to (or simulated by) steps (1)–(2.2). Namely, assume $\mathfrak{M}^+$ is defined from $\mathfrak{M}$ by using step (2.3). Assume further that $\mathfrak{M}$ has a sort U_i with more than one elements (i.e. $|U_i| > 1$). Then by using steps (1)–(2.2) one can define an expansion $\mathfrak{M}^{++}$ from $\mathfrak{M}$ such that $\mathfrak{M}^+$ is a reduct of $\mathfrak{M}^{++}$.[945]

Further:

(ii) We will not need step (2.3) in the present work. I.e. in the logical analysis of relativity, explicit definitions of form (2.3) did not come up so far.

Item (i) above shows that adding step (2.3) to the permitted steps of explicit definitions would increase the collection of sorts and relations definable over $\mathfrak{M}$ *only* in the pathological case when all universes of $\mathfrak{M}$ have cardinalities ≤ 1.

Therefore while noting that step (2.3) could be included without changing the theory of explicit definability significantly, we do *not* include it. However, sometime (in some intuitive text) when we want to get "dreamy" we might refer to explicit definability as involving four steps (1)–(2.3). $\lhd$

Remark 6.3.37 One might want to develop a more systematic understanding of what explicit definitions are. For such a more systematic understanding of explicit definitions let us rearrange the basic steps into steps (1*)–(5*) below.

(1*) Definition of new relations $\bar{R}^{new}$ explicitly the classical way (as in item (1) on p.945).

(2*) Definition of new sorts as *direct products* of old sorts together with projection functions ($U^{new} := U_i \times U_j$ etc) (as in item (2.1) on p.947).

(3*) Definition of new sorts as *disjoint unions* of old sorts together with inclusion functions ($U^{new} := U_i \,\dot{\cup}\, U_j$ etc) (as in item (2.3) on p.980).

(4*) Definition of a new sort as a definable *subset* of an old sort together with an inclusion function. I.e.

$$U^{new} := \{\, x \in U_i \;:\; \mathfrak{M} \models \psi(x) \,\}$$

and $i_{new} : U^{new} \longrightarrow U_i$ is the usual inclusion function. The expanded model is $\mathfrak{M}^+ = \langle \mathfrak{M}, U^{new}; i_{new} \rangle$.

(5*) Definition of a new sort as a definable *quotient* of an old sort exactly as in item (2.2) on p.949 (i.e. $U^{new} = U_i/R$ etc).

[945] More precisely there is a *unique* isomorphism h between $\mathfrak{M}^+$ and this reduct of $\mathfrak{M}^{++}$ such that $h \restriction \mathfrak{M}$ is the identity function.

Now, an explicit definition in the new sense is given by an arbitrary sequence (i.e. iteration) of steps (1^*)–(5^*) above.

If we disregard the trivial case when all sorts are singletons or empty, then explicit definitions in the new sense are equivalent with explicit definitions as introduced in §6.3.2. We leave checking this claim to the reader.

We would like to point out that explicit definitions as built up from steps (1^*)–(5^*) are not ad-hoc at all. In the category theoretic sense the formation of disjoint unions is the *dual* of the formation of direct products and the formation of sub-universes (or sub-structures) is the dual of the formation of quotients. So, we are left with two basic steps and their duals.

It is interesting to note that our steps (2^*)–(5^*) correspond to basic operations producing new models from old ones. (Indeed if U_i is a universe of $\mathfrak{M}$ then we can restrict $\mathfrak{M}$ to U_i and then we obtain a one-sorted reduct of $\mathfrak{M}$ with universe U_i. Hence creating new sorts from old ones is not unrelated to creating new models from old ones. All the same, we do not want to stretch this analogy too far.)

What we would like to point out here, is that steps (2^*)–(5^*) seem to form a natural, well balanced set of basic operations, while step (1^*) has been inherited from the classical theory of definability.

Further, we note that while selecting our basic steps (e.g. steps (1^*)–(5^*) above) we had to be careful to keep them implicitly definable i.e. they should not lead to "explicitly definable things" which are not implicitly definable. Therefore operations like formation of powersets cf. Example 6.3.9(1) (or all finite subsets of a set cf. Example 6.3.9(6))[946] are ruled out from the beginning.

$\triangleleft$

Further recent results on definability theory (sometimes in algebraic form[947]) are in Madarász [173], [170], [169], Madarász-Sayed [178], Hoogland [138].

[946]Seeing that $\mathcal{P}_\omega(U_i)$ leads to problems (i.e. checking Example 6.3.9(6)) is not obvious, it is not necessary to check this for understanding this work.

[947]Cf. our section on duality theory (in particular § 6.6.7).

6.4 Weak definitional equivalence and related concepts

The present section is a continuation of §6.3.

CONVENTION 6.4.1 $f : A \longrightarrow\!\!\!\!\rightarrow\!\!\succ B$ denotes that f is a surjective function from A onto B. Further $f : A \rightarrowtail B$ denotes that f is an injective function from A into B. I.e. $\longrightarrow\!\!\!\!\rightarrow\!\!\succ$ denotes surjectiveness, while $\rightarrowtail$ denotes injectiveness. (If we combine the two then we obtain $\rightarrowtail\!\!\!\!\rightarrow\!\!\succ$ denoting bijectiveness.) When used between german letters, i.e. structures, they denote injectiveness or surjectiveness of homomorphisms the natural way. Cf. Def.6.6.3(i) on p.1008. $\lhd$

Definition 6.4.2 Let K and L be two classes of models and let $f : \mathsf{K} \longrightarrow \mathsf{L}$ be a function. We say that f is a <u>*first-order definable meta-function*</u> iff for each $\mathfrak{M} \in \mathsf{K}$ $f(\mathfrak{M})$ is first-order definable over $\mathfrak{M}$ (in the sense of §6.3.2) and the definition of $f(\mathfrak{M})$ over $\mathfrak{M}$ is uniform, i.e. is the same for all choices of $\mathfrak{M} \in \mathsf{K}$.[948]

$\lhd$

A typical example for first-order definable meta-functions will be e.g. $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$, where $\mathcal{G} : \mathfrak{M} \mapsto \mathfrak{G}_{\mathfrak{M}}$, if Th is strong enough, cf. Thm.6.3.22 (p.961). A similar example will be a kind of inverse to this function $\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$, cf. Prop.6.6.44 (p.1059) and Def.6.6.41 (p.1054).

We note that if $f : \mathsf{K} \longrightarrow\!\!\!\!\rightarrow\!\!\succ \mathsf{L}$ is a surjective first-order definable meta-function then L is definable over K; and, more generally, if $f : \mathsf{K} \longrightarrow \mathsf{L}$ is a first-order definable meta-function then $Rng(f)$ is definable over K. In the other direction, if $\mathsf{L} = \mathbf{I}\mathsf{L}$ is definable over K then there is a first-order definable meta-function $f : \mathsf{K} \longrightarrow \mathsf{L}$ such that $Rng(f)$ is L up to isomorphism. To be able to claim this for the case when $\mathsf{L} \neq \mathbf{I}\mathsf{L}$ we make the following convention.

CONVENTION 6.4.3 (Class form of the axiom of choice)
In connection with the above definition, for simplicity, throughout the present chapter we assume the <u>*class form of the axiom of choice*</u>. More concretely we assume that our set theoretic universe V is well orderable by the class $\mathsf{Ordinals}$ of ordinal numbers. I.e. there is a bijection

$$f : \mathsf{Ordinals} \rightarrowtail\!\!\!\!\rightarrow\!\!\succ V.$$

[948] A first-order definable <u>meta-function</u> (acting between classes of models) is a rather <u>different</u> kind of thing from an ordinary function like *factorial* : $N \longrightarrow N$ definable in a model, say in $\mathfrak{N} \in \mathsf{Mod}$(Peano's arithmetic), cf. Example 6.3.8(1) on p.938. (This is the reason why we call f a meta-function and not simply a function.)

This implies that any proper class is well orderable and therefore there exists a bijection between any two proper classes.

◁

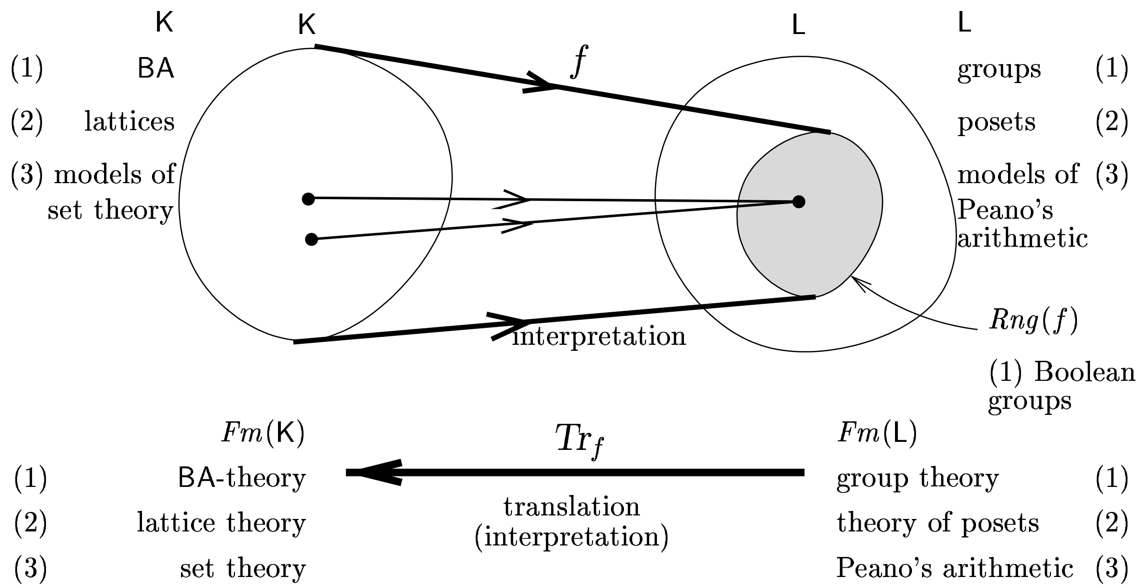


Figure 306: Examples for first-order definable meta-functions f and the induced translations between theories. For more explanation in connection with this picture cf. item (III) of Remark 6.6.4, pp. 1020–1027. The corresponding theories are labelled by the same numbers. E.g. BA is interpreted in "groups", "lattices" in "posets" etc. Here f, or the pair $\langle f, Tr_f \rangle$, or Tr_f are (often) called *interpretations*, cf. footnote 1022 on p.1023. E.g. Tr_f interprets group theory in BA-theory. Equivalently f interprets BA's in groups. (This figure also serves as an illustration for Prop.6.4.4, p.985.)

The following proposition makes connections between the following three things: (i) "interpretations" of one theory in another, (ii) first-order definable meta-functions $f : \mathsf{K} \longrightarrow \mathsf{L}$ between classes of models, and (iii) definability of a class $Rng(f)$ over another class K, see Fig.306. In this context the function Tr_f (in the proposition) below is what we call an interpretation (or translation). Cf. item (III) of Remark 6.6.4 on p.1020 and footnote 1022 on p.1023 for the intuitive idea behind interpretations.[949] In particular the proposition says that any first-order definable meta-function $f : \mathsf{K} \longrightarrow \mathsf{L}$ induces a natural syntactical translation mapping from the language $Fm(\mathsf{L})$ of L to that of K. Moreover, this translation is meaning preserving w.r.t. the semantical function f.[950]

PROPOSITION 6.4.4 *Assume $f : \mathsf{K} \longrightarrow \mathsf{L}$ is a first-order definable meta-function. Then there is a "natural" translation mapping*

$$Tr_f : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{L})$ with all free variables belonging to common sorts of K and L [951]*, $\mathfrak{A} \in \mathsf{K}$ and evaluation $\bar{a}$ of $\bar{x}$ in the common sorts (i.e. universes) of $\mathfrak{A}$ and $f(\mathfrak{A})$ the following holds.*[952]

$$f(\mathfrak{A}) \models \varphi[\bar{a}] \quad \Leftrightarrow \quad \mathfrak{A} \models Tr_f(\varphi)[\bar{a}].$$

Cf. Fig.306.

Proof: The proposition follows easily by Thm.6.3.26 (first translation theorem) on p.962. In more detail: Assume $f : \mathsf{K} \longrightarrow \mathsf{L}$ is a first-order definable meta-function. Then there is an expansion K^+ of $Rng(f)$ such that K^+ is definable over K without taking reducts. Then, by Thm.6.3.26, there is a translation mapping $Tr : Fm(\mathsf{K}^+) \longrightarrow Fm(\mathsf{K})$ such that $(\star)$ in Thm.6.3.26 holds. Let $Tr_f := Tr \restriction Fm(\mathsf{L})$. One can check that Tr_f has the desired properties. ∎

We will have results analogous to the conclusion of Prop.6.4.4 above at various points in the remaining part of this chapter, cf. e.g. Thm.6.6.45 on p.1061.

The following is a weaker form of definitional equivalence. We will use it e.g. in Thm.6.6.29 (p.1045).

[949] In the one-sorted case an interpretation $Tr : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$ is the same thing as a cylindric algebraic homomorphism between the cylindric algebras of formulas $Fm(\mathsf{L})$ and $Fm(\mathsf{K})$. I.e. if we endow $Fm(\mathsf{L})$ with the cylindric algebraic structure (of first-order formulas) and do the same with $Fm(\mathsf{K})$ then the homomorphisms between the two algebras of formulas are typical examples of interpretations.

[950] Translation functions of the type $Tr : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$ play an important role in the present work. They have two important features: (i) they are meaning preserving, and (ii) they *respect the logical structure* of the languages involved, e.g. $Tr(\neg\varphi) = \neg\, Tr(\varphi)$ and analogously for the remaining parts of our logic. (We do not discuss property (ii) explicitly, but since it is important we mention that it is discussed in the algebraic logic works e.g. in Andréka et al. [30].) In other words (ii) could be interpreted as saying that our translation mappings are grammatical, i.e. they respect the grammar of the languages involved. Cf. Remark 6.3.29 on p.969.

[951] i.e. to $Voc_0\mathsf{K} \cap Voc_0\mathsf{L}$

[952] We note that the formulas φ and $Tr_f(\varphi)$ have the same free variables (therefore the statement below makes sense).

Definition 6.4.5 (Weak definitional equivalence)
Let K, L be two classes of models. K and L are called *weakly definitionally equivalent*, in symbols

$$\mathsf{K} \equiv_{\Delta}^{w} \mathsf{L},$$

iff there are first-order definable meta-functions

$$f : \mathsf{K} \longrightarrow \mathsf{L} \quad \text{and} \quad g : \mathsf{L} \longrightarrow \mathsf{K}$$

such that for any $\mathfrak{M} \in \mathsf{K}$ and $\mathfrak{G} \in \mathsf{L}$, (i) and (ii) below hold.

(i) $(f \circ g)(\mathfrak{M}) \cong \mathfrak{M}$ and $(g \circ f)(\mathfrak{G}) \cong \mathfrak{G}$, and

(ii) moreover there is an isomorphism between the two structures $\mathfrak{M}$ and $(f \circ g)(\mathfrak{M})$ which is the identity map on the reduct $\mathfrak{M} \restriction (Voc\mathsf{K} \cap Voc\mathsf{L})$ [953] of $\mathfrak{M}$. Similarly for structures $\mathfrak{G}$ and $(g \circ f)(\mathfrak{G})$.

$\lhd$

Intuitively, K and L are weakly definitionally equivalent iff they are definable over each other and the first-order definable meta-functions induced by these definitions are inverses of each other up to isomorphism.

PROPOSITION 6.4.6 *Assume K, L are two classes of models. Then*

$$\mathsf{K} \equiv_{\Delta} \mathsf{L} \quad \Rightarrow \quad \mathsf{K} \equiv_{\Delta}^{w} \mathsf{L},$$

i.e. if K and L are definitionally equivalent then they are also weakly definitionally equivalent.

We omit the **proof**. ∎

In connection with the above proposition we note that the other direction does not hold in general, i.e.

$$\mathsf{K} \equiv_{\Delta}^{w} \mathsf{L} \quad \not\Rightarrow \quad \mathsf{K} \equiv_{\Delta} \mathsf{L}.$$

This (i.e. $\not\Rightarrow$) is so even if we assume that K and L are both axiomatizable, cf. Examples 6.4.9 (p.988) and Thm.6.6.29 (p.1045).

Examples come at the end of this section.

[953] $Voc\mathsf{K} \cap Voc\mathsf{L}$ is the common part of the vocabularies of K and L.

Remark 6.4.7 Assume that $f : \mathsf{K} \longrightarrow \mathsf{L}$ and $g : \mathsf{L} \longrightarrow \mathsf{K}$ are first-order definable meta-functions as in Def.6.4.5. Then $Rng(f)$ is L up to isomorphism and $Rng(g)$ is K up to isomorphism. Moreover, for every $\mathfrak{A} \in \mathsf{L}$ there is $\mathfrak{A}' \in Rng(f)$ such that there is an isomorphism between the structures $\mathfrak{A}$ and $\mathfrak{A}'$ which is the identity map on the reduct $\mathfrak{A} \upharpoonright (Voc\mathsf{K} \cap Voc\mathsf{L})$ of $\mathfrak{A}$; and the analogous statement holds for every $\mathfrak{B} \in \mathsf{K}$.

$\lhd$

The following proposition says that if $\mathsf{K} \equiv^w_\Delta \mathsf{L}$ then the language $Fm(\mathsf{K})$ of K can be translated into the language $Fm(\mathsf{L})$ of L in a meaning preserving way and vice-versa; more precisely these translations work well for the sentences[954] only or more generally for those formulas which contain only such free variables that range over the common sorts of K and L. Moreover these translation mappings are inverses of each other (up to logical equivalence "$\leftrightarrow$"). We note that if in addition we have $\equiv_\Delta$ in place of $\equiv^w_\Delta$ [955] then this nice, meaning preserving translation mapping extends to all formulas, cf. the end of Remark 364 on p.976.

PROPOSITION 6.4.8 *Assume* $\mathsf{K} \equiv^w_\Delta \mathsf{L}$. *Then there are "natural" translation mappings*

$$T_f : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K}) \quad and \quad T_g : Fm(\mathsf{K}) \longrightarrow Fm(\mathsf{L})$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{L})$, $\psi(\bar{y}) \in Fm(\mathsf{K})$ *with all their free variables belonging to common sorts of* K *and* L, $\mathfrak{A} \in \mathsf{L}$ *and* $\mathfrak{B} \in \mathsf{K}$, *and evaluations* $\bar{a}, \bar{b}$ *of the variables* $\bar{x}, \bar{y}$, *respectively, (i)–(iv) below hold, where* f *and* g *are as in Def.6.4.5.*

(i) $f(\mathfrak{B}) \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathfrak{B} \models T_f(\varphi)[\bar{a}]$ *and* $g(\mathfrak{A}) \models \psi[\bar{b}] \;\Leftrightarrow\; \mathfrak{A} \models T_g(\psi)[\bar{b}]$.

(ii) $\mathfrak{A} \models \varphi[\bar{a}] \;\Leftrightarrow\; g(\mathfrak{A}) \models T_f(\varphi)[\bar{a}]$ *and* $\mathfrak{B} \models \psi[\bar{b}] \;\Leftrightarrow\; f(\mathfrak{B}) \models T_g(\psi)[\bar{b}]$.

(iii) $\mathfrak{A} \models \varphi(\bar{x}) \leftrightarrow (T_f \circ T_g)(\varphi)(\bar{x})$ *and* $\mathfrak{B} \models \psi(\bar{y}) \leftrightarrow (T_g \circ T_f)(\psi)(\bar{y})$.

(iv) $\mathsf{L} \models \varphi \;\Leftrightarrow\; \mathsf{K} \models T_f(\varphi)$ *and* $\mathsf{K} \models \psi \;\Leftrightarrow\; \mathsf{L} \models T_g(\psi)$.

Proof: Item (i) of the proposition follows by Prop.6.4.4 above. Items (ii)–(iv) follow by item (i) and Remark 6.4.7. ∎

In connection with Prop.6.4.8 above cf. Remark 6.3.31 on p.973. We will have results analogous to the conclusion of Prop.6.4.8 above at various points in the remaining part of this chapter, cf. e.g. Thm.6.5.5 on p.996.

[954] *Sentence* means closed formula, i.e. formula without free variables.
[955] i.e. $\mathsf{K} \equiv_\Delta \mathsf{L}$

Examples 6.4.9 In all three examples below we state $\mathsf{K} \not\equiv_\Delta \mathsf{L}$ for some classes K, L. In all three examples we can use item (6) on p.972 to prove $\mathsf{K} \not\equiv_\Delta \mathsf{L}$.

1. Let K be the class of two-element algebras without operations. I.e.

$$\mathsf{K} = \{ A \,:\, |A| = 2 \}.$$

   Let L be the class of two-element ordered sets. Important: The sort symbol of K and the sort symbol of L are different. Then

$$\mathsf{K} \equiv^w_\Delta \mathsf{L}, \quad \text{but} \quad \mathsf{K} \not\equiv_\Delta \mathsf{L}.$$

2. Let K_2 be the same as K was in item 1. above. Let K_3 be the class of three element algebras without operations. Let the sort symbols of K_2 and K_3 be different. Then

$$\mathsf{K}_2 \equiv^w_\Delta \mathsf{K}_3, \quad \text{but} \quad \mathsf{K}_2 \not\equiv_\Delta \mathsf{K}_3.$$

3. More sophisticated example, <u>*affine structures*</u>: Let AB be the class of Abelian (i.e. commutative) groups.

   Assume $\mathfrak{G} = \langle G;\, +, -, 0\rangle \in \mathsf{AB}$.

   We define the affine relation R_+ on G as follows.

$$R_+(a,b,c,d,e,f) \quad \stackrel{\text{def}}{\iff} \quad (a-b)+(c-d) = (e-f).$$

   The <u>*affine structure*</u> associated to the group $\mathfrak{G}$ is

$$\mathfrak{A}_{\mathfrak{G}} := \langle G;\, R_+\rangle.$$

   The class of affine structures is

$$\mathsf{Af} := \{ \mathfrak{A}_{\mathfrak{G}} \,:\, \mathfrak{G} \in \mathsf{AB} \}.$$

   Let the sort symbols of AB and Af be different. Claim:

$$\mathsf{AB} \equiv^w_\Delta \mathsf{Af}, \quad \text{but} \quad \mathsf{AB} \not\equiv_\Delta \mathsf{Af}.$$

   Hint: Definability of Af over AB is trivial. Definability of AB over Af: Let $\langle G;\, R_+\rangle \in \mathsf{Af}$. We define a new relation eq as follows.

$$\langle a,b\rangle\ eq\ \langle c,d\rangle \quad \stackrel{\text{def}}{\iff} \quad R_+(a,b,a,a,c,d).$$

Let us notice that eq is an equivalence relation on $G \times G$. Now, let

$$A := G \times G/eq$$

be a new sort. Further

$$\langle a, b\rangle/eq + \langle c, d\rangle/eq = \langle e, f\rangle/eq \quad \overset{\text{def}}{\iff} \quad R_+(a, b, c, d, e, f).$$

Now, defining the rest of the Abelian group $\langle A, +, \ldots\rangle$ over the affine structure $\langle G;\ R_+\rangle$ is left to the reader.

The proof of $\not\equiv_\Delta$ is based on looking at the large number of automorphisms of the affine structure $\langle G;\ R_+\rangle$. We omit the details. (The idea is similar to that of example 1.)

$\lhd$

Remark 6.4.10 (Making $\equiv^w_\Delta$ strong by using parameters)
Consider the applications of $\equiv^w_\Delta$ in items (i), (ii) below.

(i) In Thm.6.6.26 (p.1043) it is stated that

$$\text{(Fields)} \ \equiv^w_\Delta \ (\textbf{pag}\text{-geometries}).$$

Theorems 6.6.22, 6.6.29 are analogous.

(ii) $\mathsf{Mod}(Th) \ \equiv^w_\Delta \ \mathsf{Mog}(TH)$ for certain choices of Th, TH, where the class $\mathsf{Mog}(TH)$ of geometries is defined on p.1071. We note that this is not proved or even stated in the present work, but elaborating this can be considered as a useful research exercise for the reader.

Now, if in the context (or background) of items (i), (ii) above we replace the notion of definability with parametric definability using finitely many parameters only (in the usual sense cf. p.950 and p.935, immediately below Remark 6.3.4, or e.g. Hodges [136, pp. 27–28])[956] then we will obtain that the classes in question e.g. $\mathsf{Mod}(Th)$ and $\mathsf{Mog}(TH)$ become definitionally equivalent in this weaker parametric sense. (I.e. they have a single common parametrically definable definitional expansion etc.) More concretely we could add $(n+1)$-many new constants to **pag** geometries such that

$$\text{(Fields)} \ \equiv_\Delta \ (\textbf{pag}\text{-geometries} \ + \ \text{these constants}).$$

[956] Parametric definability is a slightly weaker notion than definability.

Completely analogous improved versions of Theorems 6.6.22, 6.6.29 (pp. 1041, 1045) are also true.

Also we could add $n+1$ new constants to $\mathsf{Mog}(TH)$ and a constant (a distinguished observer) to $\mathsf{Mod}(Th)$ yielding

$$(\mathsf{Mod}(Th)\ +\ \text{new constant})\ \equiv_\Delta\ (\mathsf{Mog}(TH)\ +\ \text{new constants}),$$

for certain choices of Th and TH. This works even if we assume **Ax(eqtime)** $\in Th$ (cf. Conjecture 6.6.58 on p.1074).
It is these new auxiliary constants which are called parameters in the theory of parametric definability.

We leave elaborating the details of this parametric direction to the interested reader.

$\triangleleft$

6.5 On the connection between Tarski's language for geometry and ours (both in first-order logic), and some notational convention

In connection with the present section it might be useful for the non-logician reader to have a look at the Appendix on higher-order logic versus first-order logic.

In the discussion below we say "the language" and then instead of specifying the language we write down a typical structure of the language. We hope, this causes no confusion.

Tarski uses the language

$$\mathbf{G}_{Ta} = \langle Points;\ Col, \text{“extra relations”}\rangle,$$ [957]

while we use the language

$$\mathbf{G}_{We} = \langle Points, Lines;\ \in, \text{“extra relations”}\rangle$$ [958]

for studying geometry, where $Col \subseteq {}^3Points$ and $\in \subseteq Points \times Lines$ is the usual incidence relation. Since the "extra relations" part is essentially the same for both approaches, let us compare $\langle Points;\ Col\rangle$ and $\langle Points, Lines;\ \in\rangle$. Here, $Col \subseteq {}^3(Points)$ is a ternary relation called collinearity. Intuitively $Col(a,b,c)$ holds iff a,b,c are on the same line. Now, we claim that the two languages (that of $\mathbf{G}_{Ta}$ and $\mathbf{G}_{We}$) are of the *same expressive power* i.e. they are definitionally equivalent[959], under some very mild conditions, cf. Example 6.3.16 (p.954), and cf. Thm.6.5.3. Intuitively, $\mathbf{G}_{Ta} = \langle Points;\ Col\rangle$ is definable in $\mathbf{G}_{We} = \langle Points, Lines;\ \in\rangle$ by saying that

$$Col(a,b,c) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists \ell \in Lines)\, a,b,c \in \ell.$$

[957] Actually instead of Col Tarski uses Bw, but Col is definable from Bw (in Tarski's geometries).

[958] Sometimes we write "*Points*, *Lines*" instead of "*Mn*, *L*" only to sound more intuitive or more suggestive. Summing up: *Points* denotes *Mn* and *Lines* denotes *L*.

[959] More precisely the theory of the language $\langle Points, Lines;\ \in\rangle$ and another one of the language $\langle Points;\ Col\rangle$ are definitionally equivalent assuming very mild axioms on both sides. Even more precisely to any theory of the language $\langle Points; Col\rangle$ there is a definitionally equivalent one of the language $\langle Points, Lines;\ \in\rangle$, assuming some very mild assumptions on the first theory. The same holds in the other direction, too. Whenever we say something like said above (namely, that two languages are definitionally equivalent) we will mean what we explained just now. However we will not spell this out explicitly. Cf. Def.6.3.30 (p.970) for definitional equivalence.

In the other direction, we *simulate* the elements of the new sort *Lines* by pairs $\langle a, b\rangle$ of points[960] $a, b \in Points$ such that $Col(a,b,b) \wedge a \neq b$ holds. It remains to simulate the incidence relation "$\in$". We do this by postulating $c \in \langle a, b\rangle$ iff $Col(a,b,c)$ holds.[961]

A detailed explanation of the connections between our two-sorted language and structures $\langle Points, Lines; \in\rangle$ and Tarski's one-sorted version $\langle Points; Col\rangle$ is given both in our definability section §6.3 together with Theorems 6.5.3, 6.5.5 below, in Givant [102, pp.582-584], and in Appendix A of Goldblatt [108]. Cf. also the first 6 lines on p.viii of [108]. We would like to emphasize that the difference between the two languages is only "notational", cf. Remark 6.3.31 (p.973), the intuitive text above that remark and Thm.6.5.3.

To formulate the conditions which we need to prove definitional equivalence between Tarski's language and ours we introduce axiom **Det** in the language of $\langle Points, Lines; \in\rangle$ and axiom **det** in the language of $\langle Points; Col\rangle$. The acronym "**Det**" abbreviates "points determine lines". Similarly for "**det**".

Det $(\forall p, q \in Points)(\forall \ell, \ell' \in Lines)\,[\,(p \neq q \ \wedge\ p, q \in \ell \cap \ell') \ \rightarrow\ \ell = \ell'\,] \quad \wedge$
$(\forall \ell \in Lines)(\exists p, q \in Points)\,[\,p \neq q \ \wedge\ p, q \in \ell\,]$.

Intuitively, two different lines intersect each other in at most one point; and on each line there are at least two points.

PROPOSITION 6.5.1 $\mathsf{Ge}(\mathbf{Newbasax}) \not\models \mathbf{Det}$.

Proof: The proposition follows from Prop.6.5.8(i) (p.1000) below. ∎

Note that axiom **Det** is an extra possible assumption about our frame models $\mathfrak{M}$ considered in this section. At the end of this section, we will return to discussing the role of axiom **Det**.

Below we introduce axiom **det** in Tarski's language.

det $\big(\ Col(a,b,c) \quad\rightarrow\quad (\ Col(a,c,b) \ \wedge\ Col(b,a,c) \ \wedge\ Col(a,a,b)\,)\,\big)$ [962] $\quad\wedge$
$\big(\,[\,(\ Col(a,b,c) \ \wedge\ Col(a,b,d) \ \wedge\ a \neq b\,) \quad\rightarrow\quad Col(a,c,d)\,] \quad\wedge$
$[\,Col(a,a,a) \ \rightarrow\ (\exists b)(b \neq a \quad\wedge\quad Col(a,b,b))\,]\,\big)$.

[960] this goes exactly as we explained in item (2.1) of item (2) entitled explicit definability of new sorts on p.947

[961] We note that *Col* (definable from *Lines*) is slightly different from *coll* (which was defined from *Bw*, on p.818 way above). Cf. Item 6.6.39, p.1052.

[962] We note that this part of **det** implies that for any function $\pi : \{a,b,c\} \longrightarrow \{a,b,c\}$, $Col(a,b,c) \ \rightarrow\ Col(\pi(a), \pi(b), \pi(c))$.

Intuitively, **det** says two things, the second part is basically a translation of our axiom **Det** above, while the first part says that *Col* is invariant under permutations and even "transformations" of its arguments.

Now, we can formally define Tarski's class Ge_{Ta} and ours Ge_{We} as we promised way above.

Definition 6.5.2

$$\begin{aligned} \mathsf{Ge}_{Ta} \ &\overset{\text{def}}{:=} \ \{ \langle Points;\ Col\rangle \ : \ \langle Points;\ Col\rangle \models \mathbf{det} \}, \\ \mathsf{Ge}_{We} \ &\overset{\text{def}}{:=} \ \{ \langle Points, Lines;\ \in\rangle \ : \ \langle Points, Lines;\ \in\rangle \models \mathbf{Det} \}. \end{aligned}$$

$\lhd$

The following theorem says that Tarski's language and our language are definitionally equivalent, under some mild assumptions.

THEOREM 6.5.3

(i) Ge_{Ta} *and* Ge_{We} *are definitionally equivalent, i.e.*

$$\mathsf{Ge}_{Ta} \equiv_{\Delta} \mathsf{Ge}_{We}.$$

(ii) *There are first-order definable meta-functions*

$$\mathcal{W}e : \mathsf{Ge}_{Ta} \longrightarrow \mathsf{Ge}_{We} \quad \text{and} \quad \mathcal{T}a : \mathsf{Ge}_{We} \longrightarrow \mathsf{Ge}_{Ta}$$

such that for every $\mathfrak{A} \in \mathsf{Ge}_{Ta}$ *and for every* $\mathfrak{B} \in \mathsf{Ge}_{We}$

$$(\mathcal{W}e \circ \mathcal{T}a)(\mathfrak{A}) = \mathfrak{A} \quad \text{and} \quad (\mathcal{T}a \circ \mathcal{W}e)(\mathfrak{B}) \cong \mathfrak{B},$$

moreover there is an isomorphism between the two structures $\mathfrak{B}$ *and* $(\mathcal{T}a \circ \mathcal{W}e)(\mathfrak{B})$ *which is the identity function on Points.*

In the proof of Thm.6.5.3 we will use Lemma 6.5.4 below. Therefore the proof of the theorem comes below Lemma 6.5.4.

We note that pairs of functions like $(\mathcal{W}e, \mathcal{T}a)$ in Thm.6.5.3(ii) above will be introduced and studied in our section on duality theory §6.6.

The subject matter of the following lemma belongs to definability theory, i.e. to §6.3. For a similar lemma cf. Lemma 6.6.14 (p.1031).

LEMMA 6.5.4 *Let* K, L *be two classes of models. Assume that* K^+ *is a common expansion of* K *and* L. *Assume further that* $\mathbf{I}\mathsf{K}^+$ *is closed under taking ultraproducts and that* K^+ *is rigidly definable over* K. *Then (ii) below implies (i) below.*

(i) $\mathsf{K} \equiv_\Delta \mathsf{L}$, *i.e.* K *and* L *are definitionally equivalent.*

(ii) K^+ *is rigidly definable over* L.

The **proof** of Lemma 6.5.4 is based on Thm.6.3.28 (p.969) and can be found in Andréka-Madarász-Németi [21]. ∎

Proof of Thm.6.5.3:
<u>*Proof of (i):*</u> Let the class Ge^+_{We} be defined as follows.

$$\begin{aligned}\mathsf{Ge}^+_{We} \;\stackrel{\text{def}}{:=}\; \{\, \langle \mathit{Points}, \mathit{Lines}; \in, \mathit{Col}\rangle \;:\; &\langle \mathit{Points}, \mathit{Lines}; \in\rangle \in \mathsf{Ge}_{We}, \\ &(\forall a,b,c \in \mathit{Points})\,[\, \mathit{Col}(a,b,c) \;\leftrightarrow\; (\exists \ell \in \mathit{Lines})\, a,b,c \in \ell\,]\,\}.\end{aligned}$$

Clearly, Ge^+_{We} is rigidly definable over Ge_{We}. Further, Ge^+_{We} is an axiomatizable class, since

$$\mathbf{Det} \;+\; (\forall a,b,c \in \mathit{Points})\,[\, \mathit{Col}(a,b,c) \;\leftrightarrow\; (\exists \ell \in \mathit{Lines})\, a,b,c \in \ell\,]$$

axiomatizes Ge^+_{We}. Hence, Ge^+_{We} is closed under taking ultraproducts. Now, to prove that $\mathsf{Ge}_{We} \equiv_\Delta \mathsf{Ge}_{Ta}$, by Lemma 6.5.4 above, it is enough to prove that Ge^+_{We} is rigidly definable over Ge_{Ta}. We will prove this the following way. We will explicitly define a class Ge^+_{Ta} over Ge_{Ta} such that (a)–(c) below will hold.

(a) Ge^+_{Ta} is rigidly definable over Ge_{Ta}.

(b) $\mathsf{Ge}^+_{Ta} \subseteq \mathsf{Ge}^+_{We}$.

(c) $(\forall \mathfrak{A} \in \mathsf{Ge}^+_{We})(\exists \mathfrak{B} \in \mathsf{Ge}^+_{Ta})$ [there is an isomorphism $i : \mathfrak{A} \rightarrowtail\!\!\!\rightarrow \mathfrak{B}$ such that $i \restriction \mathit{Points}$ is the identity function].

Clearly, (a)–(c) above will imply that Ge^+_{We} is rigidly definable over Ge_{Ta} (and this will imply that $\mathsf{Ge}_{We} \equiv_\Delta \mathsf{Ge}_{Ta}$). Now, we turn to defining Ge^+_{Ta} explicitly over Ge_{Ta}. First, for every $\mathfrak{N} = \langle \mathit{Points}; \mathit{Col}\rangle \in \mathsf{Ge}_{Ta}$ we define $\mathfrak{N}^+ = \langle \mathit{Points}, \mathit{Lines}; \in, \mathit{Col}\rangle$ as follows. Let $\mathfrak{N} = \langle \mathit{Points}; \mathit{Col}\rangle \in \mathsf{Ge}_{Ta}$. First we define

$$R \stackrel{\text{def}}{:=} \{\, \langle a,b\rangle \in \mathit{Points} \times \mathit{Points} \;:\; \mathit{Col}(a,b,b),\ a \neq b\,\}$$

as a new relation. Then we define the auxiliary sort U to be R together with the projection functions pj_0, pj_1 and we forget R as a relation (but we keep it as a sort named U). Then we define a kind of incidence relation $E \subseteq Points \times U$ as follows. Let $e \in Points$ and $\ell \in U$. Then

$$e\ E\ \ell \ \stackrel{\text{def}}{\Longleftrightarrow}\ Col(pj_0(\ell), pj_1(\ell), e).$$

We define the equivalence relation $\equiv$ on U as follows. Let $\ell, \ell_1 \in U$. Then

$$\ell \equiv \ell_1 \ \stackrel{\text{def}}{\Longleftrightarrow}\ (\forall e \in Points)(e\ E\ \ell \ \leftrightarrow e\ E\ \ell_1).$$

We define the sort *Lines* to be $U/\!\equiv$ together with $\in_{U,Lines} \subseteq U \times Lines$. Finally the incidence relation $\in \subseteq Points \times Lines$ is defined as follows. Let $e \in Points$ and $\ell \in Lines$. Then

$$e \in \ell \ \stackrel{\text{def}}{\Longleftrightarrow}\ (\exists \ell' \in_{U,Lines} \ell) e\ E\ \ell'.$$

By this, the structure $\mathfrak{N}^+ = \langle Points, Lines; \in, Col\rangle$ is defined. Now, we define

$$\mathsf{Ge}^+_{Ta} :\stackrel{\text{def}}{=} \left\{ \mathfrak{N}^+ \ :\ \mathfrak{N} \in \mathsf{Ge}_{Ta} \right\}.$$

To prove the theorem it remains to prove that for Ge^+_{Ta} statements (a)–(c) above hold. Statement (a) holds because in Ge^+_{Ta} the axiom of extensionality holds for the incidence relation $\in$. Statement (b) can be proved by checking that

$$\mathsf{Ge}^+_{Ta} \ \models\ \mathbf{Det}\ +\ (\forall a, b, c \in Points)\,[\,Col(a, b, c)\ \leftrightarrow\ (\exists \ell \in Lines)\, a, b, c \in \ell\,]^{963}.$$

To prove statement (c), let $\mathfrak{A} \in \mathsf{Ge}^+_{We}$. Then, it can be checked that the reduct $\langle Points^{\mathfrak{A}}; Col^{\mathfrak{A}}\rangle$ of $\mathfrak{A}$ is a member of Ge_{Ta}. Let $\mathfrak{B} := \langle Points^{\mathfrak{A}}; Col^{\mathfrak{A}}\rangle^+ \in \mathsf{Ge}^+_{Ta}$. One can check that $\mathfrak{A} \cong \mathfrak{B}$, moreover that there is an isomorphism $i : \mathfrak{A} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{B}$ such that $i \restriction Points$ is the identity function. This completes the proof of item (i).

Proof of (ii): Let us notice that (ii) is equivalent with

$$\mathsf{Ge}_{We} \equiv^w_\Delta \mathsf{Ge}_{Ta},$$

cf. Def.6.4.5 for $\equiv^w_\Delta$. Now item (ii) follows by Prop.6.4.6 (p.986) saying that

$$\mathsf{K} \equiv_\Delta \mathsf{L} \ \Rightarrow\ \mathsf{K} \equiv^w_\Delta \mathsf{L},$$

and by item (i). For completeness we give a direct proof.

Let $\langle Points; Col\rangle \in \mathsf{Ge}_{Ta}$. Then we define the new sort *Lines* and the incidence relation $\in \subseteq Points \times Lines$ exactly as in the proof of item (i) (in the definition

[963] and by the fact that $\mathbf{Det}\ +\ (\forall a, b, c \in Points)\,[\,Col(a, b, c)\ \leftrightarrow\ (\exists \ell \in Lines)\, a, b, c \in \ell\,]$ axiomatizes Ge^+_{We}

of $\mathfrak{N}^+$ over $\mathfrak{N}$). By this, the structure $\mathcal{T}a(\langle Lines; Col\rangle) := \langle Lines, Points; \in\rangle$ is defined, and it can be checked that this structure is a member of Ge_{We}.

In the other direction let $\langle Points, Lines; \in\rangle \in \mathsf{Ge}_{We}$. Then we define $\mathcal{T}a(\langle Points, Lines; \in\rangle)$ to be $\langle Points; Col\rangle$, where $Col \subseteq {}^3Points$ is defined as follows. Let $a, b, c \in Points$. Then

$$Col(a,b,c) \stackrel{\text{def}}{\Longleftrightarrow} (\exists \ell \in Lines)\, a, b, c \in \ell.$$

It is easy to see that $\mathcal{T}a(\langle Points, Lines; \in\rangle) \in \mathsf{Ge}_{We}$. Thus

$$\mathcal{W}e : \mathsf{Ge}_{Ta} \longrightarrow \mathsf{Ge}_{We} \quad \text{and} \quad \mathcal{T}a : \mathsf{Ge}_{We} \longrightarrow \mathsf{Ge}_{Ta}$$

are first-order definable meta-functions; and it is not hard to see that they have the desired properties. This completes the proof of Thm.6.5.3. ∎

The following theorem says that the formulas of our language with free variables involving only the sort *Points* can be translated, in a meaning preserving way, to Tarski's language and vice-versa. Cf. Prop.6.4.8 (p.987).

THEOREM 6.5.5 *There are "natural" translation functions*

$$T_{\mathcal{W}e} : Fm(\mathsf{Ge}_{We}) \longrightarrow Fm(\mathsf{Ge}_{Ta}) \quad \textit{and} \quad T_{\mathcal{T}a} : Fm(\mathsf{Ge}_{Ta}) \longrightarrow Fm(\mathsf{Ge}_{We})$$

such that for every $\varphi(\bar{x}) \in Fm(\mathsf{Ge}_{We})$, $\psi(\bar{y}) \in Fm(\mathsf{Ge}_{Ta})$ with all their free variables belonging to sort Points, $\mathfrak{A} \in \mathsf{Ge}_{We}$ and $\mathfrak{B} \in \mathsf{Ge}_{Ta}$, and evaluations $\bar{a}, \bar{b}$ of the variables $\bar{x}, \bar{y}$, respectively (in the sorts Points of course), (i)–(iv) below hold, where $\mathcal{W}e$ and $\mathcal{T}a$ are as in Thm.6.5.3(ii).[964]

(i) $\mathcal{W}e(\mathfrak{B}) \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathfrak{B} \models T_{\mathcal{W}e}(\varphi)[\bar{a}]$ *and* $\mathcal{T}a(\mathfrak{A}) \models \psi[\bar{b}] \;\Leftrightarrow\; \mathfrak{A} \models T_{\mathcal{T}a}(\psi)[\bar{b}]$.

(ii) $\mathfrak{A} \models \varphi[\bar{a}] \;\Leftrightarrow\; \mathcal{T}a(\mathfrak{A}) \models T_{\mathcal{W}e}(\varphi)[\bar{a}]$ *and* $\mathfrak{B} \models \psi[\bar{b}] \;\Leftrightarrow\; \mathcal{W}e(\mathfrak{B}) \models T_{\mathcal{T}a}(\psi)[\bar{b}]$.

(iii) $\mathfrak{A} \models \varphi(\bar{x}) \leftrightarrow (T_{\mathcal{W}e} \circ T_{\mathcal{T}a})(\varphi)(\bar{x})$ *and* $\mathfrak{B} \models \psi(\bar{y}) \leftrightarrow (T_{\mathcal{W}e} \circ T_{\mathcal{T}a})(\psi)(\bar{y})$.

(iv) $\mathsf{Ge}_{We} \models \varphi \;\Leftrightarrow\; \mathsf{Ge}_{Ta} \models T_{\mathcal{W}e}(\varphi)$ *and* $\mathsf{Ge}_{Ta} \models \psi \;\Leftrightarrow\; \mathsf{Ge}_{We} \models T_{\mathcal{T}a}(\psi)$.

Proof: The theorem follows by Thm.6.5.3(ii) (p.993) and Prop.6.4.8 (p.987). Cf. also Def.6.4.5 (p.986). ∎

[964]We note that the formulas φ and $T_{\mathcal{W}e}(\varphi)$ have the same free variables (therefore (i) below makes sense). Similarly for $T_{\mathcal{T}a}$ etc.

Remark 6.5.6 Thm.6.5.5 can be generalized to formulas involving free variables of arbitrary sorts by using Thm.6.3.27 (p.965), the end of Remark 6.3.31 (p.976) and Thm.6.5.3(i) (p.993).

$\lhd$

Remark 6.5.7 By Thm.6.5.5 and Remark 6.5.6 above formulas in two-sorted language $\langle \mathit{Points}, \mathit{Lines};\ \in\rangle$ of our incidence geometries are *abbreviations* for formulas in Tarski's one-sorted language $\langle \mathit{Points};\ \mathit{Col}\rangle$. Actually, we can introduce some further useful abbreviations making our language more intuitive and more "compact". For our next definition, we need the expanded version $\langle \mathit{Points};\ \mathit{Col}, \mathit{Bw}\rangle$ of Tarski's language also due to Tarski. Besides the new sort *Lines* we can extend Tarski's language with new sorts *Planes*, *Half-lines* and the incidence relations $\in_{Pl} \subseteq \mathit{Points} \times \mathit{Planes}$ and $\in_{Hl} \subseteq \mathit{Points} \times \textit{Half-lines}$ as follows. We define

$$R \ :\overset{\text{def}}{=}\ \{\, \langle a, b, c\rangle \in {}^3\mathit{Poinst}\ :\ \neg \mathit{Col}(a, b, c)\,\} \quad \text{and}$$
$$R' \ :\overset{\text{def}}{=}\ \{\, \langle o, e\rangle \in {}^2\mathit{Points}\ :\ o \neq e,\ \ \mathit{Col}(o, e, e)\,\}$$

as new relations. Then we define the new sort U to be R together with the projection functions pj_0, pj_1, pj_2 and the new sort U' together with pj'_0, pj'_1. Intuitively, the elements of U code the planes and the elements of U' code the half-lines. I.e. $\langle a, b, c\rangle$ codes the plane containing a, b, c while $\langle o, e\rangle$ codes the half-line with origin o and containing e. We define the new incidence relations $E \subseteq \mathit{Points} \times U$ and $E' \subseteq \mathit{Points} \times U'$ as follows. Let $e \in \mathit{Points}$, $P \in U$ and $\ell \in U'$. Then

$$e\ E\ P \ \overset{\text{def}}{\iff} \bigvee_{\{i,j,k\}=\{0,1,2\}} (\exists a \in \mathit{Points})[\mathit{Col}(a, pj_i(P), pj_j(P))\ \wedge\ \mathit{Col}(e, a, pj_k(P))]$$ [965],

$$e\ E'\ \ell \ \overset{\text{def}}{\iff}\ \neg \mathit{Bw}(e, pj'_0(\ell), pj'_1(\ell))\ \wedge\ \mathit{Col}(e, pj'_0(\ell), pj'_1(\ell)).$$

The equivalence relations $\equiv$ and $\equiv'$ on U and U', respectively, are defined as follows. Let $P, P' \in U$ and $\ell, \ell' \in L$. Then

$$P \equiv P' \ \overset{\text{def}}{\iff}\ (\forall a \in \mathit{Points})(a\ E\ P\ \leftrightarrow\ a\ E\ P'),$$
$$\ell \equiv' \ell' \ \overset{\text{def}}{\iff}\ (\forall a \in \mathit{Points})(a\ E'\ \ell\ \leftrightarrow\ a\ E'\ \ell').$$

Now, the new sorts *Planes* and *Half-lines* are defined to be $U/{\equiv}$ and $U'/{\equiv'}$, respectively, together with $\in\ \subseteq U \times \mathit{Planes}$ and $\in\ \subseteq U \times \textit{Half-lines}$. (For brevity we

[965] Here the pattern

$$\bigvee_{\{i,j,k\}=\{0,1,2\}} \psi_{ijk}$$

abbreviates the disjunction $\bigvee \{\, \psi_{ijk}\ :\ \{i, j, k\} = \{0, 1, 2\}\,\}$.

omitted the subscripts of the $\in$ symbols.) Finally the incidence relations $\in_{Pl}$ and $\in_{Hl}$ are defined as follows. Let $a \in Points$, $P \in Planes$ and $\ell \in$ Half-lines. Then

$$a \in_{Pl} P \overset{\text{def}}{\iff} (\exists P' \in P)\, a\ E\ P',$$
$$a \in_{Hl} \ell \overset{\text{def}}{\iff} (\exists \ell' \in \ell)\, a\ E'\ \ell'.$$

By the above the many-sorted geometric structures (and language)

$$\langle Points, Lines, Planes, Half\text{-}lines;\ \in, \in_{Pl}, \in_{Hl}, Bw, Col\rangle$$

are *definitional expansions* of the one-sorted structures (and language) $\langle Points;\ Col, Bw\rangle$.

We note that we do not have to stop with introducing *Planes* as a convenient abbreviation. In the same spirit we can introduce the remaining geometric objects like e.g. hyper-planes or 3-dimensional subspaces, or circles, spheres etc. All these remain abbreviations only and we remain in the language $\langle Points;\ Col, Bw, eq\rangle$. In other words the expanded language $\langle Points, Lines, \ldots,$ 3-dimensional subspaces, $\ldots\rangle$ remains a definitional expansion of Tarski's original language $\langle Points;\ Col, Bw, eq\rangle$. ◁

Convention: Throughout this convention we assume the axiom **Det**. Motivated by Theorems 6.5.3, 6.5.5 above, we can *identify* our many-sorted geometry

$$\mathbf{G}_{\mathfrak{M}} = \langle Mn, L;\ L^T, L^{Ph}, \in, \prec, Bw, \perp_r, eq\rangle^{966}$$

with a one-sorted structure

$$\mathbf{G}^-_{\mathfrak{M}} = \langle Mn;\ Col, Col^T, Col^{Ph}, \prec, Bw, \perp, eq\rangle^{967}$$

where $Col^T \subseteq Mn \times Mn \times Mn$ is T-collinearity, defined from L^T the natural way, i.e. $Col^T(a,b,c) \overset{\text{def}}{\iff} (\exists \ell \in L^T)\, a,b,c \in \ell$, for $a,b,c \in Mn$, cf. the definition of Col on p.996.[968] Similarly for Col^{Ph} and Col^S. In the other direction, in $\mathbf{G}^-_{\mathfrak{M}}$, L^T is a *defined* relation and not a basic symbol, similarly L is a *defined* sort. Further in $\mathbf{G}^-_{\mathfrak{M}}$ $\perp$ is a relation between pairs of points, i.e. it is a 4-ary relation on Mn. Intuitively, $\langle a,b,c,d\rangle \in \perp$ iff the lines determined by $\langle a,b\rangle$ and $\langle c,d\rangle$ are $\perp_r$-orthogonal according to $\mathbf{G}_{\mathfrak{M}}$. We emphasize that, as it was explained in §6.3, from the point of view

[966] $\mathbf{G}_{\mathfrak{M}}$ was introduced on p.787.

[967] For completeness we note that $\mathbf{G}^-_{\mathfrak{M}}$ is a legitimate structure even in the most classical and most purist version of first-order logic.

[968] Here, Col is defined from L, cf. footnote 958 on p.991.

of first-order logic (cf. e.g. in Monk [197] the section on many-sorted logic) there is no real difference between $\mathbf{G}_{\mathfrak{M}}$ and $\mathbf{G}_{\mathfrak{M}}^-$. More precisely, the difference between $\mathbf{G}_{\mathfrak{M}}$ and $\mathbf{G}_{\mathfrak{M}}^-$ is *the same* as that between a Boolean algebra

$$\mathfrak{B}_1 = \langle B; \vee, \wedge, -, 0, 1\rangle \quad \text{and} \quad \mathfrak{B}_2 = \langle B; \vee, -, 0\rangle.$$

◁

Let us include g into $\mathbf{G}_{\mathfrak{M}}$ obtaining

$$\mathbf{G}_{\mathfrak{M}}^g = \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, \in, \prec, Bw, \perp, eq, g\rangle.$$

Let us try to make $\mathbf{G}_{\mathfrak{M}}^g$ one-sorted in the style of the above discussion. Then we obtain the following structure

$$(\mathbf{G}_{\mathfrak{M}}^g)^- = \langle Mn, \mathbf{F_1};\ Col, Col^T, Col^{Ph}, \prec, Bw, \perp, eq, g\rangle.$$

This leaves us with two problems listed in (i) and (ii) below.

(i) $(\mathbf{G}_{\mathfrak{M}}^g)^-$ remains *many-sorted* because it has two sorts Mn and F.

(ii) We can replace L with Col only when we assume axiom **Det** on our model $\mathfrak{M}$ from which the geometry is obtained. In special relativity (i.e. in the present section) we are allowed to do this and this causes no loss of generality. However in general relativity this is not allowed (because axiom **Det** would kill essential features of the theory). Cf. Fig.308 on p.1002.

We will return to the difficulty outlined in item (ii) at the end of this sub-section (p.1001)

We will extend the above identification of $\mathbf{G}_{\mathfrak{M}}^g$ with $(\mathbf{G}_{\mathfrak{M}}^g)^-$ to identifying $\mathfrak{G}_{\mathfrak{M}}$ with its variant $\mathfrak{G}_{\mathfrak{M}}^- := \langle Mn, \mathbf{F_1};\ Col, Col^T, Col^{Ph}, Col^S, \prec, Bw, \perp, eq, g, \mathcal{T}\rangle$. However, we will remain cautious with this identification in connection with generalizations toward general relativity because of item (ii) above.

We will return to the subject of identifying $\mathbf{G}_{\mathfrak{M}}$, $\mathbf{G}_{\mathfrak{M}}^g$ with $\mathbf{G}_{\mathfrak{M}}^-$, $(\mathbf{G}_{\mathfrak{M}}^g)^-$, respectively, etc. in §6.7, but cf. also §6.3.

By the above, we will consider our geometries e.g. $\mathfrak{G}_{\mathfrak{M}}$ as expansions of Tarski's geometries $\langle Mn;\ Col, \text{"extra relations"}\rangle$. Our reason for doing so is that we would like to use the insights of Tarski's school in our framework.

We note that Goldblatt [108, p.18] uses our notion $\langle Mn, L;\ \in\rangle$ and calls it an incidence structure (he uses the letters $\langle \mathcal{P}, \mathcal{L}, \mathcal{I}\rangle$ for "points", "lines", and "incidence" for our Mn, L, "$\in$"). As we already mentioned (on p.924), if $\langle Mn, L;\ \in\rangle$ satisfies some axioms, then Goldblatt [108] calls it an *affine plane* etc. So our simplest kind of geometry $\langle Mn, L;\ \in\rangle$, together with its enriched versions $\langle Mn, L;\ \in, \perp\rangle$, $\langle Mn, L;\ \in, Bw\rangle$ and $\langle Mn, L;\ \in, Bw, \perp\rangle$ are all studied in Goldblatt [108] in their present form. For completeness we note that, Goldblatt in [108] uses our kind of geometries $\langle Mn, L, Planes;\ \in, \in_{Pl}\rangle$ and he denotes them by $(\mathcal{P}, \mathcal{L}, \theta, \mathcal{I})$ on p.112, where θ is the set of planes and $\mathcal{I}$ is the incidence relation corresponding to our $\in$ and $\in_{Pl}$.

We mentioned all these things about Goldblatt [108] and the notions $\langle Mn, L;\ \in\rangle$, $\dots$, $\langle Mn, L;\ \in, Bw, \perp\rangle$ of geometries in order to clarify the connections between our notation and terminology and that of the literature. In particular, we hope that besides our definability section (§6.3) and our Theorems 6.5.3, 6.5.5, Goldblatt [108] will help the reader to see the connection between our language $\langle Mn, L;\ \in, \dots\rangle$ and that $\langle Mn;\ Col, \dots\rangle$ of the Tarski school.

In our theorems in the present section we used axiom **Det**. As we said, this restricts the class of all frame models to the smaller class

$$\mathsf{M}_{\mathrm{Det}} := \{\mathfrak{M} \in \mathsf{FM}\ :\ \mathfrak{G}_{\mathfrak{M}} \models \mathbf{Det}\}.$$

We note that $\mathsf{M}_{\mathrm{Det}}$ is axiomatizable in its original language too, this follows from Prop.6.3.18 (p.957) and Prop.6.4.4 on p.985. The investigations in Chapters 1–6 in this work do not change essentially if we restrict our attention to $\mathsf{M}_{\mathrm{Det}}$. E.g, the properties of the theories $Th \in \{\mathbf{Newbasax}, \mathbf{Bax}, \mathbf{Reich}(\mathbf{Bax}), \dots\}$ remain the same if instead of $\mathsf{Mod}(Th)$ we investigate $\mathsf{Mod}_{\mathrm{Det}}(Th) = \mathsf{M}_{\mathrm{Det}} \cap \mathsf{Mod}(Th)$.

Therefore, the geometrical counterpart of the theory developed in Chapters 1–6 of this work can be built up in the Tarskian one-sorted framework

$$\mathbf{G}^{-}_{\mathfrak{M}} = \langle Mn;\ Col, Col^{T}, Col^{Ph}, \prec, Bw, \perp, eq\rangle;$$

or if g plays an important role then in the metric version of the geometry $\mathbf{G}^{-}_{\mathfrak{M}}$, i.e. in $\mathfrak{G}^{-}_{\mathfrak{M}} = \langle \mathbf{G}^{-}_{\mathfrak{M}}, \mathbf{F_1}, Col^{S}, g, \mathcal{T}\rangle$.

To prepare the formulation of the next proposition we recall the axiom of disjoint windows (**Ax(diswind)**) from p.812.

Ax(diswind) $(\forall m, k \in Obs \cap Ib)[(m \overset{\odot}{\to} ph\ \wedge\ k \overset{\odot}{\to} ph)\ \Rightarrow\ m \overset{\odot}{\to} k]$.

PROPOSITION 6.5.8

(i) $\mathsf{Ge}(\mathbf{Newbasax}) \not\models \mathbf{Det}$*, moreover:*

There is $\mathfrak{M} \models \mathbf{Newbasax}$ *and* $\ell, \ell' \in L_{\mathfrak{M}}$ *such that* $\ell \cap \ell'$ *is an infinite set, but* $\ell \neq \ell'$.

(ii) *Assume* $\mathfrak{M} \models \mathbf{Pax} + \mathbf{Ax(diswind)}$. *Then* $\mathfrak{G}_{\mathfrak{M}} \models \mathbf{Det}$, *i.e.*

$$\mathsf{Ge}(\mathbf{Pax} + \mathbf{Ax(diswind)}) \models \mathbf{Det}.$$

Outline of proof: As a hint for the proof of (i) we include Figure 307. Item (ii) follows by Remark 6.2.80 (p.890) and item 3a of Prop.6.2.79 (p.886). ∎

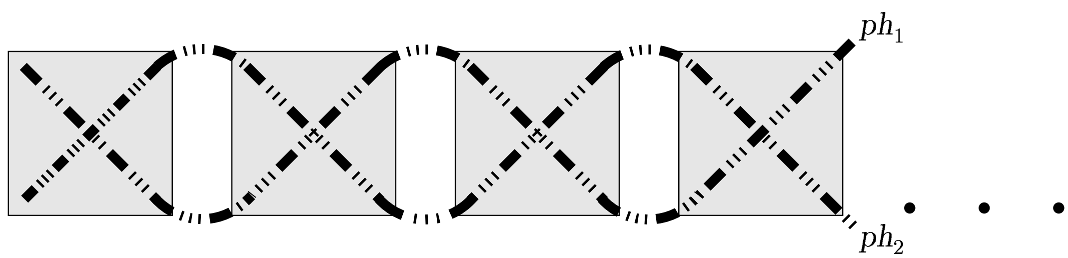


Figure 307: Illustration for the proof of Prop.6.5.8(i).

However, *when* we *generalize* our approach to *general relativity* theory then it will be essential to use many-sorted geometries of the kind $\langle Mn, L; \in\rangle$ for the following reason. As we already said, we can add axiom **Det** to our presently discussed relativity theories like **Newbasax**, **Bax** etc. without changing the essential, characteristic properties of these theories. This will *not* be the case with general relativity cf. Fig.308 (p.1002). See also Figures 355, 281 on pages 1208, 855. Namely, in general relativity it is an essential feature for lines ℓ, ℓ' that the number of intersections of ℓ and ℓ' can be arbitrarily large. That is, in general relativity, for every $n \in \omega$ it is possible to have $\ell \neq \ell'$ such that $|\ell \cap \ell'| > n$. Hence it is impossible to code lines with n-tuples of points.[969] Therefore, the way Tarski represented (or coded) lines with pairs (or n-tuples) of points does not seem to work in general relativity. Therefore, it seems to be the case, that if, for general relativity, we want to carry through the programme represented by "the geometry of Tarski's school[970]", Suppes [243], and Goldblatt [108], then we will have to develop first-order logic of geometry in the many-sorted style $\langle Points, Lines; \in\rangle$ and not in the one-sorted style $\langle Points; Col\rangle$. Of course, somebody in the future might have a new idea and reduce even general

[969] Roughly speaking, adding axiom **Det** to general relativity would basically reduce general relativity to the level of special relativity, cf. Fig.308 (p.1002). Hence we do not want to add axiom **Det** to general relativity.

[970] cf. e.g. [254, 251, 245, 237]

relativistic geometry to an elegant one-sorted language; but we feel that this is not worth the effort. Our feeling is based on the fact that many-sorted logic today is very well developed and is well known to be reducible to one-sorted logic. There seems to be a consensus that since we know how to reduce many-sorted logic to one-sorted logic, theories which are in their intuitive form many sorted are more useful when developed in many-sorted logic (as opposed to developing them in one-sorted logic). For more on these feelings we refer to Monk [197], Barwise-Feferman [43], Barwise [42, p.42 item 5.1] .

As we said before, we will discuss the interconnections between our basic relation (and function) symbols Col^T, Col^{Ph},..., eq, g (i.e. between the ingredients of $\mathfrak{G}_{\mathfrak{M}}$) in §6.7.

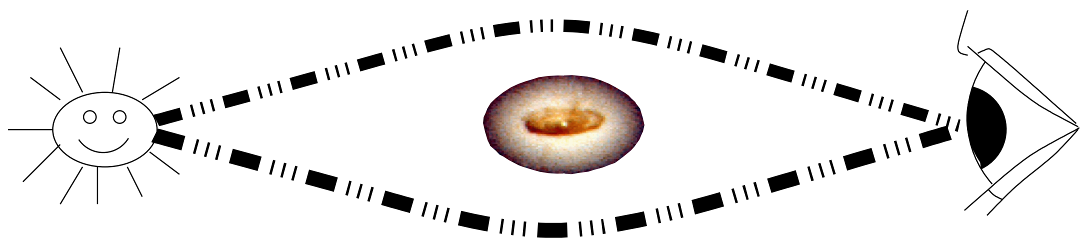


Figure 308: A massive object such as a galaxy, or even a black hole, can act as a giant lens. Light from a distant source (e.g. a quasar) is bent by the gravitational space warp surrounding the object. This effect can produce multiple images of a distant source.

6.6 Duality theory: connections between relativistic geometries ($\mathfrak{G}_{\mathfrak{M}}$) and models ($\mathfrak{M}$) of relativity

Assume we have two essentially different ways of thinking about the world. Assume, we can establish some very strong interconnections between these two ways of thinking.[971] (Call this "duality theory" between the two ways.) Then such a system of interconnections (i.e. "duality theory") can be rather useful because then we can use these two ways of thinking in a combined way, and ideas or reasonings formulated in one of these ways of thinking can be translated to the other. One could say, that such a duality theory enables us to reason about the world by using the two ways of thinking simultaneously, achieving a kind of "stereo" effect.

Figure 309: A duality theory can be viewed as a bridge connecting two worlds of mathematics, permitting two-way traffic. The bridge idea is explained in Andréka et al. [30] so much that the title of §II there is "Bridge . . . ". Cf. also [29] and Mikulás [195, §1.3 ("Bridge between logics and algebras")].

[971] Later we will refer to this interconnection, in a figurative way of speaking, as a bridge, cf. Fig.309.

A second, equally important motivation for duality theories is the following. Duality theories often establish two-way "translations"

$$\mathsf{World}_1 \quad \underset{T_2}{\overset{T_1}{\rightleftarrows}} \quad \mathsf{World}_2$$

between two "worlds"[972] of mathematics such that problems formulated in World_1 are often easier to solve the following way: (i) translate "problem" to World_2, then solve T_1(problem) in World_2 and translate the result back along T_2 to World_1. With certain other problems (originating from World_2) the other direction might sometimes work better. See Fig.309. With this "pragmatic view" we do *not* mean to diminish the importance of the intellectual pleasure and scientific value of integrating World_1 and World_2 into a unified perspective, we only want to emphasize that this pragmatic, problem-solving oriented motivation is there, too. An example is the

$$\text{"proof theory"} \ \rightleftarrows \ \text{"model theory"}$$

duality built on Gödel's completeness theorem: some proof theoretic problems like proving $Th \nvdash \varphi$ are easier to solve in the world of model theory by constructing a model $\mathfrak{M} \in \mathsf{Mod}(Th)$ with $\mathfrak{M} \not\models \varphi$. Cf. p.1019 (above item III), p.1096 ("Motivation for ... equivalence of categories ..."), p.777 item (ii) in §6.1.

Before starting our particular application of this idea (i.e. of duality theories) we note that we will list widely used examples of duality theories and motivation for duality theories on pp. 1014–1027, in Remark 6.6.61 (p.1078) and in §§ 6.6.5–6.6.7 (pp. 1078–1107). Familiarity with the examples mentioned there or below is not needed for reading and understanding this work. Further we note that such an example used in mathematical physics is the duality between commutative C^*-algebras and locally compact topological spaces, cf. item (2) on p.1100. For C^*-algebras in physics cf. e.g. Rédei [218, p.62, Chapter 6 (von Neumann Lattices)], [219]. Cf. end of §6.6.6 p.1100 for an outline of what C^*-algebras are. More motivation and examples for the uses of duality theories in mathematics and theoretical physics are collected e.g. in the books Mac Lane [168], Barr-Wells [40] under the name "adjoint situations". Cf. also the references on adjointness in item (4) of §6.6.6, pp. 1104–1107. Further duality theories in physics are e.g. in Varadarajan [270], Lawvere-Schanuel [163].

Summing up: The present section (§6.6) contains a "distributed sub-section" discussing the subject matter of duality theories (in general) throughout mathematics and mathematical physics. This distributed sub-section is spread out on pp.

[972] One world can be a branch like Boolean algebras while the other world can be another branch of mathematics like topological spaces. But on p.1103 we will see that these worlds can be arbitrarily far apart, e.g. one can be a part of analysis while the other a part of algebra (Laplace transformation).

1003–1005, pp. 1014–1027, pp. 1078–1081, pp. 1090–1093, pp. 1095–1107. (This distributed sub-section does not contain the duality theories elaborated and used in the present work, instead it intends to provide a perspective and background for them.)

So much for duality theories in general. In the present section we will investigate certain concrete duality theories. More concretely, the subject matter of the present section concerns the connections between the "observation-oriented" models $\mathsf{Mod}(Th)$ and the "theoretically-oriented" models $\mathsf{Ge}(Th)$.[973] The investigation of such connections has already been proposed by Reichenbach [223] and has been pursued to some extent in a model-theoretic spirit (similar to ours, in many respects) in Friedman [90] § VI.3 (p.236) under the title "Theoretical Structure and Theoretical Unification".[974] (In that title "theoretical structure" can be interpreted as referring to the structures in $\mathsf{Ge}(Th)$,[975] while "theoretical unification" can refer to a unified study of $\mathsf{Ge}(Th)$ and $\mathsf{Mod}(Th)$ and their interconnections [like e.g. what we do in the present section].) Cf. also $\mathsf{Mod}(Th)^+$ in Def.6.6.88 (p.1108). Cf. the introduction to the present chapter, i.e. §6.1 (p.776).

Among others, in this section we will prove that our "observation-oriented" models $\mathsf{Mod}(Th)$ are definitionally equivalent with our relativistic geometries $\mathsf{Ge}(Th)$, assuming Th is strong enough. Formally,

$$\mathsf{Mod}(Th) \;\equiv_\Delta\; \mathsf{Ge}(Th),$$

under some assumptions, cf. Thm.6.6.13 (p.1031). Besides this we will also elaborate duality theories between the worlds $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$, cf. Fig.309 (p.1003) and e.g. §§ 6.6.1, 6.6.3, 6.6.6. We note that a duality theory between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ means a weaker connection than definitional equivalence. Hence, duality theories (between $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$) are more general in that they hold under milder assumptions on Th. (Actually $\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th)$ implies isomorphism between the categories[976] $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ if we choose elementary embeddings as morphisms; which seems to be the strongest possible form of duality, cf. item (5) on p.971.)

[973] As the reader might expect at this point, this connection will appear in the form of a duality theory.

[974] In passing we note that the emphasis on model theory (in connection with studying relativity, of course), characteristic of the present work, is not without precursors, e.g. the relativity theory book Friedman [90] puts quite a bit of emphasis on using model theory in a spirit similar to ours. Cf. e.g. our reference to Friedman's $\mathcal{A}$ and $\mathcal{B}$ on p.776 herein.

[975] Or more "literally" as referring to a common expansion $\langle \mathfrak{M}, \mathfrak{G}_{\mathfrak{M}} \rangle$ of $\mathfrak{M}$ and $\mathfrak{G}_{\mathfrak{M}}$, but we are closer to the spirit of the connections between [90] and the present work if we interpret "theoretical structure" as $\mathfrak{G}_{\mathfrak{M}}$ or equivalently $\mathsf{Ge}(Th)$.

[976] Categories will be introduced later, cf. §6.6.6 (p.1084).

The following convention is made only to have a nicer duality theory between the frame models and the observer independent geometries.

CONVENTION 6.6.1

(i) Throughout the present chapter ("Observer independent geometry") we postulate that the *empty model*[977] similar to our frame models is a frame model too (i.e. is a member of FM). Further we postulate that for any $\mathfrak{M} \in \mathsf{FM}$

$$Obs^{\mathfrak{M}} = \emptyset \quad \Rightarrow \quad (\mathfrak{M} \text{ is the empty model}).$$

In the present convention the definition of the class of frame models FM was modified. The definition of $\mathsf{Mod}(Th)$ is modified accordingly, for any set Th of formulas in our frame language.

(ii) Deviating from the convention usually made in Algebra, in the present chapter, in accordance with item (i), algebraic structures with empty universes are allowed, e.g. $\langle \emptyset; +, \cdot \rangle$ with $+, \cdot$ binary operations on $\emptyset$ is a field.[978]

$\lhd$

Let us recall that by a *relativistic geometry* we understand an *isomorphic copy* of $\mathfrak{G}_{\mathfrak{M}}$, for some frame model $\mathfrak{M}$. Let us also recall that for any set Th of formulas in our frame language for relativity theory we defined

$$\mathsf{Ge}(Th) :\stackrel{\text{def}}{=} \{\mathfrak{G} \; : \; (\exists \mathfrak{M} \in \mathsf{Mod}(Th)) \; \mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}\}.$$

Let $\mathfrak{G} = \langle Mn, \ldots, L, \in, \ldots \rangle \in \mathsf{Ge}(\emptyset)$. Then we recall that we assumed that the relation $\in$ between Mn and L is the, real, set-theoretic membership relation, and that this does not cause loss of generality.

[977]We call a model empty if all its sorts (i.e. universes) are empty.

[978]The convention of allowing empty algebras and empty models comes from category theory cf. e.g. Adámek et al. [2, p.15, item 3.3(2)(e)]. Also the *motivation* for allowing such structures comes from category theoretic results; but cf. also the model theory book Hodges [136, §1.1, p.2] which does permit empty models, cf. Exercise 10 on p.11 (§1.2) in [136].

[979]By the first version we mean the $(\mathcal{M}, \mathcal{G})$-duality to be introduced soon while by the second version we mean the $(\mathcal{M}o, \mathcal{G}o)$-duality to be introduced in §6.6.4 much later.

6.6.1 A duality theory between models and geometries (first part of the first version[979])

Given a relativistic geometry $\mathfrak{G} \cong \mathfrak{G}_{\mathfrak{M}}$ for a frame model $\mathfrak{M}$, it is a natural question to ask whether we can reconstruct $\mathfrak{M}$ (up to isomorphism) from $\mathfrak{G}$.[980] A possible answer to such a question consists of elaborating a duality theory[981] acting between the geometrical world $\mathsf{Ge}(\emptyset)$ and the world $\mathsf{Mod}(\emptyset)$ of our frame models. This consists of two functions

$$\mathcal{G} : \mathsf{Mod}(\emptyset) \longrightarrow \mathsf{Ge}(\emptyset) \quad \text{and} \quad \mathcal{M} : \mathsf{Ge}(\emptyset) \longrightarrow \mathsf{Mod}(\emptyset),$$ [982]

see Figure 310. We define $\mathcal{G}$ to be the function $\mathfrak{M} \mapsto \mathfrak{G}_{\mathfrak{M}}$ (specified in Def.6.2.2

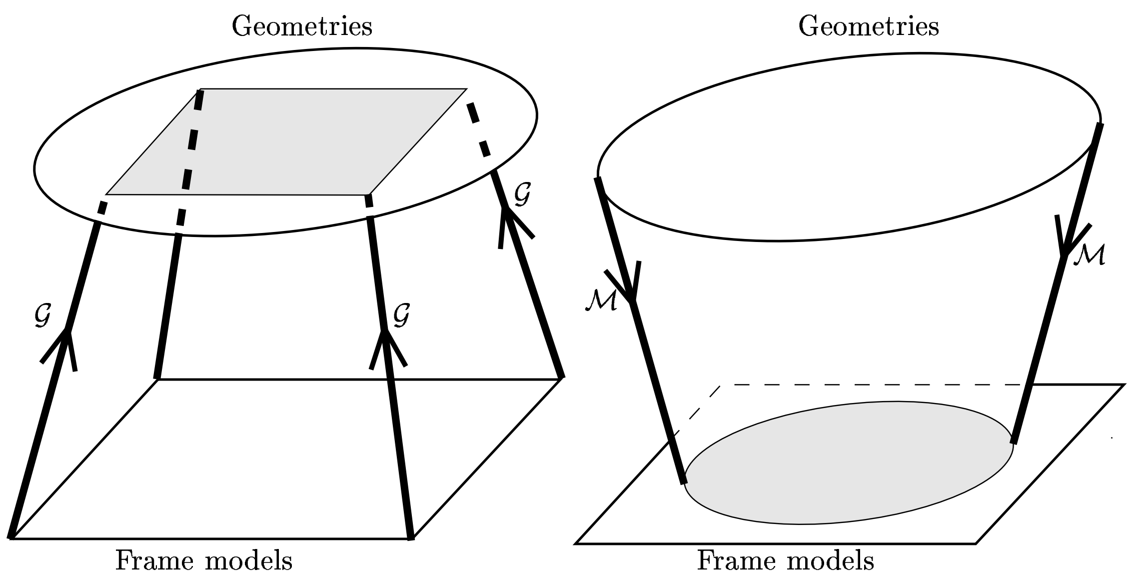


Figure 310: Connecting two worlds, namely the world of frame models and the world of geometries.

[980]Here, the emphasis is on the case when $\mathfrak{G} \neq \mathfrak{G}_{\mathfrak{M}}$; cf. Remark 6.2.4 (p.801).

[981]For duality theories cf. pp. 1003–1005, Remark 6.6.4 (pp. 1014–1027), pp. 1078–1081, pp. 1090–1093, pp. 1095–1107.

[982]Despite of the fact that $\mathcal{G}$ and $\mathcal{M}$ are only proper classes of ordered pairs (as opposed to being a set of ordered pairs) we call them functions.

way above). The function $\mathcal{M}$ will be defined later, in §6.6.3. Sometimes we call $\mathcal{G}$ and $\mathcal{M}$ *functors* because (i) they connect classes of structures, (ii) they preserve certain connections between structures, e.g. isomorphims and embeddability, and (iii) the corresponding "things" in Stone duality theory are called functors for category theoretic reasons. (Cf. item (II) in Remark 6.6.4 on p.1015 for Stone duality.) Actually, $\mathcal{M}$ and $\mathcal{G}$ will become "real" functors in §6.6.6 way below.

CONVENTION 6.6.2 If f is a function and $H \subseteq Dom(f)$ then the notation "$f : H \longrightarrow K$" means that $f \restriction H : H \longrightarrow K$.

$\lhd$

In the spirit of the above convention $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$, for any set Th of formulas in our frame language.

Besides defining $\mathcal{M}$, a duality theory is supposed to prove some theorems stating that the functors $\mathcal{G}$ and $\mathcal{M}$ behave nicely in some sense. In order to prove such theorems we assume some axioms on our models $\mathfrak{M}$. Therefore the duality theory will be of the form:

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th) \quad \text{and} \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th),$$

i.e.

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th)$$

and our theorems will be of the form (A)–(I) below, and they will be stated for certain choices of Th, see Figure 311. Motivation for discussing theorem schemas (A)–(I) can be found in §6.6.6 (p.1084) and Remark 6.6.4 (p.1014). For formulating items (A)–(I) we will need the following definition.

Definition 6.6.3 (Embeddability, weak submodel) Assume $\mathfrak{A}$ and $\mathfrak{B}$ are similar models.

(i) We say that $\mathfrak{A}$ is *embeddable* into $\mathfrak{B}$, in symbols $\mathfrak{A} \rightarrowtail \mathfrak{B}$ (or $\mathfrak{B} \leftarrowtail \mathfrak{A}$) iff there is an injective homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$. Cf. Convention 6.4.1 (p.983).

(ii) $\mathfrak{A}$ is a *weak submodel* of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B}$ iff $A \subseteq B$ and the identity function Id_A is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.[983] Hence weak submodels are always embeddable. Further, the definition for the many-sorted case is

[983] $\mathfrak{A}$ is a *strong submodel* of $\mathfrak{B}$ if every weak submodel $\mathfrak{C}$ of $\mathfrak{B}$ with the same universe as that of $\mathfrak{A}$ (i.e. with $C = A$) is a weak submodel of $\mathfrak{A}$, too. In other chapters of the present work we write

completely analogous. I.e. the existence of an *identical embedding* of say $\mathfrak{M}$ into $\mathfrak{N}$ is equivalent with $\mathfrak{M}$ being a weak submodel of $\mathfrak{N}$.[984]

$\triangleleft$

(A) Assume $\mathfrak{M} \in \mathsf{Mod}(Th)$. Then
$\mathfrak{M}$ is embeddable into $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$, i.e.

$$\mathfrak{M} \rightarrowtail (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}),$$

cf. Fig.318 (p.1035) and Fig.311.

In duality theories similar to our $(\mathcal{G}, \mathcal{M})$-duality, in addition to item (A) it is sometimes required that the embedding (or morphism) "$\rightarrowtail$" occurring in (A) is the "shortest one" in some intuitive sense, cf. Fig.312 (p.1013). This will be made precise in Definitions 6.6.78 (p.1090) and 6.6.79 (p.1091) in our category theoretic sub-section §6.6.6. An analogous remark applies to item (B) below.

(B) Assume $\mathfrak{G} \in \mathsf{Ge}(Th)$. Then
$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$ is embeddable into $\mathfrak{G}$, i.e.

$$\mathfrak{G} \leftarrowtail (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}),$$

cf. Fig.311.

We will have two kinds of dualities one represented by $(\mathcal{M}, \mathcal{G})$ and the other represented by $(\mathcal{M}o, \mathcal{G}o)$. In the first case (i.e. in the case of $\mathcal{M}, \mathcal{G}$) the (B)-type theorems will become degenerate in that they will be of the form (D) below.[985]

(C) $\mathcal{G} \circ \mathcal{M}$ has a strong fixed-point property in the sense that for any $\mathfrak{M} \in \mathsf{Mod}(Th)$

$$(\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \cong \mathfrak{M},^{986}$$

cf. the right-hand side of Fig.315 (p.1031) and Fig.311.

simply ***"submodel"*** for "strong submodel". Further, the definition (of weak and strong submodels) for the many-sorted case is completely analogous with the above one. For more on the distinction between strong and weak submodels cf. e.g. [53] or [27]. We note that if $\mathfrak{A} \subseteq \mathfrak{B}$, i.e. if $\mathfrak{A}$ is a strong submodel of $\mathfrak{B}$ then $\mathfrak{A}$ is also a weak submodel of $\mathfrak{B}$, i.e. $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B}$. The other direction does not hold in general, i.e. $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B} \ \Leftarrow\ \mathfrak{A} \subseteq \mathfrak{B}$ but $\mathfrak{A} \subseteq_{\mathsf{w}} \mathfrak{B} \ \not\Rightarrow\ \mathfrak{A} \subseteq \mathfrak{B}$.

[984] Using the notation $Uv(\mathfrak{M})$ on p.929 (§6.3), we could say that $\mathfrak{M}$ is a *weak submodel* of $\mathfrak{N}$ if the inclusion function of $Uv(\mathfrak{M})$ in $Uv(\mathfrak{N})$ is an embedding of $\mathfrak{M}$ in $\mathfrak{N}$.

[985] I.e. as a side-effect of our choice $\mathcal{M}$ and $\mathcal{G}$ we will have (B)$\Rightarrow$(D). This effect will disappear when we turn to $\mathcal{M}o$ and $\mathcal{G}o$ (i.e. to our second duality theory).

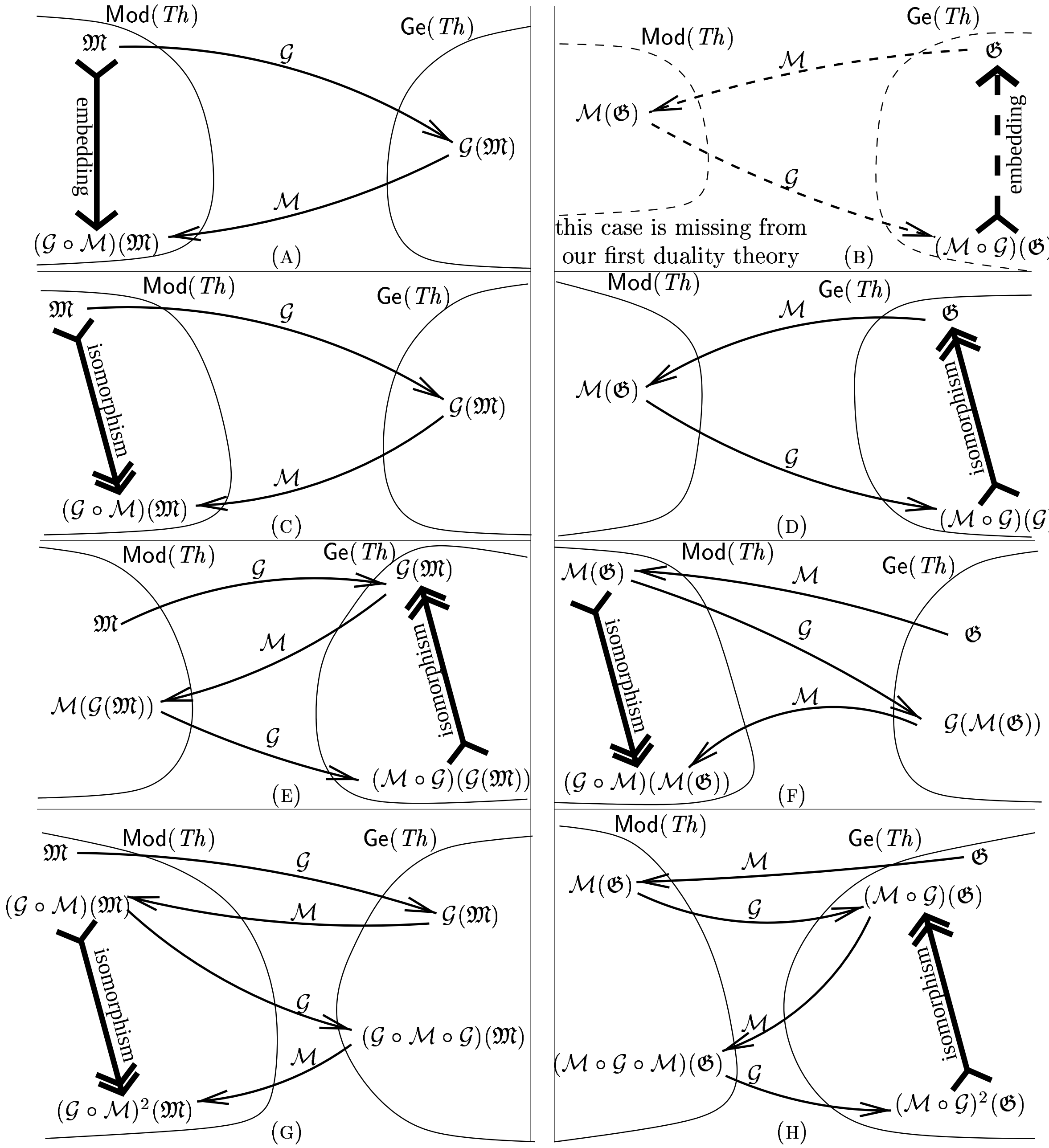


Figure 311: Illustration for theorem schemas (A)–(H) for duality theory.

$\mathcal{M} \circ \mathcal{G}$ has a strong fixed-point property in the sense that for any $\mathfrak{G} \in \mathsf{Ge}(Th)$

(D)
$$(\mathcal{M} \circ \mathcal{G})(\mathfrak{G}) \cong \mathfrak{G},$$

cf. the left-hand side of Fig.315 (p.1031) and Fig.311.

The members of the range of $\mathcal{G}$ are fixed-points[987] of $\mathcal{M} \circ \mathcal{G}$, formally: For any $\mathfrak{M} \in \mathsf{Mod}(Th)$

(E)
$$(\mathcal{M} \circ \mathcal{G})(\mathcal{G}(\mathfrak{M})) \cong \mathcal{G}(\mathfrak{M}),^{988}$$

cf. Fig.311.

The members of the range of $\mathcal{M}$ are fixed-points of $\mathcal{G} \circ \mathcal{M}$, formally: For any $\mathfrak{G} \in \mathsf{Ge}(Th)$

(F)
$$(\mathcal{G} \circ \mathcal{M})(\mathcal{M}(\mathfrak{G})) \cong \mathcal{M}(\mathfrak{G}),$$

cf. Fig.311.

For any function f, $f^2 :\stackrel{\text{def}}{=} f \circ f$.

$\mathcal{G} \circ \mathcal{M}$ has a fixed-point property in the sense that for any $\mathfrak{M} \in \mathsf{Mod}(Th)$

(G)
$$(\mathcal{G} \circ \mathcal{M})^2(\mathfrak{M}) \cong (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}),$$

cf. the left-hand side of Fig.317 (p.1035) and Fig.311.

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[986] i.e. $\mathfrak{M}$ is a fixxed-point up to isomorphism of $\mathcal{G} \circ \mathcal{M}$

[987] i.e. $\mathcal{G}(\mathfrak{M})$ is a fixed-point up to isomorphism of $\mathcal{M} \circ \mathcal{G}$.

[988] This is the typical form of basic statements of Galois connections[989], e.g. $\mathsf{Th}(\mathsf{Mod}(\mathsf{Th}(\mathsf{K}))) = \mathsf{Th}(\mathsf{K})$, or in the case of Galois theory of field extensions $\Delta(H(\Delta(\mathsf{G}))) = \Delta(\mathsf{G})$, cf. items (I), (IV) of Remark 6.6.4.

[989] The definition of Galois connection is in Def.6.6.62 (p.1080) and motivation for Galois connection is in Remark 6.6.61 (p.1078).

$\mathcal{M} \circ \mathcal{G}$ has a fixed-point property in the sense that for any $\mathfrak{G} \in \mathsf{Ge}(Th)$

(H) $$(\mathcal{M} \circ \mathcal{G})^2(\mathfrak{G}) \cong (\mathcal{M} \circ \mathcal{G})(\mathfrak{G}),$$

cf. the right-hand side of Fig.317 (p.1035) and Fig.311.

For any $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}(Th)$ and $\mathfrak{G}, \mathfrak{H} \in \mathsf{Ge}(Th)$

(I) $$\begin{aligned} \mathfrak{M} \rightarrowtail \mathfrak{N} \quad &\Rightarrow \quad \mathcal{G}(\mathfrak{M}) \rightarrowtail \mathcal{G}(\mathfrak{N}), \quad \text{and}^{990} \\ \mathfrak{G} \rightarrowtail \mathfrak{H} \quad &\Rightarrow \quad \mathcal{M}(\mathfrak{G}) \rightarrowtail \mathcal{M}(\mathfrak{H}).^{991} \end{aligned}$$

We will refer to items (A)–(I) above as theorem-schemas for our duality theories.

Figure 312 intends to illustrate our $(\mathcal{M}, \mathcal{G})$-duality[992], theorem schemas (A), (B), and the idea of a shortest "$\rightarrowtail$" in the explanation below (A). The figure itself uses the terminology of category theory which will be explained in §6.6.6 (p.1084). It also intends to serve as a complement for Fig.311.

We note that in the case of $(\mathcal{M}, \mathcal{G})$, i.e. in our first duality theory,

$$(\text{C}) \;\Rightarrow\; (\text{D}) \;\Leftrightarrow\; (\text{E}) \;\Rightarrow\; (\text{F}) \;\Leftrightarrow\; (\text{G}) \;\Rightarrow\; (\text{H}).^{993}$$

Items (C) and (D) above imply

$$\mathsf{Mod}(Th) \equiv^{w}_{\Delta} \mathsf{Ge}(Th),$$

i.e. that $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are weakly definitionally equivalent[994], for any Th in our frame language, assuming $\mathcal{M}, \mathcal{G}$ are first-order definable meta-functions[995] with

[990] Later (e.g. on p.1015, p.1016) we will see that the functors like $\mathcal{M}$, $\mathcal{G}$ can be arrow reversing. This means that the $\mathcal{M}$ image of a pattern $\mathfrak{A} \rightarrowtail \mathfrak{B}$ is of the form $\mathcal{M}(\mathfrak{A}) \twoheadleftarrow\!\!\prec \mathcal{M}(\mathfrak{B})$. For such arrow reversing dualities schema (I) obtains the form

$$\begin{aligned} \mathfrak{M} \rightarrowtail \mathfrak{N} \quad &\Rightarrow \quad \mathcal{G}(\mathfrak{M}) \twoheadleftarrow\!\!\prec \mathcal{G}(\mathfrak{N}) \\ \mathfrak{M} \twoheadrightarrow\!\!\succ \mathfrak{N} \quad &\Rightarrow \quad \mathcal{G}(\mathfrak{M}) \leftarrowtail \mathcal{G}(\mathfrak{N}) \end{aligned}$$

etc.

[991] (I) implies that

$$\mathfrak{M} \rightarrowtail \mathfrak{N} \quad \Rightarrow \quad (\mathcal{G} \circ \mathcal{M})(\mathfrak{M}) \rightarrowtail (\mathcal{G} \circ \mathcal{M})(\mathfrak{N})$$

which corresponds to closure operators (induced by Galois connections) being order preserving cf. footnote 996 on p.1013 and p.1080 (§6.6.5).

[992] Sometimes we write $(\mathcal{G}, \mathcal{M})$-duality for $(\mathcal{M}, \mathcal{G})$-duality. They are the same thing.

[993] This is so because, if $\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th)$ then, $Rng(\mathcal{G})$ is $\mathsf{Ge}(Th)$ up to isomorphism.

[994] cf. Def.6.4.5 (p.986) for the notion of weak definitional equivalence

[995] in the sense of Def.6.4.2 (p.983)

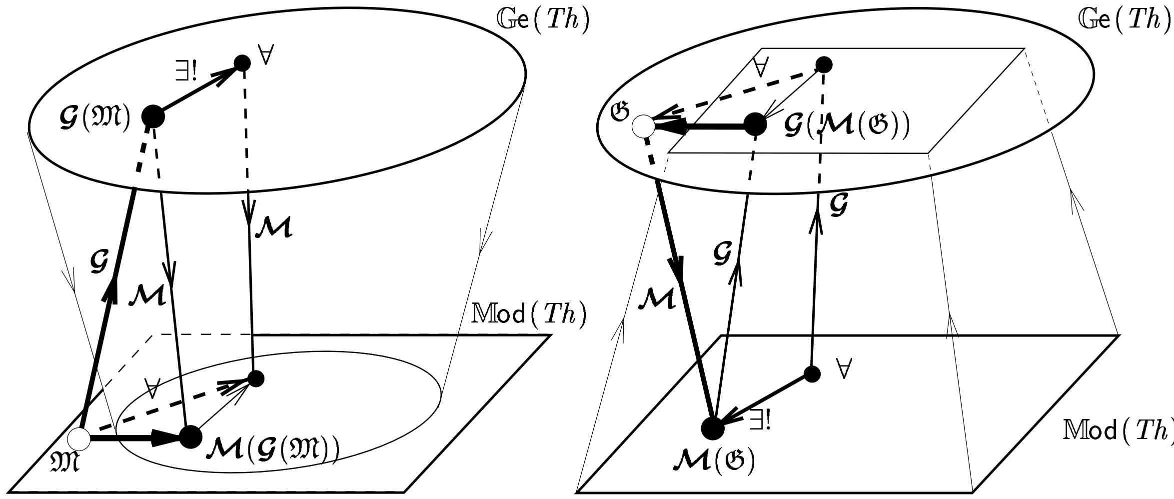

Figure 312: $(\mathcal{M}, \mathcal{G})$ is an adjoint pair of functors, under certain conditions. For the missing definitions (e.g. $\mathbb{M}\mathsf{od}(Th)$, $\mathbb{G}\mathsf{e}(Th)$) cf. §6.6.6 (p.1084).

$\mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$ such that the isomorphisms mentioned in (C) and (D) can be chosen such that they are identity functions on the sort F.

Further, if Th is strong enough, then $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ will turn out to be definitionally equivalent, in symbols

$$\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th),$$

cf. Thm.6.6.13 (p.1031). For the intuitive meaning and methodological importance of this cf. the text above Thm.6.6.13 on p.1030.

If for Th items (A) and (G) above hold, then we will say that $\mathcal{G} \circ \mathcal{M}$ is a *closure operator*[996] on $\langle \mathsf{Mod}(Th), \subseteq_{\mathsf{w}} \rangle$ *up to isomorphism*[997] (and the values of $\mathcal{G} \circ \mathcal{M}$ are fixed-points up to isomorphism), assuming it preserves the partial order $\subseteq_{\mathsf{w}}$ up to isomorphism, cf. Fig.313. In this case, we call $(\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$ the *closure* (or the $(\mathcal{G}, \mathcal{M})$-closure) of $\mathfrak{M}$. Further, if for Th items (B) and (H) hold, then we will say

[996]Let $\langle P, \leq \rangle$ be a partially ordered set (or class) and $f : P \longrightarrow P$. Then f is a *closure operator* on $\langle P, \leq \rangle$ iff for all $x, y \in P$, $x \leq f(x) = f^2(x)$ and $(\, x \leq y \;\Rightarrow\; f(x) \leq f(y)\,)$. (In passing we note that this notion admits a natural generalization to pre-ordered sets in place of partially ordered ones.)

[997]The up to isomorphism part is important, because what we know of $\mathfrak{M} \in Rng(\mathcal{G} \circ \mathcal{M})$ is that it is a fixed point of $\mathcal{G} \circ \mathcal{M}$ *only* up to isomorphism and for $\mathfrak{M} \in \mathsf{Mod}(Th)$ "$\mathfrak{M} \subseteq_{\mathsf{w}} (\mathcal{G} \circ \mathcal{M})(\mathfrak{M})$" holds only up to isomorphism.

that $\mathcal{M} \circ \mathcal{G}$ is a closure operator on $\langle \mathsf{Ge}(Th), {}_{\mathsf{w}}\supseteq \rangle$ up to isomorphism, assuming it preserves ${}_{\mathsf{w}}\supseteq$ up to isomorphism, where $\mathfrak{G}_{\mathsf{w}}\supseteq \mathfrak{H}$ iff $\mathfrak{H} \subseteq_{\mathsf{w}} \mathfrak{G}$. In such situations, sometimes, $(\mathcal{M} \circ \mathcal{G})(\mathfrak{G})$ is called the *interior*, which means *dual-closure*, of $\mathfrak{G}$.

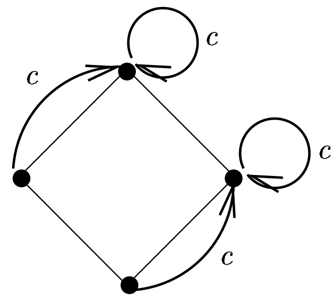

Figure 313: A possible closure operator.

Below our next item (Remark 6.6.4), beginning with p.1027 we will start developing such a duality theory.

Remark 6.6.4 (Galois theories, Galois connections, duality theories all over mathematics, in analogy with the ones in the present work)
In connection with "theorem patterns" (A)–(I) above there is an analogy between our present functors $\mathcal{G}$ and $\mathcal{M}$ and the various Galois theories, duality theories, Galois connections in abstract algebra in the sense outlined below,[998] cf. Def.6.6.62 (p.1080) for Galois connections and Remark 6.6.61 (p.1078) for motivation for studying Galois connections.

(I) *Analogy with Galois theory of fields:* Let $\mathbf{M}|\mathbf{K}$ be a field-extension, i.e. $\mathbf{M}$ is a field and $\mathbf{K}$ is a subfield of $\mathbf{M}$. Let us consider those automorphisms of $\mathbf{M}$ that leave the universe of $\mathbf{K}$ pointwise fixed. These automorphisms form a group under composition "$\circ$" and taking inverse "${}^{-1}$". This group is called the *Galois group* of the field extension $\mathbf{M}|\mathbf{K}$ and is denoted by $\mathsf{G}(\mathbf{M}|\mathbf{K})$. Let the functions

$$H : \{\, \mathbf{L} : \mathbf{L} \text{ is a field, } \mathbf{K} \subseteq \mathbf{L} \subseteq \mathbf{M} \,\} \longrightarrow \{\, \mathsf{G} : \mathsf{G} \text{ is a subgroup of } \mathsf{G}(\mathbf{M}|\mathbf{K}) \} ,$$
$$\Delta : \{\, \mathsf{G} : \mathsf{G} \text{ is a subgroup of } \mathsf{G}(\mathbf{M}|\mathbf{K}) \} \longrightarrow \{\, \mathbf{L} : \mathbf{L} \text{ is a field, } \mathbf{K} \subseteq \mathbf{L} \subseteq \mathbf{M} \,\}$$

be defined as follows.

[998]The reader not familiar with abstract algebra may safely skip this discussion of connections with Galois theory.

Let $\mathbf{L}$ be a field and $\mathbf{K} \subseteq \mathbf{L} \subseteq \mathbf{M}$. Then we define $H(\mathbf{L})$ to be the Galois group $\mathsf{G}(\mathbf{M}|\mathbf{L})$ of the field extension $\mathbf{M}|\mathbf{L}$.

In the other direction, if G is a subgroup of $\mathsf{G}(\mathbf{M}|\mathbf{K})$ then we define $\Delta(\mathsf{G})$ to be the greatest subfield $\mathbf{L}$ of $\mathbf{M}$ such that each member of G leaves the universe of $\mathbf{L}$ pointwise fixed.

The above sketched ideas lead up to a branch of abstract algebra which is called Galois theory of fields, cf. any textbook on abstract algebra e.g. Shafarevich [238]. We are recalling this because that theory[999] is proved rather useful in various parts of mathematics[1000], and now we want to point out an analogy between that theory and our present duality theories.

Now, our functions $\mathcal{G}$ and $\mathcal{M}$ are in analogy with H and Δ. Further, $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ correspond to $\{\, \mathbf{L} : \mathbf{K} \subseteq \mathbf{L} \subseteq \mathbf{M} \,\}$ and $\{\, \mathsf{G} : \mathsf{G} \subseteq \mathsf{G}(\mathbf{M}|\mathbf{K}) \,\}$, respectively. In particular, we have $\mathbf{L} \rightarrowtail (H \circ \Delta)(\mathbf{L})$ and $\mathsf{G} \rightarrowtail (\Delta \circ H)(\mathsf{G})$ in analogy with our theorem schemas (A) and (B). We call $(H \circ \Delta)(\mathbf{L})$ the (Galois) closure of $\mathbf{L}$ and similarly $(\Delta \circ H)(\mathsf{G})$ is the closure of G. (If the reader feels that the arrows go in the wrong direction when comparing $\mathsf{G} \rightarrowtail (\Delta \circ H)(\mathsf{G})$ with theorem schema (B), *then* we note that the "functors" H, Δ are order-reversing. For more on this cf. the explanation in item (II) below.)

(II) *Analogy with Stone duality theory:* First we will introduce a duality theory more general than Stone duality. Then, beginning with p.1019 we will introduce Stone duality (relying on the more general one introduced first).

Let

$$\begin{aligned} \mathcal{S} &: \text{lattices} \longrightarrow \text{topological spaces} \\ \mathcal{L} &: \text{topological spaces} \longrightarrow \text{lattices} \end{aligned}$$ [1001]

be defined as follows.

Let $\mathbf{L} = \langle L; \wedge, \vee \rangle$ be a lattice.[1002] Then we define $\mathcal{S}(\mathbf{L})$ to be the topological space $\mathbf{X} = \langle X, \mathcal{O} \rangle$ such that the set of points X consists of the proper prime ideals of the lattice $\mathbf{L}$, and the collection of open sets $\mathcal{O}$ is generated by the following collection of sets

$$\{\, \{P \in X : a \notin P\} : a \in L \,\},$$

[999] i.e. the Galois theory of fields

[1000] cf. e.g. p.1027 for applications in cylindric algebras.

[1001] Cf. p.870 for the definition of topological spaces.

[1002] For lattices, proper prime ideals etc. cf. any book on abstract algebra or universal algebra e.g. McKenzie-McNulty-Taylor [192] or Adámek et al. [2] or Davey-Priestley [68]. A *lattice* is a partially ordered set L in which any two $x, y \in L$ has a supremum[1003] $x \vee y$ and an infimum $x \wedge y$.

[1003] smallest upper bound

i.e. the latter set is a subbase[1004] for the topological space $\mathbf{X}$.

In the other direction, let $\mathbf{X} = \langle X, \mathcal{O}\rangle$ be a topological space. Then

$$\mathcal{L}(\mathbf{X}) \stackrel{\text{def}}{=} \langle \mathcal{O}; \cap, \cup\rangle.$$

Now our functors $\mathcal{G}$ and $\mathcal{M}$ are in analogy with $\mathcal{S}$ and $\mathcal{L}$. Further $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ correspond to the class of lattices and the class of topological spaces, respectively. In this connection we note the following. If Th satisfies certain conditions then $\mathfrak{M}$ is embeddable into $(\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$, i.e. $\mathfrak{M} \rightarrowtail (\mathcal{G}\circ\mathcal{M})(\mathfrak{M})$. Analogously in the case of lattices $\mathbf{L} \longrightarrow (\mathcal{S}\circ\mathcal{L})(\mathbf{L})$, i.e. there is a homomorphism going from the lattice $\mathbf{L}$ into its "closure" $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$.[1005] The existence of such a homomorphism is analogous with our theorem schema (A). On the topology side for any topology $\mathbf{X}$ there is a continuous function $f : \mathbf{X} \longrightarrow (\mathcal{L}\circ\mathcal{S})(\mathbf{X})$. In general this function need not be injective or surjective. The existence of this function is analogous with our theorem schema (B). (The reader may feel that the arrows go in the wrong direction when comparing our f with (B). However this is not a problem because our functors $\mathcal{S}$ and $\mathcal{L}$ are "contravariant", i.e. arrow reversing; and this is why schema (B) appears here in a reversed form.)

We note that members of $Rng(\mathcal{L})$ are distributive lattices, i.e.

$$Rng(\mathcal{L}) \subseteq \text{(distributive lattices)}.$$

The reason why we chose lattices in general instead of choosing just distributive lattices is to make the analogy with our $(\mathcal{G}, \mathcal{M})$ duality stronger. In analogy with $Rng(\mathcal{L}) \subseteq$ (distributive lattices) we note that

$$Rng(\mathcal{S}) \subseteq (\mathrm{T}_0 \text{ topological spaces}).^{1006}$$

[1004]Let $\mathbf{X} = \langle X, \mathcal{O}\rangle$ be a topological space. By a ***subbase*** for $\mathbf{X}$ we understand a set $\mathcal{H} \subseteq \mathcal{O}$ such that $\mathcal{H}$ generates $\mathcal{O}$ by finite intersections and infinite unions i.e.
$\mathcal{O} := \{\bigcup Y \;:\; Y \subseteq \{\bigcap H \;:\; H \text{ is a finite subset of } \mathcal{H}\}\}$.

[1005]In passing we note the following. Let $\mathbf{L}$ be an arbitrary lattice. Let $h : \mathbf{L} \longrightarrow (\mathcal{S}\circ\mathcal{L})(\mathbf{L})$ be the "canonical" homomorphism corresponding to our theorem schema (A). Then the range $Rng(h)$ of h is a sublattice of $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$. $Rng(h)$ is the same as the so called reflection of $\mathbf{L}$ in the variety of distributive lattices. This means that $Rng(h)$ is that distributive lattice which is "closest" to $\mathbf{L}$ in a sense which can be made precise by using category theory (cf. Def.6.6.78 on p.1090). Usually, $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$ is bigger than $Rng(h)$, e.g. $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$ is a complete lattice while $Rng(h)$ need not be such. (As a curiosity we note that if $\mathbf{L}$ is infinite, then $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$ is bigger than $Rng(h)$, hence $(\mathcal{S}\circ\mathcal{L})^{n+1}(\mathbf{L})$ is different from $(\mathcal{S}\circ\mathcal{L})^{n}(\mathbf{L})$.) We call $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$ the ***"closure"*** of $\mathbf{L}$ only to emphasize the connections with closure operators discussed on p.1013 (below schemas (A)–(I)), in Fig.313 and in item (I) above. Strictly speaking, we should call $(\mathcal{S}\circ\mathcal{L})(\mathbf{L})$ the $(\mathcal{S},\mathcal{L})$-closure of $\mathbf{L}$ only if the following three conditions hold (1) $\mathbf{L} \rightarrowtail (\mathcal{S}\circ\mathcal{L})(\mathbf{L})$, (2) $(\mathcal{S}\circ\mathcal{L})^2(\mathbf{L}) \cong (\mathcal{S}\circ\mathcal{L})(\mathbf{L})$, for all $\mathbf{L} \in Dom(\mathcal{S})$, and (3) $\mathcal{S}\circ\mathcal{L}$ preserves embeddability, i.e. "$\rightarrowtail$".

Therefore our $(\mathcal{S}, \mathcal{L})$ duality also yields a "stronger" duality

$$\text{distributive lattices} \quad \underset{\mathcal{L}}{\overset{\mathcal{S}}{\rightleftarrows}} \quad \mathrm{T}_0 \text{ topological spaces.}$$

This duality too is analogous with our duality

$$\mathsf{Mod}(Th) \quad \underset{\mathcal{M}}{\overset{\mathcal{G}}{\rightleftarrows}} \quad \mathsf{Ge}(Th)$$

elaborated in §§ 6.6.1, 6.6.3 cf. e.g. pp. 1007–1014. Hence $\mathsf{Mod}(Th)$ corresponds to distributive lattices, $\mathcal{G}$ to $\mathcal{S}$ etc.

Boolean algebras: Boolean algebras, BA's for short, are very important examples of distributive lattices. Here we treat BA's as special distributive lattices, hence instead of $\langle B; \cup, \cap, -\rangle$ here a BA is of the form $\langle B; \cup, \cap\rangle$, i.e. we throw away complementation as a distinguished operation.

Our above discussed $(\mathcal{S}, \mathcal{L})$ duality is too simple minded for working for BA's. Namely for any infinite distributive lattice $\mathbf{L}$ we have that $(\mathcal{S} \circ \mathcal{L})(\mathbf{L})$ is not a BA. In particular for any infinite BA, $\mathbf{B}$ we have that $(\mathcal{S} \circ \mathcal{L})(\mathbf{B})$ is not a BA. So our $(\mathcal{S}, \mathcal{L})$ duality does not *automatically* yield a

$$\mathsf{BA} \quad \rightleftarrows \quad \text{some kinds of topologies}$$

type duality. However, it is easy to modify our $(\mathcal{S}, \mathcal{L})$ to obtain a duality for BA's. For this we define

$$\mathcal{L}_B : \text{ topological spaces } \longrightarrow \text{ distributive lattices}$$

as follows. Let $\mathbf{X} = \langle X, \mathcal{O}\rangle$ be a topological space. Those subsets of X which are both open and closed (i.e. the members of $\mathcal{O}$ whose complements are also in $\mathcal{O}$) are called *clopen* sets of $\mathbf{X}$. Now, we define

$$\mathcal{L}_B(\mathbf{X}) \overset{\text{def}}{=} \langle \text{clopen sets of } \mathbf{X};\ \cup, \cap\rangle.$$

[1006]For completeness we note that $Rng(\mathcal{L} \circ \mathcal{S}) \subseteq$ (compact[1007] T_0 topological spaces), hence $\mathcal{L} \circ \mathcal{S}$: (topological spaces) $\longrightarrow$ (compact T_0 topological spaces). Further $\langle X, \mathcal{O}\rangle$ is a *T_0-space* iff $(\forall p, q \in X)(\exists Y \in \mathcal{O})$ such that Y distinguishes p from q i.e. either $p \in Y \not\ni q$ or vice versa.

[1007]For the notion of compact topological spaces cf. footnote 1008 on p.1018.

It is easy to check that

$$\mathcal{L}_B : \text{ topological spaces } \longrightarrow \mathsf{BA}$$

where BA denotes the class of BA's.

Now we have a very general duality for BA's and topologies, namely

$$\mathsf{BA} \quad \underset{\mathcal{L}_B}{\overset{\mathcal{S}}{\rightleftarrows}} \quad \text{topological spaces.}$$

This has the properties

(i) $\mathbf{B} \cong (\mathcal{S} \circ \mathcal{L}_B)(\mathbf{B})$,

(ii) $\mathbf{X} \xrightarrow{f} (\mathcal{L}_B \circ \mathcal{S})(\mathbf{X})$, for some continuous function f, and

(iii) $(\mathcal{L}_B \circ \mathcal{S})^2(\mathbf{X}) = (\mathcal{L}_B \circ \mathcal{S})(\mathbf{X})$.

So, our theorem schemas (B), (C), (H) hold for the present analogy where BA corresponds to $\mathsf{Mod}(Th)$ etc.

Next, in order to obtain a "tighter" duality, let us look at $\mathcal{S}[\mathsf{BA}] \subseteq$ topologies. One can prove that $\mathcal{S}[\mathsf{BA}]$ consists exactly of the so called *Boolean spaces* which in turn are exactly the compact[1008] , T_2 (i.e. Hausdorff)[1009] spaces having a clopen base.[1010] One can regard $(\mathcal{L}_B \circ \mathcal{S})(\mathbf{X})$ as the Boolean space "closest" to the original $\mathbf{X}$ (in some sense). Consider

$$\mathbf{X} \xrightarrow{f} (\mathcal{L}_B \circ \mathcal{S})(\mathbf{X}).^{1011}$$

Then, very roughly speaking, what f does to $\mathbf{X}$ is that (i) it collapses those points which cannot be distinguished by clopen sets, (ii) forgets those open sets which are not unions of clopen sets, and (iii) adds new points to X to make the new topology compact.[1012]

[1008] A topological space $\langle X, \mathcal{O}\rangle$ is called *compact* if for any $\mathcal{H} \subseteq \mathcal{O}$, if $X = \bigcup \mathcal{H}$ then there is a finite $\mathcal{H}_0 \subseteq \mathcal{H}$ with $X = \bigcup \mathcal{H}_0$. Cf. also footnote 1104 on p.1100.

[1009] A topological space $\mathbf{X} = \langle X, \mathcal{O}\rangle$ is called T_2 or equivalently *Hausdorff space* iff

$$(\forall p, q \in X)\,[\,p \neq q \;\Rightarrow\; (\exists A, B \in \mathcal{O})\ (A \cap B = \emptyset \text{ and } p \in A \text{ while } q \in B\,)\,].$$

[1010] For understanding the rest of this work it is not necessary to understand these topological concepts.

[1011] Later, in the category theoretical part we will call $(\mathcal{L}_B \circ \mathcal{S})(\mathbf{X})$ or "$\xrightarrow{f} (\mathcal{L}_B \circ \mathcal{S})(\mathbf{X})$" the *reflection* of $\mathbf{X}$ in the subcategory "Boolean spaces", cf. Def.6.6.78 (p.1090).

[1012] (iii) is called compactification.

Let $\mathcal{L}_{\mathsf{BA}} := \mathcal{L}_B \restriction \text{Boolean spaces}$ be the restriction of $\mathcal{L}_B$ to Boolean spaces. Then we obtain a very strong duality:

$$\mathsf{BA} \quad \underset{\mathcal{L}_{\mathsf{BA}}}{\overset{\mathcal{S}}{\rightleftarrows}} \quad \text{Boolean topological spaces}$$

with the properties

$$\begin{aligned} \mathbf{B} &\cong (\mathcal{S} \circ \mathcal{L}_{\mathsf{BA}})(\mathbf{B}), \\ \mathbf{X} &\cong (\mathcal{L}_{\mathsf{BA}} \circ \mathcal{S})(\mathbf{X}), \end{aligned}$$

for all $\mathbf{B} \in \mathsf{BA}$ and $\mathbf{X} \in$ Boolean spaces. The duality $(\mathcal{S}, \mathcal{L}_{\mathsf{BA}})$ is called *Stone duality*, and it satisfies our theorem schemas (A)–(H).[1013] Later, in the category theoretic sub-section we will see that such dualities are called *equivalences between categories*. In particular, $(\mathcal{S}, \mathcal{L}_{\mathsf{BA}})$ is an equivalence between the categories of BA's and of Boolean spaces, cf. Def.6.6.82 (p.1094). (This notion is weaker than isomorphism of categories, but this weaker, more flexible notion is generally considered more adequate for studying categories and their natural properties.)

In connection with our category theoretic sub-section way below (p.1084) we note that our functions

$$\mathcal{S} : \text{lattices} \longrightarrow \text{topological spaces} \quad \text{and} \quad \mathcal{L} : \text{topological spaces} \longrightarrow \text{lattices}$$

naturally extend from "objects" to "morphisms" (e.g. from lattices to lattice-homomorphisms and from topological spaces to continuous functions). This extension ensures that our functions become functors in the category theoretic sense, which in turn makes our duality theories more comprehensive in a way to be discussed in our category theoretic sub-section.

For more on Stone duality (for BA's and Boolean spaces) we refer to e.g. Davey-Priestley [68] and/or Burris-Sankappanavar [54].

Before continuing, we note the following. As we already mentioned in the introduction of §6.6 (cf. Fig.309 on p.1003 and $\mathsf{World}_1 \rightleftarrows \mathsf{World}_2$ on p.1004) one of the main uses of duality theories is that they connect two "worlds" of mathematics (like $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$) and they make it possible to translate problems from one world to the other, obtain solutions in this second world and translate the result back. (The main idea is that *some* problems are easier to solve in one world,

[1013] In a sense, it also satisfies (I) too but in a dual form, i.e. in an arrow-reversing form, cf. footnote 990 on p.1012. (Recall that the functor $\mathcal{S}$ is arrow-reversing i.e. contravariant cf. p.1016 and p.1015 immediately above item (II).)

while *others* are easier in the other world.) Indeed, several problems of BA-theory have been solved by translating them to topological problems via Stone duality theory (and then translating the result back, of course). Cf. also item in §6.6.6 entitled "Motivation for studying equivalences of categories, adjoint situations, etc" on p.1096.

(III) *Connections of Stone duality with the (syntax, semantics)-duality in logic and in particular with parts of definability theory discussed in §6.3 (p.928):*

In connection with Fig.306 (p.984) and Fig.314 (p.1021), for the interested reader, we note that the (syntax, semantics)-duality as discussed in this work is an organic part of algebraic logic. Therefore if the reader wants to learn more about this duality he can find more information in works usually classified as algebraic logic (or sometimes as its category-theoretic oriented parts).

Notation: For any first-order theory Th,

$$Fm(Th) \stackrel{\text{def}}{:=} Fm(\mathsf{Mod}(Th)),$$

i.e. $Fm(Th)$ is the set of formulas of the language of the theory Th. In this definition we assume that the vocabulary of Th is somehow determined by Th. I.e. when specifying a theory one has to specify its vocabulary, too. (We often leave this to context).

Convention: In the present remark (explaining duality theories etc.), we treat interpretations in a somewhat simpler way/form than in the definability section §6.3. The difference is that in the duality item we consider only one-sorted theories. I.e. the objects of the category Theories in Fig.314 are one-sorted theories. In the definability section we concentrated on many-sorted theories. Hence there interpretations were understood between many-sorted theories which made them slightly more complicated objects than interpretations in the present part.[1014] Cf. p.1023 footnote 1022, p.984, p.968 and footnote 936 (p.968).

Here the analogy is between two duality theories. One of them is Stone duality, while the other duality acts between the category of first-order theories (and translation mappings between them as morphisms) and the category of axiomatizable model classes (and first-order definable meta-functions i.e. interpretations between them). (For the latter duality see Fig.306 on p.984, while for the analogy with Stone duality cf. Fig.314.) In more detail, the category of Boolean algebras is put

[1014] E.g. in §6.3 an interpretation consisted of a function Tr together with something called "*code*" (cf. p.968). In the present item we do not need "*code*" but only Tr.

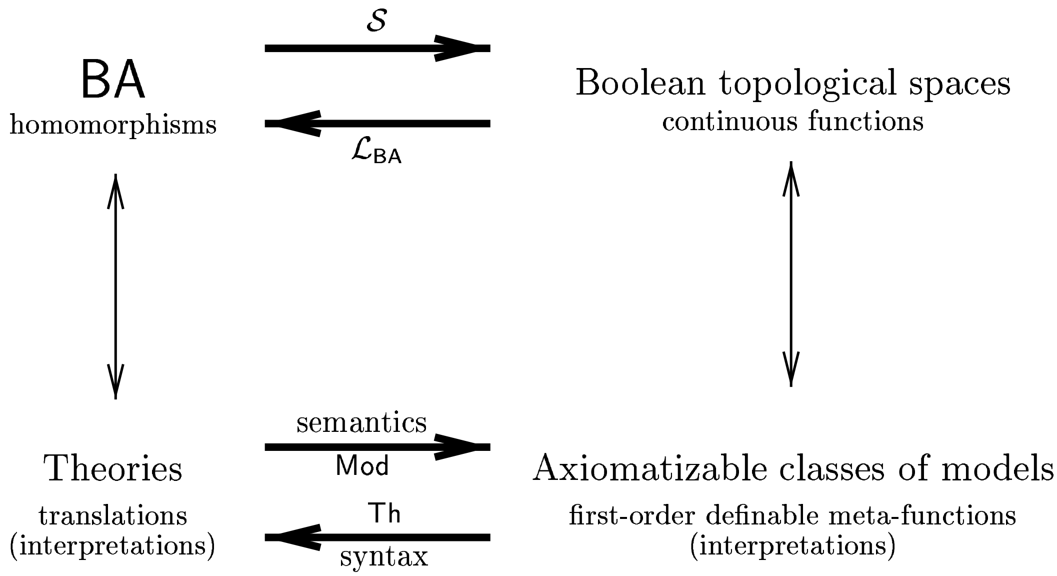

Figure 314: The analogy between Stone duality and (syntax, semantics)-duality.

into analogy with the category of (first-order) theories (in the syntactical sense) whose objects are the $\langle Fm(Th), Th\rangle$ pairs. The morphisms of this category are the translation mappings or interpretations like Tr_f in Fig.306 on p.984 cf. also footnote 1022 on p.1023, Prop.6.4.4 (p.985) and Theorems 6.3.26, 6.3.27 (pp. 962, 965). (These translation mappings are often called interpretations.[1015]) E.g. we can consider $\langle Fm(Th_1), Th_1\rangle$, $\langle Fm(Th_2), Th_2\rangle$ as two BA's and any translation mapping Tr from Th_1 to Th_2 will be a homomorphism between these BA's. Let, now, K_1 and K_2 be two axiomatizable classes of models. Let $f : \mathsf{K}_2 \longrightarrow \mathsf{K}_1$ be a first-order definable meta-function[1016]. Then f induces a translation mapping $Tr_f : Fm(\mathsf{K}_1) \longrightarrow Fm(\mathsf{K}_2)$ satisfying the conclusion of Prop.6.4.4, p.985. Let us notice that $Fm(\mathsf{K}_i)$ are theories hence they correspond to BA's (of equivalence classes of formulas) and Tr_f turns out to be a BA-homomorphism. Further K_2, K_1 are Boolean topological spaces[1017] and f is a continuous function. The details of these claims will be given soon. Now

[1015] The choice between using "translation mapping" or "interpretation" depends only on which aspects or which perspective/background we want to emphasize (and also on with which part of the literature we want to emphasize the connections cf. footnote 1022 on p.1023).
[1016] cf. p.983, Def.6.4.2
[1017] if we collapse the elementarily equivalent models

if we apply Stone duality to the BA-homomorphism Tr_f then we will obtain the continuous function f as its dual. See Fig.314.

In more detail the duality between classes of models $\mathsf{Mod}(Th)$ and sets of formulas $Fm(Th)$ of first-order logic[1018] (i.e. the duality at the heart of the theory of translation mappings $Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$, or equivalently $Tr : Fm(\mathsf{K}_1) \longrightarrow Fm(\mathsf{K}_2)$, cf. e.g. p.965 or Prop.6.4.4 on p.985) is, basically, a *special case* of our above outlined duality

$$\mathsf{BA} \quad \underset{\mathcal{L}_{\mathsf{BA}}}{\overset{\mathcal{S}}{\rightleftarrows}} \quad \text{Boolean topological spaces}$$

(or more generally of our $(\mathcal{S}, \mathcal{L}_B)$-duality). The (syntax, semantics)-duality of logic acts between the syntactical category $\{\, \langle Fm(Th), Th\rangle : Th$ is a theory of our logic$\,\}$ of theories and semantical category $\{\, \mathsf{Mod}(Th) : Th$ is a theory of our logic$\,\}$ of axiomatizable classes of models. It is of the pattern

$$\{\, \langle Fm(Th), Th\rangle \ : \ Th \text{ is a theory}\,\} \ \rightleftarrows \ \{\, \mathsf{Mod}(Th) \ : \ Th \text{ is a theory}\,\}.$$

It is important to emphasize that the vocabularies of different choices of Th may be completely different. E.g. the vocabulary of that of Th_1 may be different from that of Th_2. In these categories the emphasis is on the possible interpretations or translations between two <u>different</u> languages. Such a translation is of the form $Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$ which we already encountered in Definability theory, cf. Theorems 6.3.26, 6.3.27 (pp. 962, 965).

We claim that the above (syntax, semantics)-duality is very strongly related to Stone duality of BA's. Indeed, for any theory Th (of first-order logic) the set $Fm(Th)$ of formulas of Th forms a Boolean algebra if we identify semantically equivalent formulas (modulo Th), while $\mathsf{Mod}(Th)$ forms a compact topological space having a clopen base.[1019] Actually, $\mathsf{Mod}(Th)$ becomes a Boolean space if we identify elementarily equivalent models (obtaining $\langle \mathsf{Mod}(Th)/{\equiv_{ee}}, \mathcal{O}\rangle$ etc.). The set of *points* of the topology "$\mathsf{Mod}(Th)$" is $\mathsf{Mod}(Th)$ while the set of its closed sets is the collection of axiomatizable subclasses of $\mathsf{Mod}(Th)$.

We can view $\mathsf{Mod}(Th)$ as the dual $\mathcal{S}(Fm(Th))$ of the BA of formulas $Fm(Th)$ and similarly we can view the BA $Fm(Th)$ as the

[1018] An analogous duality claim holds for <u>any</u> logic (in place of first-order logic), cf. e.g. Andréka et al. [30], for the notion of an arbitrary logic, and see Marti-Oliet et al. [186] (in [94]) for a (syntax, semantics)-duality for such arbitrary logics.

[1019] A subclass $\mathsf{K} \subseteq \mathsf{Mod}(Th)$ is clopen iff K is finitely axiomatizable.

dual $\mathcal{L}_{\mathsf{BA}}(\langle \mathsf{Mod}(Th), \text{"complements of axiomatizable classes"}\rangle)$ of the topology $\langle \mathsf{Mod}(Th), \ldots \rangle$.[1020]

Recall that our duality functor $\mathcal{S} : \mathsf{BA} \longrightarrow$ topologies is "contravariant", i.e. it *reverses* the *directions* of our morphisms.[1021] This corresponds to the fact that if we have a (logical) translation mapping

$$Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$$

between two theories Th_1 and Th_2, then on the semantic side this will induce a function

$$\mathcal{M}_{Tr} : \mathsf{Mod}(Th_2) \longrightarrow \mathsf{Mod}(Th_1)$$

going in the *opposite* direction. Cf. e.g. Prop.6.4.4 (p.985), Theorems 6.3.26 (p.962), 6.3.27 (p.965), 6.5.5 (p.996), 6.6.16 (p.1033), 6.6.45 (p.1061), 6.6.59 (p.1075). Typical examples for such semantical functions $\mathcal{M}_{Tr}$ are the first-order definable meta-functions on pp. 983, 1030, 1059, 1073, more concretely $\mathcal{M}$, $\mathcal{G}$, $\mathcal{G}o$, $\mathcal{M}o$ etc.

Let $\equiv_{Th}$ be the equivalence relation on formulas defined by $\varphi \equiv_{Th} \psi$ iff $Th \models \varphi \leftrightarrow \psi$. We write $Fm(Th)/Th$ for $Fm(Th)/\equiv_{Th}$. Then, a closer analysis of such situations reveals that, actually, Tr is a Boolean homomorphism from the BA $Fm(Th_1)/Th_1$ to $Fm(Th_2)/Th_2$, and that $\mathcal{M}_{Tr}$ is a continuous function $\langle \mathsf{Mod}(Th_2), \mathcal{O}_2\rangle \longrightarrow \langle \mathsf{Mod}(Th_1), \mathcal{O}_1\rangle$ where $\mathcal{O}_i$ ="complements of axiomatizable classes" acting between the classes of models (viewed as topological spaces).

In general, if we have an *interpretation*[1022] of Th_1 in Th_2, then this interpretation induces a Boolean homomorphism, of the kind $Fm(Th_1) \longrightarrow Fm(Th_2)$ while[1023] on the semantical level of abstraction it induces a continuous function $\mathsf{Mod}(Th_2) \longrightarrow \mathsf{Mod}(Th_1)$.

[1020] To be precise we should write here $Fm(Th)/\equiv_{Th}$ in place of $Fm(Th)$, but this notation will be introduced only a little bit later. Further, if we concentrated on the subalgebra $Fmclosed(Th)/\equiv_{Th}$ of closed formulas (of $Fm(Th)$) then the formation of the dual $\mathcal{S}(Fm(Th)/\equiv_{Th})$ of $Fm(Th)/\ldots$ would even more closely correspond to our earlier definition of the Stone functor $\mathcal{S}$. However we stick with our original BA $Fm(Th)/\equiv_{Th}$ which we will sloppily denote as $Fm(Th)$.

[1021] i.e. it sends $f : A \longrightarrow B$ to $\mathcal{S}(f) : \mathcal{S}(B) \longrightarrow \mathcal{S}(A)$.

[1022] We do not define "interpretations" carefully, but they are basically the same what we called "translation mappings" e.g. on p.984, Fig.306 or Prop.6.4.4 on p.985. Cf. Goguen-Burstall [105], Andréka et al. [11, 12], Henkin-Monk-Tarski [129, Part II, pp. 165-171, 176, item 4.368(4)] and Gergely [100].

[1023] When referring to $Fm(Th)/Th$ as a Boolean algebra we deliberately omit the "$/Th$" part in order to put the emphasis on the intuitive ideas.

Our continuous function $\mathcal{M}_{Tr}$ is strongly related to the meta-function $f : \mathsf{K} \longrightarrow \mathsf{L}$ in item (5) of "Discussion of the definition of $\equiv_\Delta$" on p.971 (definability theory) as well as to the meta-functions $\mathcal{G}$, $\mathcal{M}$, $\mathcal{G}o$, $\mathcal{M}o$, etc. as indicated above. Typical examples of interpretations $Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$ are Tr_1, Tr_2, T_1, T_2 on pp. 973–976; $T_{\mathcal{W}e}$, $T_{\mathcal{T}a}$ in Thm.6.5.5 on p.996; $T_{\mathcal{M}}$, $T_{\mathcal{G}}$ in Thm.6.6.16 on p.1033; and $T_{\mathcal{M}o}$ in Thm.6.6.59 on p.1075.

For simplicity, assume that we are in one-sorted logic (and we do not define new sorts). Then, the special case when

$$\mathcal{M}_{Tr} : \mathsf{Mod}(Th_2) \longrightarrow\!\!\!\!\!\rightarrow \mathsf{Mod}(Th_1)$$

is *surjective* corresponds exactly to the case when $\mathsf{Mod}(Th_1)$ is being *defined* over $\mathsf{Mod}(Th_2)$ in the sense of §6.3. The translation mapping associated to this "definition of $\mathsf{Mod}(Th_1)$" in Thm.6.3.26 is exactly the Boolean homomorphism

$$Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$$

discussed in Thm.6.3.26 and Prop.6.4.4. Now, by duality theory we get that

$$Fm(Th_1)/Th_1 \rightarrowtail Fm(Th_2)/Th_2$$

is a Boolean embedding (on the algebras of formulas).

Interpretations $Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$ are more general than definitions since if Tr is an interpretation then $\mathcal{M}_{Tr} : \mathsf{Mod}(Th_2) \longrightarrow \mathsf{Mod}(Th_1)$ need not be surjective (while in case of definitions it is surjective, cf. item (2) on p.938). However, if Tr is an interpretation, then some axiomatizable subclass[1024] of $\mathsf{Mod}(Th_1)$ *is definable* over $\mathsf{Mod}(Th_2)$. We do not need this level of generality here, so we do not discuss interpretations in more detail, but for more on the theory of interpretations cf. the references in footnote 1022. For completeness, we note that the interpretation of the theory Th_1 of groups in the theory Th_2 of BA's is a typical example of an interpretation. Here the group operation $+$ is interpreted as the derived operation symmetric difference $\oplus$ of BA's (where $x \oplus y \stackrel{\text{def}}{=} (x \cap -y) \cup (y \cap -x)$). The interesting aspect of this example is that $\mathcal{M}_{Tr} : \mathsf{BA} \longrightarrow \mathsf{Groups}$ is neither surjective nor injective. (Here groups are not defined over BA's, but an axiomatizable subclass of groups called Boolean groups is being defined over BA's.)

* * *

[1024]this subclass is the class $\mathcal{M}_{Tr}[\mathsf{Mod}(Th_2)]$, which is actually a closed set of the topology $\mathsf{Mod}(Th_1)$

In the above discussion we had to put more emphasis on the "morphism" part of Stone duality than we did in the presentation of Stone duality. This "gap" will be filled in in §6.6.6 (1084). In connection with the above discussion (of theories *Th* and their models $\mathsf{Mod}(Th)$) we also refer to item (IV) below and the lattice of first-order theories in Chapter 4 beginning with p.451.

Summing up, what we tried to say in the above discussion is that Stone duality is, basically, the same thing as the (key idea of) (syntax, semantics)-duality of logic which was used implicitly in §6.3 (and which, in particular, makes translation mappings between formulas go in the opposite direction as they go between the models).[1025]

* * *

The translation mappings denoted in the above discussion[1026] as $Tr : Fm(Th_1) \longrightarrow Fm(Th_2)$ are often called *theory morphisms* in the literature. Accordingly the category with objects of the form $\langle Fm(Th), Th\rangle$ is often called the *category of theory morphisms* (this is on the left-hand side of the above duality).

Connections with physics: A closer inspection of the category of theory morphisms (or equivalently of interpretations) is not only a category but also a so called 2-dimensional category cf. e.g. Zlatos [277]. Interestingly the categories applied in physics are also 2-dimensional categories or more generally n-dimensional ones. Cf. Baez-Dolan [36], Crane [63], Freed [87], Fröhlich-Kerler [91].

In passing, we note that the Theories $\overrightarrow{\longleftarrow}$ Models duality outlined way above (under the name (syntax, semantics)-duality) is more fully represented if we

[1025] Stone duality concentrates on the category of BA's. Syntax-semantics duality concentrates on the category of theories. But a theory *Th* induces a Boolean algebra $Fm(Th)/Th$. This gives us a connection ...etc.
Else: In passing we note that $Fm(Th)/Th$ is a slightly more complex object than a plain BA. Therefore (in logical (syntax, semantics)-duality) when forming the dual $\mathcal{S}(Fm(Th)/Th)$ of $Fm(Th)/\dots$ we do not take all prime ideals of $Fm(Th)/Th$, but only those ones, say P, whose complements $-P$ form consistent theories of our logic. (To this end we have to view $-P$ as a subset of $Fm(Th)$.) Equivalently, we could use the prime ideals of the subalgebra $Fmclosed(Th)/Th$, but we think that requiring $-P$ to be a consistent theory is more helpful in building good logical intuition.
For completeness: To make the connection with Stone duality even closer, we have the following extra option: We can stick with $Fm(Th)/Th$ on the BA side (using all prime ideals) and on the topology side use model-evaluation pairs $\langle \mathfrak{M}, \bar{a}\rangle$ as points of our topology "$\mathsf{Mod}(Th)$". In this setting the analogy with Stone duality is perfect. This train of thought when pushed to the extreme leads eventually to cylindric algebras (CA's) in place of BA's, and to represented CA's in place of represented BA's (which are nothing but Boolean spaces).

[1026] of the (syntax, semantics)-duality