of m. Let P be the plane determined by $\bar{y}$ and ph, i.e.

$$P := \mathsf{Plane}(\bar{y}, tr_m(ph)),$$

cf. the upper picture in Figure 272. Let $k \in Obs$ be such that m sees that k passes through $\bar{0}$ with nonzero speed and lies in $\mathsf{Plane}(\bar{t}, \bar{y})$, i.e. $\bar{0} \in tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{y})$ and $v_m(k) \neq 0$. Such a k exists by **Ax5**. Without loss of generality we can assume that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ because of $\mathbf{Ax(}Triv_t\mathbf{)^-}$. Let

$$\bar{y}_k := \mathsf{f}_{km}[S] \cap P,$$

i.e. in the world-view of m $\bar{y}_k$ is the intersection of k's space part with plane P. Clearly, $\bar{y}_k \in \mathsf{Eucl}$ and $\bar{y}_k, \bar{y}, tr_m(ph)$ are pairwise distinct, since k lies in $\mathsf{Plane}(\bar{t}, \bar{y})$, is of nonzero speed as seen by m and since in the direction of movement clocks get out of synchronism. Without loss of generality, by $\mathbf{Ax(}Triv_t\mathbf{)^-}$, we can assume that the $\bar{y}$-axis of k as seen by m is y_k, formally

$$\mathsf{f}_{km}[\bar{y}] = \bar{y}_k.$$

Let us switch over from the world-view of m to the world-view of k. We claim that k sees ph moving in the spatial direction orthogonal to $\bar{y}$ (in the Euclidean sense). To prove this claim, let P' be the f_{mk} image of P, cf. Figure 272. Then $\bar{y} \subseteq P'$. Since f_{mk} takes $\mathsf{LightCone}(\bar{0})$, P, $tr_m(ph)$ to $\mathsf{LightCone}(\bar{0})$, P', $tr_k(ph)$, respectively and since $\mathsf{LightCone}(\bar{0}) \cap P = tr_m(ph)$ we get that

$$\mathsf{LightCone}(\bar{0}) \cap P' = tr_k(ph).$$

This and $\bar{y} \subseteq P'$ imply that $\bar{y} \perp_e tr_k(ph)$, proving our claim.

Then, by $\mathbf{Ax(}Triv_t\mathbf{)^-}$, we can assume that k sees ph in $\mathsf{Plane}(\bar{t}, \bar{x})$, i.e. $tr_k(ph) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

Then

$$w_m[\bar{y}] \perp_r^1 \ell \quad \text{and} \quad w_k[\bar{y}] \perp_r^1 \ell,$$

see Figure 272. By this, by $w_m[\bar{y}_k] = w_k[\bar{y}]$ and by $\bar{y}, \bar{y}_k \subseteq P$, we have

$$(*) \qquad \ell \perp_r^2 w_m[\bar{y}] \quad \text{and} \quad \ell \perp_r^2 w_m[\bar{y}_k] \quad \text{and} \quad w_m[\bar{y}], w_m[\bar{y}_k] \subseteq w_m[P].$$

See the upper picture in Figure 272. By item 5b of Prop.6.2.79 (p.889), we have

$$Plane'(w_m[\bar{y}], w_m[\bar{y}_k]) = w_m[P].$$

This, $(*)$ and $\ell \subseteq w_m[P]$ imply $\ell \perp_r^3 \ell$, which completes the proof of (a).

Proof of (b): Assume **Basax** + **Ax**$(Triv_t)^-$ + **Ax**$(\sqrt{\ })$. It is easy to check that $\perp'_r$ has properties 1, 2, 4 in Def.6.2.17. So it remains to prove that $\perp'_r$ has property 3 in Def.6.2.17. To prove this we will use Minkowskian orthogonality $\perp_\mu \subseteq \mathsf{Eucl} \times \mathsf{Eucl}$ which will be introduced in Def.6.2.58 (p.859). Now, by (I)–(II) below and item 5b of Prop.6.2.79, it can be checked that $\perp'_r$ has property 3 in Def.6.2.17; where (I) holds by item (d) in the proof of Claim 6.2.11 (p.816) and by the def. of $\perp'_r$, and (II) can be checked by the definition of Minkowskian orthogonality.

(I) Let $\ell, \ell' \in L$. Then $\quad \ell \perp'_r \ell' \quad \iff \quad (\forall m)(w_m^{-1}[\ell] \perp_\mu w_m^{-1}[\ell'])$.

(II) Minkowskian orthogonality has property 3 in Def.6.2.17, i.e. if lines ℓ, ℓ_1, ℓ_2 ($\in \mathsf{Eucl}$) concur at point p ($\in {}^nF$), with $\ell_1 \neq \ell_2$ and ℓ is Minkowski-orthogonal to both ℓ_1 and ℓ_2, then ℓ is Minkowski-orthogonal to every line through p in $\mathsf{Plane}(\ell_1, \ell_2)$, cf. Figure 270.

At this point Thm.6.2.19 is fully proved. ∎

Question for future research 6.2.21 The definitions of $\perp_r, \perp'_r, \perp''_r, \perp'''_r, \perp^\omega_r$ do what we have in mind only if we assume the axiom **Ax(diswind)** of disjoint-windows. It would be nice to refine these definitions such that they work without this axiom, too.

◁

Let us recall that eq is a 4-ary relation on the set of points Mn of an observer-independent geometry $\mathfrak{G}_{\mathfrak{M}}$ and was defined in item 12 of Def.6.2.2(I) (p.793). Further, eq was defined to be the transitive closure of the relation eq_0 which was first-order logic defined (in the expanded frame-model $\mathfrak{M}^+$ defined in Remark 6.2.8 on p.807); and eq_i was defined to be the "i-long-transitive closure" of eq_0. As we have already said in Remark 6.2.8, each one of eq_i's is first-order defined (in $\mathfrak{M}^+$).[745]

The next two theorems (6.2.22 and 6.2.23) say that eq is first-order definable in $\mathfrak{M}^+$ under certain conditions.

THEOREM 6.2.22 *Assume* **Basax** + **Ax**$(Triv_t)^-$ + **Ax**$(\sqrt{\ })$. *Then* $eq_2 = eq$, *therefore eq is first-order definable*[746].

[745]First-order definable is the same as first-order logic definable (which in turn is the same as definable, at least in the present work).

[746]we mean, definable over $\mathsf{Mod}(\mathbf{Basax} + \ldots)$, of course. First one defines Mn over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then eq over $\mathfrak{M}$ and Mn.

A **proof** will be given in §6.2.6 on p.906.

To formulate our next theorem, we introduce a weakened version $\mathbf{Ax}(\|)^-$ of $\mathbf{Ax}(\|)$.

$\mathbf{Ax}(\|)^-$ $(\forall m, k \in Obs \cap Ib)$
$[\, tr_m(k) = \bar{t} \;\Rightarrow\; (\mathsf{f}_{mk} = h \circ I, \quad \text{for some expansion } h \text{ and isometry } I)\,]$.[747]

Assuming **Bax**, $\mathbf{Ax}(\|)^-$ is equivalent with the following: If two observers, say m and k, have the same life-line (i.e. $tr_m(k) = \bar{t}$) then they agree on the speed of light (i.e. $c_m = c_k$) and the world-view transformation f_{mk} is an affine transformation, i.e. there is no field automorphism involved in f_{mk} (cf. Fact 4.7.7).

The essential feature of $\mathbf{Ax}(\|)^-$ is that it does not exlude the "ant and the elephant version of relativity" mentioned in Remark 4.2.1, while $\mathbf{Ax}(\|)$ does.

Let

$$Th^{+-} := \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^- + \mathbf{Ax}(Triv_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)}.$$

This theory Th^{+-} will play an essential role in the following theorems and propositions: Thm's 6.2.23 (p.829), 6.2.44(iii) (p.847), 6.6.13 (p.1031), 6.6.114 (p.1130), and Prop's 6.2.88 (p.895), 6.2.92 (p.901). Because of this, we point out a few intuitive and helpful properties of Th^{+-} (which eventually will be proved as parts of various later theorems). We collect these properties in items 1–4 below. In 1–4 below $n > 2$ is assumed.

1. The reduct
$$\langle Mn, L;\ L^T, L^{Ph}, L^S, \in, Bw, \perp_r \rangle$$
of $\mathsf{Ge}(Th^{+-})$ is a disjoint union[748] of (the similar reducts of) Minkowskian geometries[749].

2. eq behaves well in Th^{+-}, in the following sense. Whenever a, b, c in Fig.273 exist then d also exists. Further the arrangement in Fig.274 cannot happen. Formal statements of these are in Prop's 6.2.88 (p.895), 6.2.92 (p.901).

[747] Though $\mathbf{Ax}(\|)^-$ is not a first-order formula in its present form, it can be easily reformulated in the first-order frame language cf. footnote 316 on p.350.
[748] Cf. pp. 870, 873 for disjoint union of geometries.
[749] Cf. Def.6.2.58 (p.859) for Minkowskian geometries.

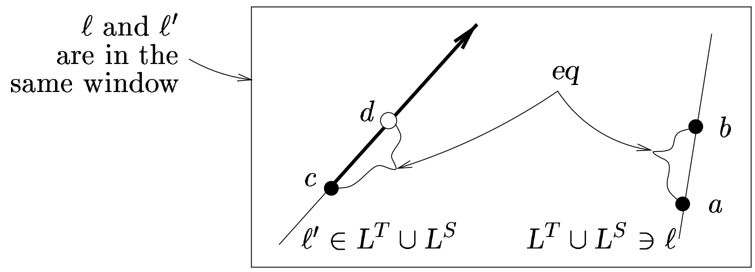

Figure 273: $(\forall a, b, c)(\exists d$ as in the figure).

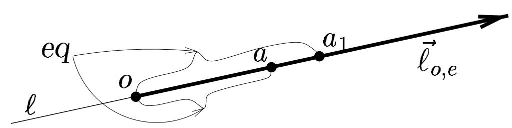

Figure 274: This cannot happen.

3. The space-like hyper-planes of the $\langle Mn, L; \in, Bw, \perp_r, eq\rangle$ reducts[750] of the elements of $\mathsf{Ge}(Th^{+-})$ are Euclidean geometries, assuming **Ax(eqtime)**, cf. Thm.6.6.114 (p.1130).

4. This theory Th^{+-}, despite of having all the nice properties in items 1–3 above, is not too strong e.g. we will see that even a strengthened version of Th^{+-} does not imply **Flxbasax**, i.e.

$$Th^{+-} \ + \ \text{“extra axioms”} \quad \not\models \mathbf{Flxbasax}$$

cf. Prop.6.2.101 (p.912) and the intuitive text below it on p.912.

THEOREM 6.2.23 *Assume* $n > 2$ *and* $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^{-} + \mathbf{Ax}(Triv_t)^{-} + \mathbf{Ax}(\sqrt{\ })$. *Then* $eq_2 = eq$*, therefore eq is first-order definable*[751].

A **proof** will be given in §6.2.6 on p.906.

In connection with the theorem below, cf. Proposition 6.2.96 on p.907.

[750]We will call these reducts Goldblatt-Tarski reducts or $GT_{\mathfrak{M}}$'s on p.923.

[751]we mean, definable over $\mathsf{Mod}(\mathbf{Bax}^{\oplus} + \ldots)$, of course. First one defines Mn over $\mathfrak{M} \in \mathsf{Mod}(\ldots)$ and then eq over $\mathfrak{M}$ and Mn.

THEOREM 6.2.24

(i) *Theorem 6.2.22 does not generalize from* **Basax** *to* $\mathbf{Bax}^{\oplus}$ *(and the assumption* $\mathbf{Ax}(||)^{-}$ *cannot be omitted from Thm.6.2.23).* *Moreover:*

For any $n > 1$*, there is* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }))$ *such that eq is not first-order definable in the expanded frame model* $\mathfrak{M}^{+} := \langle \mathfrak{M};\ Mn_{\mathfrak{M}}, \in \rangle$.

(ii) *Theorem 6.2.23 does not generalize to* $n = 2$. *Moreover:*

There is $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(||) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }))$ *such that eq is not first-order definable in the expanded frame model* $\mathfrak{M}^{+} := \langle \mathfrak{M};\ Mn_{\mathfrak{M}}, \in \rangle$.

Proof:
Outline of the proof: We choose $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }))$ (for the case of (ii) $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(||) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\sqrt{\ }))$ such that $\mathfrak{M}$ has properties (a)–(c) formulated below.

(a) $\mathfrak{F}^{\mathfrak{M}}$ is a real-closed field.

(b) $\langle \mathfrak{M};\ Mn, \in \rangle$ is first-order definable (in the sense of §6.3.2) over $\mathfrak{F}^{\mathfrak{M}}$.

(c) The subset $\{\, 2^i \ :\ i \in \mathrm{Z} \,\}$ [752] of $F^{\mathfrak{M}}$ is first-order definable over $\langle \mathfrak{M};\ Mn, \in, eq \rangle$.

Since $\langle \mathfrak{M};\ Mn, \in \rangle$ is definable over $\mathfrak{F}^{\mathfrak{M}}$, a subset A of $F^{\mathfrak{M}}$ is definable over $\langle \mathfrak{M}; Mn, \in \rangle$ iff it is definable over $\mathfrak{F}^{\mathfrak{M}}$ (cf. Thm.6.3.26, p.962). If *eq* was definable over $\langle \mathfrak{M}; Mn, \in \rangle$ then by property (c) the set $\{\, 2^i \ :\ i \in \mathrm{Z} \,\}$ would be definable over $\mathfrak{F}^{\mathfrak{M}}$. We will prove that the set $\{\, 2^i \ :\ i \in \mathrm{Z} \,\}$ is not definable over $\mathfrak{F}^{\mathfrak{M}}$ as a corollary of Lemma 6.2.28 way below. Hence, *eq* is not definable over $\langle \mathfrak{M};\ Mn, \in \rangle$.

Details of the proof:

Case of (i): Let $\mathfrak{F}$ be a real-closed field. Let $\mathfrak{M}$ be the frame-model over $\mathfrak{F}$ obtained from the Minkowski model[753] $\mathfrak{M}^M_{\mathfrak{F}}$ as follows. Intuitively, for each observer m of $\mathfrak{M}^M_{\mathfrak{F}}$ we include a new observer k such that clock of k runs twice slower than that of m and in all other properties m and k agree (i.e. $w_m(p) = w_k(p_0/2, p_1, \dots, p_{n-1})$, for all $p \in {}^nF$). The speed of light for new observers is 2^2, while the speed of light for the old observers is 1. Formally, $\mathfrak{M}$ is defined over

$$\mathfrak{M}^M_{\mathfrak{F}} \ = \ \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle \qquad \text{as follows:}$$
$$\mathfrak{M} \ \stackrel{\text{def}}{:=} \ \langle (B';\ Obs', Ph', Ib'), \mathfrak{F}, G;\ \in, W' \rangle, \quad \text{where}$$

[752] Recall that Z denotes the set of all integers.
[753] cf. Def.3.8.42 on p.331 for Minkowski models

$$\begin{aligned}
Obs' &\stackrel{\text{def}}{:=} Obs \times \{1,2\},\\
Ph' &\stackrel{\text{def}}{:=} Ph \times \{1,2\},^{754}\\
B' &\stackrel{\text{def}}{:=} Ib' \stackrel{\text{def}}{:=} Obs' \cup Ph',\\
W' &\stackrel{\text{def}}{:=} \Big\{ \Big\langle \langle m,i\rangle, p, \langle b,j\rangle \Big\rangle \in Obs' \times {}^nF \times B' \,:\, W(m, ip_0, p_1, \ldots, p_{n-1}, b) \Big\}.
\end{aligned}$$

We note that the speed of light for observers of the form $\langle m,1\rangle$ is 1 while for observers of the form $\langle m,2\rangle$ is 2^2.

It can be checked that $\mathfrak{M} \models \mathbf{Bax}^{\oplus} + \mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax}(\sqrt{\ })$.

Further, it can be checked that the Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ is first-order definable over $\mathfrak{F}$ in the sense of §6.3.2. *Hint:* The observers of $\mathfrak{M}^M_{\mathfrak{F}}$ can be identified with special Poincaré transformations of ${}^n\mathbf{F}$, namely with elements of PT^M (cf. Prop.3.8.63 on p.346 and Def.'s 3.8.38, 3.8.42). Since all these are affine transformations, they can be represented by matrices together with a vector. But a matrix together with a vector can be identified with a sequence (of length $n\cdot n+n$) of elements of $\mathfrak{F}$. The rest of defining $\mathfrak{M}^M_{\mathfrak{F}}$ over $\mathfrak{F}$ goes in the style of §6.3.2 using the "concrete construction" given for $\mathfrak{M}^M_{\mathfrak{F}}$ in Def.3.8.38 (p.325) and Def.3.8.42 (p.331).

Since $\mathfrak{M}$ was first-order defined (in the sense of §6.3.2) over $\mathfrak{M}^M_{\mathfrak{F}}$, we conclude that $\mathfrak{M}$ is first-order definable over $\mathfrak{F}$. Therefore, by Prop.6.3.18 (p.957), $\langle \mathfrak{M}; Mn, \in\rangle$ is first-order definable over $\mathfrak{F}$.

By these $\mathfrak{M}$ has properties (a) and (b) (formulated on p.830). Next we turn to proving that $\mathfrak{M}$ has property (c).

Let

$$H \stackrel{\text{def}}{:=} \left\{ x \in {}^+F : (\exists m \in Obs')(c_m = 1 \;\wedge\; \langle w_m(\bar{0}), w_m(1_t)\rangle \; eq \; \langle w_m(\bar{0}), w_m(x \cdot 1_t)\rangle) \right\}.$$

Claim **6.2.25** $H = \{ 2^i \,:\, i \in \mathrm{Z} \}$.

Proof: The proof of $\{ 2^i \,:\, i \in \mathrm{Z} \} \subseteq H$ is depicted in Figure 275. In the figure $m, k \in Obs'$ are such that the speed of light for m is 1, while the speed of light for k is 2^2, m and k are "brothers" in the sense that $m = \langle h,1\rangle$ and $k = \langle h,2\rangle$, for some $h \in Obs$.

The proof of $H \subseteq \{ 2^i \,:\, i \in \mathrm{Z} \}$ goes as follows. We will use the Minkowski distance $g_\mu : {}^nF \times {}^nF \longrightarrow F$ which will be defined in Definition 6.2.58 (p.860). It can be easily checked, e.g. by the proof of Claim 6.2.84 (p.892), that

$$(\forall m \in Obs')(\forall p,q,r,s \in {}^nF)\Big((\, c_m = 1 \;\wedge\; \langle w_m(p), w_m(q)\rangle \; eq_0 \; \langle w_m(r), w_m(s)\rangle \,) \;\Rightarrow\; (\, g_\mu(p,q) = 2^i g_\mu(r,s), \text{ for some } i \in \{-1,0,1\} \,)\Big).$$

[754] We defined Ph' as $Ph \times \{1,2\}$ only for technical reason.

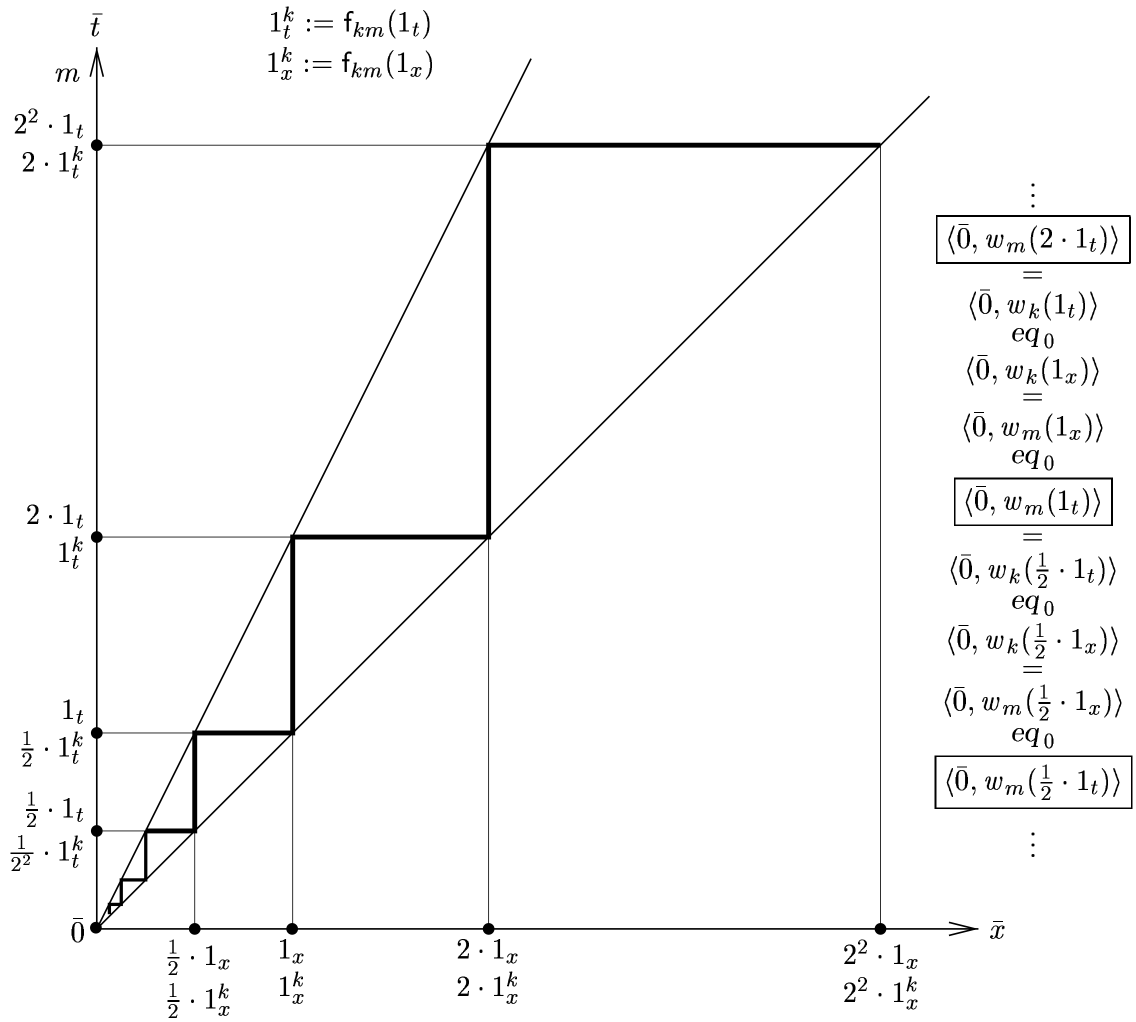

Figure 275: Proof of $\{ 2^i : i \in \mathrm{Z} \} \subseteq H$. The right-hand side column illustrates the computational part of why m thinks that $\langle \bar{0}, 1_t \rangle$ is "eq-related" to $\langle \bar{0}, 2^i \cdot 1_t \rangle$ (which means $2^i \in H$ by the definition of H).

Since eq was defined to be the transitive closure of eq_0, the above implies that

$$(\forall m \in Obs')(\forall p, q, r, s \in {}^nF)\Big((\, c_m = 1 \ \wedge\ \langle w_m(p), w_m(q)\rangle\ eq\ \langle w_m(r), w_m(s)\rangle\,) \ \Rightarrow \\ (\, g_\mu(p,q) = 2^i g_\mu(r,s),\ \text{for some } i \in \mathrm{Z}\,)\Big).$$

By this, it can be easily checked that $H \subseteq \{\, 2^i \ :\ i \in \mathrm{Z}\,\}$ indeed holds.

QED (Claim 6.2.25)

By Claim 6.2.25 (and by the definition of H), we have that property (c) holds for $\mathfrak{M}$. To complete the proof for item (i) it remains to prove that the subset $\{\, 2^i \ :\ i \in \mathrm{Z}\}$ of F is not first-order definable over $\mathfrak{F}$. This will be an immediate corollary of Lemma 6.2.28 way below.

Case of (ii): The proof of item (ii) is similar to that of (i). We will construct a model $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ }))$ such that $\mathfrak{M}$ has properties (a)–(c) formulated on p.830. Let $\mathfrak{F}$ be a real-closed field. Let $\mathfrak{M}$ be a model over $\mathfrak{F}$ obtained from the 2-dimensional Minkowski model $\mathfrak{M}^M_{\mathfrak{F}}$ as follows. Intuitively, for each observer m of $\mathfrak{M}^M_{\mathfrak{F}}$ we include a new observer k such that

$$\mathsf{f}_{km}(1_x) = 1_t \quad \text{and} \quad \mathsf{f}_{km}(1_t) = 2 \cdot 1_x, \quad \text{see Figure 276.}$$

The speed of light for new observers is 2^2 while for the old ones it is 1. Further, the

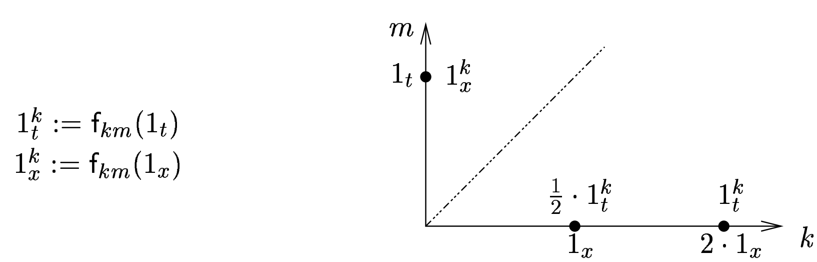

Figure 276: The picture represents the world-view of observer m.

new observers are FTL observers relative to the old ones. Formally, $\mathfrak{M}$ is defined over $\mathfrak{M}^M_{\mathfrak{F}} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W\,\rangle$ as follows:

$$\mathfrak{M} \ :\stackrel{\text{def}}{=}\ \langle (B';\ Obs', Ph', Ib'), \mathfrak{F}, G;\ \in, W'\,\rangle, \quad \text{where}$$
$$Obs' \ :\stackrel{\text{def}}{=}\ Obs \times \{1, 2\},$$

$$\begin{aligned} Ph' &\stackrel{\text{def}}{:=} Ph \times \{1,2\}, \\ B' &\stackrel{\text{def}}{:=} Ib' \stackrel{\text{def}}{:=} Obs' \cup Ph', \\ W' &\stackrel{\text{def}}{:=} \left\{ \left\langle \langle m,i\rangle, p_0, p_1, \langle b,j\rangle \right\rangle \in Obs' \times F \times F \times B' \ : \ W(m,p_1,ip_0,b) \right\}. \end{aligned}$$

We note that the speed of light for observers of the form $\langle m,1\rangle$ is 1 while for observers of the form $\langle m,2\rangle$ is 2^2.

It can be checked that $\mathfrak{M} \models \mathbf{Bax}^{\oplus}(2) + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ })$. The rest of the proof goes similarly to the proof given for item (i), i.e. we define H exactly the same way as in the proof of item (i); it can proved that H coincides with $\{\, 2^i \ : \ i \in \mathrm{Z} \,\}$, etc. The details of this part of the proof are left to the reader.

To complete the proof it remains to prove that $\{\, 2^i \ : \ i \in \mathrm{Z} \,\}$ is not definable over $\mathfrak{F}$. A generalized version of this will be proved as Lemma 6.2.28 below. Thus the theorem is proved modulo Lemma 6.2.28. ∎

For stating Lemma 6.2.28 we need a convention and a definition.

CONVENTION 6.2.26 From now on $\mathbb{Q}$ denotes the ordered field of rational numbers. Throughout we identify $\mathbb{Q}$ with its universe. $\mathbb{Q}$ is embeddable in a natural way into every ordered field $\mathfrak{F}$. When discussing an ordered field $\mathfrak{F}$ we will pretend that $\mathbb{Q}$ is a subfield of $\mathfrak{F}$. I.e. we identify $\mathbb{Q}$ with its unique isomorphic copy sitting inside $\mathfrak{F}$.

By an <u>*algebraic*</u> element of $\mathfrak{F}$ we understand an element which is algebraic over $\mathbb{Q}$.[755]

$\lhd$

Definition 6.2.27 Let $\mathfrak{F}$ be an ordered field. Let $H \subseteq F$. We call H <u>*gapy*</u> in $\mathfrak{F}$ iff

$$\Big(H \neq \emptyset \quad \text{and} \quad (\forall \text{ algebraic } a \in H)(\exists b,c \in F)(a < b < c \ \wedge \ b \notin H \ \wedge \ c \in H) \Big),$$

see Figure 277.

$\lhd$

<u>*Examples:*</u> Z, ω and $\{\, 2^i \ : \ i \in \omega \,\}$ are gapy subsets in $\mathfrak{F}$, for any ordered field $\mathfrak{F}$.

LEMMA 6.2.28 *Assume $\mathfrak{F}$ is a real-closed field. Then <u>no</u> gapy subset $H \subseteq F$ in $\mathfrak{F}$ is definable over $\mathfrak{F}$.*

[755] For completeness we recall that an element of $\mathfrak{F}$ is algebraic over $\mathbb{Q}$ iff it is a root of a nonzero polynomial with coefficients in $\mathbb{Q}$. (A root of a polynomial $p(x)$ is the same as a solution of the equation $p(x) = 0$.)

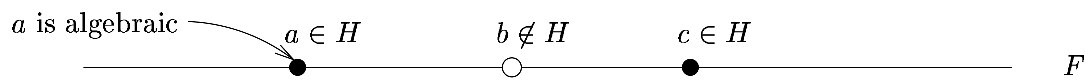

Figure 277: $H \subseteq F$ is gapy in $\mathfrak{F}$ iff it is nonempty and
$(\forall$ algebraic $a \in H)(\exists b, c$ as in the figure$)$.

Proof: Assume $\mathfrak{F}$ is a real-closed field. Throughout the proof we will use the following fact from field theory.

Fact **6.2.29** Let $p(x)$ be a unary term in the language of $\mathfrak{F}$ extended with the unary operation symbol "$-$". Then (i) and (ii) below hold.

(i) Assume that $p(x) = 0$ is a nontrivial[756] equation. Then this equation has only finitely many solutions. Further, the solutions of $p(x) = 0$ in $\mathfrak{F}$ are algebraic elements of $\mathfrak{F}$.

(ii) The intermediate value theorem holds for the function defined by $p(x)$, i.e. if $p(a) \cdot p(b) < 0$ then $p(c) = 0$ for some c strictly between a and b.[757]

Proof: Assume $p(x)$ is as above. Item (i) follows by the fact that $p(x)$ is a nonzero polynomial with coefficients in Z. Hence $p(x)$ has finitely many roots[758] and the roots of $p(x)$ are algebraic over $\mathbb{Q}$. For item (ii) cf. [136, Fact 8.4.5, p.386].

QED (Fact 6.2.29)

Now we turn to proving Lemma 6.2.28. The proof goes by contradiction. Assume that $H \subseteq F$ is gapy in $\mathfrak{F}$ and that H is definable over $\mathfrak{F}$. Then there is a first-order formula $\varphi(x)$ in the language of $\mathfrak{F}$ such that $H = \{\, a \in F \,:\, \mathfrak{F} \models \varphi[a] \,\}$. By Tarski's elimination of quantifiers Theorem for real-closed fields, i.e. by Thm.8.4.4 on p.385 of [136] and line 9 on p.376 of [136], $\varphi(x)$ is equivalent in $\mathfrak{F}$ with a *quantifier free* formula $\psi(x)$, i.e. $\mathfrak{F} \models \forall x(\varphi(x) \leftrightarrow \psi(x))$. Then $\psi(x)$ defines H, i.e.

$$H = \{\, a \in F \,:\, \mathfrak{F} \models \psi[a] \,\}\,.$$

Since ψ is quantifier free, it is a Boolean combination of atomic formulas. It is not hard to see that ψ is equivalent with a *disjunction* of formulas of the form[759]

$$(+) \qquad p_0(x) = 0 \ \wedge\ \ldots\ \wedge\ p_{k-1}(x) = 0 \ \wedge\ q_0(x) > 0 \ \wedge\ \ldots\ \wedge\ q_{m-1}(x) > 0,$$

[756] $p(x) = 0$ is called trivial in $\mathfrak{F}$ iff $\mathfrak{F} \models p(x) = 0$.

[757] This can be memorized by e.g. thinking of the Bolzano Theorem from elementary calculus.

[758] That each nonzero polynomial in $\mathfrak{F}$ has only finitely many roots is a well known property of ordered fields.

[759] using facts like $\tau(x) < \sigma(x) \Leftrightarrow \sigma(x) - \tau(x) > 0$, or $\neg(\tau < \sigma) \Leftrightarrow (\tau = \sigma \vee \sigma < \tau)$ etc.

where $m, k \in \omega$ and $p_i(x)$, $q_j(x)$ $(i \in k,\ j \in m)$ are unary terms in the language of $\mathfrak{F}$ extended with the operation symbol "$-$". *Warning:* here we include the unary operation "$-$" in the language of $\mathfrak{F}$. But then H is a finite union of sets definable by formulas of the form (+). Then one of these sets must be gapy in $\mathfrak{F}$ since H is gapy in $\mathfrak{F}$.[760] Therefore there is $H' \subseteq H$ such that H' is gapy in $\mathfrak{F}$ and H' is definable by a formula of the form (+). We may assume that this formula is exactly the one displayed in (+).

If one of the $(p_i(x) = 0)$'s is a nontrivial equation, then it has only finitely many solutions in $\mathfrak{F}$ and these solutions are algebraic elements of $\mathfrak{F}$ (by Fact 6.2.29(i)), hence H' is a finite set of algebraic elements of $\mathfrak{F}$ which contradicts the fact that H' is gapy in $\mathfrak{F}$. Therefore we may assume $k = 0$. Thus

$$(*) \qquad H' = \{\, a \in F \;:\; q_0(a) > 0 \ \wedge\ \ldots\ \wedge\ q_{m-1}(a) > 0 \,\}.$$

We may assume that none of the $(q_i(x) = 0)$'s is trivial. Therefore the set

$$Sol :\stackrel{\text{def}}{=} \{\, d \in F \;:\; (\exists i \in m) q_i(d) = 0\}$$

(of solutions) is finite by Fact 6.2.29(i).

Claim **6.2.30** $(\forall \text{ algebraic } a \in H')(\exists b, c \in F)$

$$\Big((c \text{ is algebraic}) \ \wedge\ a < b < c \ \wedge\ b \notin H' \ \wedge\ c \in H'\Big).$$

Proof: Let $a \in H'$ be such that a is an algebraic element of $\mathfrak{F}$. We have to prove that there are $b, c \in F$ such that $a < b < c$, $b \notin H'$, $c \in H'$ and c is algebraic. Let $b, c' \in F$ be such that $a < b < c'$, $b \notin H'$ and $c' \in H'$. Since H' is gapy in $\mathfrak{F}$ such b and c' exist. To prove the claim it is enough to prove that there is an algebraic $c \in H$ such that $b < c$. Clearly,

$$q_i(c') > 0, \quad \text{for all } i \in m$$

by $(*)$ and by $c' \in H'$. See Figure 278. Further, by $b \notin H'$ and $(*)$, there is $j \in m$ such that $q_j(b) \leq 0$. Let such a j be fixed. Thus, by Fact 6.2.29(ii) (and

[760] In more detail: $H = \bigcup_{i \in n} H_i$ for some $n \in \omega$ and each H_i is definable by a formula of the form (+). Then one of the H_i's is gapy in $\mathfrak{F}$ because of the following. Assume that none of the H_i's is gapy in $\mathfrak{F}$. Without loss of generality we can assume that each H_i is nonempty. Then, for all $i \in n$

$$(\exists \text{ algebraic } a_i \in H_i)(\{\, y \,:\, y > a_i \,\} \subseteq H_i \ \vee\ \{\, y \,:\, y > a_i \,\} \subseteq F \setminus H_i).$$

But then, for $a := \max\{\, a_i \,:\, i \in n \,\}$ we have that $a \in H$ and a is algebraic, further

$$\{\, y \,:\, y > a \,\} \subseteq H \ \vee\ \{\, y \,:\, y > a \,\} \subseteq F \setminus H.$$

This contradicts our assumption that H is gapy in $\mathfrak{F}$. Therefore one of the H_i's is gapy in $\mathfrak{F}$.

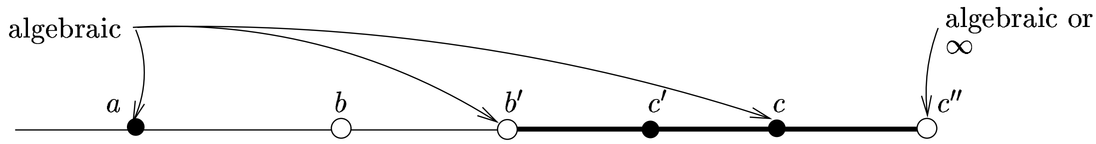

Figure 278:

by $b < c'$), there is $d \in F$ such that $b \leq d < c'$ and $q_j(d) = 0$. Therefore the set $\{ d \in Sol \,:\, d < c' \}$ is nonempty (and is finite), and

$$b \leq \max\{ d \in Sol \,:\, d < c' \}.$$

Let

$$b' :\overset{\text{def}}{=} \max\{ d \in Sol \,:\, d < c' \}.$$

Let

$$c'' :\overset{\text{def}}{=} \begin{cases} \min\{ d \in Sol \,:\, d > c' \} & \text{if } (\exists d \in Sol) d > c' \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, $b \leq b' < c' < c''$ and none of the equations $q_0(x) = 0, \ldots, q_{m-1}(x) = 0$ has a solution in the open interval $(b', c'') := \{d \in F \,:\, b' < d < c''\}$, cf. Figure 278 (recall that $q_i(c') > 0$, for all $i \in \omega$). By this, by Fact 6.2.29(ii), by $(*)$ and by $c' \in H'$, we conclude that $(b', c'') \subseteq H'$. Further (by Fact 6.2.29(i)) b' is an algebraic element of $\mathfrak{F}$ and c'' is either an algebraic element of $\mathfrak{F}$ or is ∞. Thus there is an algebraic element c of $\mathfrak{F}$ such that $c \in (b', c'') \subseteq H'$. For this choice of c we have $b < c$, $c \in H'$ and c is an algebraic element of $\mathfrak{F}$.

QED (Claim 6.2.30)

Let $a^i, b^i \in F$ $(i \in \omega)$ be such that for all $i \in \omega$, a^i is an algebraic element of $\mathfrak{F}$, $a^i \in H'$, $b^i \notin H'$, and

$$a^i < b^i < a^{i+1} < b^{i+1}.$$

By Claim 6.2.30, such a^i's and b^i's exist. By $(*)$,[761] there are $j \in m$ and an infinite subset I of ω such that

$$(\forall i \in I)\,(\, q_j(b^i) \leq 0 \;\;\wedge\;\; q_j(a^i) > 0\,).$$

Let such j and I be fixed. Let $h : \omega \rightarrowtail\!\!\!\twoheadrightarrow I$ be an order preserving bijection. Then clearly,

$$(\forall i \in \omega)(\, q_j(a^{h(i)}) > 0 \;\wedge\; q_j(b^{h(i)}) \leq 0 \;\wedge\; a^{h(i)} < b^{h(i)} < a^{h(i+1)} < b^{h(i+1)}\,).$$

[761] and by $a^i \in H'$, $b^i \notin H'$

Thus, by Fact 6.2.29(ii), for every $i \in \omega$ there is $c^{h(i)} \in F$ such that $a^{h(i)} < c^{h(i)} \leq b^{h(i)}$ and $q_j(c^{h(i)}) = 0$. By the above we conclude that the equation $q_j(x) = 0$ has infinitely many solutions, and this contradicts item (i) of Fact 6.2.29. ∎

At this point all parts of the proof of Thm.6.2.24 has been taken care of.

One of the reasons for looking at the alternative notions like $\perp'_r$, $\perp''_r$, eq_2 is that they can behave better from the point of view of definability issues. (There are of course other reasons, too, for experimenting with alternative concepts.) Similarly, we will look at alternative definitions of the topology part $\mathcal{T}$ of our geometries. Namely $\mathcal{T}'$ will be based on Bw while $\mathcal{T}''$ will be based on causality $\prec$.

Definition 6.2.31 (Alternatives $\mathcal{T}'$, $\mathcal{T}''$ for topology $\mathcal{T}$)
Assume $n > 1$. Let $\mathfrak{M}$ be a frame model of dimension n. $Mn, Bw, \prec$ are defined in items 3, 8, 7 of Def.6.2.2(I). We define the topologies $\mathcal{T}'$ and $\mathcal{T}''$ on Mn in items (i) and (ii) below, respectively.

(i) Intuitively, first by using Bw we define interiors of simplexes,[762] cf. the left-hand side of Figure 279. Then by using these (as a subbase) we define the topology $\mathcal{T}'$ on Mn the natural way, formally:

For every $H \subseteq Mn$ the *convex hull* $Ch(H)$ of H is the smallest subset of Mn having properties 1 and 2 below.[763]

1. $H \subseteq Ch(H)$.
2. $(\, a, b \in Ch(H) \;\wedge\; Bw(a, c, b)\,) \quad \Rightarrow \quad c \in Ch(H)$.

We define the collection $\mathsf{simplexes} \subseteq \mathcal{P}(Mn)$ as follows.

$$\mathsf{simplexes} :\stackrel{\text{def}}{=} \{\, H \subseteq Mn \;:\; |H| = n+1,\; (\exists m \in Obs)\, Plane(H) = Rng(w_m)\,\}\,.$$

Let $H \in \mathsf{simplexes}$. Then, intuitively, the neighborhood $S'(H)$ is defined to be the "interior" of the convex hull $Ch(H)$ of H; formally:

$$S'(H) :\stackrel{\text{def}}{=} Ch(H) \setminus \bigcup_{e \in H} Plane(H \setminus \{e\}),$$

see the left-hand side of Figure 279. Now, the topology $\mathcal{T}' \subseteq \mathcal{P}(Mn)$ is the

[762]We note that if $n = 2$ the simplexes are the triangles and if $n = 3$ the simplexes are the tetrahedra.
[763]The usual notation in the literature is "$co(H)$" for our $Ch(H)$.

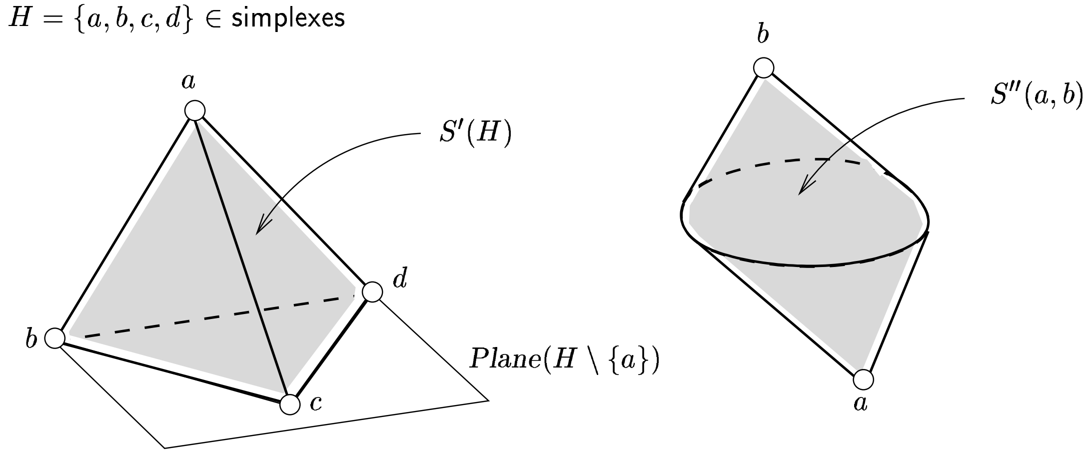

Figure 279: In the figure $n = 3$.

one generated by T'_0 below, i.e. T'_0 is a subbase for $\mathcal{T}'$.

$$T'_0 :\stackrel{\text{def}}{=} \{\, S'(H) \;:\; H \in \mathsf{simplexes}\,\}\,.$$

We note that, assuming $\mathbf{Bax}^{-} + \mathbf{Ax}(\sqrt{\ })$, T'_0 is a *base* for $\mathcal{T}'$, cf. Figure 279 (and the proof of Thm.6.2.34).

(ii) For every $a, b \in Mn$ with $a \prec b$ we define the neighborhood

$$S''(a, b) :\stackrel{\text{def}}{=} \{c \in Mn \;:\; a \prec c \prec b\,\}\,,$$

see the right-hand side of Figure 279. Now, the topology $\mathcal{T}'' \subseteq \mathcal{P}(Mn)$ is the one generated by T''_0 below, i.e. T''_0 is a subbase for $\mathcal{T}''$.

$$T''_0 :\stackrel{\text{def}}{=} \{\, S''(a, b) \;:\; a, b \in Mn,\; a \prec b\,\}\,.$$

We note that, assuming $\mathbf{Bax}^{-\oplus} + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$ and $[\,(\forall m \in Obs)\,(\,m$ thinks that there is an upper bound for the speed of light$\,)$[764] <u>or</u> $\mathfrak{F} = \mathfrak{R}\,]$, T''_0 is a *base* for $\mathcal{T}''$, where $\mathbf{Ax}(\uparrow\uparrow_0)$ is defined below, cf. Figure 279 (and the proof of Thm.6.2.34).

$\triangleleft$

[764] formally: $(\exists\lambda \in F)(\forall d \in \mathsf{directions})\, \mathsf{c}_m(d) < \lambda$.

Theorems 6.2.34, 6.2.41 and 6.2.42 below say that topologies $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ coincide, under some assumptions. For stating these theorems we introduce weakened versions $\mathbf{Ax(\uparrow\uparrow_0)}$ and $\mathbf{Ax(\uparrow\uparrow_{00})}$ of our axiom $\mathbf{Ax(\uparrow\uparrow)}$ saying that each observer sees any other observer's time flow forwards. The reason for introducing $\mathbf{Ax(\uparrow\uparrow_0)}$ is that $\mathbf{Ax(\uparrow\uparrow)}$ blurs the distinction between **Basax** and **Newbasax**, while the reason for introducing $\mathbf{Ax(\uparrow\uparrow_{00})}$ is that $\mathbf{Bax}^- + \mathbf{Ax(\uparrow\uparrow_0)} + \mathbf{Ax}(\sqrt{\ })$ excludes FTL observers already in two dimensions, cf. Prop.6.2.32. Recall from Def.4.2.6 (p.460) that $m\ STL\ k$ means that m sees k moving slower than light.

$\mathbf{Ax(\uparrow\uparrow_0)}$ $(\forall m, k \in Obs)\,(m \overset{\odot}{\to} k \ \rightarrow\ m \uparrow k)$.

Intuitively, if m sees k then k's clock runs forwards as seen by m.

$\mathbf{Ax(\uparrow\uparrow_{00})}$ $(\forall m, k \in Obs)\,(m\ STL\ k \ \rightarrow\ m \uparrow k)$.

Intuitively, if m sees k moving slower than light then k's clock runs forwards as seen by m.

PROPOSITION 6.2.32 *For any $n > 1$*

$$\mathbf{Bax}^- + \mathbf{Ax(\uparrow\uparrow_0)} + \mathbf{Ax}(\sqrt{\ }) \models \text{“}\nexists\ \textbf{FTL observers}\text{”}.$$

Proof: The proof goes by contradiction.

Assume that there is $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax(\uparrow\uparrow_0)} + \mathbf{Ax}(\sqrt{\ }))$ with $m, k \in Obs$ such that m sees k moving FTL. Without loss of generality we may assume that $\bar{0} \in tr_m(k)$ by **Ax4** and Thm.4.3.17 (p.488). Let $k' \in Obs$ be obtained from m and k exactly as in the proof of Thm.4.3.24 (p.497), cf. Figures 153, 154 (p.499). Then m sees k' moving with infinite speed, cf. the right-hand side of Figure 154 (p.499). So $\neg(m \uparrow k')$. This contradicts $\mathbf{Ax(\uparrow\uparrow_0)}$. ∎

Question for future research 6.2.33 In which ones of the theorems involving $\mathbf{Ax(\uparrow\uparrow)}$ or $\mathbf{Ax(\uparrow\uparrow_0)}$ can one replace $\mathbf{Ax(\uparrow\uparrow)}$ or $\mathbf{Ax(\uparrow\uparrow_0)}$ with $\mathbf{Ax(\uparrow\uparrow_{00})}$?

◁

THEOREM 6.2.34 *Assume* $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(\uparrow\uparrow_0)} + \mathbf{Ax}(\sqrt{\ })$. *Assume that* $(\forall m \in Obs)$ (*m thinks that there is an upper bound for the speed of light*)[765] *<u>or</u>* $\mathfrak{F} = \mathfrak{R}$. *Then (i) and (ii) below hold.*

(i) *The topologies $\mathcal{T}'$ and $\mathcal{T}''$ coincide.*

(ii) *The topology $\mathcal{T}' = \mathcal{T}''$ is a Euclidean one in the following sense:*

[765]formally: $(\exists \lambda \in F)(\forall d \in \mathsf{directions})\, \mathsf{c}_m(d) < \lambda$.

(a) *For any* $m \in Obs$, $\{ w_m^{-1}[H] \;:\; H \in \mathcal{T}' \}$ *is the usual Euclidean topology on* ${}^nF$*, i.e. the one with base* $\{ S(p,\varepsilon) \;:\; p \in {}^nF,\ \varepsilon \in {}^+F \}$*, cf. p.189 for* $S(p,\varepsilon)$*.*

(b) $\mathcal{T}'$ *is homeomorphic to a sum topology (i.e. a coproduct)*[766] *of usual Euclidean topologies on* ${}^nF$*.*

Proof: Assume the assumptions of the theorem. By Thm.4.3.11, we have that: the visibility relation $\overset{\odot}{\to}$ is an equivalence relation when restricted to Obs, and if $m \overset{\odot}{\to} k$ then $Rng(w_m) = Rng(w_k)$, otherwise $Rng(w_m) \cap Rng(w_k) = \emptyset$.

Let $O \subseteq Obs$ be a class of representatives for the equivalence relation $\overset{\odot}{\to}$.[767] Then

(∗) $\quad Mn$ is the disjoint union of the family $\langle\, Rng(w_m) \;:\; m \in O \,\rangle$ (and the members of this family are mutually disjoint).

It is easy to check that for every $m \in Obs$

$$(**) \qquad Rng(w_m) \in \mathcal{T}' \qquad \text{and} \qquad Rng(w_m) \in \mathcal{T}'',$$

i.e. $Rng(w_m)$ is an open set w.r.t. both topologies. For every $m \in Obs$, let $\mathcal{T}' \restriction Rng(w_m)$ and $\mathcal{T}'' \restriction Rng(w_m)$ be the subspace topologies of $\mathcal{T}'$ and $\mathcal{T}''$ on $Rng(w_m)$,[768] respectively, i.e.

$$\mathcal{T}' \restriction Rng(w_m) \;\overset{\text{def}}{:=}\; \{ H \cap Rng(w_m) \;:\; H \in \mathcal{T}' \} = \{ H \in \mathcal{T}' \;:\; H \subseteq Rng(w_m) \},$$
$$\mathcal{T}'' \restriction Rng(w_m) \;\overset{\text{def}}{:=}\; \{ H \cap Rng(w_m) \;:\; H \in \mathcal{T}'' \} = \{ H \in \mathcal{T}'' \;:\; H \subseteq Rng(w_m) \};$$

further let $\mathcal{T}'_m$ and $\mathcal{T}''_m$ be the topologies on the coordinate system ${}^nF$ defined as follows.

$$\mathcal{T}'_m \;\overset{\text{def}}{:=}\; \{ w_m^{-1}[H] \;:\; H \in \mathcal{T}' \}.$$
$$\mathcal{T}''_m \;\overset{\text{def}}{:=}\; \{ w_m^{-1}[H] \;:\; H \in \mathcal{T}'' \}.$$

It is easy to see that for every $m \in Obs$

(∗∗∗) $\quad w_m : {}^nF \longrightarrow Rng(w_m)$ is a homeomorphism between $\mathcal{T}'_m$ and $\mathcal{T}' \restriction Rng(w_m)$ and between $\mathcal{T}''_m$ and $\mathcal{T}'' \restriction Rng(w_m)$.

To prove item (i) of the theorem, by (∗), (∗∗), (∗∗∗) above it is enough to prove that for each m, $\mathcal{T}'_m$ and $\mathcal{T}''_m$ coincide. This holds by Claim 6.2.35 below.

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[766]Cf. p.870 for coproduct of topological spaces. Cf. also Engelking [83] under the name "sum of spaces".

[767]I.e. $(\forall m \in Obs)\, |O \cap m/\overset{\odot}{\to}\,| = 1$, where $m/\overset{\odot}{\to}$ is the equivalence class of m w.r.t. $\overset{\odot}{\to}$, as usual.

[768]i.e. they are the restrictions to $Rng(w_m)$ of $\mathcal{T}'$ and $\mathcal{T}''$, respectively

Claim **6.2.35** Let $m \in Obs$. Then (a) and (b) below hold.

(a) $\mathcal{T}'_m$ is the Euclidean topology on ${}^nF$, i.e. the one with base $\{ S(p,\varepsilon) \ : \ p \in {}^nF,\ \varepsilon \in {}^+F \}$, cf. p.189 for $S(p,\varepsilon)$.

(b) $\mathcal{T}''_m$ is the Euclidean topology on ${}^nF$.

<u>*Proof:*</u>

<u>Proof of (a):</u> A set $H \subseteq {}^nF$ is called a <u>*simplex*</u> iff $|H| = n+1$ and for each $p \in H$, $\{ q-p : q \in H,\ q \neq p \}$ is a basis[769] for the vector space ${}^n\mathbf{F}$, cf. the left-hand side of Figure 279.[770]

Clearly, a subbase for $\mathcal{T}'_m$ is

$$T'_m :\stackrel{\text{def}}{=} \{ w_m^{-1}[H] \ : \ H \in T'_0,\ w_m^{-1}[H] \neq \emptyset \} ;$$

where recall that T'_0 is the subbase of $\mathcal{T}'$. Since the world-view transformations are betweenness preserving collineations[771] it can be checked (by item 1f of Prop.6.2.79) that T'_m consists of the interiors of the convex hulls of the simplexes, where interiors of sets are defined via the Euclidean topology, and convex hulls of sets are defined in Def.4.3.28(iii) (p.509).

T'_m is a base for the Euclidean topology (on ${}^nF$) because of the following. Let H be an open set of the Euclidean topology. Then for any $p \in H$, there is a "neighborhood" of p in T'_m which is contained in H. Hence H is a union of members of T'_m.

But then, $\mathcal{T}'_m$ is the Euclidean topology on ${}^nF$.

<u>Proof of (b):</u> Let $\prec_m$ be a binary relation on ${}^nF$ defined as follows.

$$\prec_m \ :\stackrel{\text{def}}{=} \ \{ \langle p,q \rangle \in {}^nF \times {}^nF \ : \ w_m(p) \prec w_m(q) \} .$$

For every $p \in {}^nF$, let

$$\mathsf{Future}_p \ :\stackrel{\text{def}}{=} \ \{ q \in {}^nF \ : \ p \prec_m q \},$$

$$\mathsf{Past}_p \ :\stackrel{\text{def}}{=} \ \{ q \in {}^nF \ : \ q \prec_m p \}.$$

Clearly, a subbase for $\mathcal{T}''_m$ is

$$T''_m \ :\stackrel{\text{def}}{=} \ \{ w_m^{-1}[H] \ : \ H \in T''_0,\ w_m^{-1}[H] \neq \emptyset \} ,$$

where recall that T''_0 is the subbase of $\mathcal{T}''$. It is easy to see that

[769] i.e. a minimal set of generators

[770] This is practically the same notion as "simplexes" in Def.6.2.31, the only difference being that now we are in ${}^nF$ while there we were in $\langle Mn, \ldots \rangle$.

[771] by Thm.4.3.11 (p.481), Fact 4.7.7 (p.617) and Remark 3.6.7 (p.268)

(349) $T''_m \;=\; \{\,\mathsf{Future}_p \cap \mathsf{Past}_q \;:\; p,q \in {}^nF,\; p \prec_m q\,\}.$

By item 1h of Prop.6.2.79 (p.884), we have that

(350) $p \prec_m q \;\;\Leftrightarrow\;\; [\,p_t < q_t \;\wedge\; (\exists k \in Obs)\, p,q \in tr_m(k)\,].$

There are no FTL observers, by Prop.6.2.32. Thus, by Thm.4.3.29 (p.510), by (350) and by $\mathbf{Ax5_{Obs}}$, we have that for any $p \in {}^nF$

(351) Future_p is the interior of the convex hull of $\{\,q \in \mathsf{Cone}_{m,p} \;:\; p_t < q_t\,\}$, and

(352) Past_p is the interior of the convex hull of $\{\,q \in \mathsf{Cone}_{m,p} \;:\; p_t > q_t\,\}$;

where interiors of sets are defined via the Euclidean topology, and convex hulls of sets are defined in Def.4.3.28(iii) (p.509). By (349), (351), (352) and Thm.4.3.29 (p.510), we conclude that T''_m is a base for the Euclidean topology (on ${}^nF$), cf. the right-hand side of Figure 279. Hence, $\mathcal{T}''_m$ is the Euclidean topology.

QED (Claim 6.2.35)

By this item (i) of our theorem is proved. Item (ii) follows by $(*)$, $(**)$, $(***)$ and Claim 6.2.35. Namely by $(*)$, $(**)$ we have that $\mathcal{T}'$ is the sum topology (i.e. the coproduct) of the family $\langle\, \mathcal{T}' \restriction Rng(w_m) \;:\; m \in O\,\rangle$ which in turn, by $(***)$, is homeomorphic to the sum topology (i.e. the coproduct) of the family $\langle\, \mathcal{T}'_m : m \in O\,\rangle$; while by Claim 6.2.35 we have that each $\mathcal{T}'_m$ is the Euclidean topology on ${}^nF$. ∎

PROPOSITION 6.2.36 *Assume* $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$*. Then the topology* $\mathcal{T}'$ *is the Euclidean one in the sense of Thm.6.2.34(ii), i.e. it has properties (a) and (b) in the formulation of Thm.6.2.34(ii).*

Moreover T'_0 *is a base for* $\mathcal{T}'$*.*

Proof: The proposition is a corollary of the proof of Thm.6.2.34. ∎

Recall that we have introduced a strong symmetry principle $\mathbf{Ax}(\omega)$ in §3.9 (cf. p.351). Below we introduce four *weak* variants $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax}(\omega)^{00}$, $\mathbf{Ax}(\omega)^\sharp$, $\mathbf{Ax}(\omega)^{\sharp\sharp}$ of $\mathbf{Ax}(\omega)$, where $\mathbf{Ax}(\omega)^0$ and $\mathbf{Ax}(\omega)^{00}$ can be considered as natural *weakened versions* of $\mathbf{Ax}(\omega)$; while $\mathbf{Ax}(\omega)^\sharp$ and $\mathbf{Ax}(\omega)^{\sharp\sharp}$ can be considered as natural *weakened versions* of $\mathbf{Ax}(\omega) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$. I.e.

$$\begin{array}{ccccc} [\mathbf{Ax}(\omega) + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })] & > & \mathbf{Ax}(\omega)^\sharp & > & \mathbf{Ax}(\omega)^{\sharp\sharp} \\ \vee & & \vee & & \vee \\ \mathbf{Ax}(\omega) & > & \mathbf{Ax}(\omega)^0 & > & \mathbf{Ax}(\omega)^{00}. \end{array}$$

We will use these axioms in formulating some of our theorems.

Definition 6.2.37

$\mathbf{Ax}(\omega)^0$ is defined to be the <u>disjunction</u> of the following symmetry axioms: $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle 1}$+$\mathbf{Ax(eqtime)}$, $\mathbf{Ax\triangle 2}$, $\mathbf{Ax\square 1}$+$\mathbf{Ax(eqtime)}$, $\mathbf{Ax\square 2}$.[772]

$\mathbf{Ax}(\omega)^\sharp$ is defined to be $\mathbf{Ax}(\omega)^0$+$\mathbf{Ax}(\mathit{Triv}_t)^-$+$\mathbf{Ax}(\sqrt{\ })$.

$\mathbf{Ax}(\omega)^{00}$ is defined to be the <u>disjunction</u> of the following symmetry axioms: $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}$+$\mathbf{Ax}(\mathit{Triv}_t)^-$.[773]

$\mathbf{Ax}(\omega)^{\sharp\sharp}$ is defined to be $\mathbf{Ax}(\omega)^{00} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ })$.

$\lhd$

The following three propositions and Theorems 6.2.59 (p.861), 6.2.60 (p.862) show that our weak symmetry axioms $\mathbf{Ax}(\omega)^0, \mathbf{Ax}(\omega)^{00}, \mathbf{Ax}(\omega)^\sharp, \mathbf{Ax}(\omega)^{\sharp\sharp}$ are strong enough[774] under assuming **Basax**. In connection with the following proposition recall that $g_\mu^2 : {}^nF \times {}^nF \longrightarrow F$ is the square of the Minkowski-distance defined in Def.2.9.1. In the next proposition we use the notation $g_\mu(p,q) := \sqrt{g_\mu^2(p,q)}$.[775]

PROPOSITION 6.2.38 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^0 + \mathbf{Ax}(\sqrt{\ }))$, *<u>or</u> that* $n > 2$ *and* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^{00} + \mathbf{Ax}(\sqrt{\ }))$. *Then for any* $e, e' \in Mn$ *and* $m \in Obs$

$$g(e,e') = \begin{cases} g_\mu(w_m^{-1}(e), w_m^{-1}(e')) & \text{if } e \equiv^T e' \text{ or } e \equiv^{Ph} e' \text{ or } e \equiv^S e' \\ \text{is undefined} & \text{otherwise.} \end{cases}$$

The **proof** is available from Judit Madarász. ∎

Our next two propositions are not about geometry. They are here to help us become familiar with the (basic properties of the) new axioms introduced above.

[772] We note that, assuming $\mathbf{Flxbasax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$ these symmetry axioms are equivalent to one another, cf. Thm.6.2.98 (p.910), cf. also Thm.3.9.11 (p.356), Thm.2.8.17 (p.138) and [174].

[773] We note that, assuming $n > 2$ and $\mathbf{Flxbasax}^\oplus + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\uparrow\uparrow_0) + \mathbf{Ax}(\sqrt{\ })$ the symmetry axioms involved in $\mathbf{Ax}(\omega)^0$ and $\mathbf{Ax}(\omega)^{00}$ are equivalent to one another.

[774] in that they ensure that our g is (the square of) the usual Minkowski distance

[775] In connection with this definition we note that our symbol g_μ^2 (introduced on p.152) is *<u>not</u>* the square of something denoted by g_μ, but instead it is a basic symbol, like, say γ. Then, g_μ counts as a brand new symbol unrelated to g_μ^2 and our definition $g_\mu(\ldots) = \sqrt{g_\mu^2(\ldots)}$ should be understood like $g_\mu(p,q) = \sqrt{\gamma(p,q)}$. (The reason for treating g_μ^2 as basic symbol [instead of e.g. g_μ] is explained in footnote 61, p.46.)

Recall that Poincaré transformations were introduced in Def.2.9.1 (p.152) and that generalized Poincaré transformations were introduced in Def.5.0.67 (p.728).

PROPOSITION 6.2.39

(i) $\mathbf{Basax} + \mathbf{Ax}(\omega)^0 + \mathbf{Ax}(\surd\,) \models$ (*the* f_{mk}*'s are Poincaré transformations*), *and*

$\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\omega)^0 + \mathbf{Ax}(\surd\,) + \mathbf{Ax6} \models$
(*the* f_{mk}*'s are generalized Poincaré transformations*)

(ii) *Assume* $n > 2$. *Then the statement in (i) remains true if we replace the assumption* $\mathbf{Ax}(\omega)^0$ *with* $\mathbf{Ax}(\omega)^{00}$.

The **proof** is available from Judit Madarász. ∎

Recall from §3.8 that for any Euclidean $\mathfrak{F}$, the axiom system **BaCo** admits exactly one model whose ordered field reduct is $\mathfrak{F}$, up to isomorphism, and this model is the standard Minkowskian one.

PROPOSITION 6.2.40

(i) $\mathbf{Basax} + \mathbf{Ax}(\omega)^{\sharp} + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax(ext)} + \mathbf{Ax}\heartsuit \;\;=\!\!|\!\!\models\;\; \mathbf{BaCo} + \mathbf{Ax}(\surd\,)$.

(ii) *Assume* $n > 2$. *Then (i) remains true if* $\mathbf{Ax}(\omega)^{\sharp}$ *is replaced by* $\mathbf{Ax}(\omega)^{\sharp\sharp}$ *in it.*

The **proof** is available from Judit Madarász. ∎

In connection with the following theorem recall that

$$\mathbf{Basax} \models \mathbf{Newbasax} \models \mathbf{Flxbasax}^{\oplus}.$$

Let Th^+ be the theory

$$\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\omega)^{\sharp} + \mathbf{Ax(diswind)}$$

which will occur in Thm.6.2.41 below. This theory or its variants with **Basax** or **Newbasax** in place of $\mathbf{Flxbasax}^{\oplus}$ will often occur in our subsequent theorems. Therefore we note that by our previous 3 results (items 6.2.38–6.2.40), Th^+ is almost equivalent with "official special relativity" with disjoint windows allowed.[776]

[776]By "official special relativity" we refer to **Specrel**.

THEOREM 6.2.41 *Assume* $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\omega)^{\sharp} + \mathbf{Ax(diswind)}$. *Then (i) and (ii) below hold.*

(i) $\mathcal{T}$ *and* $\mathcal{T}'$ *coincide.*

(ii) *Assume* $\mathbf{Ax}(\uparrow\uparrow_0)$. *Then* $\mathcal{T}$, $\mathcal{T}'$ *and* $\mathcal{T}''$ *coincide.*[777]

Further the topology $\mathcal{T} = \mathcal{T}' = \mathcal{T}''$ *is the Euclidean one in the sense of Thm.6.2.34(ii).*

The **proof** is available from Judit Madarász. ∎

Since $\mathbf{Ax}(\omega)^{\sharp}$ was designed to be weak, Theorems 6.2.41, 6.2.42 say that $\mathbf{Flxbasax}^{\oplus}$ + (some mild assumptions) suffice for $\mathcal{T} = \mathcal{T}' = \mathcal{T}''$.

The next theorem says that if $n > 2$ then in the above theorem we could use the weaker $\mathbf{Ax}(\omega)^{\sharp\sharp}$ in place of $\mathbf{Ax}(\omega)^{\sharp}$.

THEOREM 6.2.42 *Assume* $n > 2$ *and* $\mathbf{Flxbasax}^{\oplus} + \mathbf{Ax}(\omega)^{\sharp\sharp} + \mathbf{Ax(diswind)}$. *Then (i) and (ii) in Thm.6.2.41 hold.*

The **proof** is available from Judit Madarász. ∎

Theorems 6.2.10 (p.813), 6.2.22 (p.827), 6.2.23 (p.827), 6.2.34 (p.840) and 6.2.41 motivate the following definition.

Definition 6.2.43
(Alternatives $\mathfrak{G}'_{\mathfrak{M}}$, $\mathfrak{G}''_{\mathfrak{M}}$ and $\mathsf{Ge}'(Th)$, $\mathsf{Ge}''(Th)$ for $\mathfrak{G}_{\mathfrak{M}}$ and $\mathsf{Ge}(Th)$)

(i) Assume $\mathfrak{M}$ is a frame model. Then we define $\mathfrak{G}'_{\mathfrak{M}}$ to be the geometry obtained from $\mathfrak{G}_{\mathfrak{M}} = \langle Mn, \mathbf{F_1}, \ldots\rangle$ by replacing $\perp_r$, eq with $\perp'_r$, eq_2, respectively, i.e.

$$\mathfrak{G}'_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp'_r, eq_2, g, \mathcal{T}\rangle.$$

We define $\mathfrak{G}''_{\mathfrak{M}}$ to be the geometry obtained from $\mathfrak{G}'_{\mathfrak{M}}$ by replacing the topology $\mathcal{T}$ with $\mathcal{T}'$, i.e.

$$\mathfrak{G}''_{\mathfrak{M}} :\stackrel{\text{def}}{=} \langle Mn, \mathbf{F_1}, L;\ L^T, L^{Ph}, L^S, \in, \prec, Bw, \perp'_r, eq_2, g, \mathcal{T}'\rangle.$$

[777] A physical consequence of Thm.6.2.41 is that for the various definitions of our topology (i.e. $\mathcal{T}, \mathcal{T}', \mathcal{T}''$) the so called measurable sets remain the same (under the assumptions of the theorem). The reason for this is that the measurable sets are usually derived from the topology. In principle results like this might be relevant for recent theories of physical measurement (where the notion of measurement is related to measurable sets) cf. Attila Andai personal communication. Cf. e.g. Misner-Thorne-Wheeler [196, p.1184 (lower part of the page)]. Cf. also Andai [8, Chap.4, §5] and Pulmanová [216].

(ii) Let Th be a set of formulas in our frame language for relativity theory. Then the classes of relativistic geometries $\mathsf{Ge}'(Th)$ and $\mathsf{Ge}''(Th)$ associated to Th are defined as follows.

$$\mathsf{Ge}'(Th) \quad :\stackrel{\text{def}}{=} \quad \mathbf{I}\{\mathfrak{G}'_{\mathfrak{M}} \ : \ \mathfrak{M} \in \mathsf{Mod}(Th)\},$$
$$\mathsf{Ge}''(Th) \quad :\stackrel{\text{def}}{=} \quad \mathbf{I}\{\mathfrak{G}''_{\mathfrak{M}} \ : \ \mathfrak{M} \in \mathsf{Mod}(Th)\},$$

where for taking isomorphic copies of our geometries we apply Convention 6.2.3 (i.e. we stick with the "real" membership relation "$\in$"). $\lhd$

Our next theorem says, roughly, that our class $\mathsf{Ge}(Th)$ of relativistic geometries is definable over the corresponding class of observational models.

In Theorem 6.2.44 below instead of definability of the topology part we claim definability of only a subbase for the topology. An exception is item (ii) of Thm.6.2.44, because there a base T'_0 will be definable, too. The content of Thm.6.2.44 below will be presented (discussed etc.) in a greater detail in §6.3 (cf. the proof of Thm.6.2.44).

THEOREM 6.2.44

(i) *The class* $\mathsf{Ge}'(Th)$ *is uniformly first-order* *definable*[778] *over the class* $\mathsf{Mod}(Th)$, *for any set* Th *of formulas in our frame language.*[779]

(ii) $\mathsf{Ge}''(Th)$ *is uniformly first-order* *definable* *over* $\mathsf{Mod}(Th)$, *assuming*

$$Th \models \mathbf{Bax}^- + \mathbf{Ax}(\surd).$$

(iii) $\mathsf{Ge}(Th)$ *is uniformly first-order* *definable* *over* $\mathsf{Mod}(Th)$, *assuming* $n > 2$ *and*

$$Th \models \mathbf{Bax}^{\oplus} + \mathbf{Ax}(\|)^- + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax(diswind)} + \mathbf{Ax}(\surd).$$

(iv) $\mathsf{Ge}(Th)$ *is uniformly first-order* *definable* *over the class* $\mathsf{Mod}(Th)$, *assuming*

$$Th \models \mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\surd).$$

Proof: The theorem is restated and is proved in §6.3 as Theorems 6.3.24 (p.962), 6.3.22 (p.961) and 6.3.23. ∎

[778] cf. ($\star$) in Remark 6.2.8 on p.807 (or for greater detail §6.3)

[779] With the exception of §6.3 Th is in our frame language (i.e. Th denotes an arbitrary set of formulas in our frame language).

We will see that more is true, namely $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are definitionally equivalent[780], in symbols

$$\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th),$$

assuming Th is strong enough[781], cf. Thm.6.6.13 (p.1031).

On the conditions of Thm.6.2.44(iii): The assumption $n > 2$ cannot be omitted by (the proof of) Thm.6.2.24(ii) (p.830). The assumption $\mathbf{Ax(||)}^-$ is needed because of (the proof of) Thm.6.2.24(i). Further we conjecture that **Ax(diswind)** cannot be omitted, cf. Conjecture 6.6.15 on p.1033 and Fig.316 on p.1033.

6.2.3 On the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$

Recall that $\mathbf{Ax}(\mathit{Triv}_t)^-$ is a weakened version of $\mathbf{Ax}(\mathit{Triv}_t)$ and $\mathbf{Ax}(\mathit{Triv})$, and it was introduced on p.812 in the present section. We will need $\mathbf{Ax}(\mathit{Triv}_t)^-$ and $\mathbf{Ax}(\mathit{Triv})$ quite often for the following reason. We defined, roughly speaking, the set L of lines such that something is a line if it "coincides" with a coordinate axis of some inertial observer. Therefore we have rather few lines, i.e. to have enough lines we need $\mathbf{Ax}(\mathit{Triv}_t)^-$. We could have defined lines as sets "parallel" either with the time-axis $\bar{t}$ or with a Euclidean line in the space part S of our space-time for some inertial observer. In that case we would not need $\mathbf{Ax}(\mathit{Triv}_t)^-$ so often. The *only* reason why we did *not* include $\mathbf{Ax}(\mathit{Triv}_t)^-$ into our basic theories like **Basax** or **Basax** + **Ax(symm)** is that we could derive our main theorems (e.g. no FTL observers, Twin Paradox) *even* without $\mathbf{Ax}(\mathit{Triv}_t)^-$. But whenever we need $\mathbf{Ax}(\mathit{Triv}_t)^-$ for something, we will assume it without a second thought.[782]

We will also need **Ax(eqm)** often, where **Ax(eqm)** was defined on p.796. The reason for our needing **Ax(eqm)** is the following: Without **Ax(eqm)**, g could easily become degenerate because g was defined via "min". Further, failure of **Ax(eqm)** can produce strange things, e.g. $(eq(a,b,c,d) \;\Rightarrow\; g(a,b) = g(c,d))$ can fail even

[780] The notion of definitional equivalence will be discussed in §6.3.

[781] The conditions of Thm.6.2.44(iii) together with $\mathbf{Ax}\heartsuit + \mathbf{Ax(ext)} + \mathbf{Ax(eqtime)}$ are sufficient.

[782] Omitting (or weakening) certain axioms of a theory (like $\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp$) of special relativity lead to exciting questions (such an axiom is e.g. **AxE**) but for some other axioms (like e.g. "$tr_m(m) = \bar{t}$" or the other axiom $[\forall p\,(p \in \ell \leftrightarrow p \in \ell_1) \rightarrow \ell = \ell_1]$) this does not seem to be the case. It is our impression that $\mathbf{Ax}(\mathit{Triv}_t)^-$ might belong to this second kind of axioms (though we did not think much about this, so we may be wrong).

in **Basax** without **Ax(eqm)**. (Connections between **Ax(eqm)** and some earlier introduced axioms will be discussed in §6.2.7.)

Discussion of the intuitive meaning of the geometry $\mathfrak{G}_{\mathfrak{M}}$: Intuitively, the points of $\mathfrak{G}_{\mathfrak{M}}$ are the events. The L^T-lines are the life-lines of inertial observers. The L^{Ph}-lines are the life-lines of photons. Intuitively, one could say that the set of space-like lines L^S consists of the life-lines of the potential faster than light *inertial* bodies (which are called tachions in the literature). However, these bodies need not exist in our model $\mathfrak{M}$. But certainly, if there exists an FTL *inertial* body b in a model $\mathfrak{M}$, then the life-line $\{ e \in Mn : b \in e \}$ of b is in L^S, under some assumptions on $\mathfrak{M}$, [783] cf. Prop.6.2.55 (p.858). Two events are $\equiv^T$-related if there is an inertial observer, whose life-line contains both events. This is equivalent to saying that there is an inertial observer who sees them happening at the same place, under mild assumptions[784], cf. Prop.6.2.56(i) (p.858). Two events are $\equiv^{Ph}$-related if they are connected by a photon. Two events are $\equiv^S$-related iff there is an inertial observer who sees them happening at the same time, if we assume **Ax(*Triv*)+Ax($\sqrt{\ }$)**, cf. Prop.6.2.56(ii) (p.858). Assuming **Ax(*Triv*)+Ax($\sqrt{\ }$)+Ax(eqm)+Ax4+Ax6$_{00}$**, the g-distance $g(e, e_1)$ between two events e, e_1 is (i) the Euclidean distance between them if they are simultaneous for some inertial observer, is (ii) the time elapsed between e, e_1 if they are on the life-line of some inertial observer, is (iii) zero if a photon connects them and is (iv) undefined if no inertial observer can see both of them (under some mild assumptions).

Remark 6.2.45 We have seen in earlier sections that (assuming **Ax1**, **Ax2**, **Ax3$_0$**, **Ax4**, **AxE$_{01}$**, **Ax6$_{00}$**) the irreflexive parts of $\equiv^T$ and $\equiv^{Ph}$ are disjoint because no observer moves with the speed of light,[785] hence $(e \neq e_1 \ \wedge \ e \equiv^T e_1) \ \Rightarrow \ e \not\equiv^{Ph} e_1$.

For completeness, we note that there is a tradition in the literature which codes g, $\equiv^T$, $\equiv^{Ph}$ up into one complex-valued (pseudo-metric) function

$$g^+(e, e_1) = \begin{cases} g(e, e_1) & \text{if } e \equiv^T e_1 \text{ or } g(e, e_1) \text{ is undefined} \\ 0 & \text{if } e \equiv^{Ph} e_1 \\ i \cdot g(e, e_1) & \text{otherwise.} \end{cases}$$

Here $i = \sqrt{-1}$ and $g^+ : Mn \times Mn \longrightarrow \mathbf{C}(\mathfrak{F})$, where $\mathbf{C}(\mathfrak{F}) = \mathfrak{F}(i)$ is the field of complex numbers over $\mathfrak{F}$.

[783] like e.g. $\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ })$

[784] e.g. **Ax(*Triv*)+Ax4+Ax6$_{00}$** suffices

[785] More precisely, no observer has the same life-line as a photon.

However, in the present work we will not need g^+ because the information carried by g^+ is recoverable from our structure $\langle Mn, \mathbf{F_1};\, g, \equiv^T, \equiv^{Ph}\rangle$.[786]

$\triangleleft$

Below we continue the discussion of the intuitive meaning of the parts of our geometries. Intuitively, two lines ℓ, ℓ_1 are orthogonal in the relativistic sense (i.e. $\perp_r$-related) if there is an inertial observer m who thinks that they are parallel with two different coordinate axes. There is a slight problem with this intuitive definition because in most of our models $\mathfrak{M}$ no line will be orthogonal to photon-like lines. To help this we introduced a limit construction in our definition of $\perp_r$. We refer to Remark 6.2.6 (pp. 802–805) for intuitive motivation (and cosiderations) for our using limits in the definition of $\perp_r$. If we assume $\mathbf{Bax}^{\oplus}$ and some mild assumptions then our relativistic orthogonality gets very close to the usual Minkowskian orthogonality, cf. Thm.6.2.64 (p.866). On the other hand if we do not assume $\mathbf{Bax}^{\oplus}$, then the relativistic orthogonality $\perp_r$ can behave in quite interesting, unusual ways. E.g. in **NewtK** geometries two lines are orthogonal iff at least one of them is space-like. Further, there is a $\mathbf{Bax}^{-\oplus}$ geometry with two parallel space-like lines which are $\perp_r$-orthogonal, see Figure 280. (The "meanings" of $\perp_r, L^T, L^S, \equiv^T, \equiv^S, \ldots$ will be discussed in items 6.2.48–6.2.57 below, pp. 854–858). Betweenness Bw and equidistance eq are the usual geometric relations used e.g. by Hilbert [133, 134]. $Bw(a,b,c)$ means that some inertial observer thinks that event b is between events a and c. Intuitively, $eq_0(a,b,c,d)$ means[787] that segments $\langle a,b\rangle$ and $\langle c,d\rangle$ have the same length, for some inertial observer (and this observer sees these segments on coordinate axes). Further, $eq(a,b,c,d)$ means that there is a finite chain of inertial observers such that they together (in a kind of collaboration) think that segments $\langle a,b\rangle$ and $\langle c,d\rangle$ have the same length, see Figure 266 on p.795. Further, $a \prec b$ means that there is an inertial observer who thinks that a happened earlier than b *and* who sees both a and b on his life-line.

The reader may ask what the role of the constant $1 \in \mathbf{F_1}$ is in the geometry $\mathfrak{G}_{\mathfrak{M}}$. Clearly the role of $\mathbf{F_1}$ is to represent the range of g as a special sort (or universe), but for this purpose the additive group $\mathbf{F_0} := \langle F;\, 0, +, \leq\rangle$ would be sufficient. The answer is the following. Later, in §6.6, we will experiment with reconstructing the "observational-oriented" models $\mathfrak{M}$ from the observer-independent geometries $\mathfrak{G}_{\mathfrak{M}}$.

[786] In the relativity book d'Inverno [75, pp. 107-108], our g^+ is called a *Minkowski metric* (and is denoted as η_{ab}). More precisely, the square $(g^+)^2$ of g^+ is called there a Minkowski metric, we guess that this is done there in order to avoid complex numbers. (It is important to note that a Minkowski metric is not a metric [in the usual sense] cf. footnote 669 on p.797.)

[787] Recall that eq was defined as the transitive closure of eq_0. Hence eq_0 can be considered as a kind of "core" of eq.

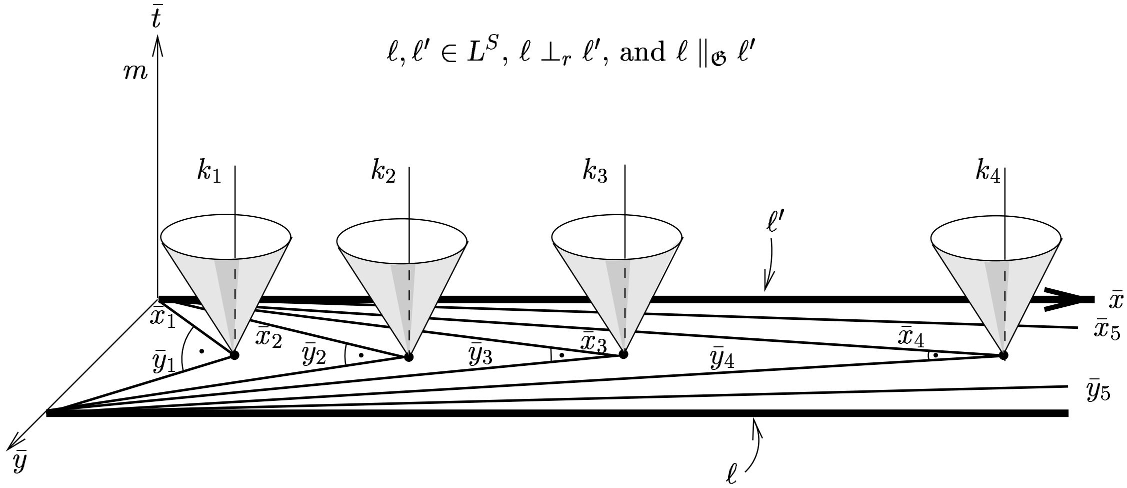

Figure 280: $\mathbf{Bax}^{-\oplus}$ geometry with two parallel space-like lines which are $\perp_r$-orthogonal.

The role of the constant 1 is to help us to reconstruct the "units of measurement" or in other words "the size of a hydrogen atom" (cf. p.139) in $\mathfrak{M}$ from $\mathfrak{G}_{\mathfrak{M}}$, at least to some extent (and under some conditions). E.g., under assuming **Ax(eqm)**, we can reconstruct the units of measurement of $\mathfrak{M}$ from $\mathfrak{G}_{\mathfrak{M}}$, cf. e.g. Thm.6.6.12 (p.1030). In passing we note that as "patterns" (A)-(E) on p.1009 suggest, there will be stronger results of "recoverability" than the just quoted one, in later parts of §6.6.1. We will return to the present subject (the role of "1" etc) in more detail in §6.6. (Cf. e.g. Remark 6.6.51, p.1065.) In particular, we will discuss how much of $\mathfrak{M}$ is recoverable from $\mathfrak{G}_{\mathfrak{M}}$ without using 1 in the form of a "duality theory" called in 6.6.4 (p.1069) $(\mathcal{G}o, \mathcal{M}o)$-duality.[788]

Summing up, the role of $1 \in \mathbf{F_1}$ is to help us to recover the units of measurement (in $\mathfrak{M}$) from the geometry $\mathfrak{G}_{\mathfrak{M}}$. (Referring back to the intuitive explanation using hydrogen atoms in §2.8 on p.139 [about justification of **Ax(symm)**], we could say that the constant "1" helps us to remember in $\mathfrak{G}_{\mathfrak{M}}$ what the "size of a hydrogen atom" was in $\mathfrak{M}$.)

Let us turn to discussing why we "celebrate" the observer-independent character

[788]Forgetting 1 from $\mathfrak{G}_{\mathfrak{M}}$ is related to what we called in Remark 4.2.1 on p.458 "ant and elephant version of relativity" which we plan to outline in some future work.

of $\mathfrak{G}_{\mathfrak{M}}$. In answering this question we will deliberately mix talking about $\mathfrak{G}_{\mathfrak{M}}$ and its (first-order logic) theory $\mathsf{Th}(\mathfrak{G}_{\mathfrak{M}})$.[789]

(1) Much of what we should say about this was already said in the introduction §6.1 (of this chapter). We will not repeat those thoughts here, the reader is asked to have a look in §6.1.

(2) Clearly $\mathfrak{G}_{\mathfrak{M}}$ is the same for all observers.

(3) By the duality theory to be developed later in §6.6, all the information available in $\mathfrak{M}$ is also available in $\mathfrak{G}_{\mathfrak{M}}$,[790] so we do not loose information when switching to $\mathfrak{G}_{\mathfrak{M}}$.

(4) $\mathfrak{G}_{\mathfrak{M}}$ satisfies certain important, desirable philosophical principles (e.g. the one saying that all our concepts should be definable from observational ones, associated to Occam's razor[791]). These principles were already satisfied by $\mathfrak{M}$, and $\mathfrak{G}_{\mathfrak{M}}$ inherits from $\mathfrak{M}$ because $\mathfrak{G}_{\mathfrak{M}}$ is first-order logic definable over $\mathfrak{M}$ (under some conditions).[792]

(5) We will see around the end of this chapter that $\mathfrak{G}_{\mathfrak{M}}$ admits mathematically elegant streamlined versions (cf. e.g. the time-like-metric geometry $\langle Mn, \mathbf{F_1}; g^{\prec}\rangle$ in §6.7.3 p.1169 as an example). These streamlined versions of $\mathfrak{G}_{\mathfrak{M}}$ provide us with a simple, mathematically elegant and transparent picture of the world (which in many regards is simpler and more elegant than $\mathfrak{M}$).

(6) $\mathfrak{G}_{\mathfrak{M}}$ provides us with a stepping-stone towards theories admitting accelerated observers and beyond that towards general relativity. Cf. e.g. §6.8 on geodesics.

(7) In some sense one feels that $\mathfrak{G}_{\mathfrak{M}}$ represents "deeper" more essential aspects of the world than $\mathfrak{M}$ does. One could say that the ingredients of $\mathfrak{M}$ are the things one sees on the "surface" of the phenomena or reality being studied while $\mathfrak{G}_{\mathfrak{M}}$ contains ingredients which make these surface phenomena "tick". In some sense one could say that $\mathfrak{G}_{\mathfrak{M}}$ contains something which could be regarded as "explanation" for $\mathfrak{M}$ (where explanation is understood in the sense of Friedman [90]). Cf. footnote 627 on p.776.

(8) The various reducts of $\mathfrak{G}_{\mathfrak{M}}$ provide us with aspects of the world which we can contemplate. So for a while we may decide to concentrate on one aspect (represented by one reduct) and ignore the others. Then we can experiment with how far we can get by concentrating on this aspect. Later we may concentrate on some other aspect (reduct). Eventually we can compare the results (and try to obtain insight into what aspect is responsible for what effect). In other words this provides us

[789] Or more precisely $\mathsf{Th}(\{\, \mathfrak{G}_{\mathfrak{M}} : \mathfrak{M} \models Th_1 \,\})$ for some fixed Th_1.

[790] under some mild conditions

[791] For further desirable philosophical principles satisfied by $\mathfrak{G}_{\mathfrak{M}}$ we refer the reader to the introduction of the present chapter (§6.1).

[792] Cf. Thm.6.2.44 (p.847). (In this respect we do not gain over $\mathfrak{M}$ but we do not loose either.)

with the machinery of "abstraction".[793] For more on this ("decomposing" the world into reducts etc.) cf. the first 5 lines of §6.6.4 (p.1069), pp. 1134–1135, and p.1124. (Notice, that the same kind of "decomposability" is not available in the original structures the $\mathfrak{M}$'s.)

(9) $\mathfrak{G}_{\mathfrak{M}}$ may be helpful in comparing the various observers, seeing their relationships with each other. We feel that this is so because in $\mathfrak{G}_{\mathfrak{M}}$ when, say, we are thinking about e.g. 3 inertial observers simultaneously we are not forced to do this from the world-view of some particular observer, instead we can look at our 3 observers from, so to speak, the "objective" perspective of $\mathfrak{G}_{\mathfrak{M}}$. As a contrast when working in $\mathfrak{M}$ we always have to choose an observer and we have to describe things from his particular perspective. This may make e.g. proofs longer (because we might have to switch perspectives).

(10) For more on why we celebrate the observer independent character of $\mathfrak{G}_{\mathfrak{M}}$ we refer to the book Matolcsi [190].

At this point we stop listing values of $\mathfrak{G}_{\mathfrak{M}}$.[794]

Remark 6.2.46 (On the philosophy of our using inertial and not necessarily inertial observers in the definition of $\mathfrak{G}_{\mathfrak{M}}$ above.)
Before starting, we note that later we will have so called windows in $\mathfrak{G}_{\mathfrak{M}}$. Roughly a window is a part of *Mn* visible for one observer.

Now, what we want to say about the "philosophy ..." is the following: (i) Everything that is "measured" (like e.g. g or $\perp_r$) (by observers of course) is defined

[793] Decompose the world into aspects, study the aspects separately and in their interaction and then put together the results.

[794] For completeness we note the following:

Many of the so called thought experiments can be translated to the language of $\mathfrak{G}_{\mathfrak{M}}$, and the outcome of the thought experiment can be predicted by knowing $\mathfrak{G}_{\mathfrak{M}}$, cf. "laws of nature" part of the introduction to the present chapter (p.778). An example for this is the so called twin paradox, assuming e.g. $n > 2$ and $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(eqtime)}$. For the case $\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp}+\mathbf{Ax}(\uparrow\uparrow)$,[795] the importance of $\mathfrak{G}_{\mathfrak{M}}$ is further elaborated in e.g. Misner-Thorne-Wheeler [196, pp. 3–47, 163–175].

The usefulness of $\mathfrak{G}_{\mathfrak{M}}$ will be especially apparent when we will turn to discussing non-inertial observers. As an illustration, let us assume, that we have a body b whose life-line is not in L^T. Assume, we would like to raise b to the level of being an observer. For simplicity, assume $n = 2$. Then b would like to coordinatize the "events" *Mn*. That is we would like to define a function $w_b : {}^2F \longrightarrow Mn$. Using $\mathfrak{G}_{\mathfrak{M}}$, there is a natural way for doing this,[796] cf. e.g. Misner-Thorne-Wheeler [196, pp. 163–175].

[795] We note that (for $n > 2$) the members of $\mathsf{Ge}(\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp}+\mathbf{Ax}(\uparrow\uparrow))$ are the Minkowskian geometries, up to isomorphism, cf. Def.6.2.58 (p.859) and Thm.6.2.59 (p.861).

[796] This does not contradict what we will say in §6.6(V) on pp. 1111–1120 (... Gödel incompleteness) about undefinability of non-inertial bodies. (The reason for this is that these two claims about definability "live" on two different levels of abstraction.)

via $Obs \cap Ib$. As a contrast; (ii) windows, existence of events (ontology of Mn) are defined via Obs (i.e. all observers).

(iii) Cf. also the definition of $\mathfrak{G}^*_{\mathfrak{M}}$ in §6.6.9 p.1111.

$\triangleleft$

We will discuss the connections with the standard Minkowskian geometry beginning with p.859 in §6.2.4.

Remark 6.2.47 (On Figure 281 [view from the black hole]) Later, in generalizations towards general relativity, our geometry $\mathfrak{G}$ will be more sophisticated than the present $\mathfrak{G}_{\mathfrak{M}}$. E.g. life-lines of photons (and other inertial bodies) will be so called geodesics instead of Euclidean lines. Geodesics will be discussed in § 6.8, pp.1177-1209. Figure 281 on p.855 represents some spectacular effect caused by geodesics being curved by a black hole.

$\triangleleft$

In items 6.2.48–6.2.57 below we continue discussing the meanings of L, L^T, L^S, $\equiv^T$, $\equiv^S$, $\perp_r$ etc. (These items can be considered as warm-up exercises for later work.) The reader may safely skip the remaining part of the present sub-section. Our next sub-section (§6.2.4) begins on p.859.

The proposition below says that a pre-image of a line along a world-view, say w_m, is a Euclidean line in ${}^nF$ or empty, under some assumptions. I.e. it connects lines of Mn to lines of ${}^nF$.

PROPOSITION 6.2.48 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$. *Let* $\ell \in L_{\mathfrak{M}}$ *and* $m \in Obs$. *Then* $w_m^{-1}[\ell] \in (\mathsf{Eucl} \cup \{\emptyset\})$.

On the proof: Cf. item 1a of Prop.6.2.79 (p.884) and the proof of Prop.6.2.79 (p.889). ∎

The following proposition says that the lines of ${}^nF$ correspond to lines of $\mathfrak{G}_{\mathfrak{M}}$ (along all world-views).

PROPOSITION 6.2.49 *Assume*
$\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Then*

$$(\forall m \in Obs)(\forall \ell_e \in \mathsf{Eucl})\ (\exists \ell \in L_{\mathfrak{M}}) \quad w_m[\ell_e] = \ell \quad \textit{and}$$

$$(\forall \ell \in L_{\mathfrak{M}})\ (\exists m \in Obs)(\exists \ell_e \in \mathsf{Eucl}) \quad w_m[\ell_e] = \ell.$$

Therefore $\quad (\forall e, e_1 \in Mn) \Big(e \sim e_1 \ \rightarrow \ (e \equiv^T e_1 \ \vee \ e \equiv^{Ph} e_1 \ \vee \ e \equiv^S e_1)\Big).$

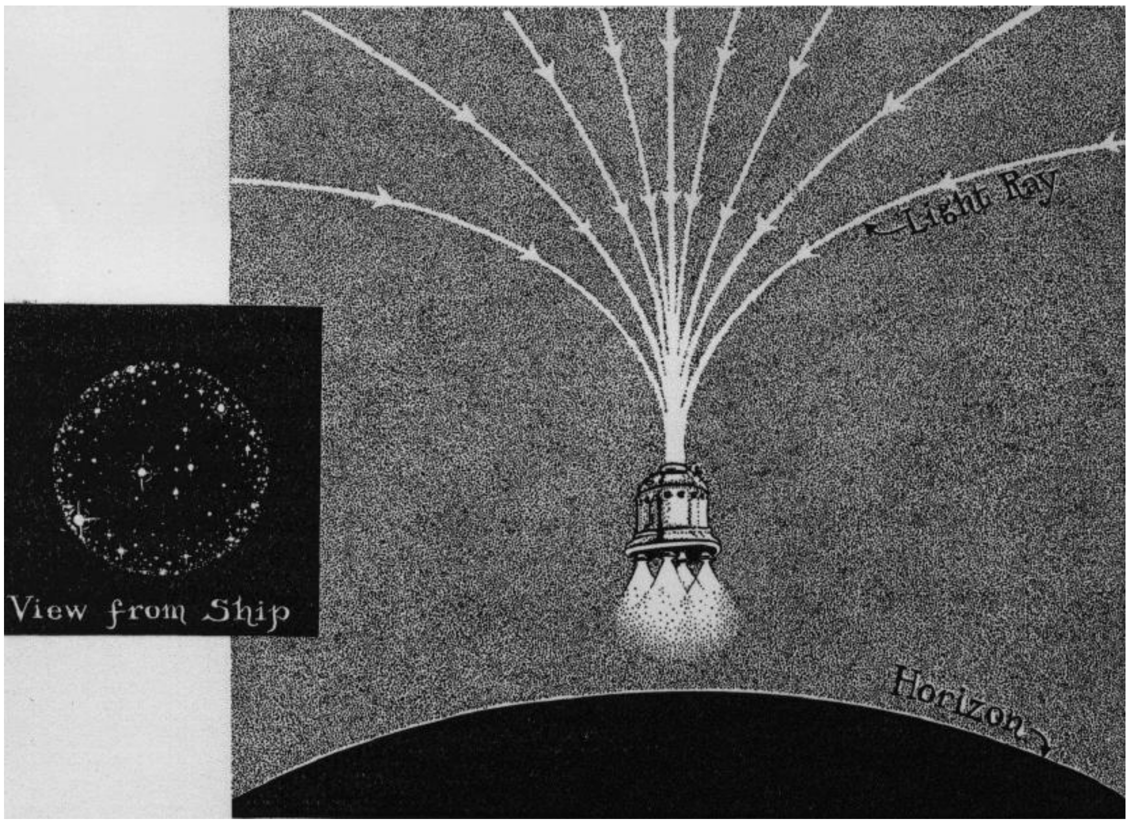

Figure 281: The starship hovering above the black-hole horizon, and the trajectories along which light travels to it from distant galaxies (the light rays). The hole's gravity deflects the light rays downward ("gravitational lens effect"), causing humans on the starship to see all the light concentrated in a bright, circular spot overhead.

Proof: By the beginning of the second proof given for Thm.6.2.10 on p.815, to prove the proposition it is enough to prove its conclusion for $\mathbf{Basax}+\mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^-+\mathbf{Ax}(\sqrt{\ })$ models. We leave this to the reader. ∎

The next proposition says that two lines of $\mathfrak{G}_{\mathfrak{M}}$ are parallel iff each observer who sees them thinks that they are parallel.

PROPOSITION 6.2.50 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$. *Let* $\ell, \ell' \in L_{\mathfrak{M}}$. *Then*

$$\ell \parallel_{\mathfrak{G}} \ell' \iff (\forall m \in Obs)\Big((\, w_m^{-1}[\ell] \neq \emptyset \ \wedge\ w_m^{-1}[\ell'] \neq \emptyset \,) \ \Rightarrow\ w_m^{-1}[\ell] \parallel w_m^{-1}[\ell'] \Big).$$

On the proof: Cf. item 5a of Prop.6.2.79 (p.884) and the proof of Prop.6.2.79 on p.889. ∎

Cf. item 5a of Prop.6.2.79 in connection with the above proposition.

Let us recall that $S := \{0\} \times^{n-1} F$ is the space-part of our coordinate system ${}^nF$.

In connection with Proposition 6.2.51 below let us recall that $\perp_0$ was the key part in the definition of $\perp_r$, cf. item 11 of Def.6.2.2 (p.790). Therefore in order to characterize $\perp_r$ it is enough to characterize $\perp_0$.

PROPOSITION 6.2.51 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax}(\sqrt{\ }))$. *Let* $\ell, \ell' \in L_{\mathfrak{M}}$. *Assume* $\ell \cap \ell' \neq \emptyset$. *Then* $\ell \perp_0 \ell'$ *iff* $(\star)$ *below holds.*

$(\star)$ $\quad (\exists m \in Obs \cap Ib)$ $\Big[$ *m thinks that* $\Big($ *ℓ and ℓ' are Euclidean lines such that they are orthogonal in the Euclidean sense and* $(\ell \parallel \bar{t} \vee \ell \parallel S)$ *and* $(\ell' \parallel \bar{t} \vee \ell' \parallel S)$ $\Big)$ [797] $\Big]$.

We omit the easy **proof**. ∎

[797] Formally, we want to say that $(\exists \ell_e, \ell'_e \in \mathsf{Eucl})\ [w_m[\ell_e] = \ell \ \wedge\ w_m[\ell'_e] = \ell' \ \wedge\ \ell_e \perp_e \ell'_e \ \wedge\ (\ell_e \parallel \bar{t} \vee \ell_e \parallel S) \ \wedge\ (\ell'_e \parallel \bar{t} \vee \ell'_e \parallel S)]$, where for each $\ell_e \in \mathsf{Eucl}$, $\ell_e \parallel S$ means that ℓ_e is parallel with a line in S.

PROPOSITION 6.2.52 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax(}\mathit{Triv}_t\mathbf{)}^{-} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(diswind)})$. *Let* $\ell, \ell' \in L_{\mathfrak{M}}$. *Then*

$$\ell \perp_r \ell'$$
$$\Updownarrow$$
$$(\exists \ell_1 \in L_{\mathfrak{M}}) \Big(\ell \parallel_{\mathfrak{G}} \ell_1 \quad \wedge \quad (\textit{one of (i)-(iii) below holds})\Big).$$

(i) There is $m \in Obs$ *and* $i, j \in n$ *with* $i \neq j$ *such that* $w_m[\bar{x}_i] = \ell_1$ *and* $w_m[\bar{x}_j] = \ell'$.

(ii) $\ell_1 \in L^{Ph}$ *and there is a 2-dimensional plane*[798] P *tangent to a light-cone*[799] *such that* $\ell_1, \ell' \subseteq P$.

(iii) Same as (ii) but with ℓ_1, ℓ' *interchanged.*

On the proof: A proof can be obtained by Thm.6.2.10 (p.813) and by the proof of Thm.6.2.19 (p.823). Cf. Figure 272 (p.825). ∎

PROPOSITION 6.2.53 *Proposition 6.2.52 remains true if we replace (i) in Prop.6.2.52 with* ($\star$) *of Prop.6.2.51.*

We omit the **proof**. ∎

We suggest that the reader compare items 6.2.51–6.2.53 with items 6.2.9, 6.2.10 (pp. 810–813).

PROPOSITION 6.2.54 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Ax(}\mathit{Triv}\mathbf{)})$. *Let* $\ell \in L_{\mathfrak{M}}$. *Then (i) and (ii) below hold.*

(i) *Assume* $\mathbf{Ax4} + \mathbf{Ax6_{00}}$. *Then*

$$\ell \in L^T_{\mathfrak{M}} \quad \Leftrightarrow \quad (\exists m \in Obs \cap Ib)\,(m \textit{ thinks that } \ell \textit{ is parallel with the time axis } \bar{t}\,)^{800}.$$

(ii) *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Then*

$$\ell \in L^S_{\mathfrak{M}} \;\Leftrightarrow\; (\exists m \in Obs \cap Ib)\,(m \textit{ thinks that } \ell \textit{ is parallel with } S)^{801}.$$

We omit the easy **proof**. ∎

---

[798] The notion of a 2-dimensional plane is defined as follows. P is a 2-dimensional plane iff there are distinct $a, b, c \in Mn$ such that they are pairwise connected (i.e. $\sim$-related), $\neg coll(a, b, c)$, and $P = Plane(\{a, b, c\})$.

[799] This property of P can be formalized as follows:
$$(\exists \ell \in L^{Ph})\,[\,\ell \subseteq P \;\wedge\; (\forall \ell' \in L^{Ph})\,(\ell' \subseteq P \;\rightarrow\; \ell' \parallel_{\mathfrak{G}} \ell)\,].$$

[800] Formally, $(\exists \ell_e \in \mathsf{Eucl})\,(w_m[\ell_e] = \ell \;\wedge\; \ell_e \parallel \bar{t}\,)$.

[801] Formally, $(\exists \ell_e \in \mathsf{Eucl})\,(w_m[\ell_e] = \ell \;\wedge\; \ell_e \parallel S)$.

PROPOSITION 6.2.55 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^{\oplus} + \mathbf{Ax}(\mathit{Triv}_t)^{-} + \mathbf{Ax}(\sqrt{\ }))$. *Assume* b *is an FTL inertial body of* $\mathfrak{M}$, *i.e.* $v_m(b) > c_m$, *for some observer* m. *Let this* m *be fixed. Then (i) and (ii) below hold.*

(i) $w_m[tr_m(b)] \in L^S_{\mathfrak{M}}$.

(ii) *Assume* **Ax6** *or that our fixed* m *and* b *are such that* $(\forall k \in Obs)\,(k \overset{\odot}{\to} b \;\Rightarrow\; m \overset{\odot}{\to} k)$. *Then*

$$\{\, e \in Mn_{\mathfrak{M}} \;:\; b \in e \,\} \in L^S_{\mathfrak{M}}.$$

We omit the easy **proof**. ∎

PROPOSITION 6.2.56 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Ax}(\mathit{Triv}))$. *Let* $e, e_1 \in Mn_{\mathfrak{M}}$. *Then (i) and (ii) below hold.*

(i) *Assume* $\mathbf{Ax4} + \mathbf{Ax6_{00}}$. *Then*

$$e \equiv^T e_1 \quad \Leftrightarrow \quad (\exists m \in Obs \cap Ib)\,(\, m \text{ sees } e \text{ and } e_1 \text{ happening at the same place}\,)$$[802].

(ii) *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Then*

$$e \equiv^S e_1 \;\Leftrightarrow\; (\exists m \in Obs \cap Ib)\,(\, m \text{ thinks that } e \text{ and } e_1 \text{ are simultaneous}\,)$$[803].

We omit the easy **proof**. ∎

The following is a corollary of Thm.3.4.19 (p.221), which says that **Bax** does not allow FTL observers, assuming $n > 2$. The corollary says that, assuming $\mathbf{Bax}^{\oplus}$, the time-like lines, the photon-like lines and the space-like lines do not run "together" anywhere.

COROLLARY 6.2.57 Assume $n > 2$. For every $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Bax}^{\oplus})$, we have that L^T, L^{Ph}, L^S are pairwise disjoint. Therefore the irreflexive parts of relations $\equiv^T, \equiv^{Ph}, \equiv^S$ are pairwise disjoint. ∎

Cf. item 4e of Prop.6.2.79 (p.889) in connection with the above corollary.

[802] Formally: $(\exists p, q \in {}^nF)\,(w_m(p) = e \;\wedge\; w_m(q) = e_1 \;\wedge\; space(p) = space(q))$.
[803] Formally: $(\exists p, q \in {}^nF)\,(w_m(p) = e \;\wedge\; w_m(q) = e_1 \;\wedge\; time(p) = time(q))$.

6.2.4 Connections with the standard Minkowskian geometry

The style of our above definition of $\mathfrak{G}_{\mathfrak{M}}$ followed a certain kind of intuition e.g. (i) events e, e_1 are defined to be spatially separated iff some inertial observer thinks that e and e_1 happened at the same time; and (ii) for events e and e_1 the relation $e \prec e_1$ is defined to hold iff some inertial observer thinks that e precedes e_1 in time (and sees e, e_1 on his life-line); etc. In general, we tried to achieve the effect that, intuitively, some relation holds between given objects iff some inertial observer thinks this is so (sometimes we had to take "min" or limits to complete the picture, but this was the general intuition).

As a contrast, in Definition 6.2.58 below, for every Euclidean $\mathfrak{F}$, we define a geometry on ${}^nF$ in a "computational" style. According to the literature[804] we call this geometry the Minkowskian geometry over $\mathfrak{F}$.

In Thm.6.2.59 below (p.861), we will see that our "intuition-oriented" definition of $\mathfrak{G}_{\mathfrak{M}}$ is equivalent with the standard Minkowskian definition mentioned above, under some assumptions on $\mathfrak{M}$. Further, if $n > 2$, the observer-independent geometries (in our sense[805]) of the Minkowski models (the latter is defined in §3.8) will turn out to coincide with the Minkowskian geometries, up to isomorphism, cf. Prop.6.2.62, p.865. (In §6.2.5 we will see that relativistic geometries corresponding to many of our theories can be obtained as "unions" of Minkowskian geometries if we concentrate on a reduct of our geometries, only. Cf. Figures 282, 283, pp. 863, 864.)

Definition 6.2.58 (Minkowskian geometry)
Assume $\mathfrak{F}$ is Euclidean. Then the n-dimensional *Minkowskian geometry* over $\mathfrak{F}$ is defined as follows.

$$Mink(n,\mathfrak{F}) :\stackrel{\text{def}}{=} Mink(\mathfrak{F}) :\stackrel{\text{def}}{=} \langle {}^nF, \mathbf{F_1}, L_\mu;\ L^T_\mu, L^{Ph}_\mu, L^S_\mu, \in, \prec_\mu, Bw_\mu, \perp_\mu, eq_\mu, g_\mu, \mathcal{T}_\mu\rangle;$$

where:

- $\mathbf{F_1} :\stackrel{\text{def}}{=} \langle F;\ 0, 1, +, \leq\rangle$, as defined in Def.6.2.2.
- $L_\mu :\stackrel{\text{def}}{=} \mathsf{Eucl}(n, \mathbf{F}) := \mathsf{Eucl}$.
- $L^T_\mu :\stackrel{\text{def}}{=} \mathsf{SlowEucl}$.

[804] cf. e.g. Kostrikin-Manin [155], cf. also Goldblatt [108]
[805] in the sense of Def.6.2.2

- $L_\mu^{Ph} :\stackrel{\text{def}}{=} \mathsf{PhtEucl}$.
- $L_\mu^S :\stackrel{\text{def}}{=} L_\mu \setminus (L_\mu^T \cup L_\mu^{Ph})$.
- $\prec_\mu$ is a binary relation on ${}^nF$ defined as follows. Let $p, q \in {}^nF$. Then

$$p \prec_\mu q \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (p_t < q_t \quad \wedge \quad \overline{pq} \in \mathsf{SlowEucl}).$$

- $Bw_\mu = \mathsf{Betw}$.
- The _Minkowskian orthogonality_ $\perp_\mu \subseteq L_\mu \times L_\mu$ is defined as follows. Let $\ell, \ell' \in L_\mu$. Then

$$\begin{gathered} \ell \perp_\mu \ell' \\ \stackrel{\text{def}}{\Longleftrightarrow} \\ (\forall \text{ distinct } p, q \in \ell)(\forall \text{ distinct } p', q' \in \ell') \\ (p_0 - q_0)(p_0' - q_0') - \left(\sum_{0<i\in n}(p_i - q_i)(p_i' - q_i')\right) = 0. \end{gathered}$$

  If $\ell \perp_\mu \ell'$ then we say that ℓ and ℓ' are _Minkowski-orthogonal_.

- Let us recall that $g_\mu^2 : {}^nF \times {}^nF \longrightarrow F$ is the square of the Minkowski-distance defined in Def.2.9.1.

  We define the _Minkowski distance_ $g_\mu : {}^nF \times {}^nF \longrightarrow F$ as follows[806]. Let $p, q \in {}^nF$. Then

$$g_\mu(p, q) :\stackrel{\text{def}}{=} \sqrt{g_\mu^2(p, q)}.$$ [807]

- eq_μ is a 4-ary relation on ${}^nF$ defined as follows. Let $p, q, p', q' \in {}^nF$. Then

$$\begin{gathered} eq_\mu(p, q, p', q') \\ \stackrel{\text{def}}{\Longleftrightarrow} \\ \Big(g_\mu(p, q) = g_\mu(p', q') \ \wedge \ [\, g_\mu(p, q) = 0 \ \Rightarrow \ (\, p = q \ \wedge \ p' = q'\,)\,]\Big). \end{gathered}$$ [808]

[806] exactly as we did above Prop.6.2.38 on p.844

[807] Cf. footnote 775 on p.844.

[808] We need the subformula "$g_\mu(p, q) = 0 \ \Rightarrow \ \ldots$" only because in our definition of eq by some accident we had the side effect that photon-like separated pairs of points are not eq-related even to themselves, cf. footnote 660 on p.793. Further, because we want to make our definition (of $\mathfrak{G}_{\mathfrak{M}}$) comparable with the Minkowskian definition (i.e. with $Mink(\mathfrak{F})$).

- $\mathcal{T}_\mu$ is defined by g_μ as described in item 14 of Def.6.2.2 (p.797).

We will sometimes omit the subscript μ from L_μ etc. because the vocabulary or similarity type of Minkowskian geometries is the same as that of relativistic geometries. $\lhd$

Assume $\mathfrak{M} \models$ **Basax**. Then for each $m \in Obs$, the bijection $w_m : {}^nF \longrightarrow Mn$ can be used to "copy" the geometry $Mink(\mathfrak{F}^{\mathfrak{M}})$ to Mn (as its new universe, i.e. as its new set of points), yielding a geometry $Mink^m_{\mathfrak{M}}$. However for different observers m, this geometry might be different (though isomorphic), because different observers might copy $Mink(\mathfrak{F}^{\mathfrak{M}})$ *differently* to Mn. Assume further $\mathfrak{M} \models \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$. Then the observers will *agree* on how to copy $Mink(\mathfrak{F}^{\mathfrak{M}})$. Formally,

$$(\forall m, k \in Obs) Mink^m_{\mathfrak{M}} = Mink^k_{\mathfrak{M}},$$

assuming $\mathfrak{M}$ satisfies the mentioned axioms. This is essentially what Thm.6.2.59 below says.[809]

Assume now $\mathfrak{M} \models \mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$. Then we could define a Minkowskian geometry on Mn as follows:

$$Mink_{\mathfrak{M}} := Mink^m_{\mathfrak{M}}$$

for an arbitrary but fixed $m \in Obs$. Our Thm.6.2.59 below says that

$$Mink_{\mathfrak{M}} = \mathfrak{G}_{\mathfrak{M}},$$

assuming $n > 2$. To keep the number of defined symbols in this work relatively small, we will *not* rely on the notation $Mink_{\mathfrak{M}}$ in the rest of this work (at least not without recalling it).

THEOREM 6.2.59 *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow))$. *Then (i)–(iii) below hold.*

(i) *Let* $n > 2$. *Then*

$$\mathfrak{G}_{\mathfrak{M}} \cong Mink(\mathfrak{F}^{\mathfrak{M}}),$$

cf. Figures 282, 283.

Moreover, for every $m \in Obs$, $w_m : {}^nF \longrightarrow Mn_{\mathfrak{M}}$ *induces an isomorphism between* $Mink(\mathfrak{F}^{\mathfrak{M}})$ *and* $\mathfrak{G}_{\mathfrak{M}}$ *the natural way.*[810]

[809] Actually, this idea of somehow identifying ${}^nF$ with Mn via some observer's world-view can be pushed through even in $\mathbf{Bax}^-$, since we have seen that the world-view transformations are line preserving, cf. Def.6.2.76 (p.880) and Prop.6.2.79 (p.884).

[810] Making this precise: Let $m \in Obs$. Let $\widetilde{w_m} : \mathsf{Eucl} \longrightarrow Mn_{\mathfrak{M}}$ be defined by $\widetilde{w_m} : \ell \mapsto w_m[\ell]$. Then $\langle w_m, \mathrm{Id} \restriction F, \widetilde{w_m} \rangle$ is a (three-sorted) isomorphism between $Mink(\mathfrak{F}^{\mathfrak{M}})$ and $\mathfrak{G}_{\mathfrak{M}}$, cf. item (II) of Def.6.2.2 (p.798) for the notion of an isomorphism between geometries.

(ii) *Let $n = 2$. Then the conclusion of (i) remains true with the exception of* eq, *i.e. instead of $\mathfrak{G}_{\mathfrak{M}}$ we have to talk about the* eq*-free reduct of $\mathfrak{G}_{\mathfrak{M}}$. The conclusion of (i) will not remain true if we do not exclude* eq *from our geometries.*

(iii) *The statement in item (i) remains true if we replace the assumption* $\mathbf{Ax}(\omega)^\sharp$ *with* $\mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}$, *where* $\mathbf{Ax}$ *is any one of* $\mathbf{Ax}(\omega)$, $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax}(\omega)^{00}$, $\mathbf{Ax}(\omega)^{\sharp\sharp}$, $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle1+Ax(eqtime)}$, $\mathbf{Ax\triangle2}$, $\mathbf{Ax\Box1+Ax(eqtime)}$, $\mathbf{Ax\Box2}$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}$.

The **proof** is available from Judit Madarász.[811] ∎

The following theorem says that, if $n > 2$, the $\prec$-free reduct of any $\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp\sharp}$ geometry coincides with the similar reduct of a Minkowskian geometry. In connection with the conditions of Theorems 6.2.59 and 6.2.60 we recall that $\mathbf{Ax}(\omega)^{\sharp\sharp}$ is weaker than $\mathbf{Ax}(\omega)^\sharp$. In Thm.6.2.59 we needed the assumption $\mathbf{Ax}(\omega)^\sharp$ for the $n = 2$ case only; for the $n > 2$ case $\mathbf{Ax}(\omega)^{\sharp\sharp}$ was sufficient.

THEOREM 6.2.60 *Assume $n > 2$. Then (i) and (ii) below hold.*

(i) *Assume $\mathfrak{G} \in \mathsf{Ge}(\mathbf{Basax}+\mathbf{Ax}(\omega)^{\sharp\sharp})$. Then the $\prec$-free reduct of $\mathfrak{G}$ coincides with the similar reduct of a Minkowskian geometry, up to isomorphism, i.e. there is a Euclidean $\mathfrak{F}$ such that*

$$(\prec\text{-free reduct of } \mathfrak{G}) \;\cong\; (\prec\text{-free reduct of } Mink(\mathfrak{F})),$$

cf. Figures 282, 283.

(ii) *The statement in item (i) remains true if we replace the assumption* $\mathbf{Ax}(\omega)^{\sharp\sharp}$ *with* $\mathbf{Ax}(\mathit{Triv}_t)^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}$, *where* $\mathbf{Ax}$ *is any one of* $\mathbf{Ax}(\omega)$, $\mathbf{Ax}(\omega)^0$, $\mathbf{Ax}(\omega)^{00}$, $\mathbf{Ax}(\omega)^\sharp$, $\mathbf{Ax(syt_0)}$, $\mathbf{Ax(symm)}$, $\mathbf{Ax(speedtime)}$, $\mathbf{Ax\triangle1+Ax(eqtime)}$, $\mathbf{Ax\triangle2}$, $\mathbf{Ax\Box1+Ax(eqtime)}$, $\mathbf{Ax\Box2}$, $\mathbf{Ax(eqspace)}$, $\mathbf{Ax(eqm)}$.

The **proof** is available from Judit Madarász. ∎

Roughly, the following proposition says that, assuming $\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow)$, the world-view transformations f_{mk} are exactly those automorphisms of the observer independent geometry $\mathfrak{G}_{\mathfrak{M}}$ which leave the sort F pointwise fixed, cf. items (iii) and (iv) of the proposition. Let us notice that this means, basically, that the world-view transformations of $\mathfrak{M}$ coincide with the (nice) automorphisms of $\mathfrak{G}_{\mathfrak{M}}$. In connection with the proposition below cf. §6.2.8.

[811] In connection with item (ii) of Thm.6.2.59 cf. the first 8 lines of the proof of Thm.6.2.22 on p.906.

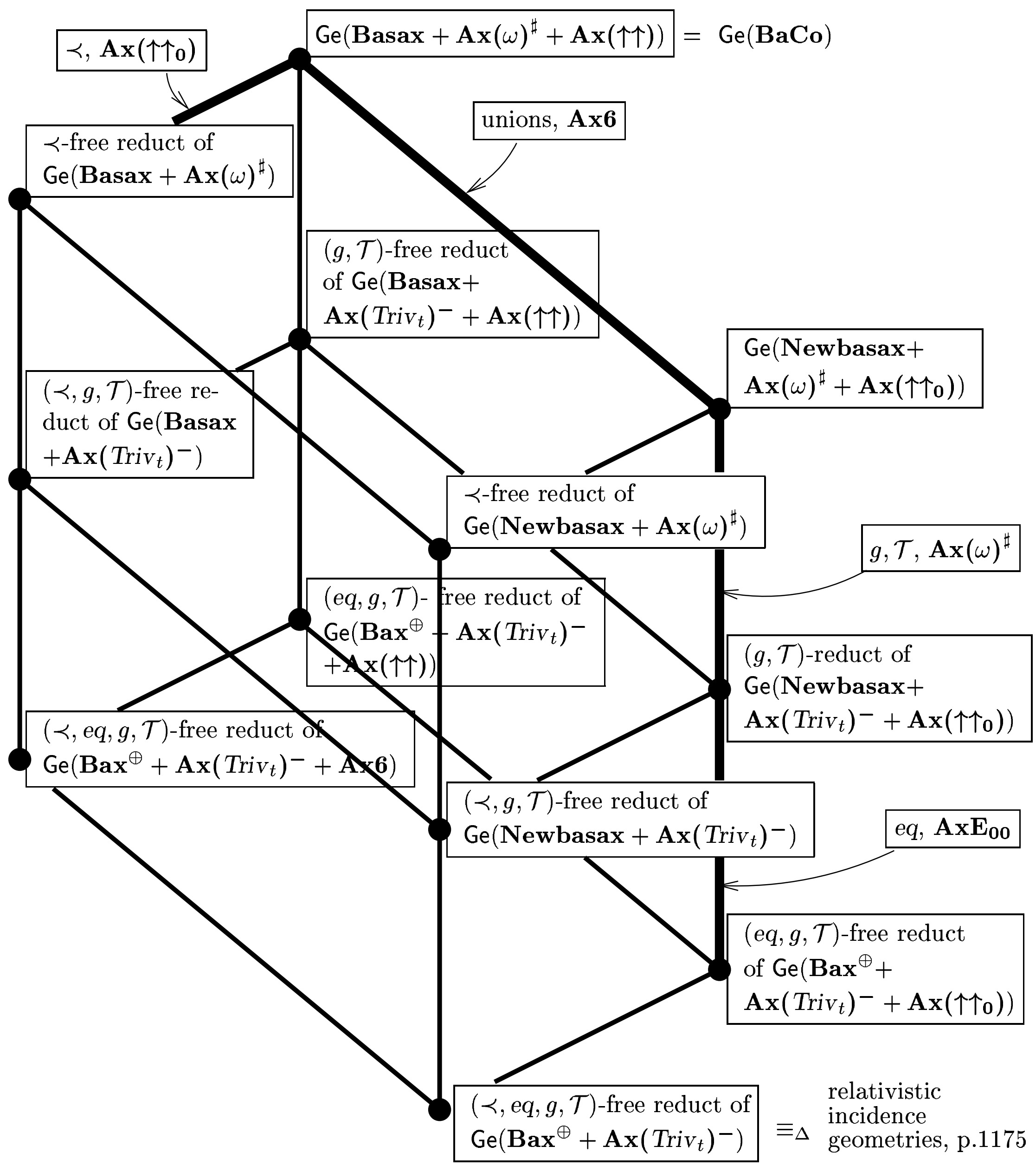

Figure 282: Reducts of geometries agreeing with the corresponding (reducts of) Minkowskian geometries. $\mathbf{Ax}(\sqrt{\ })$ and $n > 2$ are assumed. Nodes are of form $\mathsf{Rd}_{\mathsf{L}}(\mathsf{Ge}(Th))$ determined by the choice of Th and geometric sublanguage L. For detailed explanation cf. p.879. Cf. also Fig.283. For $\equiv_\triangle$ cf. p.970.

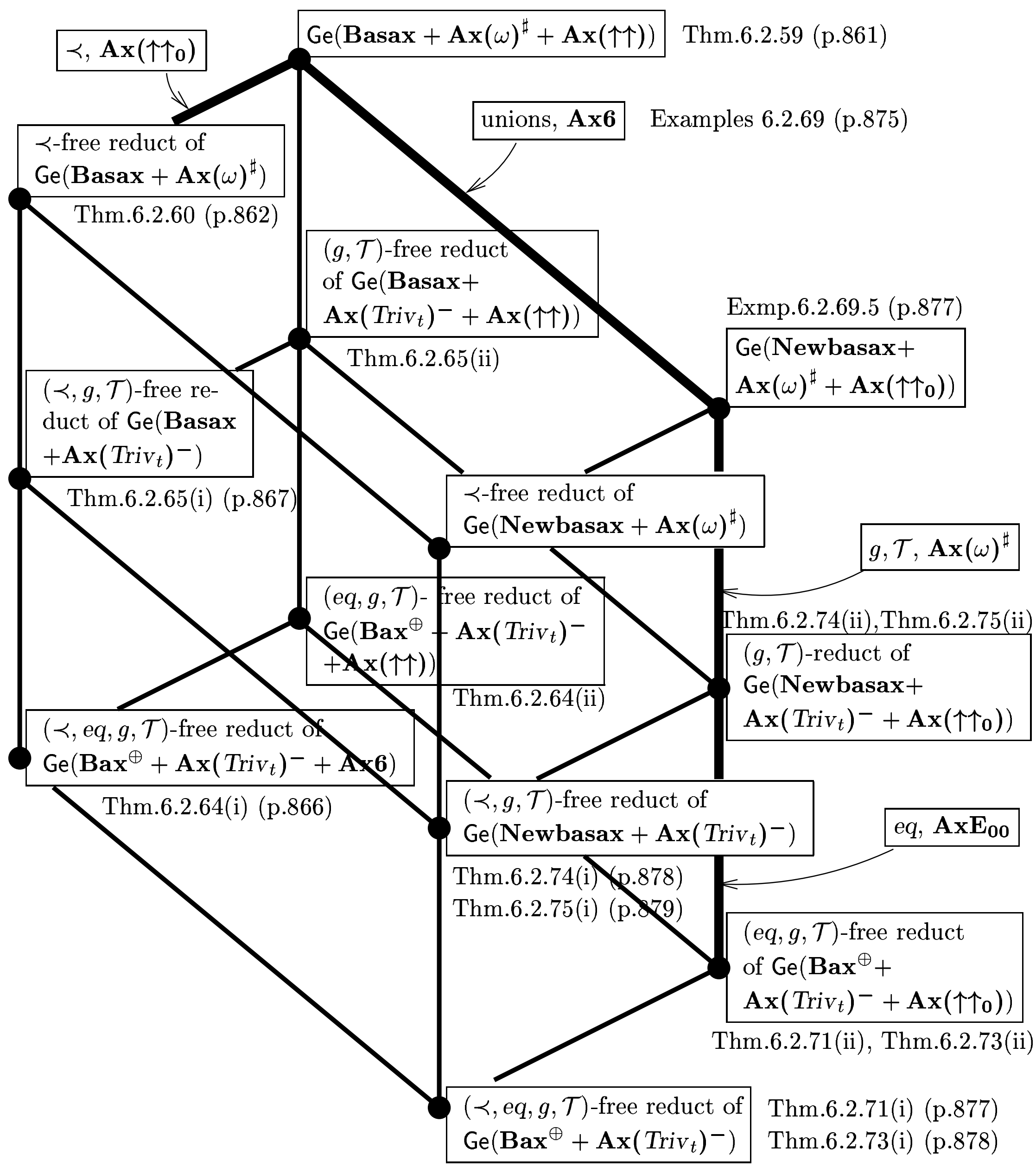

Figure 283: This is Fig.282 enriched with the names of theorems involved.

PROPOSITION 6.2.61 *Assume* $\mathfrak{M} \models (\mathbf{Basax} + \mathbf{Ax}(\omega)^\sharp + \mathbf{Ax}(\uparrow\uparrow))$. *Assume* $m, k \in Obs$. *Then (i)–(iv) below hold.*

(i) *The world-view transformation* f_{mk} *induces an automorphism of the Minkowskian geometry* $Mink(\mathfrak{F}^{\mathfrak{M}})$ *the natural way.*[812]

(ii) *For every automorphism* α *of* $Mink(\mathfrak{F}^{\mathfrak{M}})$ *which is the identity function on the sort* F, *there are* $m', k' \in Obs^{\mathfrak{M}}$ *such that* α *and* $\mathsf{f}_{m'k'}$ *coincide on* ${}^nF$.

(iii) $w_m^{-1} \circ w_k$ *induces an automorphism* $\widehat{\mathsf{f}_{mk}}$ *of the geometry* $\mathfrak{G}_{\mathfrak{M}}$, *the natural way,*[813] *where the formal definition of* $\widehat{\mathsf{f}_{mk}}$ *comes on p.914.*

(iv) *For every automorphism* α *of* $\mathfrak{G}_{\mathfrak{M}}$ *which is the identity function on the sort* F, *there are* $m', k' \in Obs^{\mathfrak{M}}$ *such that* α *and* $w_{m'}^{-1} \circ w_{k'}$ *coincide on* Mn. *I.e.* $\widehat{\mathsf{f}_{m'k'}}$ *agrees with* α.

On the proof: Items (i) and (iii), for the case $n > 2$, are corollaries of Thm.6.2.59. In the case $n = 2$, by Thm.6.2.59, we conclude that items (i) and (iii) hold for the *eq*-free reducts of the geometries. Checking that f_{mk} and $w_m^{-1} \circ w_k$ are automorphisms of the geometry reducts $\langle {}^2F; eq_\mu \rangle$ and $\langle Mn_{\mathfrak{M}}; eq_{\mathfrak{M}} \rangle$, respectively, is easy and is left to the reader. The proofs of items (ii), (iv) are available from Judit Madarász. ∎

Items (iii) and (iv) of the above proposition can be summarized, roughly, by saying that $Aut(\mathfrak{G}_{\mathfrak{M}})$ can be identified with the group $\{ \widehat{\mathsf{f}_{mk}} : m, k \in Obs \}$, which in turn can be identified by $\{ \mathsf{f}_{mk} : m, k \in Obs \}$. Cf. p.779 and §6.2.8 (p.913). Items (i) and (ii) say basically the same about $Mink(\mathfrak{F}^{\mathfrak{M}})$ in place of $\mathfrak{G}_{\mathfrak{M}}$.

Let us recall that in Definition 3.8.42 (p.331), for every Euclidean $\mathfrak{F}$, the Minkowski model $\mathfrak{M}_{\mathfrak{F}}^M$ over $\mathfrak{F}$ was defined. The proposition below says that the observer-independent geometry of the Minkowski model over $\mathfrak{F}$ is the Minkowskian geometry over $\mathfrak{F}$, up to isomorphism.

PROPOSITION 6.2.62 *Assume* $\mathfrak{F}$ *is Euclidean and* $n > 2$. *Then*

$$\mathfrak{G}_{\mathfrak{M}_{\mathfrak{F}}^M} \cong Mink(\mathfrak{F}).$$

Moreover, for every $m \in Obs^{\mathfrak{M}_{\mathfrak{F}}^M}$, $w_m : {}^nF \longrightarrow Mn$ *induces an isomorphism between* $Mink(\mathfrak{F})$ *and* $\mathfrak{G}_{\mathfrak{M}_{\mathfrak{F}}^M}$ *the natural way.*[814]

[812]Making this precise: Let $\widetilde{\mathsf{f}_{mk}} : \mathsf{Eucl} \longrightarrow \mathsf{Eucl}$ be defined by $\widetilde{\mathsf{f}_{mk}} : \ell \mapsto \mathsf{f}_{mk}[\ell]$. Then $\langle \mathsf{f}_{mk}, \mathrm{Id} \restriction F, \widetilde{\mathsf{f}_{mk}} \rangle$ is a (three-sorted) automorphism of $Mink(\mathfrak{F}^{\mathfrak{M}})$, cf. item (II) of Def.6.2.2 (p.798).

[813]Cf. footnote 812.

[814]See footnote 810.