Proof: The proposition follows by Thm.6.2.59. ∎

The following theorem says that in **Basax** models the world-view transformations f_{mk} preserve Minkowskian orthogonality.

THEOREM 6.2.63 *Assume* **Basax**. *Let* $\ell, \ell' \in \mathsf{Eucl}$ *and* $m, k \in Obs$. *Then*

$$\ell \perp_\mu \ell' \quad \Rightarrow \quad \mathsf{f}_{mk}[\ell] \perp_\mu \mathsf{f}_{mk}[\ell'].$$

The **proof** is available from Judit Madarász. ∎

Roughly, the following theorem says that the $(\prec, eq, g, \mathcal{T})$-free reduct of almost any $(\mathbf{Bax}^\oplus + \mathbf{Ax6})$-geometry coincides with the similar reduct of a Minkowskian geometry. Further, the same holds for the $(eq, q, \mathcal{T})$-free reducts of $(\mathbf{Bax}^\oplus + \mathbf{Ax6} + \mathbf{Ax(\uparrow\uparrow)})$-geometries. Stronger forms of the following theorem, not involving **Ax6**, will be stated in §6.2.5 as Theorems 6.2.71, 6.2.73.

THEOREM 6.2.64 *Assume* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^\oplus + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax6})$. *Then (i) and (ii) below hold.*

(i) *Assume $n > 2$. Then the $(\prec, \mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean $\mathfrak{F}$ such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, Bw_\mu, \perp_\mu\rangle,$$

cf. Figures 282, 283.

(The other direction also holds by Prop.6.2.62.)[815]

(ii) *Assume* $\mathbf{Ax(\uparrow\uparrow)}$. *Then the $(\mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean $\mathfrak{F}$ such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp_r\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, \prec_\mu, Bw_\mu, \perp_\mu\rangle.$$

(The other direction also holds by Prop.6.2.62.)

Proof: The theorem follows by the first proof given for Thm.6.2.10 (p.813), by Prop.6.2.62 and by Prop.6.2.32 (p.840). ∎

Roughly, the following theorems says that the $(\prec, g, \mathcal{T})$-free reduct of almost any **Basax** geometry $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry. Further, the same holds for $(g, \mathcal{T})$-free reducts of $(\mathbf{Basax} + \mathbf{Ax(\uparrow\uparrow)})$-geometries. Generalizations of the following theorem for **Newbasax** (in place of **Basax**) will be stated in §6.2.5 as Theorems 6.2.74, 6.2.75.

[815] I.e. this reduct of any Minkowskian geometry is obtainable as a reduct of a $(\mathbf{Bax}^\oplus + \ldots)$-geometry (up to isomorphism of course).

THEOREM 6.2.65 *Assume $n > 2$ and $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$. Then (i) and (ii) below hold. (Cf. Figures 282, 283.)*

(i) *The $(\prec, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean $\mathfrak{F}$ such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r, eq\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, Bw_\mu, \perp_\mu, eq_\mu\rangle.$$

(The other direction also holds by Prop.6.2.62.)

(ii) *Assume $\mathbf{Ax}(\uparrow\uparrow)$. Then the $(g, \mathcal{T})$-free reduct of $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean $\mathfrak{F}$ such that*

$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \prec, \perp_r, eq\rangle \cong \langle {}^nF, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, Bw_\mu, \prec_\mu, \perp_\mu, eq_\mu\rangle.$$

(The other direction also holds by Prop.6.2.62.)

On the proof: A proof can be obtained by the first proof given for Thm.6.2.10 (p.813), by Prop.6.2.62, by Claim 6.2.84 (p.892), by Prop.6.2.88 (p.895) and Prop.6.2.32 (p.840). ∎

Remark 6.2.66

(i) In **Basax** we know that if we are given a possible life-line ℓ then ℓ completely determines the relation of simultaneity of observers living on ℓ. (By relation of simultaneity we mean a binary relation between events). This generalizes to $\mathbf{Bax}^\oplus + \mathbf{Ax}(\sqrt{\ })$, but it does not generalize e.g. to **Reich**(**Basax**).

(ii) In $\mathbf{Basax}(4) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$ we have the following property. Assume we are given four lines $\ell, \ell_1, \ell_2, \ell_3 \in L$ intersecting at one point and mutually $\perp_r$-orthogonal. Assume exactly one of them is time-like. Then there is an observer whose coordinate axes are exactly these four lines. The other direction is also true: the coordinate axes of any observer behave like $\ell, \ldots, \ell_3$.

This generalizes to $\mathbf{Bax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$.

◁

Remark 6.2.67 (Connections between our sub-sections discussing reducts of $\mathfrak{G}$, connections with the literature, interdefinability of parts of $\mathfrak{G}$, and streamlining of $\mathfrak{G}$:)

Sub-sections 6.2.4, 6.2.5, 6.2.9, 6.6.10, 6.6.11, and 6.7 are strongly connected with each other in that they are all involved with the subjects listed in the title of the present remark. We do not discuss these connections here in more detail but the reader is invited to compare these sub-sections and to combine their contents from the point of view of the subjects mentioned in the title of the present remark. Cf. Figures 282, 283 (pp. 863–864) for what is common in these.

$\triangleleft$

6.2.5 Getting familiar with our geometries; unions of geometries and models

In this section we will analyze how the geometries $\mathfrak{G}_{\mathfrak{M}}$ are "put together" i.e. how one can have a grasp on them. Roughly, we will see that $\mathfrak{G}_{\mathfrak{M}}$ is obtained from the world-views (now regarded as geometries) of inertial observers by gluing them together in some way, cf. Fig.289 (p.887). For more on the intuition behind this (or how these ideas will be implemented) see p.883 above Prop.6.2.79.

As a motivation for studying disjoint union of geometries (and generalizations of this in items 3,4,5 below) we refer the reader to Remark 6.2.81 and Figure 290 on p.888 on the connections with Penrose diagrams from general relativity.

We will use the concept of *disjoint unions of* $\mathbf{Bax}^-$ models as well as *disjoint unions of geometries* similar to our observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$. In both cases we will assume that the field reducts of the structures in question coincide.

1. **Disjoint, generalized disjoint and photon-disjoint unions of models:** Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$. Assume $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} = \emptyset$. Then the <u>*disjoint union*</u>

$$\mathfrak{M} \,\dot\cup\, \mathfrak{N} :\stackrel{\text{def}}{=} \langle B^{\mathfrak{M}} \cup B^{\mathfrak{N}}, \ldots, \mathfrak{F}, G, \in, W^{\mathfrak{M}} \cup W^{\mathfrak{N}} \rangle$$

is defined as in the statement of Theorem 3.3.12 (p.196). Then

$$\mathfrak{M} \,\dot\cup\, \mathfrak{N} \models \mathbf{Bax}^-.$$

Actually, $\mathfrak{M} \dot\cup \mathfrak{N}$ to be defined and to be a $\mathbf{Bax}^-$ model, we do not need to assume $B^\mathfrak{M} \cap B^\mathfrak{N} = \emptyset$ since the "disjointness conditions"[816] in the statement of Thm.3.3.12 are sufficient. This more general notion (using the disjointness conditions) is called *generalized disjoint union* and is denoted by $\mathfrak{M} \cup \mathfrak{N}$.

Instead of only two models, we can form the union of any class K of models (satisfying some disjointness conditions) exactly as we did in Thm.3.3.12. In particular let $\mathsf{K} \subseteq \mathsf{Mod}_\mathfrak{F}(\mathbf{Bax}^-)$. Assume

$$(\forall \text{ distinct } \mathfrak{M}, \mathfrak{N} \in \mathsf{K})\ B^\mathfrak{M} \cap B^\mathfrak{N} = \emptyset.$$

Then the *disjoint union* $\dot\bigcup \mathsf{K}$ of K is defined exactly as in Thm.3.3.12, i.e.

$$\dot\bigcup \mathsf{K} :\stackrel{\text{def}}{=} \Big\langle \bigcup_{\mathfrak{M}\in\mathsf{K}} B^\mathfrak{M}, \ldots, \mathfrak{F}, G, \in, \bigcup_{\mathfrak{M}\in\mathsf{K}} W^\mathfrak{M} \Big\rangle.$$

Then $\dot\bigcup \mathsf{K} \models \mathbf{Bax}^-$. Again (for having $\dot\bigcup \mathsf{K} \models \mathbf{Bax}^-$) instead of complete disjointness of $B^\mathfrak{M}$ and $B^\mathfrak{N}$ it is sufficient to require the milder disjointness conditions (on K) in the formulation of Thm.3.3.12. This more general kind of union is again called *generalized disjoint union* (as it was in the case of two models above) and is denoted by $\bigcup \mathsf{K}$.

We note that if $\bigcup \mathsf{K}$ is a generalized disjoint union then

$$(\forall \text{ distinct } \mathfrak{M}, \mathfrak{N} \in \mathsf{K}) Obs^\mathfrak{M} \cap Obs^\mathfrak{N} = \emptyset,$$

while this does not necessarily hold for Ph in place of Obs.

Generalized disjoint union $\bigcup \mathsf{K}$ is called *photon-disjoint union* iff

$$(\forall \text{ distinct } \mathfrak{M}, \mathfrak{N} \in \mathsf{K})\ Ph^\mathfrak{M} \cap Ph^\mathfrak{N} = \emptyset.$$

Note that disjoint unions form a special case of photon-disjoint unions, and photon-disjoint unions form a special case of generalized disjoint unions.

2. **Disjoint unions of non-body-disjoint models:**

   Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_\mathfrak{F}(\mathbf{Bax}^-)$ be such that $B^\mathfrak{M} \cap B^\mathfrak{N} \neq \emptyset$.[817] The *disjoint union* $\mathfrak{M} \dot\cup \mathfrak{N}$ of $\mathfrak{M}$ and $\mathfrak{N}$ is defined as follows. Let $\mathfrak{N}' \in \mathsf{Mod}_\mathfrak{F}(\mathbf{Bax}^-)$ be an isomorphic copy of $\mathfrak{N}$ such that (a) and (b) below hold.

   (a) There is an isomorphism between $\mathfrak{N}$ and $\mathfrak{N}'$ which is the identity function on the sort F.

[816] these conditions were $Obs^\mathfrak{M} \cap B^\mathfrak{N} = \emptyset$, $Ph^\mathfrak{M} \cap B^\mathfrak{N} \subseteq Ph^\mathfrak{N}$, $Ib^\mathfrak{M} \cap B^\mathfrak{N} \subseteq Ib^\mathfrak{N}$, together with the same conditions but with $\mathfrak{M}$ and $\mathfrak{N}$ interchanged.

[817] The condition $B^\mathfrak{M} \cap B^\mathfrak{N} \neq \emptyset$ is in principle superfluous but we did not want the present definition of $\mathfrak{M} \dot\cup \mathfrak{N}$ overwrite the one in item 1 (approximately previous page).

(b) $\mathfrak{N}'$ is body-disjoint from $\mathfrak{M}$, i.e. $B^{\mathfrak{M}} \cap B^{\mathfrak{N}'} = \emptyset$.

Now,

$$\mathfrak{M} \dot{\cup} \mathfrak{N} :\overset{\text{def}}{=} \mathfrak{M} \dot{\cup} \mathfrak{N}',$$

where $\mathfrak{M} \dot{\cup} \mathfrak{N}'$ has already been defined.

The <u>*disjoint-union*</u> of an arbitrary class $\mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$ of *non-body-disjoint*[818] models is defined analogously to the case of two models and is denoted by $\dot{\bigcup} \mathsf{K}$.

We note that disjoint unions of (non-body-disjoint) models are determined only up to isomorphism (but this should be no disadvantage, moreover this can be easily avoided if someone wanted to).

3. **Disjoint unions of geometries:**

In the definition of disjoint unions of geometries we will use the following notions from topology.

<u>*Topological spaces:*</u> By a <u>*topological space*</u> we understand a pair $\mathbf{X} = \langle X, \mathcal{O} \rangle$ with $\mathcal{O} \subseteq \mathcal{P}(X)$ closed under finite intersections and infinite unions, and such that $\emptyset, X \in \mathcal{O}$. X is the set of <u>*points*</u> of $\mathbf{X}$ while $\mathcal{O}$ is the set of <u>*open sets*</u> of $\mathbf{X}$. If $Y \in \mathcal{O}$ then $(X \setminus Y)$ is called a <u>*closed set*</u>. Hence the closed sets are the complements of the open ones.

<u>*Coproduct of topologies:*</u> Assume $\mathbf{X}_0 = \langle X_0, \mathcal{O}_0 \rangle$ and $\mathbf{X}_1 = \langle X_1, \mathcal{O}_1 \rangle$ are disjoint topological spaces, i.e. $X_0 \cap X_1 = \emptyset$. Let us recall from topology that the <u>*coproduct*</u> (i.e. sum)[819] $\mathbf{X}_0 \coprod \mathbf{X}_1$ of the topological spaces $\mathbf{X}_0$ and $\mathbf{X}_1$ is defined as follows.

$$\begin{aligned} \mathbf{X}_0 \textstyle\coprod \mathbf{X}_1 &:\overset{\text{def}}{=} \langle X_0 \cup X_1, \mathcal{O}_0 \textstyle\coprod \mathcal{O}_1 \rangle, \quad \text{where} \\ \mathcal{O}_0 \textstyle\coprod \mathcal{O}_1 &:\overset{\text{def}}{=} \{\, U_0 \cup U_1 \;:\; U_0 \in \mathcal{O}_0,\ U_1 \in \mathcal{O}_1 \,\}. \end{aligned}$$

Assume $\mathbf{X}_i = \langle X_i, \mathcal{O}_i \rangle$ are topological spaces, for $i \in I$ with fixed set I. Assume that $\mathbf{X}_i$'s are pairwise disjoint, i.e. that $X_i \cap X_j = \emptyset$, for $i \neq j$. Then the <u>*coproduct*</u> $\coprod_{i \in I} \mathbf{X}_i$ of the family $\langle \mathbf{X}_i : i \in I \rangle$ is defined as follows.

$$\begin{aligned} \coprod_{i \in I} \mathbf{X}_i &:\overset{\text{def}}{=} \langle \bigcup_{i \in I} X_i, \coprod_{i \in I} \mathcal{O}_i \rangle, \quad \text{where} \\ \coprod_{i \in I} \mathcal{O}_i &:\overset{\text{def}}{=} \{ \bigcup_{i \in I} U_i \;:\; \langle U_i \;:\; i \in I \rangle \in \mathbf{P}_{i \in I} \mathcal{O}_i \}, \end{aligned}$$

[818] K is non-body-disjoint if there are distinct $\mathfrak{M}, \mathfrak{N} \in \mathsf{K}$ such that $B^{\mathfrak{M}} \cap B^{\mathfrak{N}} \neq \emptyset$.

[819] Cf. Engelking [83] under the name "sum of spaces".

where $\mathbf{P}_{i\in I}\mathcal{O}_i$ is the usual Cartesian product of the sets $\mathcal{O}_i$, $i \in I$. ($\mathbf{P}_{i\in I}\mathcal{O}_i$ is the generalization of the direct product $\mathcal{O}_0 \times \mathcal{O}_1$). To help the intuition we note that

$$\coprod_{i\in I} \mathcal{O}_i \;=\; \{ \bigcup_{i\in I} U_i \;:\; (\forall i \in I) U_i \in \mathcal{O}_i \}.$$

Note that the "coproduct" $\coprod_{i\in I} \mathcal{O}_i$ of $\mathcal{O}_i$'s has been defined, too.

<u>*Disjoint unions of geometries:*</u> Disjoint unions of geometries in $\mathsf{Ge}(\emptyset)$ are defined similarly to the case of models (in item 1 above), as follows.

Assume, $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots, g_i, \mathcal{T}_i\rangle \in \mathsf{Ge}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$.[820] Assume, $Mn_i \cap Mn_j = \emptyset$, for $i \neq j$ $(i, j \in I)$. The <u>*disjoint union*</u> of $\mathfrak{G}_0, \mathfrak{G}_1$ is defined by

$$\mathfrak{G}_0 \,\dot\cup\, \mathfrak{G}_1 :\stackrel{\text{def}}{=} \langle Mn_0 \cup Mn_1, \mathbf{F_1}, L_0 \cup L_1; \ldots, g_0 \cup g_1, \mathcal{T}_0 \coprod \mathcal{T}_1\rangle.$$

For the general case, the <u>*disjoint union*</u> of the family $\langle \mathfrak{G}_i \,:\, i \in I\rangle$ is

$$\dot{\bigcup_{i\in I}} \mathfrak{G}_i :\stackrel{\text{def}}{=} \Big\langle \bigcup_{i\in I} Mn_i, \mathbf{F_1}, \bigcup_{i\in I} L_i; \ldots, \bigcup_{i\in I} g_i, \coprod_{i\in I} \mathcal{T}_i \Big\rangle.$$

4. **Geometry $\mathfrak{G}^{\perp_0}_{\mathfrak{M}}$ and the class $\mathsf{Ge}^{\perp_0}(Th)$:**

For every frame model $\mathfrak{M}$ we define $\mathfrak{G}^{\perp_0}_{\mathfrak{M}}$ to be the geometry obtained from $\mathfrak{G}_{\mathfrak{M}}$ by replacing the orthogonality $\perp_r$ with the basic orthogonality $\perp_0$ (cf. p.791 for $\perp_0$). Further, for any set Th of formulas in our frame language we define

$$\mathsf{Ge}^{\perp_0}(Th) :\stackrel{\text{def}}{=} \left\{ \mathfrak{G} \;:\; (\exists \mathfrak{M} \in \mathsf{Mod}(Th))\; \mathfrak{G} \cong \mathfrak{G}^{\perp_0}_{\mathfrak{M}} \right\}.$$

<u>*A note to the reader:*</u> At a first reading, the reader may skip item 5 ("Photon-glued ...") below, in such a way that later whenever "photon-glued disjoint unions" are mentioned then the expression "photon-glued ..." should be replaced by "disjoint unions" <u>and</u> **Ax(diswind)** should be added to the assumptions. This is possible because if we assume **Ax(diswind)** then photon-glued disjoint unions become plain disjoint unions. I.e. in the remaining part of this material using photon-glued disjoint unions can be avoided under the expense of assuming **Ax(diswind)**. (We are mentioning this <u>only</u> to help those readers who do not have enough time to read the whole material.)

[820] I.e. the "field" reducts of $\mathfrak{G}_i$ and $\mathfrak{G}_j$ coincide, for all i, j.

5. **Photon-glued disjoint unions of geometries:**

In the present item we concentrate on the $\perp_0$-versions of our geometries because of the following. The point is that in the $\perp_0$-versions if two lines are orthogonal then they are in $L^T \cup L^S$. This enables us to define $\perp_0$ in the photon-glued disjoint unions to be the same as it was in the "ordinary" disjoint unions. (If we tried to extend this to $\perp_r$ then we would face the nontrivial task of defining $\perp_r$-orthogonality between the new lines obtained by "gluing" photon-like lines.)

Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots \rangle \in \mathsf{Ge}^{\perp_0}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$. Assume $Mn_i \cap Mn_j = \emptyset$, for $i \neq j$ $(i, j \in I)$. Then the disjoint union $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ is defined analogously to the case of $\mathsf{Ge}(\emptyset)$ in item 3.

When forming a disjoint union $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ of geometries ($\mathfrak{G}_i \in \mathsf{Ge}^{\perp_0}(\emptyset)$) sometimes we might want to glue certain photon-like lines together into a single, new, longer photon-like line. The idea is the following. We choose a parameter $H \subseteq L^{Ph} = \bigcup_{i\in I} L_i^{Ph}$ with $|H \cap L_i^{Ph}| \leq 1$ for all $i \in I$. Then

$$\mathrm{Glue}_H\Big(\dot{\bigcup_{i\in I}} \mathfrak{G}_i\Big)$$

is obtained from $\dot{\bigcup}_{i\in I} \mathfrak{G}_i$ by adding the new, "long" line $\bigcup H$ to L^{Ph} and throwing away (all the "old" lines in the set) H from L^{Ph}, and by adjusting $L, g, \mathcal{T}$ to the new set of photon-like lines. In more detail: The new sets of photon-like lines and lines are[821]

$$\begin{aligned} L^{Ph}_{\mathrm{Glue}} &\stackrel{\mathrm{def}}{:=} (L^{Ph} \setminus H) \cup \{\textstyle\bigcup H\} \\ L_{\mathrm{Glue}} &\stackrel{\mathrm{def}}{:=} L^{Ph}_{\mathrm{Glue}} \cup L^T \cup L^S, \quad \text{where} \end{aligned}$$

$L^T = \bigcup_{i\in I} L_i^T$ and $L^S = \bigcup_{i\in I} L_i^S$; <u>and</u>, letting $Mn = \bigcup_{i\in I} Mn_i$, the pseudometric and the topology (of the new geometry) are

$$g_{\mathrm{Glue}} \stackrel{\mathrm{def}}{:=} g \cup \{\, \langle e, e_1, \lambda\rangle \in Mn \times Mn \times F \;:\; (\exists \ell \in L^{Ph}_{\mathrm{Glue}}) e, e_1 \in \ell,\ \lambda = 0 \,\},$$

[821]For the nonspecialist of set theory, we would like to illuminate the intuitive content of the expression $(L^{Ph} \setminus H) \cup \{\bigcup H\}$. Assume $L^{Ph} = \{\, \ell, \{a\}, \{b\} \,\}$ and $H = \{\, \{a\}, \{b\} \,\}$. Then $\bigcup H = \{a, b\}$, $\{\bigcup H\} = \{\, \{a, b\} \,\}$. Hence $(L^{Ph} \setminus H) \cup \{\bigcup H\} = \{\, \ell, \{a, b\} \,\}$. Intuitively, this is what we wanted, we wanted to glue together the photon-like lines $\{a\}$, $\{b\}$ into a single new line $\{a, b\}$, and then to replace the old "short" photon-like lines $\{a\}$, $\{b\}$ with the single new line $\{a, b\}$. Summing up: $\bigcup H$ is the new long photon-like line obtained by gluing; and H is the <u>set</u> of the old short lines which we want to throw away since they are replaced by their longer version $\bigcup H$. Important: $\bigcup H$ is a line, while H is not. (It is a <u>set</u> of lines.)

$\mathcal{T}_{\mathrm{Glue}}$	is the topology on Mn determined by g_{Glue}

as described in item 14 on p.797. The rest of the ingredients of the new geometry are the same as those of $\dot{\bigcup}_{i\in I}\mathfrak{G}_i$.[822]

We may glue together more than one sequence H of photon-like lines. Namely let $\mathcal{H}\subseteq\mathcal{P}(L^{Ph})$ be given such that

$$(\forall H\in\mathcal{H})\ (\forall i\in I)|H\cap L_i^{Ph}|\leq 1.$$

Now we apply the above outlined gluing procedure for each $H\in\mathcal{H}$. Formally, we obtain

$$\mathrm{Glue}_{\mathcal{H}}\Big(\dot{\bigcup_{i\in I}}\mathfrak{G}_i\Big)$$

which differs from $\dot{\bigcup}_{i\in I}\mathfrak{G}_i$ only in L^{Ph}, L, g and $\mathcal{T}$, where the new sets of photon-like lines and lines are

$$L^{Ph}_{\mathrm{Glue}(\mathcal{H})} \stackrel{\mathrm{def}}{:=} (L^{Ph}\setminus\bigcup\mathcal{H})\cup\{\bigcup H\ :\ H\in\mathcal{H}\},$$
$$L_{\mathrm{Glue}(\mathcal{H})} \stackrel{\mathrm{def}}{:=} L^{Ph}_{\mathrm{Glue}(\mathcal{H})}\cup L^T\cup L^S;$$

<u>and</u> the pseudo-metric and the topology (of the new geometry) are

$$g_{\mathrm{Glue}(\mathcal{H})} \stackrel{\mathrm{def}}{:=} g\cup\{\langle e,e_1,\lambda\rangle\in Mn\times Mn\times F\ :\ (\exists\ell\in L^{Ph}_{\mathrm{Glue}(\mathcal{H})})e,e_1\in\ell,\ \lambda=0\},$$

$\mathcal{T}_{\mathrm{Glue}(\mathcal{H})}$	is the topology on Mn determined by $g_{\mathrm{Glue}(\mathcal{H})}$.

For a representation of this "glued" $\dot{\bigcup}_{i\in I}\mathfrak{G}_i$ see Figure 307 (p.1001) and the lower picture in Figure 289 (p.887). We call the above defined

$$\mathrm{Glue}_{\mathcal{H}}\Big(\dot{\bigcup_{i\in I}}\mathfrak{G}_i\Big)$$

a *<u>photon-glued disjoint union</u>* of the family $\langle\,\mathfrak{G}_i\ :\ i\in I\,\rangle$ of geometries.

6. **Disjoint and photon-glued disjoint unions of non-disjoint geometries:**

Disjoint unions of non-disjoint geometries are defined analogously to the case of non-body-disjoint models (in item 2 above), as follows.

[822]Let us notice that $\dot{\bigcup}_{i\in I}\mathfrak{G}_i\ \models\ (\,\ell\perp_0\ell'\ \to\ \ell,\ell'\in L^T\cup L^S\,)$ by the definition of $\perp_0$.

Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots \rangle \in \mathsf{Ge}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$ and assume that $\mathfrak{G}_i$'s are non-disjoint, i.e. $Mn_i \cap Mn_j \neq \emptyset$ for some distinct $i, j \in I$. Let $\mathfrak{G}'_i = \langle Mn'_i, \mathbf{F_1}, L'_i; \ldots \rangle \in \mathsf{Ge}(\emptyset)$, for $i \in I$ be such that (a) and (b) below hold.[823]

(a) There is an isomorphism beteen $\mathfrak{G}_i$ and $\mathfrak{G}'_i$ which is the identity function on the sort F.

(b) $(\forall$ distinct $i, j \in I)\ Mn'_i \cap Mn'_j = \emptyset$.

Now, the *disjoint union* of the family $\langle \mathfrak{G}_i \ : \ i \in I \rangle$ is defined to be the disjoint union of the family $\langle \mathfrak{G}'_i \ : \ i \in I \rangle$ (which in turn has already been defined in item 3), and is denoted by $\dot{\bigcup}_{i \in I} \mathfrak{G}_i$.

Assume $\mathfrak{G}_i = \langle Mn_i, \mathbf{F_1}, L_i; \ldots \rangle \in \mathsf{Ge}^{\perp_0}(\emptyset)$, for $i \in I$ with any fixed set I and with a common "field" reduct $\mathbf{F_1}$ and assume that $\mathfrak{G}_i$'s are non-disjoint. Let $\mathfrak{G}'_i = \langle Mn'_i, \mathbf{F_1}, L'_i; \ldots \rangle \in \mathsf{Ge}^{\perp_0}(\emptyset)$, for $i \in I$ be such that (a) and (b) above hold. By a *photon-glued disjoint union* of the family $\langle \mathfrak{G}_i \ : \ i \in I \rangle$ we understand a photon-glued disjoint union of the family $\langle \mathfrak{G}'_i \ : \ i \in I \rangle$.

We note that disjoint unions and photon-glued disjoint unions of (non-disjoint) geometries are determined only up to isomorphism (but this should be no disadvantage, moreover this can be easily avoided, cf. footnote 823).

Remark 6.2.68 We note that unions *commute* with "geometrization" in the following sense.

Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Bax}^-)$. Assume that they satisfy the disjointness conditions[824] in Thm.3.3.12, i.e. that $\mathfrak{M} \cup \mathfrak{N}$ is a generalized disjoint union.

Then

$$\mathfrak{G}^{\perp_0}_{(\mathfrak{M} \cup \mathfrak{N})} = \text{"a photon-glued disjoint union of } \mathfrak{G}^{\perp_0}_{\mathfrak{M}} \text{ and } \mathfrak{G}^{\perp_0}_{\mathfrak{N}}\text{"}.$$

Intuitively, a *generalized disjoint union* in the "observational world" correspond to a *photon-glued disjoint union* in the "geometry world", cf. Figure 284.

Assume in addition that $Ph^{\mathfrak{M}} \cap Ph^{\mathfrak{N}} = \emptyset$, i.e. that $\mathfrak{M} \cup \mathfrak{N}$ is a photon-disjoint union. Then

$$\mathfrak{G}_{(\mathfrak{M} \cup \mathfrak{N})} = \mathfrak{G}_{\mathfrak{M}} \mathbin{\dot{\cup}} \mathfrak{G}_{\mathfrak{N}}.$$

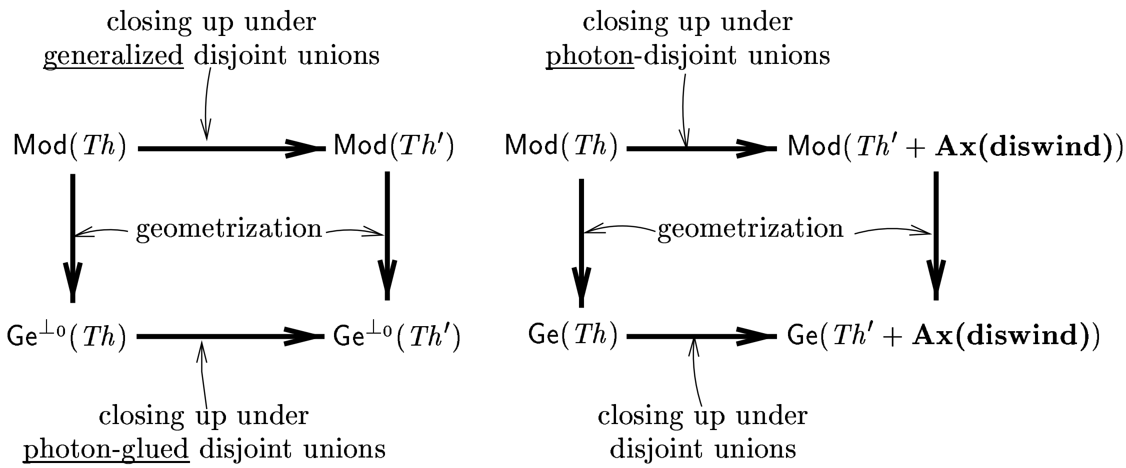
---

[823] Concrete construction of the family of geometries $\langle \mathfrak{G}'_i \ : \ i \in I \rangle$ satisfying (a) and (b): Let $i \in I$. Let $Mn'_i := Mn \times \{i\}$. Let $h_i : Mn \rightarrowtail\!\!\!\twoheadrightarrow Mn'_i$ be the bijection defined by $h_i : e \mapsto \langle e, i \rangle$. Let $h^+_i = \langle h_i, \mathrm{Id} \restriction F, \widetilde{h_i} \rangle$, where $\widetilde{h_i} : L_i \longrightarrow \{ h_i[\ell] \ : \ \ell \in L_i \}$ is defined by $\widetilde{h_i} : \ell \mapsto h_i[\ell]$. Now we define $\mathfrak{G}'_i$ to be the isomorphic copy of $\mathfrak{G}_i$ along h^+_i (i.e. it is the unique structure for which $h^+_i : \mathfrak{G}_i \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{G}'_i$ is an isomorphism).

[824] cf. footnote 816 on p.869

Intuitively, a *photon-disjoint union* in the "observational world" correspond to a *disjoint union* in the "geometry world", cf. Figure 284.

$\triangleleft$



$$\langle\, Th,\ Th' \,\rangle \in \{\, \langle\, \mathbf{Basax},\ \mathbf{Newbasax} \rangle,\ \langle\, \mathbf{Specrel},\ \mathbf{Newbasax} + \mathbf{Ax(symm)}^{\dagger} \,\rangle,$$
$$\langle\, \mathbf{Reich}(\mathbf{Basax}),\ \mathbf{Reich}(\mathbf{Newbasax}) \,\rangle,\ \langle\, \mathbf{Bax}^{-} + \mathbf{Ax6},\ \mathbf{Bax}^{-} \,\rangle \,\}$$

Figure 284: Generalized disjoint unions of models correspond to photon-glued disjoint unions of geometries, while photon-disjoint unions of models correspond to disjoint unions of geometries. (Further, the above diagrams commute in the sense of Remark 6.2.68.)

Examples 6.2.69

1. Let $\mathfrak{M}, \mathfrak{N} \in \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax})$ with $B^{\mathfrak{M}} \cap\ B^{\mathfrak{N}} = \emptyset$. Then

   $$\mathfrak{M} \,\dot{\cup}\, \mathfrak{N} \in \mathsf{Mod}(\mathbf{Newbasax}).$$

   Similarly for any class $\mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(\mathbf{Basax})$. This remains true for generalized disjoint unions of **Basax** models.

   $\mathsf{Mod}(\mathbf{Newbasax})$ is the class of all generalized disjoint unions of members of $\mathsf{Mod}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking generalized disjoint unions and contains $\mathsf{Mod}(\mathbf{Basax})$.

$\mathsf{Mod}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$ is the class of all photon-disjoint unions of members of $\mathsf{Mod}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking photon-disjoint unions and contains $\mathsf{Mod}(\mathbf{Basax})$.

2. The examples in item 1 above show up in the "geometry world" in the following "shape". See Figure 284.

   Let $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathsf{Ge}(\mathbf{Basax})$ with a common "field" reduct. Then

   $$\mathfrak{G}_1 \dot{\cup} \mathfrak{G}_2 \in \mathsf{Ge}(\mathbf{Newbasax}).$$

   Similarly for any family $\langle \mathfrak{G}_i : i \in I \rangle$ of **Basax** geometries. This remains true for photon-glued disjoint unions of **Basax** geometries, i.e. the photon-glued disjoint unions of geometries from $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$ are in $\mathsf{Ge}^{\perp_0}(\mathbf{Newbasax})$.

   $\mathsf{Ge}^{\perp_0}(\mathbf{Newbasax})$ is the class of all photon-glued disjoint unions of members of $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking photon-glued disjoint unions and contains $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$.

   $\mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax(diswind)})$ is the class of all disjoint unions of members of $\mathsf{Ge}(\mathbf{Basax})$. Further, it is the smallest class which is closed under taking disjoint unions and contains $\mathsf{Ge}(\mathbf{Basax})$.

   (If we formed the *non-disjoint* union of two **Basax** geometries say $\mathfrak{G}_0, \mathfrak{G}_1$ then we could obtain a geometry $\mathfrak{G}_0 \cup \mathfrak{G}_1$ which is not even a $\mathbf{Bax}^-$ geometry.)

3. Examples similar to those given in items 1 and 2 are illustrated in Figure 284.

4. Let $\mathfrak{G}_0, \mathfrak{G}_1 \in \mathsf{Ge}(\mathbf{Basax})$. Assume they are disjoint. Then in $\mathfrak{G}_0 \dot{\cup} \mathfrak{G}_1$ the parts Mn_0 and Mn_1 are sometimes called windows. Cf. Figure 307 (p.1001) and Figure 289 (p.887). Similarly for photon-glued disjoint unions of **Basax** geometries (i.e. $\mathsf{Ge}^{\perp_0}(\mathbf{Basax})$-structures).

   More generally in a **Newbasax** geometry, say $\mathfrak{G}$, the maximal "**Basax** subgeometries"[825] are called <u>*windows*</u>. (Here we use the notion of a subgeometry in an intuitive sense only, but it could be formalized such that all details would match.[826])

   In $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^-)$ two points $e, e_1 \in Mn$ are in the same <u>*window*</u> iff they are *connected*, i.e. $e \sim e_1$. If $\mathfrak{G} = \mathfrak{G}_{\mathfrak{M}}$, for some $\mathfrak{M} \models \mathbf{Bax}^-$, then these windows are exactly the $Rng(w_m)$'s, i.e. the subsets of Mn of the form $Rng(w_m)$ (with $m \in Obs$). Cf. Remark 6.2.13 (p.819).

[825] Recall that any **Newbasax** geometry $\mathfrak{G}$ is a photon-glued disjoint union of **Basax** geometries say $\mathfrak{G}_i$'s. These $\mathfrak{G}_i$'s (more precisely the Mn_i's) are called the windows of $\mathfrak{G}$.

[826] One possibility is to add **Ax(diswind)** to **Newbasax**.

5. Assume $n > 2$. Then every $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax(diswind)})$ is obtainable as a disjoint union of Minkowskian geometries.[827] Further, $\mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax(diswind)})$ is the disjoint unions closure of the class of Minkowskian geometries.

6. $\mathsf{Mod}(\mathbf{Flxbasax})$ is not closed under taking disjoint unions, but disjoint unions of **Flxbasax** models are **Bax** models.

   We did not have time to think about whether $\mathsf{Ge}(\mathbf{Flxbasax})$ is closed under taking disjoint unions but we think it is not closed.

$\triangleleft$

CONVENTION 6.2.70 Besides geometries in $\mathsf{Ge}(\emptyset)$ and in $\mathsf{Ge}^{\perp_0}(\emptyset)$ we will also discuss reducts of these (like e.g. $\mathbf{G}_{\mathfrak{M}}$, $\mathsf{G}_{\mathfrak{M}}$, etc.) and also slight variants of $\mathsf{Ge}(\emptyset)$ e.g. $\perp_r'$ or $\perp_r''$ in place of $\perp_r$.

We *extend* the above defined notions of *disjoint unions* and *photon-glued disjoint unions* to these kinds of geometries the natural (and obvious) way. (In the case of generalizing photon-glued disjoint unions we restrict attention to such geometries where relativistic orthogonality is $\perp_0$.)

$\triangleleft$

Now, having disjoint unions etc. at our hands we can state a stronger form of Theorem 6.2.64, not involving **Ax6**. Further, we will generalize Theorem 6.2.65 from **Basax** to **Newbasax**. Roughly, the just quoted theorems say that certain reducts of our geometries agree with the corresponding reducts of Minkowskian geometries, for certain choices of *Th*. Very roughly the new theorems will say that our relativistic geometries corresponding to many of our theories can be obtained as disjoint (or photon-glued disjoint) unions of Minkowskian geometries if we regard a reduct only.

THEOREM 6.2.71 *Assume* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Then (i) and (ii) below hold. (Cf. Figures 282, 283.)*

(i) *Assume $n > 2$. Then the $(\prec, \mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a disjoint union of the similar reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then the $(\mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a disjoint union of the similar reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

[827] This follows by example 1, Remark 6.2.68 (p.874) and Thm.6.2.59 (p.861).

Proof: The theorem follows by Thm.6.2.64 (p.866), Remark 6.2.68 (p.874) and by noticing that each $\mathbf{Bax}^{\oplus} + \mathbf{Ax(diswind)}$ model is a photon-disjoint union of $\mathbf{Bax}^{\oplus} + \mathbf{Ax6}$ models. ∎

Theorem 6.2.73 below is the "photon-glued" version of Theorem 6.2.71 above. For stating this theorem we define the $\perp_0$-versions of Minkowskian geometries.

Definition 6.2.72 Assume $\mathfrak{F}$ is Euclidean. Then the *$\perp_0$-version* $Mink^{\perp_0}(\mathfrak{F})$ *of the Minkowskian geometry* $Mink(\mathfrak{F})$ is defined to be the geometry obtained from $Mink(\mathfrak{F})$ by replacing $\perp_\mu$ with $(\perp_0)_\mu$ defined below.

$$(\perp_0)_\mu :\stackrel{\text{def}}{=} \left\{ \langle \ell, \ell' \rangle \in \perp_\mu \;:\; \ell, \ell' \in L^T_\mu \cup L^S_\mu,\;\; \ell \cap \ell' \neq \emptyset \right\}.$$

◁

THEOREM 6.2.73 *Assume* $\mathfrak{G} \in \mathsf{Ge}^{\perp_0}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }))$. *Then (i) and (ii) below hold. (Cf. Figures 282, 283.)*

- **(i)** *Assume $n > 2$. Then the $(\prec, \mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a photon-glued disjoint union of reducts of $\perp_0$-versions of Minkowskian geometries (up to isomorphism).*

  (The other direction also holds.)

- **(ii)** *Assume* $\mathbf{Ax(\uparrow\uparrow_0)}$. *Then the $(\mathrm{eq}, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a photon-glued disjoint union of reducts of $\perp_0$-versions of Minkowskian geometries (up to isomorphism).*

  (The other direction also holds.)

Proof: The theorem follows by Thm.6.2.64 (p.866), Remark 6.2.68 (p.874) and by noticing that each $\mathbf{Bax}^{\oplus}$ model is a generalized disjoint union of $\mathbf{Bax}^{\oplus} + \mathbf{Ax6}$ models. ∎

The following two theorems are generalizations of Theorem 6.2.65 (p.867).

THEOREM 6.2.74 *Assume $n > 2$ and* $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Newbasax} + \mathbf{Ax(}Triv_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Then (i) and (ii) below hold. (Cf. Figures 282, 283.)*

- **(i)** *The $(\prec, g, \mathcal{T})$-free reduct of $\mathfrak{G}$ is a disjoint union of reducts of Minkowskian geometries (up to isomorphism).*

  (The other direction also holds by item 5 of Examples 6.2.69.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then the* $(g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a disjoint union of reducts of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

Proof: The theorem follows by Thm.6.2.65 (p.867), Remark 6.2.68 (p.874) and by noticing that each **Newbasax** + **Ax(diswind)** model is a photon-disjoint union of **Basax** models. ∎

Theorem 6.2.75 below is the "photon-glued" version of Theorem 6.2.74 above.

THEOREM 6.2.75 *Assume* $\mathfrak{G} \in \mathsf{Ge}^{\perp_0}(\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }))$ *and* $n > 2$. *Then (i) and (ii) below hold. (Cf. Figures 282, 283.)*

(i) *The* $(\prec, g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a photon-glued disjoint union of reducts of* $\perp_0$*-versions of Minkowskian geometries (up to isomorphism).*

(The other direction also holds.)

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then the* $(g, \mathcal{T})$*-free reduct of* $\mathfrak{G}$ *is a photon-glued disjoint union of reducts of* $\perp_0$*-versions of Minkowskian geometries (up to an isomorphism).*

(The other direction also holds.)

Proof: The theorem follows by Thm.6.2.65 (p.867), Remark 6.2.68 and by Thm.3.3.12 saying that each **Newbasax** model is a generalized disjoint union of **Basax** models. ∎

Theorems 6.2.71, 6.2.73, 6.2.74, 6.2.75 above are all involved in Figures 282, 283 (pp. 863–864). Here we give an intuitive explanation for these figures.
Intuitive explanation for Figures 282, 283: The figures represent reducts of geometries agreeing with the corresponding reducts of (possibly unions of) Minkowskian geometries. Each node (in the figure) is of the form $\mathsf{Rd}_\mathsf{L}(\mathsf{Ge}(Th))$ for some relativity theory Th (observational) and subvocabulary L of the vocabulary of our relativistic geometries $\mathfrak{G}_\mathfrak{M}$. Hence, each node is characterized by two pieces of data Th and the "geometric reduct" (i.e. the geometric vocabulary) L. $\mathbf{Ax}(\sqrt{\ })$ and $n > 2$ are assumed in the figures. If we disregard the "$\mathsf{Rd}_\mathsf{L}\,\mathsf{Ge}$"-part i.e. if we consider the Th-part only then the figure becomes a subblattice of the lattice of our distinguished theories discussed on pp. 451–453, cf. also Fig.223 on p.653 and Remark 6.6.4(III) pp. 1020–1027. If we want to disregard Th, then we get a 6-element lattice of distinguished geometry-reducts of our relativistic geometries $\mathfrak{G}_\mathfrak{M}$. At the bottom of this lattice are the $\langle Mn, L; L^T, L^{Ph}, L^S, \in, Bw, \perp_r \rangle$ geometries which are basically

the same what we call relativistic incidence geometries $\mathsf{Ge}^{inc}(Th)$ in §6.7.4 (p.1175). More precisely $\mathsf{Ge}^{inc}(Th)$ is definitionally equivalent[828] with our "bottom" geometry, with Th as indicated in the figure, assuming **Ax(diswind)**, cf. Thm.6.7.31 (p.1164). The top of the lattice represents the whole of $\mathfrak{G}_{\mathfrak{M}}$'s, of course. Besides labelling the nodes, we labelled some of the edges too in Fig.282. The labels on an edge indicate (roughly) the changes that happen when moving along that edge, the same change happens when moving along parallel edges. E.g. the label [unions, **Ax6**] indicate that, intuitively, we can move from the higher end of that edge to the lower one by taking (possibly photon-glued) disjoint unions of our geometries and dropping **Ax6** from our Th, loosely speaking.

To understand our observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$ (and their connections with the original models $\mathfrak{M}$), below we introduce "observer-dependent" geometries $\mathfrak{G}_m$, for each observer $m \in Obs^{\mathfrak{M}}$. After this we will introduce restrictions $\mathfrak{G} \restriction N$ of geometries to subsets $N \subseteq Mn$ of their set of points.

Our next definition may look, at first sight, somewhat longish, but at second reading it will turn out to be just the natural thing, and it will turn out to be quite useful. E.g. in Prop.6.2.79 we will see that $\mathfrak{G}_{\mathfrak{M}}$ can be obtained from the world-views of observers i.e. from the w_m's by gluing them together (as we planned in the first 2 sentences of §6.2.5). For this, first, the w_m's have to be "geometrized". The geometrized versions of the w_m's will be the $\mathfrak{G}_m$'s defined below.

Definition 6.2.76 Let $\mathfrak{N}$ be a frame model and $\mathfrak{G}_{\mathfrak{N}} = \langle Mn, \mathbf{F_1}, L; \ldots \rangle$ be the geometry corresponding to it. Then using the world-view function w_m each observer m can copy the geometry $\mathfrak{G}_{\mathfrak{N}}$ to his coordinate system ${}^nF$, obtaining the <u>*observer-dependent geometry*</u> $\mathfrak{G}_m$ defined below, cf. Figure 285. Let $m \in Obs$. For every $\ell \in L$, throughout this definition, let

$$\ell_m :\stackrel{\text{def}}{=} w_m^{-1}[\ell].$$

Now,

$$\mathfrak{G}_m :\stackrel{\text{def}}{=} \langle {}^nF, \mathbf{F_1}, L_m; L_m^T, L_m^{Ph}, L_m^S, \in, \prec_m, Bw_m, \perp_m, eq_m, g_m, \mathcal{T}_m \rangle,$$

where

$$L_m :\stackrel{\text{def}}{=} \{ \ell_m : \ell \in L, \ \ell_m \neq \emptyset \},$$

$$L_m^T :\stackrel{\text{def}}{=} \{ \ell_m : \ell \in L^T, \ \ell_m \neq \emptyset \},$$

$$L_m^{Ph} :\stackrel{\text{def}}{=} \{ \ell_m : \ell \in L^{Ph}, \ \ell_m \neq \emptyset \},$$

[828] Cf. Def.6.3.30 on p.970 for definitional equivalence.

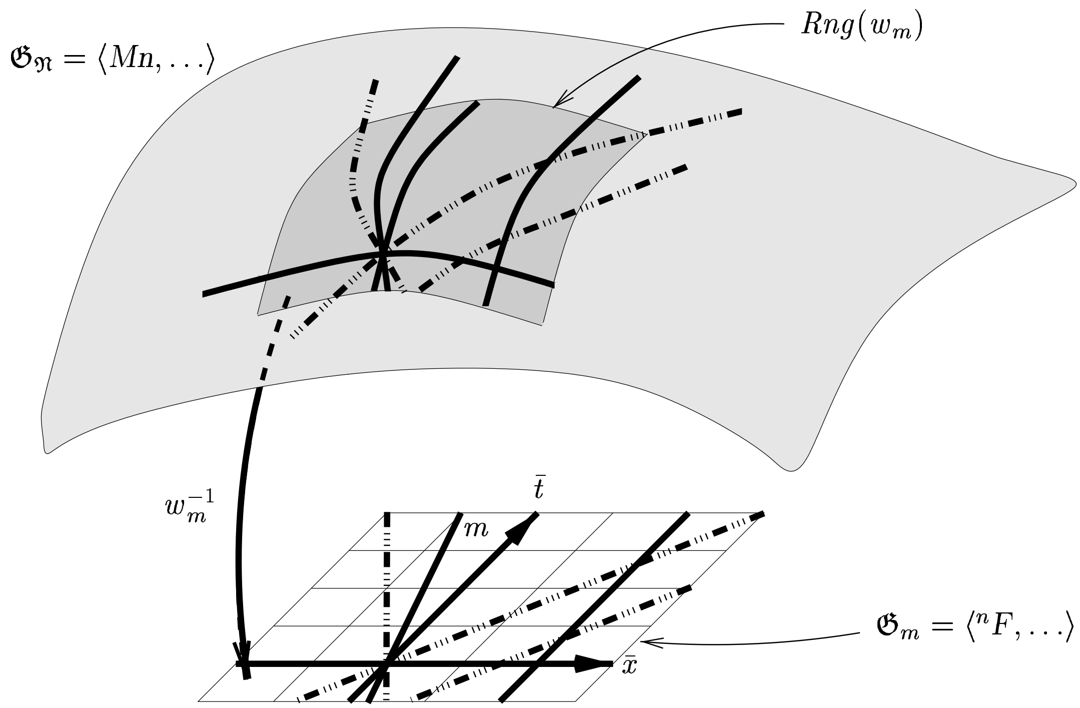


Figure 285: Using the world-view function w_m each observer m can copy the geometry $\mathfrak{G}_{\mathfrak{N}}$ to his coordinate system ${}^nF$.

$$\begin{aligned}
L^S_m &\stackrel{\text{def}}{:=} \left\{ \ell_m : \ell \in L^S,\ \ell_m \neq \emptyset \right\}, \\
&\in \text{ is the membership relation between } {}^nF \text{ and } L_m, \\
\prec_m &\stackrel{\text{def}}{:=} \{ \langle p, q \rangle \in {}^nF \times {}^nF : w_m(p) \prec w_m(q) \}, \\
Bw_m &\stackrel{\text{def}}{:=} \left\{ \langle p, q, r \rangle \in {}^3({}^nF) : Bw(w_m(p), w_m(q), w_m(r)) \right\}, \\
\perp_m &\stackrel{\text{def}}{:=} \{ \langle \ell_m, \ell'_m \rangle : \ell \perp_r \ell',\ \ell_m \neq \emptyset,\ \ell'_m \neq \emptyset \}, \\
eq_m &\stackrel{\text{def}}{:=} \left\{ \langle p, q, r, s \rangle \in {}^4({}^nF) : eq(w_m(p), w_m(q), w_m(r), w_m(s)) \right\}, \\
g_m &\stackrel{\text{def}}{:=} \{ \langle p, q, \lambda \rangle \in {}^nF \times {}^nF \times F : g(w_m(p), w_m(q)) = \lambda \}, \\
\mathcal{T}_m &\stackrel{\text{def}}{:=} \left\{ w_m^{-1}[H] : H \in \mathcal{T} \right\}.
\end{aligned}$$

We define $\mathfrak{G}_m^{\perp_0}$ to be the geometry obtained from $\mathfrak{G}_m$ by replacing $\perp_m$ with

$(\perp_0)_m$ defined below.

$$(\perp_0)_m \ :\stackrel{\text{def}}{=} \ \{ \langle \ell_m, \ell'_m \rangle \ : \ \ell \perp_0 \ell', \ \ell_m \neq \emptyset, \ \ell'_m \neq \emptyset \},$$

cf. p.791 for the definition of $\perp_0$.

$\triangleleft$

Definition 6.2.77 Let $\mathfrak{G} = \langle Mn, \mathbf{F_1}, L; \ldots, \mathcal{T} \rangle$ be an observer-independent geometry. Let $N \subseteq Mn$. Then the restrictions $\mathfrak{G} \restriction N$ and $\mathfrak{G} \restriction^+ N$ of $\mathfrak{G}$ to N are defined in (i) and (ii) below, respectively. See Figure 286, cf. also Figure 288.

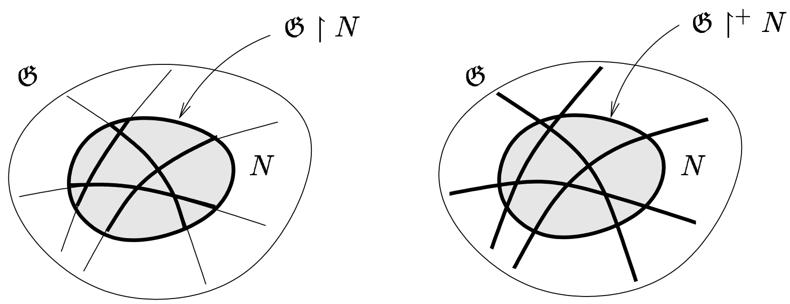


Figure 286: Illustration for Definition 6.2.77.

(i) $\mathfrak{G} \restriction N \ :\stackrel{\text{def}}{=} \ \langle N, \ \mathbf{F_1}, \ L \restriction N$ [829] $; \ L^T \restriction N, \ L^{Ph} \restriction N, \ L^S \restriction N, \ \in,$ $\prec \restriction N$ [830]$, \ Bw \restriction N, \ \perp_N, \ eq \restriction N, \ g \restriction {}^2N, \ \mathcal{T} \restriction N$ [831]$\rangle$,

where

$$\perp_N :\stackrel{\text{def}}{=} \{ \langle \ell \cap N, \ell' \cap N \rangle \ : \ \ell, \ell' \in L, \ \ell \perp_r \ell' \}.$$

(ii) We define $\mathfrak{G} \restriction^+ N$ to be the geometry obtained from $\mathfrak{G} \restriction N$ by replacing $L \restriction N, \ L^T \restriction N, \ L^{Ph} \restriction N, \ L^S \restriction N, \ \perp_N$ with $L_N, \ L^T \cap L_N, \ L^{Ph} \cap L_N, \ L^S \cap L_N, \ \perp_r \restriction L_N$, respectively, where $L_N :\stackrel{\text{def}}{=} \{ \ell \in L \ : \ \ell \cap N \neq \emptyset \}$.

[829] $L \restriction N := \{ \ell \cap N \ : \ \ell \in L \}$. This is the natural restriction of *"Lines"* to $N \subseteq$ *"Points"*. Similarly for the topology $\mathcal{T}$ in place of lines L.

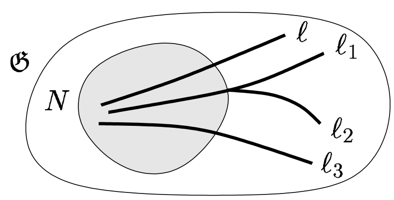
[830] We use the restriction symbol $\restriction$ for relations too the natural way. I.e. $\prec \restriction N := \prec \cap (N \times N)$. Similarly for other relations of perhaps different ranks. (Since functions are special relations our usage of $\restriction$ is ambigous. We hope context will help.)

[831] $\mathcal{T} \restriction N := \{ H \cap N \ : \ H \in \mathcal{T} \}$, cf. footnote 829.

(iii) We extend the definitions of the restrictions $\mathfrak{G} \restriction N$ and $\mathfrak{G} \restriction^+ N$ to similar geometries like e.g. $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$ the natural way. E.g. $\mathfrak{G}_{\mathfrak{N}}^{\perp_0} \restriction N$ is defined the natural way.

$\triangleleft$

Remark 6.2.78 Let $\mathfrak{G}$ be an observer-independent geometry and $N \in \mathcal{T}$, i.e. $N \subseteq Mn$ is an open set. Then $\mathfrak{G} \restriction^+ N$ is a strong submodel of $\mathfrak{G}$, in symbols $(\mathfrak{G} \restriction^+ N) \subseteq \mathfrak{G}$. $\mathfrak{G} \restriction N$ is not necessarily a submodel of $\mathfrak{G}$; moreover there is $\mathfrak{G}$ and $N \in \mathcal{T}$ such that $\mathfrak{G} \restriction N$ is not isomorphic to any submodel of $\mathfrak{G}$. Such $\mathfrak{G}$ and N are represented in Figure 287 below, cf. also item 2f of Prop.6.2.79 (p.886) and footnote 837 in it.
$\triangleleft$



$$L = \{\ell, \ell_1, \ell_2, \ell_3\}$$
$$\perp_r = \{\langle \ell, \ell_1 \rangle, \langle \ell_2, \ell_3 \rangle\}$$
$$\ell_1 \cap N = \ell_2 \cap N$$

Figure 287: $\mathfrak{G} \restriction N$ is not isomorphic to any submodel of $\mathfrak{G}$.

Item 1 of our next proposition says that, assuming $\mathbf{Bax}^-$, any observer-dependent geometry $\mathfrak{G}_m$ is basically the *familiar picture* which we often called the world-view of observer m; e.g., in $\mathfrak{G}_m$, the set of points is ${}^nF$, L consists of Euclidean lines, L^T consists of the traces (i.e. life-lines) of observers as seen by m, L^{Ph} is the set of life-lines of photons as seen by m, two lines are $\perp_0$-orthogonal iff they are two coordinate axes of some observer as seen by m, etc. For a second, let us call these familiar structures ${}^nF$-geometries. Item 3 says that any observer-independent geometry $\mathfrak{G}_{\mathfrak{N}}$ is a disjoint union of such familiar ${}^nF$-geometries, assuming $\mathbf{Bax}^- + \mathbf{Ax(diswind)}$. Formally

$$\mathfrak{G}_{\mathfrak{N}} = \dot{\bigcup_{m \in O}} \mathfrak{G}_m$$

for some $O \subseteq Obs$. $\mathbf{Ax(diswind)}$ can be omitted if we use photon-glued disjoint unions and $\perp_0$-versions of our geometries. Cf. Figure 289 (p.887).

PROPOSITION 6.2.79 (On Bax$^-$geometries)
*Let $\mathfrak{N} \models$ **Bax$^-$**. Consider the observer-independent geometry $\mathfrak{G}_{\mathfrak{N}}$. Then 1–5 below hold.*

1. *Let $m \in Obs$. Consider the observer-dependent geometry $\mathfrak{G}_m$. Then (a)–(h) below hold.*

   (a) $L_m \subseteq \mathsf{Eucl}$. *Hence,* $(\forall \ell \in L)\, w_m^{-1}[\ell] \in \mathsf{Eucl} \cup \{\emptyset\}$.[832]

   (b) $L_m^T = \{\, tr_m(k) \;:\; k \in Obs,\ m \overset{\odot}{\to} k \,\}$.

   (c) $L_m^{Ph} = \{\, tr_m(ph) \;:\; ph \in Ph,\ m \overset{\odot}{\to} ph \,\}$.

   (d) $L_m^S = \{\, \mathsf{f}_{km}[\bar{x}_i] \;:\; k \in Obs,\ m \overset{\odot}{\to} k,\ 0 < i \in n \,\}$.

   (e) $(\perp_0)_m = \{\, \langle \mathsf{f}_{km}[\bar{x}_i], \mathsf{f}_{km}[\bar{x}_j] \rangle \;:\; k \in Obs,\ m \overset{\odot}{\to} k,\ i \neq j \,\}$.

   (f) *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Then* Bw_m *and* Betw *coincide.*

   (g) $(\forall$ *distinct* $p, q, r \in {}^nF)$

   $$(\, Bw_m(p,q,r) \vee Bw_m(p,r,q) \vee Bw_m(q,p,r)\,) \;\Leftrightarrow\; (\, p,q,r \text{ are collinear}).$$[833]

   (h) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$. *Let* $p, q \in {}^nF$. *Then*

   $$p \prec_m q \quad \Longleftrightarrow \quad \Big(p_t < q_t \quad \wedge \quad (\exists k \in Obs)\, p, q \in tr_m(k)\Big).$$

2. *Let $m \in Obs$. Consider the observer-dependent geometry $\mathfrak{G}_m$. Then (a)–(g) below hold.*

   (a) *Assume* **Ax6**. *Then*

   $$\mathfrak{G}_m \cong \mathfrak{G}_{\mathfrak{N}} \quad \text{and} \quad \mathfrak{G}_m^{\perp_0} \cong \mathfrak{G}_{\mathfrak{N}}^{\perp_0}.$$

   Actually, the world-view function w_m induces an isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}}$ (and between $\mathfrak{G}_m^{\perp_0}$ and $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$) the natural way.[834]

   (b)

   $$\mathfrak{G}_m \cong (\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)) \quad \text{and} \quad \mathfrak{G}_m^{\perp_0} \cong (\mathfrak{G}_{\mathfrak{N}}^{\perp_0} \restriction Rng(w_m)),$$

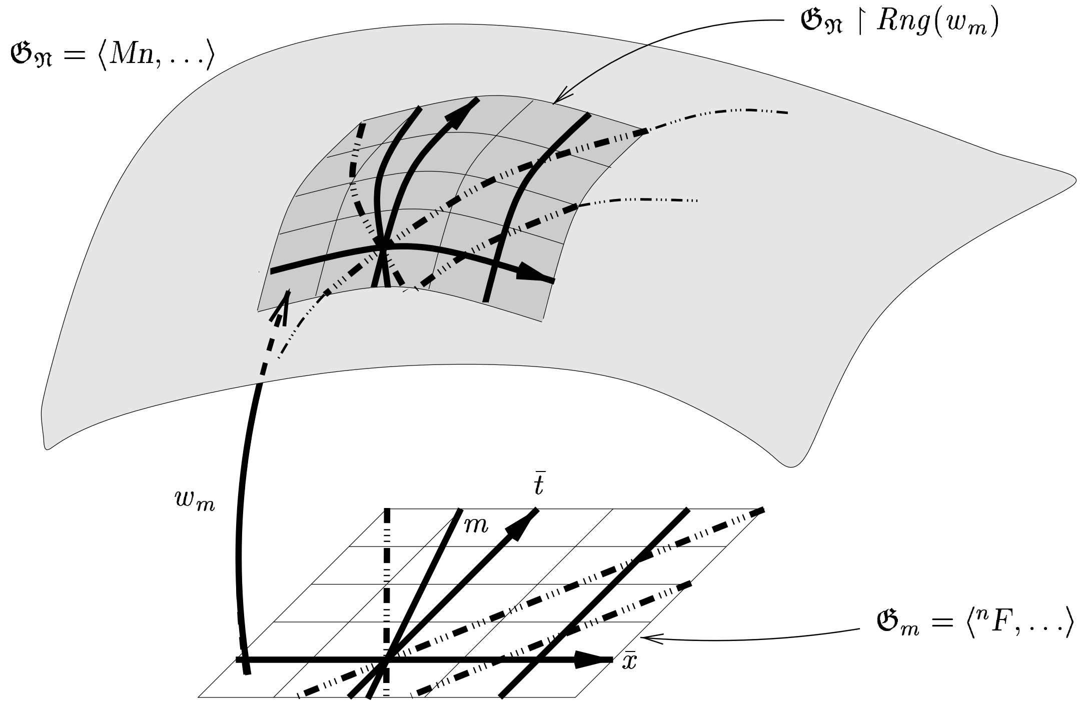


Figure 288: $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}} \upharpoonright Rng(w_m)$ are isomorphic.

see Figure 288. Actually, the world-view function w_m induces an isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}} \upharpoonright Rng(w_m)$ the natural way.[835]

(c) $(\forall \ell \in L^T \cup L^S)\,(\,\ell \cap Rng(w_m) \neq \emptyset \;\Rightarrow\; \ell \subseteq Rng(w_m)\,)$.
Intuitively, time-like and space-like lines do not stick out from the window $Rng(w_m)$, see Figure 288.

(d) Assume **Ax(diswind)**. *Then, intuitively, lines do not stick out from the*

[832] Cf. Prop.6.2.48 (p.854).

[833] Cf. Prop.6.2.14 on p.819.

[834] Making this precise: Let $\widetilde{w_m} : L_m \longrightarrow \{\, w_m[\ell] \;:\; \ell \in L_m \,\}$ be defined by $\widetilde{w_m} : \ell \mapsto w_m[\ell]$. Then $Rng(\widetilde{w_m}) = L_{\mathfrak{N}}$ and $\langle w_m,\ \mathrm{Id} \upharpoonright F,\ \widetilde{w_m} \rangle$ is a (three-sorted) isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}}$ (and between $\mathfrak{G}_m^{\perp_0}$ and $\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$). Cf. item (II) of Def.6.2.2 (p.798) for isomorphisms between geometries.

[835] Making this precise: Let $\widetilde{w_m}$ be defined as in footnote 834. Then $\langle w_m,\ \mathrm{Id} \upharpoonright F,\ \widetilde{w_m} \rangle$ is an isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_{\mathfrak{N}} \upharpoonright Rng(w_m)$.

window $Rng(w_m)$, formally:

$$(\forall \ell \in L)\,(\,\ell \cap Rng(w_m) \neq \emptyset \;\Rightarrow\; \ell \subseteq Rng(w_m)\,),$$

cf. Figure 288 (in the figure some photon-like lines do stick out from the window $Rng(w_m)$).
Therefore

$$(\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)) = (\mathfrak{G}_{\mathfrak{N}} \restriction^{+} Rng(w_m)),$$

cf. Figure 286 (p.882).

(e) *Assume* **Ax(diswind)**. *Then $\mathfrak{G}_m$ is isomorphic to a strong submodel of $\mathfrak{G}_{\mathfrak{N}}$ (and $Rng(w_m) \in \mathcal{T}$). In more detail:*

$$\mathfrak{G}_m \cong (\mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m)) = (\mathfrak{G}_{\mathfrak{N}} \restriction^{+} Rng(w_m)) \subseteq \mathfrak{G}_{\mathfrak{N}},$$

cf. Remark 6.2.78.
The world-view function w_m induces an embedding of $\mathfrak{G}_m$ into $\mathfrak{G}_{\mathfrak{N}}$ the natural way.[836] *See Figure 288 and the the upper picture in Figure 289.*

(f) *The assumption* **Ax(diswind)** *is needed in item (e) above. I.e. there is $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$ and $k \in Obs^{\mathfrak{M}}$ such that $\mathfrak{G}_k$ is not isomorphic to any submodel of $\mathfrak{G}_{\mathfrak{M}}$.*[837]

(g) *Assume $k \in Obs$ is such that $m \overset{\odot}{\to} k$. Then the geometies $\mathfrak{G}_m$ and $\mathfrak{G}_k$ are isomorphic, i.e. $\mathfrak{G}_m \cong \mathfrak{G}_k$. Actually, the world-view transformation f_{mk} induces an isomorphism between $\mathfrak{G}_m$ and $\mathfrak{G}_k$ the natural way.*[838]

3. *By Thm.4.3.11 (and* **Ax4***), $\overset{\odot}{\to}$ is an equivalence relation when restricted to Obs.*[839] *Let $O \subseteq Obs$ be a class of representatives for the equivalence relation $\overset{\odot}{\to}$.*[840] *Then (a) and (b) below hold.*

   (a) *Assume* **Ax(diswind)**. *Then $\mathfrak{G}_{\mathfrak{N}}$ is the disjoint union of the family $\langle\, \mathfrak{G}_{\mathfrak{N}} \restriction Rng(w_m) \;:\; m \in O\,\rangle$.*

[836] Let $\widetilde{w_m}$ be defined as in footnote 834, p.885. Then
$Rng(\widetilde{w_m}) = \{\, \ell \in L_{\mathfrak{N}} \;:\; \ell \subseteq Rng(w_m)\,\} = \{\, \ell \in L_{\mathfrak{N}} \;:\; \ell \cap Rng(w_m) \neq \emptyset\,\}$;
and $\langle w_m,\ \mathrm{Id} \restriction F,\ \widetilde{w_m}\rangle$ is an embedding of $\mathfrak{G}_m$ itno $\mathfrak{G}_{\mathfrak{N}}$.

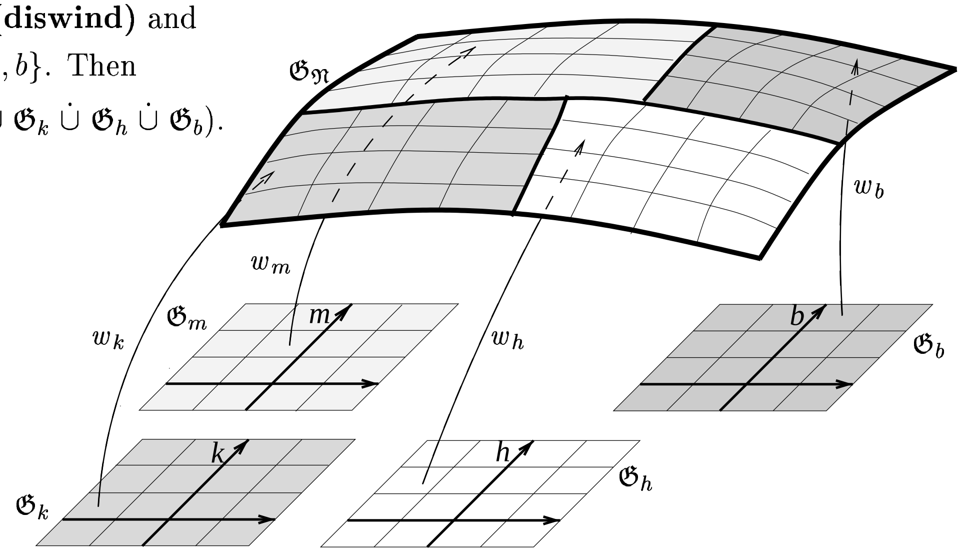
[837] E.g. let $\mathfrak{M}$ be the generalized disjoint union of two **NewtK** models $\mathfrak{M}_1$, $\mathfrak{M}_2$ with $Ph^{\mathfrak{M}_1} = Ph^{\mathfrak{M}_2}$. Then for each observer k, $L_k^{Ph} \cap L_k^{S} \neq \emptyset$, while $L^{Ph} \cap L^{S} = \emptyset$. Thus for each k, $\mathfrak{G}_k$ is not ismorphic to any submodel of $\mathfrak{G}_{\mathfrak{M}}$.

[838] Cf. footnote 812 on p.865.

[839] assuming $\mathbf{Bax}^-$ of course

[840] I.e. $(\forall m \in Obs)\,|O \cap m/\overset{\odot}{\to}\,| = 1$, where $m/\overset{\odot}{\to}$ is the equivalence class of m w.r.t. $\overset{\odot}{\to}$, as usual.

Assume **Ax(diswind)** and
$O = \{m, k, h, b\}$. Then
$\mathfrak{G}_{\mathfrak{N}} \cong (\mathfrak{G}_m \,\dot{\cup}\, \mathfrak{G}_k \,\dot{\cup}\, \mathfrak{G}_h \,\dot{\cup}\, \mathfrak{G}_b)$.



Assume $O = \{m, k, h, b\}$. Then
$\mathfrak{G}_{\mathfrak{N}}^{\perp_0}$ is a photon-glued disjoint union of $\mathfrak{G}_m^{\perp_0}, \mathfrak{G}_k^{\perp_0}, \mathfrak{G}_h^{\perp_0}, \mathfrak{G}_b^{\perp_0}$, up to isomorphism:

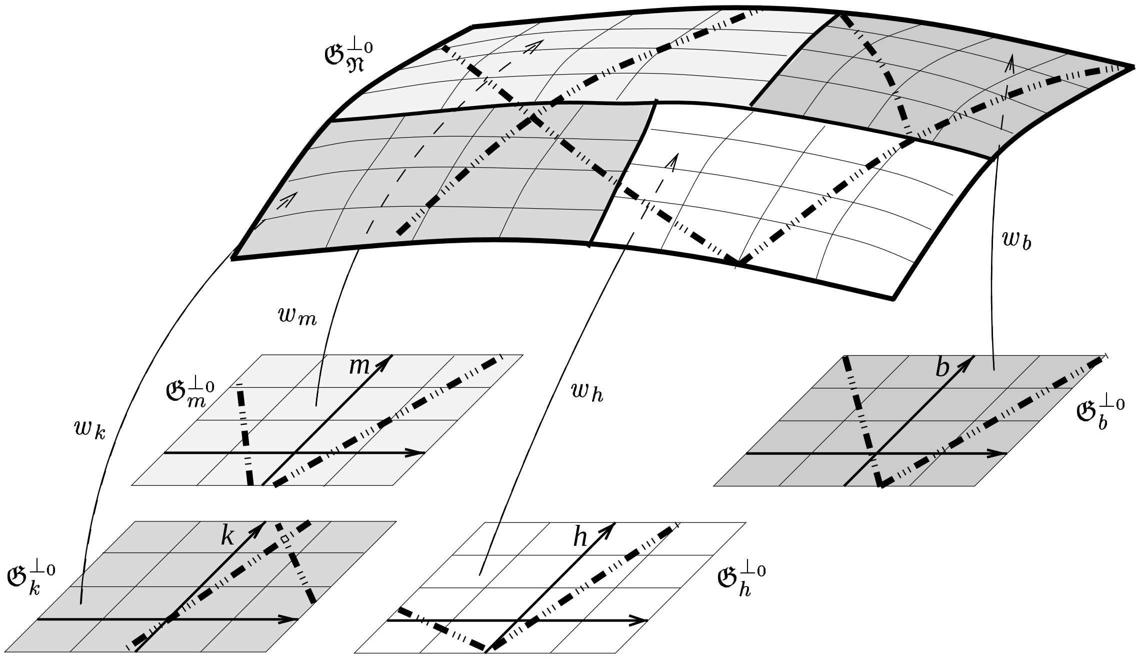


Figure 289: Notice that a “photon-line” splits up to two in the lower picture.

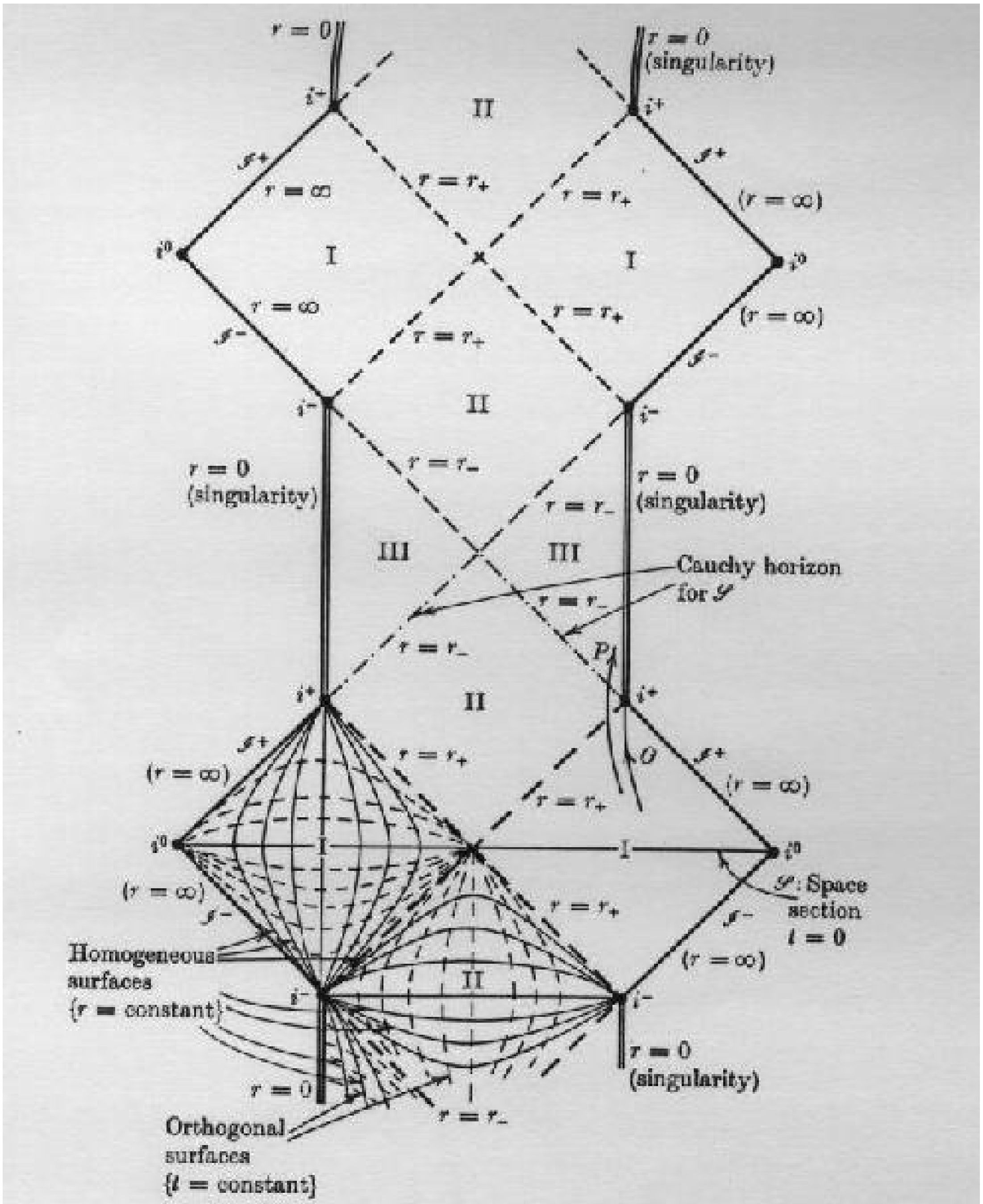


Figure 290: The geometry of a rotating black hole (general relativity) represented by a Penrose diagram.

Therefore, by item 2b, $\mathfrak{G}_{\mathfrak{N}}$ is the disjoint union of the family $\langle\, \mathfrak{G}_m \,:\, m \in O \,\rangle$, up to isomorphism, i.e.

$$\mathfrak{G}_{\mathfrak{N}} \;\cong\; \dot{\bigcup_{m\in O}} \mathfrak{G}_m,$$

see Figure 289.

(b) *$\mathfrak{G}^{\perp_0}_{\mathfrak{N}}$ is a photon-glued disjoint union of the family $\langle\, \mathfrak{G}^{\perp_0}_{\mathfrak{N}} \restriction Rng(w_m) \,:\, m \in O \,\rangle$.*
Therefore, by item 2b, $\mathfrak{G}^{\perp_0}_{\mathfrak{N}}$ is a photon-glued disjoint union of the family $\langle\, \mathfrak{G}^{\perp_0}_{m} \,:\, m \in O \,\rangle$ up to isomorphism. See Figure 289.

4. *(a)–(e) below hold.*

   (a) *Assume* **Ax(diswind)**. *Then two distinct lines meet in at most one point; formally: $(\forall \text{ distinct } \ell, \ell' \in L)\, |\ell \cap \ell'| \le 1$.*

   (b) *Assume we are given two distinct lines such that one of them is time-like or space-like. Then the two lines meet in at most one point. Formally:*

   $$(\forall \text{ distinct } \ell, \ell' \in L)\, (\ell \in L^T \cup L^S \;\Rightarrow\; |\ell \cap \ell'| \le 1).$$

   (c) $L^T \cap L^{Ph} = \emptyset$.

   (d) *Assume $\mathsf{c}_m(d) < \infty$. Then $L^S \cap L^{Ph} = \emptyset$.*

   (e) *Assume* $\mathbf{Ax}(\sqrt{\ }) + (\mathsf{c}_m(d) < \infty)$ *and* $(n > 2$ *or* $\mathbf{Ax}(\uparrow\uparrow_0))$. *Then i, ii below hold.*

      i. L^T, L^{Ph}, L^S *are pairwise disjoint.*

      ii. *The irreflexive parts of relations $\equiv^T, \equiv^{Ph}, \equiv^S$ are pairwise disjoint.*

5. *Assume* **Ax(diswind)**. *Let $m \in Obs$ and $\ell, \ell' \in L$ be such that $w_m^{-1}[\ell] \ne \emptyset$ and $w_m^{-1}[\ell'] \ne \emptyset$. Then $w_m^{-1}[\ell], w_m^{-1}[\ell'] \in \mathsf{Eucl}$ (by item 1a), and (a), (b) below hold.*

   (a) $\ell \parallel_{\mathfrak{G}} \ell' \quad \Longleftrightarrow \quad w_m^{-1}[\ell] \parallel w_m^{-1}[\ell']$.

   (b) *Assume ℓ, ℓ' are distinct and $\ell \cap \ell' \ne \emptyset$. Then*

   $$Plane(\ell, \ell') = Plane'(\ell, \ell') = w_m[\mathsf{Plane}(w_m^{-1}[\ell], w_m^{-1}[\ell'])].$$

On the proof: The proof is left to the reader as an exercise, but we note the following. Items 1b, 1c hold for arbitrary frame model, i.e. the assumption $\mathbf{Bax}^-$ is not needed in these items. The proof of the proposition is based on the following. Assume $\mathbf{Bax}^-$. Let $m, k \in Obs$. Then (i)-(vii) below hold.