# 4 Weak, flexible, more general axiom systems for relativity

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**Contents of Chapter 4**

## 4.1 Newtonian kinematics as a special case of axiomatic special relativity

Newtonian kinematics is a precursor of special relativity.[339] Therefore, it is natural to ask ourselves how the two theories are related. The usual, informal answer to this question is that the natural evolution of science represents a movement from first describing a "smaller world" and later, in the more advanced theory, describing a "bigger world". Accordingly, Newtonian kinematics describes the smaller world of slow relative motion while special relativity describes the broader world of both slow and fast relative motion. Since in the present work we are involved in the logical analysis of relativity theories, it is natural to ask ourselves whether the above quoted informal answer (concerning the relationship between the two theories) can be incorporated into our picture based on mathematical logic, in particular, whether the above informal answer can be made more tangible.[340] In the present section we will see two affirmative answers, i.e. we will see two ways of elaborating the connection between the Newtonian and the relativistic theories. In the first approach we treat the speed of light as a constant $c$ which can be chosen independently of the theory, i.e. $c$ can be regarded as a kind of parameter of our theory. Choosing this parameter amounts to choosing between the Newtonian and the relativistic theories. The second approach is connected to nonstandard analysis and is more ambitious than the first one in the respect that it integrates the two theories into a single consistent formal theory (i.e. it makes the two theories consistent even in the rigorous sense of mathematical logic).

The above already indicates that in the present section we will speak about connections between different theories (of first order logic, of course). In later sections of the present chapter this will be even more so. Therefore at the end of this section we recall from textbooks of logic what is known as the <u>lattice of first-order theories</u>.

In the first part of the present section we show that Newtonian kinematics is a special case of a very natural, slight generalization of special relativity. We get this generalization of special relativity by separating out a mere notational convention, namely that the speed of light, $c$, is 1, and we replace this with the axiom saying that the speed of light is the same for all observers. After this, we get, basically,

[339] More precisely, the kinematics of special relativity, but that is what we call specrel here.

[340] The logically minded secondary school student is often puzzled by the statement that Newtonian kinematics is valid for slow motion. The source of puzzlement is the question of how to draw the line between slow and fast. Is, for example, "slow + slow = slow" true?

Newtonian kinematics when $c = \infty$, and we get special relativity when $c < \infty$. Next we elaborate this idea in a little more detail.

In this section we will use

$$\mathbf{Specrel} = \mathbf{Basax} + \mathbf{Ax(symm)}^{\dagger}$$

as our central version of special relativity theory.[341] Consider the speed of light axiom **AxE** of **Specrel**. We will decompose **AxE** to an essential part and to an axiom of convenience. The essential part $\mathbf{Ax}(E_{ess})$ says that the speed of light is the same for all observers and all photons. I.e. it says that

$$(\forall m, k \in Obs)(\forall ph, ph_1 \in Ph)v_m(ph) = v_k(ph_1).$$

Hence there is a "constant" $c \in F \cup \{\infty\}$ such that $\forall m \forall ph(v_m(ph) = c)$. The second part of **AxE** concerns only notational convenience, namely we agree that we will choose our units of measurments such that this constant will have 1 as its value. Formally,

$$c = 1.$$

There is an essential consequence of our convention $c = 1$, namely that

$$c < \infty.$$

This is something which is hard to eliminate by changing our units of measurements. But, for the duration of the present introduction, let us forget about this "essential" consequence of $c = 1$, and after having postulated $\mathbf{Ax}(E_{ess})$, let us treat $c = 1$ as a mere convention of notational convenience. Then in this spirit we will obtain the essential part **Flxspecrel** of **Specrel** by replacing **AxE** with the essential part $\mathbf{Ax}(E_{ess})$ in **Specrel**.[342] The name **Flxspecrel** intends to abbreviate "flexible version of **Specrel**". Then we will find that Newtonian kinematics is a special case of our (flexible version) **Flxspecrel** of **Specrel**; namely we get Newtonian kinematics by assuming $c = \infty$ (and two natural simplifying assumptions). In other words, Newtonian kinematics will be seen to be equivalent with $\mathbf{Flxspecrel} + c = \infty +$ *simplifying axioms*. We will introduce our formalized version **NewtK** of Newtonian kinematics by using the axiom $c = \infty$, but later in this section we will discuss the relationship with more traditional formalizations of Newtonian kinematics where the speed of light is not mentioned at all.

It is not surprising that the theories **NewtK** and **Specrel** admit a least common generalization. Namely, any two first order theories, say, $Th_1$ and $Th_2$ admit a least

[341] More precisely, we treat **Specrel** as a representative of the strong ones of our relativity theories.
[342] We will also have to modify **Ax5** in **Specrel**.

common generalization in the form $\mathsf{Th}(\mathsf{Mod}(Th_1) \cup \mathsf{Mod}(Th_2))$. What is interesting in the present section is the fact that if we separate the "essential part" **Flxspecrel** of **Specrel** from a mere notational convention concerning the value of the constant $c$, then we arrive at a natural, very slight generalization of **Specrel** which contains all the essentail ideas[343] in **Specrel** and which admits the Newtonian theory as a special case.

At this point it is tempting to draw up the following alternative history (i.e. things could have happened in the following order). Assume, Newton summarizing the results of all the experiments made before (together with some principles of aesthetics like Occam's razor) arrives at the theory **Flxspecrel**$^+$ = **Flxspecrel** + *the simplifying assumptions we mentioned above.*[344] This leaves the value of $c$ open. Then he makes experiments trying to figure out this number. Suppose that he finds that the only thing he can determine is that $c$ is bigger than all the speeds he can measure. After a long while he makes the decision to assume $c = \infty$ as a simplifying axiom of default. I.e. he says that until somebody finds evidence against it, let us assume $c = \infty$ only in order to make our theory about what the world is like simpler.

This way Newton arrives at the (tentative) theory

$$\mathbf{NewtK} = \mathbf{Flxspecrel}^+ + c = \infty.$$

In the meantime, better instruments of measurement were developed, and (Roemer and) Michelson and Morley discover that the assumption $c = \infty$ has to be withdrawn, hence the "world" arrives at **Flxspecrel**$^+$ as the current theory. By the results of the just quoted people, the extra assumption $c < \infty$ is added. This leads to the notational convention $c = 1$ (at least in the works of certain people). This brings our alternative history to its end.[345]

### Newtonian kinematics NewtK and flexible version Flxbasax of special relativity

As a first step (in carrying out the above programme), below we formalize the theory **NewtK** (which we call Newtonian kinematics). Since in the present section we will rely on **Newbasax** and **Bax**, we recall the following facts about them (in comparison with **Basax**). In **Newbasax** we allow that different observers observe

[343] except for the idea of $c$ being finite

[344] $\mathbf{Flxspecrel}^+ \stackrel{\text{def}}{=} \mathbf{Flxspecrel} + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}\square\mathbf{1}$. We note that **Flxspecrel**$^+$ is just as good a representative of the stronger relativity theories studied here as **Flxspecrel** is. Cf. §3.8.

[345] We note that the above alternative history is different from the way things actually happened.

different events,[346] and in **Bax** in addition to this we allow that the speed of light be different for different observers (in the same model) but for a fixed observer the speed of all photons is the same and nonzero. Further, in **Bax** for each observer $m$ a constant $c_m \in F \cup \{\infty\}$ was defined such that for $m$ the speed of all visible photons is $c_m$.

The notation $m \uparrow k$ was introduced in §3.8 (p.296), and it denotes that the time of $k$ flows forwards as seen by $m$.[347] **Ax(↑↑)** introduced below expresses that each observer sees any other observer's time flow forward:

**Ax(↑↑)** $(\forall m, k \in Obs) m \uparrow k$ .

Recall that the axiom **Ax(symm)** was introduced in §§2.8, 3.9. **Ax(symm)** expresses that observers see each other the same way, modulo perhaps a coordinate-transformation. The formula $\mathbf{Ax(symm)}^{\dagger}$ denotes **Ax(symm)** + **Ax(‖)** + **Ax(*Triv*)**, where **Ax(‖)** and **Ax(*Triv*)** are axioms expressing "trivial" properties of the collection of world-view transformations, they are introduced in §2.8. The axiom **Ax□1** is a very natural axiom, it is a kind of symmetry axiom. It is introduced in §3.9 and it says that any two observers $m, k$ are equivalent in the sense that if $m$ sees an observer in a certain way, then $k$ also sees an observer in the same way, i.e.

**Ax□1** $(\forall m, k, m' \in Obs)(\exists k' \in Obs) \mathsf{f}_{mk} = \mathsf{f}_{m'k'}$ .

Now, we define the axiom system **NewtK** as follows:

$$\mathbf{NewtK}^- \stackrel{\text{def}}{=} \mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax(symm)}^{\dagger} + (\forall m \in Obs)(c_m = \infty).$$[348]

$$\mathbf{NewtK} \stackrel{\text{def}}{=} \mathbf{NewtK}^- + \mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}\Box\mathbf{1}.$$

In the next part of this section (on p.435), we will check that our formalization **NewtK** of Newtonian kinematics agrees with the traditional (say, "secondary school") version of Newtonian kinematics.

**Remark 4.1.1 (On the style of our formalization of NewtK.)**
We axiomatized Newtonian kinematics the way we did, i.e. by postulating $c_m = \infty$, only for reasons of convenience. A more traditional axiomatization would have

[346] Also recall that **Newbasax** models are basically disjoint unions of **Basax** models, cf. Figures 65 (p.195) and 307 (p.1001).

[347] Formally, $m \uparrow k$ denotes that $\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t > 0$.

[348] Let us recall that $c_m = \infty$ abbreviates the formula $(\forall ph \in Ph)\ [m \stackrel{\odot}{\to} ph \ \Rightarrow \ v_m(ph) = \infty]$, since we assumed **Bax**.

taken up more space here, — we give such traditional axiomatizations in the part beginning with p.435.

The most distinctive feature of Newtonian kinematics is that events do not get out of synchronism, and the role of postulate $c = \infty$ in **NewtK** is to ensure this. The role of **Ax(↑↑)** is to ensure absolute time, and the role of **Ax□1** is to ensure absolute space. Cf. Propositions 4.1.15 and 4.1.16 on p.443.

We will see that the usual principles of Newtonian kinematics are indeed derivable from **NewtK**, e.g. it is derivable that the paradigmatic effects discussed in § 2.5 do *not* occur. In more detail, moving clocks do not slow down, clocks in the rear and front of a moving spaceship do not get out of synchronism (i.e. *"time is absolute"*), moving meter sticks do not shrink, see Prop.4.1.12. More on the connection between the present and the traditional formalizations of Newtonian kinematics can be found in the part beginning with p.435. ◁

**PROPOSITION 4.1.2** *(i)–(ii) below hold.*

**(i)** **Specrel** $\;=\!\mid\models\;$ $\mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax(symm)}^{\dagger} + (\forall m \in Obs) c_m = 1$.

**(ii)** $\mathbf{NewtK}^{-}$ $\;=\!\mid\models\;$ $\mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax(symm)}^{\dagger} + (\forall m \in Obs) c_m = \infty$.

We omit the easy **proof**. ∎

Proposition 4.1.2 above suggests that we take $\mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax(symm)}^{\dagger}$ as our theory **Flxspecrel**. Indeed, this is very close to it. We only need to add one more axiom.[349] Because we will use **Bax** as our basic axiom system, we introduce a formalization $\mathbf{AxE_{02}}$ of $\mathbf{Ax}(E_{ess})$ which matches **Bax** better.[350] Since we will use the axiom $\mathbf{AxE_{02}}$ only when **Bax** is already assumed, in writing up $\mathbf{AxE_{02}}$ we can use the convenient abbreviation $c_m$ available in **Bax**. This way we can make the axiom shorter.

$\mathbf{AxE_{02}}$ $\quad (\forall m, k \in Obs)\; c_m = c_k,$ $\quad$ in more general form[351] :

$$(\forall m, k \in Obs)(\forall ph, ph_1 \in Ph)\;[(m \overset{\odot}{\to} ph \wedge k \overset{\odot}{\to} ph_1) \;\Rightarrow\; v_m(ph) = v_k(ph_1)].$$

Intuitively, $\mathbf{AxE_{02}}$ says that the speed of light is the same for everyone.

[349] And we only need to add this for the case $n = 2$, see items 3.9.37, 3.9.40 on pp.386, 387.

[350] In §3.4.2 the last axiom of **Bax** was $\mathbf{AxE_{01}}$. This is why we call our new axiom $\mathbf{AxE_{02}}$.

[351] i.e. without assuming **Bax**

**Definition 4.1.3** We define our flexible versions of **Newbasax** and **Specrel** as follows.

$$\textbf{Flxbasax} \stackrel{\text{def}}{=} \textbf{Bax} + \textbf{AxE}_{\textbf{02}}\,.$$

$$\textbf{Flxspecrel} \stackrel{\text{def}}{=} \textbf{Bax} + \textbf{Ax6} + \textbf{Ax(symm)}^{\dagger} + \textbf{AxE}_{\textbf{02}}\,.$$

$\lhd$

**Definition 4.1.4** Assume **Flxbasax**. Then there is a definable constant $c$, namely we choose $c = c_m$, for some $m \in Obs$. Since for all observers $m$ and $k$, $c_m = c_k$ holds in **Flxbasax**, *we consider c to be* well *defined*. We note that $c$ is the speed of light, in models of **Flxbasax**.[352]

$\lhd$

As a curiosity, we note that while $\textbf{AxE}_{\textbf{02}}$ can be used without **Bax**, the defined constant $c$ cannot be used without **Flxbasax**. (Actually instead of **Flxbasax** $\textbf{AxE}_{\textbf{02}}$ is enough if we agree that if no observer sees any photon then $c := \infty$.)

In connection with Definition 4.1.3 above see Figures 123, 124 and 126. Figures 123 and 126 use the lattice style representation. In them, the theories further up are stronger theories in the sense that they prove more theorems. The labels $\varphi_1, \ldots, \varphi_n$ on a line connecting a lower theory $Th_1$ with a theory $Th_2$ which is further up indicate that we get $Th_2$ from $Th_1$ by adding $\varphi_1, \ldots, \varphi_n$ to it, i.e. $Th_2 \mathrel{=\!\!|\!\models} Th_1 \cup \{\varphi_1, \ldots, \varphi_n\}$. More on the lattice of theories and on how to draw lattices of theories is said in the part beginning with p.451, especially on p.453. Figure 124 is an alternative way of illustrating the relationship between the theories in this section.

**PROPOSITION 4.1.5**

**(i)** $\textbf{NewtK}^{-} \mathrel{=\!\!|\!\models} \textbf{Flxspecrel} + (c = \infty)$.

**(ii)** $\textbf{Specrel} \mathrel{=\!\!|\!\models} \textbf{Flxspecrel} + (c = 1)$.

**(iii)** $\textbf{Newbasax} \mathrel{=\!\!|\!\models} \textbf{Flxbasax} + (c = 1)$.

We omit the easy **proof**. ∎

[352] It is interesting to note that in different models of **Flxbasax** the speed of light $c$ may be different. We note that, actually, $c$ is the square of usual speed of light, cf. the definition of $v_m(k)$ in Definition 2.2.2(ii) on p.46.

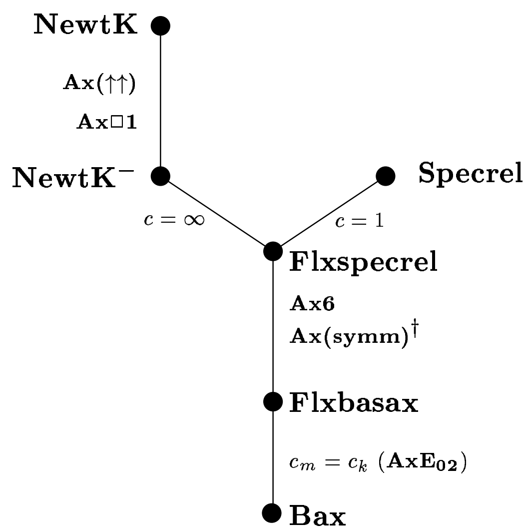



Figure 123:

At this point we suggest the reader to have a look at Figure 126 on p.433.

Below we show that **Flxbasax** can be considered as a version of **Newbasax** where we relaxed the postulate "$c = 1$" (to "$c \neq 0$"). **Flxbasax** then will have two kinds of models: one kind in which $c < \infty$, we will argue that these are very close to models of **Newbasax**, and in the other kind of models $c = \infty$, these are generalizations of models of **NewtK**.

In **Newbasax**, the only axioms which involve the speed of light are **Ax5** and $\mathbf{AxE_0}$. In the formulas $\mathbf{Ax5^c} \ldots \mathbf{AxE_0^f}$ below, $c$ is a free variable, ranging over elements of the quantity sort. Let $\mathbf{Ax5^c}$ and $\mathbf{AxE_0^c}$ be the versions of **Ax5** and $\mathbf{AxE_0}$ where we change 1 to $c$:

$\mathbf{Ax5^c}$ $ang^2(\ell) < c \;\Rightarrow\; (\exists k \in Obs)\; tr_m(k) = \ell$ and

$ang^2(\ell) = c \;\Rightarrow\; (\exists ph \in Ph)\; tr_m(ph) = \ell.$

$\mathbf{AxE_0^c}$ $tr_m(ph) \neq \emptyset \;\Rightarrow\; v_m(ph) = c.$

In connection with the contents (i.e. meanings) of $\mathbf{Ax5^f}$ and $\mathbf{AxE_0^f}$ we note that $c$ was defined to be the unique speed of light in models of **Flxbasax**, cf. Def. 4.1.4. Define

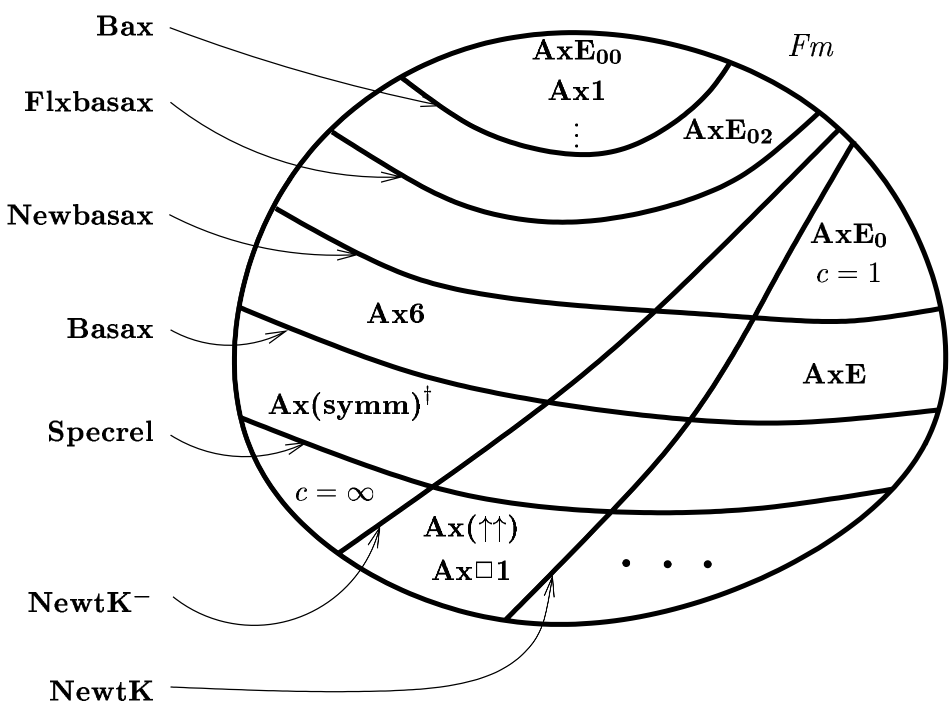



Figure 124: An alternative illustration of the relationship between the theories in this section.

**Ax5^f** $\quad [(\exists m_1)(\exists ph_1)\; c = v_{m_1}(ph_1) \;\Rightarrow\; \mathbf{Ax5^c}] \;\wedge\; Ph \neq \emptyset$ and

**AxE$_0^f$** $\quad [(\exists m_1)(\exists ph_1)\; c = v_{m_1}(ph_1) \;\Rightarrow\; \mathbf{AxE_0^c}] \;\wedge\; Ph \neq \emptyset.$

Then it is easy to see that **AxE$_0^f$** is equivalent to **AxE$_{02}$** and **Ax5$^f$** is equivalent to

$$\Big[(\exists m_1)(\exists ph_1)\; ang^2(\ell) < v_{m_1}(ph_1) \quad\Rightarrow\quad (\exists k \in Obs)\; \ell = tr_m(k) \qquad \text{and}$$
$$(\exists m_1)(\exists ph_1)\; ang^2(\ell) = v_{m_1}(ph_1) \quad\Rightarrow\quad (\exists ph \in Ph)\; \ell = tr_m(ph)\Big].$$

The next proposition says that we can get **Flxbasax** from **Newbasax** by changing **Ax5**, **AxE$_0$** to **Ax5$^f$**, **AxE$_0^f$**, and adding the axiom $c \neq 0$ (i.e. **AxE$_{01}$**). This is what corresponds to relaxing the postulate $c = 1$ to $c \neq 0$. We can get **Flxspecrel** from **Specrel** similarly. Thus, **Flxspecrel** is indeed the axiom system we talked about in the introduction to this section: **Flxspecrel** is the system we get by leaving out **AxE** from **Specrel**, and adding **AxE$_{02}$**. Of course, we had to change those axioms, too, which mentioned 1 as the speed of light (i.e. **Ax5**).

**PROPOSITION 4.1.6** *The two statemens below hold.*

$$\mathbf{Flxbasax} \;\;=\!\!|\!\!\models\;\; (\mathbf{Newbasax} \setminus \{\mathbf{Ax5},\ \mathbf{AxE_0}\}) \cup \{\mathbf{Ax5^f},\ \mathbf{AxE_0^f},\ \mathbf{AxE_{01}},\}\,.$$
$$\mathbf{Flxspecrel} \;\;=\!\!|\!\!\models\;\; (\mathbf{Specrel} \setminus \{\mathbf{Ax5},\ \mathbf{AxE}\}) \cup \{\mathbf{Ax5^f},\ \mathbf{AxE_0^f},\ \mathbf{AxE_{01}}\}\,.$$

We omit the easy **proof**. ∎

Now, we turn to discussing the connection between **Newbasax** and (**Flxbasax** $+\; c < \infty$).

**Remark 4.1.7 (Flxbasax is equivalent with Newbasax in a certain sense)** We feel that **Newbasax** is extremely close to the theory (**Flxbasax** $+\; c < \infty$). They are not equivalent in the *usual* sense since

$$(\mathbf{Flxbasax} \;+\; c < \infty) \;\not\models\; c = 1\,.$$

I.e. $c = 1$ is not provable from the new theory, while of course **Newbasax** $\models c = 1$. However, we feel that all theorems of **Newbasax** not involving the exact value of $c$ (like $c = 1$) will turn out to be provable from (**Flxbasax** $+\; c < \infty$), and vice versa. But, this statement is not very easy to formalize because a formula (theorem of **Newbasax**) might involve the value of $c$ implicitly. One way of making the meaning of the above statement precise is to compare the models of the two theories in question. Now, it can be shown[353] that every model of (**Flxbasax** $+\; c < \infty$)

[353]We have to assume **Ax**($\sqrt{\ }$). See Madarász [172] and Madarász-Németi [175].

can be obtained from a model of **Newbasax** by changing the unit of measurement for time[354]; and vice versa. (The models of **Flxbasax** $+ \ c = \infty$ cannot be obtained from models of **Newbasax** this way.) Thus, structurally the models of (**Flxbasax** $+ \ c < \infty$) are exactly like those of **Newbasax**, except that a fairly innocent "ratio" in them may vary (representing the value of $c$). Figure 125 intends to represent such a model.[355]

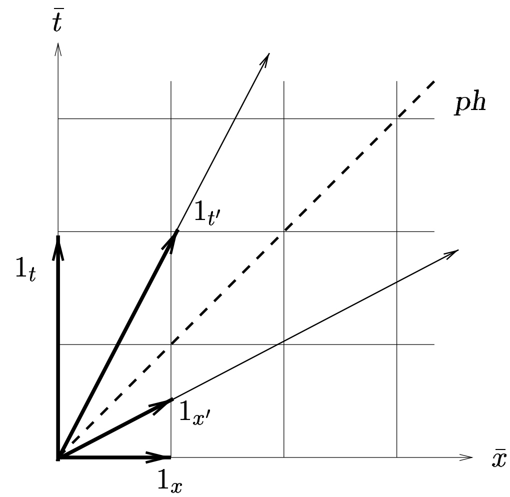



Figure 125: A model of (**Flxbasax** $+ \ c < \infty$)

In connection with the question of how close (**Flxbasax** $+ \ c < \infty$) is to **Newbasax** (i.e. to **Flxbasax** $+ \ c = 1$) we feel that the geometry chapter (§6) contains substantial useful information, cf. §6.6.

We think that the above comparison of models of (**Flxbasax** $+ \ c < \infty$) and **Newbasax** is a way of making precise that the theorems provable in (**Flxbasax** $+ \ c < \infty$) are exactly those theorems of **Newbasax** which do not use the exact value of $c$, i.e. which do not exploit the convention $c = 1$ (which was made for convenience only). This supports our claim that, in many respects,

[354]Making this precise: Let $\mathfrak{M} = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, W \rangle$ be a frame model and let $c \in {}^+F$. Define $g : B \times {}^nF \times B \longrightarrow B \times {}^nF \times B$ by $g(b, p_0, \ldots, p_{n-1}, h) = (b, 1/\sqrt{c} \cdot p_0, p_1, \ldots, p_{n-1}, h)$, and let $\mathfrak{M}^c = \langle (B;\ Obs, Ph, Ib), \mathfrak{F}, G;\ \in, g[W] \rangle$. Then we can consider $\mathfrak{M}^c$ as the model $\mathfrak{M}$ such that we only changed the unit of measurement for time.

[355]Cf. also the models of **Bax** illustrated in Figure 75 and discussed on pp. 233–243 in the present work (the point here is that understanding the models of **Bax** is relevant [though not necessary] for understanding the models of **Flxbasax**).

(**Flxbasax** $+ c < \infty$) is a better[356] version of special relativity than **Newbasax** (or **Basax**) is because it does not dwell on inessential issues like the *convention* $c = 1$. Clearly, $c = 1$ is only a convenient convention and is not essential conceptually.

◁

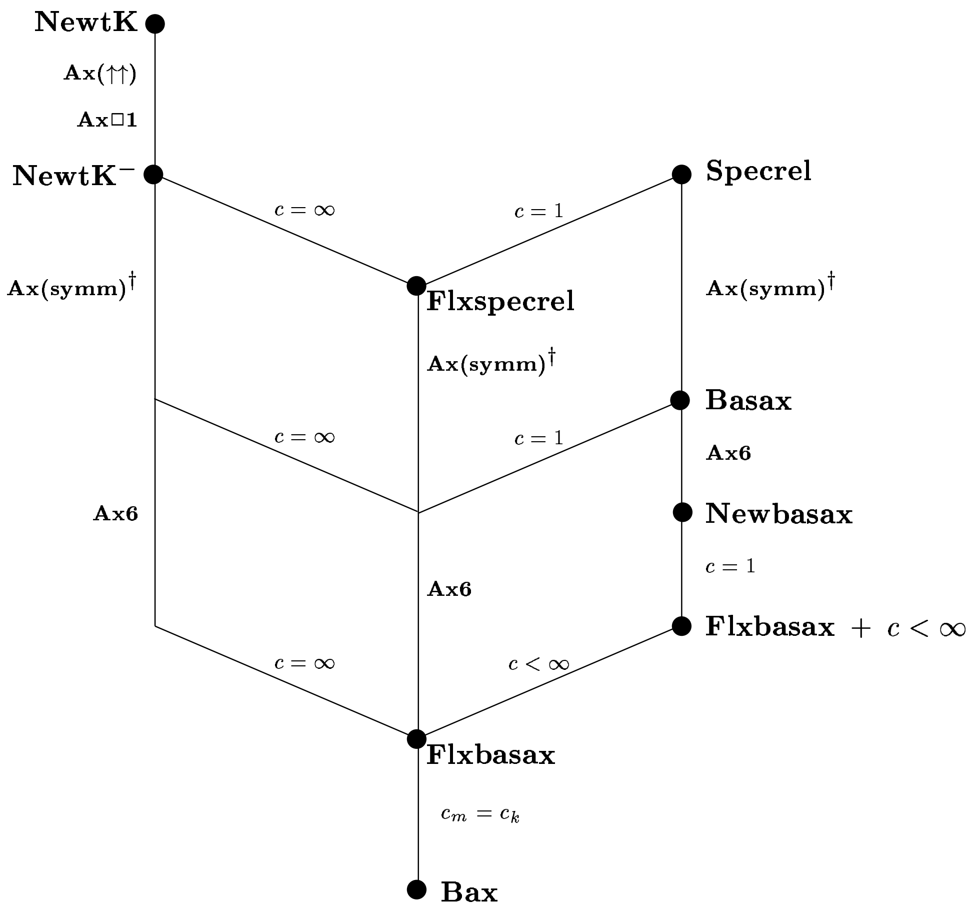



Figure 126: The lattice (or poset) of the theories in the present section. (We assume **Ax**($\sqrt{\ }$).)

In the present work we do not continue the discussion of **Flxbasax** much further, but we note that we feel that **Flxbasax** might reflect the essential ideas of special

[356]we mean, more focused on the essential ideas

relativity more faithfully than e.g. **Newbasax** or **Basax** do (at least for the purposes of e.g. conceptual analysis). In some cases, e.g. in the part beginning with p.446, this flexibility turns out to be very useful, though. On the other hand, in other cases, the difference might turn out to be not very deep, since at any point of studying **Flxbasax** one may decide to introduce the *simplifying assumption* $c < \infty \Rightarrow c = 1$ (this way excluding the irrelevant cases of $c = 2$, $c = 3$ etc). However, it might be useful to keep in mind that $c = 1$ is only a simplifying assumption.

**Remark 4.1.8** Instead of **Newbasax**, we could have chosen **Flxbasax** as our generic (or central) axiomatization of special relativity, in this work. Then we would develop (and analyze) special relativity in the form of **Flxbasax**. This would lead to an alternative version of the present work. This alternative version would look rather *similar* to the present one with two exceptions (i) our relativity theory would be a *generalization* of Newtonian kinematics, and (ii) in addition to **Ax(symm)**, we would have a second "axiom of choice" namely "$c < \infty$".[357] The latter means that, from time to time, for certain theorems we would have to assume the extra potential axiom $c < \infty$ the same way as we did with **Ax(symm)** in §2.8 for proving **Ax(Twp)**. But, for example, for the "no FTL observers theorem"s we will not need the "axiom of choice" **Ax(symm)**.

Although this would be a very natural way of developing the logical analysis of relativity, we do not carry it through in the present work. However, since the changes would be small and natural, the interested reader is invited to "build up in his minds eye" this variant of the present work (centered around **Flxbasax**, such that relativity comes out as a generalization of **NewtK**).

◁

In connection with the above remark we also note the following. In a sense, **Flxspecrel** is "the" natural <u>common generalization</u> of **NewtK** and **Specrel**, because of the following. Consider the intersection of the theories generated by **NewtK** and **Specrel**. We could call this intersection their least common generalization. This theory can be axiomatized by **Flxspecrel** extended with the extra axiom

$$(\star) \qquad c = 1 \quad \lor \quad (c = \infty \ + \ \mathbf{Ax(\uparrow\uparrow)} \ + \ \mathbf{Ax\square 1})\,.$$

Since axiom $(\star)$ seems artificial for us, we throw it away and call the *remaining* part, **Flxspecrel**, the *natural* common generalization.[358]

[357] Recall from the introduction of §2.8 that sometimes we refer to **Ax(symm)** as our "axiom of choice" (because of an analogy with set theory). Sometimes half-jokingly we call it axiom of choice because you may choose to assume it or you may choose not to.

[358] There are many analogous situations to this, let us pick one e.g. from geometry. In geometry,

**Connections with traditional formalizations of Newtonian kinematics**

The question arises why we formulated Newtonian kinematics with the postulate $c = \infty$ instead of using the traditional terminology of Newtonian kinematics like "absolute time", "absolute space", Galilean transformations. Below we give an alternative formulation which does not mention photons at all, and we state some propositions which indicate an answer to the question of why we say that **NewtK** is a formalization of Newtonian kinematics.

In **Specrel**, the axioms that mention photons are **Ax2**, **Ax5**, and **AxE**. Below we change these axioms to $\mathbf{Ax2^n}$, $\mathbf{Ax5^n}$, and $\mathbf{AxE^n}$, where the upper script "**n**" stands for "no photon version (i.e. traditional version)".

**Definition 4.1.9 (traditional version $\mathbf{NewtK^n}$)**

$\mathbf{Ax2^n}$ $\quad Obs \subseteq Ib$.

$\mathbf{Ax5^n}$ $\quad ang^2(\ell) < \infty \;\Rightarrow\; (\exists k \in Obs)\; tr_m(k) = \ell$.

$\mathbf{AxE^n}$ $\quad (\forall p, q \in {}^nF)\; [p_t = q_t \;\Rightarrow\; \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t]$.

$$\mathbf{NewtK^n} \stackrel{\text{def}}{=} (\mathbf{Specrel} \setminus \{\mathbf{Ax2}, \mathbf{Ax5}, \mathbf{AxE}\}) \cup \{\mathbf{Ax2^n},\ \mathbf{Ax5^n},\ \mathbf{AxE^n}\}\,.$$

Thus,

$$\mathbf{NewtK^n} = \{\mathbf{Ax1}, \mathbf{Ax2^n}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5^n}, \mathbf{Ax6}, \mathbf{AxE^n}, \mathbf{Ax(symm)}^{\dagger}\}.$$

◁

The proposition below says that $\mathbf{NewtK^-}$ and $\mathbf{NewtK^n}$ imply the same theorems in the language which does not mention photons. Well, there is exactly one photon-free formula in which they differ:

$(\nu)$ $$(\forall m \in Obs)(\forall \ell)[ang^2(\ell) = \infty \Rightarrow (\exists b \in Ib)\, tr_m(b) = \ell]\,.$$

$(\nu)$ says that each line $\ell$ with $ang^2(\ell) = \infty$ is the trace of an inertial body.

"Absolute Geometry" is a natural common generalization of Euclidean Geometry and Bolyai-Lobachevski Geometry. So Euclidean Geometry would correspond to **NewtK**, Bolyai-Lobachevski Geometry to **Specrel** and Absolute Geometry to **Flxspecrel**. As a second example consider the theory BA of Boolean Algebras without the operation of complementation. Now, a common generalization of BA and the theory of linearly ordered sets may be the theory of distributive lattices.

**PROPOSITION 4.1.10** *Let $\varphi$ be any formula in the frame-language in which the unary relation symbol Ph does not occur.*

**(i)** $\quad \mathbf{NewtK^-} \models \varphi \quad$ *iff* $\quad \mathbf{NewtK^n} + \nu \models \varphi.$

**(ii)** $\quad \mathbf{NewtK^-} \models \nu \quad$ *while* $\quad \mathbf{NewtK^n} \not\models \nu.$

**Proof:** It is easy to check that $\mathbf{NewtK^-} \models \mathbf{NewtK^n} + \nu$ and that $\mathbf{NewtK^n} \not\models \nu$. This also proves direction $\Leftarrow$ in (i). Let $\varphi$ be as in the hypothesis, and assume $\mathbf{NewtK^-} \models \varphi$. We want to show $\mathbf{NewtK^n} + \nu \models \varphi$. So let $\mathfrak{M}$ be frame-model such that $\mathfrak{M} \models \mathbf{NewtK^n} + \nu$. Define

$$\begin{aligned} Ph' &\stackrel{\text{def}}{=} \{\, b \in Ib \;:\; (\exists m \in Obs)\; ang^2(tr_m(b)) = \infty \,\}, \qquad \text{and} \\ \mathfrak{M}' &\stackrel{\text{def}}{=} \langle (B;\ Obs, Ph', Ib), \mathfrak{F}, G;\ \in, W \rangle . \end{aligned}$$

Then $\mathfrak{M}$ and $\mathfrak{M}'$ differ only in the interpretation of the unary relation symbol *Ph*. Therefore $\mathfrak{M} \models \varphi$ iff $\mathfrak{M}' \models \varphi$, because *Ph* does not occur in $\varphi$. It is not hard to check that $\mathfrak{M}' \models \mathbf{NewtK^-}$ by $\mathfrak{M} \models \mathbf{NewtK^n} + \nu$ and by the definition of $Ph'$. Thus $\mathfrak{M}' \models \varphi$ by $\mathbf{NewtK^-} \models \varphi$, and so $\mathfrak{M} \models \varphi$. This proves direction $\Rightarrow$ of (i). ∎

Now we turn to see what paradigmatic effects hold in **NewtK**.

Below we recall some formalizations of the paradigmatic effects from §§ 2.5, 2.8. **(clock)**, **(meter)**, **(asynch)** will denote, respectively, that "clocks slow down", "meter rods shrink", and "clocks get out of synchronism", cf. Thm's 2.8.7, 2.8.8. We copied these formulas from §2 except that we replaced 1 with $c_m$ where 1 denoted speed of light.

**(clock)** $\quad (\forall m, k \in Obs)\, [0 < v_m(k) \leq c_m \;\Rightarrow\; |\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t| > 1].$

**(meter)** $$(\forall m, k, k_1 \in Obs)\Big[(0 < v_m(k) \leq c_m \;\wedge\; \bar{0} \in tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \;\wedge\; \bar{t} \neq tr_k(k_1) \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \;\wedge\; tr_k(k_1) \parallel \bar{t}) \;\Rightarrow\; |tr_m(k_1)(0)| < |tr_k(k_1)(0)|\Big],$$

see Figure 38 on p.101 (in §2.5).

**(asynch)** $$(\forall m, k \in Obs)\Big[0 < v_m(k) \leq c_m \;\Rightarrow\; (\exists p, q \in {}^nF)\, (p_t = q_t \;\wedge\; \mathsf{f}_{mk}(p)_t \neq \mathsf{f}_{mk}(q)_t)\Big].$$

Recall that we proved in §2.8 that

$$\mathbf{Basax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{(clock)}, \mathbf{(meter)}, \mathbf{(asynch)}.$$

All these will fail in **NewtK**. Now we turn to paradigmatic effects that are true in **NewtK**. The next formula **(vel)** formalizes that "velocities add up", see Figure 127. Observers $m$ and $k$ are in strict standard configuration means that all their spatial unit vectors point in the same direction, cf. Definitions 5.0.42 on p.709 and 2.3.16 on p.71.

**(vel)** $\quad (\forall m, k \in Obs)(\forall b \in Ib)\Big( [\, m$ and $k$ are in strict standard configuration and $m \uparrow k\,] \implies [v_m(k), v_k(b) < \infty \Rightarrow \vec{v}_m(b) = \vec{v}_m(k) + \vec{v}_k(b)]\Big).$

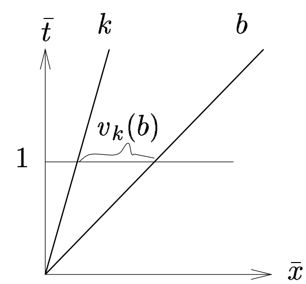



Figure 127: Velocities add up in **NewtK**

The <u>*principle of absolute time*</u> is formulated in d'Inverno [75, p.17], as: if two observers synchronize their clocks (at one event), then they will agree about the time of all events, regardless of their move relative to each other. The formula **(abstime)** is a formalization of this:

**(abstime)** $\quad (\forall m, k \in Obs)\ [(\exists p \in {}^nF)\ \mathsf{f}_{mk}(p)_t = p_t \ \Rightarrow\ (\forall p \in {}^nF)\ \mathsf{f}_{mk}(p)_t = p_t].$

Next we give a formulation of "space is absolute". The formula **(abspace)** below expresses that the spatial distance between two simultaneous events is the same for all observers.

**Notation.** For any $p \in {}^nF$, we define $\quad space(p) \stackrel{\text{def}}{=} \langle p_1, \ldots, p_n \rangle$.

**(abspace)** $\qquad (\forall m, k \in Obs)(\forall p, q \in {}^nF)\ [p_t = q_t \ \Rightarrow$
$\qquad ||space(p) - space(q)|| = ||space(\mathsf{f}_{mk}(p)) - space(\mathsf{f}_{mk}(q))||.$

See Figure 128.

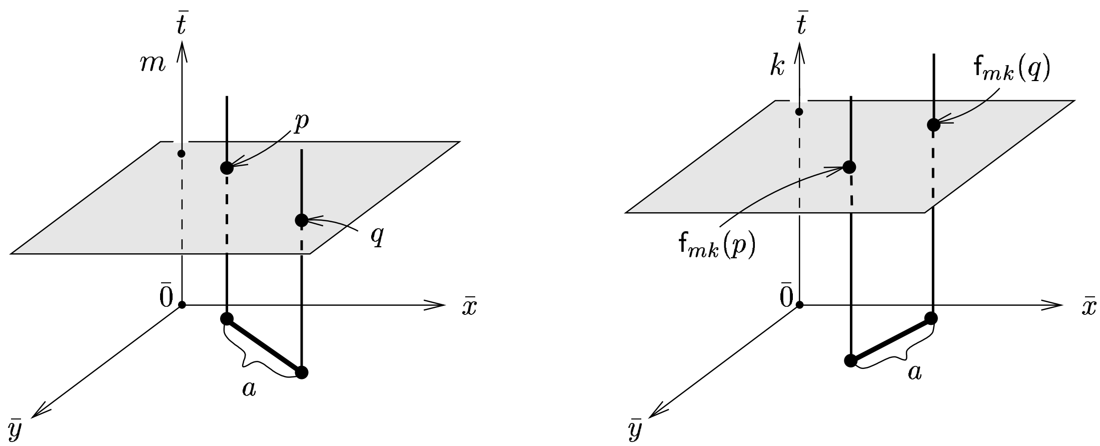



Figure 128: Illustration of **(abspace)**.

Let **(meter)**$^=$, **(meter)**$^\geq$, **(meter)**$^\leq$ denote the formulae we obtain from **(meter)** by changing the last $<$ in it to "$=$", "$\geq$", and "$\leq$" respectively. Similarly, let **(clock)**$^=$ denote the formula we get from **(clock)** by changing $>$ in it to "$=$". Now, **(meter)**$^=$ expresses that moving meter rods do not change their length, and **(meter)**$^\geq$ expresses that no moving meter rod shrinks. Thus, **(meter)**$^=$, **(meter)**$^\geq$ imply the negation of **(meter)**, while the meaning of **(meter)**$^\leq$ is very close to that of **(meter)**, and it holds in **Specrel**. Similarly, **(clock)**$^=$ implies the negation of **(clock)**, hence it fails in **Specrel**. We note that none of **(meter)**$^=$, **(meter)**$^\geq$, **(vel)**, **(abstime)**, **(abspace)** holds in **Specrel**.

We also note that $\neg$**(asynch)** expresses that "simultaneity is absolute (i.e. is the same for each observer)", and **(clock)**$^=$ expresses that "time-distance is absolute".

**Definition 4.1.11** A transformation $\mathsf{f} : {}^nF \longrightarrow {}^nF$ is called a *Galilean transformation* if it is an affine transformation which preserves Euclidean distance in the space-part $S = \{0\} \times {}^{n-1}F$ (i.e. $\|\mathsf{f}(p) - \mathsf{f}(q)\| = \|p - q\|$ for all $p, q \in S$), and there is $a \in F$ such that $\mathsf{f}(p)_t = p_t + a$ for all $p \in {}^nF$. See Figure 129. We call f a *generalized Galilean transformation* if in addition it possibly reverses the flow of time, i.e. f is a generalized Galilean transformation if either f is a Galilean transformation or else $\varepsilon \circ \mathsf{f}$ is a Galilean transformation where $\varepsilon : {}^nF \longrightarrow {}^nF$ is defined by $\varepsilon(p) \stackrel{\text{def}}{=} (-p_0, p_1, \ldots, p_{n-1})$ for all $p \in {}^nF$. $\lhd$

In other words, f is a generalized Galilean transformation if it is an affine transformation which is distance-preserving in the space-part $S$ of ${}^nF$, and also on the time-part in the sense that $\|p_t - q_t\| = \|\mathsf{f}(p)_t - \mathsf{f}(q)_t\|$, for all $p, q \in {}^nF$. See Figure 129. We note that Galilean transformations form a group (under composition of functions as operation), and every element of *Triv* is a Galilean transformation.

Proposition 4.1.12 below states that **NewtK** has the usual properties of Newtonian kinematics.

**PROPOSITION 4.1.12** *Assume* **Ax**$(\sqrt{\ })$.

**(i)** **NewtK** $\models$ $\{$**(abstime)**, **(abspace)**, **(vel)**$\}$.

**(ii)** **NewtK** $\models$ $\{\neg$**(clock)**, $\neg$**(meter)**, $\neg$**(asynch)**$\}$.

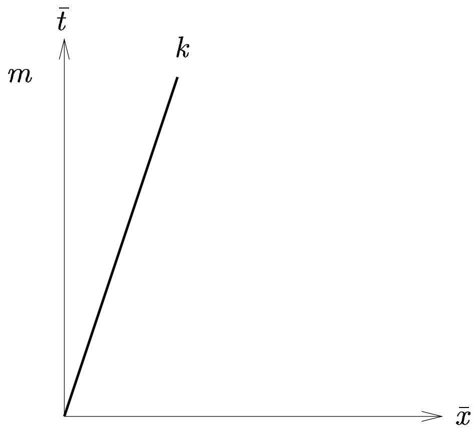
**(iii)** *The world-view transformations in models of* **NewtK** *are Galilean.*

We will prove Proposition 4.1.12 together with the next propositions, the proofs start on p.443 and end on p.446.

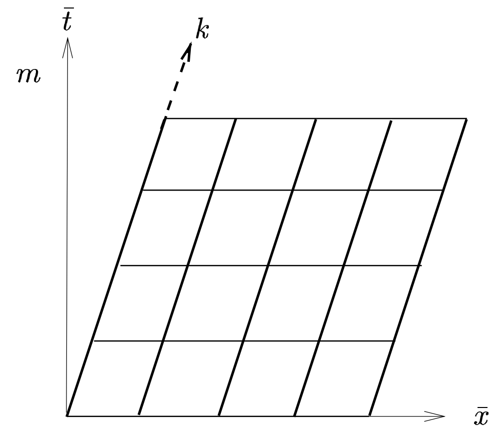
At the end of §2.4 we talked about Minkowski-circles. For $n = 2$, the Minkowski-circle of a model of **NewtK** is as in Figure 130. This follows from Proposition 4.1.12.

At this point we note the following difference between **NewtK** and **Specrel**. Let $c \in F \cup \{\infty\}$, $c \neq 0$ and let $: {}^nF \longrightarrow {}^nF$ be a linear mapping. Consider the following property of $f$:

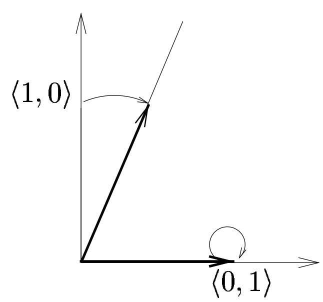
$(\star)$ $f = f^{-1}$ and $(\forall \ell \in \mathsf{Eucl})[ang^2(\ell) = c \ \Rightarrow\ ang^2(f[\ell]) = c]$.




World-view of $m$.




Coordinate-system
of $k$ as seen by $m$




Two-dimensional world-view transformation in $\mathbf{NewtK}(2)$.

Figure 129: $\mathsf{f}_{mk}$ is a Galilean transformation in models of **NewtK**.

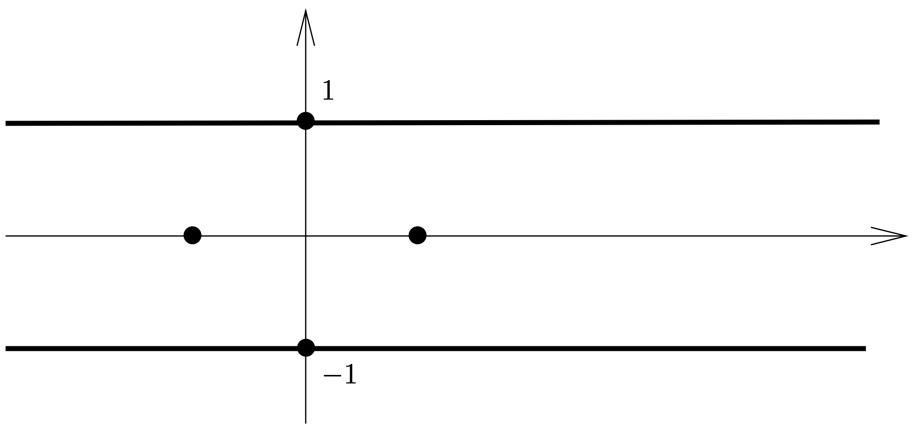



Figure 130: Minkowski-circle of a model of **NewtK**(2).

If $f$ satisfies ($\star$) then in principle $f$ can be a world-view transformation in a model of **Flxspecrel** with $c$ as speed of light. Now, if $c = 1$, then indeed $f$ is a world-view transformation in a model of **Specrel**, because if a linear transformation $f$ satisfies ($\star$) with $c = 1$, then $f$ is a Lorentz-transformation. However, for $n > 2$ there are linear transformations satisfying ($\star$) with $c = \infty$ which are not generalized Galilean. Then by Proposition 4.1.12(iii), $f$ cannot be a world-view transformation in a model of **NewtK**. The following is an example of such an $f$. Let $f : {}^3F \longrightarrow {}^3F$ be any linear mapping such that

$$f(1_t)_t = 1,\ \ f(1_x) = -1_x \ \ \text{and} \ \ f(1_y)_y = 1,\ \ f(1_y)_t = 0.$$

See Figure 131. We do not know whether the above $f$ can be a world-view transformation in a model of **NewtK**$^-$ or not. See Question 4.1.18.

Proposition 4.1.13 below gives characterizations for **NewtK** and **NewtK**$^-$ within **Flxspecrel** in terms of paradigmatic effects. It says that, within **Flxspecrel**, **NewtK** can be characterized with "time is absolute and space is absolute", while **NewtK**$^-$ can be characterized with either one of the following: "velocities add up", "simultaneity is absolute", "time-distance is absolute".

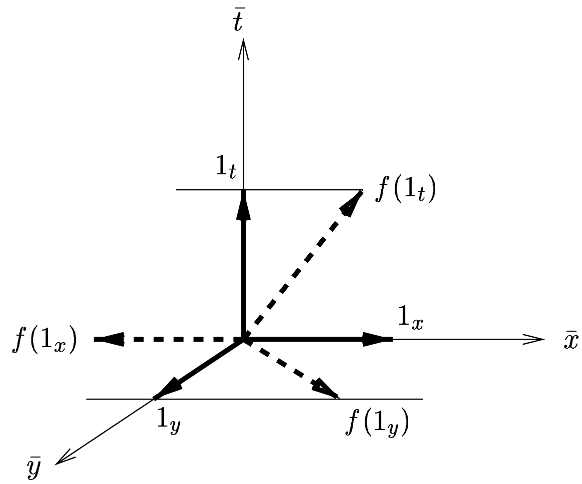



Figure 131: A linear transformation $f$ with the property that $f = f^{-1}$.

**PROPOSITION 4.1.13** *Assume* $\mathbf{Ax}(\sqrt{\ })$.

$$\begin{aligned}
\mathbf{NewtK} &\;=\!\mid\!\models\; \mathbf{Flxspecrel} + \mathbf{(abstime)} + \mathbf{(abspace)}\,,\\
\mathbf{NewtK} &\;=\!\mid\!\models\; \mathbf{Flxspecrel} + \mathbf{(abstime)} + \mathbf{(meter)}^{=}\,,\\
\mathbf{NewtK}^{-} &\;=\!\mid\!\models\; \mathbf{Flxspecrel} + \mathbf{(vel)}\,,\\
\mathbf{NewtK}^{-} &\;=\!\mid\!\models\; \mathbf{Flxspecrel} + \mathbf{(clock)}^{=}\,,\\
\mathbf{NewtK}^{-} &\;=\!\mid\!\models\; \mathbf{Flxspecrel} + \neg\mathbf{(clock)}\,,\\
\mathbf{NewtK}^{-} &\;=\!\mid\!\models\; \mathbf{Flxspecrel} + \neg\mathbf{(asynch)}\,.
\end{aligned}$$

We give the proof of Proposition 4.1.13 beginning with p.443.

The next three propositions show the roles of axioms $c = \infty$, $\mathbf{Ax}(\uparrow\uparrow)$ and $\mathbf{Ax}\square\mathbf{1}$ in **NewtK**. Proposition 4.1.14 below deals with $c = \infty$.

**PROPOSITION 4.1.14** *Assume* $\mathbf{Ax}(\sqrt{\ })$.

$$\begin{aligned}
\mathbf{Flxspecrel} &\models\; c = \infty \;\leftrightarrow\; \mathbf{(vel)}\,,\\
\mathbf{Flxspecrel} &\models\; c = \infty \;\leftrightarrow\; \mathbf{(clock)}^{=}\,,\\
\mathbf{Flxspecrel} &\models\; c = \infty \;\leftrightarrow\; \neg\mathbf{(clock)}\,,\\
\mathbf{Flxspecrel} &\models\; c = \infty \;\leftrightarrow\; \neg\mathbf{(asynch)}\,.
\end{aligned}$$

We give the proof of Proposition 4.1.14 beginning with p.443.

Propositions 4.1.15, 4.1.16 below talk about the roles of axioms $\mathbf{Ax}(\uparrow\uparrow)$, $\mathbf{Ax}\square\mathbf{1}$. Note that $\mathbf{NewtK}^- \;=\!\mid\!\models (\mathbf{Flxspecrel} + c = \infty)$.

**PROPOSITION 4.1.15** *Assume* $\mathbf{Ax}(\sqrt{\ })$.

$$\begin{aligned} \mathbf{NewtK}^- &\models \mathbf{Ax}(\uparrow\uparrow) \leftrightarrow \mathbf{(abstime)}, \\ \mathbf{NewtK}^- + \mathbf{Ax}(\uparrow\uparrow) &\models \mathbf{Ax}\square\mathbf{1} \leftrightarrow \mathbf{(abspace)}. \end{aligned}$$

**PROPOSITION 4.1.16** *Assume* $\mathbf{Ax}(\sqrt{\ })$. *In* $\mathbf{NewtK}^-$, *statements (i) and (ii) below are equivalent.*

**(i)** $\mathbf{Ax}(\uparrow\uparrow) + \mathbf{Ax}\square\mathbf{1}$

**(ii)** *The world-view transformations are Galilean.*

The next proposition and the question below deal with the connections between **NewtK** and $\mathbf{NewtK}^-$.

**PROPOSITION 4.1.17** *Assume* $\mathbf{Ax}(\sqrt{\ })$. *In* $\mathbf{NewtK}^-$, *statements (i) – (iii) below are equivalent with each other.*

**(i)** **(abspace)**

**(ii)** $(\mathbf{meter})^=$

**(iii)** *The world-view transformations are generalized Galilean.*

We give the proof of Propositions 4.1.15 – 4.1.17 beginning with p.443.

**QUESTION 4.1.18** *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Is* $\mathbf{NewtK} \;=\!\mid\!\models (\mathbf{NewtK}^- + \mathbf{Ax}(\uparrow\uparrow))$ *true? Equivalently, is* $\mathbf{NewtK}^- + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{(abspace)}$ *true? Still in other words, is* $\mathbf{Ax}\square\mathbf{1}$ *needed in (i) of Proposition 4.1.17?*

We now turn to proving Propositions 4.1.12 – 4.1.17.

<u>Proof of Propositions 4.1.12 – 4.1.17.</u>

Throughout the proof we assume $\mathbf{Ax}(\sqrt{\ })$. First we will prove statements (1) - (11), the propositions will follow easily from these.

**(1)** The world-view transformations in models of $\mathbf{NewtK}^-$ are affine transformations.

Proof of (1): The world-view transformations in models of $\mathbf{NewtK}^-$ are collineations by $\mathbf{Bax} \subseteq \mathbf{NewtK}^-$ and Theorem 3.4.36. Then to show that they are affine (i.e. no field-automorphism is involved in them) the proof of Proposition 2.9.5 on p.155 goes through by $\mathbf{Ax}(\sqrt{\ })$ and $\mathbf{Ax(symm)} \in \mathbf{NewtK}^-$. ∎

Let $a \in F$. Define

$$S(a) \stackrel{\text{def}}{=} \{p \in {}^nF : p_t = a\}\,.$$

We call $S(a)$ the <u>$a$-simultaneity</u>. We say that the <u>distance</u> between $S(a)$ and $S(b)$ is $|a-b|$.

**(2)** The world-view transformations in models of $\mathbf{NewtK}^-$ take simultaneities to simultaneities so that they preserve the distance between simultaneities.

Proof of (2): Assume $\mathfrak{M} \models \mathbf{NewtK}^- + \mathbf{Ax}(\sqrt{\ })$, $m,k \in Obs$ and $f \stackrel{\text{def}}{=} \mathsf{f}_{mk}$. By $c = \infty$ we have that $f$ takes a simultaneity to a simultaneity, i.e. $(\forall a \in F)(\exists b \in F) f[S(a)] = S(b)$. Then $b = f(a \cdot 1_t)_t$. Since $f$ is affine, it either enlarges the distance between any two simultaneities, or else it decreases the distance between any two simultaneities, and then $f^{-1}$ behaves just the opposite way.[359] By $\mathbf{Ax(symm)} + \mathbf{Ax}(\|)$ we have that $f = \sigma \circ f^{-1} \circ \delta$ for some isometric world-view transformations $\sigma, \delta$ which take $\bar{t}$ parallel to $\bar{t}$. Then $\sigma, \delta$ also take simultaneities to simultaneities and they preserve distance between simultaneities because they are isometries. Thus $f = \sigma \circ f^{-1} \circ \delta$ can hold only if $f$ both enlarges and decreases the distance between simultaneities, i.e. if $f$ preserves these distances. ∎

**(3)** In models of $\mathbf{NewtK}^-$, the world-view transformations are generalized Galilean for observers in standard configuration.

Proof of (3): We may assume $n \geq 2$. We have $|\mathsf{f}_{mk}(1_t)_t| = 1$ and $\mathsf{f}_{mk}[S(0)] = S(0)$ by (2). Let $1 < i < n$. Now we can prove $\mathsf{f}_{mk}(1_i) = 1_i$ by an argument using $\mathbf{Ax(symm)} + \mathbf{Ax}(\|)$ similar to one in the proof of (2), and using that $\mathsf{f}_{mk}[\bar{x}_i] = \bar{x}_i$, $\mathsf{f}_{mk}(1_i) = \lambda \cdot 1_i$ for some $\lambda > 0$ hold since $m,k$ are in standard configuration. ∎

**(4)** The world-view transformations in a model of $\mathbf{NewtK}$ are Galilean.

[359] This is so because e.g. if $f(1_t)_t > 1, f(\bar{0}) = \bar{0}$ and $a,b \in F$, then $|f(a \cdot 1_t)_t - f(b \cdot 1_t)_t| = |f((a-b) \cdot 1_t)_t| = |(a-b) \cdot f(1_t)_t| > |a-b|$.

Proof of (4): Theorem 5.0.49 on p.713 (§5) states that standard configurations work in $\mathbf{Relnoph}_0$. That means that $(\forall m,k \in Obs)(\exists m',k' \in Obs)$ $[tr_m(m') \in Triv,\ tr_k(k') \in Triv$ *and* $m',k'$ *are in standard configuration*$]$. The only axiom in $\mathbf{Relnoph}_0$ which does not occur in $\mathbf{NewtK} + \mathbf{Ax}(\sqrt{\ })$ is $\mathbf{Ax}\triangle\mathbf{1}$, but it is easy to check that $\mathbf{Ax}\triangle\mathbf{1}$ is not used in the proof of Theorem 5.0.49. Now (3) finishes the proof of (4). ∎

The following three statements are not hard to check, we leave them to the reader.

**(5)** If the world-view transformations are all generalized Galilean in a model $\mathfrak{M}$, then **(vel)**, **(abspace)**, $\mathbf{(meter)}^=$ hold in $\mathfrak{M}$.

**(6)** If the world-view transformations are all Galilean in a model $\mathfrak{M}$, then $\mathbf{Ax}(\uparrow\uparrow)$, **(abstime)** hold in $\mathfrak{M}$.

**(7)** In models of $\mathbf{NewtK}^-$, the statements **(abspace)**, $\mathbf{(meter)}^=$, and "all the world-view transformations are generalized Galilean" are equivalent.

**(8)** In models of **Flxspecrel**, either one of statements **(vel)**, **(abstime)**, $\neg$**(clock)**, $\neg$**(asynch)** implies $c = \infty$.

Proof of (8): Let $\psi$ be any one of the statements **(vel)**, ..., $\neg$**(asynch)** and let $\mathfrak{M}$ be any model of $\mathbf{Flxspecrel} + \mathbf{Ax}(\sqrt{\ })$. We will prove $\mathfrak{M} \models (c \neq \infty \ \Rightarrow \neg\psi)$. Assume $\mathfrak{M} \models c \neq \infty$. Then, as we stated in Remark 4.1.7, $\mathfrak{M}$ is like a model $\mathfrak{M}'$ of **Newbasax**, except that the unit of measurement of time may be different. One can check that $\mathfrak{M}' \models \mathbf{Ax6}+\mathbf{Ax(symm)}+\mathbf{Ax}(\|)$, by $\mathfrak{M} \models \mathbf{Ax6}+\mathbf{Ax(symm)}+\mathbf{Ax}(\|)$. Then $\neg\psi$ is true in $\mathfrak{M}'$ by the theorems in §2.8. One can check that then $\neg\psi$ will be true in $\mathfrak{M}$ also. ∎

**(9)** $\mathbf{Flxspecrel} \models \mathbf{(abstime)} \rightarrow \mathbf{Ax}(\uparrow\uparrow)$.

Proof of (9): First we prove $\mathbf{(abstime)} \rightarrow \mathbf{Ax}(\uparrow)$. Assume $tr_m(k) = \bar{t}$ and $m \downarrow k$. We will derive a contradiction. Let $a \stackrel{\text{def}}{=} \mathsf{f}_{km}(\bar{0})_t$. Now by **Ax(eqtime)**, $m \downarrow k$, and $tr_m(k) = \bar{t}$ we have that both $m$ and $k$'s clocks show $a/2$ at $a/2 \cdot 1_t$, and this contrdicts **(abstime)** and $m \downarrow k$.

Assume now that $m,k \in Obs$ are arbitrary. Let $k' \in Obs$ be such that $tr_k(k') = \bar{t}$ and $\mathsf{f}_{k'm}(\bar{t})_t = 0$. Then $m \uparrow k'$ by **(abstime)**, and $k' \uparrow k$ by the above. Hence $m \uparrow k$. ∎

The next statement, (10), is not difficult to check, we leave it to the reader.

**(10)** In models of $\mathbf{NewtK^-}+\mathbf{Ax(\uparrow\uparrow)}+\mathbf{(abspace)}$ the world-view transformations are Galilean.

**(11)** $\mathbf{NewtK^-}+\mathbf{Ax(\uparrow\uparrow)} \models \mathbf{(abspace)} \to \mathbf{Ax\Box 1}$.

Proof of (11): Assume $\mathfrak{M} \models \mathbf{NewtK^-}+\mathbf{Ax(\uparrow\uparrow)}+\mathbf{(abspace)}$, and let $m, k, m' \in Obs$. We may assume $\bar{0} \in tr_m(k)$. Let $k' \in Obs$ be such that $tr_{m'}(k') = tr_m(k)$. Then by (10), both $\mathsf{f}_{mk}$ and $\mathsf{f}_{m'k'}$ are Galilean. Let $\delta$ and $\sigma$ denote $\mathsf{f}_{mk}$ and $\mathsf{f}_{m'k'}$ restricted to $S(0)$, respectively. Then $\delta, \sigma$ are isometries of $S(0)$. Let $k'' \in Obs$ be such that $\bar{0} \in tr_{k'}(k'') = \bar{t}$ and $\mathsf{f}_{k'k''}$ restricted to $S(0)$ is $\sigma^{-1} \circ \delta$. Such a $k''$ exists by $\mathbf{Ax}(\mathit{Triv})$. Now, $\mathsf{f}_{mk}$ and $\mathsf{f}_{m'k''}$ agree on $S(0)$. Since both are Galilean, $tr_m(k) = tr_{m'}(k'')$, and $\mathbf{Ax(\uparrow\uparrow)}$ holds, they are equal by the definition of a Galilean transformation. ∎

Now, it is not difficult to check that Propositions 4.1.12 – 4.1.17 follow from statements (1) – (11). By this, we finished the proofs of Propositions 4.1.12 – 4.1.17.

### Newtonian kinematics is true for small velocities

In the introduction we mentioned the "view" that Newtonian kinematics describes a smaller world (world of small relative velocities) while special relativity describes a bigger world extending "Newton's smaller world". This brings up the question: Does the old theory of the small world remain true as a part of the new theory or is the old theory completely inconsistent with the new one? Below we will show that it is possible to formalize special relativity such that Newtonian kinematics will remain a part of it (in a precise sense of mathematical logic).

To carry out the above, we use non-Archimedean fields.

**Definition 4.1.19** An ordered field $\mathfrak{F}$ is called <u>*Archimedean*</u> if the "integers are cofinal in $\mathfrak{F}$", i.e. if

$$(\forall x \in F)(\exists n \in \omega) x < n.$$

$\lhd$

We note that we consider integers (elements of $\omega$) and rational numbers ($n/m$ for $n, m \in \omega$) as elements of any field: $n \in F$ denotes the element of $F$ which we obtain by adding the unit of $\mathfrak{F}$, 1, to itself $n-1$ times. I.e.

$$n = \underbrace{1 + \ldots + 1}_{n \text{ times}} .$$

It is known in algebra that the Archimedean fields are exactly the isomorphic copies of subfields of the reals $\mathfrak{R}$. Note that being Archimedean is not a first-order expressible property.[360]

Assume that $\mathfrak{F}$ is non-Archimedean. Then there is $x \in F$ such that $x > n$ for all $n \in \omega$. This means that the reciprocal $y \stackrel{\text{def}}{=} 1/x$ of $x$ is smaller than each rational number. Thus, $y$ is an "infinitely small number" (or element). We axiomatize the notion of "small" as follows.

**Definition 4.1.20** Let $\mathfrak{F}$ be an ordered field and let $I \subseteq F$. We say that $I$ is an *"ideal of infinitely small numbers"* if $I$ satisfies the following:

(i) $(x \in I \text{ and } |y| < |x|\,) \implies y \in I$.

(ii) $x, y \in I \implies x + y \in I$.

(iii) $x \in I \implies 1/x \notin I$.

(iv) $(x \in I \text{ and } 1/y \notin I\,) \implies x \cdot y \in I$.

We say that $x$ is *infinitely small* if $x \in I$. We say that $x$ is *infinitely large* if $1/x \in I$. We say that $x$ is *finite* (or small) if $1/x \notin I$. ◁

Figure 132 represents these notions.

Assume that $I$ is an ideal of infinitely small elements of $\mathfrak{F}$. Then each nonzero rational number is a finite, not infinitely small element of $\mathfrak{F}$.

Let $\mathfrak{M}$ be a model of **Flxspecrel**, let $\mathfrak{F}$ be the field-part of $\mathfrak{M}$, and let $c$ be the (square of) speed of light in $\mathfrak{M}$. (Cf. Def. 4.1.4.) Let $I$ be an ideal of infinitely small elements in $\mathfrak{F}$, and assume that $1/c \in I$. We are going to define a new model $\mathfrak{M}_0/I$ from $\mathfrak{M}$ such that we keep only observers with finite speed, and we consider everything only up to infinitely small quantities. We turn to defining $\mathfrak{M}_0/I$ now.

Set

$$F_0 \stackrel{\text{def}}{=} F \setminus \{1/x : x \in I\}.$$

$F_0$ is the set of *finite* elements of $F$. For all $x, y \in F$ we define

$$x \cong_I y \iff (x - y) \in I.$$

[360]E.g. no nonprincipal ultrapower of a field is Archimedean.

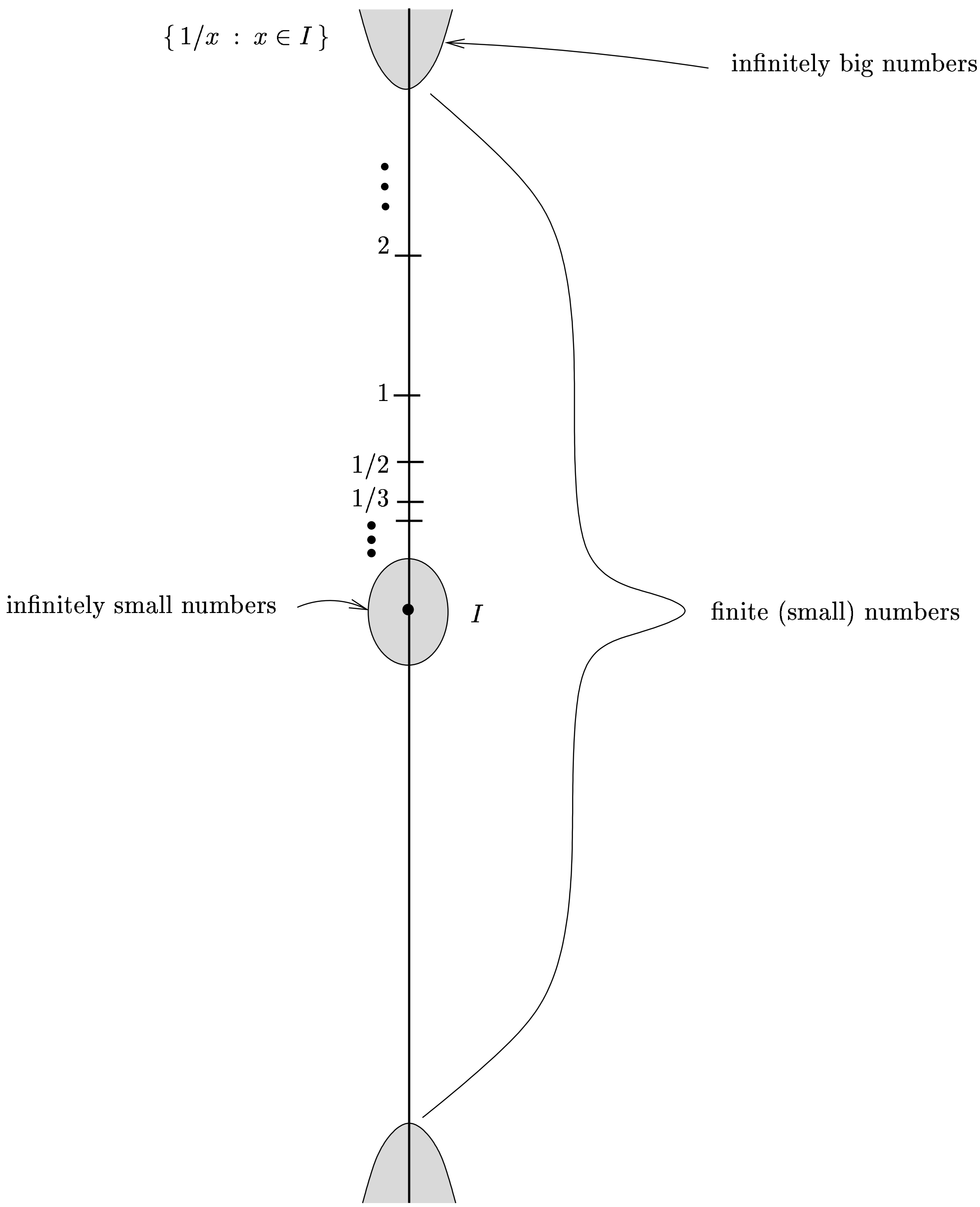



Figure 132: A non-Archimedean field in which $I$ is an ideal of infinitely small elements.

Then $\cong_I$ is an equivalence relation determined by the ideal $I$ the usual way. Now, $F_0$ (together with the operations of $\mathfrak{F}$) is not a field but only a ring. However, $\mathfrak{F}_0/{\cong_I} \stackrel{\text{def}}{=} \mathfrak{F}_0/I$ is a field already. We note that the universe of $\mathfrak{F}_0/I$ is $F_0/I \stackrel{\text{def}}{=} \{[x]_I : x \in F_0\}$, where $[x]_I \stackrel{\text{def}}{=} \{y \in F_0 : y \cong_I x\}$ denotes the equivalence class of $\cong_I$ $x$ is in.

Let $m_0 \in Obs$ be arbitrarily fixed. Define

$$Obs_0 \stackrel{\text{def}}{=} \{m \in Obs : v_{m_0}(m) \in F_0\},$$

$$G_0 \stackrel{\text{def}}{=} \mathsf{Eucl}(n, \mathfrak{F}_0/I),$$

$$W/I \stackrel{\text{def}}{=} \{\langle m, [p_0]_I, \dots, [p_{n-1}]_I, b\rangle : \langle m, p_0, \dots, p_{n-1}, b\rangle \in W, p_0, \dots, p_{n-1} \in F_0\},$$

$$\mathfrak{M}_0 \stackrel{\text{def}}{=} \langle (B, Obs_0, Ph, Ib), \mathfrak{F}_0/I, G_0, \epsilon, W/I\rangle.$$

By the above, the model $\mathfrak{M}_0/I$ has been defined. The next proposition says that this model is a model of $\mathbf{NewtK}^-$. Proposition 4.1.21 below expresses, to our minds, that in special relativity, Newtonian kinematics is true for small velocities, with negligible error. Here "small" means "finite element of $\mathfrak{F}$", and "negligible" means "element of $I$".

**PROPOSITION 4.1.21** $\mathfrak{M}_0/I \models \mathbf{NewtK}^-$.

**On the proof:** First we show that the speed of all photons is infinite in the new model $\mathfrak{M}_0/I$. The proof is illustrated in Figure 133. Let $m \in Obs_0, ph \in Ph$. Then $v_m(ph) = c$ in the original model $\mathfrak{M}$. This means that in one time-unit (minute), $ph$ travels $c$ space-units (kilometers). Let $x \in F_0$. Then it takes $x/c$ minutes for $ph$ to travel $x$ kilometers. Now, $x/c \in I$ by our assumptions $x \in F_0, 1/c \in I$ and by condition (iv) in the definition of an ideal of infinitely small numbers. Thus, $tr_m(ph)$ is a straight line with infinite slope in $\mathfrak{M}_0/I$. See Figure 133.

To show that **Ax6** holds in $\mathfrak{M}$, we use that $\mathsf{f}_{mk}$ is continuous in $\mathfrak{M}$, and thus $p_i \cong_I q_i$ for all $i < n$ implies that $\mathsf{f}_{mk}(p)_i \cong_I \mathsf{f}_{mk}(q)_i$ for all $i < n$. We omit the rest of the proof. ∎

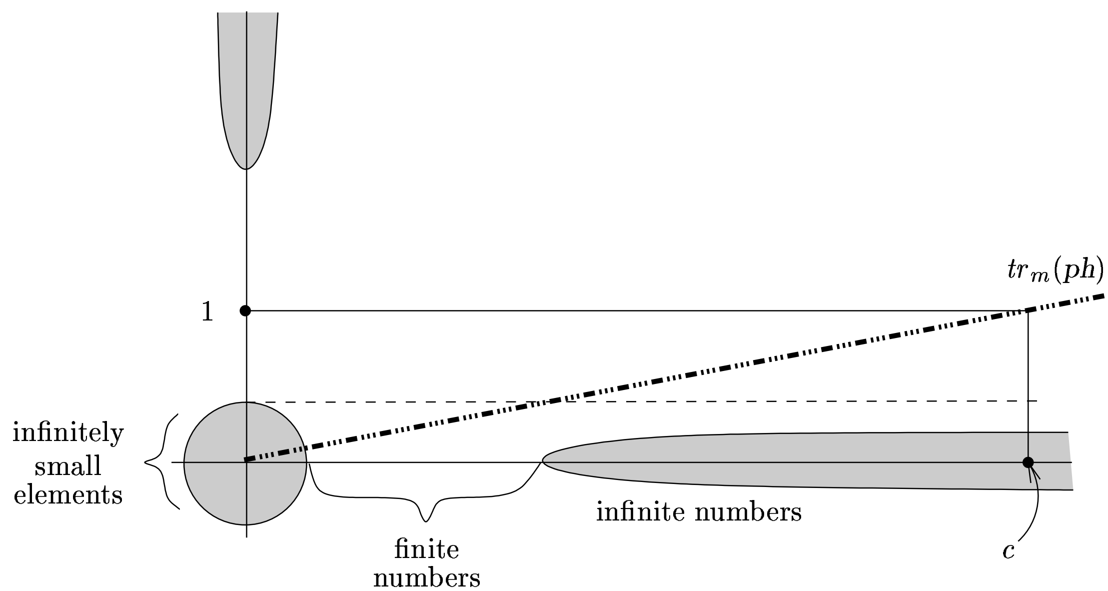



Figure 133: It takes infinitely small time for a photon $ph$ to travel a finite distance: the speed of photons is infinite in $\mathfrak{M}_0/I$.

## On the lattice of first order theories

As Tarski wrote (cf. [250]), deductively closed theories are those "organic units" which, among other things, are at the center of the investigations in logic. Therefore, "larger" structures like e.g. the lattice of theories whose elements are these organic units are important for studying logic.

Recall that $\mathsf{FM} = \mathsf{Mod}(\emptyset)$ is the class of models of our frame language for relativity, cf. §2.1, p.35. Let

$$\mathsf{TH} \stackrel{\text{def}}{=} \{\mathsf{Th}(\mathsf{K}) : \mathsf{K} \subseteq \mathsf{FM}\}$$

be the class of ***deductively closed theories*** in our frame language. Similarly,

$$\mathsf{EC}_\Delta \stackrel{\text{def}}{=} \{\mathsf{Mod}(\Gamma) : \Gamma \text{ is a set of formulas in our frame language}\}$$

is the collection of axiomatizable classes of models (in our language).[361] Set theoretical inclusion "$\subseteq$" makes these two collections partially ordered structures:

$$\mathbb{TH}_0 \stackrel{\text{def}}{=} \langle \mathsf{TH}, \subseteq \rangle \quad \text{and}$$

$$\mathbb{EC}_{\Delta,0} \stackrel{\text{def}}{=} \langle \mathsf{EC}_\Delta, \supseteq \rangle.$$

Such structures are called ***posets*** (for partially ordered sets). Our operator Mod is a ***dual isomorphism*** between these two structures

$$\mathsf{Mod} : \mathbb{TH}_0 \rightarrowtail\!\!\!\twoheadrightarrow \mathbb{EC}_{\Delta,0} \quad ;$$

in particular,

$$Th_1 \supseteq Th_2 \quad \Longleftrightarrow \quad \mathsf{Mod}(Th_1) \subseteq \mathsf{Mod}(Th_2).$$

Recall that in a partially ordered set $\mathsf{sup}(x, y)$ and $\mathsf{inf}(x, y)$ denote the supremum and infimum of two elements $x, y$ if these exist. I.e. $\mathsf{sup}(x, y)$ exists if $x$ and $y$ have a least upper bound in the poset, and then $\mathsf{sup}(x, y)$ denotes this least upper bound. Similarly, $\mathsf{inf}(x, y)$ denotes the largest lower bound of $x, y$ if this exists. Thus sup and inf are partial binary operations on posets.

**FACT 4.1.22** *Posets $\mathbb{TH}_0, \mathbb{EC}_{\Delta,0}$ turn out to be lattices, i.e. the binary operations* sup *and* inf *are definable in them (and they always exist).* $\lhd$

[361] Officially, "axiomatizable classes" are called "elementary classes" and they are denoted as $\mathsf{EC}_\Delta$ where "$\Delta$" indicates that the classes in question need not be finitely axiomatizable.

This way we obtain the lattices

$$\mathbb{TH} \stackrel{\text{def}}{=} \langle \mathsf{TH}, \mathsf{sup}, \mathsf{inf}, \subseteq \rangle \quad \text{and}$$

$$\mathbb{EC}_\Delta \stackrel{\text{def}}{=} \langle \mathsf{EC}_\Delta, \mathsf{inf}, \mathsf{sup}, \supseteq \rangle.$$

$\mathsf{Mod} : \mathbb{TH} \rightarrowtail\!\!\twoheadrightarrow \mathbb{EC}_\Delta$ is a dual isomorphism between these two lattices, e.g.

$$\mathsf{Mod}(Th_1 \ \mathsf{sup}\ Th_2) = \mathsf{Mod}(Th_1)\ \mathsf{inf}\ \mathsf{Mod}(Th_2).$$

**PROPOSITION 4.1.23** *In* $\mathbb{EC}_\Delta$ $\mathsf{inf}$ *and* $\mathsf{sup}$ *coincide with the usual set theoretic operations "$\cap$" and "$\cup$". Hence*

$$\mathbb{EC}_\Delta = \langle \mathsf{EC}_\Delta, \cap, \cup, \supseteq \rangle.$$

∎

**PROPOSITION 4.1.24** *In the lattice* $\mathbb{TH}$*, we have* $Th_1\,\mathsf{inf}\,Th_2 = Th_1 \cap Th_2$*, while* $Th_1\,\mathsf{sup}\,Th_2 = \mathsf{Th}(\mathsf{Mod}(Th_1 \cup Th_2))$. ∎

Summing up:

$$\langle \mathsf{TH}, \mathsf{sup}, \cap, \subseteq \rangle \quad \underset{Th}{\overset{\mathsf{Mod}}{\rightleftarrows}} \quad \langle \mathsf{EC}_\Delta, \cap, \cup, \supseteq \rangle$$

represents the structure(s) associated to the lattice of first order theories (elaborated for our frame language[362] ).

We can think about $\mathbb{EC}_\Delta = \langle \mathsf{EC}_\Delta, \cap, \cup, \supseteq \rangle$ as a <u>*representation*</u> of our lattice $\mathbb{TH}$ of theories in the sense that in $\mathbb{EC}_\Delta$ all the operations are concrete, set-theoretic ones. However, $\mathbb{EC}_\Delta$ is more important than just a representation for $\mathbb{TH}$, namely $\mathbb{EC}_\Delta$ is the <u>*semantic form*</u> or semantic version of our lattice of theories.

This lattice helps us in working with several theories at the same time. E.g.: <u>*the least common generalization*</u> of two deductively closed theories $Th_1$, $Th_2$ is defined to be

$$Th_1\,\mathsf{inf}\,Th_2 = Th_1 \cap Th_2 = \mathsf{Th}(\mathsf{Mod}(Th_1) \cup \mathsf{Mod}(Th_2)).$$

**CONVENTION 4.1.25** If we apply this notion to theories which are not deductively closed, then what *we mean* is the least common generalization of *their deductive closures.* Then

$$\text{least common generalization}(Th_1, Th_2) \stackrel{\text{def}}{=} \mathsf{Th}(\mathsf{Mod}(Th_1) \cup \mathsf{Mod}(Th_2)).$$

◁

[362] The same ideas work for any language of (many-sorted) first order logic.

**CONVENTION 4.1.26** Whenever we discuss theories which are not deductively closed, as soon as we discuss their place in the lattice of theories, we *automatically* switch over to the deductive closures of the theories in question. Further, suppose we are discussing theories $Th_1, \ldots, Th_k$. Then by the lattice of these theories, we understand the sublattice of $\mathbb{TH}$ generated by the deductive closures of $Th_1, \ldots, Th_k$. $\lhd$

For more on the lattice of theories (and on its generalizations, e.g. the category of theories) we refer to [12], [105]. In this connection we note that in standard logic books the lattice $\mathbb{TH}$ is usually introduced via discussing the so called *Galois connection* induced by the binary relation "$\models$" ($\subseteq \mathsf{FM} \times Fm$) between the posets $\langle \mathcal{P}(Fm), \subseteq \rangle$ and $\langle \mathcal{P}(\mathsf{FM}), \subseteq \rangle$.[363] This Galois connection induces the two functions $\mathsf{Mod} : \mathcal{P}(Fm) \longrightarrow \mathcal{P}(\mathsf{FM})$ and $\mathsf{Th} : \mathcal{P}(\mathsf{FM}) \longrightarrow \mathcal{P}(Fm)$, connecting the "world of models" with the "world of formulas" and vice versa. The theory of this Galois connection is called *"syntax-semantics duality"*. Other source books discuss $\mathbb{EC}_\triangle$ first and might only briefly mention[364] that it represents the lattice $\mathbb{TH}$ of theories. Besides the above references, we also refer to P. M. Cohn [60], Bell & Slomson [45, Chap.7, pp.140-160], Henkin, Monk, & Tarski [129].

**On drawing the lattice $\mathbb{TH}$ of all theories:** Several figures in this work represent sublattices of $\mathbb{TH}$. Such figures are e.g. on pages 429, 433, 552, 583, 593, 653. Although occassionally we refer (intuitively) to these figures as representing sublattices of $\mathbb{TH}$, in reality they represent only subposets of $\mathbb{TH}_0$ because e.g. in Figure 180, $\mathbf{Bax}^{--}$ is the largest theory indicated which is lower than both $\mathbf{Bax}_{\mathbf{3}}^{--}$ and $\mathbf{Bax}_{++}^{--}$, yet we do not want to claim that $\mathbf{Bax}^{--}$ would be the infimum of $\mathbf{Bax}_{\mathbf{3}}^{--}$ and $\mathbf{Bax}_{++}^{--}$.[365] In these figures, the theories further up are stonger ones, and the theories lower down are weaker ones (we call a theory the stronger the more theorems it proves). In accordance with our Convention 4.1.25, in the figures the axiom systems represent their deductive closures. Thus, if $Th_1$ and $Th_2$ are connected with a line and $Th_2$ is further up, then this means that $Th_2 \models Th_1$.

Sometimes we say that in the figure we assume, say, $\mathbf{Ax}(\sqrt{\ })$. By this we mean that if in the figure we add $\mathbf{Ax}(\sqrt{\ })$ to all the theories, then we get a subposet of $\mathbb{TH}_0$ the above way.

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[363]Here $\mathsf{FM}$ is the class of our models and $Fm$ is the set of our (frame) formulas, cf. §2.1.

[364]some books forget to mention this explicitly

[365]This should create no confusion since each subposet generates a unique sublattice (so a figure representing a subposet indirectly represents a sublattice, too). Our preference for the expression "sublattice" (over subposet) comes only from our impression that sublattices are more broadly known than subposets.

## 4.2 Weakening the symmetry principle Ax(symm) corresponding to Einstein's Special Principle of Relativity (SPR)

The present section (§4.2) is a kind of continuation of §2.8 ("Some symmetry axioms") and of §3.9 (Symmetry axioms).

In the present chapter (§4) we want to investigate weak sub-systems of **Basax** (such as e.g. **Bax**). Some of these will be of philosophical significance connected either to the Reichenbach-Grünbaum version of relativity or to Friedman's conceptual analysis of relativity. As we indicated earlier, when investigating a sub-system (like e.g. **Bax**) of **Basax**, we also want to investigate what happens if we add to the sub-system in question a symmetry principle like **Ax(symm)** corresponding to Einstein's Special Principle of Relativity, SPR.[366] E.g. when investigating **Bax** as a possible relativity theory, we also intend to investigate **Bax+Ax(symm)** or something like this as say, the "symmetry-enriched" version of **Bax**.

So far, in earlier parts of the present work, we have introduced the theories **Basax**, **Newbasax**, **Bax**, **Flxbasax**. In a similar spirit we will introduce a theory **Reich**(**Bax**) in §4.5. **Reich**(**Bax**) will be what we call the Reichenbach-Grünbaum version of **Bax**, and we will write about it in the present section without recalling[367] it from §4.5 (Def.4.5.3, p.562). The above theories form a hierarchy

($\star$) **Basax** > **Newbasax** > **Flxbasax** > **Bax** > **Reich**(**Bax**).

Since we introduced the *symmetry principle* **Ax(symm)** in §2.8, we can study the *symmetric versions*

($\star\star$) **Basax+Ax(symm)**, **Newbasax+Ax(symm)**,
**Flxbasax+Ax(symm)**,

etc. of all these theories.

In the pattern "*Th* + **Ax(symm)**" we call *Th* the <u>*core theory part*</u>, and **Ax(symm)** the <u>*symmetry principle part*</u> of the "symmetrized" theory *Th* + **Ax(symm)**.

[366]This is why we compared **Ax(symm)** to the axiom of choice of set theory (namely you may choose to add it to your system or you may choose not to).

[367]Explaining the philosophy of **Reich**(**Bax**) would take up quite some space and for the present purposes it is enough to know that **Reich**(**Bax**) is much weaker than **Bax**.

Our *general strategy* (concerning symmetry principles) is the following. We introduce core theories $Th_1, Th_2, \dots, Th_n, \dots$ analogously to the ones listed in ($\star$) above. Then for each core theory $Th_n$ we want to *define* and study its *symmetric version* $Th_n$ + "symmetry principle" analogously to the symmetric versions listed in ($\star\star$) above. The intuitive motivation *why* we use this dichotomy (or decomposition)

$$(\star\star\star) \qquad \text{"core theory"} + \text{"symmetry principle"}$$

and how we choose the symmetry principle for a given core theory will be discussed in Remark 4.2.18 around the end of this section (§4.2). For the time being, it is enough to know that we want to build up (or decompose) our theories according to this pattern (cf. ($\star\star\star$) above) and that we want to do this in a systematic way. We hope that the example of the symmetric version **Basax+Ax(symm)** introduced in §2.8 of **Basax** will provide sufficient intuition for what we want to do next.

**Ax(symm)** is suitable for defining the symmetric versions of **Basax**, **Newbasax**, and **Flxbasax** in the style of ($\star\star\star$) above. However, for many of our *weaker* theories $Th_1, \dots, Th_n, \dots$ to be introduced in later parts of the present chapter (§4), **Ax(symm)** will turn out to be *too strong* for defining the symmetric version of $Th_n$ in the style of ($\star\star\star$). What do we mean by saying that **Ax(symm)** might be too strong for $Th_n$? A more careful answer will be given in Remark 4.2.18 way below, but the basic idea is the following.

**Ax(symm)** might "blur" the distinction between $Th_n$ and $Th_{n-1}$. A strong form of *"blurring"* this distinction is the case if

$$Th_n + \mathbf{Ax(symm)} \mathrel{=\!\!|\!\!\models} Th_{n-1} + \mathbf{Ax(symm)}.$$

If this happens, if $Th_n < Th_{n-1}$, and if the distinction between $Th_{n-1}$ and $Th_n$ *was* important in introducing $Th_n$, then we say that **Ax(symm)** is too strong for studying $Th_n$, or equivalently too strong for defining the symmetric version of $Th_n$. Further, if

$$Th_n + \mathbf{Ax(symm)} + \text{"some auxiliary axioms"} \models Th_{n-1}$$

would turn out to be the case where $Th_n$ was *intended* to be a subtheory of **Basax** *strictly weaker* than $Th_{n-1}$ (in some subtle but *essential* respect), then we will again say that **Ax(symm)** is too strong for studying $Th_n$ (since it blurs distinctions between $Th_n$ and other theories), *and* in such cases we will use "more refined" symmetry principles like e.g. **Ax(syt)**, to be introduced soon, in place of **Ax(symm)**.[368] We will also use the expression that a symmetry principle is *not adequate* for $Th_n$,

[368]for creating the symmetric version of *Th*.

meaning that it is too strong or that it blurs some important distinction in the above indicated sense.[369] Indeed, in Thm.4.2.4(ii) below we will find that under mild conditions

$$\mathbf{Bax} + \mathbf{Ax(symm)} \models \mathbf{Flxbasax}.$$

In view of the above discussion, this means that **Ax(symm)** is too strong for **Bax**. This also means that we consider **Ax(symm)** as not suitable for defining the symmetric version of **Bax**, hence **Bax+Ax(symm)** is *not* the symmetric version of **Bax** (in our terminology).[370]

This leads us to the following *research task*.

We look at our weak theories

$$\mathbf{Bax},\ \mathbf{Reich}(\mathbf{Bax}), \dots, Th_n$$

and we *search for a symmetry* principle which is *adequate* for the weak theory in question, say, $Th_n$. Roughly, this means a triple task: The principle, call it *sym*, is *adequate* for $Th_n$ if

(i) is *weak enough* so that it does not blur distinctions important in the definition of $Th_n$,

(ii) is *strong* enough for proving interesting theorems from $Th_n + sym$, and

(iii) is formulated in a "spirit" compatible with the spirit of the definition of $Th_n$.[371]

As an illustration of (i)–(iii) above we note the following. For the choice of $Th_n$ = **Bax**, and symmetry principle **Ax(syt)** to be introduced soon, part (i) of adequateness is "achieved" by Thm.4.2.2, part (ii) by Thm.4.2.9, and part (iii) by the discussion at the end of Remark 4.2.3.

As the reader might already guess on the basis of the above example, in the present section we will carry out the above outlined "triple task" for the case of the theory **Bax**. I.e. we will seek out a symmetry principle **Ax(syt)** adequate for **Bax**. After defining **Ax(syt)**, we will prove that **Ax(syt)** is indeed adequate for **Bax** (i.e. that it satisfies conditions (i)–(iii) above, when they are made precise in a certain way[372]). Then, we study the relationship between **Bax** and its symmetric version

[369] Roughly speaking, "not adequate" is the same as "too strong" (the difference is only in what aspect or "intuition" we want to emphasize).

[370] Again, we refer to Remark 4.2.18 for motivation.

[371] Goals (i)–(iii) above may look vague at this point, but they will be made more tangible in Remark 4.2.18 way below and in §4.7.

[372] E.g. to prove (ii), we show that under mild assumptions, the symmetric version **Bax+Ax(syt)** of **Bax** proves the twin paradox **Ax(TwP)**.

**Bax+Ax(syt)**. After this we briefly discuss related questions, e.g. how the new symmetry principle **Ax(syt)** relates to the things that we have studied before, for example that it can be considered as a weakened version of **Ax(symm)**, in the sense that under mild assumptions **Bax** $\models$ **Ax(symm)** $\rightarrow$ **Ax(syt)**, cf. Thm.4.2.13 and **Bax** $\models$ **Ax(syt$_0$)** $\rightarrow$ **Ax(TwP)**, see Thm.4.2.9. This will complete our discussion of the case of the core theory **Bax**.

Finding the *symmetry principles adequate* for our *further* weak theories like **Reich**(**Bax**) will be addressed in later sections, cf. e.g. Remark 4.2.18 and §4.7. However, we emphasize one thing already here; namely that **Basax** $>$ **Bax** $>$ $Th_n$ does not necessarily imply that the symmetry principle, call it $sym_n$, adequate for $Th_n$ should be weaker than, or even comparable with, the principles **Ax(symm)** or **Ax(syt)** adequate for **Basax** and **Bax** respectively.[373]

Let us turn to introducing **Ax(syt)**, the symmetry principle which we will consider to be adequate for **Bax**. We note that the "name" **Ax(syt)** intends to refer to "symmetry of time". First we recall **Ax(syt$_0$)** and **Ax(||)** from §2.8.

Intuitively, **Ax(syt$_0$)** says that

"as I see your clocks slowing down (because of your speed relative to me) so do you see my clocks (because of my speed relative to you) slowing down".

**Ax(syt$_0$)** $\qquad m \overset{\odot}{\rightarrow} k \quad \Rightarrow \quad (\forall p \in \bar{t}\,)\ |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{mk}(\bar{0})_t| = |\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(\bar{0})_t|.$

The above will be the first part of **Ax(syt)**. The second part of **Ax(syt)** will be the auxiliary axiom **Ax(||)** to be recalled below.

**Ax(||)** $\quad tr_m(k) \parallel \bar{t} \quad \Rightarrow \quad$ ($\mathsf{f}_{mk}$ is an isometry[374]).

Now we are ready to define our weak symmetry principle.

$$\mathbf{Ax(syt)} \overset{\text{def}}{=} \mathbf{Ax(syt_0)} + \mathbf{Ax(\|)}\,.$$

[373] The only thing that we can expect on this level of generality is that $sym_n$ will be "more subtle" or "more refined" than **Ax(symm)** or **Ax(syt)**, assuming that **Ax(syt)** is not adequate for $Th_n$. The reason for this is that if **Ax(syt)** is not adequate for $Th_n$ then it presumably blurs some distinctions. Hence for $sym_n$ *not to blur* these distinctions, we guess, that $sym_n$ will have to be more refined (or subtle) in some sense.

[374] I.e. $\mathsf{f}_{mk}$ preserves (square of) Euclidean distances, i.e. $(\forall p, q \in {}^nF)\|p-q\| = \|\mathsf{f}_{mk}(p) - \mathsf{f}_{mk}(q)\|$, cf. Def.3.9.3 on p.349.

**Remark 4.2.1** (On the intuitive content of **Ax(||)**.)
**Ax(||)** is a very natural axiom which is always assumed in all theories of motion like e.g. relativity theory. The *only* reason why we did not assume it already in **Basax** is that we did not need it yet. We plan to elaborate a variant of relativity called the "Ant and the elephant version" where we will not assume **Ax(||)**, but that will be a different story.[375] Till then, **Ax(||)** counts as one of those auxiliary axioms which from the intuitive physical point of view count as trivially true.

◁

**THEOREM 4.2.2**

**(i)** $\mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt)} \not\models \mathbf{Flxbasax}$, *moreover*

**(ii)** $\mathbf{Bax} + \mathbf{Ax6} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt)} + (c_m < \infty) \not\models c_m = c_k$.

**On the proof:** For the idea of the proof we refer the reader to the proof-idea given for item 3.9.42 in §3.9. ∎

The above theorem points in the direction that **Ax(syt)** does not "kill" the typically interesting "unorthodox" or "nonstandard" models of **Bax**.

**Remark 4.2.3** The above theorem can be interpreted as implying that **Ax(syt)** is a symmetry principle[376] *adequate* for **Bax** in the sense that it does not blur the distinction between **Bax** and the stronger core theories **Newbasax** or even **Flxbasax**. I.e. **Ax(syt)** is *weak enough* to be adequate. Thm.4.2.9 way below will point in the direction that **Ax(syt)** is also *strong enough* i.e. has interesting consequences (when added to **Bax** of course). Therefore part (i) of adequateness (as described in the introduction) is satisfied by **Ax(syt)** for **Bax**. To see that part (iii) is also satisfied, we make the following observation. The essential feature of **Bax**, permitting different observers "believing" in different speeds of light is not removed (and is not even "restricted") by adding **Ax(syt)** (even in the presence of auxiliary axioms). Therefore **Ax(syt)** seems to be *adequate* for **Bax**, even from the point of view of part (iii) of adequateness as described in the introduction.

◁

**THEOREM 4.2.4** *Assume $n > 2$. Then*

**(i)** $\mathbf{Bax} + \mathbf{Ax(||)} + \mathbf{Ax(symm)} \models (m \overset{\odot}{\to} k) \;\Rightarrow\; c_m = c_k$. *Therefore,*

**(ii)** $\mathbf{Bax} + \mathbf{Ax(||)} + \mathbf{Ax6} + \mathbf{Ax(symm)} \models \mathbf{Flxbasax}$.

---

[375] There we will look into "shrinking observers" like in Asimov's book "Fantastic voyage".
[376] i.e. instance of Einstein's special principle of relativity, SPR

**On the proof:** This is fully proved in §3.9 as Prop.3.9.37. ∎

We conjecture that Thm.4.2.4 above remains true without the condition $n > 2$.[377]

**PROPOSITION 4.2.5** **Bax + Ax6 + Ax($\sqrt{\ }$) + Ax(*Triv*) + Ax(symm)** $\not\models$ **Flxbasax**. *I.e.* **Ax(||)** *is needed in Thm.4.2.4 above.*

**On the proof:** Let $\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G, \in, W\rangle$ be a model of **Basax+Ax($\sqrt{\ }$)+Ax(*Triv*)+Ax(symm)**. Let $\mathfrak{M}^*$ be a frame model obtained from $\mathfrak{M}$ the following way. (The formal construction of $\mathfrak{M}^*$ will be given at the end of the proof.) For every observer $m$ of $\mathfrak{M}$ we include a new observer $m^*$ such that for every $p \in {}^nF$, $\mathsf{f}_{mm^*}(p) = \langle p_0, \frac{1}{2}p_1, \frac{1}{2}p_2, \ldots, \frac{1}{2}p_{n-1}\rangle$. Then for every $m \in Obs$:

- The life line of $m^*$ coincides with that of $m$.
- The same events are simultaneous for $m$ and $m^*$; moreover $m$ and $m^*$ agree on the time coordinates of the events.
- The meter rods of $m^*$ are twice as long as those of $m$.

It is not hard to check that $\mathfrak{M}^* \models$ **Bax+Ax6+Ax($\sqrt{\ }$)+Ax(*Triv*)+Ax(symm)**. For every $m \in Obs$, $c_{m^*} = \frac{1}{4}$ while $c_m = 1$. Hence $\mathfrak{M}^* \not\models$ **Flxbasax**. Checking all these are left to the reader. For completeness we include the formal definition of $\mathfrak{M}^*$.

$$\begin{aligned}
\mathfrak{M}^* &\stackrel{\text{def}}{=} \langle (B^*, Obs^*, Ph^*, Ib^*), \mathfrak{F}, G \in, W^*\rangle, \text{ where}\\
B^* &\stackrel{\text{def}}{=} B \times \{1,2\},\\
Obs^* &\stackrel{\text{def}}{=} Obs \times \{1,2\},\\
Ib^* &\stackrel{\text{def}}{=} Ib \times \{1,2\},\\
W^* &\stackrel{\text{def}}{=} \Big\{ \Big\langle \langle m,i\rangle, p, \langle b,j\rangle \Big\rangle \in Obs^* \times {}^nF \times B^* \; : \\
&\qquad\qquad \Big\langle m, \langle p_0, ip_1, ip_2, \ldots, ip_{n-1}\rangle, b \Big\rangle \in W \Big\}. \quad ∎
\end{aligned}$$

All the same, by Thm.4.2.4, **Ax(symm)** is too strong for studying **Bax**.

[377] *On the idea of a possible proof for this conjecture:* The intuitive content of **Ax(symm)** is that
(∗) "as I see you, so do you see me"
(of course the formalization slightly distorts this idea). Now, assume $m$ sees that the speed of $k$ causes $k$ think that the speed of light $c_k$ is smaller than $m$'s speed of light $c_m$. Then by (∗) above $k$ should think that the speed $v_k(m)$ of $m$ causes $m$ to think that the speed of light $c_m$ is smaller than $k$'s speed of light $c_k$. But then $c_k \leq c_m \leq c_k$ implies $c_k = c_m$. We did not check whether this idea works.

We interpret Thm.4.2.4 above as pointing in the direction that **Ax(symm)** as a symmetry principle is *not adequate* for studying **Bax**. Namely, **Ax(symm)** is too strong for **Bax** as it blurs the distinction between **Bax** and the theory **Flxbasax**, under relatively mild assumptions.

Summing up, by items 4.2.4, 4.2.2, and 4.2.9 **Ax(symm)** is not adequate for **Bax** while **Ax(syt)** is such (as this was "predicted" in the introduction of the present section).

To formulate theorems to the effect that **Ax(syt)** is also strong enough[378] (for **Bax**) first we need some definitions.

**Definition 4.2.6** The *twin paradox* **Ax(TwP)** is defined as in §2, p.140, with the only *change* that the subformula $m\ STL\ k$ pronounced as "$m$ sees $k$ moving *slower than light*" is re-defined the following, more general way.

Below we will use the notation $\mathsf{c}_m(d)$ to be introduced later (cf. §4.3, p.490). Intuitively, $\mathsf{c}_m(d)$ is the speed of photons *moving in direction* $d$ as observed by $m$.

$$m\ STL\ k \stackrel{\text{def}}{\iff} [(v_m(k) < \infty \text{ and } v_m(k) < \mathsf{c}_m(\vec{v}_m(k))) \text{ or } v_m(k) = 0].$$

I.e. $m\ STL\ k$ holds if the speed $v_m(k)$ of $k$ is smaller than that of light in the direction $\vec{v}_m(k)$ in which $k$ is moving as observed by $m$.

$\lhd$

**Remark 4.2.7** At the present level of generality we could have written $c_m$ in place of $\mathsf{c}_m(\vec{v}_m(k))$ but in later sections we will have theories which allow the speed of light to be different in different directions.

$\lhd$

**Definition 4.2.8** Next we define the *existential version* **Ax(∃TwP)** of the twin paradox as follows. First, let us recall that **Ax(TwP)** is a formula of the pattern

$$(\forall m \ldots)(\forall p \ldots)\Big([\ldots] \ \Rightarrow |\ldots| > |\ldots|\Big);$$

cf. p.140, above Thm.2.8.18. Let $\psi_1, \psi_2$ be formulas such that **Ax(TwP)** is the formula

$$(\forall m, k_1, k_2 \in Obs)(\forall p, q, r \in {}^nF)(\psi_1 \Rightarrow \psi_2).$$

Now we define **Ax(∃TwP)** as follows.

[378] This was part (ii) of adequateness in the introduction.

**Ax(∃TwP)** $(\exists m, k_1, k_2 \in Obs)(\forall p, q, r \in {}^nF)[\psi_1 \ \wedge \ \psi_2]$.

Intuitively, instead of saying that for each pair of twins the clocks of the accelerating twin runs slower, we say *only* that there exist twins such that the clocks of the accelerating twin runs slower. Cf. also **Ax(Twinp)** and **Ax(Twinp$_0$)** in the geometry chapter (§6) around p.**??** (see the whole text between items **??** and **??**).

◁

Theorem 4.2.9 below says that under mild conditions (i.e. under assuming $c_m < \infty +$ **Ax**($\sqrt{\ }$)), **Bax** $\models$ **Ax(syt$_0$)** $\rightarrow$ **Ax(TwP)**. This is an analogon of Thm.2.8.18 saying that **Basax** + **Ax(symm)** + **Ax**($\sqrt{\ }$) $\models$ **Ax(TwP)**.

**THEOREM 4.2.9** **Bax** + $(c_m < \infty)$ + **Ax**($\sqrt{\ }$) + **Ax(syt$_0$)** $\models$ **Ax(TwP)**.

**On the proof:** First, one proves that the assumptions of the theorem imply that moving clocks slow down, i.e.
**Bax** + $(c_m < \infty)$ + **Ax**($\sqrt{\ }$) + **Ax(syt$_0$)** $\models$

$$0 < v_m(k) < c_m \ \rightarrow \ (\forall p, q \in tr_m(k)) \ |q_t - p_t| < |\mathsf{f}_{mk}(q)_t - \mathsf{f}_{mk}(p)_t|.$$

Then it is not hard to check that this implies Thm.4.2.9. ∎

**QUESTION 4.2.10** *Under what weaker assumptions (than the ones in Thm.4.2.9) is* **Ax(∃TwP)** *provable ? See also Proposition 4.2.14.*

◁

In view of the above results, conjectures and discussions,

*we define* a[379] <u>*symmetric version of* **Bax**</u> to be **Bax+Ax(syt)**.

**QUESTION 4.2.11**

**(i)** *Investigate the models of* **Bax** + **Ax(syt)**.

**(ii)** *Look briefly into the models of* **Bax+Ax(symm)***, but cf. Thm.4.2.4 in this connection.*[380]

[379] We wrote "a symmetric version" instead of "the symmetric version" because other symmetric versions (of **Bax**) are also possible, and we do not want to go into discussing them, here.

[380] We already decided that they are not important, so this is only a question of mathematical curiosity.

**(iii)** *Study the models of* **Flxbasax + Ax(symm)** *and those of* **Flxbasax + Ax6** + $c < \infty$ + **Ax(symm)**.

◁

**QUESTION 4.2.12** *At the end of each one of §§ 4.3–4.5, we suggest that the reader spend a little time for figuring out what the* symmetric versions *of the subtheories of* **Basax** *introduced in* that *section look like and how they behave. (Also try to figure out what the* adequate *symmetry principle for the subtheory of* **Basax** *in question might be; is it* **Ax(symm)**, **Ax(syt)** *or perhaps something else.)*

◁

Before turning to related issues, let us look at the connection between our symmetry principles **Ax(symm)** and **Ax(syt)**.

**THEOREM 4.2.13** [381] **Bax** + **Ax(||)** $\models$ **Ax(symm)** $\to$ **Ax(syt)**.

**Proof:** Assume **Bax** + **Ax(||)** + **Ax(symm)**. Since **Ax(||)** is assumed, it is enough to prove $\mathbf{Ax(syt_0)}$. Let $m, k \in Obs$. Let $m', k' \in Obs$ such that $tr_m(m') = tr_k(k') = \bar{t}$ and $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$. Such $m', k'$ exist by **Ax(symm)**. Let $M := \mathsf{f}_{m'm}$ and $N := \mathsf{f}_{kk'}$. Then, we have that $\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ M$. $N$ and $M$ are isometries by **Ax(||)**. At this point we ask the reader to consult Figure 134 and to see for himself that $\mathbf{Ax(syt_0)}$ is true for $m$ and $k$. For completeness we include the proof of this.

Let $p \in \bar{t}$. Since $N$ is an isometry with $N[\bar{t}\,] = \bar{t}$ we have that

$$\|p\| = \|N(p) - N(\bar{0})\| \quad \text{and} \quad N(\bar{0}), N(p) \in \bar{t}.$$

Since $\mathsf{f}_{km}$ is a bijective collineation and segments $\bar{0}p$ and $N(\bar{0})N(p)$ are of the same length and lying on the same line (i.e. on the $\bar{t}$ axis) we have that $\mathsf{f}_{km}$ takes these two segments to the same line and to segments of the same length, i.e. $(\exists \ell \in \mathsf{Eucl})\ \mathsf{f}_{km}(\bar{0}), \mathsf{f}_{km}(p), \mathsf{f}_{km}(N(\bar{0})), \mathsf{f}_{km}(N(p)) \in \ell$ and

$$\|\mathsf{f}_{km}(p) - \mathsf{f}_{km}(\bar{0})\| = \|\mathsf{f}_{km}(N(p)) - \mathsf{f}_{km}(N(\bar{0}))\|.$$

By these, we have

$$|\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(\bar{0})_t| = |\mathsf{f}_{km}(N(p))_t - \mathsf{f}_{km}(N(\bar{0}))_t|. \tag{283}$$

Since $M$ is an isometry preserving $\bar{t}$, we have

$$|\mathsf{f}_{km}(N(p))_t - \mathsf{f}_{km}(N(\bar{0}))_t| = |M(\mathsf{f}_{km}(N(p))_t - M(\mathsf{f}_{km}(N(\bar{0}))_t|.$$

[381] This theorem remains true for the theory $\mathbf{Bax}^-$ in place of **Bax**; where $\mathbf{Bax}^-$ will be introduced in Def.4.3.7, p.479.

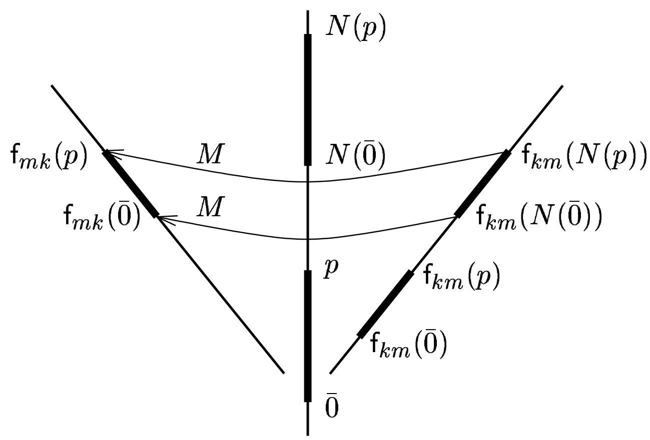



Figure 134:

By this, (283) and $\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ M$ we have

$$|\mathsf{f}_{km}(p)_t - \mathsf{f}_{km}(\bar{0})_t| = |\mathsf{f}_{mk}(p)_t - \mathsf{f}_{km}(\bar{0})_t|.$$

Thus $\mathbf{Ax(syt_0)}$ holds for $m$ and $k$. ∎

In connection with Thm.4.2.13 above, we note that

$$\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) \models \mathbf{Ax(symm)} \leftrightarrow \mathbf{Ax(syt)},$$

c.f. items 2.8.13 (p.135), 3.9.47 (p.391).

Below we will state that even the existential twin paradox $\mathbf{Ax(\exists TwP)}$ cannot be proved without assuming some symmetry principle. Then we ask some questions about the connections between the twin paradox and the symmetry principles discussed in this section.

**PROPOSITION 4.2.14** $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) \not\models \mathbf{Ax(\exists TwP)}$.

**On the proof:**
It is not hard to construct a model $\mathfrak{M}$ of $\mathbf{Basax}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax}(\mathit{Triv})+\mathbf{Ax}(\|)$ such that there is an observer $m_0 \in \mathit{Obs}$ such that for every $k \in \mathit{Obs}$ and $p, q \in tr_{m_0}(k)$ the time elapsed between events $w_m(p)$ and $w_m(q)$ for $k$ is exactly the Euclidean distance between $p$ and $q$, that is

$$|\mathsf{f}_{m_0k}(q) - \mathsf{f}_{m_0k}(p)| = |q - p|.$$

Then it is easy to check that for such $\mathfrak{M}$, $\mathfrak{M} \not\models$ **Ax(∃TwP)** since in Euclidean geometry the sum of the lengths of two sides of a triangle is always greater than the length of the third side of this triangle. ∎

**QUESTION 4.2.15** *Find an interesting theory proving* **Ax(∃TwP)** *but not* **Ax(TwP)***. I.e. is* **Ax(∃TwP)** *useful?*

◁

**QUESTION 4.2.16** *Assume* **Basax** + **Ax**($\sqrt{\ }$) + **Ax(eqtime)**.

**(i)** *How much weaker is* **Ax(TwP)** *than* **Ax(syt$_0$)***. I.e. how big the gaps are in the hierarchy*

$$\mathbf{Ax(\exists TwP)} < \mathbf{Ax(TwP)} < \mathbf{Ax(syt_0)} < \mathbf{Ax(symm)}?$$

**(ii)** *What is the answer to the above question if we assume* **Ax(***Triv***)** *(in addition to* **Basax** *+etc.)?*

◁

**Question for future research 4.2.17** Is **Basax** + **Ax**($\sqrt{\ }$) + **Ax(TwP)** $\models$ **Ax(syt$_0$)** true?

◁

**Remark 4.2.18 (On symmetry axioms, principles of parsimony etc. in weak theories of relativity)**

Let us recall that **Ax(symm)** is an axiom of different status than the rest of axioms collected in **Basax** (or in its weaker versions like e.g. **Bax** or **Reich**(**Bax**)). Namely, axioms in **Basax** like e.g. **AxE** can be called "experiments-motivated" in that we assume them because we think that some experiments conducted in the past can be interpreted as suggesting that they might be true; *while* **Ax(symm)** can be called *"aesthetics-motivated"* in that **Ax(symm)** is a simplifying principle which we assume if and when we want to make our mental model of the world simpler. Therefore, we could call **Ax(symm)** a *principle of parsimony*[382], and the same applies to the similar symmetry axioms discussed in §3.9. In the literature, the expression Occam's razor is often used when referring to such principles of parsimony. (We could quote here e.g. Webster's Dictionary or Friedman [90] which in turn also mentions Leibniz and Mach in the present connection).[383]

[382]i.e. a principle of economy in terms of theoretical concepts.

[383]The just outlined distinction between simplifying axioms like **Ax(symm)** and experiment-motivated axioms like **AxE** is *only* of a *heuristic* value, and is not an absolute distinction. (In §2.8 we show how some parts of **Ax(symm)** can be made testable, and in §4.5 we will argue that parts of **AxE** might not be testable, after all.) More on the "relative" status of this distinction will be said in Remark 4.2.19 following the present remark.

As we saw in §2.8 and in the introduction to the present section (§4.2), a relativity theory like e.g.

$$\mathbf{Basax} + \mathbf{Ax(symm)}$$

can be decomposed to a *core part* **Basax** and to a *simplifying principles* part **Ax(symm)**.

In the present work, we begin our studies with the core part and then, having gained some understanding of the core part, we study the question of which simplifying principles fit our core theory the best way. For example §3.9 is entirely devoted to the issue of simplifying assumptions (these are often called "symmetry axioms" in the present work). For more on this subject we refer the reader to the introductions of §§2.8, 4.7, 3.9.

To distinguish the *core part* of a theory *Th* from its simplifying assumptions part, in the present work we use the following rule of thumb. Given *Th*, the core part of *Th* consists of those axioms of *Th* which are provable from $\mathbf{Basax} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\sqrt{\ })$.[384] The rest of *Th* is the simplifying part of *Th*. There are exceptions when this rule of thumb does not work, such an exception is **NewtK** (introduced in §4.1).

Because of the above rule of thumb, we develop the hierarchy of our core theories as a hierarchy of sub-theories of **Basax**, cf. Figure 180 on p.552. Then later, for each core-theory $Cth \leq \mathbf{Basax}$ we ask ourselves which simplifying assumptions (or equivalently symmetry axioms) would be the most adequate if and when we want to apply a principle of parsimony (i.e. Occam's razor) to the theory *Cth* in question. For the core theories $Cth \geq \mathbf{Bax}$ which are strong enough we usually find either **Ax(symm)**[385] or its weaker version **Ax(syt)** adequate. Hence the parsimonious version of

| | | | |
|---|---|---|---|
| **Basax** | is | **Basax + Ax(symm)**, | that of |
| **Flxbasax** | is | **Flxbasax + Ax(symm)**, | that of |
| **Bax** | is | **Bax + Ax(syt)**, | etc. |

However, in the present chapter (§4) we will study theories which do not contain **Bax** (i.e. in which **Bax** is not provable). An important such theory is the Reichenbach-Grünbaum version of special relativity introduced and discussed in §4.5. (This theory is extensively studied in the literature and is of remarkable philosophical significance). To each one *Th* of our theories containing **Bax**, in §4.5 we will associate

[384] It is debatable whether $\mathbf{Ax}(\mathit{Triv})$ and $\mathbf{Ax}(\sqrt{\ })$ should count as core. Perhaps we should introduce a third category called *auxiliary axioms*. Then $\mathbf{Ax}(\mathit{Triv})$, $\mathbf{Ax}(\sqrt{\ })$, $\mathbf{Ax}(\|)$ etc. would count as auxiliaries. However we do not go into this here any further, since it does not seem to be an important issue. All the same, sometimes we will refer to "auxiliary axioms" meaning the above indicated group.

[385] and its variants studied in §3.9

its *Reichenbach-Grünbaum version* **Reich**(*Th*). This way we will obtain new core theories **Reich**(**Basax**), ..., **Reich**(**Bax**).

Finding adequate symmetry principles for core theories **Reich**(*Th*) will turn out to be a more delicate matter (than the cases of **Basax**, **Flxbasax**, or even **Bax**). In more detail:

As we already experienced in the case of **Bax**, for core theories *weaker* than **Flxbasax**, **Ax(symm)** might be too strong, it might be *no longer adequate*.[386] Similarly we will see that for some important theories weaker than **Bax**, the symmetry principle **Ax(syt)** will be no longer adequate (though it was adequate for **Bax**). Therefore a new research topic appears, namely searching for that principle of parsimony which is adequate to a weak theory, say, **Reich**(*Th*), ($Th \geq$ **Bax**). After we found the principle of parsimony, call it $\mathbf{Ax(symm)_{Reich}}$, which is adequate for **Reich**(*Th*), we can define the *parsimonious version* (or symmetric version) of the core theory **Reich**(*Th*) to be **Reich**(*Th*)+$\mathbf{Ax(symm)_{Reich}}$. The just outlined research task (i.e. finding $\mathbf{Ax(symm)_{Reich}}$ etc.) is the main subject of our section §4.7.

◁

**Remark 4.2.19** In this work we often emphasize that **Ax(symm)** is a principle of *different nature* than e.g. the axioms of **Basax** or **Newbasax**. In explaining how **Ax(symm)** differs from the rest of the axioms we usually say that it is a principle of parsimony, or a principle of aesthetics, or a "simplifying principle" as opposed to being motivated by the outcome of past experiments like e.g. **AxE**. In this remark we would like to explain that while the above distinction is a useful guiding light for understanding the "logic" or a philosophy of the present work, it contains an element of *oversimplification*. We hope that this oversimplification will cause no misunderstandings. The oversimplification is the following: The "simplifying principle" contra "experiment-motivated axiom" distinction might suggest that we consider **Ax(symm)** as something completely subjective (or conventional). This is *not* so, namely in §2.8 we explained that by bringing hydrogen atoms into our picture of the world, some "part" of **Ax(symm)** becomes experimentally testable. Indeed, such testable "parts" of **Ax(symm)** will be identified in[387] §4.7, an example for these is the testable symmetry principle $\mathbf{R(Ax\ syt_0)}$ introduced in that section.[388] Having explained what might be confusing or misleading in our calling **Ax(symm)** a simplifying principle, we will go on using this distinction because (i) we hope

[386] As an example for this we refer to Thm.4.2.4(ii).

[387] Actually, finding these can be considered as one of the main goals of that section.

[388] The reason why we call $\mathbf{R(Ax\ syt_0)}$ a "part" of **Ax(symm)** is that $\mathbf{Basax} + \mathbf{Ax(symm)} \models \mathbf{R(Ax\ syt_0)}$ will be proved in §4.7, moreover $\mathbf{R(Ax\ syt_0)}$ is the "testable version" of a weaker version $\mathbf{Ax(syt_0)}$ of **Ax(symm)**.

that the above explanation ensures that the reader does not misunderstand our intentions, and (ii) we hope that this distinction helps to understand why (and in what sense) we treat the symmetry principles differently from the rest of the axioms. ◁

## 4.3 Relaxing isotropy; connections with Friedman's conceptual analysis

In §1.1 (introduction of this work) we indicated that we intend to investigate a *hierarchy* of *weak subsystems* (or subtheories) of the axiomatic theory/theories of relativity we are developing in the present work. For motivation to do this cf. e.g. §1.1 items (VI), (I), (II), (III), (V), (X). Have we started looking into weak systems? The answer is yes, *but* only a little bit. Namely, to study why there are no FTL observers in **Newbasax**, we introduced the weak subtheory **Bax** of **Newbasax**, in §3.4.2 (p.219). Up to this point, **Bax** has been the weakest axiom system studied in this work. In the present section we introduce a weakened version $\mathbf{Bax}^-$ of **Bax**. Let us recall that **Bax** was obtained from **Newbasax** by permitting the speed of light to be different for different observers. At the same time **Bax** retained the requirement of **Newbasax** that for any fixed observer photons moving in *different directions must have the same speed*.[389] The latter requirement was formulated as $\mathbf{AxE_{00}}$, and $\mathbf{AxE_{00}}$ is the key axiom of **Bax**. In what follows we will introduce a weaker (than $\mathbf{AxE_{00}}$) axiom **AxP1** which[390] will allow the speed of light to be different in different directions (in contrast with $\mathbf{AxE_{00}}$ where this is not allowed). Then (among other changes) we will replace the speed-of-light axiom $\mathbf{AxE_{00}}$ in **Bax** with the new axiom **AxP1**.[391] The key axiom of our new axiom system $\mathbf{Bax}^-$ to be introduced below will be **AxP1**.[392] Roughly, **AxP1** will say that for an observer $m$, the speed of a photon $ph$ may *depend only* on the point $p \in {}^nF$ where $m$ sees $ph$ and on the spatial direction in which $ph$ is moving. (As a contrast, in **Bax** this speed was not allowed to vary either with $p$ or with the direction.)

[389]The property that something does not depend on spatial directions is called *isotropy*, cf. Remark 3.4.15.

[390]The name **AxP1** originates from Friedman's conceptual analysis of relativity (cf. e.g. §1.1 item (V) and Remark 4.3.40 on p.522), where he introduced a principle (P1) concerning the speed of light, and that principle corresponds roughly to our **AxP1**. Friedman's (P1) will be recalled in Remark 4.3.40.

[391]After we replace $\mathbf{AxE_{00}}$ with **AxP1** in **Bax**, we will have to adjust the remaining axioms to the change, since they were designed to be companions of $\mathbf{AxE_{00}}$ and not of **AxP1**.

[392]We will see that a key difference between **Bax** and $\mathbf{Bax}^-$ is that **Bax** postulates a kind of isotropy while $\mathbf{Bax}^-$ does not, cf. item (i) in the discussion of WPI on p.219. (As we said, isotropy means that certain things do not depend on spatial direction.)

With introducing $\mathbf{Bax}^-$ we have purposes 1–4 below in mind.

1. To seriously begin the study of weak systems (which will be more fully developed in §4.4 below) in order to answer the "why" type questions.

2. The goals in §1.1 items (I–V), (X), in connection with weak systems.

3. To prepare the ground for developing a first-order logic study of the Reichenbach-Grünbaum version of relativity (cf. e.g. Szabó [244], Friedman [90]) mentioned in the introductions of §§ 3, 3.4. The first-order axiomatizations, e.g. **Reich**(**Basax**), **Reich**(**Bax**), of this version of relativity are so different from the "Einsteinian" theories that they cannot be based on **Bax**. However, $\mathbf{Bax}^-$ will be flexible enough to serve as a *common core* of both kinds of theories e.g. both of **Reich**(**Basax**) and **Newbasax** (or **Bax** etc). Therefore $\mathbf{Bax}^-$ will also serve as a common *"platform" for comparing* the Einsteinian and the other (e.g. Reichenbachian) versions.

4. To prepare the ground for elaborating a logically precise version of *conceptual analysis* of relativity where (the original, informal version of) the latter was briefly recalled from the literature in §1.1 (V), p.8. A detailed discussion of how $\mathbf{Bax}^-$ and **AxP1** are used (in the present work) for the purposes of conceptual analysis is given in Remark 4.3.40 (pp.522–524). Here we only mention that Friedman [90] systematizes the various principles concerning the speed of light which can be found in the literature of relativity into three principles which he calls (P1), (P2), and (P3). We will recall (P1)–(P3) in Remark 4.3.40, where we will briefly *recall Friedman's conceptual analysis* and will discuss how we formalize it here. We will see that our **AxP1** can be considered as a possible formalized version of Friedman's (P1), and we will elaborate on the connections between our axiom systems and Friedman's principles, in Remark 4.3.40. We will return to discussing the question of how careful or how faithful our formalization of Friedman's principle (P1) is, in §4.4.

   In passing, we note that at the present point we already have the axioms which we will need to formalize Friedman's (P2) in our frame language. What we will have to work for below is elaborating the axioms using which we will be able to formalize Friedman's (P1), too, in our frame language. These axioms will be collected into $\mathbf{Bax}^-$.

As we said, the key axiom of $\mathbf{Bax}^-$ will be **AxP1**. To introduce **AxP1**, we need to formulate some definitions and a convention on terminology. (These will be Items 4.3.1–4.3.3.)

**CONVENTION 4.3.1** (On *terminology:*) As we emphasized in §2, we call ${}^nF$ the *coordinate-system* of our model $\mathfrak{M}$ and not *space-time*. Space-time of $\mathfrak{M}$ will be introduced in §6 and it will be something else; namely a structure $\langle Mn, \dots \rangle$ whose universe $Mn$ is a subset of $\mathcal{P}(B)$. Cf. Item 6.2.5 in §6. Cf. also Matolcsi [190, §II.1.2, p.151]. Despite of this, occasionally we use the word "space-time" for ${}^nF$, for reasons of convenience. Namely, ${}^nF$ contains a time-axis and $n-1$ space axes. Therefore it is handy to speak about the *space-part* $S = \{0\} \times {}^{n-1}F$, the *time-part* $\bar{t}$ $(= F \times {}^{n-1}\{0\})$, of ${}^nF$ and to call the rest of ${}^nF$ space-time part (since it involves both space and time coordinates). We hope, this will cause no confusion, and that the reader will remember that we do not intend to regard ${}^nF$ as space-time.

Sometimes, ${}^nF$ is called "*relative space-time*" because the observer "splits" space-time to a space-part and a time-part as in ${}^nF$. Cf. Matolcsi [190], e.g. bottom of p.154, p.165, and §II.1.7.

◁

Below, we will define two functions *time* and *space* such that for any point $p$ in our coordinate-system $time(p)$ and $space(p)$ are the time coordinate and the space "coordinate" of $p$, respectively.

**Definition 4.3.2** We define functions $time : {}^nF \longrightarrow F$ and $space : {}^nF \longrightarrow {}^{n-1}F$ as follows.

$$(\forall p \in {}^nF)(time(p) :\stackrel{\text{def}}{=} p_0 \quad \wedge \quad space(p) :\stackrel{\text{def}}{=} \langle p_1, p_2, \dots, p_{n-1} \rangle).$$

◁

**Definition 4.3.3 (direction, moving forwards, backwards)**

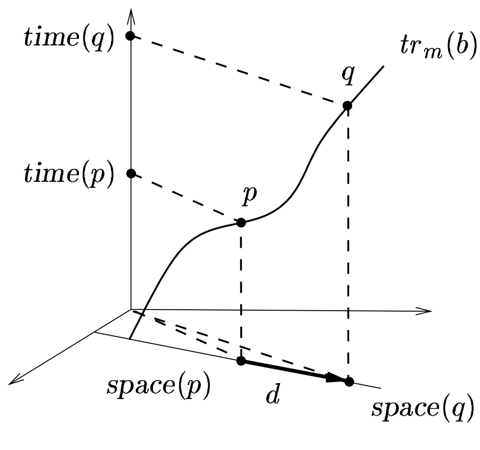
**(i)** By a *spatial direction* or simply by a *direction* we understand a *space*-vector $d \in {}^{n-1}F$, with $d \neq \bar{0}$.

**(ii)** $\mathsf{directions} :\stackrel{\text{def}}{=} \{d \in {}^{n-1}F \;:\; d \neq \bar{0}\}$.

**(iii)** Let $\mathfrak{M}$ be a frame model. Then body $b$ is said to *move in direction* $d$ (as seen by observer $m$) iff

$$(\forall p, q \in tr_m(b))(\exists \lambda \in F)(space(q) - space(p) = \lambda \cdot d).$$

Body $b$ is said to move *forwards* in direction $d$ (as seen by observer $m$) iff



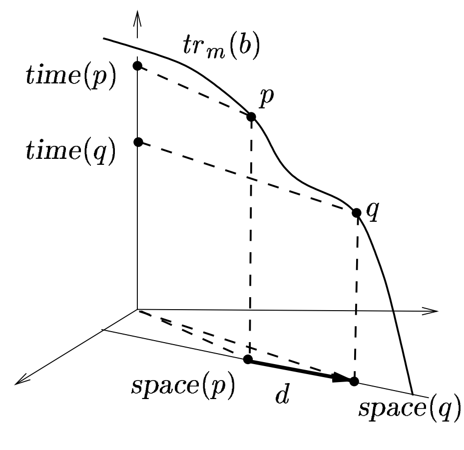




Figure 135: Illustration for Def.4.3.3.

$$\Big([b \text{ moves in direction } d] \qquad \text{and}$$
$$[(\forall p, q \in tr_m(b))(\exists 0 \leq \lambda \in F)$$
$$(time(p) < time(q) \ \Rightarrow\ space(q) - space(p) = \lambda \cdot d)]\Big),$$ see Figure 135.

Body $b$ is said to move *backwards* in direction $d$ (as seen by observer $m$) iff

$$\Big([b \text{ moves in direction } d] \qquad \text{and}$$
$$[(\forall p, q \in tr_m(b))(\exists 0 \leq \lambda \in F)$$
$$(time(p) > time(q) \ \Rightarrow\ space(q) - space(p) = \lambda \cdot d)]\Big),$$ see Figure 135.

**(iv)** When $d \in S, d \neq \bar{0}$, we say that body $b$ moves in direction $d$ (forwards, backwards), if $b$ moves in direction $space(d)$ (forwards, backwards).

**(v)** We extend the notion of being parallel to directions, and in more general, to space-vectors as follows. If $d, d_1 \in {}^{n-1}F$, then $d \parallel d_1$ denotes that $d = \lambda \cdot d_1$ or $d_1 = \lambda \cdot d$ for some $\lambda \in F$. $\lhd$

We note that "$b$ moves in direction $d$" means that if we look only at the space-part of $b$'s trace, then this is a straight line $\ell'$ parallel with $\ell_d = \{\lambda \cdot d : \lambda \in F\}$; $b$ moves along $\ell'$ but maybe with different speeds at each moment, and backwards or forwards in each moment. If we look at the trace of $b$ in the whole space-time, then "$b$ moves in direction $d$" means that $tr_m(b)$ lies in a plane parallel with $\mathsf{Plane}(\bar{t}, \ell_d)$; $tr_m(b)$ will be a straight line in this plane if $b$ moves with a constant speed.

Up to this point, when we said that body $b$ moves in a certain direction, then we meant to say that $b$ moves *forwards* in that direction. From the present point on we will indicate whether $b$ moves forwards or backwards in a certain direction, except when there is no danger of confusion. (We note that an inertial body $b$ moves both forwards and backwards in a given direction $d$ iff $[v_m(b) = \infty$ or $v_m(b) = 0]$. We also note that if $v_m(b) = 0$ then $b$ moves in all directions.)

Now, we are ready to formulate the key axiom of $\mathbf{Bax}^-$.

**AxP1** $(\forall m \in Obs)(\forall ph_1, ph_2 \in Ph)(\forall d \in \mathsf{directions})$ $\Big(\big(ph_1$ and $ph_2$ are moving forwards in direction $d$ as seen by $m$ and $tr_m(ph_1) \cap tr_m(ph_2) \neq \emptyset\big) \Rightarrow$

$$tr_m(ph_1) = tr_m(ph_2)\Big).$$

Intuitively, photons "emitted" at a point of space-time[393] in the same direction (forwards) have the same speed (as seen by observer $m$, of course). In other words: Starting out from one point $p$ of space-time, in every direction (forwards) there is at most one "speed of light" (i.e. photon-trace). Yet in other words: photons moving forwards in the same direction do not race with one another.

In an intuitive language, **AxP1** says that the speed of a photon may depend *only* on the point $p$ of space-time where it was emitted, and on the direction $d$ in which it was emitted (forwards).[394]

**AxP1** in itself allows that the "speed of light" at point $q \in {}^{n-1}F$ of *space* in direction $d$ varies with time. E.g. it allows that at a time $t$ the speed of light in direction $d$ is 1, while 5 minutes later at the same point of space this speed of light is bigger. However, if we assume that the traces of photons are straight lines, then this cannot happen: the speed of light at point $p$ of space-time in direction $d$ does not depend on the time-coordinate of $p$. Moreover, the other axioms of $\mathbf{Bax}^-$ will imply that the speed of a photon depends only on the direction $d$ it was emitted

[393] In reality ${}^nF$ is not "space-time" but only our coordinate-system, so here we should say "point of the coordinate-system ${}^nF$". We wrote "space-time" above *only* because in the present context it sounds more suggestive.

[394] Here *"ph is emitted at p"* means only that $p$ is on the trace of $ph$.

in, and it does not depend on the point $p$ of space-time where it was emitted. See Thm.4.3.17, Figure 146.

**Definition 4.3.4 (light-cone)**

$$\mathsf{Cone}_{m,p} := \bigcup\{tr_m(ph) \ : \ ph \in Ph \ \& \ p \in tr_m(ph)\}.$$

We call $\mathsf{Cone}_{m,p}$ the *light-cone starting at p as seen by m*. ◁

**Remark 4.3.5 (On AxP1 and light-cones)**
If we think a little more on the intuitive content of **AxP1**, we will see that it postulates the existence of a partial function $c_m : {}^nF \times \mathsf{directions} \longrightarrow F$ to each observer $m$ in such a way that (i) $c_m(p,d)$ is defined iff $m$ sees a photon at point $p$ moving forwards in direction $d$, (ii) $c_m(p,d)$ is the speed of every photon $m$ sees at $p$ moving forwards in direction $d$,[395] and (iii) for equivalent directions $c_m(p,d)$ is the same.[396]

Now, if we try to visualize this $c_m(p,d)$ function, then we will see that this means that to every point $p$ of ${}^nF$, we associate a so called *light-cone* (representing the speeds of photons at that point going in all the possible directions). For example pictures like in Figure 136 become possible.

Looking into e.g. d'Inverno [75, pp.216–224 or pp.259–262] or Friedman [90, pp.186–187] or Penrose [213, pp.221–223] we notice that associating light-cones to points of space-time is an essential and powerful tool of general relativity. At this point we do not discuss this connection any further; but it is interesting to notice that **AxP1** says that, for every $m$, there are light-cones glued to each point of ${}^nF$ and the motion of photons is regulated by these light-cones. Let us assume for a moment that **AxP1** is used in a context where our "no FTL observers theorem" is true.[397] Then, **AxP1** together with this "no FTL principle" says that the movement of both photons and observers is regulated by the light-cones glued to the points of ${}^nF$ (from the point of view of some observer $m$).

But then this gives us quite a nice, pleasantly visualizable road-map of "space-time". The road-map is defined by gluing light-cones to the points of space-time and by postulating the "rule" that observers can move only inside of these light-cones.

In the above intuitive discussion we tacitly assumed properties of the light-cones which are not provable from **AxP1** alone; so, in principle, we should return to the above outlined intuitive picture later, when we have more information on the light-cones etc. An adequate place for this seems to be the part of our next section §4.4

[395] We will return to discussing this $c_m(p,d)$ function in §4.4 on p.535.
[396] Formally, (iii) says the following:$(\forall\lambda \in {}^+F)[c_m(p,d)$ is defined $\Rightarrow$ $c_m(p,d) = c_m(p,\lambda\cdot d)]$.
[397] This means that the traces of observers are inside the light-cones.

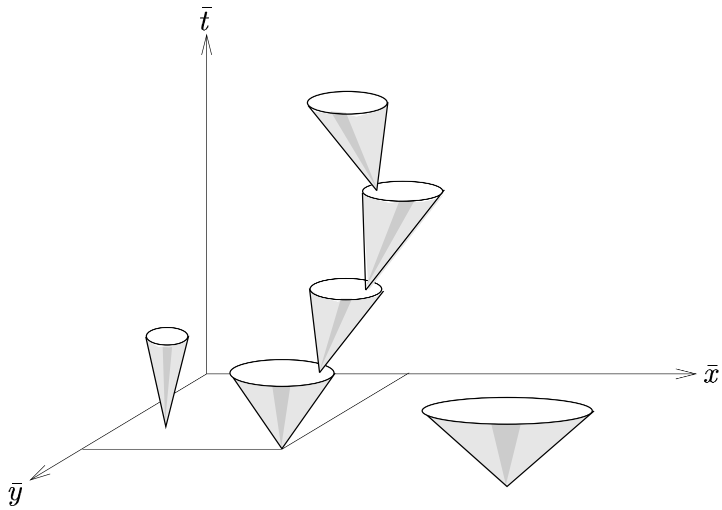



Figure 136: To every point $p$ of ${}^nF$, we associate a so called light-cone.

beginning with p.535 (cf. e.g. the discussion of $c_m(p,d)$ there). Till then, interpreting **AxP1** as an axiom postulating that there is a light-cone "road-map" of space-time remains an informal idea about the intuitive meaning of **AxP1** (and of similar axioms coming in the next section).

We will return to discussing the function $c_m(p,d)$ and drawing light-cones etc. close to the end of §4.4 beginning with p.535 (cf. e.g. principle $(*)$ on p.535).

$\lhd$

At this point (and also in connection with the above remark) it might be of interest to note the following. While **AxE** (or even $\mathbf{AxE_{00}}$) will not survive the transition from special relativity to general relativity, **AxP1** (or a variant of it[398]) does. This can be already anticipated by looking into Chapter 8 on accelerated observers where **AxE** will have to be restricted to inertial observers while **AxP1** (in a slightly refined form) will remain true for all observers.

Let us recall that

$$\mathbf{Bax} = \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}}\}\,.$$

Of these axioms $\mathbf{Ax5^{Obs}}$, $\mathbf{Ax5^{Ph}}$ and $\mathbf{AxE_{00}}$ will be of special importance for us. (They all contain an element of the principle of isotropy.) Recall that our purpose is to define a system $\mathbf{Bax^-}$ weaker than **Bax** by using **AxP1**.

[398] That variant of **AxP1** is the following. $(\forall m \in Obs)(\forall p \in Dom(w_m^-))(\forall ph_1, ph_2 \in Ph)$ $[(m$ sees $ph_1$ and $ph_2$ at point $p$ moving in direction $d$ forwards$) \Rightarrow\ tr_m(ph_1) = tr_m(ph_2)]$.

Next we start looking into the (somewhat technical) question of how we can weaken **Bax** by using **AxP1** in place of $\mathbf{AxE_{00}}$ and adjusting (to the change) the rest of the axioms. The following proposition states that the theory generated by **Bax** remains the same if we simply replace $\mathbf{AxE_{00}}$ with **AxP1**.

**PROPOSITION 4.3.6**

**(i)** $\mathbf{Bax} \ \dashv\models\ (\mathbf{Bax} \setminus \{\mathbf{AxE_{00}}\}) + \mathbf{AxP1}$.[399]

**(ii)** *Assume* $\mathbf{Ax1}$–$\mathbf{Ax3_0}$, $\mathbf{Ax5^{Ph}}$, $\mathbf{Ax}(\sqrt{\ })$. *Then* $\mathbf{AxE_{00}} \dashv\models \mathbf{AxP1}$.

**Proof:**

Proof of (ii): The proof of "direction $\models$" is obvious. To prove "direction $\dashv$" let $\mathfrak{M} \models \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{AxP1}, \mathbf{Ax5^{Ph}}, \mathbf{Ax}(\sqrt{\ })\}$. We have to prove that $\mathfrak{M} \models \mathbf{AxE_{00}}$. To see this let $m \in Obs$, $d \in \mathsf{directions}$, and $ph_1, ph_2 \in Ph$ such that $m \overset{\odot}{\to} ph_1$ and $ph_2$ moves forwards in direction $d$ as seen by $m$. Let $ph \in Ph$ such that $ph$ moves forwards in direction $d$ as seen by $m$, $tr_m(ph) \cap tr_m(ph_2) \neq \emptyset$, and $v_m(ph) = v_m(ph_1)$. Such a $ph$ exists by $\mathbf{Ax5^{Ph}}$ and $\mathbf{Ax}(\sqrt{\ })$. Since $ph$ and $ph_2$ move forwards in direction $d$ as seen by $m$ and $tr_m(ph) \cap tr_m(ph_2) \neq \emptyset$, we have $tr_m(ph) = tr_m(ph_2)$ by **AxP1**. But this implies $v_m(ph_2) = v_m(ph) = v_m(ph_1)$, and this completes the proof of (ii).

Outline of the proof of (i): Throughout the proof of (i) the reader is asked to consult Figure 137. We will prove "direction $\dashv$", because the proof of the other direction is straightforward. To prove direction $\dashv$ let $\mathfrak{M}$ be a frame model of $(\mathbf{Bax} \setminus \{\mathbf{AxE_{00}}\}) \cup \mathbf{AxP1}$. We have to prove $\mathfrak{M} \models \mathbf{AxE_{00}}$. In the proof of (ii) we have seen that $\mathfrak{M} \models \mathbf{AxE_{00}}$ holds under assuming $\mathbf{Ax}(\sqrt{\ })$. Now without assuming $\mathbf{Ax}(\sqrt{\ })$ the proof is not so obvious. Let $m \in Obs$. Let $ph \in Ph$ such that

$$(284)\quad m \overset{\odot}{\to} ph \ \land\ (\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < v_m(ph) \ \Rightarrow\ (\exists k \in Obs)\, tr_m(k) = \ell).$$

Such a $ph$ exists by $\mathbf{Ax5^{Obs}}$. Let $ph' \in Ph$ be arbitrary. To prove $\mathfrak{M} \models \mathbf{AxE_{00}}$ it is enough to prove that $v_m(ph) = v_m(ph')$. Let

$$\mathsf{LightCone} \overset{\text{def}}{=} \bigcup \{\ell \in \mathsf{Eucl} \ :\ \bar{0} \in \ell \ \&\ ang^2(\ell) = v_m(ph)\},$$
$$\mathsf{LightCone}' \overset{\text{def}}{=} \bigcup \{\ell \in \mathsf{Eucl} \ :\ \bar{0} \in \ell \ \&\ ang^2(\ell) = v_m(ph')\}.$$

To prove $v_m(ph) = v_m(ph')$ it is enough to prove that $\mathsf{LightCone} = \mathsf{LightCone}'$. The proof of this goes by contradiction. Assume $\mathsf{LightCone} \neq \mathsf{LightCone}'$. It is not hard

[399] We think that this is also true without $\mathbf{AxE_{01}}$.

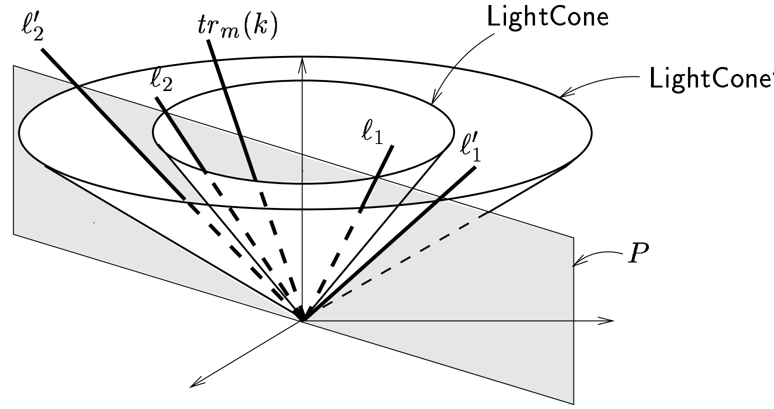



Figure 137: Illustration for the proof of Prop.4.3.6(i).

to check that there is a plane $P$ such that $\bar{0} \in P$, $P \cap \mathsf{LightCone} = \ell_1 \cup \ell_2$ and $P \cap \mathsf{LightCone}' = \ell_1' \cup \ell_2'$, for some pairwise different $\ell_1, \ell_2, \ell_1', \ell_2' \in \mathsf{Eucl}$. Let such $P, \ell_1, \ell_2, \ell_1', \ell_2'$ be fixed. Now by $\mathbf{Ax5^{Ph}}$ there are $ph_1, ph_2, ph_1', ph_2' \in Ph$ such that $tr_m(ph_1) = \ell_1$, $tr_m(ph_2) = \ell_2$, $tr_m(ph_1') = \ell_1'$ and $tr_m(ph_2') = \ell_2'$. Now by (284), there is $k \in Obs$ with $\bar{0} \in tr_m(k) \subseteq P$. Checking the details is left to the reader.
$\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF$ is a bijection taking lines to lines by Thm.4.3.11 way below. But then

$$tr_k(ph_1) \cap tr_k(ph_2) \cap tr_k(ph_1') \cap tr_k(ph_2') \neq \emptyset,$$
$$(\exists d \in \mathsf{directions})\ ph_1, ph_2, ph_1', ph_2' \text{ move in direction } d \text{ as seen by } k, \text{ and}$$
$$tr_k(ph_1),\ tr_k(ph_2),\ tr_k(ph_1'),\ tr_k(ph_2') \text{ are pairwise different.}$$

But this contradicts **AxP1**. Hence $\mathsf{LightCone} = \mathsf{LightCone}'$ and $v_m(ph) = v_m(ph')$. This completes the proof. ∎

The reason why the above proposition is true is that the Weak Principle of Isotropy ($\mathbf{Ax5^{Ph}}$) was included into **Bax** and is so strong that it makes **AxP1** equivalent with $\mathbf{AxE_{00}}$ (under assuming $\mathbf{Ax}(\sqrt{\ })$ and that the traces of photons are straight lines [or empty]). Moreover, according to our plans, the difference between the philosophies of **Bax** and $\mathbf{Bax^-}$ is that $\mathbf{Bax^-}$ will not assume isotropy (while **Bax** does, via $\mathbf{Ax5^{Ph}}$). Therefore we first have to fine-tune the axioms in $\mathbf{Bax} \setminus \{\mathbf{AxE_{00}}\}$ and only then try to use **AxP1**.

By looking at Prop.4.3.6(ii) above, we notice that we have to weaken $\mathbf{Ax5^{Ph}}$. This weakened version of $\mathbf{Ax5^{Ph}}$ is $\mathbf{Ax5_{Ph}}$ below. We will also change $\mathbf{Ax5^{Obs}}$ to $\mathbf{Ax5_{Obs}}$ below, because $\mathbf{Ax5^{Obs}}$ does not fit the "philosophy" or paradigm of (P1).[400]

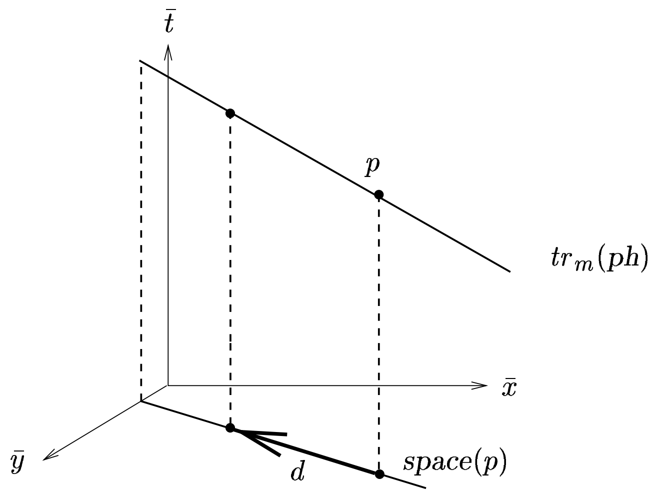



Figure 138: Illustration for $\mathbf{Ax5_{Ph}}$.

$\mathbf{Ax5_{Ph}}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})(\exists ph \in Ph)$
$[p \in tr_m(ph) \ \land \ (ph \text{ is moving forwards in direction } d \text{ as seen by } m)]$.

See Figure 138.

Intuitively, from any point $p$ of space-time in any direction there is a photon moving forwards in that direction.

We use the convention that $\infty$ is bigger than any element of $F$, i.e. that $\lambda < \infty$ for every $\lambda \in F$.

$\mathbf{Ax5_{Obs}}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})$
$\Big(\Big[(\exists ph \in Ph)(p \in tr_m(ph) \ \land \ (ph \text{ is moving forwards in } d \text{ as seen by } m)\Big] \ \Rightarrow$
$\Big[(\exists ph \in Ph)\Big(p \in tr_m(ph) \ \land \ (ph \text{ is moving forwards in } d \text{ as seen by } m) \ \land$

400 A kind of isotropy is somehow implicit in the formulation of $\mathbf{Ax5^{Obs}}$. This is why we have to replace that axiom with a more careful version.

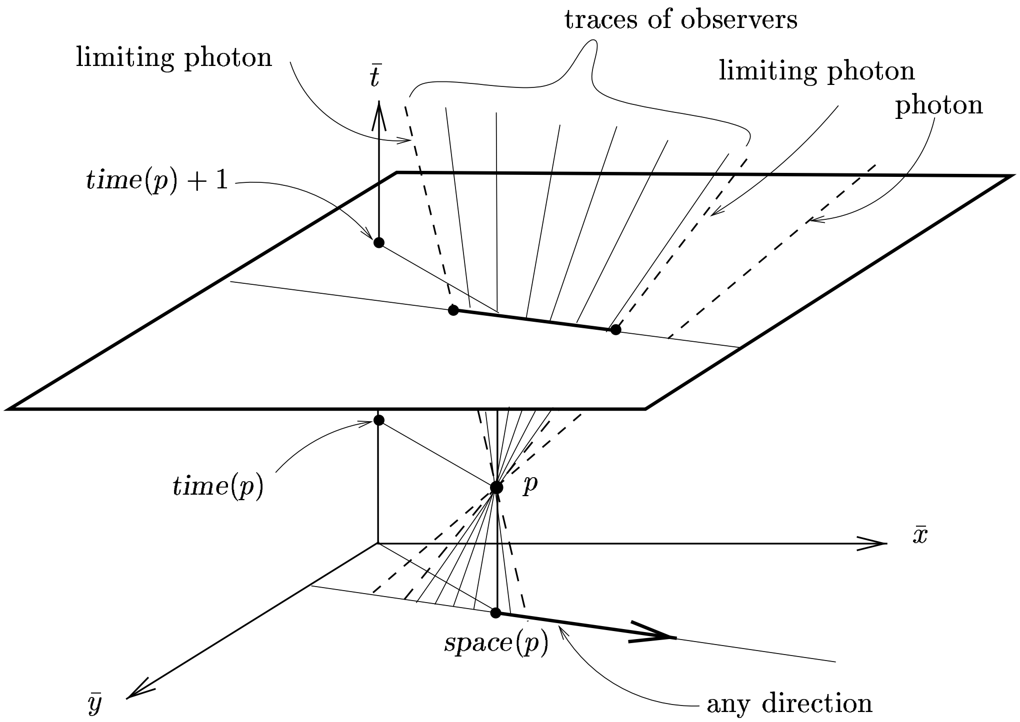



Figure 139: Illustration for $\mathbf{Ax5_{Obs}}$.

$(\forall\lambda \in F)(0 \leq \lambda < v_m(ph) \Rightarrow (\exists k \in Obs)(p \in tr_m(k) \quad\wedge\quad v_m(k) = \lambda \quad\wedge$ $(k$ is moving forwards in direction $d$ as seen by $m))\Big)\Big]\Big)$.[401] See Figure 139.

Intuitively: Let us fix an observer $m$, a direction $d$, and a point $p$ of space-time. We will speak about things moving forwards in direction $d$ through point $p$ as seen by $m$ (without mentioning all this data). Assume there is a photon moving in direction $d$. Then there is a photon in the same direction which is limiting in the following sense: For all speeds slower than this limiting photon, there is an observer moving with this speed.

Now, we are ready to define our weak system $\mathbf{Bax}^-$.

**Definition 4.3.7** We define

$$\mathbf{Bax}^- \stackrel{\text{def}}{=} \Big(\mathbf{Bax} \setminus \{\mathbf{Ax5^{Ph}}, \mathbf{Ax5^{Obs}}, \mathbf{AxE_{00}}\}\Big) \cup \{\mathbf{Ax5_{Obs}}, \mathbf{Ax5_{Ph}}, \mathbf{AxP1}\},$$

where $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax5_{Ph}}$ and $\mathbf{AxP1}$ are defined above. Therefore

$$\mathbf{Bax}^- = \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Ph}}, \mathbf{Ax5_{Obs}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{01}}, \mathbf{AxP1}\}.$$

$\lhd$

Next we state that (under assuming $\mathbf{Ax}(\sqrt{\ }))$ $\mathbf{Bax}^-$ is indeed strictly weaker than $\mathbf{Bax}$ (as we wanted).

**PROPOSITION 4.3.8**

**(i)** $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) \not\models \mathbf{Bax}$.[402]

**(ii)** *Assume* $n \leq 3$. *Then* $\mathbf{Bax} \models \mathbf{Bax}^-$.

**Proof:**

(i) $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{Bax}^-$ is not hard to check, we leave it to the reader. We prove here $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) \not\models \mathbf{Bax}$. To prove this, we have to show the existence of a model $\mathfrak{M} \models \mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$ in which $\mathbf{Bax}$ is not true.

The idea of constructing $\mathfrak{M}$ is the following. We start out of the world-view of an observer $m$ in a model, say, of $\mathbf{Basax}$. Then we define the worldview-transformations $\mathsf{f}_{km}$ to be Galilean transformations. See Figure 140.

In more detail: Let $\mathfrak{F}^{\mathfrak{M}}$ be any Euclidean field. Then $\mathbf{Ax}(\sqrt{\ })$ will hold in $\mathfrak{M}$. Define

[401] It is a task of future research to find natural (and short) versions of $\mathbf{Ax5_{Obs}}$ and $\mathbf{Ax5_{Ph}}$ which are provable from $\mathbf{Bax}$.

[402] I.e. $\mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ })) \subsetneq \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }))$.

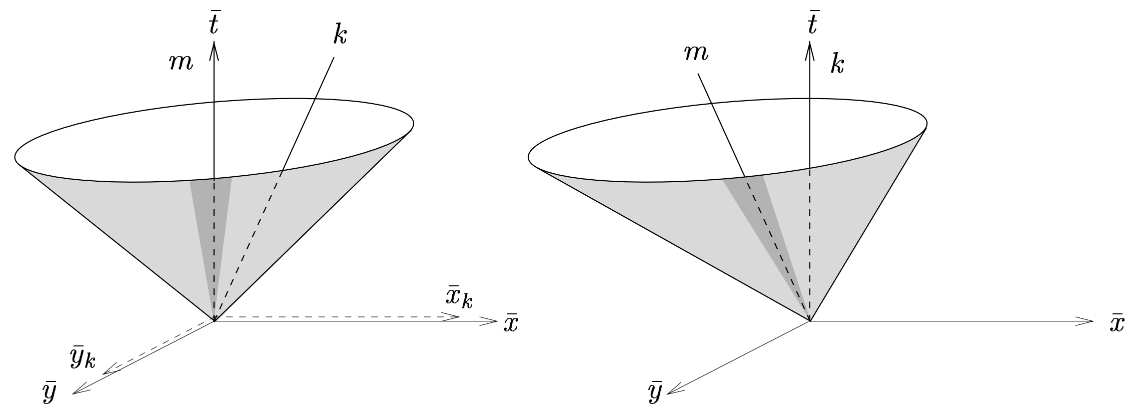



Figure 140: An anisotropic model of $\mathbf{Bax}^-$, the $\mathsf{f}_{mk}$'s are Galilean.

$$B \stackrel{\text{def}}{=} \mathsf{Eucl}(n, \mathbf{F}),$$

$$Obs \stackrel{\text{def}}{=} \mathsf{SlowEucl},$$

$$Ph \stackrel{\text{def}}{=} \mathsf{PhtEucl}.$$

As in §2.4, let $m \stackrel{\text{def}}{=} \bar{t}$ and define the world-view of $m$ such that $tr_m(b) = b$, for all $b \in B$. We now only have to define $\mathsf{f}_{km}$ for $k \in Obs$. We define $\mathsf{f}_{km}$ to be an affine transformation with the following properties:

$$\mathsf{f}_{km}(\bar{0}) \in k \quad \text{and} \quad \mathsf{f}_{km}(\bar{0})_t = 0,$$

$$\mathsf{f}_{km}(1_t) \in k \quad \text{and} \quad \mathsf{f}_{km}(1_t)_t = 1,$$

$$\mathsf{f}_{km}(1_i) = 1_i + \mathsf{f}_{km}(\bar{0})_i \quad \text{for } 0 < i < n.$$

It is not difficult to check that the model we have just defined is a model of $\mathbf{Bax}^-$, and it is not a model of $\mathbf{Bax}$, because for any $k \in Obs$, $k \neq \bar{t}$ there is a direction $d$ such that the speed of light as seen by $k$ is not the same in directons $d$ and $-d$.

(ii) For $n < 3$, the proof is easy. Assume $n = 3$. In the proof of Thm.3.4.19 for $n = 3$ beginning with p.233, we transform each $\mathbf{Bax}(3)$-model $\mathfrak{M}$ to a $\mathbf{Newbasax}(3)$ model. By Thm.3.6.17 (in §3.6 way below), we conclude that $\mathbf{Ax}(\sqrt{\ })$ is true in these

**Newbasax** models. Since the construction did not change the field $\mathfrak{F}^{\mathfrak{M}}$, we conclude that

(285) $$\mathbf{Bax}(3) \models \mathbf{Ax}(\sqrt{\ }).$$

∎

**Remark 4.3.9** One might wonder why $\mathbf{Ax}(\sqrt{\ })$ is needed in (i) of the above proposition. The reason for this is that in **Bax** we did not assume $\mathbf{Ax}(\sqrt{\ })$ and therefore in certain models of **Bax** there may exist (spatial) directions, say $d$, in which no photon moves.[403] This may happen despite of $\mathbf{Ax5^{Ph}}$ because in some direction $d$ there need not exist a line $\ell$ with $ang^2(\ell) =$ "speed of light".[404] On the other hand $\mathbf{Bax^-}$ does postulate that in every (spatial) direction there is a photon moving. Hence we do not know whether there is $n > 3$ with $\mathbf{Bax}(n) \not\models \mathbf{Bax^-}(n)$.

◁

**CONVENTION 4.3.10** In the present work when comparing refinements of **Bax** we will almost always assume $\mathbf{Ax}(\sqrt{\ })$. E.g. if we say that $\mathbf{Bax^-}$ is weaker than **Bax** then we really mean to say that $\mathbf{Bax^-} + \mathbf{Ax}(\sqrt{\ })$ is weaker than $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ })$.

◁

The following theorem is an analogous counterpart of Thm.3.1.1 (§3.1).

**THEOREM 4.3.11** $\mathbf{Bax^-} \models (\forall m, k \in Obs)\Big(m \overset{\odot}{\to} k \;\Rightarrow\; (\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF$ *is a bijective collineation*$)\Big)$.

We will prove a stronger theorem, namely we will prove that in Theorem 4.3.11 we can replace $\mathbf{Bax^-}$ with a much weaker axiom system **Pax**. Let $\mathbf{Ax5_{Obs}}^{--}$ be the following axiom:

$\mathbf{Ax5_{Obs}}^{--}$ $(\forall m \in Obs)(\forall d \in \mathsf{directions})(\forall p \in {}^nF)(\exists \lambda \in {}^+F)$
$(\forall q \in {}^nF)\Big[space(p) - space(q) = \delta \cdot d$ for some $\delta \in F \;\Rightarrow\; (\forall 0 \le \varepsilon < \lambda)$
$(\exists k \in Obs)(k$ moves forwards in direction $d$ with speed $\varepsilon$ and $q \in tr_m(k))\Big]$.

Intuitively, $\mathbf{Ax5_{Obs}}^{--}$ says that for each direction $d$ there is a $\lambda$ such that through any point there are observers moving forwards in direction $d$ with all speeds smaller than $\lambda$. $\mathbf{Ax5_{Obs}}^{--}$ allows that these $\lambda$'s be different for points of different planes parallel with $\bar{t}$.

---

[403]We do not know if there are such models of **Bax**.

[404]The same applies to **Basax** and **Newbasax** in place of **Bax**.

**LEMMA 4.3.12** $\mathbf{Bax}^- \models \mathbf{Ax5_{Obs}}^{--}$.

**Proof.** Assume $\mathbf{Bax}^-$. Let $m, d, p$ be as in $\mathbf{Ax5_{Obs}}^{--}$. There is a unique photon $ph$ going through $p$ and moving forwards in direction $d$, by $\mathbf{Bax}^-$. Let $\lambda \stackrel{\text{def}}{=} v_m(ph)$. Let $q$ be as in $\mathbf{Ax5_{Obs}}^{--}$, and let $ph_1$ be a photon going through $q$ and moving forwards in direction $d$. Assume that $v_m(ph) \neq v_m(ph_1)$. Then $ph$ and $ph_1$ would meet in a point $q_1$, because $tr_m(ph)$ and $tr_m(ph_1)$ are both in the plane $P$ parallel with $\bar{t}$ and containing $p, q$ (by our assumptions). But then $tr_m(ph), tr_m(ph_1)$ would be two photon-traces going through $q_1$ forwards in direction $d$, and this contradicts **AxP1**. Thus $v_m(ph_1) = v_m(ph) = \lambda$. Let $0 \leq \varepsilon < \lambda$. By $\mathbf{Bax}^-$ then there is an observer $k$ going forwards in direction $d$ with speed $\varepsilon$, and going through $q$. ∎

Let **Pax** denote the axiom system we obtain from $\mathbf{Bax}^-$ by replacing $\mathbf{Ax5_{Obs}}$ in it with $\mathbf{Ax5_{Obs}}^{--}$ and omitting all axioms mentioning photons (except for **Ax2**). I.e.

$$\mathbf{Pax} \stackrel{\text{def}}{=} \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Obs}}^{--}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}\}.$$

**THEOREM 4.3.13** $\mathbf{Pax} \models (\forall m, k \in Obs)\Big(m \stackrel{\odot}{\to} k \;\Rightarrow\; (\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF$ *is a bijective collineation*$)\Big)$.

**Proof.** In the proof we use the methods of the proofs of Thm.3.1.1 (p.160) and Lemma 3.3.16 (p.198).

Assume $\mathfrak{M} \models \mathbf{Pax}$, $m, k \in Obs$ and $m \stackrel{\odot}{\to} k$. If $p \in Dom(\mathsf{f}_{mk})$, then we say that <u>*k sees p*</u>.

**Claim 4.3.14** Assume $v_m(m') = 0$. Then if $k$ sees at least one point on $tr_m(m')$, then there is at most one point on $tr_m(m')$ which $k$ does not see.

**Proof of Claim 4.3.14.** Assume that $p \in tr_m(m') \cap Dom(\mathsf{f}_{mk})$. Let $P$ be a plane parallel with $\bar{t}$ which contains $tr_m(m')$. We will "work" in $P$. See Figure 141. Let $S$ be a neighbourhood of $p$ such that $p \in S \subseteq Dom(\mathsf{f}_{mk})$. Such a neighbourhood exists by $\mathbf{Ax6_{01}}$. Let $d$ be a direction such that all straight lines lying in $P$ move in direction $d$. Let $\lambda$ belong to $p$ and $d$ according to $\mathbf{Ax5_{Obs}}^{--}$. I.e. from all points $q \in P$ and for all $\varepsilon < \lambda$ there are observers moving through $q$ forwards in direction $d$ with speed $\varepsilon$.

Assume that $q, r \in tr_m(m')$, $q \neq r$ such that $k$ sees neither $q$ nor $r$. Let us choose observers $m_1, \ldots, m_4$ according to Figure 141. This is possible by $\mathbf{Ax5_{Obs}}^{--}$: we choose these observers so that they all go forwards in direction $d$ and their speeds

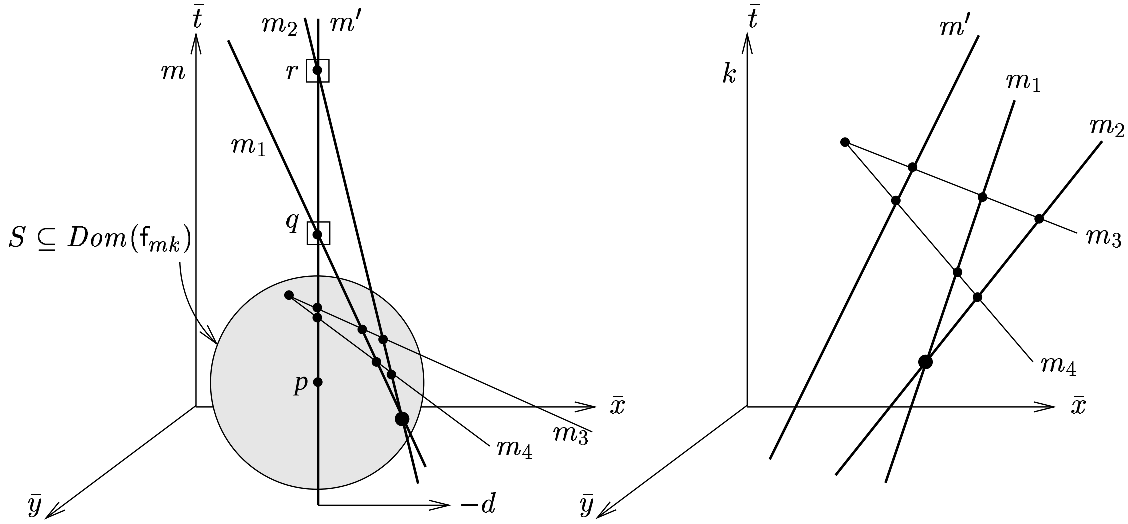



Figure 141:

are sufficiently small. The important thing is that all the indicated meeting points are inside $S$, i.e. $k$ sees all these meeting points (except for $q, r$ which $k$ does not see). Then $k$ sees all the observers $m', m_1, \ldots, m_4$ with those meeting points which are inside $S$. Thus in $k$'s worldview, the traces of the observers $m', m_1, \ldots, m_4$ are all in one plane, i.e. they are coplanar. On the other hand, we will show that $q \notin Dom(\mathsf{f}_{mk})$ implies that in $k$'s worldview, $m_1$ does not meet $m'$. Indeed, assume that $m_1$ and $m'$ meet in $k$'s worldview, say in point $s$. Then by $\mathbf{Ax6_{00}}$, $m'$ sees the event $w_k(s)$, say $w_k(s) = w_{m'}(s_1)$. Also by $\mathbf{Ax6_{00}}$, $m'$ sees the event $w_m(q)$, say $w_m(q) = w_{m'}(s_2)$. But both $m'$ and $m_1$ are present in both events $w_{m'}(s_1)$ and $w_{m'}(s_2)$, so $s_1$ must equal $s_2$, since the traces of $m'$ and $m_1$ meet only in one point, since they are different in $m'$'s worldview (also by $\mathbf{Ax6_{00}}$, since e.g. in $m$'s worldview there is an event on $m'$'s trace in which $m_1$ is not present). Since $s_1 = s_2$, we then have $w_m(q) = w_k(s_1)$, which contradicts our assumption $q \notin Dom(\mathsf{f}_{mk})$. Similarly, in $k$'s worldview, $m_2$ does not meet $m'$ because $r \notin Dom(\mathsf{f}_{mk})$. Thus, in $k$'s worldview both $tr_k(m_1)$ and $tr_k(m_2)$ are parallel with $tr_k(m')$, though $m_1$ and $m_2$ meet. This contradicts the fact that in a plane to each line $\ell$ and point $u$ there is only one line parallel to $\ell$ which goes through $u$. This finishes the proof of Claim 4.3.14. ∎

We say that an observer $m_1 \in Obs$ is slow if, in $m$'s worldview, $m_1$ moves forwards in direction $d$ in a plane with less speed than the $\lambda$ belonging to this plane according to $\mathbf{Ax5_{Obs}}^{--}$. The following claim is analogous with Lemma 3.3.16.

**Claim 4.3.15** Assume that $m_1$ is a slow observer. If $k$ sees a point on $tr_m(m_1)$, then $k$ sees all points on $tr_m(m_1)$.

**Proof of Claim 4.3.15.** Let $p \in tr_m(m_1) \cap Dom(\mathsf{f}_{mk})$ and let $S$ be a neighbourhood of $p$ such that $S \subseteq Dom(\mathsf{f}_{mk}) \cap Dom(\mathsf{f}_{mm_1})$. Such a neighbourhood exists by $\mathbf{Ax6_{01}}$. Let $q \in tr_m(m_1)$ be arbitrary. See Figure 142.

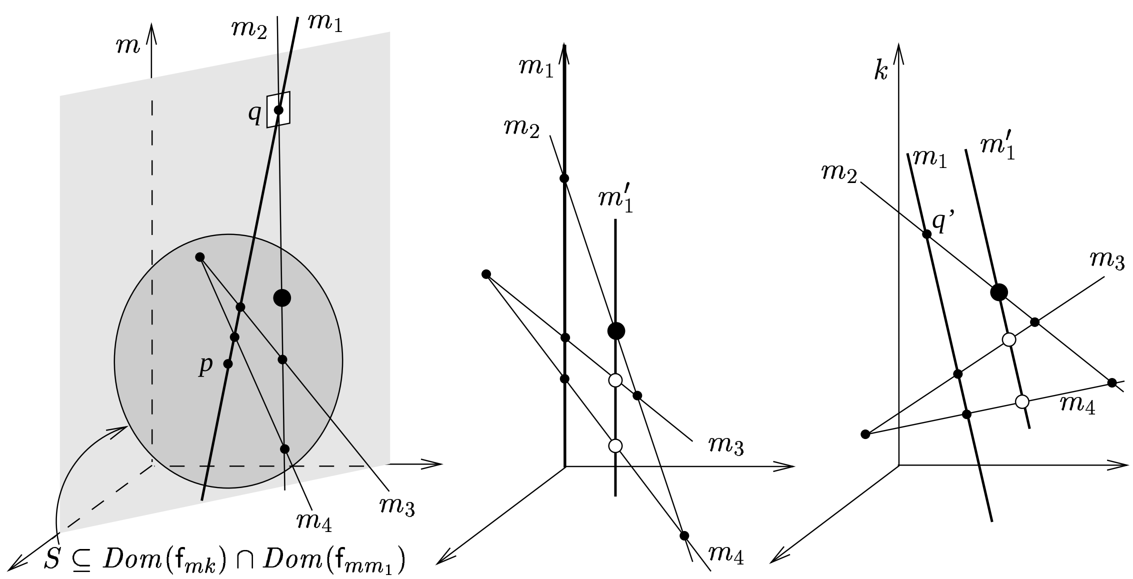



Figure 142:

Let $m_2, m_3, m_4$ be as in Figure 142: $m_2$ meets $m_1$ at $q$, and $m_3, m_4$ meet $m_1$ and $m_2$ and each other inside $S$. Further, $m_1, \ldots, m_4$ have diffferent traces in $m$'s worldview. Such observers exist by $\mathbf{Ax5_{Obs}}^{--}$ as in the proof of the previous claim. Let $r$ be a point on $tr_m(m_2)$ inside $S$, but different from the meeting points with $m_3, m_4$. (This is the fat point in Figure 142.)

Let us move into the worldview of $m_1$. $m_1$ sees all the meeting points that are inside $S$, because $S \subseteq Dom(\mathsf{f}_{mm_1})$. Thus, the traces of $m_3, m_4, m_1, m_2$ are all in one plane in $m_1$'s worldview. Let $m_1'$ be an observer such that $tr_{m_1}(m_1')$ is a straight line parallel with $\bar{t}$ in this plane, which goes through $r' \stackrel{\text{def}}{=} \mathsf{f}_{mm_1}(r)$. Such an observer exists by $\mathbf{Ax5_{Obs}}^{--}$.

Let us move now into the worldview of $k$. By $r \in S \subseteq Dom(\mathsf{f}_{mm_1}) \cap Dom(\mathsf{f}_{mk})$, $k$ sees the event on $m_1$'s trace which is at $r'$. In $m_1$'s worldview, the meeting points of $m_3$ and $m_4$ with $m_1'$ are different from each other and from $r'$, because the traces of $m_1', m_3, m_4$ are all different in $m_1$'s worldview. Therefore, by Claim 4.3.14, $k$ does not see at most one of these points, and thus $k$ sees one of the meeting points of $m_1'$ with $m_3$ or $m_4$. Therefore, the trace of $m_1'$ in $k$'s worldview is also coplanar with the traces of $m_1, \ldots, m_4$. Also, $m_1'$ and $m_1$ cannot meet in $k$'s worldview by $\mathbf{Ax6_{00}}$, because they do not meet in $m_1$'s worldview. Now, in $k$'s worldview, the traces of $m_2$ and $m_1'$ meet, they are coplanar with the trace of $m_1$, and $m_1$ and $m_1'$ do not meet. Thus $m_2$ must meet $m_1$ (as before, because on a plane through a point there is only one straight line parallel with a given one). Let us assume that $m_2$ and $m_1$ meet in $k$'s worldview at $q'$.

It remains to show that the event $e$ in $m$'s worldview at $q$ is the same as the event $e'$ in $k$'s worldview at $q'$. To show this we will go back to $m_1$'s worldview. By $\mathbf{Ax6_{00}}$, $m_1$ sees both events $e, e'$ and he must see them on his own life-line, because $m_1 \in e \cap e'$. On the other hand, also $m_2 \in e \cap e'$, and on $m_1$'s life-line there is only one point where $m_2$ is present, namely in the meeting point of $m_1$ with $m_2$, because the traces of $m_1$ and $m_2$ are different (e.g. by $\mathbf{Ax6_{00}}$, because these traces are different in $m$'s worldview). Thus $e = e_1$ and this finishes the proof of Claim 4.3.15. ∎

**Claim 4.3.16** If $Dom(\mathsf{f}_{mk}) \neq \emptyset$, then $Dom(\mathsf{f}_{mk}) = {}^nF$.

**Proof of Claim 4.3.16.** We are in the worldview of $m$. We will connect any two points of ${}^nF$ with traces of slow observers. Let $p, q \in {}^nF$ be arbitrary. See Figure 143.

If $space(p) = space(q)$, then there is an observer with speed 0 whose trace connects $p$ and $q$. Assume therefore $space(p) \neq space(q)$, and let $d = space(q) - space(p)$. Then $d \in \mathsf{directions}$. Let $m_1$ be any slow observer moving forwards in direction $d$ with nonzero speed, and through $p$, and let $m_2$ be another observer which is at rest at point $space(q)$. Such observers exists by $\mathbf{Ax5_{Obs}}^{--}$. Then $m_1$ and $m_2$ will meet, say in point $r$. Then by Claim 4.3.15, $p \in Dom(\mathsf{f}_{mk})$ implies $q \in Dom(\mathsf{f}_{mk})$. This finishes the proof of Claim 4.3.16. ∎

From here on the proof is basically the same as the proof of Theorem 3.1.1. One of the changes we make is that we replace "slow lines" with "traces of slow observers". For completeness, we briefly include the rest of the proof.

Assume that $m, k \in Obs$, $m \overset{\odot}{\to} k$.

First we show that $\mathsf{f}_{mk}$ takes midpoints on the trace of an observer to midpoints, this is an analogon of Lemma 3.1.10, see Figure 144.

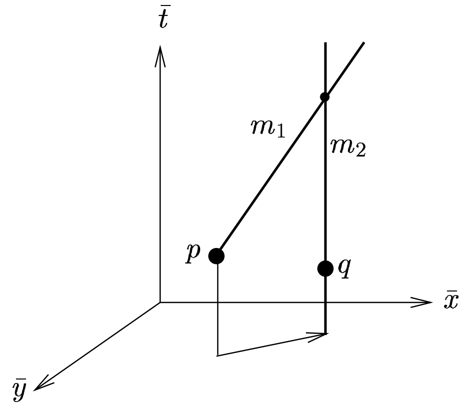



Figure 143:

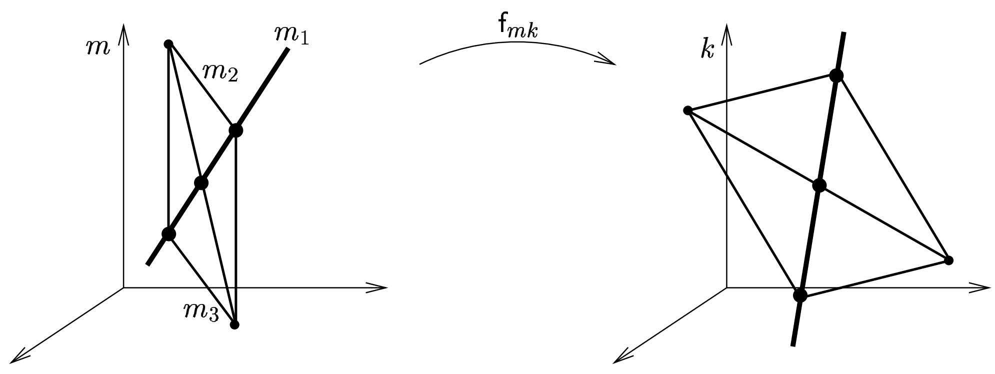



Figure 144:

Let $m_1 \in Obs$ and let $p, q, r \in tr_m(m_1)$ be such that $q$ is the midpoint of $p$ and $r$. In $m$'s worldview, let us choose slow observers $m_2, \dots, m_6$ as in Figure 144: the traces of $m_1, \dots, m_5$ are coplanar, $m_2$ and $m_3$ do not meet, and similarly $m_4$ and $m_5$ do not meet, $m_2$ and $m_5$ meet at $r$, $m_3$ and $m_4$ meet at $p$, $m_6$ and $m_1$ meet at $q$ and $m_2, m_6, m_4$ all meet in one point, and $m_3, m_6, m_5$ all meet in one point. Informally, $m_1, \dots, m_6$ form a paralelogram as in Figure 144. Since $\mathsf{f}_{mk}$ is everywhere defined by Claim 4.3.16, $k$ also sees all these meeting points. Thus, in $k$'s worldview $m_1, \dots, m_6$ also form a paralelogram. Since the diagonals of a paralelogram bisect each other, $\mathsf{f}_{mk}(q)$ is the midpoint of $\mathsf{f}_{mk}(p)$ and $\mathsf{f}_{mk}(r)$.

Assume that $\ell \in \mathsf{Eucl}$ and $p, q, r \in \ell$. In $m$'s worldview we choose slow observers $m_1, \dots, m_4$ as in Figure 145.

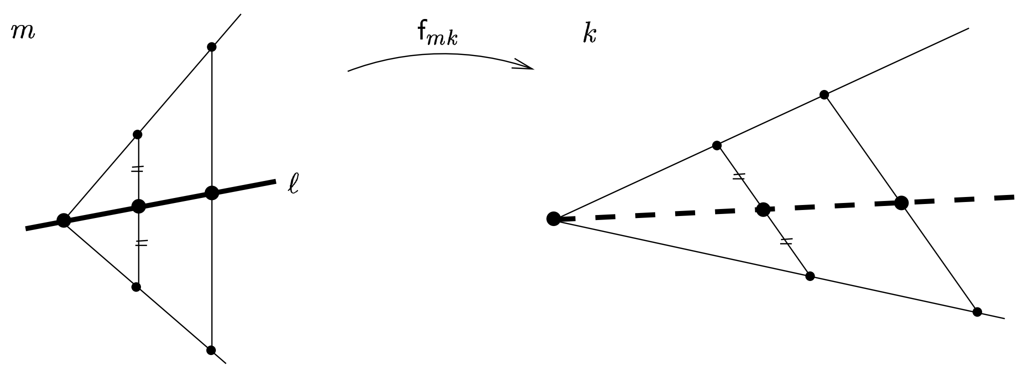



Figure 145:

From here on the proof is practically the same as on pages 169-170. Also the proof of $\mathsf{f}_{mk}$ being an injection is the same as that of Claim 2.3.7 on p.60, because in that proof we used only the consequence of **Ax5** that through each point there move at least two different observers, and $\mathbf{Ax5_{Obs}}^{--}$ also implies this fact. By this, Theorem 4.3.13 has been proved. ∎

The next theorem says that in models of $\mathbf{Bax}^-$, all light-cones are alike in the world-view of an observer $m$. See Figure 146. This means that the "speed of light", according to $m$, depends only on the spatial direction $d$ in which the light particle was emitted, and it does not depend on the point of space-time $p$ where it was emitted. Using the notation $c_m(p, d)$ introduced in Remark 4.3.5, this means that $c_m(p, d) = c_m(q, d)$ for any $p, q \in {}^nF$. Yet in other words, this means that if $\ell$ is parallel with the trace of a photon, then $\ell$ itself is the trace of a photon. (We note that the same is true for "observer" in place of "photon", in models of $\mathbf{Bax}^-$.)

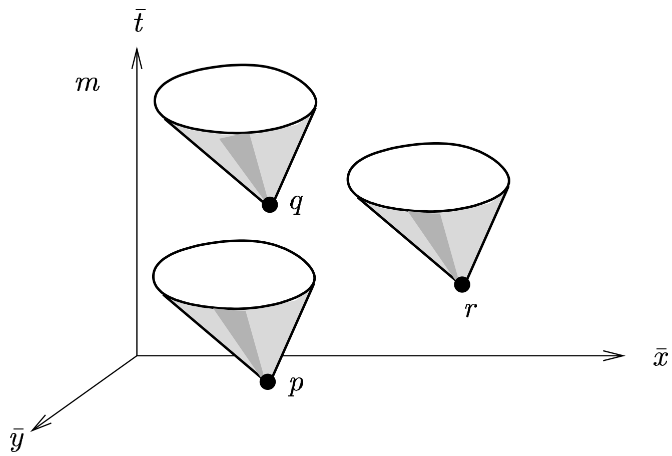



Figure 146: All light-cones are alike in a world-view of a model of $\mathbf{Bax}^-$.

**THEOREM 4.3.17 (light-cones are alike)** *Assume* $\mathbf{Bax}^-$. *Then*

$$c_m(p,d) = c_m(q,d)$$

*for all observers* $m$, *points* $p, q \in {}^nF$ *and directions* $d$.

**Proof.** Let $ph_1$ and $ph_2$ be photons moving forwards in direction $d$ as seen by $m$, and $p \in tr_m(ph_1)$, $q \in tr_m(ph_2)$. Such photons exist by $\mathbf{Ax5_{Ph}}$. Assume that $v_m(ph_1) \neq v_m(ph_2)$, say $v_m(ph_1) < v_m(ph_2)$. Let $k$ be an observer moving in direction $d$, through $q$, and with speed $v_m(ph_1)$. See Figure 147. Such an observer $k$ exists by $\mathbf{Ax5_{Obs}}$ and $\mathbf{AxP1}$. Then $tr_m(k)$ is parallel with $tr_m(ph_1)$ because $k$ and $ph_1$ move forwards in the same direction and with the same speed. But this implies that $ph_1$ is at rest in $k$'s world-view, because $\mathsf{f}_{mk}$ is a collineation by Theorem 4.3.11 and collineations take parallel lines to parallel ones. This contradicts $\mathbf{AxE_{01}} \in \mathbf{Bax}^-$. ∎

Since in models of $\mathbf{Bax}^-$, the speed $c_m(p,d)$ does not depend on $p$, we introduce a notation which reflects this:

$$\mathsf{c}_m(d) \stackrel{\text{def}}{=} c_m(\bar{0}, d).$$

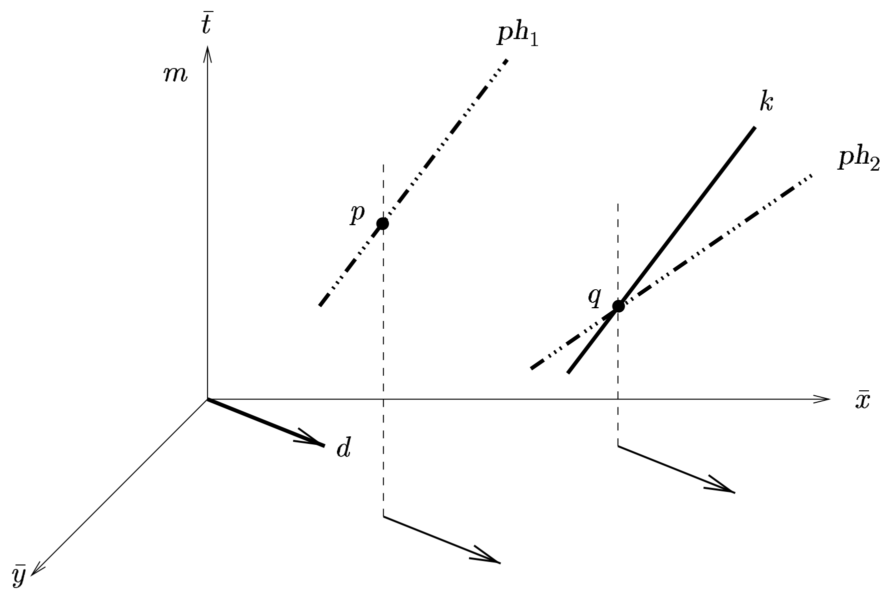



Figure 147: Illustration for the proof of Thm.4.3.17: if $ph_1$ and $ph_2$ move forwards in the same direction but with different speed, then there is an observer $k$ who sees one of them at rest.

Recall that one of our symmetry axioms is $\mathbf{Ax\Box 1}$, and it says that $(\forall m, k, m')(\exists k')\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$. The following theorem states that if we assume this symmetry principle (together with mild auxiliary axioms), then we can derive **Flxspecrel** from $\mathbf{Bax}^-$. In natural words this means that $\mathbf{Ax\Box 1}$ together with mild axioms and $\mathbf{Bax}^-$ imply that there is a uniform speed $c$ of light such that for each observer, in each direction, light moves with speed $c$ (and also all observers see the same events).

**THEOREM 4.3.18** *Assume $n > 2$. Let* $\mathbf{Ax} \stackrel{\text{def}}{=} \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax6}$. *Then*

$$\mathbf{Bax}^- + \mathbf{Ax\Box 1} + \mathbf{Ax} \models \mathbf{Flxspecrel}.$$

A proof outline for Theorem 4.3.18 is given on p.759, where this theorem is re-stated as Theorem 5.2.17. ∎

The following considerations add interest to the above theorem. $\mathbf{Bax}^-$ says very little about photons, it contains only two very natural assumptions about photons (**AxP1** and $\mathbf{Ax5_{Ph}}$). The point of the above theorem is that these two natural postulates together with finiteness of speed of photons are sufficient for deriving special relativity (if one is willing to use the symmetry principle $\mathbf{Ax\Box 1}$, together with some simple and very convincing axioms). (We refer to Remark 4.1.7 where we argued that $\mathbf{Flxbasax} + c_m(p, d) < \infty$ is very close to **Basax**.)

**Questions for future research 4.3.19**

Partial answers to the questions below can be found in Thm.4.3.29 at the end of this section.

1. Investigate the models of $\mathbf{Bax}^-(n)$, for various choices of $n$. E.g. what *kinds* of models does $\mathbf{Bax}^-$ have? In this connection see Theorems 4.3.17, 4.3.29, 4.5.8, and the proofs of Theorems 4.3.21, 4.3.25, 4.5.8.

2. Assume $\mathbf{Bax}^-(3)$ and that $v_m(ph) \neq \infty$. Are then the light-cones "coherent" surfaces in the following sense? Let $\mathsf{Cone}_{m,\bar{0}}$ be the light-cone starting in the origin $\bar{0}$ (as seen by an observer $m$), cf. Def.4.3.4 on p.473. Cf. also Figure 175 on p.539.

   Consider the simultaneity $P = \{\, q \in {}^nF \;:\; q_t = 1 \,\}$.

   (a) Is then $\mathsf{Cone}_{m,\bar{0}} \cap P$ a curve in some sense?

   (b) If yes, is then it a closed curve? Is it strongly continuous in the sense of Definition 4.4.8?

(c) Is $P \cap \mathsf{Cone}_{m,\bar{0}}$ homeomorphic to a circle?

See Theorem 4.3.29 in this connection.

3. The same as in item 3 above but for $\mathbf{Bax}^-(4)$.

$\lhd$

**$\mathbf{Bax}^-$ and the paradigmatic effects of relativity, e.g. FTL.**

**Notation 4.3.20** $c_m(p,d) < \infty$ abbreviates the formula stating that the speed of light is finite for all observers $m$, at all points $p$ and in all directions $d$.

$\lhd$

Since Newtonian Kinematics is a special case of $\mathbf{Bax}^-$, i.e. since $\mathsf{Mod}(\mathbf{NewtK}) \subseteq \mathsf{Mod}(\mathbf{Bax}^-)$, we cannot expect *any* relativistic effect to be provable from $\mathbf{Bax}^-$ (except perhaps the "$\nexists$ FTL observers" effect), c.f. Thm.4.1.12. E.g. let $\psi$ be a relativistic effect like "moving clocks slow down" or "meter-rods shrink". Then we can be sure that

$$\mathbf{Bax}^- \not\models \psi\,.$$

However, we may ask ourselves, which relativistic effects are provable from ($\mathbf{Bax}^-$ + $c_m(p,d) < \infty$). Newtonian Kinematics is excluded by $c_m(p,d) < \infty$, hence we may think of ($\mathbf{Bax}^-$ + $c_m(p,d) < \infty$) as the "purely relativistic part" of $\mathbf{Bax}^-$.

Still, the paradigmatic effects in §2.5 are not provable from ($\mathbf{Bax}^-$ + $c_m(p,d) < \infty$). The reason is that $\mathbf{Bax}^-$ has models $\mathfrak{M}$ in which the speed of light is finite in each direction, yet $\mathfrak{M}$ is basically a model of $\mathbf{NewtK}$. In fact, all models of $\mathbf{NewtK}$ can be made to satisfy ($\mathbf{Bax}^-$ + $c_m(p,d) < \infty$) with extremely little change, as follows. Let us start out from any model of Newtonian Kinematics and postulate in each direction a finite speed of light (satisfying some mild requirements) in the world-view of an observer $m$. After this, we "throw away" the observers that so became faster than light (according to $m$), and we keep the old world-view transformations. This way we arrive at a model of $\mathbf{Bax}^- + c_m(p,d) < \infty$. The model that we gave in the proof of Proposition 4.3.8(i) is of this kind, so its construction also proves Theorem 4.3.21 below. We give a slight modification of that construction in the proof below because the general construction will be interesting for us later, too.

**THEOREM 4.3.21 (no basic paradigmatic effects hold in $\mathbf{Bax}^-$)**

**(i)** $\mathbf{Bax}^- + c_m(p,d) < \infty \not\models$ (some of the) moving clocks slow down, *i.e. Thm.2.5.2(i) becomes false if we replace* **Basax** *in it with* $(\mathbf{Bax}^- + c_m(p,d) < \infty)$.

**(ii)** $\mathbf{Bax}^- + c_m(p,d) < \infty \not\models$ (some of the) moving meter-rods shrink, *i.e. Thm.2.5.9 becomes false if we write* $(\mathbf{Bax}^- + c_m(p,d) < \infty)$ *in place of* **Basax**.

**(iii)** $\mathbf{Bax}^- + c_m(p,d) < \infty \not\models$ (moving clocks get out of synchronism), *i.e. Thm.2.5.6 is false for* $(\mathbf{Bax}^- + c_m(p,d) < \infty)$.

**Proof.** Let $K$ be a convex, open, bounded subset of $S_1 \stackrel{\text{def}}{=} \{1\} \times {}^{n-1}F$ in the sense that for any $\ell \in \mathsf{Eucl}$ such that $\ell \subseteq S_1$, $\ell \cap K$ is a bounded open interval, i.e. there are two "bounding" points on $\ell$ such that the points of $\ell$ which are in $K$ are exactly those that are strictly in between these two points,[405] formally: $(\exists p, q \in \ell)(\forall r \in \ell)[r \in K \quad \text{iff} \quad \mathsf{Betw}(p,r,q)]$. Assume further that $1_t \in K$. Let $\mathbb{C} \subseteq S_1$ be the boundary of $K$, i.e. $\mathbb{C}$ consists of these bounding points. Formally:

$$\mathbb{C} = \{p \in S_1 : (\exists \ell \in \mathsf{Eucl})(\exists q \in \ell \cap S_1, q \neq p)(\forall r \in \ell)[r \in K \quad \text{iff} \quad \mathsf{Betw}(p,r,q)]\}.$$

For any ordered field there is such a set $K$. E.g. if $n = 3$, then we can choose $K$ to be the interior of a square. See Figure 148.

We will use $K$ and $\mathbb{C}$ for defining the observers and photons of our model.

$$Obs_0 \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} : \bar{0} \in \ell, \ell \cap K \neq \emptyset\},$$

$$Ph_0 \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} : \bar{0} \in \ell, \ell \cap \mathbb{C} \neq \emptyset\},$$

$$Obs \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} : (\exists \ell' \in Obs_0)\ell \parallel \ell'\},$$

$$Ph \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} : (\exists \ell' \in Ph_0)\ell \parallel \ell'\}.$$

$$B \stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} Obs \cup Ph.$$

Let $m \stackrel{\text{def}}{=} \bar{t} \in Obs_0$. We define the world-view $w_m$ of $m$ such that $(\forall \ell \in Obs \cup Ph)\ell = tr_m(\ell)$. It remains to define the world-view transformations $\mathsf{f}_{km}$ for $k \in Obs$. For $k \in Obs_0$ we define $\mathsf{f}_{km}$ to be a collineation of ${}^n\mathfrak{F}$ such that

[405] $\mathsf{Betw}(p,r,q)$ means that $p, r, q$ are collinear points of ${}^nF$ and $r$ is strictly in between $p$ and $q$, formally: $r \neq p, q$ and $r = p + \lambda \cdot (q - p)$ for some $0 < \lambda < 1$. Cf. Figure 161 on p.510.

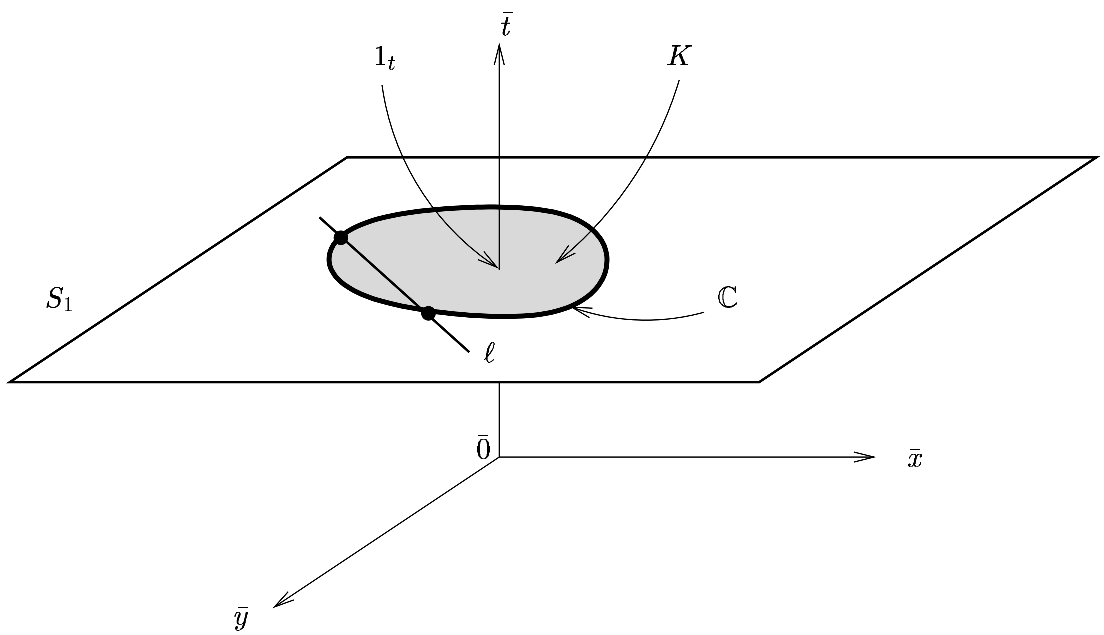



Figure 148: A convex subset $K$ of $S_1$ and its boundary $\mathbb{C}$.

$$\mathsf{f}_{km}(1_t) = p \quad \text{where } \{p\} = k \cap K,$$

$$\mathsf{f}_{km}(\bar{0}) = \bar{0} \quad \text{and}$$

$$\mathsf{f}_{km}(1_i) = 1_i \quad \text{for } 0 < i < n.$$

Thus $\mathsf{f}_{km}$ is a Galilean transformation taking $\bar{t}$ to $k$. See Figure 149.

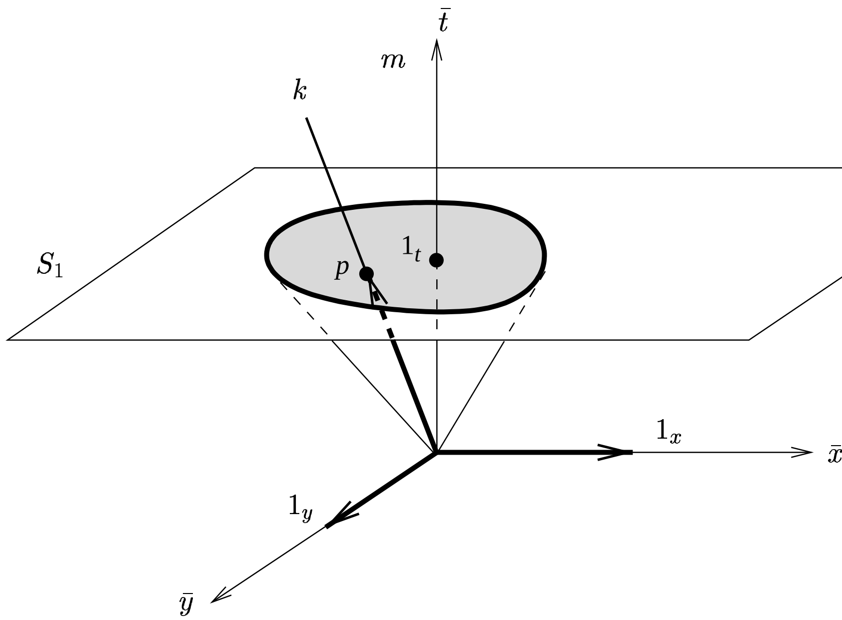
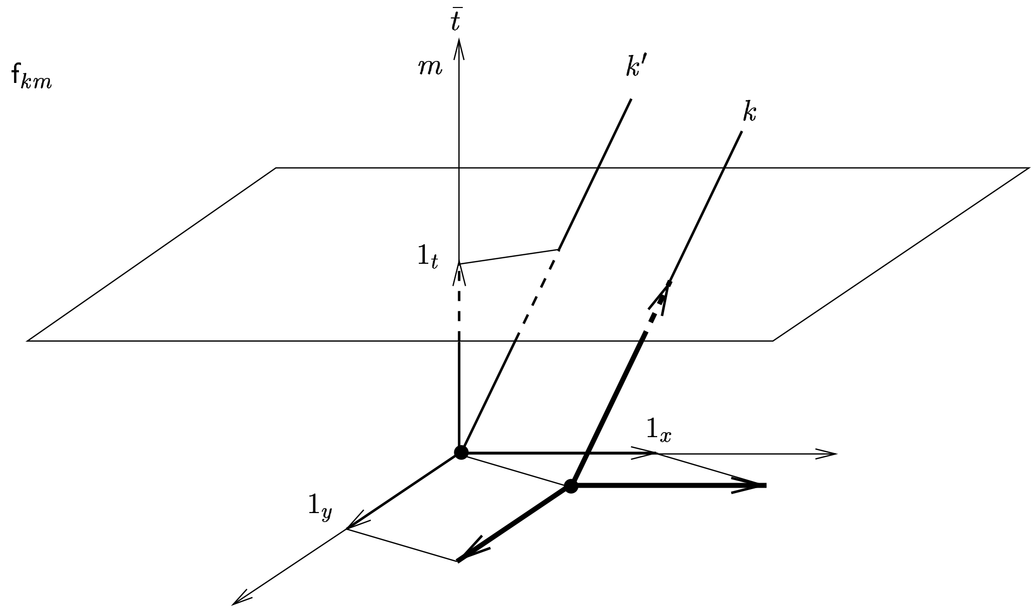



Figure 149: $\mathsf{f}_{km}$ takes the unit vectors to the bold-face ones.

Assume now $k \in Obs \setminus Obs_0$. Let $k' \in Obs_0$ be such that $k \parallel k'$, and let $p \in S(= \{0\} \times {}^{n-1}F)$ be such that $p \in k$. We define $\mathsf{f}_{kk'}$ to be the translation of ${}^nF$ which takes $\bar{0}$ to $p$, and we define $\mathsf{f}_{km} = \mathsf{f}_{kk'} \circ \mathsf{f}_{k'm}$, see Figure 150.

By these, our model $\mathfrak{M} \stackrel{\text{def}}{=} \langle (B, Obs, Ph, Ib), \mathfrak{F}, \mathsf{Eucl}(n, \mathbf{F}), \epsilon, w_k \rangle_{k \in Obs}$ has been defined. It is not difficult to check that $\mathfrak{M} \models \mathbf{Bax}^- + c_m(p,d) < \infty$. (We use the properties of $K$ in proving that $\mathfrak{M} \models \{\mathbf{AxP1}, \mathbf{Ax5_{Obs}}, \mathbf{Ax5_{Ph}}, c_m(p,d) < \infty\}$.)

Also it is not difficult to check that the following three statements are valid in $\mathfrak{M}$ for all $m, k \in Obs_0$ and $k' \in Obs, k' \parallel k$:



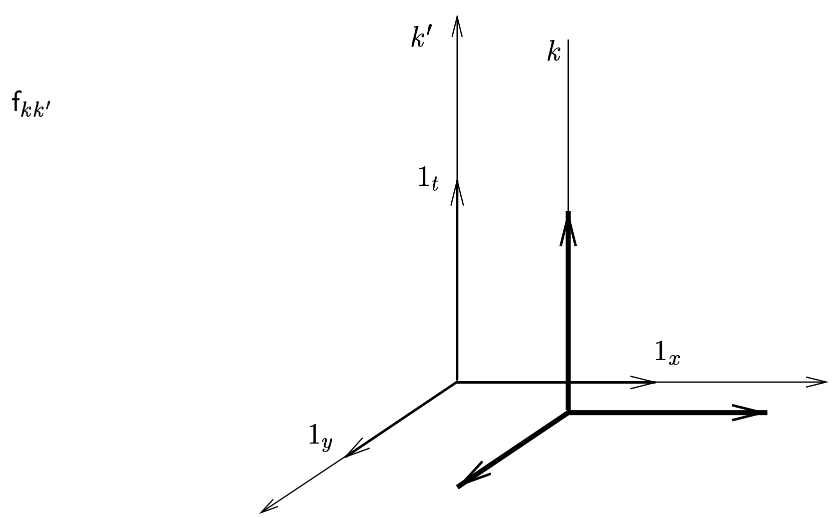




Figure 150: Galilean transformations.

(i) $\mathsf{f}_{km}(1_t)_t - \mathsf{f}_{km}(\bar{0})_t = 1$,

(ii) events $e, e'$ are simultaneous for $m$ implies that $e, e'$ are simultaneous for $k$, too,

(iii) $m$ thinks that the distance between $k, k'$ is $\|p\|$ implies that $k$ thinks that the distance between $k$ and $k'$ is $\|p\|$. See Figure 151.

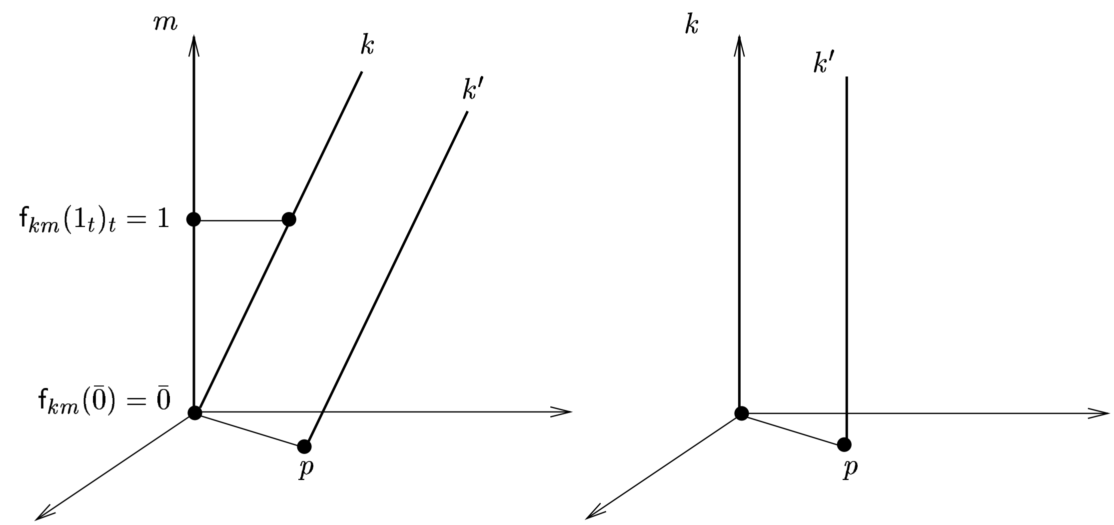



Figure 151: $m$ and $k$ see spatial distances to be the same.

But then, using the definition of $tr_k(k')$ for $k \parallel k'$, we obtain that (i)-(iii) are true for any $m, k, k' \in Obs, k \parallel k'$. Now (i)-(iii) are just the negations of the three basic paradigmatic effects in §2.5. ∎

The next theorem states that the twin paradox fails in a strong form (i.e. even **Ax(∃TwP)** fails) under assuming **Ax(symm)**. However, we will see that if we replace **Ax(symm)** with other symmetry axioms, then **Ax(TwP)** becomes provable. Cf. Theorems 4.3.23, 4.7.15 on pages 497, 622.

**THEOREM 4.3.22** *Let* $\mathbf{Ax} \stackrel{\text{def}}{=} \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax6}$. *Then*

$$\mathbf{Bax}^- \ + \ c_m(p,d) < \infty \ + \ \mathbf{Ax(symm)} \ + \ \mathbf{Ax} \ \not\models \ \mathbf{Ax(\exists TwP)}\,.$$

**Proof.** Let us take a model $\mathfrak{M}$ of $\mathbf{Bax}^- + c_m(p,d) < \infty + \mathbf{Ax}(\sqrt{\ })$ as constructed in the previous proof of Thm.4.3.21. Add observers to $\mathfrak{M}$ such that in the so enlarged model $\mathfrak{M}'$ , $\mathbf{Ax}(\mathit{Triv})$ will be true, and still all the world-view transformations are Galilean. (This amounts to including all "brothers and sisters" of observers as in the proof of Thm.2.8.2 on p.127 (§2.8).) Since this model $\mathfrak{M}'$ is still basically a model of Newtonian Kinematics (i.e. since the world-view transformations are Galilean), $\mathbf{Ax(symm)}$, $\mathbf{Ax}$ and the negation of $\mathbf{Ax(\exists TwP)}$ hold in $\mathfrak{M}'$. ∎

There *are* some paradigmatic effects which remain true in our weak system.

**THEOREM 4.3.23** *If we replace* **Ax(symm)** *with* **Ax□1** *in Theorem 4.3.22 above, then* **Ax(TwP)** *becomes provable. I.e.*

$$\mathbf{Bax}^- \ + \ c_m(p,d) < \infty \ + \ \mathbf{Ax\Box 1} \ + \ \mathbf{Ax} \ \models \ \mathbf{Ax(TwP)}\,.$$

**Proof.** $\mathbf{Bax}^- + \mathbf{Ax\Box 1} + \mathbf{Ax}$ implies **Flxspecrel** by Theorem 4.3.18. It is not difficult to check that **Ax(TwP)** holds in $\mathbf{Flxspecrel} + c_m(p,d) < \infty$. To check this, one can use Remark 4.1.7 where we argued that $\mathbf{Flxbasax} + c_m(p,d) < \infty$ is very close to **Newbasax**. Thus $\mathbf{Flxspecrel} + c_m(p,d) < \infty$ is very close to $\mathbf{Basax} + \mathbf{Ax(symm)}$, and in the latter **Ax(TwP)** holds. ∎

We will see later that Theorem 4.3.23 above remains true if we replace **Ax□1** in it with the symmetry axiom $\mathbf{R}(\mathbf{Ax\ syt_0})$. Cf. Theorem 4.7.15 on p.622.

**THEOREM 4.3.24** *Assume* $\mathbf{Ax}(\sqrt{\ })$ *and* $n > 2$. *Then*

**(i)** $\mathbf{Bax}^- \ + \ c_m(p,d) < \infty \ \models$ "$\nexists$ FTL observers",
*and therefore*

**(ii)** $\mathbf{Bax}^- \ + \ c_m(p,d) < \infty \ \models$ "there is a speed limit for moving observers, in some sense".

**(iii)** $\mathbf{Bax}^- \ + \ c_m(p,d) < \infty \ \models$ "velocities of observers do not add up the usual Newtonian way".

**Proof.** The main idea of the proof of **Basax** $\models$ "$\nexists$ FTL observers" (i.e. Thm.3.4.1 on p.203) can be pushed through to prove (i). The rest, (ii), (iii) follow from (i).

We give the proof for (i) in more detail. We will prove that if $m$ sees an FTL observer, then $m$ sees also a photon with infinite speed. Assume that $k$ is an FTL observer in $m$'s world-view, i.e. $k$'s speed is bigger than the speed of the photon going in the same direction as $k$. See Figure 152.

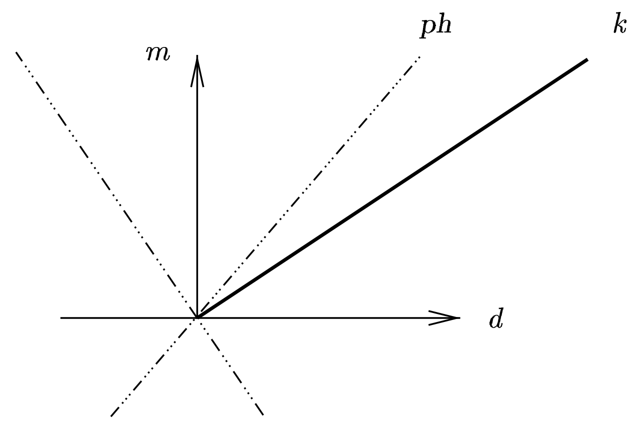



Figure 152: $k$ is a faster than light observer as seen by $m$.

Let $P$ denote the (2-dimensional) plane containing $m$ and $k$, i.e. $P \stackrel{\text{def}}{=} \mathsf{Plane}(\bar{t}, tr_m(k))$. Look at $P$ in $m$'s world-view! See Figure 153.

Then in $P$ there are two photon(trace)s, say $ph_1, ph_2 \in Ph$ (by $\mathbf{Ax5_{Ph}}$). Let $\ell$ be the intersection of $P$ with the space-part (i.e. with $S = \{0\} \times {}^{n-1}F$). Then $\ell \in \mathsf{Eucl}$. In $m$'s world-view, there is no photon-line between $k$ and $\ell$, so in $k$'s world-view there is no photon-line between $\bar{t}$ and $\ell' \stackrel{\text{def}}{=} \mathsf{f}_{mk}[\ell]$, because $\mathsf{f}_{mk}$ is betweenness-preserving (here is where we use $\mathbf{Ax}(\sqrt{\ }))$. But then $\ell'$ is an observer-trace by $\mathbf{Ax5_{Obs}}$, i.e. $\ell' = tr_k(k')$ for some $k' \in Obs$. Now let us move into the world-view of $k'$. See Figure 154.

Let $P'$ be the $\mathsf{f}_{mk'}$-image of a plane $P_1$ in $S = \{0\} \times {}^{n-1}F$ containing $\ell$. Then $P'$ is a plane going through $\bar{t}$ in the world-view of $k'$, because $\ell$ is in $P_1$. Thus there is a photon(line) in $P'$ by $\mathbf{Ax5_{Ph}}$, say $ph_3$. But then $m$ will see $ph_3$ in $S$, i.e. $m$ will see $ph_3$ as having infinite speed. See Figure 154. ∎

We also tend to conjecture that for some of our "group axioms" in §3.10.1 we will have $\mathbf{Bax}^-$ + $c_m(p,d) < \infty$ + "group axioms" $\models$ "the paradigmatic effects

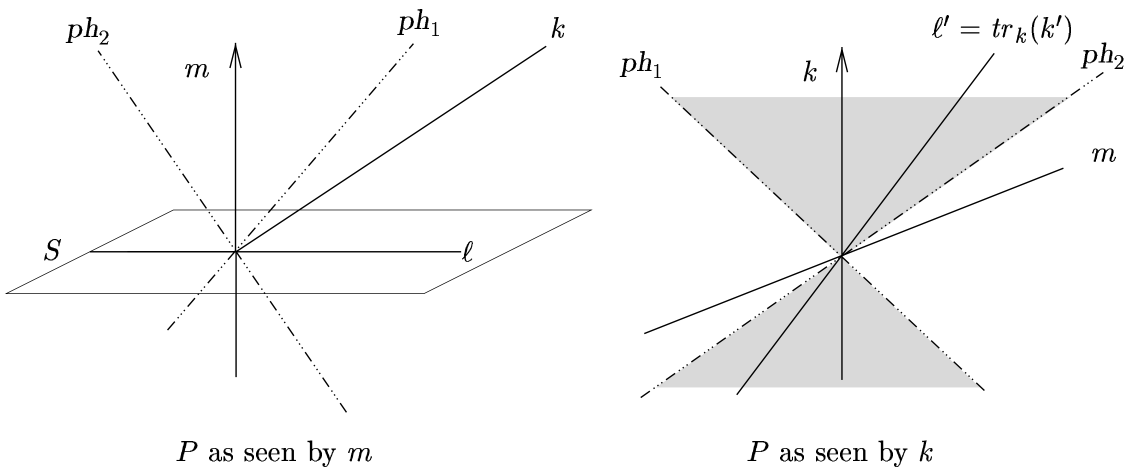



Figure 153: $k$ sees an observer on $\ell$.

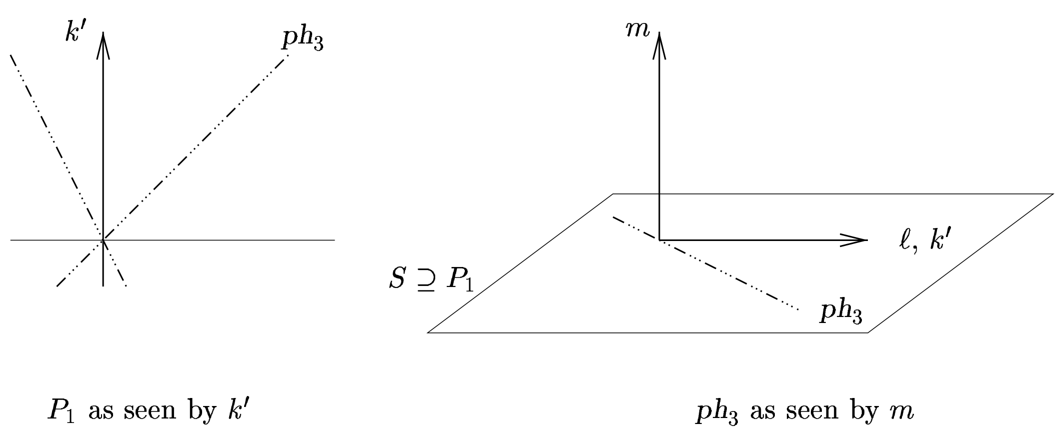



Figure 154: $m$ sees a photon $ph_3$ with infinite speed.

in §2.5". At this point we do not discuss which of the "group axioms" are needed for this conjecture.

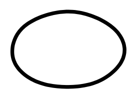
Next we investigate the "limits" of the "no FTL theorem" (cf. Thm 4.3.24). Our next theorem states that if we allow photons with infinite speed at least in one direction, then we can have observers $k$ moving faster than light (the speed of light in direction of $k$'s movement will be finite, of course). We note that, independently of us, Attila Andai obtained similar results, see Andai [7].

**THEOREM 4.3.25**

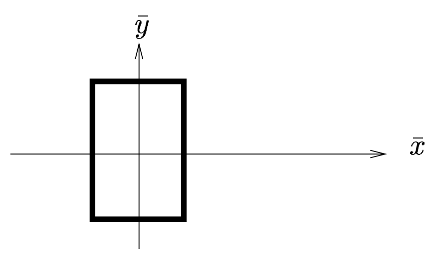
$$\mathbf{Bax}^- \not\models \text{“}\not\exists\ \textbf{FTL observers”}, \qquad \textit{for all } n.$$

*I.e. existence of FTL observers is consistent with* $\mathbf{Bax}^-$.

*Intuitive idea of the proof:* Let $n = 3$. The intersection of "the light-cone" with a simultaneity is usually like this:

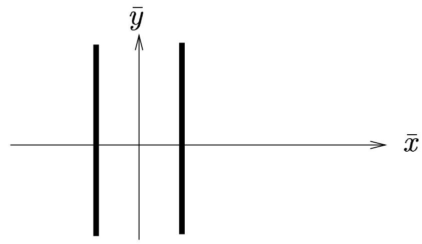


Cf. Figure 178 at the end of §4.4 on p.548. According to Figure 178 (or with the proof of Thm.4.3.21) the following light-cone section[406] is consistent with $\mathbf{Bax}^-$.

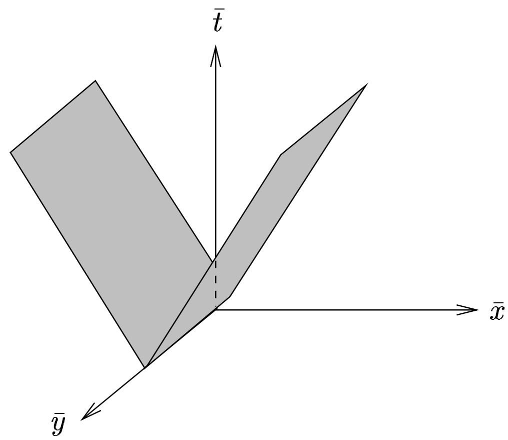



Now, let us make the $y$-sides of this light-cone section grow to infinity. We obtain the light-cone section:

[406]By a *cone-section* we understand the intersection of a cone and a plane, which is usually called a "conic-section". Similarly for a *light-cone section*.




This seems to be consistent with $\mathbf{Bax}^-$, the only peculiarity being that the speed of light in the $1_y$ and $-1_y$ directions is $\infty$ while in all other directions it is finite. Therefore the light-cone (starting from the origin), in this model of $\mathbf{Bax}^-$, will look like this:




I.e. the light-cone (starting from $\bar{0}$) consists of two intersecting *planes*.

But then the *inside* of the cone and the *outside* of the cone are not so extremely different as they were in the case of **Bax**. Indeed, if these two planes contain the two lines in $\mathsf{Plane}(\bar{t}, \bar{x})$ with angle squared 1 ($ang^2 = 1$), then interchanging the axes $\bar{t}$ and $\bar{x}$ will leave the light-cone fixed and will interchange the inside with the outside of this cone. We now turn to giving the proof.

**Proof of Thm.4.3.25.** Based on the above ideas, we give a concrete model of $\mathbf{Bax}^-$ in which there are FTL observers. Let $\mathfrak{F}$ be any ordered field, and let $n \geq 2$.

$$B \stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} \mathsf{Eucl}(n, \mathbf{F}),$$

$$Obs_0 \stackrel{\text{def}}{=} \{\ell \in B : \bar{0} \in \ell \text{ and } (\exists p \in \ell)|p_t| \neq |p_x|\},$$

$$Ph_0 \stackrel{\text{def}}{=} \{\ell \in B : \bar{0} \in \ell \text{ and } (\exists p \in \ell \setminus \{\bar{0}\})|p_t| = |p_x|\},$$

$$Obs \stackrel{\text{def}}{=} \{\ell \in B : (\exists \ell' \in Obs_0)\ell' \parallel \ell\},$$

$$Ph \stackrel{\text{def}}{=} \{\ell \in B : (\exists \ell' \in Ph_0)\ell' \parallel \ell\}.$$

Then, of the lines going through $\bar{0}$, the ones that lie on the two planes in Figure 155 are photon-traces, all the others are observer traces.

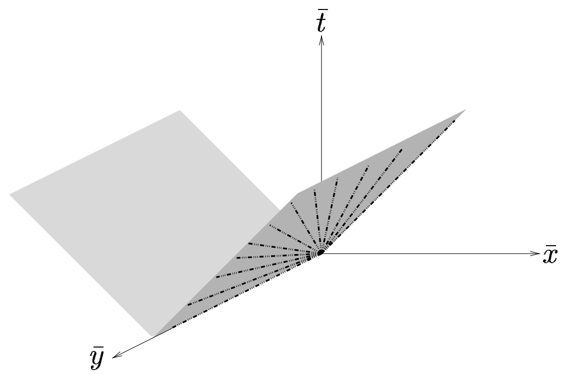



Figure 155: The photon-lines going through $\bar{0}$.

Let $m \stackrel{\text{def}}{=} \bar{t}$. We define the world-view of $m$ so that $\ell = tr_m(\ell)$ for all $\ell \in B$. Now, in $m$'s world-view, a line $\ell \in Obs_0$ is a faster-than-light observer iff $(\exists p \in \ell)|p_t| < |p_x|$. These are exactly the lines (of the ones going through $\bar{0}$) that lie outside the two "photon-planes". See Figure 156.

We will define $\mathsf{f}_{km}$ for $k \in Obs$ to be (bijective) affine transformations. Thus it will be enough to define $\mathsf{f}_{km}$ on $\bar{0}$ and on the unit-vectors $1_i$, $i < n$.

Assume that $k \in Obs_0$ is not an FTL-observer, i.e. that $k$ lies inside[407] the two photon-planes. Then we define $\mathsf{f}_{km}$ so that

$$\mathsf{f}_{km}(1_t) \in k \setminus \{\bar{0}\},$$

$$\mathsf{f}_{km}(\bar{0}) = \bar{0},$$

$$\mathsf{f}_{km}(1_i) = 1_i \quad \text{for } 0 < i < n.$$

Assume that $k \in Obs_0$ is an FTL-observer, i.e. that $k$ lies outside the two photon-planes. Then we define $\mathsf{f}_{km}$ so that

$$\mathsf{f}_{km}(1_t) \in k \setminus \{\bar{0}\},$$

[407] Here "inside" means that "in the same part where $\bar{t}$ lies".

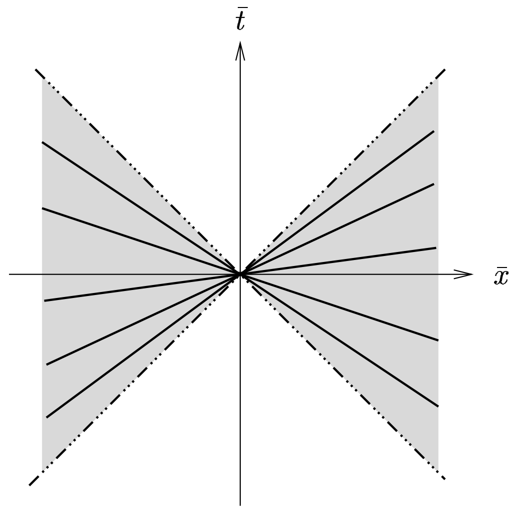



Figure 156: FTL observers in the $tx$-plane.

$$\mathsf{f}_{km}(\bar{0}) = \bar{0},$$

$$\mathsf{f}_{km}(1_x) = 1_t,$$

$$\mathsf{f}_{km}(1_i) = 1_i \quad \text{for } 1 < i < n.$$

See Figure 157.

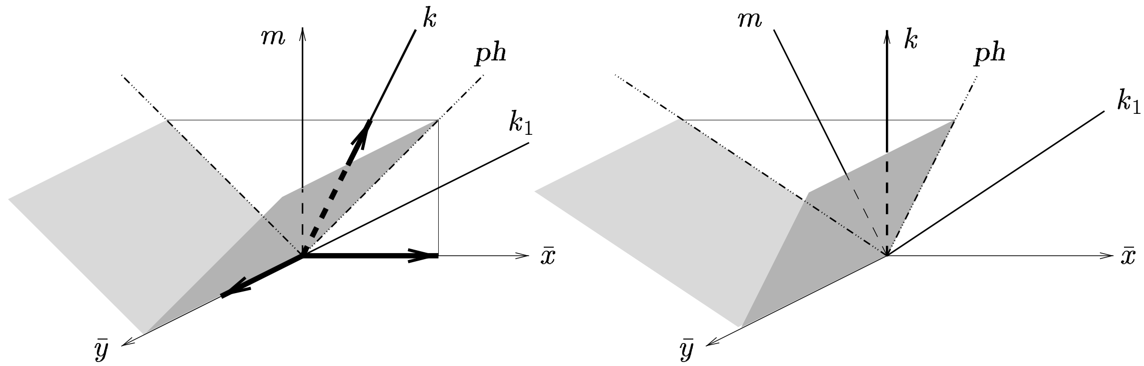
If $k \in Obs$ is parallel with $k_1 \in Obs_0$, then we define $\mathsf{f}_{kk_1}$ to be a translation as in the proof of Thm.4.3.21, and we define $\mathsf{f}_{km} = \mathsf{f}_{kk_1} \circ \mathsf{f}_{k_1m}$. It is not difficult to check that the model so obtained is a model of $\mathbf{Bax}^-$. ∎

We note that Thm.4.3.24 is in contrast with Thm.4.3.25.

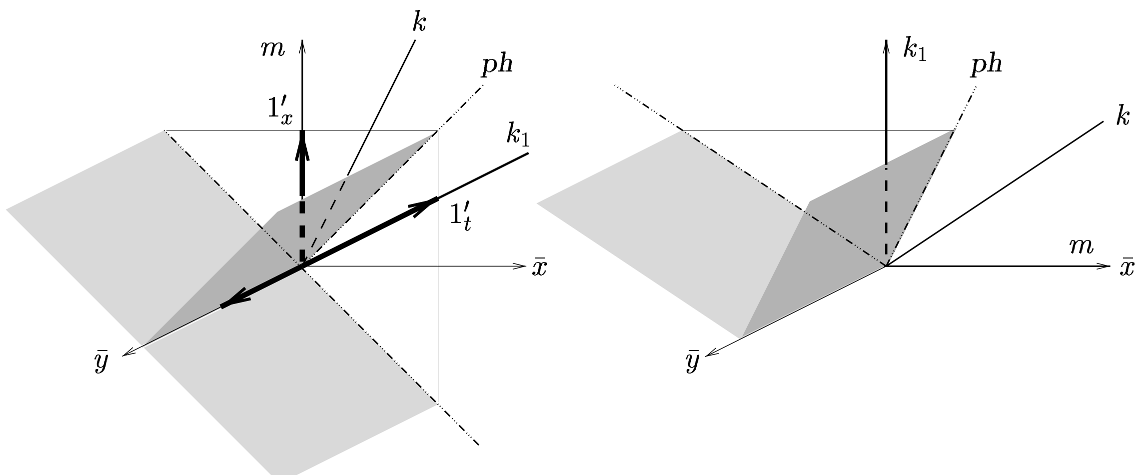
We will discuss how the paradigmatic effects behave in some important theories of $\mathbf{Bax}^-$ in §4.8. We close this part by asking some questions concerning the symmetric version of $\mathbf{Bax}^-$, in accordance with our section §4.2.

**Questions for future research 4.3.26**

**(i)** What is the theory $Th_1 \stackrel{\text{def}}{=} (\mathbf{Bax}^- + c_m(p,d) < \infty + \mathbf{Ax(symm)} + \mathbf{Ax(}Triv\mathbf{)} + \mathbf{Ax(\|)} + \mathbf{Ax}(\sqrt{\ }))$ like? Does it prove any of the potential theorems (or axioms) discussed in this work? We note that this theory still admits "Galilean models", cf. the proof of Thm.4.3.22. The proof of Theorem 4.3.22 also shows that $Th_1 \not\models \mathbf{Flxspecrel}$.




$k$'s world-view




$k_1$'s world-view

Figure 157: World-view transformations in a model with FTL-observers.

**(ii)** What is the answer if we add some of the symmetry principles in §2.8? In connection with this see Thm.4.3.18 (or Thm.5.2.16 on p.759 in §5), which states that if we replace **Ax(symm)** in $Th_1$ with **Ax□1**, then the so obtained theory already proves **Flxspecrel**.

◁

**Shapes of light-cones in models of Bax⁻.**

**Definition 4.3.27 (Photon-sphere, or light-sphere)** Assume $\mathfrak{M}$ is a frame-model and $m \in Obs, t \in F$. We define

$$\mathbb{C}_{m,t} \stackrel{\text{def}}{=} \mathsf{Cone}_{m,\bar{0}} \cap (\{t\} \times {}^{n-1}F)$$

$$= \{p \in {}^nF : p_t = t \text{ and } (\exists ph \in Ph)\{\bar{0}, p\} \subseteq tr_m(ph)\}.$$

$\mathbb{C}_m \stackrel{\text{def}}{=} \mathbb{C}_{m,1}$. See Figure 158.[408]

◁

The photon-sphere has an intuitive <u>*physical meaning*</u>: if we flash on a light at time 0 (and place (0,...,0)) for a very short time, then at a later time the photons created at time 0 will form a so-called photon-sphere. This photon-sphere at time $t$ is $\mathbb{C}_{m,t}$. Observer $m$, at time $t$, will observe the photons created at $\bar{0}$ exactly at places $space[\mathbb{C}_{m,t}]$.

**Examples.** ($n = 4$) In models of **Basax** and **Newbasax**, the photon-sphere $\mathbb{C}_m$ is a (3-dimensional) sphere with center $\bar{0}$ and radius 1. In models of **Flxbasax** and **Bax**, $\mathbb{C}_m$ is still a sphere but with an arbitrary radius, or the empty set. So far, in models of **Bax⁻** we have seen two kinds of light-spheres. In the proof of Thm.4.3.21 we gave **Bax⁻**-models where the light-sphere can be the boundary of any bounded convex set. In the model $\mathfrak{M}$ we gave in the proof of Thm.4.3.25, the light-sphere (of $\bar{t}$) is two parallel planes, see Figures 155, 157 (for $n = 3$), Figure 159 (for $n = 4$). We note that in $\mathfrak{M}$, if $m = \bar{t}$ flashes on a light at $\bar{0}$, then $m$ will see the following: at moment 0 the whole $zy$-plane will flare up, then it will separate into two "walls" of light moving left and right with speed 1.

[408] $\mathbb{C}_m$ is denoted by $C_{m,\bar{0}}$ in the next section, in the definition of **Ax(ii)** after Def.4.4.9.

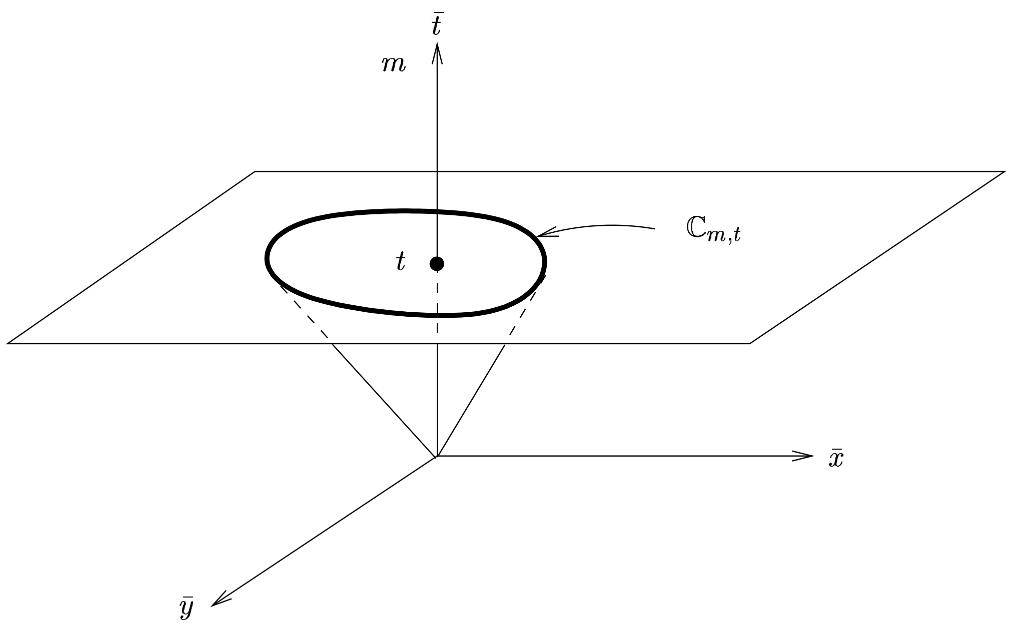



Figure 158: The "photon-sphere" at time $t$ $(n = 3)$. See also Figure 175.

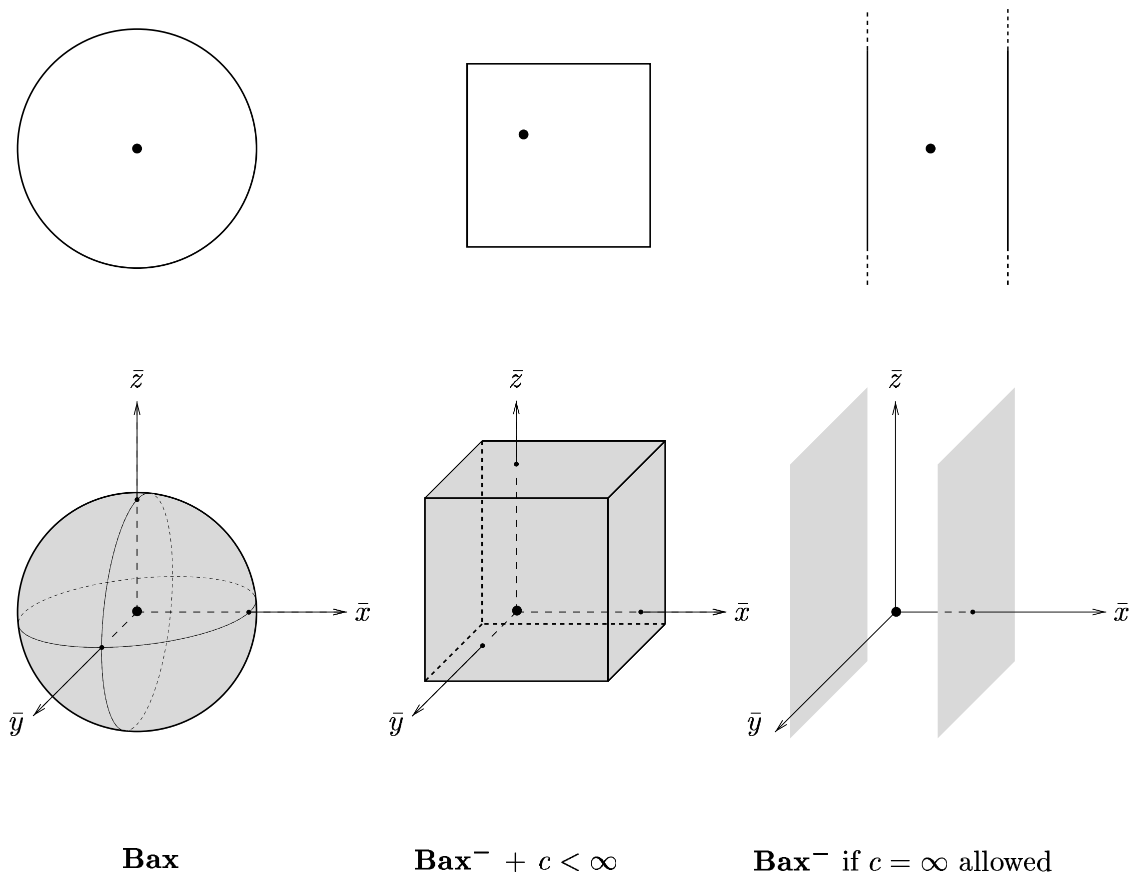



Figure 159: Shapes of light-spheres in models of **Bax$^-$**.

We note that the photon-sphere $\mathbb{C}_m$ represents the (partial) function $\mathsf{c}_m(d)$ as follows. Let $p \in \mathbb{C}_m$. Then

$$\|space(p)\| = \mathsf{c}_m(d),$$

where $d$ is the direction of the line $\overline{1_t p}$. See Figure 160.

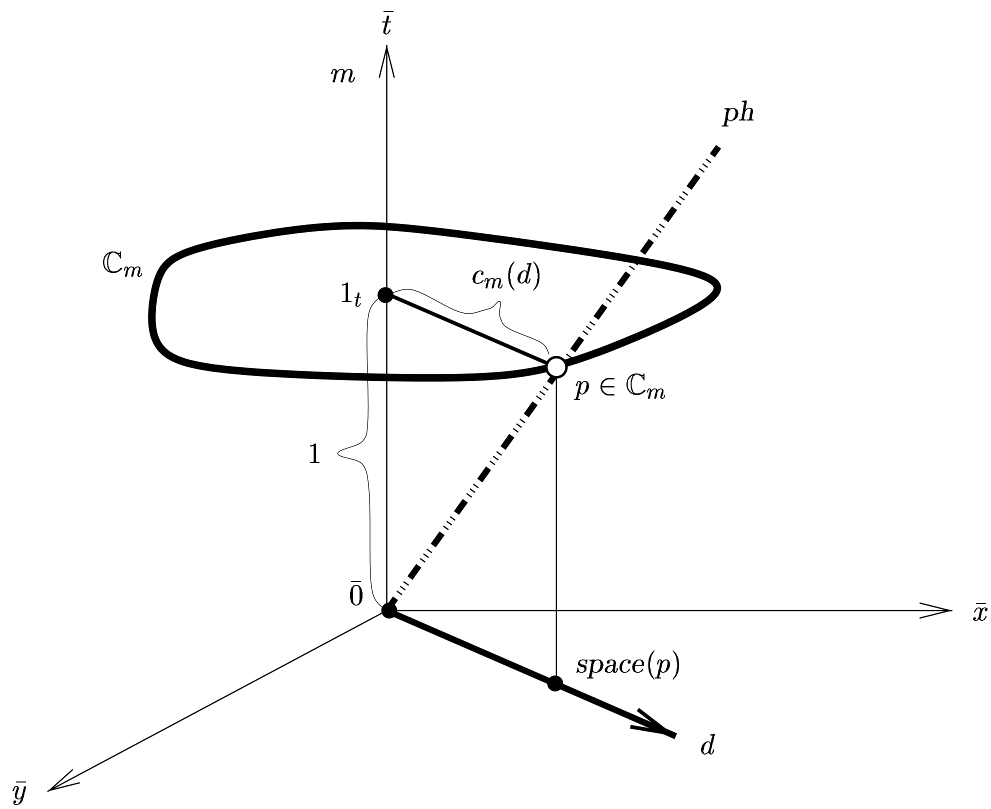
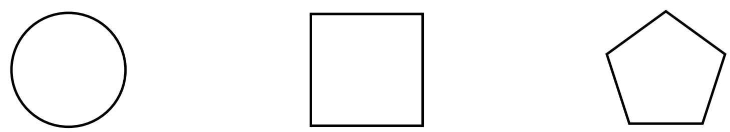



Figure 160: $\mathbb{C}_m$ represents the function $\mathsf{c}_m(d)$.

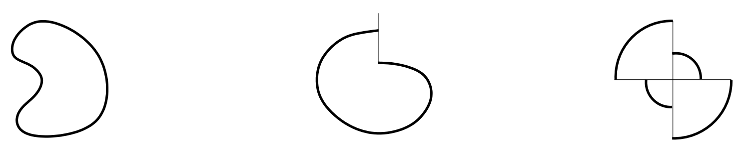
We note that $\mathbb{C}_m$ also determines the light-cone:

$$\mathsf{Cone}_{m,\bar{0}} = \bigcup\{\overline{\bar{0}p} : p \in \mathbb{C}_m\}.$$

We are going to prove that, in models of $(\mathbf{Bax}^- + c_m(p,d) < \infty)$, the light-cone is smooth to the extent that the photon-sphere is not broken, i.e. if the speed of light changes with direction, then it changes gradually, with no sudden change. But the photon-sphere still can have sharp "edges", i.e. there can be sudden changes in the above rate of change.[409] Combining Thm.4.3.29 below and the proof of Thm.4.3.21, we have that (for $n = 3$) the following photon-spheres do occur in some model of $\mathbf{Bax}^- + c_m(p,d) < \infty$:



However, the following cannot occur as photon spheres (in $\mathbf{Bax}^-$):



Next we recall some definitions from geometry.

**Definition 4.3.28 (convex, boundary, interior)** Let $p, q \in {}^nF$ and $K \subseteq {}^nF$.

**(i)** $int(p,q)$ denotes the *open interval* determined by $p$ and $q$, i.e.

$$int(p,q) \stackrel{\text{def}}{=} \{r \in \overline{pq} : \mathsf{Betw}(p,r,q)\}.$$

See Figure 161.

**(ii)** We say that $K$ is *convex* if $(\forall p, q \in K) int(p,q) \subseteq K$.

**(iii)** The *convex hull* of $K$ is the smallest convex set containing $K$, i.e. it is

$$\bigcup \{int(p,q) : p, q \in K\} \cup K.$$

**(iv)** We say that $K$ is *bounded* if there is $\lambda \in {}^+F$ such that any coordinate of any point of $K$ is between $-\lambda$ and $\lambda$.

[409] Axiom **Ax(consm)** at the end of this section will exclude such sudden changes in the rate of change.

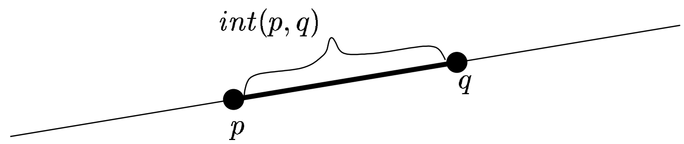



Figure 161: $int(p, q)$ denotes the open interval determined by $p$ and $q$.

**(v)** We say that $p$ is a *boundary point* of $K$ if every neighbourhood of $p$ intersects both $K$ and its complement; i.e. if $N \cap K \neq \emptyset$ and $N \setminus K \neq \emptyset$ for every neighbourhood $N$ of $p$. The *boundary* of $K$ is the set of all boundary points of $K$. The *interior* of $K$ is the set of non-boundary elements of $K$.

$\triangleleft$

Let $p \in {}^nF$ be any point distinct from $\bar{0}$. We say that $p$ is a *light-point* if $\overline{\bar{0}p}$ is the trace of a photon, and we say that $p$ is an *observer-point* if $\overline{\bar{0}p}$ is the trace of an observer (as seen by $m$). Now, $\mathsf{Cone}_{m,\bar{0}} \setminus \{\bar{0}\}$ is the set of light-points. In some sense, the light-points and the observer-points of a simultaneity determine the world-view of $m$. See Figure 149. In models of $\mathbf{Bax}^- + c_m(p,d) < \infty$, the photon-sphere itself determines the world-view, see the no FTL-theorem, Thm.4.3.24.

**THEOREM 4.3.29 (light-cones are continuous and convex)** *Assume* $\mathfrak{M} \models \mathbf{Bax}^- + c_m(p,d) < \infty + \mathbf{Ax}(\sqrt{\ })$, *and let* $m \in Obs$. *Then (i)-(iii) below hold.*

**(i)** *The boundary of* $\mathbb{C}_m$*'s convex hull is* $\mathbb{C}_m$*, the interior of* $\mathbb{C}_m$*'s convex hull is the set* $K$ *of observer-points in* $S_1 = \{1\} \times {}^{n-1}F$*. Thus* $K$ *is a convex, open set with boundary* $\mathbb{C}_m$*.*

**(ii)** $\mathsf{c}_m(d)$ *is a continuous function of* $d$*, in the sense of Def.4.4.8(i),(ii). In this sense,* $\mathbb{C}_m$ *is a continuous surface.*

**(iii)** $\mathbb{C}_m$ *is bounded, if* $\mathfrak{F} = \mathfrak{R}$*, the ordered field of reals.*

**Proof.** Let $K$ denote the set of observer-points in $S_1$. We want to prove that $K$ is convex, open, the boundary of $K$ is $\mathbb{C}_m$, $\mathbb{C}_m$'s convex hull is $\mathbb{C}_m \cup K$, and $\mathsf{c}_m$ is continuous.

In the proof we will use the following simple statements. We call a point $p \neq \bar{0}$ *empty* if it is neither an observer-point nor a light-point.

**LEMMA 4.3.30 (light-points and observer-points on a line)** *Let $\ell$ be any straight line.*

**(a)** *If there is an observer-point on $\ell$, then there are at most two light-points on $\ell$.*

**(b)** *Every observer-point on $\ell$ has a "neighbourhood" of observer-points on $\ell$, i.e. if $p \in \ell$ is an observer-point, then there are $p, q \in \ell$ such that $r \in int(p,q)$ and every point of $int(p,q)$ is an observer-point.*

**(c)** *Between an observer-point and an empty point there is a light-point.*

**(d)** *Assume that $p, q$ are distinct light-points on $\ell$, and $r$ is an observer-point in $\ell \setminus int(p,q)$. Then every point of $\ell \setminus int(p,q)$ is an observer-point.*

**(e)** *Assume that there is a light-point between the two observer-points $p$ and $q$. Then every point of $\overline{pq} \setminus int(p,q)$ is an observer-point.*

**Proofsketch.** **(a)**. Assume that $k \in \ell$ is an observer-point on $\ell$, and $ph_1, ph_2, ph_3$ are distinct light-points on $\ell$. See Figure 162. Then in $k$'s world-view, the traces of $k$ and $ph_1, ph_2, ph_3$ are in one plane (by Thm.4.3.11). Hence $ph_1, ph_2, ph_3$ are all moving (backwards or forwards) in direction $d$, for some $d$. Thus at least two of them move in the same direction, with different traces. This contradicts **AxP1**.

**(b)-(e):** Let $\ell$ be a line and let $k$ be an observer-point on $\ell$. Then in $k$'s world-view, $\ell$ is a line intersecting $\bar{t}$. $\mathsf{Plane}(\bar{t}, \mathsf{f}_{km}(\ell))$ looks like on Figure 163, by $\mathfrak{M} \models \mathbf{Bax}^- + c_m(p,d) < \infty$.

One can check that (b)-(e) hold for every line $\ell$ in this plane which does not contain the point $\mathsf{f}_{km}(\bar{0})$. ∎

Now we turn to proving Thm.4.3.29.

**Proof of Thm.4.3.29.**

K is convex: Assume that $p, q \in K$ and $r \in int(p,q)$. We have to show $r \in K$. Assume the contrary, i.e. assume that $r$ is either an empty point or a light-point. If $r$ is an empty point, then there is a light-point between $r$ and $q$. So in either case, there is a light-point between $p$ and $q$. Thus by Lemma 4.3.30(e), all points of $\overline{pq} - int(p,q)$ are observer-points. Let $\ell$ be such that $1_t \in \ell$ and $\ell \parallel \overline{pq}$. Let $s$ be an empty point on $\ell$. There is such by Thm.4.3.24. There is a light-point $u$ between $p$ and $s$ by Lemma 4.3.30(c). Let $k$ be the intersection point of $\overline{pq}$ and $\overline{1_t u}$. Then $k$ is an observer-point since $k \notin int(p,q)$. But then $k$ is a faster-than-light observer because $k \notin int(1_t, u)$. This contradicts Thm.4.3.24. See Figure 164.

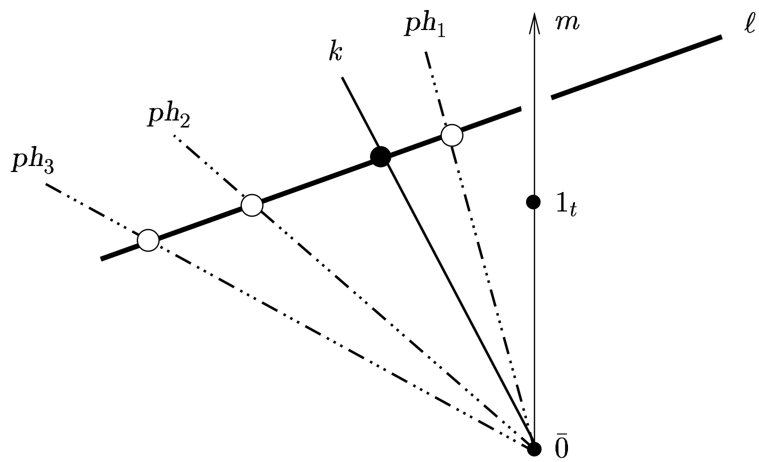



Figure 162: On a line with an observer-point there are at most two light-points.

<u>$K$ is open:</u> Recall that $S(p,\varepsilon)$ denotes the $\varepsilon$-sphere with center $p$ in ${}^nF$. Let $p \in K$, we have to show that $S(p,\varepsilon) \subseteq K$ for some $\varepsilon \in F$. We use Lemma 4.3.30(b) and convexity of $K$. Figure 165 shows the idea of the proof for $n = 3, 4$. ($\ell_1, \ell_2, \ell_3 \subseteq S_1$.) We hope that Figure 165 is sufficient for recovering the proof.

<u>$\mathbb{C}_m$ (and $\mathsf{c}_m$) is continuous:</u>[410]

Assume that $p \in \mathbb{C}_m$, and $\varepsilon \in F$. We will show that there is a $\delta$ such that for every $q \in \mathbb{C}_m$, if the angle between $\overline{1_t q}$ and $\overline{1_t p}$ is smaller than $\delta$, then $q \in S(p,\varepsilon)$. See Figure 166.
(We work in $S_1$.) Let $p' \in int(1_t, p)$ and $\varepsilon_1 \in F$ be such that $S(p',\varepsilon_1) \subseteq K \cap S(p,\varepsilon)$. Such $p', \varepsilon_1$ exist by the openness of $K$. Let $q'$ be the mirror image of $p'$ w.r.t. $p$. Then every point in $S(q',\varepsilon_1)$ is the mirror image of a point in $S(p',\varepsilon_1)$. Hence by convexity of $K$ and $p \notin K$ we have that no point in $S(q',\varepsilon_1)$ is an observer-point. Now $\delta$ as on Figure 166 will do.

<u>The boundary of $K$ is $\mathbb{C}_m$:</u> Every point of $\mathbb{C}_m$ is in the boundary, because if $p \in \mathbb{C}_m$, then every point in $int(1_t, p)$ is an observer-point while every point in $int(p, r)$ is an empty point if $p \in int(1_t, r)$.

On the other hand, assume that $p \notin \mathbb{C}_m$. If $p \notin S_1$, then $p$ has a neighbourhood

[410] The theory of convex sets is rather extensive, see e.g. Valentine [261], or any book by Victor Klee. Most likely, it is a known theorem that the boundary of a convex, open set is continuous.

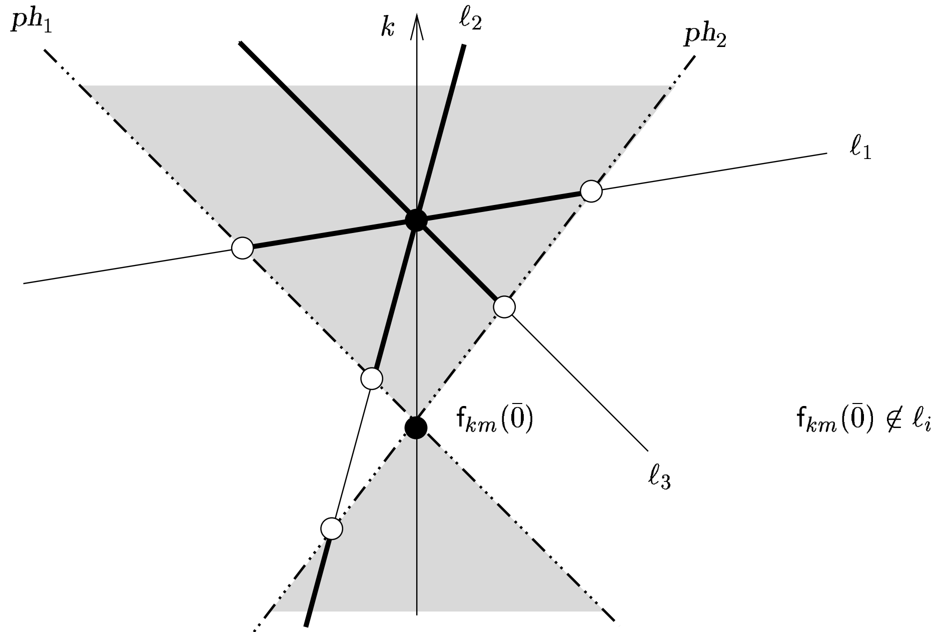



Figure 163: Types of lines with an observer-point.

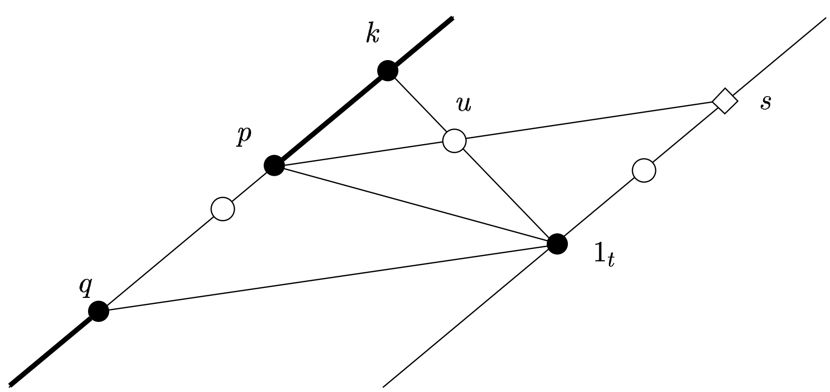



Figure 164: If every point of $\overline{pq} \setminus int(p,q)$ is an observer-point, then one of them is an FTL observer-point.

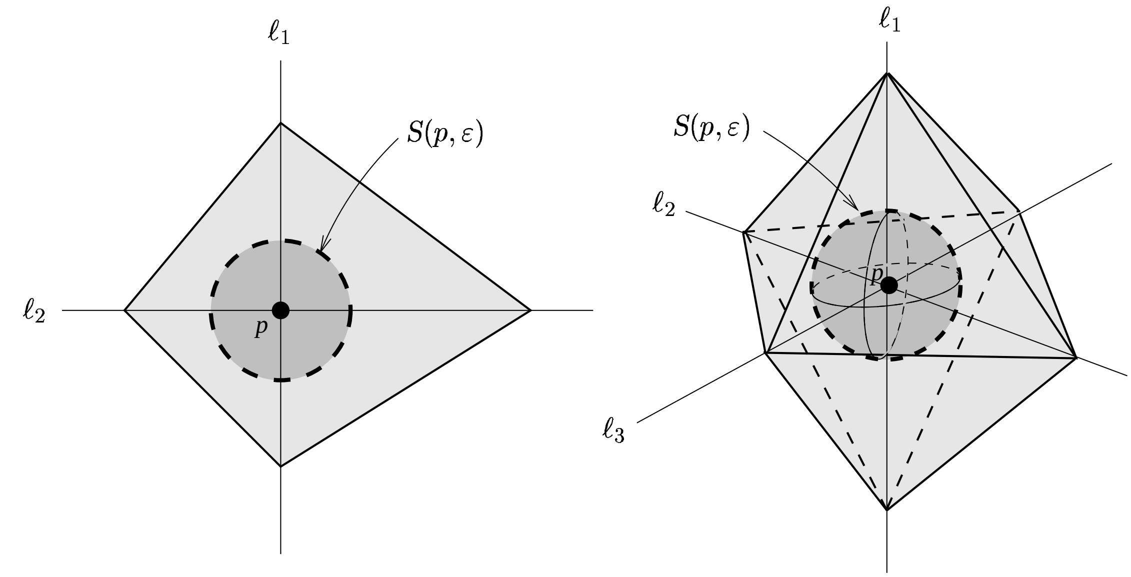



Figure 165: $K$ is open.

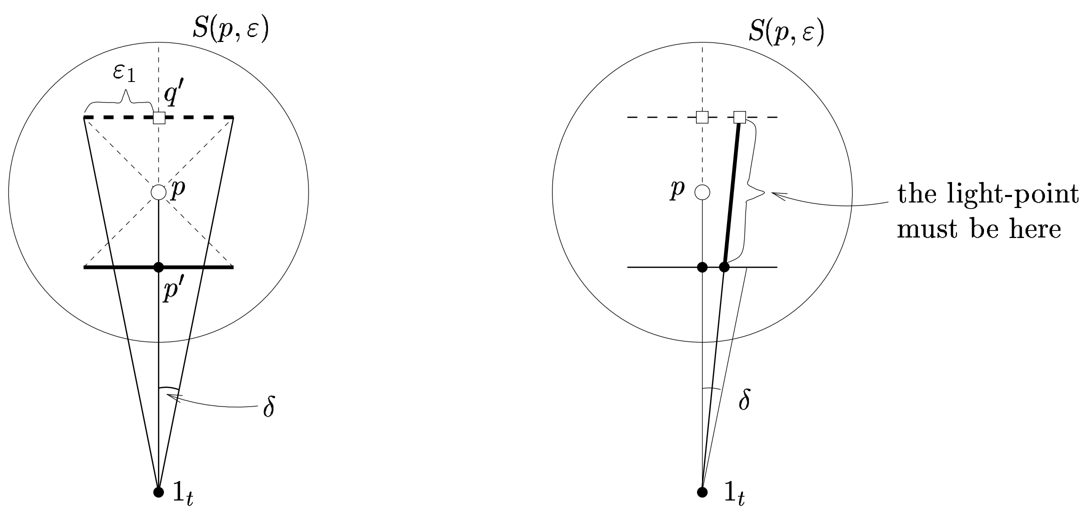



Figure 166: Illustration for proof of continuity of $\mathsf{c}_m$.

disjoint from $S_1 \supseteq K$. If $p \in K$, then $p$ has a neighbourhood which is a subset of $K$, because $K$ is open. Assume that $p$ is an empty point in $S_1$. Look at the line $\ell = \overline{1_t p}$. There are $q, r \in \ell$ such that $q \in K, r \notin K$ and $r$ is the midpoint of the segment $qp$. Let $S(q, \varepsilon) \subseteq K$, $r \notin S(q, \varepsilon)$. Then $S(p, \varepsilon)$ is disjoint from $K$, as we have seen in the proof of continuity of $\mathsf{c}_m$.

<u>The convex hull of $\mathbb{C}_m$ is $K \cup \mathbb{C}_m$:</u> This follows from the fact that $K$ is convex and $\mathbb{C}_m$ is its boundary, as follows. Let $p, q \in \mathbb{C}_m$, and $r \in int(p, q)$. Assume $r \notin K$, we will show that $r \in \mathbb{C}_m$. Since $p, q$ are in the boundary of $K$ and $K$ is convex, every neighbourhood of $r$ contains a point from $K$, see Figure 167. Since $r \notin K$, then every neighbourhood of $r$ contains a point both from $K$ and from its complement, i.e. $r$ is in the boundary of $K$. Thus $r \in \mathbb{C}_m$ as was to be shown.

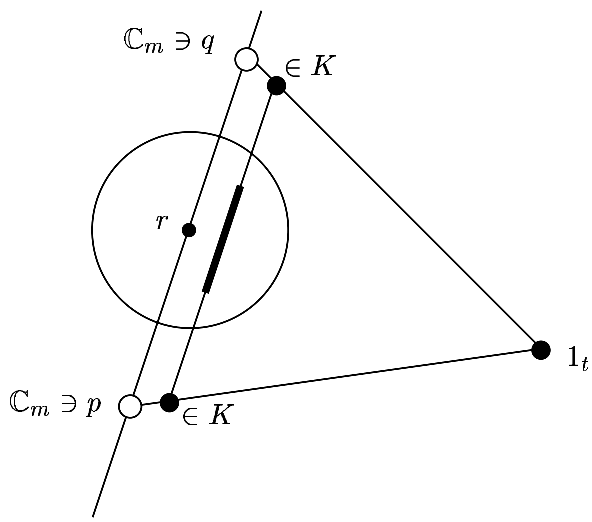



Figure 167: The convex hull of $\mathbb{C}_m$ is $\mathbb{C}_m \cup K$.

<u>$K$ is bounded if $\mathfrak{F} = \mathfrak{R}$:</u> We will use continuity of $\mathbb{C}_m$. Assume that $K$ is not bounded. Let $p \in K$. Then for each $i \in \omega$ there is $q^i \in K$ such that $|p - q^i| > i$. Then there is a direction to which the directions of $\overline{pq^i}$, $i \in \omega$ come arbitrarily close, because $\mathfrak{F} = \mathfrak{R}$.[411] Let $\overline{pq}$ have this direction. But then, by continuity of $\mathbb{C}_m$, no light-point can be on the line $\overline{pq}$. See Figure 168. ∎

[411] This can be proved by induction on $n \in \omega$, where $p, q^i \in {}^n\mathrm{R}$ for $i \in \omega$.

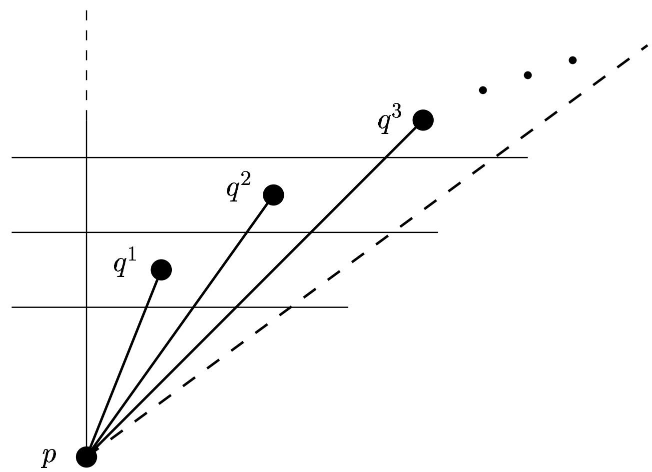



Figure 168: Illustration for the proof of boundedness of $\mathbb{C}_m$.

**Remark 4.3.31** It seems to us that the condition "$\mathfrak{F} \cong \mathfrak{R}$" is necessary in Theorem 4.3.29(iii). I.e. if $\mathfrak{F}$ is not isomorphic to $\mathfrak{R}$, then one can construct a model of $\mathbf{Bax}^- + c_m(p,d) < \infty$ with field-part $\mathfrak{F}$ where $\mathbb{C}_m$ is not bounded for some $m$. On the other hand, we conjecture that in models of $\mathbf{Bax}^-_\partial + c_m(p,d) < \infty$ to be introduced soon, the condition "$\mathfrak{F} \cong \mathfrak{R}$" is not necessary for $\mathbb{C}_m$ to be bounded. The key idea here is that an ordered field $\mathfrak{F}$ is isomorphic to the ordered field $\mathfrak{R}$ of real numbers if and only if $\mathfrak{F}$ is complete in the sense[412] that each bounded infinite subset $H \subseteq F$ has a density point (i.e. a point $p \in F$ which arbitrarily close contains an element of $H$ different from $p$). ◁

**Question for future research 4.3.32** Is it true that the boundary of $\mathbb{C}_m$'s convex hull is $\mathbb{C}_m$ in models of $\mathbf{Bax}^-$, without the assumption $c_m(p,d) < \infty$? ◁

### Improving our theory $\mathbf{Bax}^-$ to $\mathbf{Bax}^-_\partial$ (smoothing out the light-cone)

We will base some of the (relatively important) future theories like $\mathbf{Reich}(\mathbf{Bax})$ on $\mathbf{Bax}^-$, therefore, it is worthwhile to ask ourselves whether $\mathbf{Bax}^-$ is strong enough in some *natural* respects. This will yield the reinforced version $\mathbf{Bax}^-_\partial$ of $\mathbf{Bax}^-$ which

[412]It can be proved that this notion of completeness is equivalent to Dedekind completeness.

we will need e.g. in answering some of our why-type questions, cf. §4.8 (Thm.4.8.9 on p.649).

We already proved that $\mathsf{c}_m$ is continuous in models of $\mathbf{Bax}^- + c_m(p,d) < \infty$, and we mentioned that we did not know whether e.g. *strong continuity*[413] was provable. Questions like this motivate the following definitions. Also, clearly $c_m(p,d)$ does not have a derivative in some of the models of the above mentioned theory, cf. Figure 159. From the physical point of view, it is very natural to assume that $c_m(p,d)$ is strongly continuous and has a derivative. This is, among others, what we will require in $\mathbf{Bax}^-_\partial$.

First we need to spell out some (otherwise well known) definitions in first-order logic, i.e. in our frame language.

**Notation 4.3.33** $f : A \overset{\circ}{\longrightarrow} B$ abbreviates $A \supseteq Dom(f) \overset{f}{\longrightarrow} B$, i.e. it abbreviates that $f$ is a <u>*partial function*</u> from $A$ to $B$.

◁

**Definition 4.3.34 (derivative $f'$ of $f$.)**

**(i)** Let $\mathfrak{M}$ be a frame model and assume $f : F \longrightarrow F$ is a function (first-order) definable in $\mathfrak{M}$ with possibly using parameters. Then the <u>*derivative*</u> $f' : F \overset{\circ}{\longrightarrow} F$ is another definable partial function, defined (from $f$) by the *usual* first-order formula.[414] That is:

$$(\forall x, y \in F)\Big(f'(x) = y \quad \Leftrightarrow \\ (\forall \varepsilon \in {}^+F)(\exists \delta \in {}^+F)(\forall \Delta \in F)[0 < |\Delta| < \delta \quad \Rightarrow \\ \Rightarrow \quad y - \varepsilon < \tfrac{f(x+\Delta)-f(x)}{\Delta} < y + \varepsilon]\Big).$$

**(ii)** The definition is extended to the case $f : F \longrightarrow F \cup \{\infty\}$ the natural way: we let $\infty - x = x - \infty = \infty$, and $\infty / x = \infty$, further, $\infty - \infty = 0$, for $x \in F$. (But $Rng(f') \subseteq F$ by definition.)

◁

Let us recall that for $m \in Obs^{\mathfrak{M}}$ the function $\mathsf{c}_m : \mathsf{directions} \longrightarrow F \cup \{\infty\}$ was defined by $\mathsf{c}_m(d) = c_m(\bar{0}, d)$. Therefore $\mathsf{c}_m : {}^{n-1}F \overset{\circ}{\longrightarrow} F^\infty$ is a typical $n-1$-ary function (undefined on $\bar{0}$ but defined everywhere else).

This motivates our discussing $k$-ary functions $g : {}^kF \rightarrow F$, with $k \in \omega$. For simplicity, assume $k = 3$. From category theory we borrow the notation $g(-, y, z)$

[413]strong continuity is defined in Definition 4.4.8 on p.536.
[414]Of the usual notions of a derivative, ours is only but one.

for the unary function $g(-,y,z) \stackrel{\text{def}}{=} \langle g(x,y,z) : x \in F\rangle$ which is obtained from $g$ by fixing its arguments $y$ and $z$ such that the resulting function has only one argument $x$. The function $g(-,y,z)$ is specified by three data: $g, y, z$. Similarly, $g(x,-,z) \stackrel{\text{def}}{=} \langle g(x,y,z) : y \in F\rangle$ etc. Thus, for any fixed $g : {}^3F \to F$, and $y, z \in F$, we have $g(-,y,z) : F \to F$.

**Definition 4.3.35 (partial derivative $\partial_i f$ of $f$)**

**(i)** Assume $g : {}^3F \to F$ is definable (with parameters) in $\mathfrak{M}$ (analogously with the situation in Def.4.3.34). The <u>*partial derivative*</u> $\partial_1 g \stackrel{\text{def}}{=} \partial_x g$ is a partial function

$$(\partial_1 g) : {}^3F \stackrel{\circ}{\longrightarrow} F$$

defined by $(\partial_1 g)(x,y,z) \stackrel{\text{def}}{=} (g(-,y,z))'(x)$. I.e. for any fixed $y, z \in F$ we have $(\partial_1 g)(-,y,z) = (g(-,y,z))'$ which is the derivative (cf. Def.4.3.34) of the unary function $g(-,y,z) : F \to F$.

Similarly, $(\partial_2 g)(x,y,z) \stackrel{\text{def}}{=} (g(x,-,z))'(y)$ etc.

Summing up, $\partial_i g : {}^3F \stackrel{\circ}{\longrightarrow} F$ is a partial function defined by a first-order formula in the style of Definition 4.3.34.

**(ii)** The generalization to $g : {}^kF \to F$, with $k \in \omega$ is the obvious one.

**(iii)** The reason that we write $\partial_1$ for $\partial_x$ (instead of writing $\partial_0$) is that in §2, we numbered our "coordinates" $t, x, y, z$ by $0, 1, 2, 3$ (so that the space part $\langle x,y,z\rangle$ of a "space-time" vector $\langle t,x,y,z\rangle$ gets conveniently numbered by $1\ldots$ [instead of, say, something like $2\ldots$]).

$\triangleleft$

Now we are ready for defining our axiom **Ax(consm)** stating that the light-cone $\mathsf{Cone}_{m,\bar{0}}$ is of a reasonably smooth shape (i.e. is not "broken", contains no discontinuities etc).

Recall that $\mathsf{c}_m : \mathsf{directions} \longrightarrow F \cup \{\infty\}$ is a function with domain $\mathsf{directions} = {}^{n-1}F \setminus \{\bar{0}\}$. Therefore, when we say things like $\mathsf{c}_m$ is everywhere "nice" (where nice may be "continuous", or "has a derivative" etc), then we do not claim that it is nice at value $\bar{0}$ (since $\bar{0} \notin \mathsf{directions}$).

$$\mathbf{Ax(consm)} \stackrel{\text{def}}{=} \mathbf{Ax(cnsm_0)} + \mathbf{Ax(cnsm_1)} + \mathbf{Ax(cnsm_2)},$$

where the latter are defined below.[415]

Let us notice that "$\mathsf{c}_m$ is nice" as an axiom means "$(\forall m \in Obs)\mathsf{c}_m$ is nice". The notion of strong continuity is defined in Def.4.4.8 on p.536 (in §4.4).

**Ax(cnsm$_0$)** $\mathsf{c}_m$ is a strongly continuous function defined on directions.

**Ax(cnsm$_1$)** For all $0 < i < n$, the partial derivative $(\partial_i \mathsf{c}_m)$ : directions $\to F$ is everywhere defined on the domain directions[416] (of $\mathsf{c}_m$).

**Ax(cnsm$_2$)** For all $0 < i < n$, $\partial_i \mathsf{c}_m$ is strongly continuous on the domain directions.

Intuitively, our new axiom **Ax(consm)** says that $\mathsf{c}_m$ is a strongly continuous function having a derivative which is also strongly continuous (on the domain directions). This means that the "light-sphere" $\mathbb{C}_m$ cannot be like any of these:

But it may be anything like this:

[415]In naming this axiom, "consm" abbreviates the expression "cone-smooth". The full name of this axiom is *axiom of cone-smoothness* .

[416]Since directions $\subseteq {}^{n-1}F$ and $\partial_i \mathsf{c}_m$ was defined to be a partial function on ${}^{n-1}F$, the only special thing about $\bar{0}$ here is that $\partial_1 \mathsf{c}_m(0,0,0)$ is not defined. But this whole thing about $\bar{0}$ is only an "administrative" issue here, since the important arguments $d$ for $\mathsf{c}_m$ are vectors of length 1.

**Definition 4.3.36** (**complete ordered field**) Let $\mathfrak{F}$ be an ordered field. $\mathfrak{F}$ is said to be <u>*complete*</u> iff for any ordinal $\alpha$, all Cauchy sequences of elements of $\mathfrak{F}$ indexed by $\alpha$ have limits in $\mathfrak{F}$. Formally:

$$(\forall \alpha \in Ord)(\forall s \in {}^{\alpha}F)[s \text{ is Cauchy } \Rightarrow \lim_{i\in\alpha} s_i \text{ exists in } \mathfrak{F}].$$

$\lhd$

We note that the above notion of completeness is strictly weaker than Dedekind completeness, namely there is a proper class of non-elementarily-equivalent complete ordered fields (this is not true for Dedekind complete ordered fields).

If our field $\mathfrak{F}^{\mathfrak{M}}$ is incomplete (which is allowed even in **Basax**), then the light-sphere in Figure 169 seems to be consistent with **Ax(consm)**.

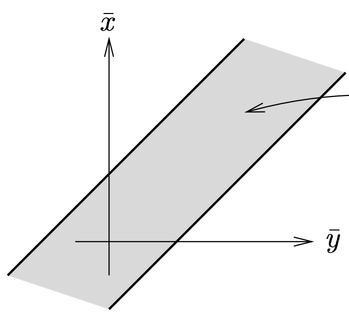



Figure 169:

In the figure, the direction $d$ parallel with the sides of our light-sphere does not exist in our ${}^{n-1}F$. This can be imagined by embedding $\mathfrak{F}$ into a bigger $\mathfrak{F}^+$ and choosing $d \in {}^{n-1}F^+$ such that $|d| = 1$ and $d \notin {}^{n-1}F$.

Let us also notice that in **Ax(consm)** we do not require the existence of a second derivative of $\mathsf{c}_m$ representing the light-sphere. Roughly, our axiom requires that at every point on the light-sphere, there is a <u>*tangent-line*</u> touching the light-sphere and that the "slope" of this tangent-line changes gradually as we move along the "sphere".

As we mentioned after Definition 4.3.27, the photon-sphere does have an intuitive physical meaning: If we flash on a light at time 0 for a very short time, then at a later time the photons created at time 0 will form a so called photon-sphere. Our axiom says that this photon-sphere is not “broken” and has no sharp “edges”. In other words, if the speed of light changes with direction, then it changes gradually, with no sudden changes, and moreover with no sudden changes in the rate of change, either. As we said, we claim that this axiom is a very mild and reasonable assumption from the physical point of view.

**Definition 4.3.37 (The smooth versions of our theories)**

**(i)** $\mathbf{Bax}^-_\partial \stackrel{\text{def}}{=} \mathbf{Bax}^- + \mathbf{Ax(consm)}$.

**(ii)** Let *Th* be a theory in our frame language. Then

$$Th_\partial \stackrel{\text{def}}{=} Th + \mathbf{Ax(consm)}$$

is called the <u>*cone-smooth version*</u> of the theory *Th*.

◁

**THEOREM 4.3.38** *Let $n > 2$. Then*

**(i)** $\mathbf{Bax}^- \not\models \mathbf{Bax}^-_\partial$.

**(ii)** $\mathbf{Bax}^- + c_m(p,d) < \infty \not\models \mathbf{Bax}^-_\partial$.

**Idea of proof.** We saw in the proof of Thm.4.3.21 that the photon-sphere $\mathbb{C}_m$ can be a rectangle in models of $\mathbf{Bax}^- + c_m(p,d) < \infty$. ∎

**Questions for future research 4.3.39**

**(i)** How much of $\mathbf{Ax(cnsm}_i\mathbf{)}$, $i < 3$ follow from $\mathbf{Bax}^- + c_m(p,d) < \infty$? We know that continuity of $\mathsf{c}_m$ follows, but we do not know what the situation with strong continuity is.

**(ii)** The theory **Reich**(**Bax**) will be introduced in section 4.5. What is the answer to **(i)** with **Reich**(**Bax**) in place of $\mathbf{Bax}^-$?

◁

We will see in §4.8 that axiom **Ax(consm)** can help in proving relativistic effects for a theory. In particular, about the Reichenbachian theory $\mathbf{Reich}_0(\mathbf{Bax})_\partial$ we will be able to prove more relativistic effects than about $\mathbf{Reich}_0(\mathbf{Bax})$.[417] See Thm.4.8.9 on p.649. Discussing the "why-type" questions in the framework of the lattice of our distinguished theories (cf. e.g. Figure 180), we will find that $\mathbf{Reich}_0(\mathbf{Bax})_\partial$ is the first (or weakest) theory where real relativistic effects (in the sense of §2.5) begin to appear. E.g. in $\mathbf{Bax}^-_\partial + c_m(p,d) < \infty$ we have no paradigmatic effects (except for FTL), because in the proof of Thm.4.3.21 we said that we can take any convex set, and therefore here it is enough to observe that then we can take a very smooth one, too.[418]

*      *      *

### Conceptual analysis of relativity, connections with the literature

**Remark 4.3.40** **(Connections with Friedman [90])**
Our introduction (and study) of **Bax** and $\mathbf{Bax}^-$ can be viewed as a continuation of the conceptual analysis of relativity started in Friedman [90] p.159 §IV.6. Namely Friedman [90, p.159] introduces informal axioms (P1), (P2) and (P3) for the purposes of conceptual analysis of the usual axioms about the speed of light, showing up in various versions of relativity theory. Let us recall Friedman's principles (P1), (P2) and (P3) concerning the speed of light.

(P1) The constancy of the velocity[419] of light: Light is propagated with a constant velocity $c$ independent of the velocity of its source.

(P2) The invariance of the velocity of light: Light has the same constant velocity $c$ in all inertial reference frames.

(P3) The limiting character of the velocity of light: No "causal" signal can propagate with velocity greater than that of light.

[417] However, this may be our fault only, i.e. we do not have a counterexample.

[418] A more detailed study of the paradigmatic effects in terms of our hierarchy of theories comes in §4.8.

[419] Friedman uses the word "velocity". In certain contexts we will use the word "speed" instead (cf. Gardner [98, p.7]) and cf. p.48 herein.

(P1) says that photons move rather like sound moves and not like bullets (emitted from guns) move: the velocity of the gun from which a bullet is emitted adds to the velocity of bullet, while the velocity of sound depends only on the medium in which it is propagated. When saying that the velocity of light is independent of the velocity of its source, we mean that photons emitted at a point $p$ of space-time, in direction $d$ by various sources of light, like e.g. by a moving light-bulb and another non-moving light-bulb, have the same speed. By saying that this speed is "constant", we mean that all light bulbs at a particular point of space-time in a particular direction can emit photons with the same single speed only.[420] Since we do not talk in our language about "sources" of light, or "emission" of light, it seems that it is a good formalization of (P1) if we say that at any point $p$ of space-time in any direction $d$, there is at most one photon trace. This is what **AxP1** says. So, the velocity of light-particles depends only on two data: (i) the point $p$ of space-time where the light-particle is emitted, and (ii) the direction $d$ in which the light-particle goes (and this velocity does not depend on other things, e.g. not on which light-bulb emitted the photon). At the beginning of the present section, in Remark 4.3.5 we denoted the speed of this photon as seen by observer $m$ as $c_m(p, d)$. We will start using this notation again in the next section (§4.4) beginning with p.535 in item ($*$) there. Then (P2) says that $c_m(p, d)$ does not depend on $m$, $p$ or $d$. We consider $\mathbf{Bax}^-$ (cf. Def.4.3.7) as the completely formalized[421] counterpart of (P1). We consider **Bax** as the formalized counterpart of (P1 + Weak Principle of Isotropy), where Weak Principle of Isotropy (WPI) is formalized as $\mathbf{Ax5^{Ph}}$ above 3.4.16 (on p.219). Further we consider **Flxbasax** as the formalized counterpart of (P2).[422] We note that (P1+WPI) $\not\models$ (P2). When one uses a principle like (P1), usually one takes as granted some background axioms. In special relativity such a background axiom is e.g. that the traces of inertial bodies are straight lines. We will take as background axioms **Ax1**, **Ax2**, $\mathbf{Ax3_0}$, **Ax4**, $\mathbf{Ax5_{Ph}}$, $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$, $\mathbf{AxE_{01}}$ which seem to be implicitly assumed in all versions of special relativity in Friedman [90]. We will call the collection of these "trivial" axioms $\mathrm{SPR}_0$, where the abbreviation $\mathrm{SPR}_0$ refers to "the trivial part" of the Special Principle of Relativity (SPR) in the sense of Friedman [90, p.149, principle (R)]. Now the formalized version of (P1+$\mathrm{SPR}_0$) is $\mathbf{Bax}^-$, the formalized version of (P1+WPI+$\mathrm{SPR}_0$) is **Bax**, and **Flxbasax** will turn out to be the formalized version of (P2+$\mathrm{SPR}_0$) [cf. Propositions 4.3.41, 4.3.42]. For

[420]This is in accordance with [90, p.160], where it is said that one of the most important consequences of (P1) is that we have a so-called light-cone (in each point). We will elaborate on the light-cone aspect more in §4.4, but cf. also Remark 4.3.5 (p.473).

[421]in first order logic

[422]However, occasionally we will use **Newbasax** instead of **Flxbasax** as the counterpart of (P2). We are allowed to do this because **Newbasax** is very close to **Flxbasax**. We will be motivated to do this because **Newbasax** is one of the "main characters" of this work.

completeness, we note that $(P2+SPR_0) \models WPI$. We admit that **Bax** is only one of the possible formalizations of $(P1+WPI+SPR_0)$. Another possible formalization of $(P1+WPI+SPR_0)$ is **Reich**(**Bax**) in §4.5 way below (at least in a certain sense). We sum this up in the following table:

| | | | | |
|---|---|---|---|---|
| $\mathbf{Bax}^-$ | is the formalization of | (P1) | | $+ SPR_0$. |
| **Bax** | is the formalization of | (P1) + (WPI) | | $+ SPR_0$. |
| **Flxbasax** | is the formalization of | (P2) | | $+ SPR_0$. |

**Newbasax** is very close to **Flxbasax**, therefore **Newbasax** is very close to being the formalized counterpart of $(P2+SPR_0)$. We note this because **Newbasax** is one of the "main characters" of the present work, while **Flxbasax** is not. Therefore if we want the formal counterparts of Friedman's principles in terms of main characters of the present work then we get $\mathbf{Bax}^-$, **Bax**, **Newbasax** for (P1), (P1+WPI), (P2), respectively, cf. the above table.

For completeness, we note that we never assume Friedman's (P3) as an axiom for the following reason: Part of (P3) turns out to be a theorem of our **Newbasax** (and also of **Bax**) (hence of Friedman's (P2) + Special Principle of Relativity, too) [cf. Theorems 3.4.2, 3.4.19 herein], while the other part of (P3) concerning bodies which are not observers does not seem to be needed in any part of developing the theory. Actually, we do have some philosophical reasons for not assuming this second part of (P3).

We should emphasize that our principle $SPR_0$ is strictly weaker than Special Principle of Relativity in Einstein's 1905 paper. Therefore we do not call our $SPR_0$ "Special Principle of Relativity" outside this remark. Our reason for not using the original Special Principle of Relativity is that (it is so strong that) it would blur the distinction between (P1) and (P2) (as indeed is pointed out in Friedman [90, p.160]).[423]

As we said, we will return to more careful (or thorough) considerations concerning the first-order formalization(s) of Friedman's principle (P1) in §4.4 way below.

$\triangleleft$

Propositions 4.3.41, 4.3.42 below serve to illuminate parts of Remark 4.3.40 above. Principle $SPR_0$ ($= \mathbf{Bax}^- \setminus \{\mathbf{AxP1}\}$) was introduced in that remark.

[423] In the present work, we study formalized instances of Einstein's SPR under the name "symmetry axioms". An example is **Ax(symm)** in §2.8. Cf. also §§ 5, 3.8, 3.9. Usually, into these symmetry principles we do not include the trivial part $SPR_0$. The reason for not including $SPR_0$ is the goal of "decomposability" of our theories into weaker subtheories formulated e.g. in §1.1.

Whenever we state a proposition beginning with "assume $\mathbf{Ax}(\sqrt{\ })$" this means that $\mathbf{Th_1} \models \mathbf{Th_2}$ abbreviates the longer statement

$$\mathbf{Th_1} + \mathbf{Ax}(\sqrt{\ }) \ \models \ \mathbf{Th_2} + \mathbf{Ax}(\sqrt{\ }).$$

**PROPOSITION 4.3.41** *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Then (i)–(iii) below hold.*

**(i)** $\mathrm{SPR}_0 + \mathbf{AxE_0} \ \models\!\!\!\dashv \ \mathbf{Newbasax}$.

**(ii)** $\mathrm{SPR}_0 + \mathbf{AxE_{02}} \ \models\!\!\!\dashv \ \mathbf{Flxbasax}$.

**(iii)** $\mathrm{SPR}_0 + \mathbf{Ax5^{Ph}} + \mathbf{AxP1} \ \models\!\!\!\dashv \ \mathbf{Bax}$.

**On the proof:** The proofs of items (i), (ii) are straightforward. Item (iii) follows by Prop.4.3.6(ii). ∎

**PROPOSITION 4.3.42**

**(i)** $(\mathrm{SPR}_0 \setminus \{\mathbf{AxE_{01}}\}) + \mathbf{AxE_0} \ \models \ \mathbf{Newbasax}$.

**(ii)** $\mathrm{SPR}_0 + \mathbf{Ax5^{Ph}} + \mathbf{AxP1} \ \models \ \mathbf{Bax}$.

**On the proof:** The proof of item (i) is straightforward. The proof of item (ii) is similar to the proof of Prop.4.3.6. ∎

**Remark 4.3.43** (On a possible more balanced formulation of our axiom systems studied so far)
We could have chosen the speed-of-light-free part of **Bax** and of **Newbasax** to be $\mathrm{SPR}_0$. In more detail, let

$$\begin{aligned}
\mathbf{Basax'} &:= \mathrm{SPR}_0 + \mathbf{AxE} + \mathbf{Ax6},\\
\mathbf{Newbasax'} &:= \mathrm{SPR}_0 + \mathbf{AxE_0},\\
\mathbf{Flxbasax'} &:= \mathrm{SPR}_0 + \mathbf{AxE_{02}},\\
\mathbf{Bax'} &:= \mathrm{SPR}_0 + \mathbf{AxE_{00}},\\
\mathbf{Bax^-} &:= \mathrm{SPR}_0 + \mathbf{AxP1}.
\end{aligned}$$

These, "more balanced" versions are equivalent with the originals as Prop.4.3.44 below says. (We note that Prop.4.3.44 below is an organic part of the present remark [Rmk.4.3.43].)

**Proposition 4.3.44** *For simplicity assume* $\mathbf{Ax}(\sqrt{\ })$. *Then (i)-(iv) below hold.*

$$\begin{array}{lrcl}
\text{(i)} & \mathbf{Basax} & \models\!\!\!\dashv & \mathbf{Basax'}.\\
\text{(ii)} & \mathbf{Newbasax} & \models\!\!\!\dashv & \mathbf{Newbasax'}.\\
\text{(iii)} & \mathbf{Flxbasax} & \models\!\!\!\dashv & \mathbf{Flxbasax'}.\\
\text{(iv)} & \mathbf{Bax} & \models\!\!\!\dashv & \mathbf{Bax'}.
\end{array}$$

We omit the *proof*. ∎

Let us disregard $\mathbf{Ax}(\sqrt{\ })$ for a while. Then it would have been a possibility to define **Newbasax** as **Newbasax′** is defined now. Then the hierarchy

$$\mathbf{Newbasax} \mapsto \mathbf{Flxbasax} \mapsto \mathbf{Bax} \mapsto \mathbf{Bax}^-$$

could have been developed (i.e. defined) by gradually weakening *only* the speed of light axiom $\mathbf{AxE}, \mathbf{AxE_0}, \ldots, \mathbf{AxP1}$.

Another advantage of switching to the $\mathrm{SPR}_0$ based systems **Newbasax′** etc. would be that probably the new systems **Newbasax′**, **Flxbasax′**, **Bax′** would be "logically independent in a greater extent" in the intuitive sense, that they would be more "balanced" as the notion of being balanced[424] was explained in item (ii) of the introduction to §3.3. However, to outline the above plan of "streamlining" or "balancing" our hierarchy $\mathbf{Newbasax} \mapsto \ldots \mapsto \mathbf{Bax}^-$ of axiom systems, we had to ignore $\mathbf{Ax}(\sqrt{\ })$. This is so because $\mathbf{Ax}(\sqrt{\ })$ was needed in Prop.4.3.44. However this need for $\mathbf{Ax}(\sqrt{\ })$ is not very deep, namely it was caused by a fairly arbitrary decision we made when formulating $\mathbf{Bax}^-$ (namely we assumed that there are photons moving in every direction).

It could be an entertaining experiment (for the future) to refine our axiom systems such that the above plan would be realized (i) in a natural fashion[425] and (ii) without having to disregard $\mathbf{Ax}(\sqrt{\ })$. However, here we do not go into this experiment any further.

◁

[424]Balanced = "the proof theoretic power is evenly distributed among the axioms". What we mean by balanced here is the same what we call "having good decomposability" in footnote 423 on p.524.

[425]In their present form, $\mathbf{Ax5_{Ph}}$ and $\mathbf{Ax5_{Obs}}$ would look somewhat artificial for the reader who would meet them when first seeing **Newbasax** at the beginning of §3.3.

## 4.4 On the careful formalization of Friedman's principle (P1); a hierarchy of weak, general axiom-systems

Here, among other things, we will experiment with pushing the process of *weakening* our speed of light axiom (**AxE** $\mapsto \ldots \mapsto$ **AxP1**) to the extreme.[426] (For this of course we will have to adjust those other axioms too which involve photons.) This section has a slightly different nature from the previous ones. Much of what we do in this section is of an experimental character and is less polished than the other parts of this work. The reason for this is that the main purpose of this section is to *broaden* the scope of our *imagination*. We will not use in later parts the axioms and axiom-systems introduced here.

**Bax**$^{-}$ contains only two very natural assumptions about photons, which seem to be acceptable even if one knows nothing about the Michelson-Morley experiment. These assumptions are: (i) Photons are not like bullets, in the sense that photons moving in the same direction have the same speed (they cannot overtake one another).[427] (ii) Photons are *not* like sound, in the sense that if an observer $m$ points his flashlight in a direction $d$, then the photons emitted by the flash-light will move forwards in direction $d$ (as observed by $m$, of course). I.e. no "ether-wind" can "blow" all the emitted photons backwards (i.e. no "ether-wind" can "blow" the light-cone completely off the time axis $\bar{t}$ as illustrated in Figure 170).[428] In this respect, light behaves differently from sound. In the previous section we saw that these two natural weak assumptions on photons suffice to prove quite a lot, especially if we add the postulates of finiteness of speed of light, or a symmetry principle. (E.g. the worldview transformations are collineations, light-cones are alike (at each time and place), there are no FTL observers if the speed of light is finite, and we can derive the flexible version of special relativity if we assume a symmetry principle and finiteness of speed of light. Cf. items 4.3.11, 4.3.17, 4.3.18, 4.3.24.) In this section we remove the second assumption on photons, and therefore we will make many analogies with sound. One can also view this section as a discussion of what the consequences of light being different from sound are.[429] Although it is not

[426] In a sense, this process will culminate in our chapter §5 where we will not mention the speed of light (or photons for that matter) at all. However, this way of connecting §5 to the present section might be slightly misleading, because in the present section we are preparing our imagination for the "patterns" which we might encounter in our chapter 8 ("Accelerated Observers") and eventually in general relativity while §5 has no such ambitions.

[427] This is formulated as axiom **AxP1**.

[428] This is formulated as $\mathbf{Ax5_{Ph}}$.

[429] One can do this by comparing the theorems in the present section with those in the previous

customary to compare light and sound, here comparing them will provide a useful analogy.

In this section we will be more "thorough" in formalizing Friedman's principle (P1), arriving at axioms weaker than **AxP1**.[430] The main new idea is that, drawing from the analogy with speed of sound, we allow that a photon supposed to move forwards in direction $d$, actually moves backwards in direction $d$ (in the analogy with sound the reason may be either that there is a wind, or that our observer — e.g. a supersonic airplane — moves faster in direction $d$ than sound).[431] In §4.3 we already had "tilted" light-cones, in the present section we will allow light-cones tilt so much that the time-axis will no longer stay inside the cone. See Figure 170.

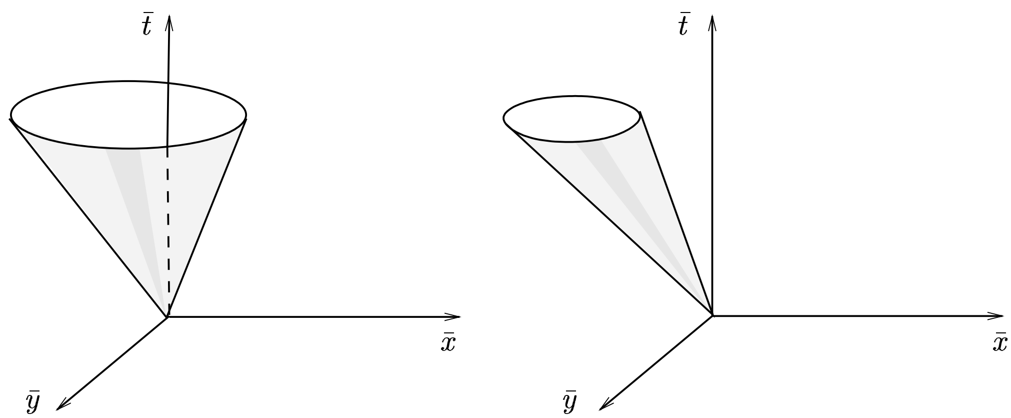



Figure 170: We will allow light-cones tilt so much that the time-axis will no longer stay inside the cone.

In our previous formalization $\mathbf{Bax}^-$ of (P1), for any observer $m$, and for any direction, say the direction marked by the vector $1_x$, there is a photon moving forwards in direction $1_x$. If we keep the analogy with sound in mind, and if we do not want to exclude FTL observers (at least not a priori), then one might imagine that for some observer $m$ a photon supposed to move forwards in direction $1_x$ might seem to be moving (slowly) forwards in the opposite direction $-1_x$. Then $m$ would see two photons moving along the $\bar{x}$ axis, one moving forwards with speed say 0.1 in

one.

[430](P1) was recalled in Remark 4.3.40 in §4.3.

[431]In the literature of relativity, this kind of hypothetical "wind" is often called ether-wind.

direction $-1_x$ while the other moving forwards with speed say 1.1 also in direction $-1_x$.

In our first refined formalization of (P1), in $\mathbf{AxP1^-}$, we will still require that from any point $p$ of space-time, in any direction there are at most two photon-traces (moving forwards or backwards in this direction). This way we will arrive at a very weak system $\mathbf{Bax^{--}}$. Then we will add various restrictions concerning (i) the shapes and "behaviors" of light-cones and (ii) the "local relationship" between the light-cones and the observers.[432] These additional restrictions (i.e. axioms) will lead to a hierarchy of axiom systems stronger than $\mathbf{Bax^{--}}$ but weaker than $\mathbf{Bax}$. We refer the reader to Figure 180 on p.552 for this hierarchy.

Let us turn to formulating the speed of light axiom, $\mathbf{AxP1^-}$ of $\mathbf{Bax^{--}}$.

$\mathbf{AxP1^-}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \mathsf{directions})$
$\Big(\big|\{tr_m(ph) \,:\, p \in tr_m(ph) \;\&\; ph \text{ is moving in direction}^{433}\; d \;\&\; ph \in Ph\}\big| \leq 2\Big).$

Intuitively: For any observer $m$ and point $p$, in any *spatial direction* $d$, $m$ will observe at most two kinds of photons "starting out from $p$", one supposed to move forwards in this direction, while the other one supposed to move backwards. See Figure 171. Thinking further on the analogy with supersonic airplanes (and the velocity of sound), we conclude that in some directions there might be only one photon, and still in other directions there might be none. To illustrate this, consider Figure 171. Since the light-cone (or sound-cone for airplanes) only touches the plane $\mathsf{Plane}(\bar{t}, \bar{y})$ there will be only one photon in direction $1_y$. This corresponds to the case when the supersonic airplane moves exactly with the speed of sound.[434] If the airplane goes a bit faster, then there will be no "photon" (or sound wave) in direction $1_y$.

Our $\mathbf{AxP1^-}$ is still a formalization of Friedman's (P1) which says that the speed of light does not depend on the velocity of its source. We turn to defining the axiom system $\mathbf{Bax^{--}}$, which could be considered as a careful (or almost finicky) formalization of Friedman's (P1). To do this we will replace $\mathbf{AxP1}$ by $\mathbf{AxP1^-}$ in $\mathbf{Bax^-}$. We note that $(\mathbf{Bax^-} \setminus \{\mathbf{AxP1}\}) + \{\mathbf{AxP1^-}\}$ is equivalent with $\mathbf{Bax^-}$ under the assumption that there are no photons with infinite speed.[435] To obtain a weaker axiom system than $\mathbf{Bax^-}$ we have to replace $\mathbf{Ax5_{Ph}}$ and $\mathbf{Ax5_{Obs}}$ by $\mathbf{Ax5^-_{Ph}}$ and

[432] These type (ii) restrictions can be regarded as being motivated by features of general relativity.
[433] as seen by $m$
[434] Here we disregard the fact that this might destroy the plane.
[435] This is so because $\{\mathbf{Ax5_{Ph}}, \mathbf{AxP1^-}, \mathbf{AxE_{01}}\} \models \mathbf{AxP1}$ under the assumption that there are no photons with infinite speed.

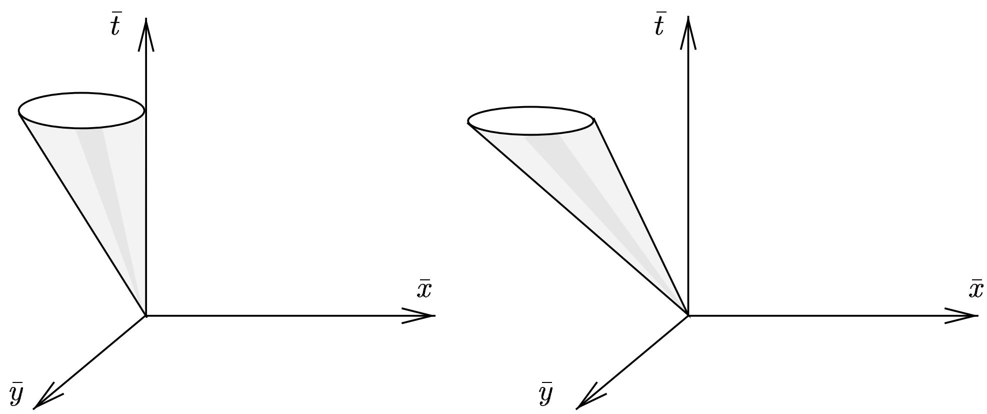



Figure 171: Illustration for **AxP1$^-$**: In some directions there might be only one photon, and still in other directions there might be none.

**Ax5$^-_{\mathbf{Obs}}$** below, respectively. (This is natural, since the mentioned axioms do involve photons).

**Ax5$^-_{\mathbf{Ph}}$** Assume observer $m$ sees photons $ph_1$ and $ph_2$ moving forwards in directions $d_1$ and $d_2$, respectively, through point $p$ of space-time. Then for any direction $d_3$ in between $d_1$ and $d_2$, and distinct from $d_1$ and $d_2$, $m$ sees at least two photons with different life-lines, one of which is moving forwards in direction $d_3$. Here we say that $d_3$ is between $d_1$ and $d_2$ iff there are $\mu, \lambda \in {}^+F$ such that $d_3 = \mu \cdot d_1 + \lambda \cdot d_2$. Every observer $m$ at every point $p$ of space-time sees at least two photons, which are moving in different directions.

To formulate **Ax5$^-_{\mathbf{Obs}}$** we will use Def.4.4.1 below.

**Definition 4.4.1** Let $\ell_1, \ell_2 \in \mathsf{Eucl}$ and $p \in {}^nF$ such that $\ell_1 \cap \ell_2 = \{p\}$. Let $\ell \in \mathsf{Eucl}$. Then $\ell$ is *between* $\ell_1$ and $\ell_2$ iff $p \in \ell$ and there is $\ell' \in \mathsf{Eucl}$ such that $\ell \parallel \ell'$ and there are $q \in \ell' \cap \ell_1$ and $r \in \ell' \cap \ell_2$ such that $q \neq p$ and $time(q) \leq time(p) \leq time(r)$, see Figure 172.

$\triangleleft$

**Ax5$^-_{\mathbf{Obs}}$** Assume observer $m$ sees photons $ph_1$ and $ph_2$ moving in direction $d$ through point $p$ of space-time and $tr_m(ph_1) \neq tr_m(ph_2)$. Assume $\ell \in \mathsf{Eucl}$ such that $\ell$ is between $tr_m(ph_1)$ and $tr_m(ph_2)$ . Then there is an observer $k$ such that $tr_m(k) = \ell$.

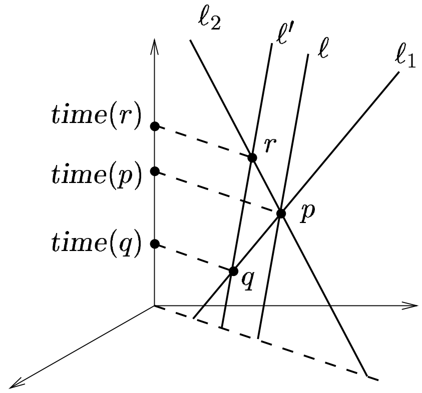



Figure 172: Illustration for Def.4.4.1: $\ell$ is between $\ell_1$ and $\ell_2$.

**Definition 4.4.2**

$$\mathbf{Bax}^{--} \stackrel{\text{def}}{=} (\mathbf{Bax}^{-} \setminus \{\mathbf{AxE_{01}}, \mathbf{Ax5_{Ph}}, \mathbf{Ax5_{Obs}}, \mathbf{AxP1}\}) \cup \{\mathbf{Ax5^{-}_{Ph}}, \mathbf{Ax5^{-}_{Obs}}, \mathbf{AxP1^{-}}\}.$$

◁

Notice that we omitted $\mathbf{AxE_{01}}$ which says that the speed of a photon is not 0.

Beginning with $\mathbf{AxP1^{-}_{1}}$ below, and ending with item 4.4.5 on p.534, we will experiment with adding to $\mathbf{Bax}^{--}$ a *particular kind* of axioms which concern the "local relationship" between light-cones and observers.[436] After this part (i.e. after item 4.4.5) we will look into *different* kinds of axioms which concern the "shape" of the light-cones. Since these two different kinds of axioms can be used independently of each other, the lattice on p.552 will branch out above $\mathbf{Bax}^{--}$. The left hand side will represent the local relationship axioms while the right hand side, the shape of light-cones axioms.

Next, we consider two potential axioms $\mathbf{AxP1^{-}_{1}}$ and $\mathbf{AxP1^{-}_{2}}$ which could be added to $\mathbf{Bax}^{--}$ to make it a stronger version of Friedman's (P1).

[436] These "local relationship" restrictions are motivated by the following feature of general relativity. According to some space-time diagrams in general relativity, an observer far away from point $p$ might think that the light-cone at point $p$ is tilted very-very much, but there will be a "local" observer $m_p$ whose life-line contains $p$ and who will "think" that the light-cone at $p$ is not tilted so much i.e. $m_p$ will think that the time axis is inside the light-cone in question. However, these analogies with general relativity have to be treated with caution, cf. footnote 438 on p.533.

**AxP1$_1^-$** $(\forall m \in Obs)(\forall p \in {}^nF)(\exists k \in Obs \cap w_m(p))(\forall d \in \mathsf{directions})$ $(\exists ph \in Ph \cap w_m(p))(ph$ is moving forwards in direction $d$ as seen by $k)$.

That is, for every event $E$ there is an observer $k \in E$ such that $k$ sees in any direction $d$ a photon $ph \in E$ moving forwards in direction $d$.

Intuitively, this means the following (when there is no photon with infinite speed). In any event $E$ there is an observer $k$ which thinks that his *life-line is inside the light-cone* starting at event $E$. In other words, $k$ thinks that the time-axis $\bar{t}$ is inside the light-cone starting at event $E$.

By $\mathsf{Planes} = \mathsf{Planes}(n, \mathbf{F})$ we denote the set of all planes of ${}^n\mathbf{F}$.

**AxP1$_2^-$** $(\forall m \in Obs)(\forall p \in {}^nF)(\exists \ell \in \mathsf{Eucl})\Big((\forall P \in \mathsf{Planes})(\ell \subseteq P \Rightarrow$ $(\exists ph_1, ph_2 \in Ph)\Big(tr_m(ph_1), tr_m(ph_2) \subseteq P \ \wedge \ tr_m(ph_1) \cap tr_m(ph_2) = \{p\} \wedge$ $(\ell$ is between $tr_m(ph_1)$ and $tr_m(ph_2))\Big)\Big)$. See Figure 173.

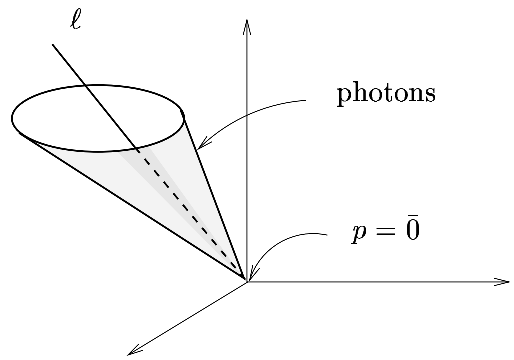



Figure 173: Illustration for **AxP1$_2^-$**.

The reader may ask: what is the role of the line $\ell$ in **AxP1$_2^-$**? The answer is this: Let us think about the analogy with the speed of sound. If there is a wind, then "against the wind" sound goes slower, while "with the wind" it goes faster. Hence, in direction $1_x$ speed of light $c_x$ may be small while in direction $-1_x$ the speed $c_{-x}$ might be very large.[437] If we think of airplanes moving

[437]For simplicity, in this explanation we write $c_x$ instead of $c_m(p, 1_x)$.

faster than the speed of sound, we realize that (in theory) it is reasonable to allow $c_x$ to be a negative number. Imagine e.g. $c_x = -0.1$ and $c_{-x} = 1.1$. Then, moving along the $\bar{x}$ axis we see two kinds of photons, one with speed 0.1, the other with 1.1 and <u>*both*</u> moving forwards in direction $-1_x$. So, in principle, the photon $ph_x$ moving in direction $1_x$ might have a negative speed $(-0.1)$ and therefore $ph_x$ might appear to the observer as if it was moving forwards in the direction $-1_x$. All this is quite natural, if we think of sound in place of light and if our observer is a supersonic airplane. If we push these ideas further, we will arrive at the above formulation of $\mathbf{AxP1_2^-}$: Here, the role of $\ell$ is analogous with the role of observer $k$ in $\mathbf{AxP1_1^-}$. We can think of $\ell$ as the life-line of a leaf drifting in the wind.

We note that $\mathbf{AxP1_2^-} \not\models \mathbf{AxP1^-}$ and, similarly, $\mathbf{AxP1_1^-} \not\models \mathbf{AxP1^-}$. We turn to defining axiom systems $\mathbf{Bax_1^{--}}$ and $\mathbf{Bax_2^{--}}$ by adding $\mathbf{AxP1_1^-}$ and $\mathbf{AxP1_2^-}$ to $\mathbf{Bax^{--}}$, respectively.

**Definition 4.4.3**

$$\mathbf{Bax_1^{--}} \stackrel{\text{def}}{=} \mathbf{Bax^{--}} + \mathbf{AxP1_1^-}.$$
$$\mathbf{Bax_2^{--}} \stackrel{\text{def}}{=} \mathbf{Bax^{--}} + \mathbf{AxP1_2^-}.$$

◁

We note that in $\mathbf{Bax_1^{--}}$, the two axioms $\mathbf{AxP1^-}$ and $\mathbf{AxP1_1^-}$ might have the following, natural, joint effect (when $c_m(p,d) \neq 0, \infty$ is also valid in the model).

(+) For every event $E$, there is an observer $k \in E$, such that for every direction $d$ $k$ sees *exactly one* photon trace going through $E$ and moving in direction $d$ forwards.

(The existence part of (+) seems to be coming from $\mathbf{AxP1_1^-}$ while the uniqueness part from $\mathbf{AxP1^-}$.)

Next, we consider an axiom $\mathbf{AxP1_3^-}$ which could be added to $\mathbf{Bax_1^{--}}$. The axiom system obtained in such a way will be called $\mathbf{Bax_3^{--}}$. We note that in a certain sense $\mathbf{Bax_3^{--}}$ is motivated, indirectly, by Figure 16.10 on p.220 of d'Inverno [75].[438]

[438] One has to be careful with these analogies, because in general relativity if a light-cone is tilted too much (or is unusual in some other way) then that light-cone has to be far away from the time-axis $\bar{t}$ i.e. from the life-line of that observer whose world-view is being represented on the space-time diagram in question.

**AxP1$_3^-$** $(\forall m \in Obs)(\forall p \in \bar{t}\,)(\forall d \in \mathsf{directions})(\exists ph \in Ph)$
$(p \in tr_m(ph)$ and $ph$ is moving forwards in direction $d$ as seen by $m)$.

That is, each observer $m$ through any point $p$ of its life-line in any direction $d$ sees a photon moving forwards in direction $d$. Intuitively, this means (when there is no photon with infinite speed) that each observer thinks that he moves slower than light (in the sense that his life-line is inside the light-cone).

**Definition 4.4.4**

$$\mathbf{Bax}_3^{--} \stackrel{\text{def}}{=} \mathbf{Bax}_1^{--} + \mathbf{AxP1}_3^-.$$

◁

**Question for future research 4.4.5** Investigate axiom systems $\mathbf{Bax}^{--}$, $\mathbf{Bax}_1^{--}$, $\mathbf{Bax}_2^{--}$, $\mathbf{Bax}_3^{--}$ in the same spirit as we investigated e.g. **Bax** or **Basax**. Further, compare them with our weak axiom systems like $\mathbf{Bax}^-$ or **Rel**(**noph**), where the latter will be introduced in § 5. In particular, it would be interesting to know whether $\mathbf{Bax}_3^{--} \models \mathbf{Bax}^-$ holds or not. ◁

So far we studied what we call "local relationship between light-cones and observers" style axioms ($\mathbf{AxP1}_1^-$ etc.). (In Figure 180 they appear on the left hand side of the lattice.) Next we turn to discussing what we call the "shape of light-cones" style axioms. (As we said, they can be used *independently* from the "local relationship" axioms. This is why Figure 180 branches out above $\mathbf{Bax}^{--}$).

To formulate these assumptions, first we make it explicit that the velocity of a photon $ph$ depends only on (i) the point $p$ of space-time where $ph$ was created and (ii) on the direction $d$ in which (according to $m$) $ph$ is moving. The formal version of this is condition $(*)$ below. For formulating $(*)$ we first need to formulate items 4.4.6-4.4.8 below.

The axiom systems introduced in the rest of this section are of an *experimental character*. They are not polished carefully, their translatability to our first-order frame language is not double-checked (therefore some of them might need a minor adjustment) etc. (All the same, the intuitive idea behind all of them should be sound.)

**Definition 4.4.6 (Directional speed)** Let $\mathfrak{M}$ be a frame model satisfying **Ax1**, **Ax2**, $\mathbf{Ax3_0}$, i.e. in which traces of inertial bodies are straight lines (or empty). Let $m \in Obs$, $b \in Ib$ and $d \in \mathsf{directions}$ such that body $b$ moves in direction $d$ (as seen by $m$). Then the *speed* of body $b$ *in direction* $d$ (as seen by $m$) is $v_m(b)$ if body $b$ moves forwards in direction $d$ and is $-v_m(b)$ otherwise. We note that the speed of $b$ in direction $d$ may be $\infty$. We make the convention that $-\infty = \infty$. ◁

**Notation 4.4.7** Let $\mathfrak{F}$ be an ordered field. Then

1. Recall from §2, p.46 that $\infty$ denotes an element *not* in $\mathfrak{F}$.

2. $F^\infty := F \cup \{\infty\}$,

   the topology on $F^\infty$ is the usual one, i.e. a sequence $p = \langle p_i : i < \lambda \rangle$ with $\lambda$ an ordinal has a limit in this new topology either if it has a limit in the natural topology connected to $\mathfrak{F}$, or if $p$ is cofinal in the sense that $(\forall r \in F)(\exists \eta < \lambda)(\forall \eta < i < \lambda) p_i > r$ (and in the latter case the limit is $\infty$). Hence $F$ is *not* a closed subset of $F^\infty$. Also, $\{\infty\}$ is a closed set, but the interval $(x, \infty]$ is an open (moreover a clopen) subset of $F^\infty$.[439]

3. $\mathfrak{F}^\infty$ denotes the extension of $\mathfrak{F}$ with the single element $\infty$, where the operations on $\infty$ are the usual ones, i.e. $(\forall x \in F)\ x < \infty$, $\infty = -\infty$, etc. We note that $\mathfrak{F}^\infty$ is not a field.

$\triangleleft$

The formulation and discussion of $(*)$ below will be a continuation of Remark 4.3.5 "On **AxP1** and light-cones" (p.473), in §4.3. The definition of $c_m(p, d)$ below will be a little bit more general (or more flexible) than the definition of basically the same function in Remark 4.3.5 (§4.3). The reason is that in Remark 4.3.5 we had **AxP1** which ensured, roughly, that a photon "emitted" in direction $d$ forwards, would move forwards. Therefore, we were allowed to say that $c_m(p, d)$ is defined iff there is a photon moving forwards in direction $d$ (at point $p$). Cf. e.g. Figure 173. As a contrast, now, by Figure 173, a photon emitted forwards in direction $d$ might move backwards. Therefore, now $c_m(p, d)$ might be *defined* and be negative. This was not possible in Remark 4.3.5 (i.e. in **Bax⁻**). Now we turn to formulating condition $(*)$ promised way above.

To each observer $m$ there is a partial function

$$c_m : {}^nF \times \mathsf{directions} \longrightarrow F^\infty$$

such that (I), (II) below hold.

$(*)$

(I) $c_m(p, d)$ is defined iff there exists a photon $ph$ moving in direction $d$ (as seen by $m$) and $p \in tr_m(ph)$.

(II) $c_m(p, d) = \max\{s \in F^\infty : s$ is the <u>*speed*</u> of $ph$ <u>*in direction*</u>[440] $d$ (as seen by $m$), *and* $ph \in Ph$ is moving in direction $d\}$.[441]

[439] We note that these conditions uniquely determine a topology on $F^\infty$.

Note that $c_m(p,d)$ may be infinite, too.

We often refer to the function $c_m$, discussed above, as "the $c_m(p,d)$ function" to indicate explicitly what the arguments of $c_m$ are. This is somewhat ambiguous since $c_m(p,d)$ should be a value in $F$ and the function is just $c_m$ but we hope context will help.

Since we want to use $(*)$ only in models of $\mathbf{Bax}^{--}$, instead of calling $(*)$ a condition on our models we could regard it simply as a *definition* of the function $c_m$. The reason for this is that in all models of $\mathbf{Bax}^{--}$ there exists a unique function $c_m$ satisfying $(*)$. (But this is not true in arbitrary frame models, or even in $\mathbf{Bax}^{--}$ *without* **AxP1⁻**.)

Let us notice that $c_m(p,d)$ may be *negative*. Therefore observer $m$ may see (or have the illusion, so to speak) that a photon $ph$ which was expected to be moving forwards in direction $d$ is actually moving backwards in *direction* $d$. This is why in axiom **AxP1⁻** we said *only* that there are at most two photon velocities corresponding to a space-time point $p$ and a direction $d$ and did *not* require that these two photons should move in opposite directions.

Summing up the "genesis" of (our principles and) $\mathbf{Bax}^{--}$, principle $(*)$ is derived from Friedman's (P1), and axiom **AxP1⁻** is in turn derived from (or justified by) $(*)$. This way, we obtained $\mathbf{Bax}^{--}$. However, $\mathbf{Bax}^{--}$ is very weak because we did not say anything about *how* $c_m(p,d)$ depends on its arguments $p$ and $d$. This motivates the definition of $\mathbf{Bax}_{+}^{--}$ below, where we *will* say something about how $c_m(p,d)$ depends on its arguments.

To define $\mathbf{Bax}_{+}^{--}$, first we need Def.4.4.8 below (which is of an auxiliary nature).

**Definition 4.4.8 (strong continuity)**

**(i)** A function $f : F \longrightarrow F^{\infty}$ is called *continuous* iff it satisfies the usual first-order formula defining continuity i.e. iff

$$(+)\quad \begin{array}{l}(\forall x \in Dom(f))(\forall \text{neighborhood } N \text{ of } f(x))\\ (\exists \text{ neighborhood } H \text{ of } x)\ f[H] \subseteq N.\end{array}$$ [442]

---

[440] "speed in direction $d$" (is not the same as "speed" and) was defined in Def.4.4.6.

[441] Let us notice, that condition $(*)$ i.e. the existence of such a $c_m$ function implies that if there is a photon moving in direction $d$ (at $p$ etc) then there is one whose *speed in direction* $d$ is maximal.

[442] By a *neighborhood of $x$* we understand an open interval containing $x$, but cf. also $\varepsilon$-neighborhood $S(p,\varepsilon)$ in item 3.3.1 on p.189. An open interval $N$ contains $\infty$ iff $N = \{y : y > a\}$ for some $a \in F$.

**(ii)** In similar situations, when a natural metric is available on the domain and range of $f$ we use the same definition. E.g. let $n, k \in \omega$. Then a function $f : {}^nF \longrightarrow {}^kF$ is called continuous iff it satisfies the natural counterpart of condition (+), using the square of the Euclidean distance in place of $|x - y|$ in defining neighborhoods etc. In particular a neighborhood in ${}^nF$ is an open "ball" (i.e. sphere, cf. item 3.3.1 on p.189).

**(iii)** Let $f$ : $\mathsf{directions} \longrightarrow \mathfrak{F}^\infty$. Then, we consider $f$ continuous if it is such in the sense of (ii) above.

**(iv)**

$$f : F \longrightarrow F^\infty \text{ is called } \underline{\textit{strongly continuous}}$$

$$\Updownarrow$$

$$\Big[\text{it is continuous and } (\forall x, y \in F)(\forall z \in F)\Big(z \text{ is between } f(x) \text{ and } f(y) \quad \Rightarrow$$
$$(\exists w \text{ between } x \text{ and } y)\ f(w) = z\Big)\Big].$$

More formally, the second condition says,

$$(\forall x, y, z)\Big(f(x) \le z \le f(y) \ \Rightarrow\ \exists w\, [x \le w \le y \ \ \wedge \ \ f(w) = z]\Big).$$

**(v)** Let $f$ : $\mathsf{directions} \longrightarrow F^\infty$ be a continuous function. Then the definition of strong continuity is analogous with the one in (iv) above, as follows. We call $f$ *strongly continuous* iff $(\forall \varepsilon \in {}^+F)(\forall d, d_1 \in \mathsf{directions})$ [(the square of the angle[443] between $d$ and $d_1$)$< \varepsilon \ \Rightarrow \quad (\forall z$ between $f(d)$ and $f(d_1))(\exists d_2 \in \mathsf{directions})(d_2$ is between[444] $d$ and $d_1 \ \wedge\ f(d_2) = z)$].[445]

**(vi)** Let $f : F \overset{\circ}{\longrightarrow} F^\infty$ be a *partial* function. Then $f$ is called *continuous* iff it satisfies formula (+) in item (i) above. To clarify this definition, we note that the neighbourhoods $N$ and $H$ (quantified over in formula (+)) are understood in $\mathfrak{F}$. (Hence they need not be subsets of $Dom(f)$.) Further, $f$ is *strongly continuous* iff $\Big[$it is continuous and $(\forall x, y, z \in F)([f(x) < z < f(y) \ \wedge\ int(x, y) \subseteq Dom(f)] \Rightarrow z \in f[int(x, y)])\Big]$.[446]

---

[443] This can be defined analogously with our $ang^2(\ell)$ on p.46, e.g. we choose to use the directed line determined by $d$ as we used the $\bar{t}$ axis and we use the directed line of $d_1$ as we used $\ell$ etc., cf. the footnote for item (vii) below.

[444] This was defined in the formulation of $\mathbf{Ax5^-_{Ph}}$ on p.530.

[445] this is expressible by a first-order formula

[446] The only difference with the definition for total functions (in (iv) above) is the extra condition "$int(x, y) \subseteq Dom(f)$".

**(vii)** The generalization of (vi) to partial functions of directions is the natural one, as follows. Assume $f : \mathsf{directions} \overset{\circ}{\longrightarrow} F^\infty$ is a partial continuous function. Then $f$ is called *strongly continuous* iff[447] $(\exists\varepsilon \in {}^+F)(\forall d, d' \in \mathsf{directions})$ $\Big([ang^2(d,d') < \varepsilon \ \wedge \ (\forall d'' \in \mathsf{directions})\ (d'' \text{ is between } d \text{ and } d'' \Rightarrow d'' \in Dom(f))] \Longrightarrow (\forall z \in int(f(d), f(d')))(\exists d'' \text{ between } d \text{ and } d')z = f(d'')\Big)$.

◁

As an illustration, we note that in the case of ordinary functions $f : F \to F$, with $\mathfrak{F}$ not complete, continuity without strong continuity looks like in Figure 174.The missing point indicated in the figure can be e.g. $\sqrt{2}$ and $\mathfrak{F}$ may be the field of rationals. We then can define $f$ e.g. by $f(x) = 2$ if $x > \sqrt{2}$ and $f(x) = 1$ if $x < \sqrt{2}$. For *complete* fields "continuity"="strong continuity".

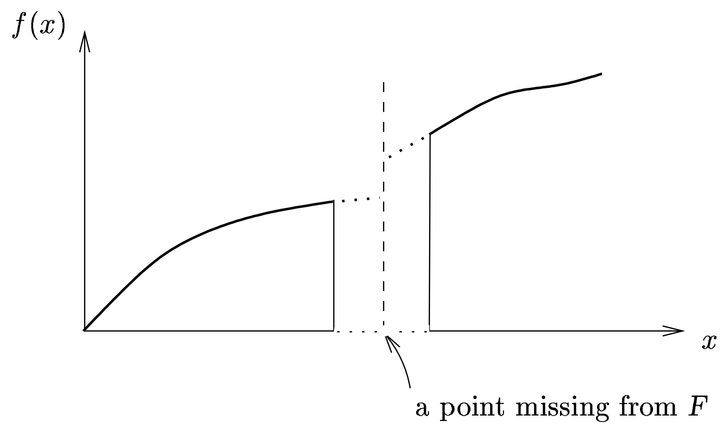



Figure 174: A continuous but not strongly continuous function

**Definition 4.4.9** Let

$$\mathbf{Bax}_+^{--} \overset{\text{def}}{=} \mathbf{Bax}^{--} \ + \ \text{postulates } \mathbf{Ax(i)} \text{ and } \mathbf{Ax(ii)} \ \underline{\textit{defined below}}.$$

◁

[447]As already indicated in a footnote for item (v), $ang^2(d, d')$ is the square of the (tangent of the) angle between vectors $d, d'$ defined in the spirit of the definition on p.46. Indeed, let $p, q \in {}^nF \setminus \{\bar{0}\}$. Let $H \overset{\text{def}}{=} \{\|p - \lambda \cdot q\| : \lambda \in {}^+F\}$. If $min(H)$ exists, we define $ang^2(p, q) \overset{\text{def}}{=} \min(H)$. Otherwise, $ang^2(p, q) \overset{\text{def}}{=} \infty$. (This definition does not "grasp" angles greater than $90^o$, but we do not need them. If later such "big" angles would be needed, the definition is easy to adjust.

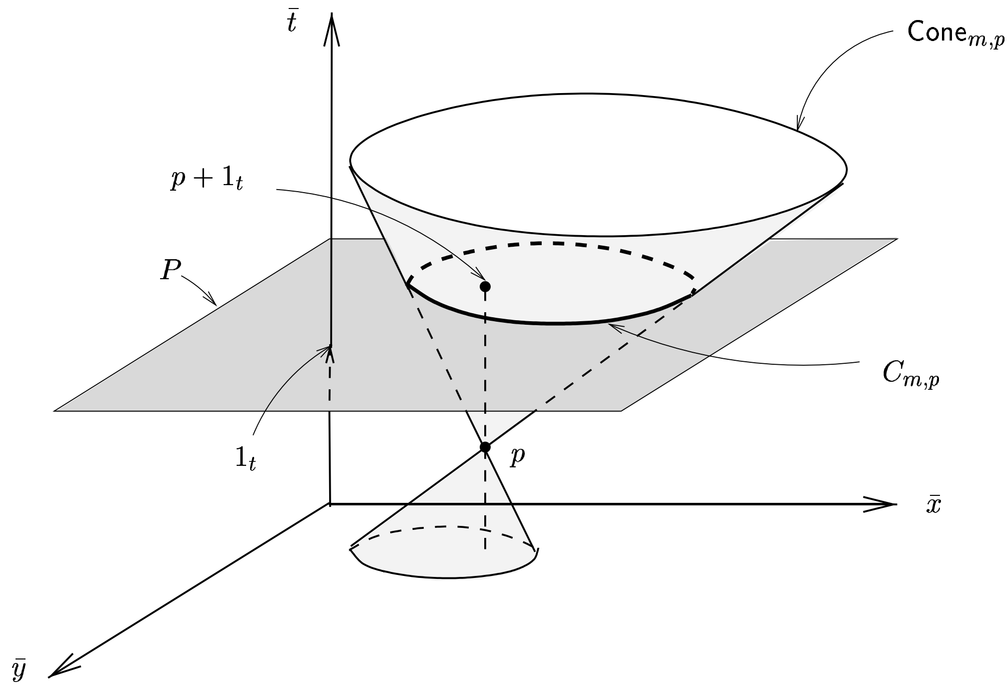



Figure 175: Illustration for **Ax(ii)**.

**Ax(i)** There exists a partial function $c_m$ satisfying $(*)$ above such that the following holds. $c_m(p,d)$ is a *strongly continuous* function of both its arguments ($p$ and $d$).

**Ax(ii)** First we formalize this condition for the case $n = 3$. Let $P$ be the plane $P := \{q + (1_t + p) \; : \; q \in S\}$. Let $C_{m,p} = P \cap \mathsf{Cone}_{m,p}$. Now, we first postulate that $C_{m,p}$ is *homeomorphic*[448] with a circle in plane $S$ which condition will be translated to our first-order frame language in Remark 4.4.10 below.[449] This homeomorphism is defined via the usual topology inherited from the space ${}^n\mathbf{F}$. Let us turn to the case of arbitrary $n$. Now, $C_{m,p}$ is defined as above and we postulate that $C_{m,p}$ is homeomorphic with the $(n-1)$-sphere $\{q \in S : |q| = 1\}$. See Figure 175. Further, we postulate that $(\exists p \in P)$

$$(\forall \text{ line } \ell \subseteq P)[p \in \ell \;\Rightarrow\; \ell \text{ intersects } C_{m,p} \text{ at most}^{450} \text{ in two points}].^{451}$$

By some accident, the above homeomorphism condition excludes the case when $c_m(p,d)$ is *infinite* in some (or all) directions $d$. We did *not* want to exclude this, but it would make the formulation of **Ax(ii)** too complicated to remove this undesirable side effect (one possibility would be to add the line (or a circle) of infinitely distant points to plane $P$ etc.) Anyway, instead of "formal manipulations" we informally declare here that we do want to allow $c_m(p,d) = \infty$. Indeed, the first-order version of **Ax(ii)** in Remark 4.4.10 below will allow $c_m(p,d) = \infty$ (and will still require that, in some sense, $C_{m,p}$ is a "closed curve" like a circle is).

**Remark 4.4.10 (First-order formulation or approximation of axiom Ax(ii) in $\mathbf{Bax}_+^{--}$)**

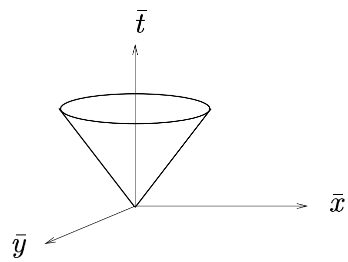
1. On the intuitive idea of the first-order formalization of **Ax(ii)**. If the light-cone contains $\bar{t}$ like this:

[448] By a homeomorphism (between two topological spaces) we understand a continuous bijection $h$ whose inverse $h^{-1}$ is also continuous.

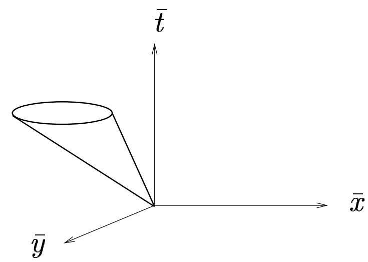
[449] This translation will be only an approximation which however seems to work in the most important situations.

[450] We write "at most two" instead of exactly two in order to allow the speed of light to be infinite in some directions.

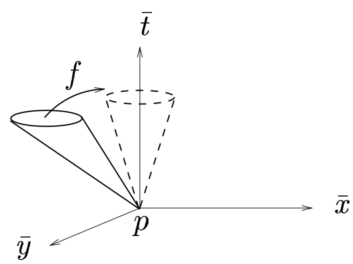
[451] If $\bar{t}$ is inside the light-cone, then this condition follows from $\mathbf{AxP1^-}$. But if $\bar{t}$ is outside then $\mathbf{AxP1^-}$ is not enough.




then it is relatively easy to formulate in first-order language what we want. However, if it is like in the case of a supersonic airplane, i.e. like this,




then formalizing what we want becomes harder. Therefore, we will *first* use an affine transformation $f$ which will rotate the light-cone around $p$ such that $\bar{t}$ gets inside the cone like this:




Then, whatever we wanted to say about the *shape* of the original $\mathsf{Cone}_{m,p}$, we will say about the $f$-image $f[\mathsf{Cone}_{m,p}]$ of the cone. Since affine transformations are homeomorphisms, if a topological property holds for the $f$-image of the cone then it will hold for the original cone too. Well, this is the intuitive idea, let us turn to implementing it.

2. First we observe, that in the definition of $c_m(p,d)$ we did not use the observer $m$ in itself, instead we used the set

$$\mathsf{Pth}_{m,p} := \{\, tr_m(ph) \;:\; ph \in Ph \quad \wedge \quad p \in tr_m(ph)\,\}$$

of life-lines of photons. Since $\mathsf{Cone}_{m,p} = \bigcup \mathsf{Pth}_{m,p}$, we can say that $\mathsf{Pth}_{m,p}$ is also the *light-cone* (at point $p$ as seen by $m$) but in a *different form*.

So, in defining $c_m(p,d)$ we used $\mathsf{Pth}_{m,p}$ (together with ${}^n\mathfrak{F}$). Let $f : {}^nF \longrightarrow {}^nF$ be an affine transformation leaving $p$ fixed. Then we define the binary function $c^f_m(p,d)$ the same way as $c_m(p,d)$ was defined, but now using the *f-image*

$$f[\mathsf{Pth}_{m,p}] := \{\, f[\ell] \ : \ \ell \in \mathsf{Pth}_{m,p}\}$$

*of the light-cone* $\mathsf{Pth}_{m,p}$ instead of the original $\mathsf{Pth}_{m,p}$. Let us notice that the speed $v_m(b)$ of an inertial body $b$ (as seen by $m$) was defined via the line $tr_m(b)$. Therefore we may speak (if we want) about the *speed of a line* $\ell$. Now, we define

$c^f_m(p,d) :=$
 $\max \{\, s \in F^\infty \ : \ s$ is the *speed* of line $\ell$ *in direction d* and $\ell \in f[\mathsf{Pth}_{m,p}] \,\}$.

3. The first-order version $\mathbf{Ax(ii)}^+$ of (the second-order) $\mathbf{Ax(ii)}$ says the following. Recall that the hyperplane

$$P = \{q + 1_t + p \ : \ q \in S\}$$

was defined in $\mathbf{Ax(ii)}$.

$\mathbf{Ax(ii)}^+$ $(\forall m)(\forall p)(\exists$ affine transformation $f$ of ${}^nF)[f(p) = p \ \wedge \ f[P] = P \ \wedge \ (\forall d \in \mathsf{directions})\ c^f_m(p,d)$ is defined $\wedge$ $c^f_m(p,d)$ (or more precisely $\langle c^f_m(p,d) \ : \ d \in \mathsf{directions}\rangle$) is a *strongly continuous*[452] function of $d$].[453] This completes the definition of the first-order approximation $\mathbf{Ax(ii)}^+$ of $\mathbf{Ax(ii)}$.

4. Now, the *purely first-order* formulation of

$\mathbf{Bax}^{--}_{+}$ is defined to be $\quad \mathbf{Bax}^{--} + \mathbf{Ax(i)} + \mathbf{Ax(ii)}^+$.

We did not check how well the first-order axiom $\mathbf{Ax(ii)}^+$ approximates the second-order one $\mathbf{Ax(ii)}$ i.e. how the first-order version of $\mathbf{Bax}^{--}_{+}$ approximates its original, second-order version. Throughout the discussion below we assume $\mathbf{Bax}^{--}_{+} \setminus \{\mathbf{Ax(ii)}\}$. We *conjecture* that if $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$ then $\mathbf{Ax(ii)}^+ \Leftrightarrow \mathbf{Ax(ii)}$. We also guess, that whenever $\mathfrak{F}^{\mathfrak{M}}$ is *complete* then $\mathbf{Ax(ii)}^+$ is a

[452]Strong continuity is defined in Def.4.4.8.

[453]We note that without the condition $f[P] = P$ we would get an interesting, very permissive version of $\mathbf{Ax(ii)}^+$ which can be considered as remotely similar to Swartzshild coordinatization of black-holes in general relativity.

reasonably good approximation of $\mathbf{Ax(ii)}$. We are not sure whether this approximation is good when $\mathfrak{F}^{\mathfrak{M}}$ is an arbitrary Euclidean field. (We mean, we did not check this).

5. $\mathbf{Bax}_{+}^{--}$ is our first axiom system which has a second-order version (using $\mathbf{Ax(ii)}$) and a first-order version (using $\mathbf{Ax(ii)}^{+}$). It remains a research task for the future to (i) figure out how well the first-order version of $\mathbf{Bax}_{+}^{--}$ approximates the second-order version, and (ii) to *improve* the first-order version of $\mathbf{Bax}_{+}^{--}$ such that it would become as close to the original intuition as possible.

   Following the rules of the game we set to ourselves in the present work, we should consider the first-order version of $\mathbf{Bax}_{+}^{--}$ the "official" one, and whenever we speak about $\mathbf{Bax}_{+}^{--}$ we should mean its first-order version (using $\mathbf{Ax(ii)}^{+}$). However, using the experimental character of the present section as an excuse, when speaking about $\mathbf{Bax}_{+}^{--}$, on the *intuitive level* of thought, we will have in mind the first (i.e. second-order) version. All the same, we will try to avoid explicitly stating theorems which would fail for "the official" i.e. first-order version of $\mathbf{Bax}_{+}^{--}$.

$\triangleleft$

**Conjecture 4.4.11**

**(i)** *In* $\mathbf{Bax}_{+}^{--}$ *something like a "light-cone" already exists. Indeed* $\mathsf{Cone}_{m,p}$ *can be visualized like a cone-like surface e.g. like in Figure 176.*[454]

**(ii)** $\mathbf{Bax}_{+}^{--} \models \{\mathsf{Cone}_{m,p}$ *solidifies into something like a cone-like surface. For* $n = 3$ *the horizontal intersections of this "cone" are closed curves but need not be circles or even ellipses.*$\}$
*It is left to the reader to formalize this statement for* $n > 3$.

$\triangleleft$

[454]For this conjecture we might (or might not) need to make some assumptions on $\mathfrak{F}^{\mathfrak{M}}$ (like completeness or being Archimedian).

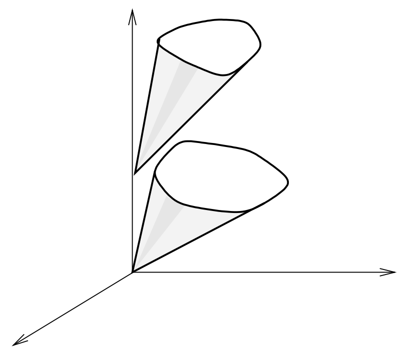


Figure 176: $\mathsf{Cone}_{m,p}$ can be visualized like a cone-like surface.

Let us strengthen $\mathbf{Bax}_{+}^{--}$ a little more:

**Definition 4.4.12**

$$\mathbf{Bax}_{++}^{--} \stackrel{\text{def}}{=} \mathbf{Bax}_{+}^{--} + \Big\{ c_m(p,d) \text{ does not depend on time, i.e.}$$
$$(\forall \Delta t \in \bar{t}\,)(c_m(p,d) \text{ is defined} \Rightarrow c_m(p,d) = c_m(p+\Delta t, d)) \Big\}.$$

◁

Now $\mathbf{Bax}_{++}^{--}$ implies that the speed of light going in direction $d$ is the same everywhere in the plane $\mathsf{Plane}(\bar{t}, d)$ determined by $\bar{t}$ and direction $d$.

**Definition 4.4.13** Let

$$\mathbf{Bax}(\mathbf{P1}) \stackrel{\text{def}}{=} \mathbf{Bax}_{++}^{--} + \Big\{ c_m(p,d) \text{ does not depend on } p, \text{ i.e.}$$
$$(c_m(p,d) \text{ is defined} \Rightarrow c_m(p,d) = c_m(p',d) \text{ for all } p,p' \in {}^nF) \Big\}.$$

◁

We note that

$$\mathbf{Bax}_{++}^{--} < \mathbf{Bax}(\mathbf{P1}) < \mathbf{Reich}(\mathbf{Bax}).$$

That is, our $\mathbf{Bax}^{--}_{++}$ is still compatible[455] with Reichenbachian relativity where our formalization of the latter will be introduced in §4.5 of this work (cf. Friedman [90, pp.165-176].)

We now show that $\mathbf{Bax}(\mathbf{P1})$ is weaker than $\mathbf{Bax}^-$ in the sense that while $\mathbf{Bax}^- + c_m(p,d) < \infty$ does not allow faster-than-light observers, $\mathbf{Bax}(\mathbf{P1}) + c_m(p,d) < \infty$ already allows the existence of faster-than-light observers. Thus, in a sense, Proposition 4.4.14 below shows why Theorem 4.3.24 is true. It implies further that the requirement that each observer sees photons in each direction moving forwards is quite a strong requirement.

**PROPOSITION 4.4.14** *Assume $n > 2$. No paradigmatic effects hold in $\mathbf{Bax}(\mathbf{P1}) + c_m(p,d) < \infty +$ the $\mathsf{f}_{mk}$'s are collineations. In particular,*

$$\mathbf{Bax}(\mathbf{P1}) + c_m(p,d) < \infty + \textit{the } \mathsf{f}_{mk}\textit{'s are collineations} \not\models \nexists \textit{ FTL observer.}$$

**Proof.** To show that the three basic paradigmatic effects – i.e. clocks slow down, meter-rods shrink, and events get out of synchronism – do not hold in $Th \stackrel{\text{def}}{=}$ "$\mathbf{Bax}(\mathbf{P1}) + c_m(p,d) < \infty +$ the $\mathsf{f}_{mk}$'s are collineations" , notice that the model (in which these effects fail) constructed in the proof of Thm.4.3.21 is also a model of $Th$ if we choose the parameter $K$ to be e.g. a sphere. The same idea shows that $\mathbf{Ax(symm)} \to \mathbf{Ax(\exists TwP)}$ fails in $Th$ (but using the proof of Thm.4.3.22 in place of Thm.4.3.21).

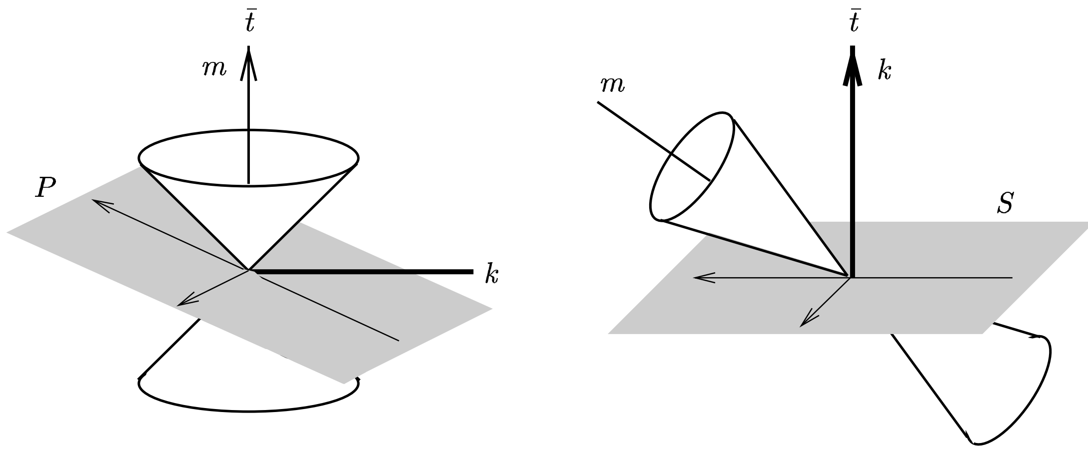
To show that the existence of faster-than-light observers is compatible with $Th$, we have to give a separate proof, since in this respect $Th$ behaves differently from $\mathbf{Bax}^- + c_m(p,d) < \infty$ (cf. Thm.4.3.24).

Let $\mathfrak{F} \stackrel{\text{def}}{=} \mathfrak{R}$,
$B \stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} \mathsf{Eucl}$,
$Ph \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} : ang^2(\ell) = 1\}$,
$Obs \stackrel{\text{def}}{=} B \setminus Ph$.

As usual, let $m \stackrel{\text{def}}{=} \bar{t}$ and define the world-view $w_m$ of $m$ such that $tr_m(b) = b$ for all $b \in B$. For any $k \in Obs$ we now define $\mathsf{f}_{mk}$. Let $P$ be any "space-like" hyper-plane, i.e. such that $(\forall ph \in Ph) ph \not\subseteq P$; and also $k \not\subseteq P$. We now choose $\mathsf{f}_{mk}$ to be a collineation that takes $S$ to $P$ and $\bar{t}$ to $k$. See Figure 177.

Then $\mathsf{f}_{mk}$ defines the world-view of $k$, and it is not difficult to check that the relevant axioms of $\mathbf{Bax}(\mathbf{P1})$ are true in $w_k$. Let $\mathfrak{M} \stackrel{\text{def}}{=} \langle (B, Obs, Ph, Ib), \mathfrak{R}, \mathsf{Eucl}(n, \mathbf{F}), \epsilon, w_m \rangle_{m \in Obs}$. It is not difficult to check that $\mathfrak{M} \models Th$, and there are faster than light observers in $\mathfrak{M}$. ∎

[455] in the sense that $\mathbf{Reich}(\mathbf{Bax})$ is a special case of $\mathbf{Bax}^{--}_{++}$.




$m$'s world-view $\qquad$ $k$'s world-view

Figure 177: Illustration for the proof of Proposition 4.4.14.

We consider $\mathbf{Bax(P1)}$ and $\mathbf{Bax}^{--}_{++}$ as the very "faithfully" (or carefully) formalized[456] counterparts of Friedman's (P1), while we consider our $\mathbf{Bax}^{-}$ as a more "pragmatically" formalized version of (P1). It seems that $\mathbf{Bax}^{-}$ is an adequate formalization of Friedman's (P1) for the purposes of the present work, at least for a first systematization of the subject. For a future, second (or third) refined systematization, the more "finicky" formalizations $\mathbf{Bax(P1)}$ and $\mathbf{Bax}^{--}_{++}$ will probably prove useful.

For the present investigation (for the time being) we will stick with $\mathbf{Bax}^{-}$ and we will treat $\mathbf{Bax}^{--}$, $\mathbf{Bax(P1)}$, $\mathbf{Bax}^{--}_{++}$ as potential future research subjects.

In the table below (i.e. in Figure 178), we summarize our hierarchy

$$\mathbf{Bax}^{--} < \mathbf{Bax}^{-} < \mathbf{Bax} < \mathbf{Newbasax}$$

of theories from the point of view of the shape of light-cones and $c_m(p,d)$. Explanation for the table: Throughout the table $n = 3$. We are looking at a light-cone starting out from point $p$ of space-time. I.e. we are looking at $\mathsf{Cone}_{m,p}$. In column 3, we see the intersection of the light-cone with the "simultaneity" i.e. plane $P = S + (p + 1_t)$, cf. Def.4.4.9 and Figure 175. The "circle" is the intersection of the

[456]if we disregard the fact that these axiom systems are in an experimental stage only and they are not polished.

cone with the plane $P$, i.e. it is $C_{m,p}$, in other words it is the set of intersections of photons (of finite speeds) starting from $p$ with plane $P$. The shaded area is the set of intersection points of observers going through point $p$ (with $P$). As one can see, in the first three rows, there is no shaded area outside the circles; this fact is caused by our "no FTL observers" theorems. The "fat" point represents the intersection (with $P$) of the coordinate-line parallel with $\bar{t}$ and going through $p$. In the 3-rd and 4-th rows ($\mathbf{Bax}^-$, $\mathbf{Bax}^{--}$) there are more than one figures some further up and some lower down. The upward figures represent the "ideal" cases we have in mind while the downward figures represent pathological cases which *might*[457] not be excluded by the axioms.

We note that Thm.4.3.29 says that the shaded area in the $\mathbf{Bax}^-$-row is a convex open set with a continuous boundary.

Figure 179 is a continuation of Figure 178. But while Figure 178 contains 3 columns, Figure 179 is a continuation only of the middle column of Figure 178. (The reason is that we have more axiom systems in Figure 179 than in Figure 178, therefore we had to "economize" somewhere. The interested reader is invited to restore the missing columns.)

**Questions for future research 4.4.15**

**(i)** Investigate the axiom system $\mathbf{Bax}_+^{--}$ in the same spirit as we investigated e.g. **Basax**. Is $\mathbf{Bax}_+^{--} \models (\mathsf{f}_{mk}$ preserves Euclidean lines) true? What are the models of $\mathbf{Bax}_+^{--}$ like? Do light-cones at each point look the same in them, i.e. is it true that if a straight line $\ell$ is parallel with a trace of a photon, then $\ell$ itself is a trace of a photon (in an observer's world-view)? Is $\mathbf{Bax}_+^{--} \models (\ell \parallel tr_m(k) \Rightarrow (\exists k' \in Obs)\ell = tr_m(k'))$ true? What are the models of $\mathbf{Bax}_+^{--} + \mathbf{Ax(symm)}$ like? Compare $\mathbf{Bax}_+^{--}$ with our other weak axiom systems like $\mathbf{Bax}^-$ or **Relnoph**.

**(ii)** Elaborate and investigate **Bax(P1)** in the same spirit.

**(iii)** Do the above for all axiom systems introduced in this section.

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[457] Perhaps some of these are excluded but we did not prove it yet.

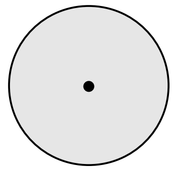
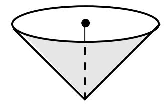
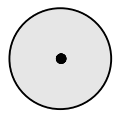
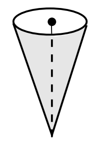
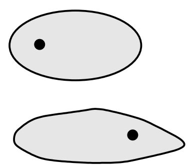
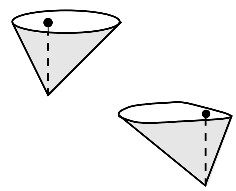
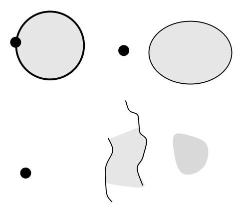
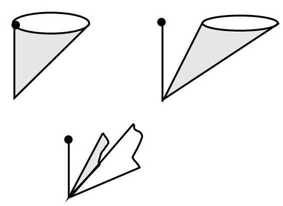
| | | | |
|---|---|---|---|
| **Newbasax** | $c_m(p,d) = 1$ |  |  |
| **Bax** | $c_m(p,d) = c_m$ |  |  |
| **Bax$^-$** | $c_m(p,d) > 0$ |  |  |
| **Bax$^{--}$** and versions | $c_m(p,d)$ ($\leq 0$ allowed) |  |  |

Figure 178: Shapes of light-cones in our theories.

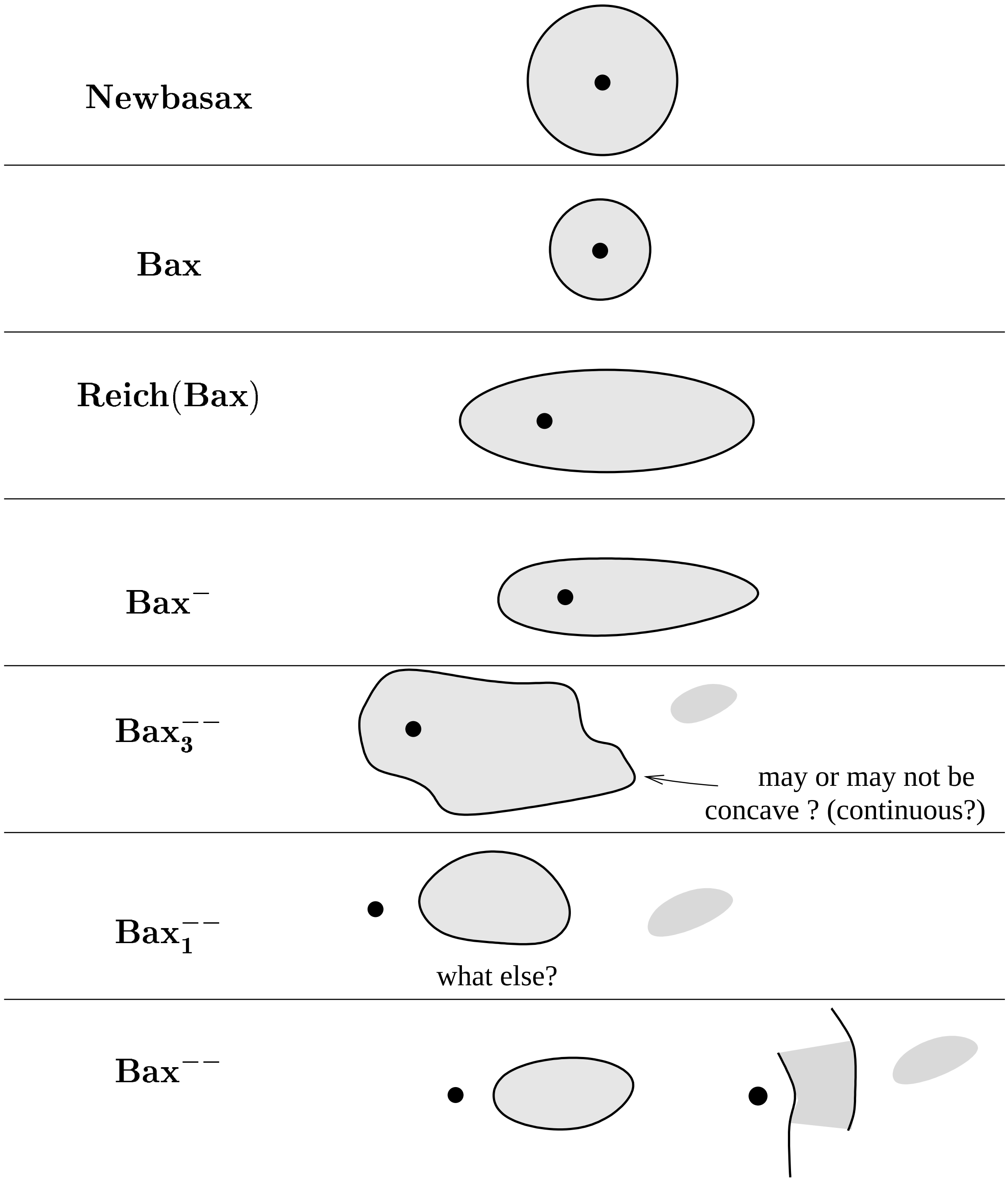



Figure 179: A continuation of Figure 178, shapes of light-cones in more of our theories.

**On Figure 180:** We collected (most of) the axiom systems introduced so far in a lattice shown in Figure 180. We call a theory the stronger the more theorems it proves. In the lattice, the theories further up are stronger ones, and the theories lower down are weaker ones. This lattice represents a partial order on the theories. Namely, if in the lattice $Th_1$ and $Th_2$ are connected with a line and $Th_2$ is further up than $Th_1$, then this means that $Th_2 \geq Th_1$ in the sense that

$$Th_2 + c_m(p,d) < \infty + \mathbf{Ax}(\surd) \models Th_1.$$

See also the part on the lattice of theories beginning with p.451. Using the terminology of that part, if we add $c_m(p,d) < \infty + \mathbf{Ax}(\surd)$ to all the theories represented in Figure 180, then we get a subposet of the poset $\mathbb{TH}_0$ of all theories.

In the lattice we indicated one theory, $\mathbf{Reich}(\mathbf{Bax})_\partial$, which will be introduced later. We indicated this theory because of its central place in the lattice. In drawing the lattice in Figure 180 we pretended that we know that $\mathbf{Reich}(\mathbf{Bax})_\partial + c_m(p,d) < \infty \models \mathbf{Bax(P1)}$. The only questionable part of this seems to be

$$(\star) \qquad \mathbf{Reich}(\mathbf{Bax})_\partial + c_m(p,d) < \infty \models \mathbf{Ax(ii)}^+$$

where $\mathbf{Ax(ii)}^+$ is one of the axioms of $\mathbf{Bax}_+^{--}$ (on p.538). We do not know whether $(\star)$ holds, however, the following is our excuse. When formulating $\mathbf{Ax(ii)}^+$, we intended to express a property of the light-cone which, in our opinion, is implied by the axiom $\mathbf{Ax(consm)}$ of $\mathbf{Reich}(\mathbf{Bax})_\partial$ (together with the rest of the axioms of $\mathbf{Reich}(\mathbf{Bax})_\partial$). However, to formulate the intuitive idea of $\mathbf{Ax(ii)}^+$ in first-order language consizely required some compromises. Therefore, as an accident, now, $\mathbf{Ax(ii)}^+$ might be slightly stronger than what follows from $\mathbf{Reich}(\mathbf{Bax})_\partial + c_m(p,d) < \infty$ ; but then this is something which has to be "smoothed out" in the future. Anyway, we guess that the original form $\mathbf{Ax(ii)}$ of this axiom will follow from $\mathbf{Reich}(\mathbf{Bax})_\partial + c_m(p,d) < \infty$ (even if something would go wrong with the present form of $\mathbf{Ax(ii)}^+$).

**On answering the "why" type questions discussed in the introduction:** We now indicate how we can use our poset of theories in answering why-type questions. As an example, we will use paradigmatic effects. If we take a paradigmatic effect, e.g. that "moving clocks slow down" and indicate on the poset in Figure 180 which theories prove this effect and which do not, then with this we made a first step in answering "*why* the paradigmatic effect in question is true". This is what we do in §4.8, see Figure 223 on p.653. Such an answer can have slightly more "structure" to it than simply figuring out which axiom (or axioms) of **Newbasax** (or of **Newbasax**+

**Ax(symm)**) is responsible for the effect in question. We note that for certain effects (i.e. for certain predictions of relativity) this kind of search for an answer to the "why" question may lead to enriching the poset on Figure 180 with new theories. We also note that such an answer might be "multi-dimensional" in the sense that it may say that the effect in question becomes provable iff we are both strictly above, say, $\mathbf{Bax}^-$ *and* strictly to the right from, say, **Flxbasax**.[458]

**Questions for future research 4.4.16** (in connection with Figure 180)

**(i)** Include the remaining axiom systems for relativity studied in this work into the hierarchy in Figure 180.

**(ii)** In Figure 180 the symbol "$\not\equiv$" means that we know that the axiom systems involved are not equivalent, while "$? \equiv$" indicate that it could be interesting to check whether they are equivalent (but we did not have time to think about these).

**(iii)** Which ones of our theories in Figure 180 are "blurred" into each other by adding **Ax(symm)** to them? In other words, what does the lattice in Figure 180 look like if to each theory in it we add **Ax(symm)**? Are there $Th_1$, $Th_2$ in the hierarchy in Figure 180 such that $Th_1 \not\equiv Th_2$ but $Th_1 + \mathbf{Ax(symm)} \equiv Th_2 + \mathbf{Ax(symm)}$? Or if they do not become completely equivalent, do then $Th_1 + \mathbf{Ax(symm)}$ and $Th_2 + \mathbf{Ax(symm)}$ become very close to each other, in some sense?
Which theories imply $\mathbf{Ax(symm)} \rightarrow \mathbf{Ax(\exists TwP)}$? What is the situation with other symmetry axioms? In this connection see §4.2.

◁

[458]We deliberately did not make it precise what we mean by being to the right from some element of the poset.

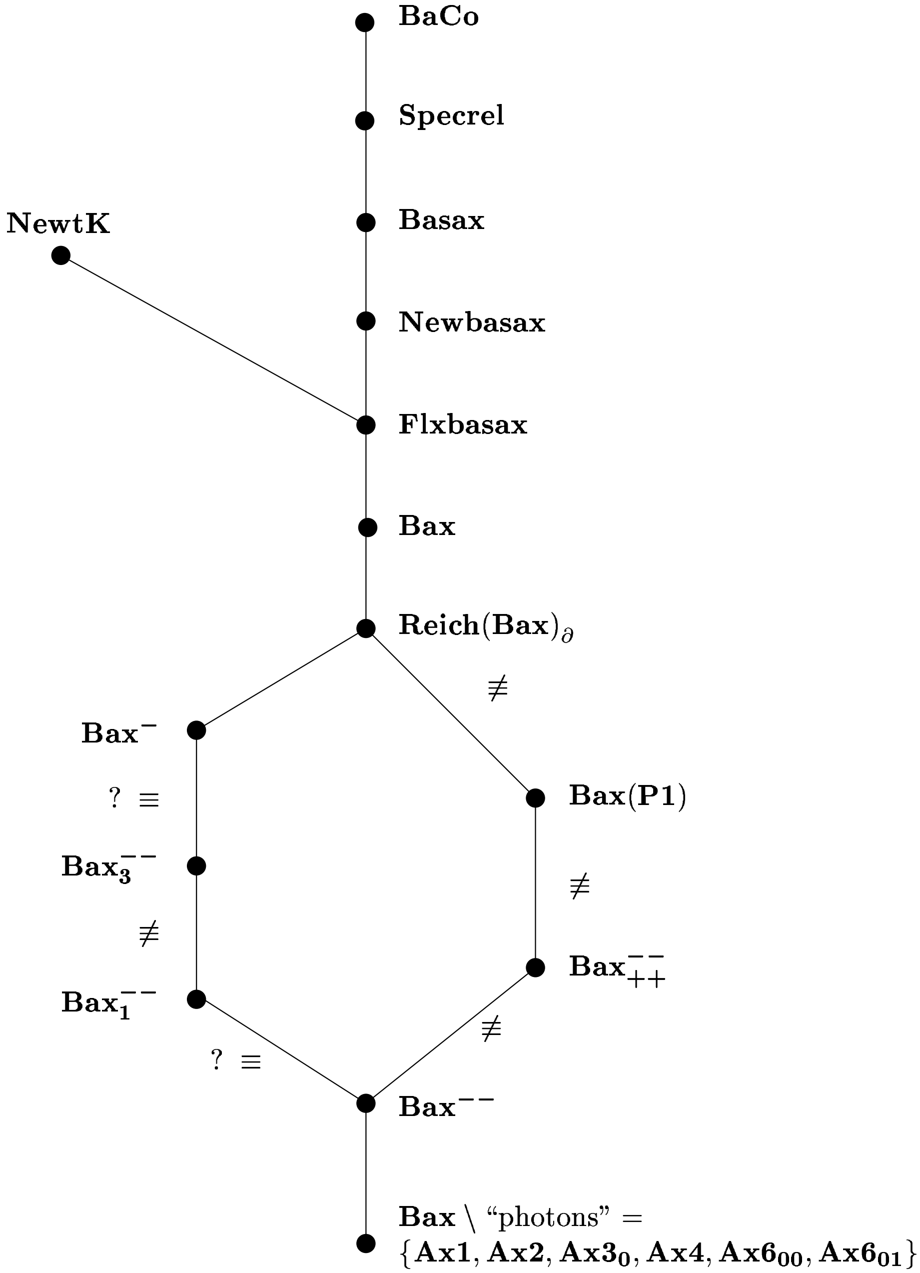



Figure 180: The lattice of our theories introduced so far, where $Th_1 < Th_2$ means $Th_2 + c_m(p, d) < \infty + \mathbf{Ax}(\sqrt{\ }) \models Th_1$. Some parts of this lattice represent conjectures only (while others are theorems).

## 4.5 Reichenbachian version of relativity (nonstandard simultaneities)

What we call the Reichenbachian version of (special) relativity theory starts out with the idea that our speed of light axiom **AxE** (or even $\mathbf{AxE_{00}}$) has *never* been *confirmed by experiment*. Moreover, Reichenbach, Grünbaum, Salmon and others argue that for logical or philosophical reasons, **AxE** cannot be tested by experiment (not even in principle). The idea is the following.

We can send a photon to the *Moon*, bounce it back with a mirror and measure the time when it arrives back to the *Earth* by a clock on the *Earth*. What we are measuring this way is the time needed for the *round-trip Earth* $\mapsto$ *Moon* $\mapsto$ *Earth*. Knowing the spatial distance between the *Earth* and the *Moon*, we can compute the *average speed* $c$ of the photon during the *Earth* $\mapsto$ *Moon* $\mapsto$ *Earth* round trip. However, knowing the average speed does *not* tell us what the speed of the *Earth* $\mapsto$ *Moon* trip was. In principle, it is possible that during the *Earth* $\mapsto$ *Moon* trip the photon went faster than $c$ while backwards (during *Moon* $\mapsto$ *Earth*) it came slower than $c$. In other words, knowing that the time (duration) of the *Earth* $\mapsto$ *Moon* $\mapsto$ *Earth* round trip was $\Delta t$ does not imply that the time of the *one-way* trip *Earth* $\mapsto$ *Moon* was $\Delta t/2$. Possibly, *Earth* $\mapsto$ *Moon* lasted for $\Delta t/2 - \varepsilon$ while *Moon* $\mapsto$ *Earth* lasted for $\Delta t/2 + \varepsilon$.

We could exclude this uncertainty by putting a clock on the *Moon* and synchronizing it with the one on the *Earth* etc. *However*, to synchronize the *Moon*-clock with the *Earth*-clock, the only reasonable idea seems to be to use light signals which in turn amounts to *assuming* that we know something about the *one-way* (e.g. *Earth* $\mapsto$ *Moon*) speed of light. But the point is that it is exactly this one-way speed what we are trying to measure, hence we *cannot assume* that we know it *before* measuring it.

For simplicity, we will use the expression "*two-way speed*" for the average speed of the round-trip like the *Earth* $\mapsto$ *Moon* $\mapsto$ *Earth* round-trip. Given a spatial direction $d$, by the "two-way speed of light in direction $d$" we will mean the average speed of an *Earth* $\mapsto$ *Moon* $\mapsto$ *Earth* style round-trip of a photon where the spatial line connecting *Earth* and *Moon* is parallel with $d$.

As we indicated above, Reichenbach, Grünbaum and their followers argue that only the two-way speed of light is subject to experiment. The debate started more than 50 years ago, and in 1977 Salmon [233] collected and investigated the ideas that came up in the meantime to measure the one-way speed of light, and he concludes that none of them seems to work. (In particular, this applies to the Michelson-Morley experiment too, of course.) Cf. also L. E. Szabó [244].

The above sketched ideas and developments led to the fact that today there exists a fairly broad and well established direction in the literature which discusses/promotes what we call the Reichenbachian version of (special) relativity, cf. e.g. Salmon [233], Friedman [90, p.165-320], Winnie [275], L. E. Szabó [244] where further references can be found.[459]

The key idea in the Reichenbachian version is that in **AxE** (or $\mathbf{AxE_{00}}$) we should speak about the *two-way* speeds of photons only and keep silent about their one-way speeds. Now, we can utilize the "lego-character" or "modular character" of the logic approach to relativity by simply "pulling out" the axiom **AxE** (or its variants like $\mathbf{AxE_0}$) from our theories and "plugging into their place" a version of **AxE** which mentions only two-way speeds (cf. Remark 4.5.5). This way we can obtain a theory $\mathbf{Reich}_0(\mathbf{Basax})$ from **Basax**, $\mathbf{Reich}_0(\mathbf{Bax})$ from **Bax** and in general $\mathbf{Reich}_0(Th)$ from theory *Th* where *Th* is any one of our relativity theories (summarized in Figure 180).[460] We will call $\mathbf{Reich}_0(Th)$ the *Reichenbachian version* of the (special) relativity theory *Th*.

There is a further motivation, *independent* of philosophical considerations like the "Reichenbach-Grünbaum approach" outlined above. Namely, for studying the world-views (or coordinate-frames) of so called rotating observers [cf. our chapter on accelerated observers], one needs a theory about inertial observers without gravitation (i.e. one needs a special relativity) which is like $\mathbf{Reich}_0(\mathbf{Basax})$ as opposed to being like **Basax**. In particular, concerning the inertial observer which is co-moving with (a spatial point of) the accelerated one, the *one-way* speed of light as measured by the inertial observer $k$ has to be allowed to be *nonstandard*. I.e. our **AxE** will be violated while its Reichenbachian version $\mathbf{R}(\mathbf{AxE})$ will remain valid. Cf. Matolcsi [191] for more on this.

Throughout this section we will heavily rely on $\mathbf{Ax}(\sqrt{\ })$. The reason for this is that some definitions will be much simpler and much more intuitive with using $\mathbf{Ax}(\sqrt{\ })$ than without using $\mathbf{Ax}(\sqrt{\ })$. E.g. Def. 4.5.1 is even meaningless without

[459]The distinction between *one-way speed* and *two-way speed of light* and its connections with what we call "nonstandard simultaneities" or "artificial simultaneities" below is also discussed in Matolcsi [190] and in the references therein. (Matolcsi uses the expression "nonstandard synchronization" for what is called a "nonstandard simultaneity" in e.g. Friedman [90].)

[460]More precisely, we will do this only for those choices of *Th* where $Th \geq \mathbf{Bax}$. The subscript "0" in $\mathbf{Reich}_0(Th)$ indicates that we will have a "fuller" Reichenbachian version $\mathbf{Reich}(Th)$ of *Th*.

$\mathbf{Ax}(\sqrt{\ })$. For this reason, we will include $\mathbf{Ax}(\sqrt{\ })$ into the definition of our Reichenbachian theories (cf. the definitions of $\mathbf{R}(\mathbf{AxE}) - \mathbf{R}(\mathbf{AxE_{00}})$). Since we did not include $\mathbf{Ax}(\sqrt{\ })$ into our earlier theories, e.g. we did not include $\mathbf{Ax}(\sqrt{\ })$ into **Basax**, this will entail that in many places $\mathbf{Ax}(\sqrt{\ })$ will "pop up", cf. e.g. Def.4.5.7. However, we could have completely avoided the use of $\mathbf{Ax}(\sqrt{\ })$ in this section (on the expense of simplicity). In Remark 4.5.2 we show how to formulate our axioms, and thus our Reichenbachian theories, without using $\mathbf{Ax}(\sqrt{\ })$. The general philosophy behind all this is that in the present section §4 we want to concentrate on the case when $\mathbf{Ax}(\sqrt{\ })$ is assumed everywhere (to avoid generating too much "side-tracking"). At first reading, the reader is invited to assume $\mathbf{Ax}(\sqrt{\ })$ at the beginning of the present section and then to *ignore* all references to $\mathbf{Ax}(\sqrt{\ })$ (in this section).

Let us turn to obtaining the Reichenbachian version $\mathbf{R}(\mathbf{AxE})$ of **AxE**. Assume $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })$. Recall that $\mathsf{c}_m(d)$ is the square of the speed of light in direction $d$. Since "speed = distance/time", for discussing round-trips it will be more convenient to use its reciprocal "1/ speed = time/distance".

**Notation 4.5.1** Let $m \in Obs$ and $d \in \mathsf{directions}$. Then

$$T_m(d) \overset{\text{def}}{=} \begin{cases} 1/\sqrt{\mathsf{c}_m(d)} & \text{if } 0 \neq \mathsf{c}_m(d) < \infty, \\ \infty & \text{if } \mathsf{c}_m(d) = 0,^{461} \\ 0 & \text{if } \mathsf{c}_m(d) = \infty. \end{cases}$$

◁

Since we assume $\mathbf{Ax}(\sqrt{\ })$, $T_m(d)$ exists. Intuitively, $T_m(d)$ is the time needed for a photon moving in direction $d$ to cover a distance of unit length. $T_m(d)$ is illustrated in Figure 181.

Now, we are ready to define the Reichenbachian versions $\mathbf{R}(\mathbf{AxE})$, $\mathbf{R}(\mathbf{AxE_{00}})$ etc. of our <u>*speed of light axioms*</u>.

To simplify the discussion, we treat the speed of light axioms of **Basax** and **Newbasax** in a unified way. This is justified by noting that

$$(\mathbf{Basax} \setminus \{\mathbf{AxE}\}) + (\mathbf{AxE_0}) \models \mathbf{Basax}.$$

[461] In $\mathbf{Bax}^-$ we have $c_m(d) \neq 0$.

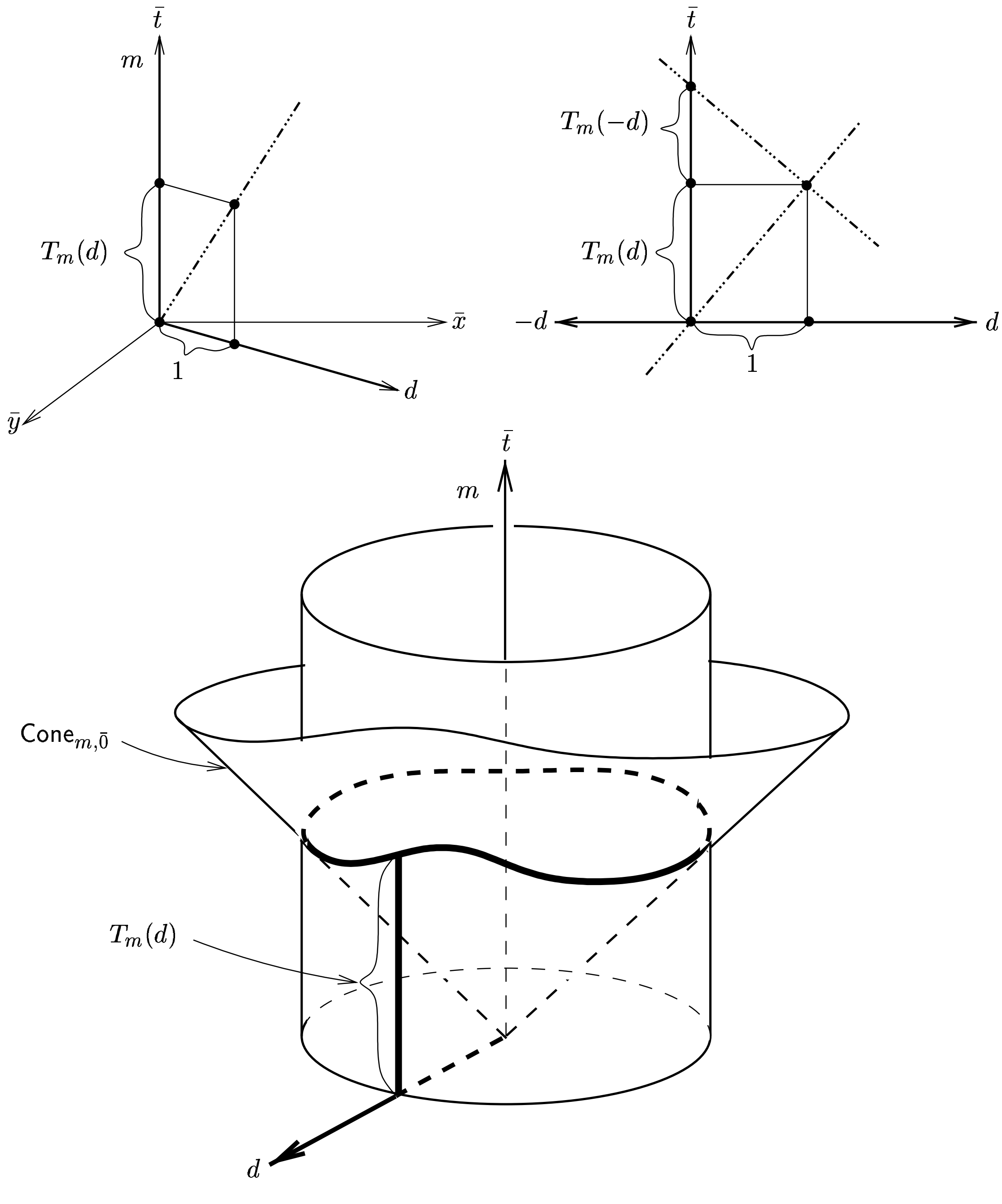



Figure 181: The function $T_m(d)$. Consider the intersection of the light-cone $\mathsf{Cone}_{m,\bar{0}}$ with the cylinder of radius 1 around $\bar{t}$. $T_m(d)$ is the height of this intersection curve in direction $d$.

$\mathbf{R}(\mathbf{AxE})$, which is defined to be the same as $\mathbf{R}(\mathbf{AxE_0})$, says the following.

$\mathbf{Ax}(\sqrt{\ })$ and $(\forall m \in Obs)(\forall d \in \mathsf{directions})T_m(d) + T_m(-d) = 2.$

Intuitively, the time $T_m(d)$ needed for a photon to cover a distance of length 1 together with the time $T_m(-d)$ to come back is 2. Hence the average speed of the photon-round-trip is 1. Figures 183 and 184 illustrate $\mathbf{R}(\mathbf{AxE})$.

Let us turn to the speed of light axiom $\mathbf{AxE_{02}}$ of **Flxbasax** (cf. p.428).

$\mathbf{R}(\mathbf{AxE_{02}})$

$(\forall m, k \in Obs)(\forall d, d_1 \in \mathsf{directions})T_m(d) + T_m(-d) = T_k(d_1) + T_k(-d_1)$, and $\mathbf{Ax}(\sqrt{\ })$.

Intuitively, for all observers and all directions, the round-trip covering a distance of length 1 takes the same time. Clearly, this is equivalent with saying that the average speed of a photon-round-trip is the same for all observers and all directions.

The speed-of-light axiom $\mathbf{AxE_{00}}$ of **Bax** is "Reichenbachized" as follows:

$\mathbf{R}(\mathbf{AxE_{00}})$

$(\forall d, d_1 \in \mathsf{directions})T_m(d) + T_m(-d) = T_m(d_1) + T_m(-d_1)$ and $\mathbf{Ax}(\sqrt{\ })$.

Intuitively, for any observer $m$, the two-way speed of light is isotropic in the sense that it is the same in all directions.

**Remark 4.5.2 (Reichenbachian versions without $\mathbf{Ax}(\sqrt{\ })$)** Axioms $\mathbf{R}(\mathbf{AxE})$ – $\mathbf{R}(\mathbf{AxE_{00}})$ above can be formulated without assuming $\mathbf{Ax}(\sqrt{\ })$. We denote the $\mathbf{Ax}(\sqrt{\ })$-free version of $\mathbf{R}(\ldots)$ with $\mathbf{R}(\ldots)^-$.

$\mathbf{R}(\mathbf{AxE})^-$

$\forall m(\forall ph, ph_1 \in Ph)(\forall p, q)\Big([p \notin \bar{t} \wedge q \in \bar{t} \wedge \overline{\bar{0}p} = tr_m(ph) \wedge \overline{pq} = tr_m(ph_1)] \Rightarrow \|q\|/\|space(p)\| = 4\Big).$

See Figure 182. Let us notice that $\|q\| = (\Delta t)^2$ and $\|space(p)\| = (\Delta s)^2$, hence what the axiom says is that $(\Delta t)^2/(\Delta s)^2 = (\Delta t/\Delta s)^2 =$
$(1/\text{velocity}_1 + 1/\text{velocity}_2)^2 = (2/\text{averagevelocity})^2 = (2/1)^2 = 4.$

Similarly,

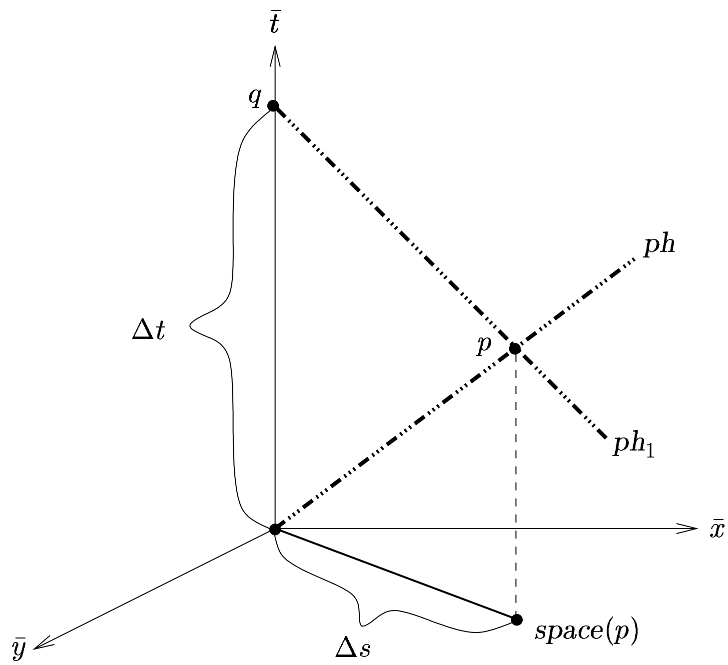



Figure 182: Illustration for $\mathbf{R}(\mathbf{AxE})^{-}$.

$\mathbf{R(AxE_{02})^-}$

$$(\exists r \in F)\forall m(\forall ph, ph_1 \in Ph)$$
$$\Big([p \notin \bar{t} \ni q \ \wedge\ \overline{\bar{0}p} \in tr_m(ph) \ \wedge\ \overline{pq} = tr_m(ph_1)] \ \Rightarrow\ \|q\|/\|space(p)\| = r\Big).$$

$\mathbf{R(AxE_{00})^-}$ is formulated analogously. We leave the details to the interested reader. ◁

**On the visual meaning of R(AxE).** **R(AxE)** says that if we first take the intersection of the two light-cones $\mathsf{Cone}_{m,\bar{0}}$ and $\mathsf{Cone}_{m,(2,0,\dots)}$, and then project the so obtained set to the space-part $S$, then we obtain the sphere of radius one and center $\bar{0}$. See Figure 183. $\mathbf{R(AxE_{00})}$ says that the projection (to $S$) of the intersection of the two light-cones is a sphere, but perhaps with radius different than 1, $\mathbf{R(AxE_{02})}$ permits that the radii of these spheres in the different world-views be different.

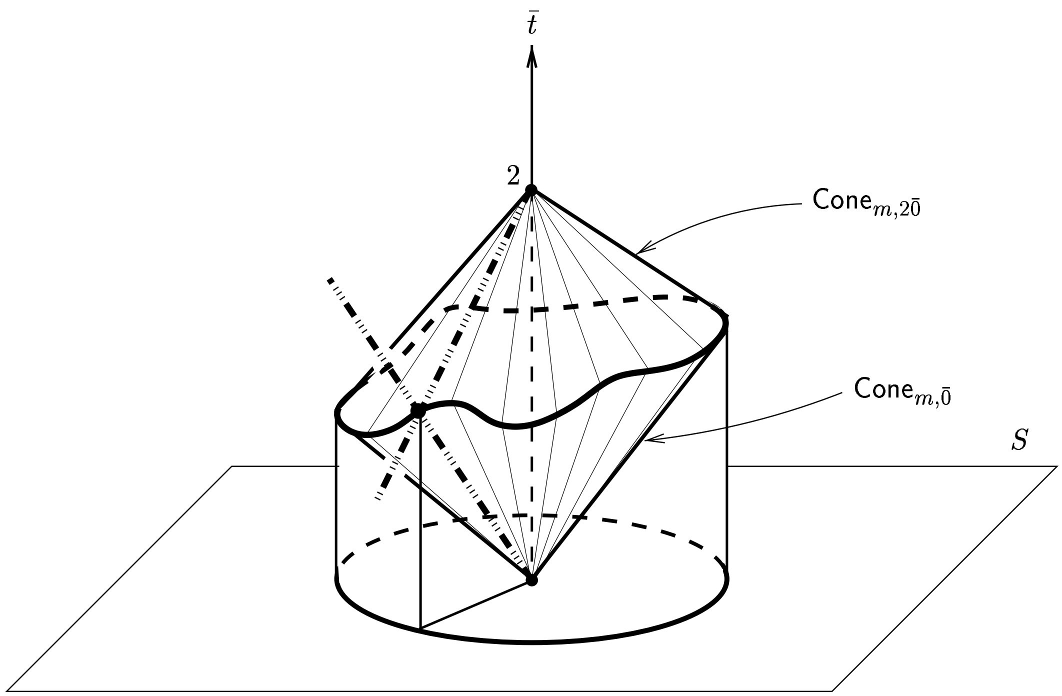



Figure 183: Illustration of **R(AxE)**.

Another "visual formulation" of **R(AxE)** says that the intersection of the light-cone $\mathsf{Cone}_{m,\bar{0}}$ with the cylinder $\mathcal{C}$ of radius 1 around $\bar{t}$ consists of two "curves" which are translations of each other (i.e. in some sense they are parallels), the translation is made by a vector of length 2 and parallel to $\bar{t}$. See Figure 184.

If we do not assume $\mathbf{Ax}(\sqrt{\ })$, then the intersection $\mathcal{T}$ of the cone $\mathsf{Cone}_{m,\bar{0}}$ and the cylinder $\mathcal{C}$ may be "partial" in the sense that $\mathcal{T}$ need not contain a point in each direction (i.e. on each plane containing $\bar{t}$). However, the $\mathbf{Ax}(\sqrt{\ })$-free formulation $\mathbf{R}(\mathbf{AxE})^-$ can be visually formulated, too: $\mathbf{R}(\mathbf{AxE})^-$ says that *for all $r \in F^+$*, if we project the (earlier discussed) intersection $\mathcal{T}_r$ of the light-cone $\mathsf{Cone}_{m,\bar{0}}$ with the light-cone $\mathsf{Cone}_{m,(2r,0,\dots)}$, then we get the sphere of radius $r$ around $\bar{0}$.

**On the physical meaning of $\mathbf{R}(\mathbf{AxE})$.** Let us put a sphere around $\bar{0}$ with radius 1, such that the inside surface of this sphere is a reflecting one. Let us switch on a light at $\bar{0}$ for a very short time. $\mathbf{R}(\mathbf{AxE})$ postulates that all the photons (i.e. the photons from all directions) will arrive back (from the inside surface of the sphere) at time instance 2. Thus, *optically* (i.e. via photons) the observer will *see* that after switching on the light, for 2 minutes nothing happens, and then at time instance 2 the whole sphere flares up (lightens up). On the other hand, if the photon-sphere $\mathbb{C}_m$ is not the sphere of radius 1 and with center $1_t$ – we will see later that this is possible – then the observer will "think" (or "observe") that the photons reached the inside of the sphere at different time-instances. Figure 185 illustrates this on a concrete example ($n = 3$): In the figure, according to $m$'s world-view, the photon that reaches $\mathcal{C}$ earliest is the one that reaches it at point A, then come the two "neighbours" of A, and so on, while the photon that reaches $\mathcal{C}$ latest reaches it at point B, and by this time all the other photons have reached $\mathcal{C}$. Thus, according to this world-view, $m$ will "think" (or "observe") that first A flares up, then two luminous points go left and right from A on the circle, they both reach B at the same time, and then these two luminous points blink out. But, as we said above, *optically* $m$ will see that the points of $\mathcal{C}$ flare up at the same time, because all the photons come back to $m$'s eye at the same time (at time instance 2). This illustrates that "seeing via photons" and "observing via the world-views" are different things.[462]

We note that in the above formulations of $\mathbf{R}(\mathbf{AxE})^-$ etc. we concentrated on photons "emitted" at the origin $\bar{0}$ because we will use these axioms together with $\mathbf{Bax}^-$, and we already saw (in the section devoted to $\mathbf{Bax}^-$) that the speed $c_m(p, d)$ of light does not depend on the point $p$ where the photon is "emitted". Now we can turn to defining the Reichenbachian versions $\mathbf{Reich}_0(Th)$ of our theories $Th$.

[462] The Reichenbachian philosophy can be interpreted as saying that we should take seriously only the facts "observable via photons". Cf. also the definition of our later "view-function", Def.4.7.5.

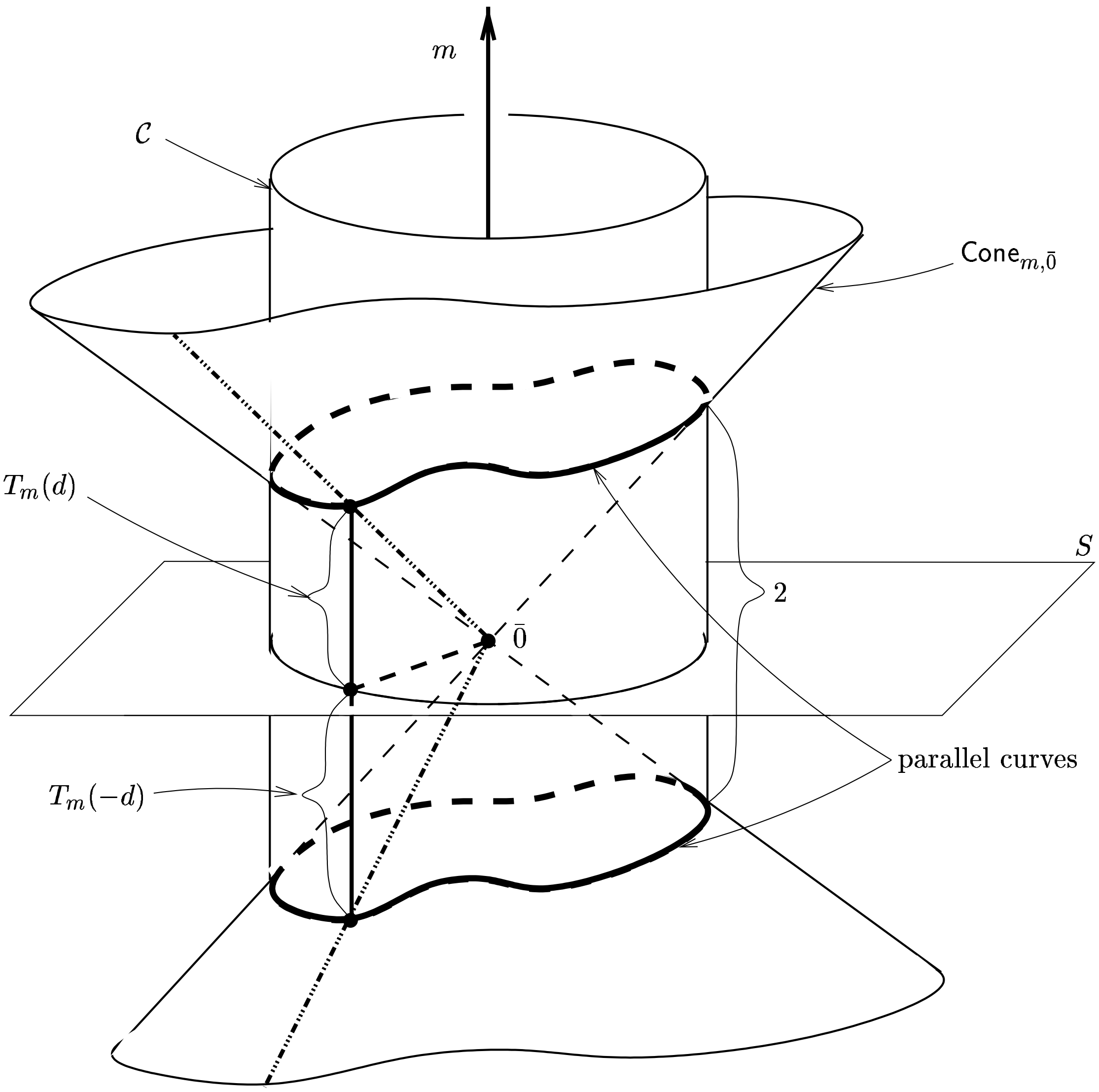



Figure 184: $\mathbf{R}(\mathbf{AxE})$ states that the two curves constituting $\mathcal{C} \cap \mathsf{Cone}_{m,\bar{0}}$ are translations of each other by 2.

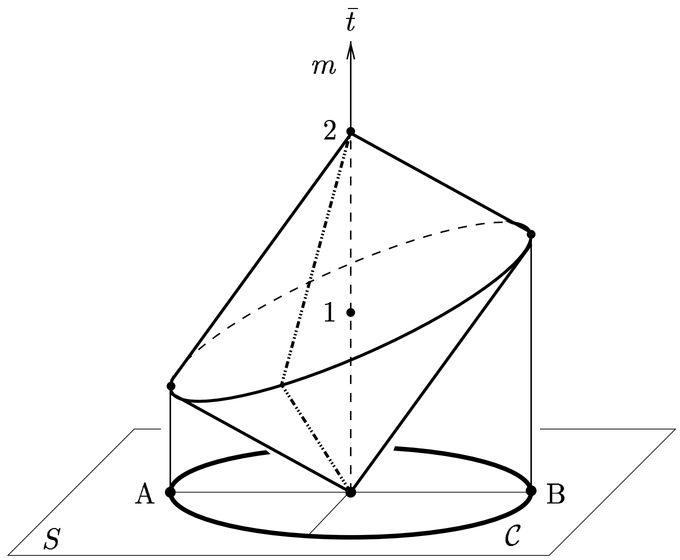



Figure 185: $m$ will optically see all points of $\mathcal{C}$ light up at the same time. But $m$ will observe A light up earlier than B.

**Definition 4.5.3** (Reichenbachian versions of our theories)

$$\mathbf{Reich}_0(\mathbf{Bax}) \stackrel{\text{def}}{=} \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE_{00}}),$$

$$\mathbf{Reich}_0(\mathbf{Flxbasax}) \stackrel{\text{def}}{=} \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE_{02}}),$$

$$\mathbf{Reich}_0(\mathbf{Newbasax}) \stackrel{\text{def}}{=} \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE}),$$

$$\mathbf{Reich}_0(\mathbf{Basax}) \stackrel{\text{def}}{=} \mathbf{Reich}_0(\mathbf{Newbasax}) + \mathbf{Ax6},$$

$$\mathbf{Reich}_0(\mathbf{Bax})_\partial \stackrel{\text{def}}{=} \mathbf{Bax}^-_\partial + \mathbf{R}(\mathbf{AxE_{00}}).$$

$\lhd$

For completeness we note that we could have introduced $\mathbf{Reich}_0(Th)_\partial$ for any $Th \in \{\mathbf{Bax}^-, \ldots, \mathbf{Basax}\}$, but to save space, we did not go into this.

**PROPOSITION 4.5.4 (isotropy "de-Reichenbachizes" theories)**
*Assume* $\mathbf{Ax}(\sqrt{\ })$. *Let* $Th \in \{\mathbf{Bax}, \mathbf{Flxbasax}, \mathbf{Newbasax}, \mathbf{Basax}\}$. *Then*

$$\Big(\mathbf{Reich}_0(Th) + (\forall d \in \mathsf{directions})\mathsf{c}_m(d) = \mathsf{c}_m(-d)\Big) \;=\!\!|\!\!\models\; Th.$$

$\blacksquare$

Let us notice that, at least in some sense, the idea of "Reichenbachizing" a theory $Th$ is to eliminate the (implicit) assumption $\mathsf{c}_m(d) = \mathsf{c}_m(-d)$ from the theory. Therefore, Proposition 4.5.4 above expresses something in the direction of saying that, of all the possible theories, it is $Th$ to which $\mathbf{Reich}_0(Th)$ is connected in the sense that if we "invert the process of Reichenbachization", then we obtain $Th$ from $\mathbf{Reich}_0(Th)$.[463]

In connection with Definition 4.5.3 above we note the following. If we wanted to discuss the $\mathbf{Ax}(\sqrt{\ })$-free version $\mathbf{Reich}_0(Th)^-$ of $Th$, then we could use the $\mathbf{Ax}(\sqrt{\ })$-free versions $\mathbf{R}(\text{speed-of-light-axiom})^-$ of the speed-of-light axiom of $Th$. E.g. we define $\mathbf{Reich}_0(\mathbf{Bax})^- = \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE_{00}})^-$. $\mathbf{Reich}_0(\mathbf{Bax})^-_\partial$, $\mathbf{Reich}_0(\mathbf{Flxbasax})^-$, $\mathbf{Reich}_0(\mathbf{Newbasax})^-$ are defined analogously. $\mathbf{Reich}_0(\mathbf{Basax})^- \stackrel{\text{def}}{=} \mathbf{Reich}_0(\mathbf{Newbasax}) + \mathbf{Ax6}$. We do not plan to study these $\mathbf{Ax}(\sqrt{\ })$-free versions in the present work, we included them for completeness and also to support future research.

**Remark 4.5.5 (On the style of defining $\mathbf{Reich}_0(Th)$)** Intuitively, we think of "$\mathbf{Reich}_0$" as a general operator which to any one of our relativity theories $Th$ associates its Reichenbachian version $\mathbf{Reich}_0(Th)$. On this intuitive level, the general "plan" of defining the operator $\mathbf{Reich}_0$ is summarized in items (i), (ii) below.

**(i)** First we recall that in Remark 4.3.43 we reformulated our distinguished theories $Th$ to the form

$$Th' = \mathrm{SPR}_0 + (\text{extra principles}) + (\text{the speed of light axiom of } Th').$$

Then, under assuming $\mathbf{Ax}(\sqrt{\ })$, we proved the "equivalence-statement"

$$Th \;\; =\!\mid\!\models \;\; Th'$$

for our distinguished choices of $Th$, cf. Proposition 4.3.44. Here, the "extra principles" do not involve the speed of light while "the speed of light axiom of $Th'$" might (in principle) be slightly different from that of $Th$ (for technical reasons). So, the common core of our (re-formulated) theories was $\mathrm{SPR}_0$ in Remark 4.3.43. In the present item, we choose $\mathbf{Bax}^- = (\mathrm{SPR}_0 + \mathbf{AxP1})$ as the *common core* of our theories, i.e. we use $\mathbf{Bax}^-$ in place of $\mathrm{SPR}_0$.[464] Then

[463] The reason why Proposition 4.5.4 has to be stated (as opposed to be completely self-evident) is the style of our Definition 4.5.3 above. Cf. Remark 4.5.5 below.

[464] Although **AxP1** does involve one-way movement of photons, it is completely consistent with what we call Reichenbachian philosophy, since it is testable by thought experiments, and it only says that $c_m(p,d)$ is indeed the speed of light (at point $p$ in direction $d$ etc) as opposed to being merely the supremum of the possible photon speeds at point $p$ moving in direction $d$, cf. the definition of $c_m(p,d)$ on p.535. Cf. also Remark 4.3.5, p.473 on the philosophical meaning of **AxP1**.

we imitate what happened in Remark 4.3.43, with $\mathbf{Bax}^-$ in place of $\mathrm{SPR}_0$. So, first we re-formulate our distinguished theories $Th$ to the form

(+) $Th^\star =$
$\mathbf{Bax}^- +$ (extra principles of $Th^\star$) + (the speed of light axiom of $Th^\star$).

As it was implicit already in Remark 4.3.43, we define the speed of light axioms of **Basax**, **Newbasax**, **Flxbasax**, **Bax** to be **AxE**, $\mathbf{AxE_0}$, $\mathbf{AxE_{02}}$, $\mathbf{AxE_{00}}$ respectively.[465] Then, the speed of light axiom of $Th^\star$ is defined to be the same as that of $Th$. (In Remark 4.3.43, these axioms were displayed as the "central column".) Analogously with Proposition 4.3.44, then, assuming $\mathbf{Ax}(\sqrt{\ })$ as usual, we prove our new equivalence statement

$$(++) \qquad Th \mathrel{=\!\!\mid\!\models} Th^\star$$

for the above four choices[466] of $Th$.

**(ii)** Next, we define the Reichenbachian version $\mathbf{Reich}_0{}^\star(Th^\star)$ of $Th^\star$ as follows.

($\star$) $\mathbf{Reich}_0{}^\star(Th^\star) \stackrel{\text{def}}{=}$
$(Th^\star \setminus \{$speed of light axiom of $Th^\star\}) + \mathbf{R}($speed of light axiom of $Th^\star)$,

i.e. $\mathbf{Reich}_0{}^\star(Th^\star)$ is obtained by replacing (in $Th^\star$) the speed of light axiom of $Th^\star$ with its Reichenbachian version.[467] Then we prove

$$(\star\star) \qquad \mathbf{Reich}_0(Th) = \mathbf{Reich}_0{}^\star(Th^\star)$$

for our distinguished theories $Th$. (Recall that $\mathbf{Reich}_0(Th)$ was defined in Definition 4.5.3. So ($\star\star$) above means that our ad-hoc looking definition in Def.4.5.3 is identical with the systematic definition in the present item.)

For completeness, we note that what we call the speed of light part of $Th^\star$ can be identified (without causing any harm) with the collection of those axioms of $Th$ which were denoted as $\mathbf{AxE_{index}}$ where "**index**" is some subscript including the empty subscript, too. (The only deviation from our definition on p.564 above would then be that we would regard $\mathbf{AxE_{01}}$ saying $\mathsf{c}_m(d) \neq 0$ as a speed of light axiom and would replace it with its two-way form $\mathbf{R}(\mathbf{AxE_{01}}) \stackrel{\text{def}}{=} (T_m(d) + T_m(-d) < \infty)$.)

[465] We will return to discussing this definition soon (on p.564).

[466] $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$

[467] which has been defined on p.557. The key idea in defining $\mathbf{R}$(speed of light ...) is to replace all references to the one-way speed of light with references to its two-way speed, cf. p.557.

In the above plan, statement $(\star)$ represents the *uniform pattern*[468] of Reichenbachizing our theories. Before formulating pattern $(\star)$, we had to bring our theories *Th* to the "normal form" $Th^\star$ because of the following. Without switching to the normal form first, the "core part" of our theory could contain an innocent-looking implicit assumption about the one-way speed of light as indeed **Basax** $\setminus$ {speed of light axiom of **Basax**} contains **Ax5** (if we consider the speed of light axiom of **Basax** to be **AxE** as we do both in Remark 4.3.43 and here). But we do not want **Ax5** (in its present form) in any Reichenbachian theory. Further, we had to add **AxP1** to the core part, because **AxP1** is a one-way assumption which *is* experimentally testable, hence it is included in the Reichenbachian versions. However, the "mechanical" Reichenbachization of, say, **AxE** described above would eliminate **AxP1** from $\mathbf{R}(\mathbf{AxE})$, i.e. $\mathrm{SPR}_0 + \mathbf{Ax6} + \mathbf{R}(\mathbf{AxE}) \not\models \mathbf{AxP1}$. Hence we had to include **AxP1** into the core part of $\mathbf{Reich_0}^\star(Th^\star)$ to ensure $\mathbf{Reich_0}^\star(Th^\star) \models \mathbf{AxP1}$.

Let us recall that $\mathbf{Reich_0}(Th)$ is that formulation of the Reichenbachian version of *Th* which was given in Definition 4.5.3. Hence statemens $(\star)$ and $(\star\star)$ above explain how we obtained the formulation of "$\mathbf{Reich_0}$" in Def.4.5.3. In other words, $(\star)$ and $(\star\star)$ show that the formulation in Def.4.5.3 is obtained in accordance with the systematic, natural (and uniform) general plan[469] outlined in items (i), (ii) above. $\triangleleft$

In the present section we will not investigate $\mathbf{Reich_0}(\mathbf{Bax})_\partial$. However, in section 4.8, $\mathbf{Reich_0}(\mathbf{Bax})_\partial$ will become interesting because a certain interesting natural paradigmatic effect $(\mathrm{E5})^{+\exists}$ becomes true at this point. See Thm.4.8.9.

[468] or "algorythm"

[469] The key points of this plan are summarized in items $(+)$, $(++)$, $(\star)$, $(\star\star)$ above.

**The class of standard models of $\mathbf{Reich}_0(\mathbf{Basax})$, $\mathbf{Reich}_0(\mathbf{Bax})$ etc. Changing simultaneities in a model $\mathfrak{M}$.**

Below we construct a large class of models for our Reichenbachian theories, e.g. for $\mathbf{Reich}_0(\mathbf{Basax})$. It remains an open problem to decide whether these are all the models of $\mathbf{Reich}_0(\mathbf{Basax})$.

Let $\mathfrak{M}$ be a frame model and $m \in Obs$. Consider a hyper-plane[470] $P \subseteq {}^nF$ with $\bar{0} \in P$. We call $P$ m-space like iff

$$(\forall \ell \in \mathsf{Eucl})\Big[(\bar{0} \in \ell \subseteq P \ \wedge\ ang^2(\ell) < \infty) \ \Rightarrow \\ \ell \text{ is not the life-line either of a slow observer or of a photon}\Big].$$ [471]

Intuitively, a hyper-plane $P$ is $m$-space like, if (in $m$'s world-view) $P$ does not contain the life-line of any slow observer or "slow photon" (where a photon is slow iff its speed is not $\infty$.) We note that if $\mathfrak{M} \models \mathbf{Bax}^-$, then the space-part $S$ of ${}^nF$ is an $m$-space like hyper-plane for every $m \in Obs$.

We will prove that if we change the simultaneity in a world-view of an observer $m$ in a model of $\mathbf{Reich}_0(\mathbf{Bax})$ to an $m$-space like hyper-plane, then in the so obtained new model $\mathbf{Reich}_0(\mathbf{Bax})$ will remain true. Now we describe the construction of changing simultaneities in more detail. First we show the idea in figures for the case $n = 2, 3$.

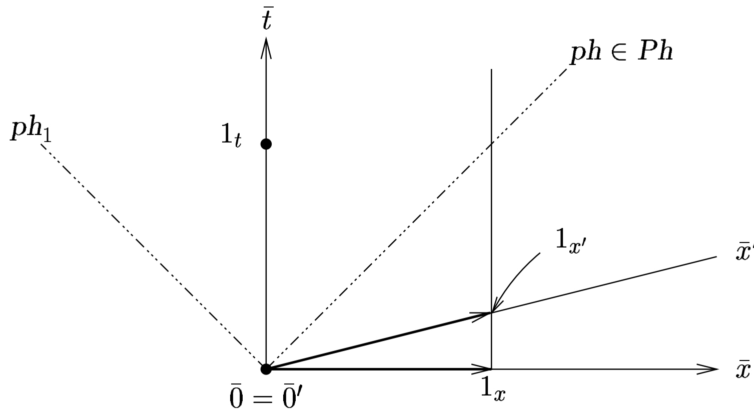
Let $\mathfrak{M}$ be a frame-model and let $m \in Obs$. We change the world-view $w_m : {}^nF \to \mathcal{P}(B)$ of $m$ to $w_m^+$ as described below.

The new world-view $w_m^+$:

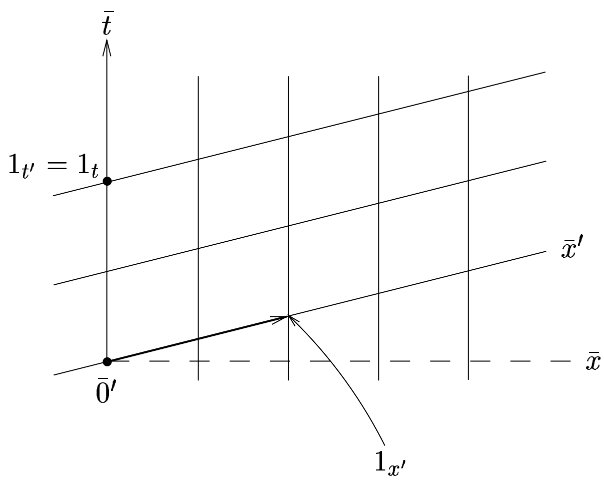
[470] i.e. $(n-1)$-dimensional plane in the sense of Def.3.1.8 on p.164.

[471] i.e. $(\forall k \in Obs)[(\infty \neq v_m(k) \ \wedge\ v_m(k) < \mathsf{c}_m(\vec{v}_m(k))) \ \Rightarrow\ \ell \neq tr_m(k)] \ \wedge\ (\forall ph \in Ph)\ell \neq tr_m(ph)$.

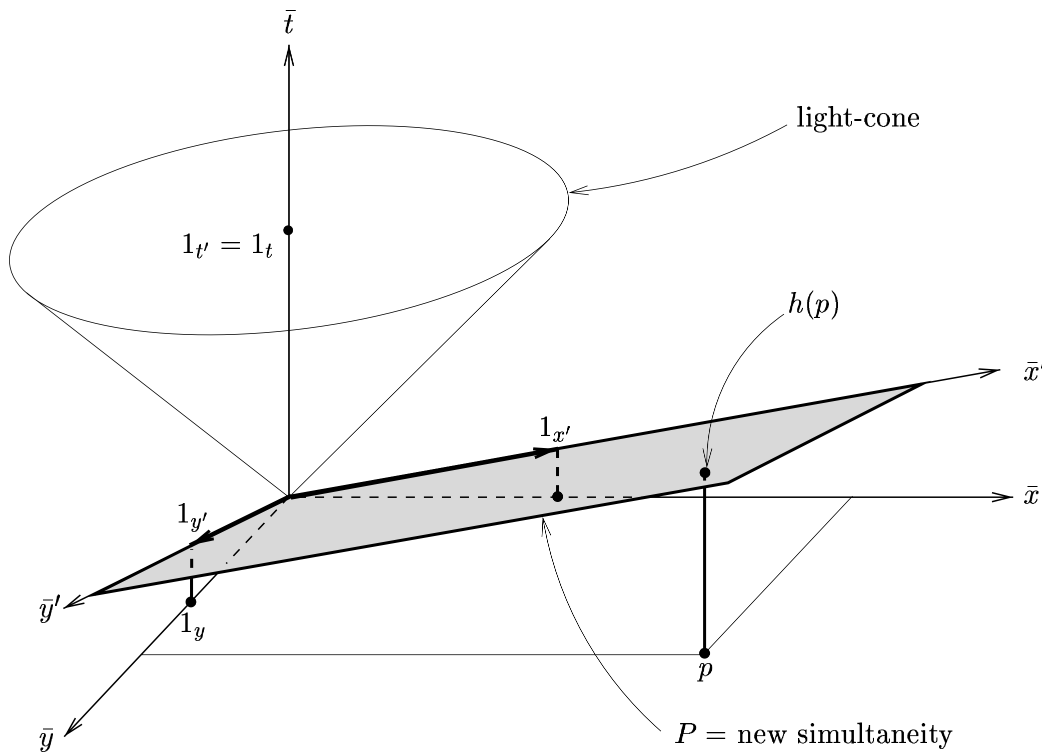
(1) Assume $n = 2$. We replace the old simultaneity represented by $\bar{x}$ with a new simultaneity $\bar{x}'$ as represented in the figure below.




The new coordinate-grid representing $w_m^+$ looks like the following.



(2) Assume $n = 3$. The new simultaneity is an arbitrary $m$-space like plane $P$:




**Definition 4.5.6 (artificial-simultaneity models)**

Let $\mathfrak{M}$ be any frame-model and let $n$ be arbitrary. Recall that $S = \{p \in {}^nF : p_0 = 0\}$ is the space hyper-plane of ${}^nF$. For all $m \in Obs$, let $P_m$ be an $m$-space like hyper-plane containing $\bar{0}$, and let $P \stackrel{\text{def}}{=} \langle P_m : m \in Obs \rangle$. Let $m \in Obs$ and let $h_m : {}^nF \to {}^nF$ be the linear transformation taking $S$ to $P_m$ such that $(\forall p \in {}^nF) h_m(p) - p \in \bar{t}$ and $h_m(1_t) = 1_t$. I.e. if $p \in S$ then $h_m(p)$ is the point of $P_m$ above (or below) $p$. (Hence $(\forall p \in \bar{t}) h_m(p) = p$.) Now we define

$$w_m^+ \stackrel{\text{def}}{=} h_m \circ w_m.$$

The new model is obtained from $\mathfrak{M}$ by replacing all world-view functions $w_m$ with their new versions $w_m^+$ defined via $P_m$ as above. We denote the new model by $\mathfrak{M}/P$.

We call $\mathfrak{M}/P$ an *artificial-simultaneity version* (*art-sim* for short) of $\mathfrak{M}$. We also call $\mathfrak{M}/P$ the model $\mathfrak{M}$ *relativized* with artificial simultaneities $P$, or just the model $\mathfrak{M}$ relativized with $P$. $\triangleleft$

On the intuitive (or physical) meaning of artificial-simultaneity models: Given $m$'s world-view $w_m$, $m$ can imagine that at each point $s \in {}^{n-1}F$ of space[472] there is a clock, and $w_m(t,s) = e$ means that event $e$ happened at place $s$ when this clock showed time $t$. Assume that $m$ re-sets each clock, i.e. he changes when they show 0, but does not change the rate of their ticking. Then $m$ can base his new world-view on these newly set clocks, and this way he obtains a new world-view function $w_m^+$ (i.e. $w_m^+(t,s) = e$ means that event $e$ happened at place $s$ when the newly set clock showed time $t$). It is this new world-view $w_m^+$ what we got in Definition 4.5.6; and the intuitive meanings of $P_m$ and $h_m$ occurring in the definition are as follows. For $s \in {}^{n-1}F$, let $t_s$ be the time what the old clock at $s$ showed when $m$ re-set it to 0. Then $h_m(0,s) = (t_s, s)$ and this is the intuitive meaning of the function $h_m$ in Def.4.5.6. The intuitive meaning of $P_m$ in the same definition is that $w_m[P_m]$ is the set of events that became simultaneous with event $w_m(\bar{0})$, according to the new world-view function $w_m^+$. Thus one can conceive relativization as an act of re-synchronization of clocks: at each place $s \in {}^{n-1}F$, $m$ has a brother, and all these brothers agree that they re-set their clocks so that the newly set clocks will show 0 exactly at events $w_m[P_m]$.

The definition of $\mathfrak{M}/P$ makes sense even when $P_m$ is an arbitrary surface in the sense of Def.4.5.25, see p.594. The reason why we required $P_m$ to satisfy further properties is the following. We required $P_m$ to be a hyper-plane in order that the life-lines of inertial bodies remain straight lines in the relativized model, cf. Conjecture 4.5.26. At the end of this section we discuss the reasons for wanting the life-lines of inertial bodies remain straight lines, cf. Remark 4.5.29. We required $P_m$ to be $m$-space like in order that $\mathbf{Bax}^-$ be preserved by relativization, see the footnote 475 on p.571. On the other hand, we required $\bar{0} \in P_m$ only for convenience (nothing would change if we dropped this condition, only the definitions would get longer in a trivial way).

We note that it is not difficult to see that the relation "art-sim version of" is symmetric, i.e. if $\mathfrak{M}$ is an art-sim version of $\mathfrak{M}^+$, then $\mathfrak{M}^+$ also is an art-sim version of $\mathfrak{M}$. Actually, "art-sim version of" is an equivalence relation.

**Definition 4.5.7 (standard models of $\mathbf{Reich}_0(Th)$)** Let $Th$ be a theory in our frame-language such that $Th \geq \mathbf{Bax}$.

[472] This is not really space, but see Convention 4.3.1.

**(i)** We say that $\mathfrak{M}$ is a <u>*standard model of*</u> $\mathbf{Reich}_0(Th)$ if $\mathfrak{M}$ is an art-sim version of a model of $Th + \mathbf{Ax}(\sqrt{\ })$.

**(ii)** Let $\mathsf{K}$ be a class of frame-models. Then $\mathbf{Asim}(\mathsf{K})$ denotes the class of all artificial-simultaneity versions of elements of $\mathsf{K}$, i.e.

$$\mathbf{Asim}(\mathsf{K}) \stackrel{\text{def}}{=} \{\mathfrak{M}/P : \mathfrak{M} \in (\mathsf{K} \cap \mathsf{Mod}(\mathbf{Ax}(\sqrt{\ }))) \text{ and } (\forall m \in Obs)[P_m \text{ is an } m\text{-space like hyper-plane containing } \bar{0}]\}.$$

$\mathbf{Asim}(\mathsf{K})$ is called the <u>*art-sim hull*</u> of $\mathsf{K}$.[473]

Note that $\mathsf{K} \cap \mathsf{Mod}(\mathbf{Ax}(\sqrt{\ })) \subseteq \mathbf{Asim}(\mathsf{K})$.

**(iii)**
$$\mathbf{Asim}(Th) \stackrel{\text{def}}{=} \mathbf{Asim}(\mathsf{Mod}(Th + \mathbf{Ax}(\sqrt{\ }))).$$

The members of $\mathbf{Asim}(Th)$ are called both <u>*standard models*</u> of **Reich**(Th) and <u>*art-sim*</u> (versions of) <u>*models*</u> of *Th*. Occasionally we might call the members of $\mathbf{Asim}(Th)$ <u>*relativized*</u> (versions of) models of *Th*. The reason for this is an analogy with algebraic logic which we do not discuss here; but intuitively, we think of an art-sim version $\mathfrak{M}/P$ of the model $\mathfrak{M}$ as being the same as $\mathfrak{M}$ except that it is "relativized" to the new, artificial simultaneities $P_m, m \in Obs^{\mathfrak{M}}$.

◁

**THEOREM 4.5.8 (Changing simultaneities preserves $\mathbf{Reich}_0(Th)$)**
*Let $\mathfrak{M}$ be a frame-model, $m \in Obs^{\mathfrak{M}}$ and $P : Obs \to \mathcal{P}({}^nF)$ such that $P_m$ is an $m$-space like hyper-plane containing $\bar{0}$ for all $m \in Obs$. Then*

$$\begin{array}{lcl}
\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Basax}) & \Rightarrow & \mathfrak{M}/P \models \mathbf{Reich}_0(\mathbf{Basax}), \\
\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Newbasax}) & \Rightarrow & \mathfrak{M}/P \models \mathbf{Reich}_0(\mathbf{Newbasax}), \\
\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Flxbasax}) & \Rightarrow & \mathfrak{M}/P \models \mathbf{Reich}_0(\mathbf{Flxbasax}), \\
\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Bax}) & \Rightarrow & \mathfrak{M}/P \models \mathbf{Reich}_0(\mathbf{Bax}), \\
\mathfrak{M} \models \mathbf{Bax}^- & \Rightarrow & \mathfrak{M}/P \models \mathbf{Bax}^-.
\end{array}$$

*I.e. the transition $\mathfrak{M} \mapsto \mathfrak{M}/P$ preserves validity of all of our distinguished Reichenbachian theories.*

[473] $\mathsf{K} \subseteq \mathbf{Asim}(\mathsf{K})$ only if $\mathsf{K} \models \mathbf{Ax}(\sqrt{\ })$. Actually, **Asim** is a so called complemented closure operation (cf. e.g. [129, p.38] or [30]). This means that for any classes $\mathsf{K}, \mathsf{L} \subseteq \mathsf{Mod}((\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ })))$ of models we have: $\mathsf{K} \subseteq \mathbf{Asim}(\mathsf{K}) = \mathbf{Asim}(\mathbf{Asim}(\mathsf{K}))$, $\mathbf{Asim}(\mathsf{K}) \subseteq \mathbf{Asim}(\mathsf{L})$ if $\mathsf{K} \subseteq \mathsf{L}$ and ($\mathsf{K} \subseteq \mathbf{Asim}(\mathsf{L})$ iff $\mathsf{L} \subseteq \mathbf{Asim}(\mathsf{K})$).

**Proof.** Checking $\mathfrak{M}/P \models \mathbf{Bax}^-$ is straightforward. It is here where one uses that $P_m$ is an $m$-space like hyper-plane. Actually, for $\mathfrak{M}/P$ to make sense, it is enough to require that for all $m \in Obs$, $P_m$ is a hyper-plane not containing $\bar{t}$.[474] This more general relativization preserves $\mathbf{Bax}^- \setminus \{\mathbf{Ax5_{Ph}}, \mathbf{Ax5_{Obs}}, \mathbf{AxP1}\}$. To prove that $\{\mathbf{Ax5_{Ph}}, \mathbf{Ax5_{Obs}}, \mathbf{AxP1}\}$ is also preserved by relativization, we use that $P_m$ is $m$-space like (for all $m \in Obs$).[475] To see that the two-way speed of light was not changed by switching from simultaneity $S$ to simultaneity $P_m$ (i.e. that $h_m$ leaves this speed unchanged), one either consults the figures above, or equivalently, one uses the fact that $h_m$ leaves $\bar{t}$ pointwise fixed (and that a "round-trip" begins and ends on $\bar{t}$) and that $h_m$ leaves the 1-cylinders around $\bar{t}$ fixed, too. See Figure 186. ∎

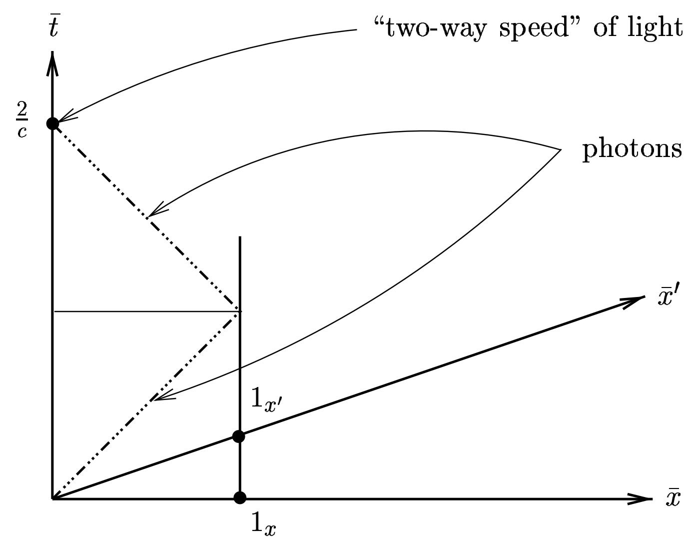



Figure 186: Illustration for the proof of Theorem 4.5.8: two-way speed of light does not change when relativizing a model.

The next corollary says that the art-sim models in $\mathbf{Asim}(Th)$ of $Th$ are suitable, in some sense, for studying the Reichenbachian version $\mathbf{Reich}_0(Th)$ of $Th$ (whenever $\mathbf{Reich}_0(Th)$ is already defined).

[474]Even less is enough. It is enough to require that $P_m$ is a surface in the sense of Def.4.5.25, cf. p.594.

[475]Actually, "$m$-space like" is the exact property we need here in the sense that for all $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax})$ and $P$ such that $P_m$ is a hyper-plane for all $m \in Obs$, we have that
$\mathfrak{M}/P \models \mathbf{Bax}^-$ iff $(\forall m)P_m$ is $m$-space like.

**COROLLARY 4.5.9**

Assume that $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$. Then

$$\mathbf{Asim}(Th) \models \mathbf{Reich}_0(Th).$$

$\triangleleft$

It remains an open question whether the above corollary holds "backwards" too, in the following sense.

**Problem 4.5.10** Let $Th$ be as in Cor.4.5.9. Is then

$$\mathbf{Asim}(Th) = \mathsf{Mod}(\mathbf{Reich}_0(Th))$$

true? In other words, is there a model of $\mathbf{Reich}_0(\mathbf{Bax})$ which cannot be obtained from a model of **Bax** by relativizing? We note that the answer to this problem for $n = 2$ is in the negative, i.e. every model of $\mathbf{Reich}_0(\mathbf{Bax}(2))$ is an art-sim version of a model of $\mathbf{Bax}(2)$. (Cf. p.577 in the proof of Thm.4.5.13.) $\triangleleft$

We will see that it is possible to add a natural axiom (which does not contradict the "Reichenbachian spirit") to $\mathbf{Reich}_0(Th)$ which makes the answer to the above problem positive, cf. Thm.4.5.13. (Of course, this does not answer the problem in the mathematical sense.)

### Do we have enough speed-of-light axioms in R(AxE)?

In formulating the <u>*observational version*</u> $\mathbf{R}(\mathbf{AxE})$ of the speed of light axiom **AxE**, we used a mirror, put it on the *Moon* and then used the mirror to measure the two-way speed of light talking about the *Earth*↦*Moon*↦*Earth* round-trip (cf. p.553 in this section). Is this the only observational aspect of the speed of light? Certainly not: instead of the two-way speed of light we could talk about its three-way speed. Namely, we can put a mirror on the *Moon*, another one on the *Mars*, and then measure the time needed for the *Earth*↦*Moon*↦ *Mars*↦*Earth* round-trip. We call this a three-way round-trip, and the average speed of this three-way round-trip is called the <u>*three-way speed of light*</u>.

Then in a new axiom, say $\mathbf{R}_3(\mathbf{AxE})$, we could do the same with the three-way speed of light what we did earlier with its two-way speed. Indeed, this is what we will do. In $\mathbf{R}_3(\mathbf{AxE})$ we will postulate that the three-way speed of light is constant (does not depend on the three chosen directions $d_1, d_2, d_3$ etc). Before formalizing $\mathbf{R}_3(\mathbf{AxE})$, let us see whether this will be useful for us. Instead of three-way speed,

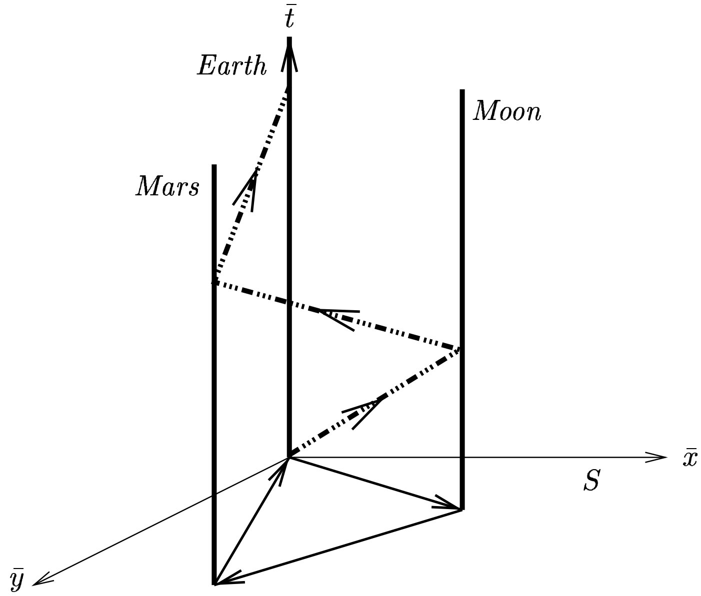



Figure 187: Three-way round-trip of photons.

we can discuss 4-way speed as well, as $k$-way speed, for $k \in \omega$. Should we add all these axioms to our theories **Reich**(*Th*)? Fortunately, we will be able to formulate a theorem to the effect that 3 is the largest number we need. Since we are on an informal level, we call this theorem Statement ($\star$) and will formulate its formal counterpart later (as Theorem 4.5.11(i)).

$$(\star) \qquad \mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{R}_3(\mathbf{AxE}) \models \mathbf{R}_k(\mathbf{AxE})$$

for any $k \in \omega$.

We could define $\mathbf{R}_3$(speed of light axiom of *Th*) for all our distinguished theories *Th* analogously. Fortunately, we do not need to do this because we can choose $\mathbf{R}_3$(speed of light axiom of *Th*) to be the same for all of our possible choices of *Th*. We will denote this unified axiom as $\mathbf{R}_\Delta(E)$.

Intuitively, $\mathbf{R}_\Delta(E)$ says that if we take the time $\Delta t$ needed for the *Earth*$\mapsto$*Moon*$\mapsto$*Mars*$\mapsto$*Earth* round trip and divide it with the spatial distance covered during the trip, then the so obtained average velocity is independent from the "choice of the *Earth, Moon, Mars*" (i.e. choosing any three other inertial bodies at relative rest[476] would yield the same result). Formally:

$$\mathbf{R}_\Delta(E) \quad (\forall m \in Obs)(\exists r \in F)(\forall d_1, d_2, d_3 \in \textsf{directions})\Big[d_1 + d_2 + d_3 = \bar{0} \Rightarrow \frac{|d_1| \cdot T_m(d_1) + |d_2| \cdot T_m(d_2) + |d_3| \cdot T_m(d_3)}{|d_1| + |d_2| + |d_3|} = r\Big].$$

The definition of $\mathbf{R}_\Delta(E)$ is a special case of the following more general definition of $\mathbf{R}_k(\mathbf{AxE})$, which states that the $k$-way speed of light is constant. Let $k \in \omega$ be any nonzero number.

$$\mathbf{R}_k(\mathbf{AxE}) \quad (\forall m \in Obs)(\exists r \in F)(\forall d_1, \ldots, d_k \in \textsf{directions})\Big[d_1 + \ldots + d_k = \bar{0} \Rightarrow \frac{|d_1| \cdot T_m(d_1) + \ldots + |d_k| \cdot T_m(d_k)}{|d_1| + \ldots + |d_k|} = r\Big].$$

We note that $\mathbf{R}_1(\mathbf{AxE})$ is a vacuous statement (hence automatically true). $\mathbf{R}_2(\mathbf{AxE})$ is equivalent to $\mathbf{R}(\mathbf{AxE_{00}})$ and $\mathbf{R}_3(\mathbf{AxE})$ is $\mathbf{R}_\Delta(E)$. The next theorem states that $\mathbf{R}_3(\mathbf{AxE})$ implies $\mathbf{R}_k(\mathbf{AxE})$ for all $k \in \omega$, in models of $\mathbf{Bax}^-$. This is why we gave it a special name ($\mathbf{R}_\Delta(E)$).[477]

**THEOREM 4.5.11** *Let $k \geq 2$.*

(i) $\mathbf{Bax}^- \models \mathbf{R}_3(\mathbf{AxE}) \to \mathbf{R}_k(\mathbf{AxE})$.

(ii) *In models of $\mathbf{Bax}^- + \mathbf{R}_3(\mathbf{AxE})$, the $k$-way speed of light is the same as the two-way speed of light.*

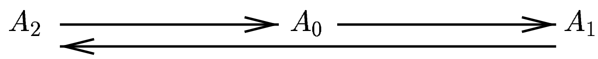
**Proof.** Assume $\mathfrak{M} \models \mathbf{Bax}^-$ and $m \in Obs^{\mathfrak{M}}$. We will work in $m$'s world-view. Let $A_0 = \bar{0}, A_1, A_2, A_3$ be four distinct points in the space-part. Let $i, j < 4$. By **AxP1** and $\mathbf{Ax5_{Ph}}$, for each time instant $t \in F$, there is a photon which starts at $t$ from $A_i$ and reaches $A_j$ some time later. Let $T_{ij}$ be the time, according to $m$, needed for this. By Theorem 4.3.17, this time is independent of the time-instant $t$ when the photon starts from $A_i$. Let $d_{ij}$ denote the (spatial) distance between $A_i$ and $A_j$. Let $v_k$ denote the $k$-way speed of light (if this is constant).

First we show $\mathfrak{M} \models \mathbf{R}_3(\mathbf{AxE}) \to \mathbf{R}_2(\mathbf{AxE})$. Assume $\mathfrak{M} \models \mathbf{R}_3(\mathbf{AxE})$, i.e. that the three-way speed of light is constant in $\mathfrak{M}$ (then it is denoted as $v_3$).

Assume that $A_0, A_1, A_2$ are collinear, $A_0$ is between $A_1$ and $A_2$ and $d_{01} = d_{02}$, as on the figure.

[476] For simplicity, we pretend that the *Earth, Moon* etc. are at relative rest (but this is not important).

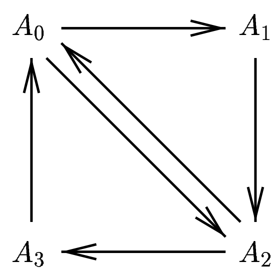
[477] We will see that Problem 4.5.10 is equivalent to asking whether $\mathbf{R}_2(\mathbf{AxE})$ implies $\mathbf{R}_k(\mathbf{AxE})$ for all $k \in \omega$, too.




Let us investigate the round-trip of photons $A_0 \mapsto A_1 \mapsto A_2 \mapsto A_0$. This is a 3-way round-trip, so $T \stackrel{\text{def}}{=} T_{01} + T_{12} + T_{20} = v_3 \cdot (d_{01} + d_{12} + d_{20})$ by $\mathbf{R}_3(\mathbf{AxE})$. Now (by Thm.4.3.17) $T_{20} = T_{01}$ and $T_{12} = T_{10} + T_{02} = 2 \cdot T_{10}$. Thus $T = 2 \cdot (T_{01} + T_{10})$ and $d \stackrel{\text{def}}{=} d_{01} + d_{12} + d_{20} = 2 \cdot (d_{01} + d_{10})$. Hence $T_{01} + T_{10} = v_3 \cdot (d_{01} + d_{10})$, showing that the average speed of the two-way round-trip $A_0 \mapsto A_1 \mapsto A_0$ is $v_3$. Since $A_1 \neq A_0$ was chosen arbitrarily, this shows that $\mathfrak{M} \models \mathbf{R}_2(\mathbf{AxE})$ and $v_2 = v_3$.

Next we show $\mathfrak{M} \models \mathbf{R}_3(\mathbf{AxE}) \to \mathbf{R}_k(\mathbf{AxE})$ and $v_3 = v_k$. We give the proof for $k = 4$, the case $k > 4$ is completely analogous. Assume $\mathfrak{M} \models \mathbf{R}_3(\mathbf{AxE})$.

To show $\mathbf{R}_4(\mathbf{AxE})$ (i.e. constancy of $v_4$) and $v_4 = v_3$, consider the round-trip of photons $A_0 \mapsto A_1 \mapsto A_2 \mapsto A_0 \mapsto A_2 \mapsto A_3 \mapsto A_0$.




Let $T \stackrel{\text{def}}{=} T_{01} + T_{12} + T_{23} + T_{30}$ and $d \stackrel{\text{def}}{=} d_{01} + d_{12} + d_{23} + d_{30}$. We want to show that $T = d \cdot v_3$. Now, by a similar argument to the previous one, by $\mathbf{R}_3(\mathbf{AxE})$ we have $T + T_{20} + T_{02} = (d_{01} + d_{21} + d_{20}) \cdot v_3 + (d_{02} + d_{23} + d_{30}) \cdot v_3 = d \cdot v_3 + 2 \cdot d_{02} \cdot v_3$. On the other hand, by $\mathbf{R}_2(\mathbf{AxE})$ and $v_3 = v_2$ we have $T_{20} + T_{02} = 2 \cdot d_{02} \cdot v_2 = 2 \cdot d_{02} \cdot v_3$. These yield $T = d \cdot v_3$, as was to be shown. ∎

It is not difficult to check that $\mathbf{Asim}(\mathbf{Bax}) \models \mathbf{R}_\Delta(E)$. Moreover, we are going to prove that $\mathbf{R}_\Delta(E)$ together with $\mathbf{Reich}_0(Th)$ axiomatizes $\mathbf{Asim}(Th)$ for our distinguished theories $Th$ (see Thm.4.5.13). This motivates our next definition of the full Reichenbachian versions $\mathbf{Reich}(Th)$ of our theories $Th$.

**Definition 4.5.12 (full Reichenbachian version of a theory)**

Let $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$. Then the <u>*full Reichenbachian version*</u> of $Th$ is defined as follows:

$$\mathbf{Reich}(Th) \stackrel{\text{def}}{=} \mathbf{Reich}_0(Th) + \mathbf{R}_\Delta(E).$$

$\lhd$

Theorem 4.5.13 below says that the full Reichenbachian version of a theory proves exactly those formulas which are true no matter how we change the simultaneities (to nonstandard ones) in the models of the theory. This says, in a way, that $\mathbf{Reich}(Th)$ is that part of the theory $Th$ which remains if we disregard simultaneities. One way of disregarding simultaneities is allowing all the possible simultaneities in models - which in principle is the same as leaving simultaneities out from the models.

Thm.4.5.13 is a so-called "axiomatization type" theorem, or "representation theorem" in logic. It gives an axiomatization for the class $\mathbf{Asim}(Th)$ of models[478] – if viewed from the models' side –, and it characterizes all models of the theory[479] $\mathbf{Reich}(Th)$ – if viewed from the theory's side.

**THEOREM 4.5.13 (axiomatization of $\mathbf{Asim}(Th)$)**

*Let* $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$. *The models of* $\mathbf{Reich}(Th)$ *are exactly the art-sim versions of the models of* $Th$, *i.e.*

$$\mathbf{Asim}(Th) = \mathsf{Mod}(\mathbf{Reich}(Th)).$$

**Proof.** Let $Th$ be as in the statement of the theorem. We have to prove $\mathbf{Asim}(Th) \models \mathbf{Reich}(Th)$ and $\mathsf{Mod}(\mathbf{Reich}(Th)) \subseteq \mathbf{Asim}(Th)$.

$\mathbf{Asim}(Th) \models \mathbf{R}_\Delta(E)$ is not difficult to check (cf. the proof of Thm.4.5.8). Thus $\mathbf{Asim}(Th) \models \mathbf{Reich}(Th)$ by Corollary 4.5.9.

To prove $\mathsf{Mod}(\mathbf{Reich}(Th)) \subseteq \mathbf{Asim}(Th)$, we will construct suitable art-sim versions. First we will show that

($\star$) every model of $\mathbf{Bax}^- + \mathbf{R}_\Delta(E)$ has an isotropic art-sim version (i.e. one in which $\mathbf{Bax}$ holds).[480]

[478] It says that $\mathfrak{M} \in \mathbf{Asim}(Th)$ iff $\mathfrak{M} \models \mathbf{Reich}(Th)$, i.e. $\mathbf{Reich}(Th)$ axiomatizes $\mathbf{Asim}(Th)$.

[479] It says that $\mathfrak{M}$ is a model of $\mathbf{Reich}(Th)$ iff $\mathfrak{M}$ is an art-sim version of a model of $Th + \mathbf{Ax}(\sqrt{\ })$.

[480] In fact, for any $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^-)$, $\mathfrak{M}$ has an isotropic art-sim version iff $\mathfrak{M} \models \mathbf{R}_\Delta(E)$.

($\star$) will imply $\mathsf{Mod}(\mathbf{Reich}(Th)) \subseteq \mathbf{Asim}(Th)$ as follows. Assume $\mathfrak{M} \models \mathbf{Reich}(Th)$. Then $\mathfrak{M} \models \mathbf{Bax}^- + \mathbf{R}_\Delta(E)$, so by ($\star$), $\mathfrak{M}$ has an art-sim version $\mathfrak{M}^+ \models \mathbf{Bax}$. By Thm.4.5.8 $\mathfrak{M}^+ \models \mathbf{Reich}_0(Th) + \mathbf{Bax}$. By $\mathbf{Bax} \models \mathsf{c}_m(d) = \mathsf{c}_m(-d)$ and Prop.4.5.4 then $\mathfrak{M}^+ \models Th$. Since "art-sim version of" is a symmetric relation, then $\mathfrak{M}$ is an art-sim version of $\mathfrak{M}^+ \in \mathsf{Mod}(Th)$.

We now turn to proving ($\star$). We will prove ($\star$) for $n = 2, n = 3$, and then for $n > 3$. For $n = 2$ we will prove a stronger statement:

Assume $n = 2$. Then $\mathsf{Mod}(\mathbf{Bax}^-) \subset \mathbf{Asim}(\mathbf{Bax})$: The proof is represented in Figure 188, see also the later Fig.193.

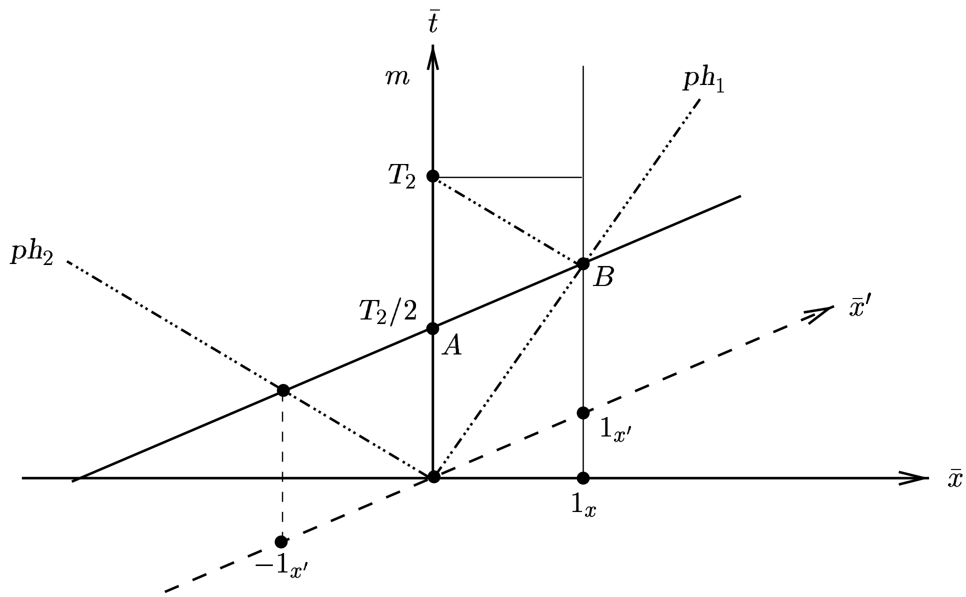



Figure 188:

Assume $\mathfrak{M} \models \mathbf{Bax}^-$ and let $m \in Obs$. Let $\bar{x}' = \bar{x}'_m$ be chosen as in Fig.188 (i.e. $\bar{0} \in \bar{x}' \parallel \overline{AB}$). Then it is not difficult to check that after relativizing with $\bar{x}'$, the speed of the two photons $ph_1, ph_2$ will be the same. Thus if we relativize with $(\bar{x}'_m : m \in Obs)$, then we get a model $\mathfrak{M}^+ \models \mathbf{Bax}$. See also Lemma 4.5.21.

Assume $n = 3$. Then $\mathsf{Mod}(\mathbf{Bax}^- + \mathbf{R}_\Delta(E)) \subset \mathbf{Asim}(\mathbf{Bax})$: The proof is represented in Figure 189. Assume $\mathfrak{M} \models \mathbf{Bax}^- + \mathbf{R}_\Delta(E)$, and let $m \in Obs$. Let $\bar{x}'$ and $\bar{y}'$ be constructed as in the case of $n = 2$ in $\mathsf{Plane}(\bar{t}, \bar{x}), \mathsf{Plane}(\bar{t}, \bar{y})$ respectively. Let $P_m \stackrel{\text{def}}{=} \mathsf{Plane}(\bar{x}', \bar{y}'), P \stackrel{\text{def}}{=} \langle P_m : m \in Obs \rangle$ and $\mathfrak{M}^+ \stackrel{\text{def}}{=} \mathfrak{M}/P$. It is not difficult to check that $\bar{t} \notin P_m$ for all $m \in Obs$, thus $\mathfrak{M}^+ = \mathfrak{M}/P$ makes sense (see the proof of Thm.4.5.8, or p.594). Instead of checking that $P_m$ is $m$-space like, we will prove directly that $\mathfrak{M}^+ \models \mathbf{Bax}$.

Let $m \in Obs$ and $d \in \mathsf{directions}$ be arbitrary. Let $ph$ be a photon moving forwards in direction $d$, in $\mathfrak{M}$. We will show that in $\mathfrak{M}^+$ also, $ph$ will move forwards in direction $d$; and moreover with speed $v_2$ where $v_2$ is the two-way speed of light in $\mathfrak{M}$. We may assume that no photon has infinite speed in $m$'s world-view in $\mathfrak{M}$ (since otherwise all photons would have infinite speed by $\mathfrak{M} \models \mathbf{R}_\Delta(E)$ and then $P_m = S$ would be the case).

Let $A \stackrel{\text{def}}{=} (0, -d)$, i.e. $A = (0, d_1, \ldots, d_{n-1})$ where $-d = (d_1, \ldots, d_{n-1})$. Let $B \in \bar{x}$ be such that $\overline{AB} \parallel \bar{y}$. Let $ph_1, ph_2, ph_3$ be photons travelling the round-trip $\bar{0} \mapsto B \mapsto A \mapsto \bar{0}$ in $\mathfrak{M}$, see Fig.189. Let $d_{0B}, d_{BA}, d_{A0}$ denote the distances between $\bar{0}, B$; $B, A$; and $A, \bar{0}$ respectively, and let $d \stackrel{\text{def}}{=} d_{0B} + d_{BA} + d_{A0}$. Then the round-trip takes $t = v_2 \cdot d$ "minutes" in $\mathfrak{M}$, i.e. $\{t\} = tr_m(ph_3) \cap \bar{t}$.

Let us now move into $\mathfrak{M}^+$. In $\mathfrak{M}^+$, the time needed for $ph_1$ to cover the distance $d_{0B}$ is $v_2 \cdot d_{0B}$ because we chose $\bar{x}'$ so that this be true. Similarly, the time needed for $ph_2$ to cover the distance $d_{AB}$ is $v_2 \cdot d_{AB}$, we chose $\bar{y}'$ so that this be true. This shows that in $\mathfrak{M}^+$, the time needed for $ph_3$ to cover the distance $d_{A0}$ is $d - (v_2 \cdot d_{0B} + v_2 \cdot d_{BA}) = v_2 \cdot d_{A0}$. Thus the speed of $ph_3$ is $v_2$, and $ph_3$ moves forwards in $\mathfrak{M}^+$ in direction $d$. Since the traces of $ph$ and $ph_3$ are parallel in $\mathfrak{M}$, so they are in $\mathfrak{M}^+$, hence $ph$ also moves in direction $d$ with speed $v_2$ in $\mathfrak{M}^+$.

The <u>case of $n > 3$</u> is completely analogous to the case of $n = 3$, we omit it. ∎

**Definition 4.5.14** (simultaneity-stable formulas, $Th$-simultaneity-stable formulas)

Let $Th$ be a set of formulas. We call $\psi$ <u>*Th-simultaneity-stable*</u> if for all models $\mathfrak{M}$ of $Th + \mathbf{Ax}(\sqrt{\ })$,

$$\mathfrak{M} \models \psi \quad \text{iff} \quad [\ \psi \text{ is valid in all art-sim versions of } \mathfrak{M}].$$

We call a formula $\psi$ of our frame-language <u>*simultaneity-stable*</u> if $\psi$ is $\mathbf{Bax}^-$-simultaneity-stable. ◁

Intuitively, "$\psi$ is simultaneity-stable" means that validity of $\psi$ is insensitive for changing simultaneities in a model, or that validity of $\psi$ does not depend on simultaneities. We note that the property "$\emptyset$-simultaneity-stable" is stronger than the property "$\mathbf{Bax}^-$-simultaneity-stable", which is stronger than e.g. "$\mathbf{Reich}(\mathbf{Bax})$-simultaneity-stable". Formulas to show this are as follows. Let $\psi$ denote the formula $c_m(p, d) = c_m(p, -d)$. Then the formula $\neg\mathbf{Bax}^- \to \psi$ is $\mathbf{Bax}^-$-simultaneity-

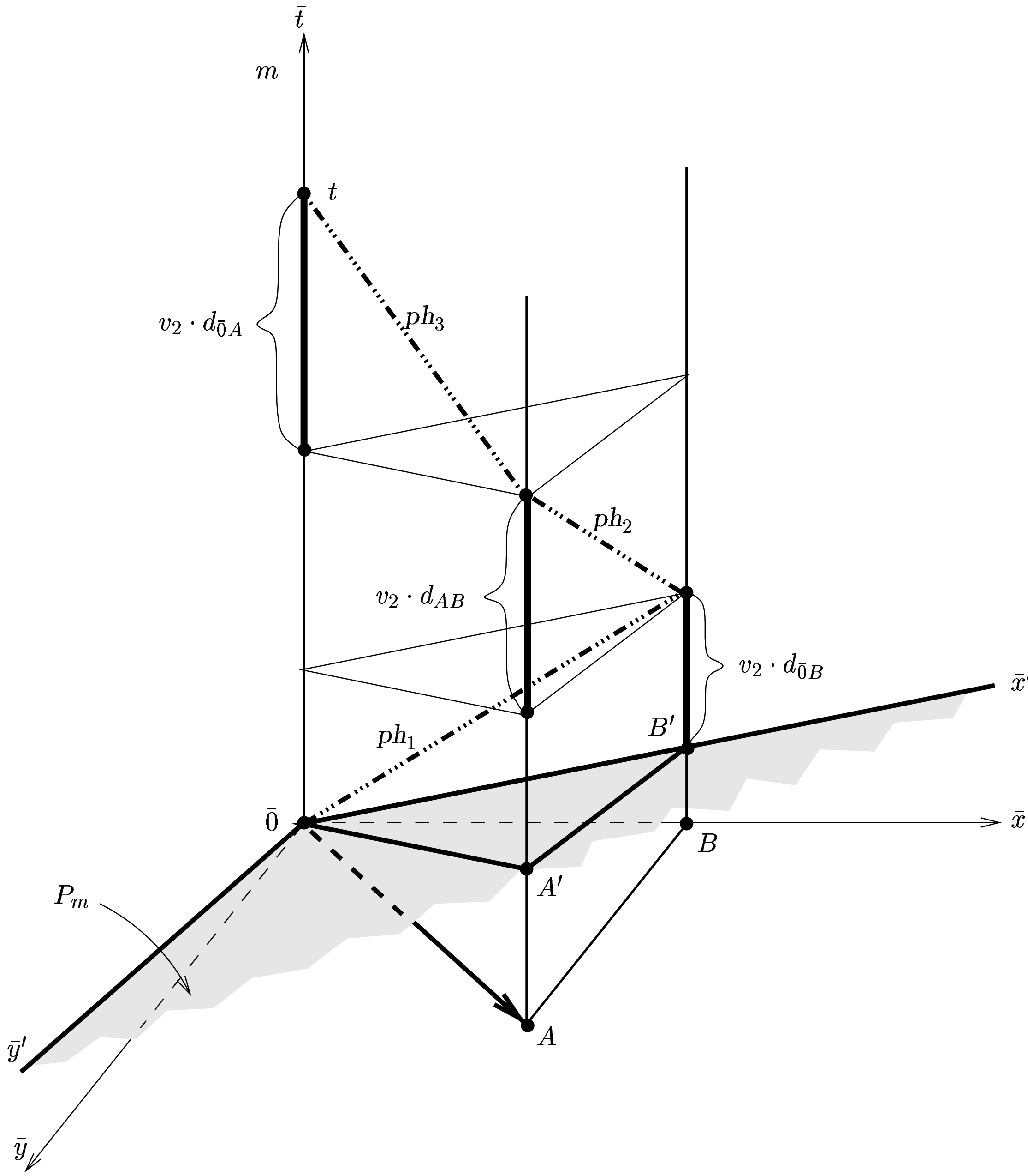



Figure 189: $ph_3$ goes in direction $-d$ with speed $v_2$ in $\mathfrak{M}^+$. Illustration for the proof of Thm.4.5.13.

stable but not $\emptyset$-simultaneity-stable, and the formula $\neg\mathbf{R}_\Delta(E) \to \psi$ is $\mathbf{Reich}(\mathbf{Bax})$-simultaneity-stable but not $\mathbf{Bax}^-$-simultaneity-stable.

**COROLLARY 4.5.15** Let $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$ and let $\varphi$ be a formula in the frame language.

**(i)** $\mathbf{Reich}(Th)$ consists of simultaneity-stable formulas.

**(ii)** $\mathbf{Reich}(Th) \models \varphi$ iff $[Th + \mathbf{Ax}(\surd) \models \varphi$ and $\varphi$ is $Th$-simultaneity-stable$]$.

**(iii)** Assume that $\varphi$ is simultaneity-stable. Then

$$\mathbf{Reich}(Th) \models \varphi \quad \text{iff} \quad Th + \mathbf{Ax}(\surd) \models \varphi.$$

$\lhd$

To our minds, Corollary 4.5.15 indicates that $\mathbf{Reich}(Th)$ is the natural choice for the Reichenbachian version of our theory $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$. The weaker form $\mathbf{Reich}_0(Th)$ discussed earlier is only a precursor for this "real thing" $\mathbf{Reich}(Th)$. However, we do not know whether these two variants are really different, see Problem 4.5.10.

We note that if we change "$Th$-simultaneity-stable" to "simultaneity-stable" in Corollary 4.5.15, then we get a false statement e.g. if $Th = \mathbf{Bax}$. The example given after Def.4.5.14 shows this.

Corollary 4.5.15 above suggests a way of defining the Reichenbachian version $\mathbf{Reich}'(Th)$ for any theory $Th$, as follows.

**Definition 4.5.16** Let $Th$ be any set of formulas. Then we define

$$\mathbf{Reich}'(Th) \stackrel{\text{def}}{=} \{\varphi : Th + \mathbf{Ax}(\surd) \models \varphi \text{ and } \varphi \text{ is simultaneity-stable}\}.$$

$\lhd$

**PROPOSITION 4.5.17**

**(i)** *For* $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$ *we have*

$$\mathbf{Reich}'(Th) \models\!\dashv \mathbf{Reich}(Th).$$

**(ii)** $\mathbf{Asim}(Th) \models \mathbf{Reich}'(Th)$.

$\lhd$

We do not know whether inclusion $\subseteq$ in Prop.4.5.17(ii) above can be changed to equality $=$.

The above also shows that "simultaneity-stable" formulas are the ones adequate for being possible additions (as possible new axioms) to our Reichenbachian theories. It would be interesting to know whether "simultaneity-stable" coincides with "experimentally testable" ("verifiable", or "observable") in some sense.[481]

We already mentioned that the relation "art-sim version of" is an equivalence relation. Now, $\mathbf{Asim}(\{\mathfrak{M}\})$ is the equivalence class of $\mathfrak{M}$ via this relation. Then "$\psi$ is simultaneity-stable" means that for any equivalence class $\mathbf{Asim}(\{\mathfrak{M}\})$, $\psi$ is either true in all members of this class or else $\psi$ is false in all members of this class. We could represent the class $\mathbf{Asim}(\{\mathfrak{M}\})$ with the model $\mathfrak{M}^-$ we get from $\mathfrak{M}$ by "removing all simultaneities from $\mathfrak{M}$". The move $\mathfrak{M} \mapsto \mathfrak{M}^-$ represents a step in abstraction, namely we abstract from simultaneities[482] while we keep the "essential parts" of $\mathfrak{M}$. A more radical step of abstraction will be done in the geometry chapter §6 where we will obtain a geometrical structure $\mathfrak{G}(\mathfrak{M})$ from the model $\mathfrak{M}$. Again, in some sense, $\mathfrak{G}(\mathfrak{M})$ will retain all the essential information about $\mathfrak{M}$ while abstracting from the non-essential or "conventional"[483] data in $\mathfrak{M}$. Actually, our present step $\mathfrak{M} \mapsto \mathfrak{M}^-$ of abstracting from parts of $\mathfrak{M}$ can be considered the first step of the abstraction process $\mathfrak{M} \mapsto \mathfrak{G}(\mathfrak{M})$ discussed in the geometry chapter.

**Question for future research 4.5.18** Give a syntactical[484] characterization of "simultaneity-stable". I.e. find a syntactical definition of, say, "simultaneity-free",

[481] This is a typical problem area of logic. The first task is to define "experimentally testable" in an intuitively satisfactory way. The idea is that in an experimentally testable formula one is not allowed to refer to the space-time coordinates of an event. Instead, one can use the view-functions defined later in Def.4.7.5 together with the purely space-locations of the "brothers" (i.e. observers whose life-lines are parallel to $m$'s one). These locations remain observable via thought-experiments, e.g. we can use the so-called "radar-distance" between $m$ and its brothers.

A further natural question comes up, namely whether the "observable formulas" (or observational formulas) are exactly those which are expressible in the language of the observer-independent geometry $\mathfrak{G}(\mathfrak{M})$ associated to the models $\mathfrak{M}$ in §6.

This train of thoughts is related to the similarly logic oriented idea of trying to identify those formulas which are testable by thought-experiments. We do not discuss these ideas in more detail in the present work.

[482] since they are considered as conventional notions by the Reichenbach-Grünbaum school of thought

[483] Cf. e.g. Friedman [90] for the definition of some part of $\mathfrak{M}$ being ***conventional*** .

[484] We call a property of formulas "***syntactical***", if the property involves only the "form" of the formulas, i.e. if the property is defined by referring to the characters occurring in the formulas. On the other hand, "***semantical***" means "meaning-oriented", usually this means "formalized via the models of the formulas". The present definition of "simultaneity-stable" is "semantical".

and prove that in $\mathbf{Bax}^-$, every simultaneity-stable formula is equivalent to a simultaneity-free one, and vice versa. $\lhd$

We now turn to the relationships between our new and old theories.

**THEOREM 4.5.19**

(i) $\mathbf{Reich}(\mathbf{Basax}) \not\models \mathbf{Bax}$, *therefore*

(ii) $\mathbf{Reich}(Th) \not\models Th$ and
$\mathbf{Reich}_0(Th) \not\models Th$

*for* $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$.

(iii) $\mathbf{Reich}_0(\mathbf{Bax})_\partial + \mathbf{Reich}(\mathbf{Basax}) \not\models \mathbf{Bax}$.

**Proof.** Using Def.4.5.6, one easily constructs standard models of $\mathbf{Reich}(\mathbf{Basax})$ in which isotropy (hence **Bax**) fails. These models satisfy $\mathbf{Reich}(\mathbf{Bax})_\partial$, too. This proves (i) and (iii), of which (ii) is a corollary. ∎

We represent the inclusion-relations between our new Reichenbachian theories and the old ones in the lattice in Figure 190. All inclusions between the theories represented in the lattice are represented also. This means that no inclusion holds between these theories that is not represented in the lattice. E.g. Thm.4.5.19 above implies that none of the inclusions $\mathbf{Reich}(Th) \geq Th$ for $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$ hold.

### Connections with Einstein's 1/2-simultaneity

We give another axiomatization of $\mathbf{Reich}(Th)$ in terms of Einstein's 1/2-simultaneity. To each observer $m$, Einstein introduced a <u>*definable*</u> simultaneity traditionally denoted as $S_m^{1/2}$. We recall this definition below.

**Definition 4.5.20** (Einstein's simultaneity)

(1) Let $p \in {}^nF$. Then $p$ is <u>*1/2-simultaneous*</u> with $\bar{0}$ iff there is $t \in \bar{t}$ for which there is a photon round-trip $(-t) \mapsto p \mapsto t$.

I.e. if we send a photon from $-t \in \bar{t}$ to $p$ and bounce it back to $\bar{t}$, then the time needed for the round-trip is $|t| \cdot 2$. Intuitively, the event on the time axis $\bar{t}$ 1/2-simultaneous with $p$ is the one which happened <u>*half-time*</u> during the time interval $int(-t, t)$ needed for the round-trip.[485] See Figure 191.

[485] This is why the literature calls this 1/2-simultaneity. Of course, in the definition everything is to be understood in an observer's world-view.

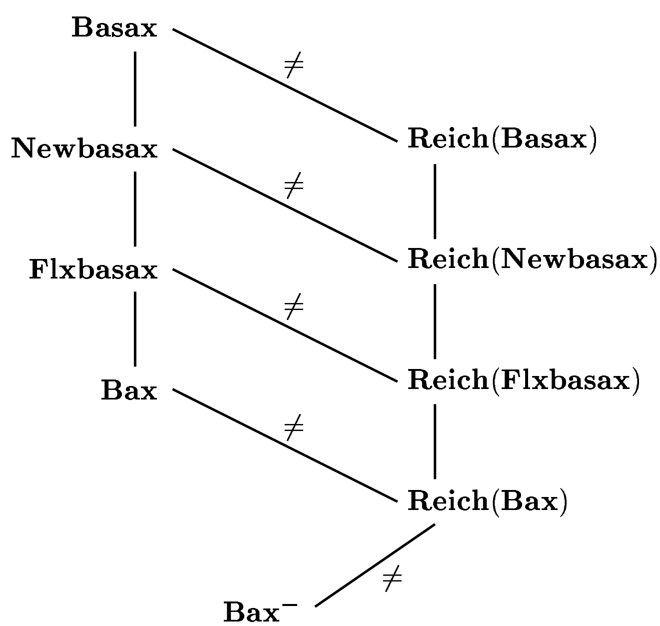



Figure 190: Reichenbachian theories in our lattice of theories under assuming $\mathbf{Ax}(\sqrt{\ })$.

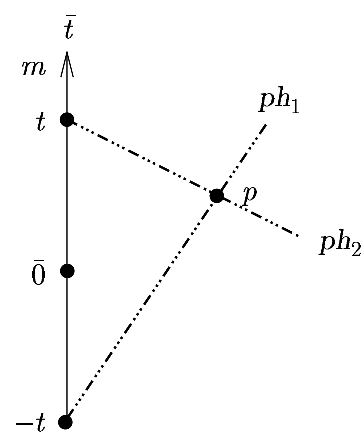



Figure 191: $p$ is 1/2-simultaneous with $\bar{0}$

(2)

$$S_m^{1/2} \stackrel{\rm def}{=} \{p \in {}^nF : p \text{ is } 1/2 - \text{simultaneous with } \bar{0}\}.$$

(3) Let $p, q \in {}^nF$. Then $p$ and $q$ are called $S_m^{1/2}$-simultaneous (or *1/2-simultaneous* for short) iff $(\exists t \in \bar{t})[p+t, q+t \in S_m^{1/2}]$.

$\triangleleft$

Clearly, being $S_m^{1/2}$-simultaneous is an equivalence relation on ${}^nF$, if we assume **Bax**$^-$. Now we are ready for defining our new axiom **AxR**$^+$ for our full Reichenbachian theories. **AxR**$^+$ is of a character which, in principle at least, is "experimentally testable" or "observable". So, assuming **AxR**$^+$ does not contradict Reichenbachian philosophy (i.e. it is of a different status than, say, speaking about the one-way speed of light is).

**AxR**$^+$ $\quad (\forall m, k \in Obs)\Big(tr_m(k) \parallel \bar{t} \Rightarrow$

$[\, p \text{ and } q \text{ are } S_m^{1/2}\text{-simultaneous}] \Leftrightarrow [\mathsf{f}_{mk}(p) \text{ and } \mathsf{f}_{mk}(q) \text{ are } S_k^{1/2}\text{-simultaneous}]\Big).$

The $S_m^{1/2}$-simultaneity relation (which was defined on the set ${}^nF$) induces an equivalence relation $E_m^{1/2}$ on the set $\mathcal{P}(B)$ of events via $w_m$. Now, **AxR**$^+$ says that $E_m^{1/2}{=}E_k^{1/2}$ if $tr_m(k) \parallel \bar{t}$. Still in other words, **AxR**$^+$ says that if $m$ and $k$ do not move relative to each other, then they "see" the same events as 1/2-simultaneous (1/2-simultaneity will be the same notion for them). We are going to see that, in models of **Bax**$^-$, **AxR**$^+$ is equivalent with each one of the following two, simpler, statements:

**AxR**$^-$ Let $m \in Obs$. Assume $p, q \in S_m^{1/2}$ and $p \neq q$. Then $\overline{pq} \subseteq S_m^{1/2}$.

**AxR$^{1/2}$** $S_m^{1/2}$ is a(n $m$-space like) hyper-plane, for all $m \in Obs$.

We will also see that, in models of **Bax**$^-$, $\mathbf{R}_\triangle(E)$ is equivalent to **AxR**$^+$+$\mathbf{R}_2$(**AxE**).

The next lemma sheds light to the relevance of $S_m^{1/2}$ in our present investigations.

**LEMMA 4.5.21** *Assume* $\mathfrak{M} \models$ **Bax**$^-$ *and let* $P = \langle P_m : m \in Obs\rangle$ *be such that* $P_m$ *is an* $m$*-space like hyper-plane containing* $\bar{0}$ *for all* $m$*. Then*

$$\mathfrak{M}/P \models \forall d(\mathsf{c}_m(d) = \mathsf{c}_m(-d)) \qquad \text{iff} \qquad P_m = S_m^{1/2}.$$

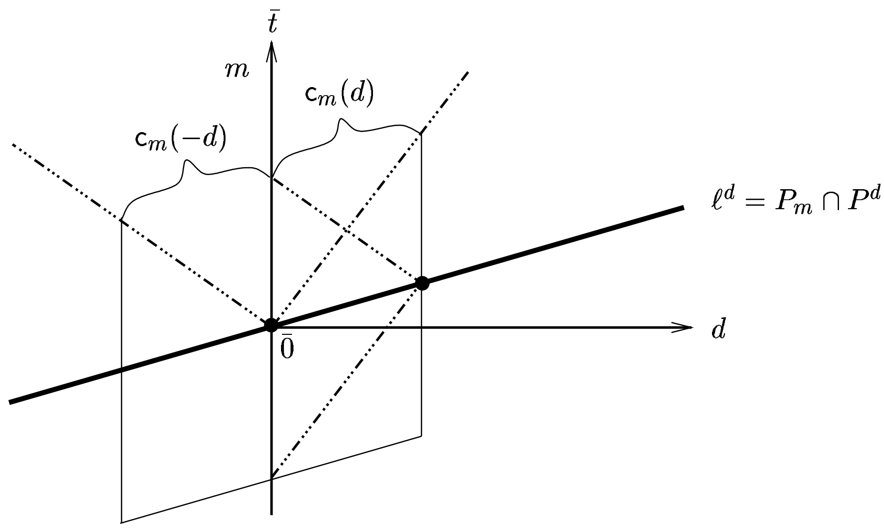



Figure 192: Illustration for the proof of Lemma 4.5.21.

**Proof.** Let $m \in Obs$ and $d \in \mathsf{directions}$ be arbitrary. Let $P^d \stackrel{\text{def}}{=} \mathsf{Plane}(\bar{t}, \{\lambda \cdot d : \lambda \in F\})$ and let $\ell^d \stackrel{\text{def}}{=} P_m \cap P^d$, see Figure 192.

Now $\mathsf{c}_m(d) = \mathsf{c}_m(-d)$ in $\mathfrak{M}/P$ iff $\ell^d \in S_m^{1/2}$, this can be read-off from Figure 192. This implies our statement. See also Figure 193. ∎

Lemma 4.5.21 shows that $\mathfrak{M}$ can have an isotropic relativization only if $S_m^{1/2}$ is an $m$-like hyper-plane in it for all $m$, i.e. if $\mathbf{AxR^{1/2}}$ holds in it. Thus in particular, $\mathbf{Asim(Bax)} \models \mathbf{AxR^{1/2}}$. We are going now to investigate when $S_m^{1/2}$ is a hyper-plane. Consider the following weaker statement:

$$\mathbf{AxR^{--}} \quad (\forall m \in Obs)(\forall p \in S_m^{1/2})\overline{\bar{0}p} \subseteq S_m^{1/2}.$$

Clearly, $\mathbf{AxR^{--}}$ is a weaker statement than $\mathbf{AxR^{-}}$, which is a weaker statement than $\mathbf{AxR^{1/2}}$. We are going to show that $\mathbf{AxR^{--}}$ always holds in $\mathbf{Bax^{-}}$, and in models of $\mathbf{Bax^{-}}$, $\mathbf{AxR^{--}}$ does not imply $\mathbf{AxR^{1/2}}$ while $\mathbf{AxR^{-}}$ already implies $\mathbf{AxR^{1/2}}$.

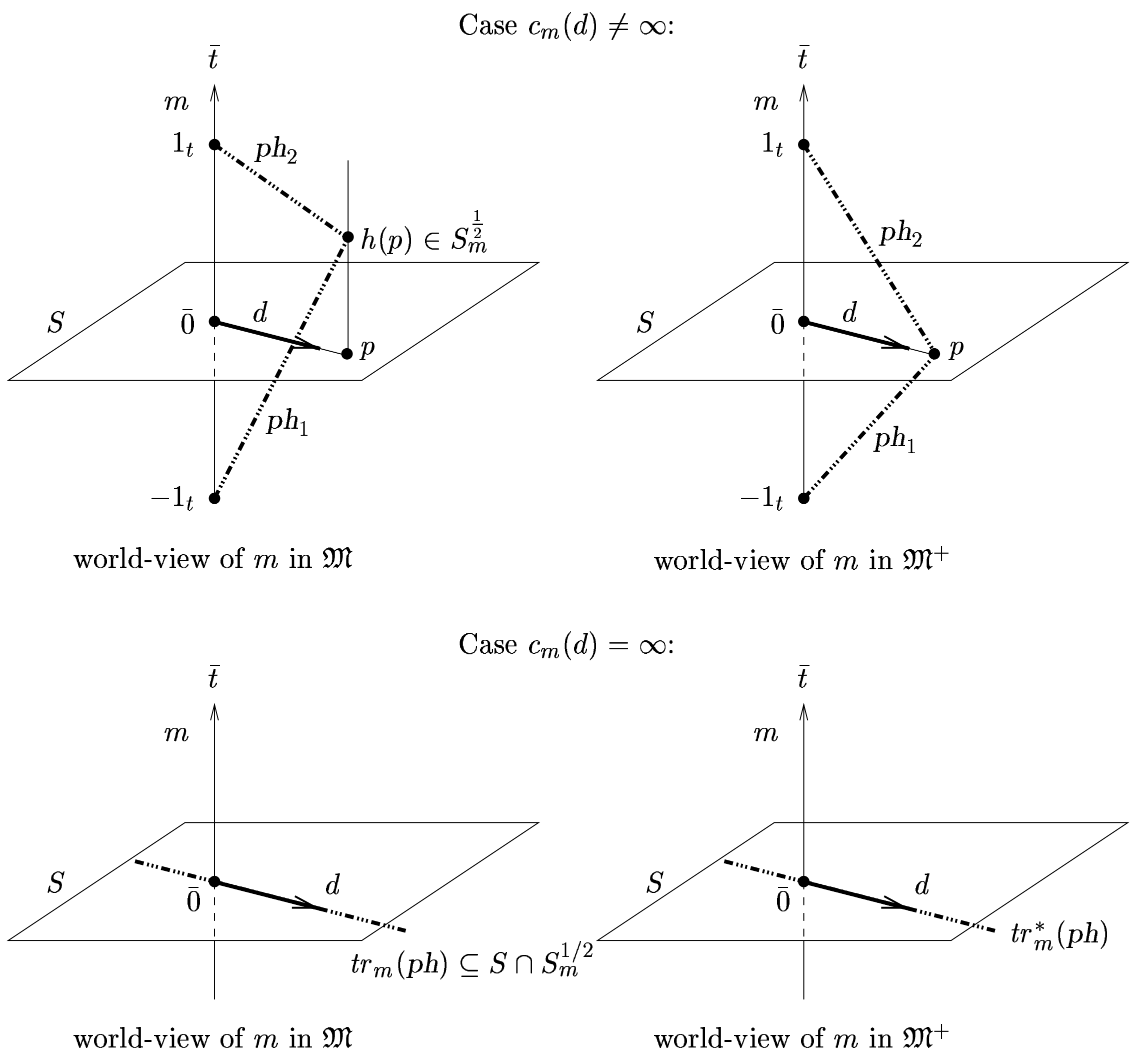



Figure 193: Illustration for the proof of Lemma 4.5.21

**LEMMA 4.5.22** *The following four statements are true.*

$\mathbf{Bax}^- \models \mathbf{AxR}^{--}$.

$\mathbf{Bax}^- \models \mathbf{AxR}^+ \leftrightarrow \mathbf{AxR}^-$.

$\mathbf{Bax}^- \models \mathbf{AxR}^+ \leftrightarrow \mathbf{AxR^{1/2}}$.

$\mathbf{Bax}^- \models \mathbf{R}_\Delta(E) \leftrightarrow (\mathbf{AxR}^+ + \mathbf{R}_2(\mathbf{AxE}))$.

**Proof.** The proof of $\mathbf{Bax}^- \models \mathbf{AxR}^{--}$ is illustrated in Figure 194, we do not write out a detailed proof.

Next we show that $\mathbf{AxR}^+$, $\mathbf{AxR}^-$, and $\mathbf{AxR^{1/2}}$ are equivalent in $\mathbf{Bax}^-$. Let us assume $\mathbf{Bax}^-$ from now on in the proof.

Proof of $\mathbf{AxR}^+ \rightarrow \mathbf{AxR}^-$: We have already seen that $\mathbf{AxR}^{--}$ holds in $\mathbf{Bax}^-$. $\mathbf{AxR}^-$ differs from $\mathbf{AxR}^{--}$ in that in $\mathbf{AxR}^{--}$, $\bar{0}$ is replaced with an arbitrary $q \in S_m^{1/2}$. $\mathbf{AxR}^+$ will be used to transform the situation to a world-view where $q$ is on $\bar{t}$, we apply $\mathbf{AxR}^{--}$ there, and then we use $\mathbf{AxR}^+$ again to transfer the result back to $m$'s world-view. In detail: Let $q, p \in S_m^{1/2}, q \neq p$. Let $k \in Obs$ be such that $q \in tr_m(k) \parallel \bar{t}$. Such a $k$ exists. Then $q' \stackrel{\text{def}}{=} \mathsf{f}_{mk}(q)$ and $p' \stackrel{\text{def}}{=} \mathsf{f}_{mk}(p)$ are $S_m^{1/2}$-simultaneous by $\mathbf{AxR}^+$. Further, $q' \in \bar{t}$. Then, by the definition of $S_k^{1/2}$-simultaneity, $p'' \stackrel{\text{def}}{=} p' - q' \in S_k^{1/2}$. Thus $\overline{\bar{0}p''} \subseteq S_k^{1/2}$ by $\mathbf{AxR}^{--}$, so any two elements of the line $\ell' \stackrel{\text{def}}{=} \overline{q'p'}$ are $S_k^{1/2}$-simultaneous. But since $\mathsf{f}_{mk}$ is a collineation by $\mathbf{Bax}^-$(Thm.4.3.11), we have that $\ell' = \mathsf{f}_{mk}[\overline{qp}]$. By $\mathbf{AxR}^+$ again, we get that any two elements of $\overline{qp}$ are $S_m^{1/2}$-simultaneous. By $q, p \in S_m^{1/2}$ this implies that $\overline{qp} \subseteq S_m^{1/2}$. See Figure 195.

Proof of $\mathbf{AxR}^- \rightarrow \mathbf{AxR^{1/2}}$: Assume $\mathbf{AxR}^-$. Then, it is known from geometry (and it is not hard to see) that $S_m^{1/2}$ is a $j$-dimensional plane of ${}^nF$ for some $j \leq n$. To prove that $S_m^{1/2}$ is a hyper-plane it remains to prove that $S_m^{1/2}$ is $(n-1)$-dimensional. For every $i \in n \setminus \{0\}$ let

$$p^i \in S_m^{1/2} \cap (\mathsf{Plane}(\bar{t}, \bar{x}_i) \setminus \bar{t}). \tag{293}$$

Such $p^i$'s exist because of the following. Let $i \in n \setminus \{0\}$. Case 1: $\mathsf{c}_m(1_i) = \infty$. Then every $p^i \in \bar{x}_i \setminus \{\bar{0}\}$ is such. Case 2: $\mathsf{c}_m(1_i) \neq \infty$. Let $ph_1, ph_2 \in Ph$ such that $-1_t \in tr_m(ph_1)$, $1_t \in tr_m(ph_2)$, $ph_1$ is moving in direction $1_i$ and $ph_2$ is moving in direction $-1_i$. Then $tr_m(ph_1) \cap tr_m(ph_2) \neq \emptyset$ by $c_m(1_i) \neq \infty$. Let $p^i \in tr_m(ph_1) \cap tr_m(ph_2)$. For this choice of $p^i$, (293) above holds.

Now, vectors in the set $\{ p^i : i \in n \setminus \{0\} \}$ are linearly independent since $p^i \in \mathsf{Plane}(\bar{t}, \bar{x}_i) \setminus \bar{t}$. Therefore $S_m^{1/2}$ is at least $(n-1)$-dimensional. It is easy to check

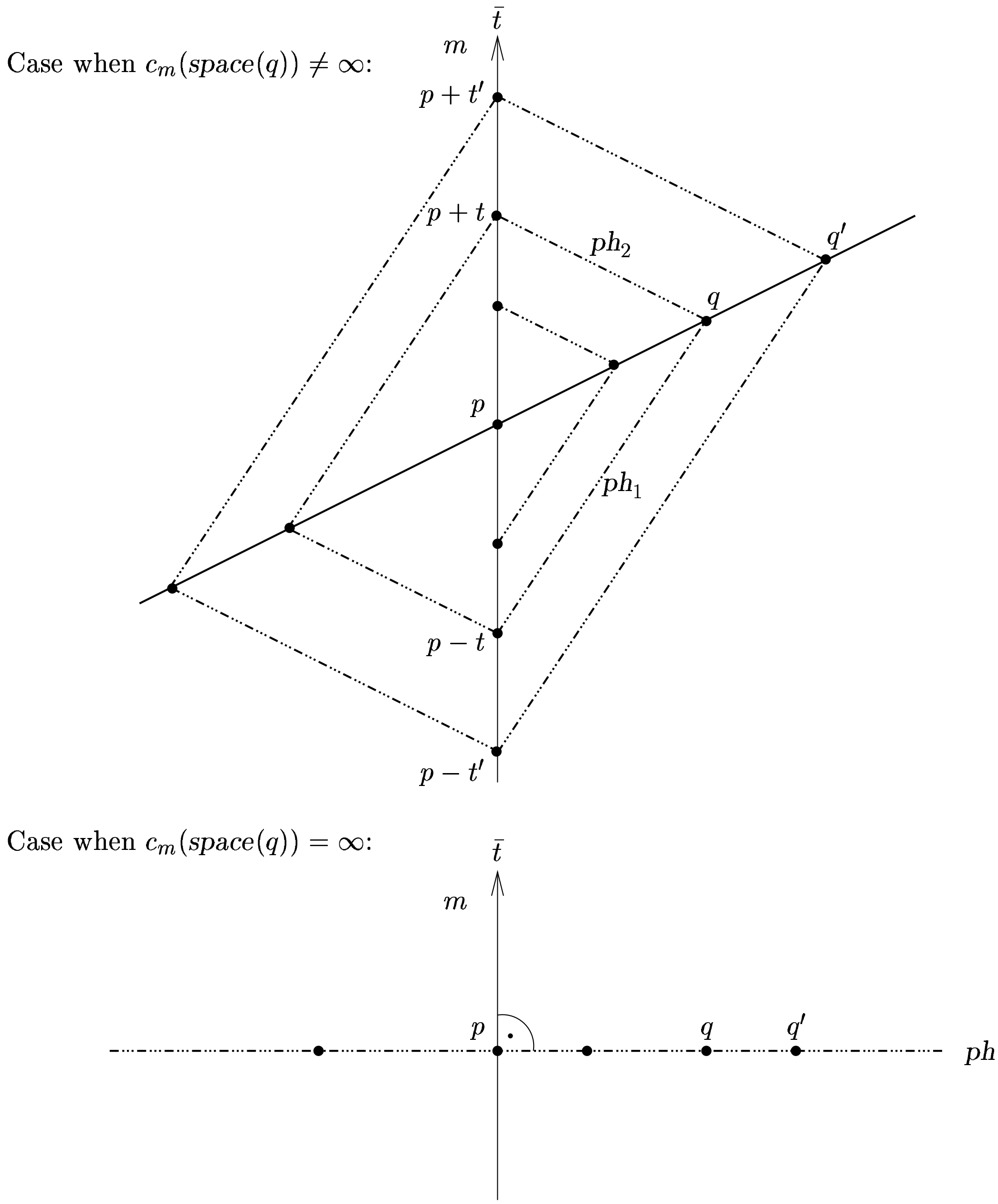



Figure 194: Illustration for the proof of $\mathbf{Bax}^- \models \mathbf{AxR}^{--}$.

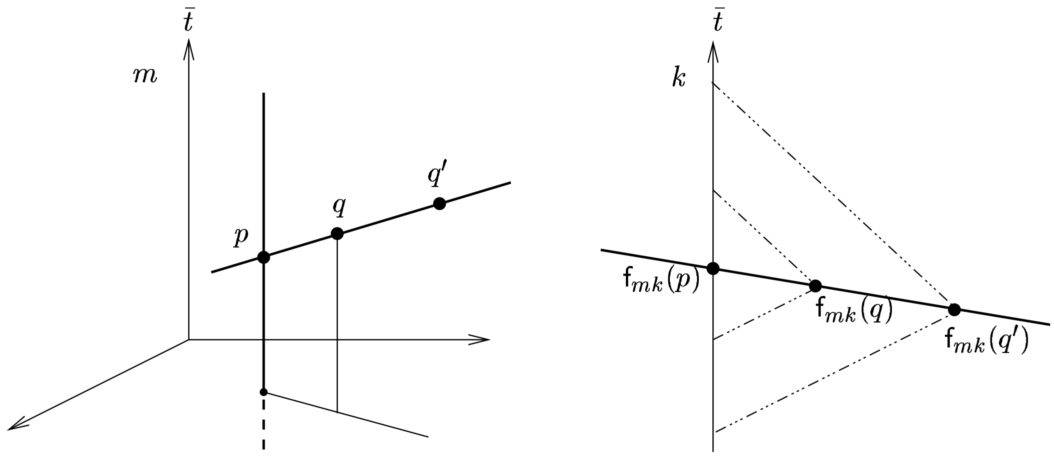



Figure 195: Illustration for the proof of $\mathbf{AxR^+} \to \mathbf{AxR^-}$.

that $S_m^{1/2} \cap \bar{t} = \{\bar{0}\}$. Hence $S_m^{1/2}$ is at most $(n-1)$-dimensional. Therefore $S_m^{1/2}$ is $(n-1)$-dimensional.

It remains to show that $S_m^{1/2}$ is $m$-space-like. To see this let $\ell \in \mathsf{Eucl}$ such that $\bar{0} \in \ell \subseteq S_m^{1/2}$ and $ang^2(\ell) \neq \infty$. Let $p \in \ell$ be such that $p \neq \bar{0}$. Such a $p$ exists because $\bar{t} \not\subseteq S_m^{1/2}$. Then for some $t \in \bar{t}$ there is a photon round-trip $-t \mapsto p \mapsto t$. But then, as it is illustrated in Figure 196, $\ell$ is not a life-line of a slow observer or of a photon.

Proof of $\mathbf{AxR^{1/2}} \to \mathbf{AxR^+}$: Assume $\mathbf{AxR^{1/2}}$, $tr_m(k) \parallel \bar{t}, p \in \bar{t}$ and $r$ is $S_m^{1/2}$-simultaneous with $p$. It is enough to show that $r' \overset{\text{def}}{=} \mathsf{f}_{mk}(r)$ is $S_k^{1/2}$-simultaneous with $p' \overset{\text{def}}{=} \mathsf{f}_{mk}(p)$. Let $q \in tr_m(k)$ be such that $p$ and $q$ are $S_m^{1/2}$-simultaneous. Such a $q$ exists. Let $q' \overset{\text{def}}{=} \mathsf{f}_{mk}(q)$. As one can see in Figure 197, then $p'$ and $q'$ are $S_k^{1/2}$-simultaneous.

Let $s$ be the fourth vertice of the paralelogram $pqrs$ and let $s' \overset{\text{def}}{=} \mathsf{f}_{mk}(s)$. See Figure 198. Then $p'r'q's'$ is a paralelogram in $k$'s world-view, too, because $\mathsf{f}_{mk}$ is a collineation. As illustrated in Figure 198, one can see that $s'$ and $q'$ are $S_k^{1/2}$-simultaneous, because $r$ and $p$ are $S_m^{1/2}$-simultaneous. By $\mathbf{AxR^{1/2}}$ then every two elements of the plane containing $p', q', s'$ are $S_k^{1/2}$-simultaneous. In particular, $r'$ and $p'$ are $S_k^{1/2}$-simultaneous, as was to be shown.

Proof of $\mathbf{R}_\Delta(E) \to (\mathbf{AxR^+} + \mathbf{R}_2(\mathbf{AxE}))$: It is not difficult to check that

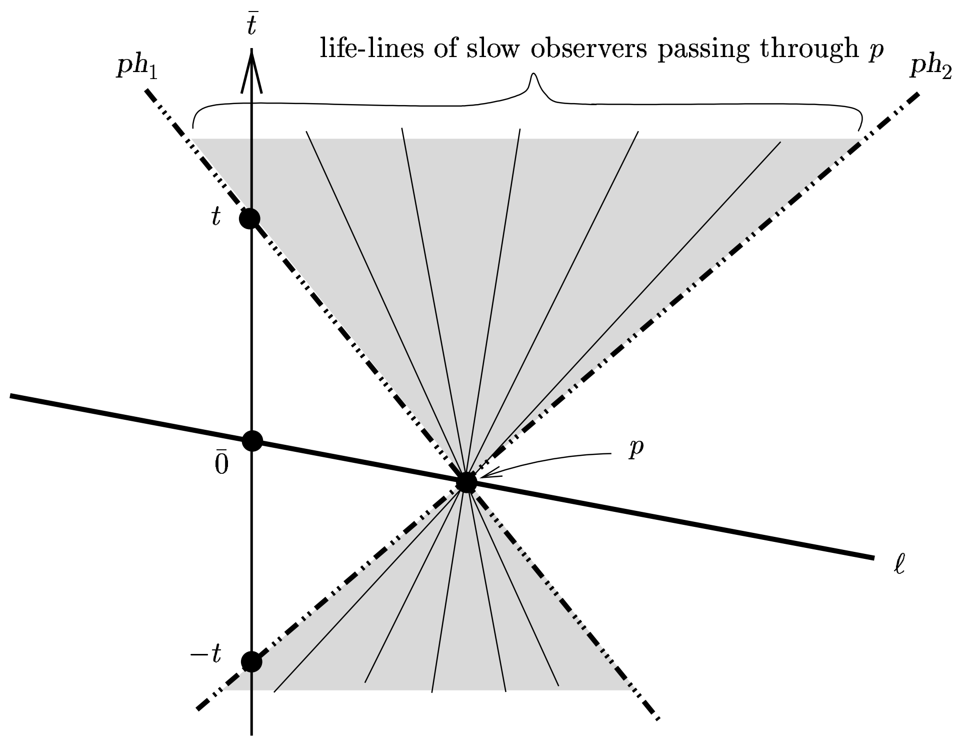



Figure 196: Illustration for the proof of $\mathbf{AxR^-} \to \mathbf{AxR^{1/2}}$.

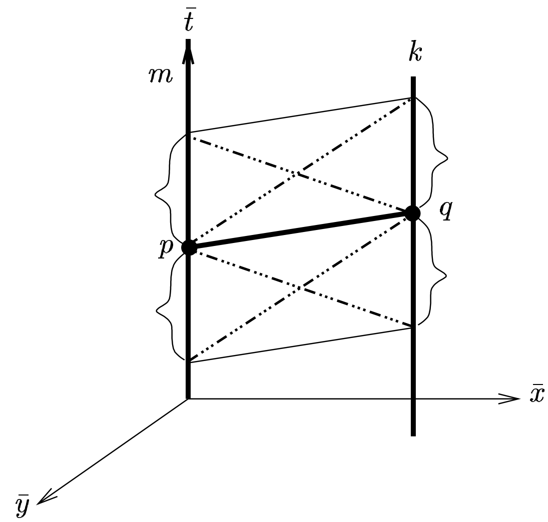



Figure 197: If $p$ and $q$ are $S_m^{1/2}$-simultaneous, then they are also $S_k^{1/2}$-simultaneous

$\mathbf{Asim}(\mathbf{Bax}) \models \mathbf{AxR^+}$. Thus $\mathbf{Bax}^- + \mathbf{R}_\Delta(E) \models \mathbf{Reich}(\mathbf{Bax}) \models \mathbf{AxR^+}$, by $\mathbf{Asim}(\mathbf{Bax}) = \mathsf{Mod}(\mathbf{Reich}(\mathbf{Bax}))$, cf. Thm.s 4.5.11, 4.5.13. We have already seen $\mathbf{R}_\Delta(E) \rightarrow \mathbf{R}_2(\mathbf{AxE})$ (Thm.4.5.11).
<u>Proof of $(\mathbf{AxR^+} + \mathbf{R}_2(\mathbf{AxE})) \rightarrow \mathbf{R}_\Delta(E)$:</u> Assume $\mathfrak{M} \models \mathbf{Bax}^- + \mathbf{AxR^+} + \mathbf{R}_2(\mathbf{AxE})$. Then $\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Bax})$ and $\mathfrak{M} \models \mathbf{AxR^{1/2}}$. Let $P \stackrel{\text{def}}{=} \langle S_m^{1/2} : m \in Obs \rangle$ and $\mathfrak{M}^+ \stackrel{\text{def}}{=} \mathfrak{M}/P$. Then $\mathfrak{M}^+ \models \mathbf{Reich}_0(\mathbf{Bax})$ by Thm.4.5.8, and $\mathfrak{M}^+ \models \mathsf{c}_m(d) = \mathsf{c}_m(-d)$ by Lemma 4.5.21. Thus $\mathfrak{M}^+ \models \mathbf{Bax}$ by Proposition 4.5.4, i.e. $\mathfrak{M} \in \mathbf{Asim}(\mathbf{Bax})$. But $\mathbf{Asim}(\mathbf{Bax}) \models \mathbf{R}_\Delta(E)$ by Thm.4.5.13.
∎

Consider the theories $\mathbf{Bax}^- + \mathbf{Ax}$ where $\mathbf{Ax}$ is one of $\mathbf{AxR^{--}}$, $\mathbf{AxR^-}$, $\mathbf{AxR^{1/2}}$, $\mathbf{AxR^+}$, $\mathbf{R}_2(\mathbf{AxE})$, $\mathbf{R}_\Delta(E)$. By Lemma 4.5.22, we get the lattice in Figure 199.
It is not hard to see that $\mathbf{Bax}^- \not\models \mathbf{R}_2(\mathbf{AxE})$ and $\mathbf{Bax}^- \not\models \mathbf{AxR^-}$, these imply the two $\neq$'s in the lattice. Two of the four questions indicated on the lattice are equivalent to Problem 4.5.10 and amount to asking whether

$$\mathbf{Bax}^- \models \mathbf{R}_2(\mathbf{AxE}) \rightarrow \mathbf{AxR^-}$$

holds; the remaining two questions are equivalent to asking whether

$$\mathbf{Bax}^- \models \mathbf{AxR^+} \rightarrow \mathbf{R}_2(\mathbf{AxE})$$

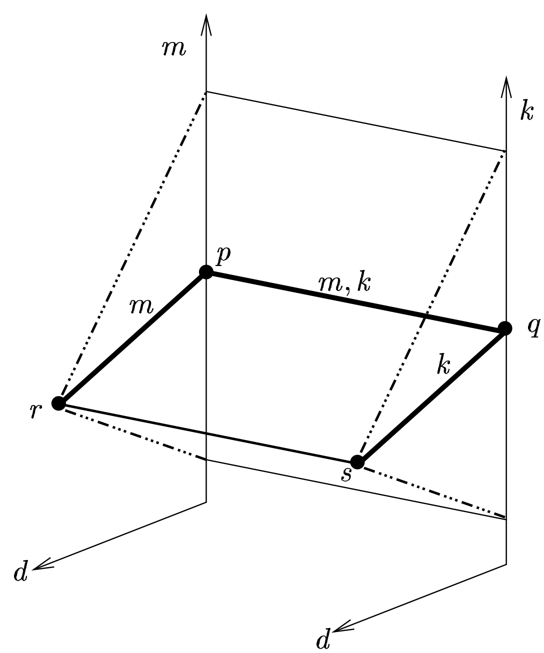



Figure 198: Illustration for proof of $\mathbf{AxR}^- \rightarrow \mathbf{AxR}^+$.

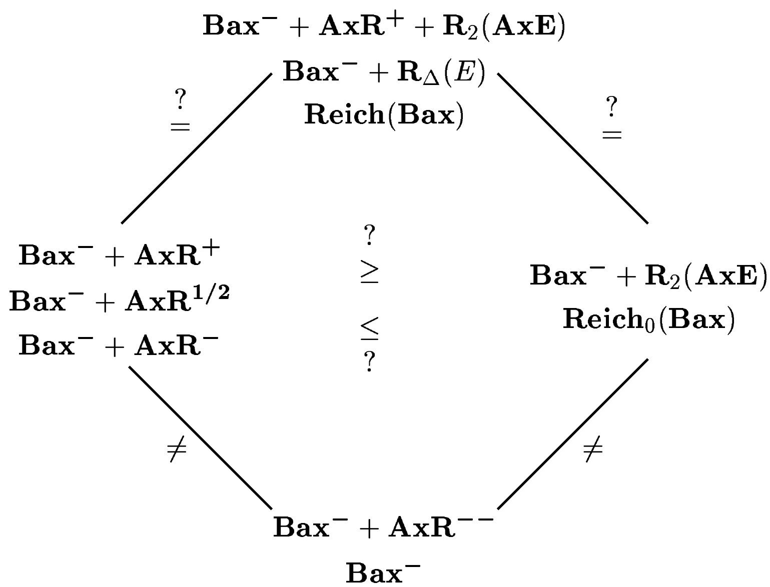



Figure 199:

holds or not. We do not even know whether the weaker statement below holds:

$$\mathbf{Bax}^- + \forall m \forall d(\mathsf{c}_m(d) = \mathsf{c}_m(-d)) \models \mathbf{Bax}.$$

The next corollary gives alternative axiomatizations for $\mathbf{Reich}(Th)$ in terms of Einstein's 1/2-simultaneities.

**COROLLARY 4.5.23** Let $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$.

$$\mathbf{Reich}(Th) \;=\!\!|\!\!\models \mathbf{Reich}_0(Th) + \mathbf{AxR}^+ \quad ,$$

$$\mathbf{Reich}(Th) \;=\!\!|\!\!\models \mathbf{Reich}_0(Th) + \mathbf{AxR}^- \quad ,$$

$$\mathbf{Reich}(Th) \;=\!\!|\!\!\models \mathbf{Reich}_0(Th) + \mathbf{AxR}^{1/2} \quad .$$

$\triangleleft$

**Problem 4.5.24** Recall the definitions of the "differentiable" versions $\mathbf{Bax}_\partial^-$ and $\mathbf{Reich}_0(\mathbf{Bax})_\partial$ from Def.4.5.3. Is

$$\mathbf{Reich}_0(\mathbf{Bax})_\partial \models \mathbf{Reich}(\mathbf{Bax})$$

or

$$(\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Bax}_\partial^-) \models \mathbf{Reich}(\mathbf{Basax})$$

true?

**Relativizing with arbitrary surfaces, connections with re-coordinatization**

**Definition 4.5.25** Let $H \subseteq {}^nF$. Then $H$ is called a (generalized) *surface* of ${}^nF$ iff there is a function $h : S \longrightarrow F$ such that $H = \{\langle h(s), s_1, \ldots, s_{n-1}\rangle : s \in S\}$, where $S$ is the space part[486] of ${}^nF$, cf. p.470. $H$ is called *continuous* if $h$ is a continuous function; etc. $\lhd$

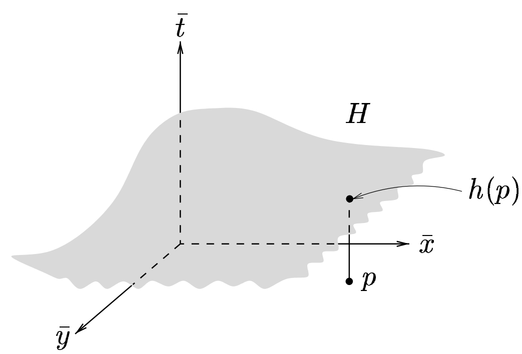



Figure 200: $H$ is a surface.

In Def.4.5.6 we used hyper-planes $P$ for artificial simultaneities in obtaining the relativized model $\mathfrak{M}/P$ from $\mathfrak{M}$. We could repeat the same definition with arbitrary surfaces $H$ in place of $P$.

[486] $S = \{0\} \times {}^{n-1}F$.

**Conjecture 4.5.26**

(i) *Assume that* $\mathfrak{M}/H$ *is obtained from* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax})$ *with relativizing to surfaces* $H_m, m \in Obs$. *Then*

$$\mathfrak{M}/H \models \mathbf{Bax}^- \;\Rightarrow\; (\forall m \in Obs)[H_m \text{ is an } m\text{-space like hyper-plane}].$$

(ii) *We conjecture the same if one assumes only* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax})$ *in place of* **Basax**.

◁

Actually, we guess that the above conjecture remains true if we replace $\mathfrak{M} \models \mathbf{Bax}^-$ in it with $\mathfrak{M} \models \mathrm{SPR}_0$, or even with $\mathbf{Bax}_3^{--}$ in place of $\mathbf{Bax}^-$. However, we did not think about this.

Our mentioning models obtained by arbitrary surfaces (as artificial simultaneities) is partly motivated by Friedman [90, pp.170-172] where Reichenbachian relativity via arbitrary surfaces is discussed.[487] Our Conjecture 4.5.26 above seems to point in the direction that arbitrary surfaces are not particularly useful in studying Reichenbachian versions of special relativity. On the other hand, in d'Inverno [75] on pp.217-220, the transition from Schwarzschild coordinates to Eddington-Finkelstein coordinates seems to suggest that in general relativity, using surfaces (instead of hyper-planes) as "artificial simultaneities" is useful.

We mention the following only because it might interest the reader, but it is not necessary for the rest of this work. Let us consider e.g. the Schwarzschild-coordinatization of spacetime outside the event-horizon of a Schwarzschild black hole as represented in Fig.16.7 (p.217) of d'Inverno [75]. That coordinatization uses the 1/2-simultaneities of observers outside the event-horizon and at relative rest w.r.t. the black hole (i.e. at rest w.r.t. the singularity). More precisely, let $m$ be such a "suspended" observer outside the event-horizon. Then using $S_m^{1/2}$ as our simutaneity (and using $m$'s internal clocks for measuring time), one obtains a coordinatization like the one on Fig.16.7 in d'Inverno. Further, if we want to obtain the (advanced) Eddington-Finkelstein coordinatization in Fig.16.10 of d'Inverno from the just discussed Schwarzschild one, then we can do this by relativizing to a space-like hypersurface $P$ of the original Schwarzschild coordinatization (or world-view). In this step, $P$ is not a hyper-plane and it cannot be replaced by a hyper-plane. I.e. we cannot choose $P$ to be "flat".

[487] A further motivation is preparing "the ground" for our discussion of the question why it is only **AxE** of the axioms of **Basax** about which we worry (from the testability point of view) when defining **Reich**(**Basax**). Cf. Remark 4.5.29 way below.

Concerning our earlier open question whether the two theories $\mathbf{Reich}_0(Th)$ and $\mathbf{Reich}(Th)$ coincide, the above conjecture implies the following. If $\mathbf{Reich}_0(\mathbf{Basax}) \not\models \mathbf{Reich}(\mathbf{Basax})$, then the counterexample (i.e. a model $\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Basax})$) is *not* obtainable from a model even of **Bax** by relativizing to *any surface* (as a new, artificial simultaneity). Hence, for any one of our four distinguished theories $Th$, if $\mathbf{Reich}_0(Th) \not\models \mathbf{Reich}(Th)$ is the case, then this cannot be proved by generalizing relativization to arbitrary surfaces as artificial simultaneities (and starting from usual models in which at least $\mathrm{SPR}_0$ or $\mathbf{Bax}^{--}$ holds).

With this, we stop discussing surfaces other than planes as artificial simultaneities.

**Remark 4.5.27** (continuation of the present subject in other sections and chapters of this work) The subject matter $\mathbf{Reich}(Th)$, $Th \geq \mathbf{Bax}$ of the present section will be further discussed in sections 4.7, 4.8 and in the Geometry chapter (§6). Inside §6, we would like to call attention to §6.6.10 where $\mathbf{Reich}(Th)$ will be especially often investigated, cf. e.g. pp.1124-1128 and Thm.s 6.6.107-6.6.110. However, other parts of §6 also study $\mathbf{Reich}(Th)$, cf. e.g. Proposition 6.7.16 on p.1145. ◁

**Questions for future research 4.5.28**

(i) Clarify the connection between $\mathbf{Reich}_0(Th)$ and our chapter on the (first-order logic) theories of accelerated observers (§8). E.g. from the point of view of rotating observers, is it better to base the theory of accelerated observers on **Reich**(**Newbasax**) in place of **Newbasax**?

(ii) Investigate which one of our paradigmatic effects, collected in the last sections of §2 as well as in §4.7 way below, hold in $\mathbf{Reich}_0(Th), \mathbf{Reich}(Th)$ for $Th \in \{\mathbf{Basax}, \mathbf{Newbasax}, \mathbf{Flxbasax}, \mathbf{Bax}\}$. Actually, we will do part of this in section 4.8.

**Remark 4.5.29** (Why exactly these axioms for $\mathbf{Reich}(Th)$, e.g. for **Reich**(**Flxbasax**)?) $\mathbf{Reich}_0(\mathbf{Flxbasax})$ was obtained from **Flxbasax** by revising one of its axioms $\mathbf{AxE_{02}}$ on the basis of the "Reichenbach-Grünbaum philosophy" saying that the one-way speed of light is not "observable". By a little abuse of terminology we could say that the Reichenbach-Grünbaum-Winnie school says that $\mathbf{AxE_{02}}$ is not observational, hence should be replaced with an observational version like e.g. $\mathbf{R}(\mathbf{AxE_{02}})$. The question naturally comes up why we revise only the speed of light axiom $\mathbf{AxE_{02}}$ and not the rest of the axioms like **Ax1** etc.

One can answer this on two levels.

(1) The first, "easy-going" answer is that $\mathbf{AxE_{02}}$ is the "fancy" axiom or the exotic, exciting axiom of **Flxbasax**, the rest of the axioms go back (more or less) to the time of Galileo (let us ignore the photon-part of **Ax5** [488]) so why should we worry about revising them.

(2) To obtain a more thorough answer, let us recall the list of axioms.

$$\mathbf{Reich}_0(\mathbf{Flxbasax}) = \mathbf{Bax}^- + \mathbf{R}(\mathbf{AxE_{02}}) = \mathbf{R}(\mathbf{AxE_{02}}) +$$

$$\{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Obs}}, \mathbf{Ax5_{Ph}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{01}}, \mathbf{AxP1}\}.$$

We claim that $\mathbf{AxE_{01}}$, **AxP1** are observational (for brevity, we do not discuss the reasons which are available in the literature). We did discuss $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$ on p.190, so let us concentrate on the rest, i.e. on $\mathbf{Ax1}$-$\mathbf{Ax5_{Ph}}$.

**Ax1-Ax2** can be regarded as notational preparations for $\mathbf{Ax3_0}$, hence we ignore them.[489] $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax5_{Ph}}$ express properties of the <u>*logic*</u> of the present work, i.e. that we can think in terms of thought-experiments. As we already indicated, this logic can be made explicit by using <u>*first-order modal logic*</u> as an expansion of our present frame language. We postpone the presentation of this modal logic (of $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax5_{Ph}}$) to a future work (it was presented at the University of Amsterdam at a series of lectures in 1998 spring; part of that can be found in Vályi [262]). Since the "**Ax5** axioms" can be regarded as a <u>*kind of logical axioms*</u>, it does not seem to be important to revise them from the point of view of observationality.[490] (They do contain tacit assumptions about "important things" like the nature of the life-lines of certain bodies, but these assumptions are only inherited from other (than **Ax5**) axioms where they do appear explicitly, hence they can be addressed when discussing those axioms).

Now, the only axiom which is left to revise is $\mathbf{Ax3_0}$. So let us turn to discussing this axiom. The discussion of $\mathbf{Ax3_0}$ is organized into items (i)-(iii) below.

**(i)** This is an observational axiom <u>relative</u> to the life-lines of photons. Namely, $\mathbf{Ax3_0}$ says that the life-lines of inertial bodies are straight lines. The part of this which says that the life-lines of such bodies are straight *relative* to the life-lines of photons is observational, hence it can be safely included in **Reich**(*Th*). It remains to discuss the part $\mathbf{Ax3_{Ph}}$ of $\mathbf{Ax3_0}$ which says that the life-lines of photons are straight.

[488] e.g. because we always adjust it to the remaining axioms whenever a change is made

[489] their revision belongs to the revision of $\mathbf{Ax3_0}$, hence is postponed

[490] To be more precise, we note that $\mathbf{Ax5_{Obs}}$ does carry some implicit assumption which is more than a "logical axiom". Namely, it implies that all velocities which are "slower" than the velocity of light are *realizable by some potential observer.* All the same, this assumption seems innocent enough. We will return to the issue of eliminating this assumption e.g. around the end of §5.

**(ii)** $\mathbf{Ax3_{Ph}}$ follows from assuming that $c_m(p,d)$ does not depend on $p$. The temporal part, $(\forall \Delta t \in \bar{t})c_m(p,d) = c_m(p+\Delta t, d)$ of this is an observational axiom[491], hence it is safely includable into $\mathbf{Reich}(Th)$. What remains to be discussed is $\mathbf{Ax3_{space,Ph}}$ which is a "fragment" of $\mathbf{Ax3_{Ph}}$ defined as:[492]

> $\mathbf{Ax3_{space,Ph}}$
>
> $(\forall \lambda \in F)c_m(p,d) = c_m(p + \lambda \cdot d, d).$

We claim that under the above discussed observational "fragments" of $\mathbf{Ax3_0}$, we have that $\mathbf{Ax3_{space,Ph}}$ implies $\mathbf{Ax3_0}$.

**(iii)** Throwing away $\mathbf{Ax3_{space,Ph}}$ would amount to allowing that very far (in "pure space") from the Earth the speed of light is different but in such a subtle way that this is impossible to detect by any kind of thought-experiment. There are two things to be said about this.

**(iii.1)** By using the methods of the present work, we could introduce a new, more radical version $\mathbf{Reich}^2$ of our operator **Reich**, such that in the so obtained hyper-Reichenbachian theories, $\mathbf{Ax3_{space,Ph}}$ is not included (but the above mentioned observational fragments of $\mathbf{Ax3_0}$ are included). The interested reader is invited to carry through this procedure as a kind of exercise. Completely analogously to our present development of the analysis of $\mathbf{Reich}(Th)$, one could elaborate the analysis of $\mathbf{Reich}^2(Th)$ and its relationship to relativization with hyper-surfaces (in contrast to our relativizations with hyper-planes). Since carrying this through on the basis of the present work seems to be more or less deterministic, we do not include it here (to improve coherence and compactness).

**(iii.2)** Studying $\mathbf{Reich}(Th)$ is well motivated by the literature[493], and has interesting applications in constructing the world-views of rotating observers[494], hence indirectly it might help someone in understanding the theory of rotating black holes. As a contrast, the present authors are not aware of such motivations for studying $\mathbf{Reich}^2(Th)$. This is a second reason for not studying $\mathbf{Reich}^2(Th)$.

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[491] The proof is available from the authors.

[492] in writing $p+\lambda\cdot d$ we assume that first $d$ is extended to an "n-vector", i.e. we mean $p+\lambda\cdot\langle 0,d\rangle$.

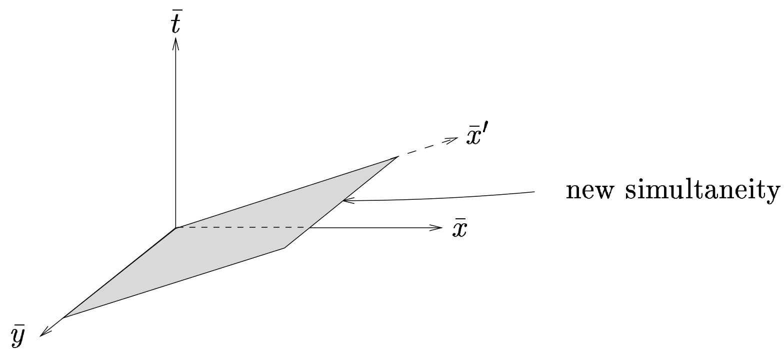
[493] Cf. e.g. Matolcsi [191].

[494] Cf. e.g. L. E. Szabó [244], Salmon [233].

In view of all the above, we are under the impression that, for the time being, $\mathbf{Ax3}_{\mathbf{space},\mathbf{Ph}}$ is a useful simplifying assumption in our thoeries $\mathbf{Reich}(Th)$.[495]

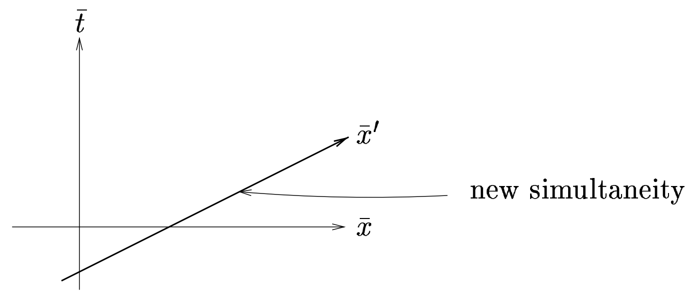
Summing up: We went through the axioms of **Reich**(**Basax**) and investigated whether some of them needs a revision similar to the one that was applied to **AxE**. (I.e. we asked ourselves "why exactly [and only] **AxE** was revised?") We found the following. The only remaining axiom that could need such a revision is the rather mild statement $\mathbf{Ax3}_{\mathbf{space},\mathbf{Ph}}$. Then we discussed whether revising that axiom here would be useful. We found that the nature of $\mathbf{Ax3}_{\mathbf{space},\mathbf{Ph}}$ is *different* from that of **AxE** to the extent that it seems better not to revise $\mathbf{Ax3}_{\mathbf{space},\mathbf{Ph}}$ in the present work.

**Remark 4.5.30** Theorem 4.5.13 shows that the models of $\mathbf{Reich}(Th)$ can be reduced to **Basax**-models via choosing *nonstandard simultaneities* which are "straight" hyper-planes, like this:

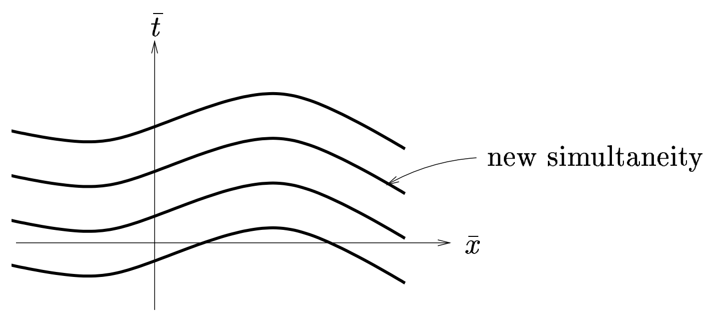



or this:

[495]There seems to be a big difference between the move $Th \mapsto \mathbf{Reich}(Th)$ and $\mathbf{Reich}(Th) \mapsto \mathbf{Reich}^2(Th)$, as follows. $Th \mapsto \mathbf{Reich}(Th)$ has significant philosophical implications e.g. because it implies things about the meaning of the English word "now". These implications in turn induce implications about our understanding of *causality* and *determinism*, cf. L. E. Szabó [244]. The present authors are not aware of similar implications in connection with $\mathbf{Reich} \mapsto \mathbf{Reich}^2$. (E.g. what commonly used word would so profoundly change its meaning as "now" does in the $Th \mapsto \mathbf{Reich}(Th)$ case?)




If we want to have (models based on) curved nonstandard simultaneities like this:




then we could achieve this by developing **Reich**$^2$(**Basax**) as outlined in the "$\mathbf{Ax3_0}$-part" (i.e. second part) of Remark 4.5.29 above. In more detail, in **Reich**(**Basax**), we could replace $\mathbf{Ax3_0}$ by $\mathbf{Ax3_{rel,Ph}}$, $\mathbf{Ax3_{time,Ph}}$ and $\mathbf{Ax3_{Ph,rest}}$ outlined below.

$\mathbf{Ax3_{rel,Ph}}$ says that life-lines of inertial bodies are straight relative to photon lines as explained in item (i) of Remark 4.5.29. (We leave the formalization of $\mathbf{Ax3_{rel,Ph}}$ to the interested reader.) $\mathbf{Ax3_{time,Ph}}$ was outlined in item (ii) of the quoted remark (it is the postulate involving $c_m(p + \Delta t, d)$.) $\mathbf{Ax3_{Ph,rest}}$ would state whatever properties of life-lines of photons we want to keep (after relaxing the original condition saying that they are straight lines). Such a condition could be that photon-lines are continuous and differentiable.

For more in this direction we refer the reader to the part beginning with item 4.5.25 and ending just before 4.5.27 (the title of that part is "Relativizing with arbitrary surfaces...").

## 4.6 Generalizing standard configurations

In §§ 4.6, 4.7 we will use the set *Triv* of trivial transformations and also axioms **Ax(***Triv***)**, **Ax(***Triv*$_t$**)** which are introduced in §§ 2.8, 3.5, 3.8. For completeness we recall the definitions.

$$Triv \stackrel{\text{def}}{=} Triv(n,\mathfrak{F}) \stackrel{\text{def}}{=} \{\, f \in Aftr \;:\; f \text{ is an isometry}^{496},\ f[\bar{t}] \parallel \bar{t},\ f(1_t)_t - f(\bar{0})_t > 0 \,\}.$$

As we explain in §3.5, the transformations in *Triv* involve *no "relativistic effects"*, one could say that they are very non-relativistic or, so to speak, trivial.

**Ax(***Triv***)** $(\forall f \in Triv)(\exists k \in Obs)\ \mathsf{f}_{mk} = f.$

> **Ax(***Triv***)** says that every observer can "re-coordinatize" his world-view by any trivial transformation. **Ax(***Triv*$_t$**)** below is a weaker form of **Ax(***Triv***)**.

**Ax(***Triv*$_t$**)** $(\forall f \in Triv)\ \Big(f[\bar{t}] = \bar{t} \quad \Rightarrow \quad (\exists k \in Obs)\ \mathsf{f}_{mk} = f\Big).$

Proposition 4.6.1 below can be regarded as the formalized version of what is said about standard configurations in usual relativity books, cf. e.g. d'Inverno [75] p.27, last 6 lines.[497] Intuitively, it says that, in $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv)$ we have that every pair $m, k$ of observers can be replaced with ones $m', k'$ in standard configuration such that the $m \mapsto m'$ and $k \mapsto k'$ differences remain "trivial", cf. the definition of *Triv* above. In other words, $m$ and $m'$ are the same observer up to trivial differences in orientation (and the same for $k$, $k'$). We defined standard configurations in Def.2.3.16, p.71.

**PROPOSITION 4.6.1** $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv) \models$
$\forall m,k\ \exists m',k'\ (\mathsf{f}_{mm'}, \mathsf{f}_{kk'} \in Triv \quad \& \quad m' \text{ and } k' \text{ are in standard configuration}).$
*In particular:*
$\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv) \models \forall m,k\ (\bar{0} \in tr_m(k) \ \Rightarrow$
$\exists m',k'\ [tr_m(m') = \bar{t} = tr_k(k') \quad \& \quad m' \text{ and } k' \text{ are in standard configuration}]).$

We omit the **proof**. ∎

It seems to us that Prop.4.6.1 above is related to Thm.3.6.4(i) saying "$PT = \{triv_0 \circ rhomb \circ triv \;:\; \ldots\}$" because the class *Rhomb* of transformations

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[496] $f : {}^nF \longrightarrow {}^nF$ is an isometry iff it preserves the square of Euclidean distances, i.e. $(\forall p,q \in {}^nF)\|p-q\| = \|f(p) - f(q)\|$, cf. p.134 in §2.8 and Def.3.9.3 on p.349.

[497] There, the word "boost" is used for $\mathsf{f}_{mk}$ where $m,k$ are in standard configuration.

is related to standard configurations. The use of Prop.4.6.1 above is in that it says that in **Basax**, the $\mathsf{f}_{mk}$'s *can be reduced* to the case when $m$ and $k$ are in *standard* configuration. This proposition can be generalized to **Bax** (cf. Prop.4.6.2), but not much further.

**Proposition 4.6.2** *Prop.4.6.1 above remains true if* **Basax** *is replaced with* **Bax** *in the following sense.*

$$\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv) \models \forall m,k \Big(m \overset{\odot}{\to} k \Rightarrow \exists m',k' \,(\mathsf{f}_{mm'},\mathsf{f}_{kk'} \in Triv \quad \& \quad m' \text{ and } k' \text{ are in standard configuration})\Big).$$

We omit the **proof**. ∎

In weak theories like $\mathbf{Bax}^-$ or **Reich**(**Basax**), we cannot hope for the conclusion of Prop.4.6.1 or Prop.4.6.2 because we cannot expect $(\mathsf{f}_{mk}(\bar{0}) = \bar{0} \quad \wedge \quad \mathsf{f}_{mk}[\bar{t}\cup\bar{x}] \subseteq \mathsf{Plane}(\bar{t},\bar{x}))$ to imply that $\mathsf{f}_{mk}[\bar{y}] \subseteq S$ where $S$ is the space-part of ${}^nF$. Therefore we need a more flexible version of the notion of a standard configuration. For completeness, we note that Prop.4.6.2 above was our main motivation for discussing standard configurations.

**PROPOSITION 4.6.3** *Assume $n > 2$. Then*

$$\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(Triv) + \mathbf{Ax}(\|) \not\models \forall m,k \Big(\bar{0} \in tr_m(k) \Rightarrow \exists m',k' \,[tr_m(m') = tr_k(k') = \bar{t} \quad \& \quad m' \text{ and } k' \text{ are in standard configuration}]\Big).$$

**On the proof:** The proof uses artificial simultaneities as in the definition of **Asim**(**Basax**) cf. Def.4.5.6, p.568. We leave the details to the reader. ∎

**Conjecture 4.6.4** *Assume $n > 2$. Then*

$$\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(Triv) \not\models \exists m,k \,(m \neq k \text{ and } m,k \text{ are in standard configuration}).$$

◁

If the above conjecture is true then in some models of **Reich**(**Basax**) + **Ax**(*Triv*) *standard configurations* simply *do not exist* (in nontrivial form).

These motivate our definition below.

**Definition 4.6.5** Let $m,k \in Obs$.

**(i)** $m$ and $k$ are said to be in <u>*pre-standard configuration*</u> iff $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ and $\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t},\bar{x})] = \mathsf{Plane}(\bar{t},\bar{x})$. See Figure 201.

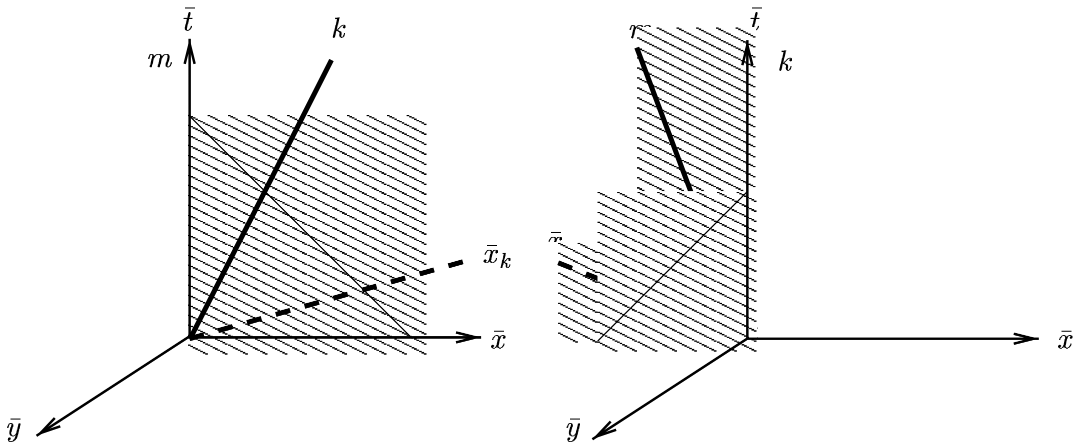



Figure 201: Pre-standard configuration.

**(ii)** $m$ and $k$ are in pre-standard *symmetric* configuration (*pre-standard-sym configuration* for short) iff in addition to the condition in (i) they also satisfy $(\star)$ and $(\star\star)$ below.

$(\star)$

$$[m \text{ sees } k \text{ moving forwards in direction } 1_x \quad \Leftrightarrow \quad k \text{ sees } m \text{ moving forwards in direction } 1_x].$$

See Figure 202.

$(\star\star)$

$$v_m(k) = 0 \Rightarrow [\mathsf{f}_{mk}(1_t)_t > 0 \text{ iff } \mathsf{f}_{mk}(1_x)_x > 0].$$

◁

Intuitively, if $v_m(k) \neq 0$, then $m$ and $k$ are in pre-standard configuration iff they meet at $\bar{0}$ and if they both see the other moving in direction $1_x$ (forwards or backwards). If $v_m(k) = 0$ then the second condition is of course granted, and we replace it by saying that they see each other's $x$-unit vectors pointing in direction $\bar{x}$.

**FACT 4.6.6** *Standard configurations are also pre-standard ones. I.e. if $m, k$ are in standard configuration then they are in pre-standard configuration too. (The other direction is of course not true in general.) But some standard configurations are* not *pre-standard-sym ones.*

◁

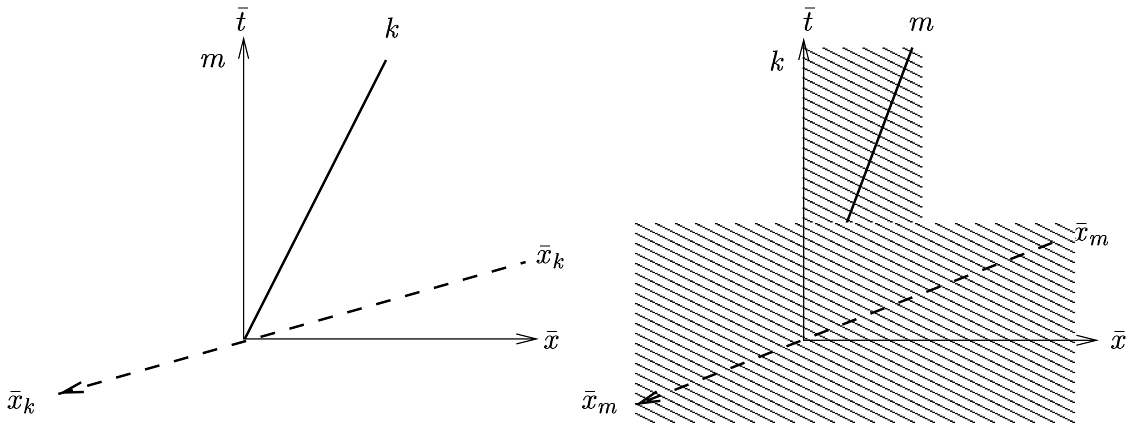



Figure 202: Pre-standard-sym configuration.

Let <u>*name-standard configuration*</u> be one of the variants of standard configuration. Then we say that <u>*name-standard configurations work*</u> in $\mathfrak{M}$ iff

$$\mathfrak{M} \models \forall m, k \Big( m \overset{\odot}{\to} k \;\Rightarrow$$

$$\exists m', k' \, (\mathsf{f}_{mm'}, \mathsf{f}_{kk'} \in \mathit{Triv} \quad \& \quad m' \text{ and } k' \text{ are in name-standard configuration})\Big).$$

Further, they <u>*work*</u> in *Th* iff they work in every model of *Th*. E.g. Prop.4.6.1 says that standard configurations work in **Basax**+**Ax**($\sqrt{\ }$)+**Ax**(*Triv*).

**PROPOSITION 4.6.7**

**Bax**$^-$ + **Ax**($\sqrt{\ }$) + **Ax**(*Triv*) $\models$ *pre-standard-sym configurations work.*[498]

*Therefore*

**Reich**(**Bax**) + **Ax**(*Triv*) $\models$ *pre-standard-sym configurations work.*

We omit the **proof**. ∎

If we added the condition $v_m(k) = v_k(m)$ to Def.4.6.5(ii) above, then the so obtained pre-standard-<u>*strongly*</u>-symmetric configurations would not work in **Reich**(**Basax**) + **Ax**($\sqrt{\ }$) + **Ax**(*Triv*) by Thm.4.7.31(ii) on p.629.[499]

[498] I.e. $\forall m, k \Big( m \overset{\odot}{\to} k \;\Rightarrow\; \exists m', k' \, (\ldots \;\&\; m', k'$ are in pre-standard-sym configuration$)\Big)$.

[499] This is so because of the following. If pre-standard-strongly-symmetric configurations worked in **Reich**(**Basax**) + **Ax**($\sqrt{\ }$) + **Ax**(*Triv*) then **Reich**(**Basax**) + **Ax**($\sqrt{\ }$) + **Ax**(*Triv*) $\models$ **Ax(sy)** would hold, where **Ax(sy)** will be introduced on p.628. But then, by Thm.4.7.31(ii), **Reich**(**Basax**) + **Ax**($\sqrt{\ }$) + **Ax**(*Triv*) $\models$ **Basax** would hold. This would be a contradiction, since there are models of **Reich**(**Basax**) + **Ax**($\sqrt{\ }$) + **Ax**(*Triv*) which are not models of **Basax**.

Pre-standard configurations do not say anything about how $k$ sees $m$'s $\bar{y}$ axis. (On the other hand, standard configurations require quite a lot from $\mathsf{f}_{km}[\bar{y}]$).

**Definition 4.6.8** Let $m, k \in Obs$.

**(i)** Assume $n \leq 4$. We say that $m, k$ are in _quasi-standard configuration_ iff they are in pre-standard configuration and $\mathsf{f}_{km}[\bar{y}] \perp_e \bar{z}$ and $\mathsf{f}_{km}[\bar{z}] \perp_e \bar{y}$.

**(ii)** Let $n$ be arbitrary. Then the definition is the natural generalization of that in (i). I.e. we require $(\forall i, j > 1)[i \neq j \;\Rightarrow\; \mathsf{f}_{km}[\bar{x}_i] \perp_e \bar{x}_j]$.

$\lhd$

**Definition 4.6.9** Let $m, k \in Obs$. We say that $m, k$ are in _quasi-standard-sym configuration_ iff (i)–(iii) below hold.

**(i)** $m, k$ are in quasi-standard configuration.

**(ii)** $m, k$ are in pre-standard-sym configuration.

**(iii)** $(\forall i > 1)\ (\mathsf{f}_{km}(1_i))_i > 0$.

$\lhd$

**PROPOSITION 4.6.10**

$$\mathbf{Reich}(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) \models \textit{quasi-standard-sym configurations work.}$$

**Proof:** The proof uses artificial simultaneities and Thm.4.5.13 (p.576). We omit the proof. ∎

## 4.7 Which symmetry principles are suitable for our Reichenbachian theories **Reich**(*Th*)?

In §4.2 we discussed the motivation for the present section.

Thm.4.7.1 below says that, as we anticipated, **Ax(symm)** is a too strong symmetry principle for studying e.g. **Reich**(**Basax**). Motivated by that theorem, first we will introduce "gentle" symmetry principles **R**(**Ax eqsp**) and **R**(**Ax syt$_0$**) which will be in harmony with the spirit of our theories **Reich**(*Th*).[500] In particular, we will see that **R**(**Ax eqsp**) does not blur the distinction between **Reich**(**Basax**) and **Basax**. Then we will see the same about **R**(**Ax syt$_0$**) too. We will define the symmetric version of **Reich**(**Basax**) to be **Reich**(**Basax**) + **R**(**sym**), where **R**(**sym**) is **R**(**Ax eqsp**) + **R**(**Ax syt$_0$**).

As we said above, to illustrate that we really need to search for new symmetry principles adequate to (the philosophy of) our theories **Reich**(*Th*), we state one of our theorems to the effect that our earlier symmetry principle **Ax(symm)** blurs the distinction between **Reich**(*Th*) and *Th*.

**THEOREM 4.7.1** $\quad$ **Reich**$_0$(**Basax**) + **Ax**(||) + **Ax(symm)** $\models$ **Basax**.

**On the proof:** We will return to proving this theorem as Thm.4.7.31(i) later. ∎

So, clearly **Ax(symm)** is *not adequate* for studying **Reich**(*Th*) because adding it to the "Reichenbachian refinement" **Reich**(*Th*) of *Th* kills the result of all our efforts to refine the theory (and implies e.g. strong statements about the one-way *speed* of light which we wanted to avoid).

An analogous theorem can be stated which implies that already **Ax(syt)** is *non-adequate* for **Reich**(*Th*), at least in some sense. Such "non-adequateness" statements about **Ax(syt)** will be stated as Theorems 4.7.20 and 4.7.21 way below. Cf. also Remark 5.0.61 on p.724.

Before going on, let us stop briefly to reflect on our usage of the word "adequate". From our above discussion it is clear that if $\psi$ is a potential symmetry axiom then $\psi$ is either adequate or is not adequate for **Reich**(**Basax**). However, we did not give an explicit definition of adequateness. Instead, "adequateness" was implicitly described

[500] The "names" **R**(**Ax eqsp**) and **R**(**Ax syt$_0$**) intend to refer to the fact that we consider these axioms as the "Reichenbachian" or "observations oriented" (i.e. testable) versions of our earlier axioms **Ax(eqspace)** and **Ax(syt$_0$)**, respectively.

in §4.2 and above (in the present section). Any simultaneity-stable formula counts as *adequate*.[501] Also if

$$\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(\mathit{Triv}) + \psi \models \mathbf{Bax}$$

then $\psi$ is called *non-adequate*. If $\psi$ would fall in between these two extremes then the definition of adequacy remains implicit.

To see that Thm.4.7.20 points in the direction of non-adequateness of $\mathbf{Ax(syt_0)}$, we recall that theorem. Thm.4.7.20 says that $\mathbf{Ax(syt_0)}$ is very far from being simultaneity-stable. Indeed, it says that any nontrivial relativization $\mathfrak{M} \mapsto \mathfrak{M}/P$ makes $\mathbf{Ax(syt_0)}$ false in $\mathfrak{M}/P$ if it was originally true in $\mathfrak{M}$, assuming $\mathfrak{M} \models \mathbf{Bax} + \mathbf{Ax}(\sqrt{\ })$. We consider items 4.7.20, 4.7.21, and 5.0.61 way below as evidence pointing in the direction that $\mathbf{Ax(syt_0)}$ is a symmetry principle not quite adequate for studying **Reich**(**Basax**).

Later we will introduce a very natural and mild-looking symmetry principle **Ax(sy)**, which is a consequence of **Ax(symm)**+**Ax(||)**. It says the following:

$$(\forall m, k \in \mathit{Obs})\ v_m(k) = v_k(m).$$

I.e. "I see you moving with the same speed as you see me moving". Then we will see that even this very mild principle **Ax(sy)** is not adequate for **Reich**(**Basax**), cf. Thm.4.7.31(ii). Moreover, **Ax(sy)** blurs the distinction between **Reich**(**Basax**) and **Basax**.

Summing up, we found that *none of* **Ax(symm)**, **Ax(syt)**, or even **Ax(sy)** is *adequate* as a possible symmetry principle for **Reich**(*Th*). This motivates our search for new symmetry principles which are adequate for **Reich**(*Th*). In this search we have two informal criteria for the new symmetry principles. These are (i), (ii) below.

(i) The principle should be in harmony with the philosophy behind **Reich**(*Th*) e.g. it should not blur[502] the distinction between *Th* and **Reich**(*Th*).

(ii) "**Reich**(*Th*) + the new symmetry principle" should yield something interesting which is not provable from **Reich**(*Th*) alone, analogously to the results in §2.8 ("A symmetry axiom") which were provable from **Basax**+**Ax(symm)** but not from **Basax** alone. An example of such statements is the twin paradox (**Ax(TwP)**) but see §2.8 for more.

[501] This is so by an act of definition.

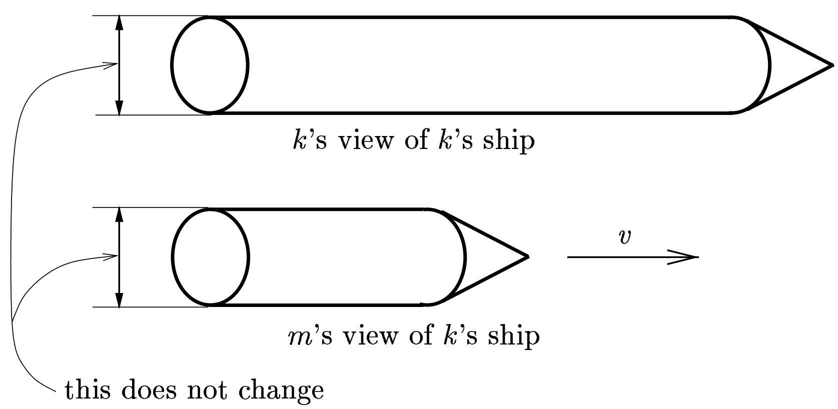
[502] in the sense explained above, cf. Thm.4.7.1 and the discussion preceding it.

Motivated by the above discussed theorems, we turn our attention to searching for symmetry principles adequate for studying our Reichenbachian theories **Reich**($Th$).

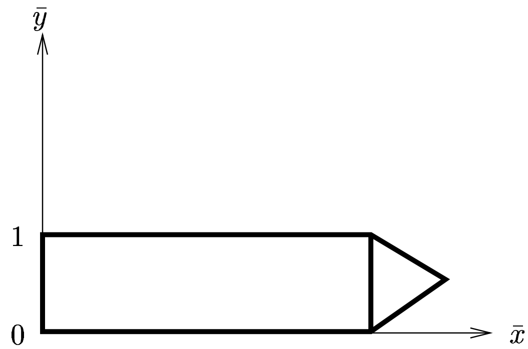
Before getting started, we note that we would like to have the new principles, call them **Ax(new)**, to be as much in harmony with the spirit of the "Reichenbach-Grünbaum philosophy" as **R**(**AxE**) is, in contrast with **AxE** (which according to the Reichenbach-Grünbaum philosophy is not testable). A possible way of doing this is to make the new principles as testable by thought-experiments as **R**(**AxE**) is. (We note that this may not be the only way.) We already said this in our above discussion of adequateness and simultaneity-stableness. When we want to emphasize the connection with Reichenbachian philosophy of relativity, we will write *Reichenbach-adequate* for what we called adequate above.

Now, we can turn to our first new symmetry principle **R**(**Ax eqsp**).

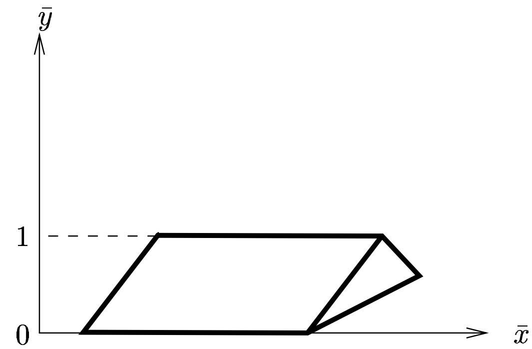
The new axiom, **R**(**Ax eqsp**) is related to the paradigmatic effects in §2.5. Intuitively, **R**(**Ax eqsp**) says that moving spaceships do not change their *thickness* as a consequence of motion cf. §2.5 for terminology. I.e. motion might cause spaceships to get shorter, clocks to slow down etc., but it will not affect width or thickness of these spaceships.




In formalizing this "thickness does not change" principle, one has to be a little bit careful, because in models of **Reich**(**Basax**) motion might distort spaceships the following way:

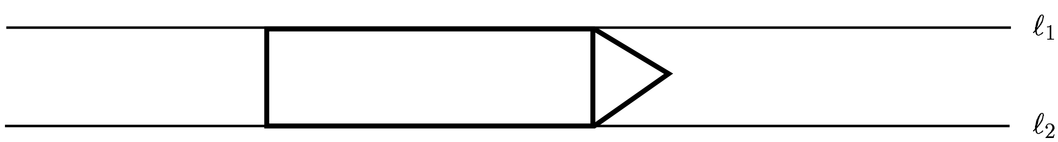



$k$'s view of $k$'s ship




$m$'s view of $k$'s ship

To represent thickness of a spaceship, we imagine two (parallel) lines $\ell_1, \ell_2$ attached to the sides of the ship like this:




Then, thickness of the ship is represented by the distance between $\ell_1$ and $\ell_2$. (The distance between $\ell_1$ and $\ell_2$ is $\min\{\, \|p-q\| \,:\, p \in \ell_1 \text{ and } q \in \ell_2 \,\}$.)

In space-time, this will look like in Figure 203. In Figure 203 the spaceship is sandwiched between two planes $\text{Plane}_1$ and $\text{Plane}_2$ both parallel with $\mathsf{Plane}(\bar{t}, \bar{x})$. We have to apologize, because the picture in Figure 203 is a little bit misleading (deliberately). Namely, the spaceship does not have a dimension parallel with $\bar{t}$, or if it does, then it looks like in Figure 204.

Anyway, on both figures $\text{Plane}_1$ and $\text{Plane}_2$ are represented. Now, thickness of the spaceship is identified with the Euclidean distance between planes $\text{Plane}_1$ and $\text{Plane}_2$. Our axiom will state that the (shortest Euclidean) distance between $\text{Plane}_1$ and $\text{Plane}_2$ is the same for both observers $m$ and $k$ where ($k$ lives inside the spaceship and) $\text{Plane}_1$, $\text{Plane}_2$, $m$ and $k$ are as in Figure 204.

**Definition 4.7.2** Let $H, H_1 \subseteq {}^nF$ be two sets of points. Then the (shortest Euclidean) _distance_ between $H$ and $H_1$ is defined as follows.

$$\mathsf{Eudist}(H, H_1) \stackrel{\text{def}}{=} \inf\{\, \|p-q\| \,:\, p \in H \text{ and } q \in H_1 \,\}\,.$$

$\lhd$

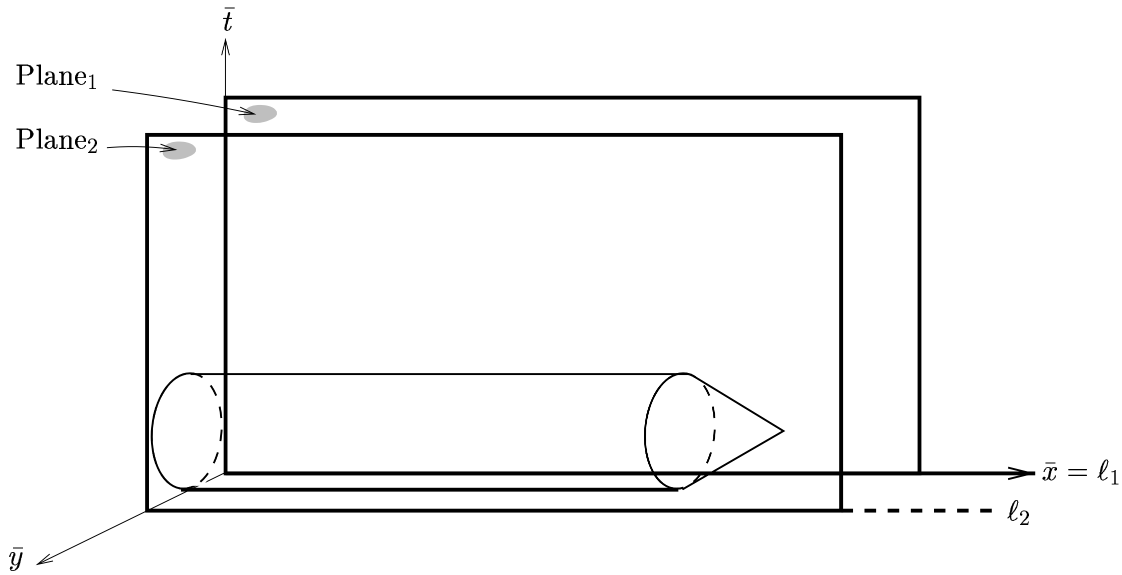



Figure 203:

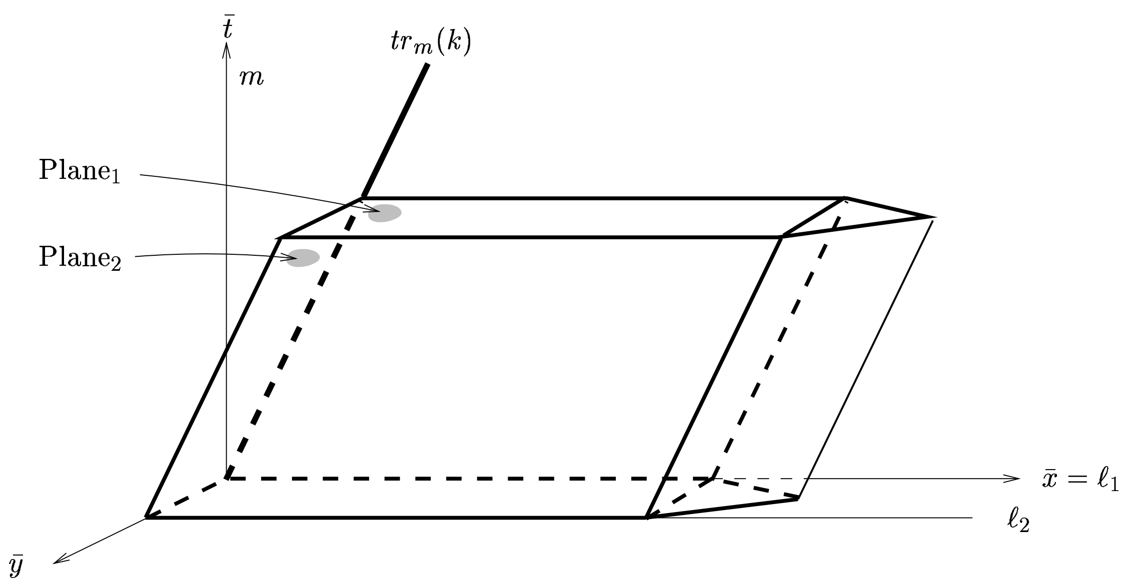



Figure 204:

We note that if $H, H_1$ are two parallel planes or lines then their distance always exists (even without **Ax**($\sqrt{\ }$)).

**R**(**Ax eqsp**) to be introduced below is in some sense the Reichenbachian counterpart of the idea coded by **Ax(eqspace)**. Cf. items 4.7.10, 4.7.13. However, **R**(**Ax eqsp**) is not obtained by a mechanical translation from **Ax(eqspace)** e.g. because the formulation of the latter heavily uses simultaneities.

**R(Ax eqsp)** Assume $m$ and $k$ are in pre-standard configuration. Let $P$ be a (2-dimensional) plane parallel with $\mathsf{Plane}(\bar{t}, \bar{x})$. Then the distance between $P$ and $\mathsf{Plane}(\bar{t}, \bar{x})$ is the same as the distance between $\mathsf{f}_{mk}[P]$ and $\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t}, \bar{x})]$. Formally,

$$(\star) \qquad \mathsf{Eudist}(P, \mathsf{Plane}(\bar{t}, \bar{x})) = \mathsf{Eudist}(\mathsf{f}_{mk}[P], \mathsf{f}_{mk}[\mathsf{Plane}(\bar{t}, \bar{x})]).$$

See Figure 205.

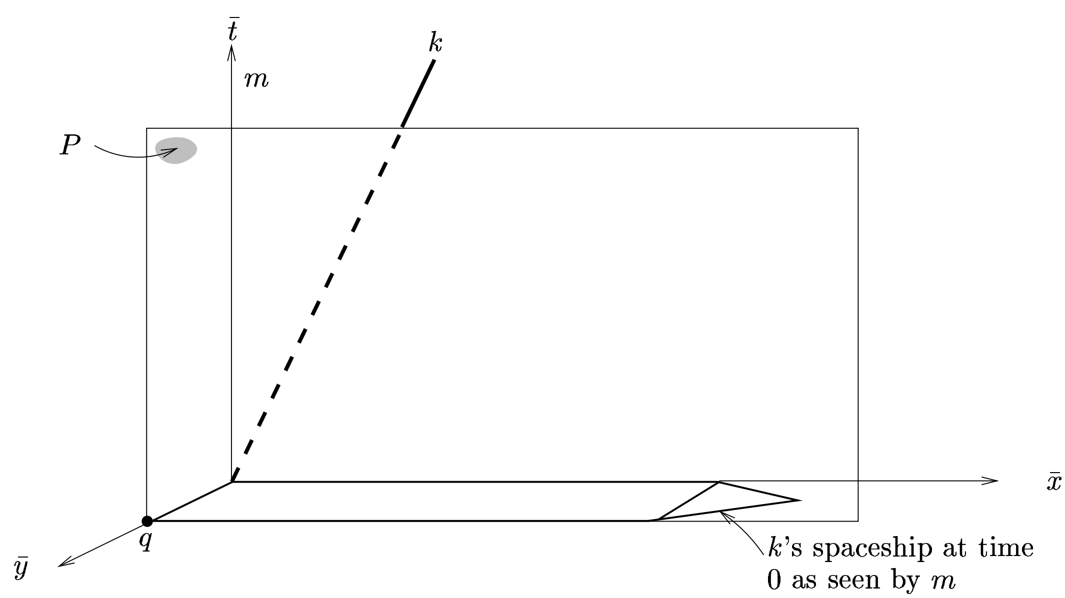



Figure 205: Illustration for **R**(**Ax eqsp**).

We hope that consulting Figures 203, 204, 205 will convince the reader that **R**(**Ax eqsp**) says what we promised to say, namely that the *thickness* of the spaceship of $k$ *is the same* for observers $k$ and $m$. Cf. also the paradigmatic effect in item 2.5.12(ii)(b) and (iii) on p.101; and paradigmatic effect (E5) in §4.8.

The reason why we consider $\mathbf{R}(\mathbf{Ax\ eqsp})$ a *symmetry* principle is the following: It is not hard to see, that $\mathbf{R}(\mathbf{Ax\ eqsp})$ says (at least in some sense) that $m$ sees the thickness of $k$'s spaceship the same way as $k$ sees the thickness of $m$'s one (assuming that both observers think that the thickness of their own ship is 1).[503]

For completeness, let us see a bit more formally why our axiom says what we claim to say. Clearly, $P$ is the "life-line" of the right-hand side[504] while $\mathsf{Plane}(\bar{t}, \bar{x})$ is the life-line of the left-hand side of the ship as seen by $k$. But why is the distance between the two sides of the ship the same as the distance between these two planes (in $k$'s world-view). Formally, thickness of the ship as seen by $k$ is $\mathsf{Eudist}(P \cap S, \mathsf{Plane}(\bar{t}, \bar{x}) \cap S)$, because thickness of the ship is a spatial distance hence we have to intersect the life-lines involved with space $= S$. However, we are lucky because we are discussing planes *parallel with* $\bar{t}$. For such planes $P, P_1$ we have $\mathsf{Eudist}(P, P_1) = \mathsf{Eudist}(P \cap S, P_1 \cap S)$.[505] In the world-view of $m$ we also have $\mathsf{f}_{mk}[P]$ and $\mathsf{f}_{mk}[\mathsf{Plane}(\bar{t}, \bar{x})] = \mathsf{Plane}(\bar{t}, \bar{x})$ are parallel with $\bar{t}$ since we assumed a pre-standard configuration (between $m$ and $k$). This argument shows, that in *both* world-views the distance between our planes coincides with the distance between their space-parts. This is why in ($\star$) it was sufficient to write "$\mathsf{Eudist}(P, \ldots)$" instead of the more complicated "$\mathsf{Eudist}(P \cap S, \ldots)$".

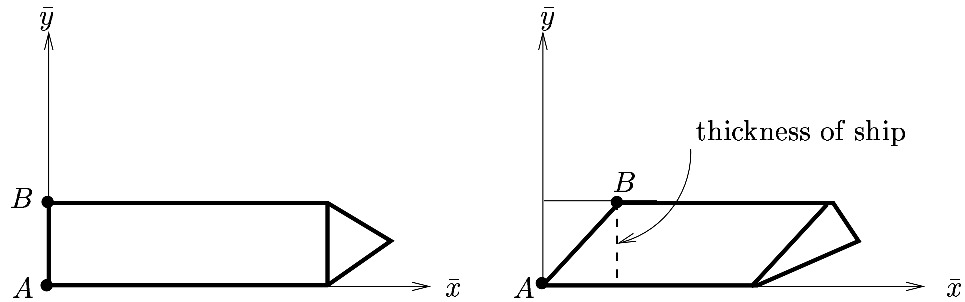
With this we proved that $\mathbf{R}(\mathbf{Ax\ eqsp})$ says indeed that thickness of ships do not change (as it was claimed). In this connection cf. Matolcsi [190, p.158, items II.1.3.7, II.1.3.8].

We note that we cannot formalize our new axiom $\mathbf{R}(\mathbf{Ax\ eqsp})$ in the form of **Ax(eqspace)** on p.136. The reason for this is the distortion effect of **Reich**(**Basax**) models mentioned on p.608 above. Namely, in **Reich**(**Basax**) models, it may happen that $m$ and $k$ see $k$'s spaceship the following way in "pure space":

[503] Besides the above reason, we also note that the original axiom **Ax(eqspace)** was already a symmetry principle since it is a consequence of **Ax(symm)** (under assuming **Bax**).

[504] We identify one side of the ship with an infinitely long stick (or rod) i.e. a line in space $S$. Hence the life-line of this long stick will be a plane in space-time.

[505] This is so because $\bar{t} \perp_e S$ and the distances we are considering are Euclidean.




$k$'s view of $k$'s ship $\qquad$ $m$'s view of $k$'s ship

We identified two "corners" of the spaceship with *bodies* $A$ and $B$. (It is important here that $A, B$ are not events but bodies extending through time.) Now, in $m$'s world-view the thickness of the ship is *not* the distance between $B$ and $A$ but rather the distance between $B$ and line $\bar{x}$.

About the intuitive picture behind the new axiom: Instead of two bodies like $A, B$ above, imagine two (infinitely long) rods attached to the two sides of the spaceship. In $k$'s world-view, at time $t = 0$ one rod can be the $\bar{x}$ axis, while the other rod can be a line $\ell$ parallel with $\bar{x}$. Then the "life-line" of the first rod is the $\mathsf{Plane}(\bar{t}, \bar{x})$ plane. (The life-line of a rod is not a line but a plane.) The life-line of rod $\ell$ is the plane $P$ parallel with $\mathsf{Plane}(\bar{t}, \bar{x})$. This is why our new axiom talks about distances between planes (instead of talking about distances between life-lines of bodies or distances between events as we did e.g. on pp.133–136.)

We will discuss soon the suitability (and basic properties) of $\mathbf{R}(\mathbf{Ax\ eqsp})$ as *our first candidate* for being a *symmetry principle* suitable (or adequate) for our Reichenbachian theories beginning with Prop.4.7.6. Among others we will see (i)–(iii) below.

(i) $\mathbf{R}(\mathbf{Ax\ eqsp})$ is *true* in the relativizations of models (of $\mathbf{Bax}$) which we considered as "symmetric" before; e.g. $\mathbf{Asim}(\mathbf{Bax} + \mathbf{Ax(symm)}) \models \mathbf{R}(\mathbf{Ax\ eqsp})$, cf. Thm.4.7.18.

(ii) $\mathbf{R}(\mathbf{Ax\ eqsp})$ *does not blur* the distinction between $\mathbf{Reich}(\mathbf{Basax})$ and $\mathbf{Basax}$ cf. Corollary 4.7.19.

(iii) $\mathbf{R}(\mathbf{Ax\ eqsp})$ has interesting consequences when added to e.g. $\mathbf{Reich}(Th)$, e.g.

$$\mathbf{Reich}(\mathbf{Flxbasax}) + (\mathsf{c}_m(d) < \infty) + \mathbf{R}(\mathbf{Ax\ eqsp}) \models \mathbf{Ax(TwP)},$$

if $n > 2$, cf. Corollary 4.7.25.

**Remark 4.7.3** ($\mathbf{R}$(**Ax eqsp**) without involving standard configurations.) In the formulation of $\mathbf{R}$(**Ax eqsp**) we assumed that $m, k$ are in pre-standard configuration. The reason for this was *only* to simplify the formulation. For completeness we include the following definition.

**$\mathbf{R}^+$(Ax eqsp)** Assume $m, k \in Obs$ such that $m \overset{\odot}{\to} k$. Assume $P, Q$ are parallel planes of ${}^nF$ such that they are parallel with both $\bar{t}$ and $tr_m(k)$. Then

$$\mathsf{Eudist}(P, Q) = \mathsf{Eudist}(\mathsf{f}_{mk}[P], \mathsf{f}_{mk}[Q]).$$

**Proposition 4.7.4**

$$\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{R}(\mathbf{Ax\ eqsp}) \leftrightarrow \mathbf{R}^+(\mathbf{Ax\ eqsp}).$$

**Proof:** The proof is straightforward and is available from Judit Madarász. ∎

When we use $\mathbf{R}$(**Ax eqsp**), we usually have to assume some auxiliary axioms like **Ax(***Triv***)** too. The advantage of $\mathbf{R}^+$(**Ax eqsp**) is that when using it we do not need these auxiliary axioms (this is somehow connected to Prop.4.7.4 above).

◁

### On temporal symmetry principles like Ax(syt)

Having studied the Reichenbach-adequate version $\mathbf{R}$(**Ax eqsp**) of our symmetry principle **Ax(eqspace)**, we turn our attention to obtaining a Reichenbach-adequate version $\mathbf{R}(\mathbf{Ax\ syt_0})$ of our temporal symmetry principle $\mathbf{Ax(syt_0)}$. Besides the pursuit of knowledge and understanding, we have a further motivation for looking into $\mathbf{R}(\mathbf{Ax\ syt_0})$. Namely, for $n = 2$, $\mathbf{R}$(**Ax eqsp**) does not say anything[506], but $\mathbf{R}(\mathbf{Ax\ syt_0})$ does have implications for $n = 2$ too.

The method of Reichenbachizing $\mathbf{Ax(syt_0)}$ remains the same as it was in the case of Reichenbachizing **AxE**. Namely, we switch attention to those things which can be checked by thought-experiments. More concretely, in the statement "as $m$ sees $k$ so does $k$ see $m$" we replace those things which an observer sees via his coordinate system with those things which the observer sees literally i.e. optically, via photons.[507]

[506] as it was expectable since **Ax(eqspace)** has this property, already.
[507] Compare with the remark on p.560 and Figure 185.

**Definition 4.7.5** Let $m \in Obs$. Then we define the relation $view_m \subseteq {}^nF \times \bar{t}$ as follows.

$$view_m \stackrel{\text{def}}{=} \{ \langle p, q \rangle \in {}^nF \times \bar{t} \, : \, p_t \leq q_t \text{ and } (\exists ph \in Ph) \; p, q \in tr_m(ph) \} .$$

Intuitively $view_m$ intends to represent (the temporal aspect of) how observer $m$ sees *literally* i.e. optically (via photons) the events "happening" in its world-view. More concretely, $p \in {}^nF$ and $q \in \bar{t}$ are $view_m$ related iff observer $m$ literally sees, via photons, event $w_m(p)$ at time $q$. We note that under assuming $\mathbf{Bax^-}$, $view_m$ is a function $view_m : {}^nF \longrightarrow \bar{t}$, cf. Figure 206. Therefore, in formulas we will use $view_m$ as if it were a unary function symbol. The translation algorithm is analogous to that given for the $\mathsf{f}_{mk}$'s in Convention 2.3.10 (p.61).

◁

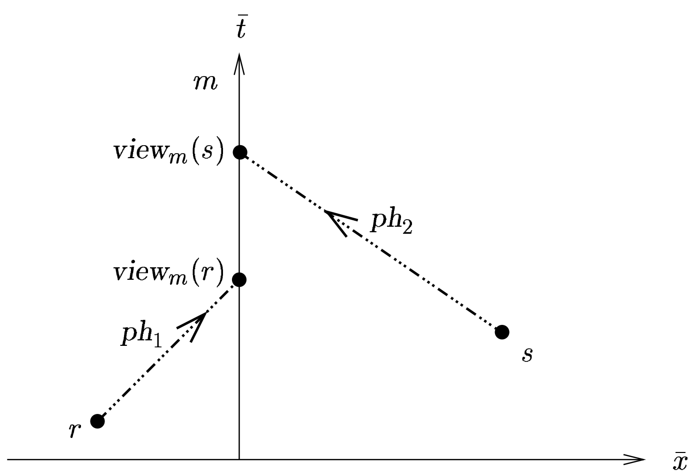



Figure 206: Illustration for Def.4.7.5.

**R(Ax syt$_0$)** $(\forall m, k \in Obs)[\mathsf{f}_{mk}(\bar{0}) = \bar{0} \;\Rightarrow$
$(\forall p \in \bar{t}\,)|view_m(\mathsf{f}_{km}(p))| = |view_k(\mathsf{f}_{mk}(p))|]$.

That is $m$ and $k$ *literally* see, via photons, each other's clocks slowing down with the same rate, see Figure 207. The main difference between **Ax(syt$_0$)** and **R**(**Ax syt$_0$**) is that in **Ax(syt$_0$)** we use the the simultaneities of $m$ and $k$ in formalizing how *they "see"* each other's clocks while in **R**(**Ax syt$_0$**) we are careful to use only what they literally see via photons (so as to avoid possible tacit assumptions about simultaneities "creeping into the picture").

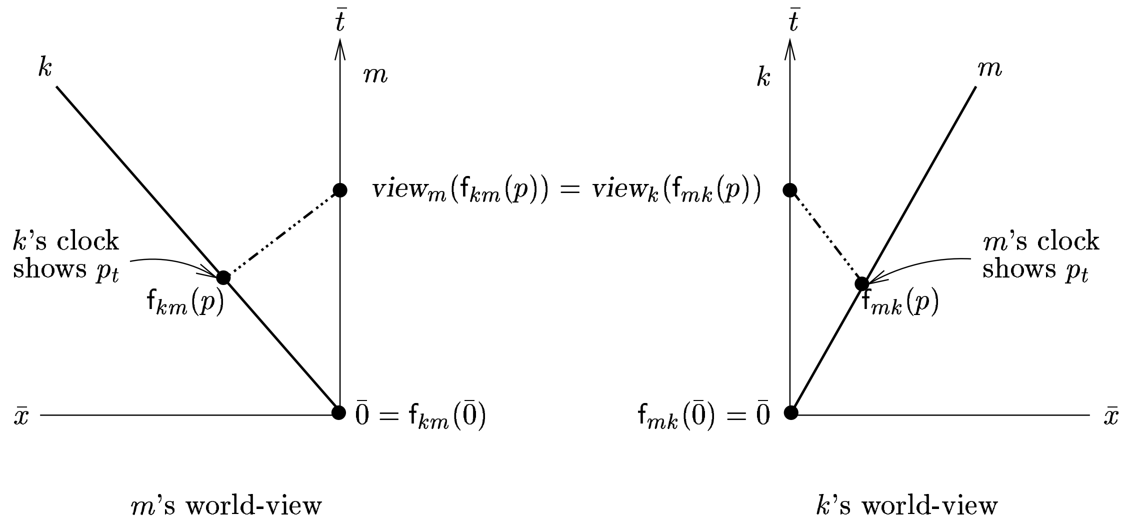



Figure 207: Illustration for $\mathbf{R}(\mathbf{Ax\ syt_0})$.

For the time being, our "fullest" Reichenbachian symmetry principle $\mathbf{R}(\mathbf{sym})$ is defined as follows.

$$\mathbf{R}(\mathbf{sym}) \stackrel{\text{def}}{=} \mathbf{R}(\mathbf{Ax\ eqsp}) + \mathbf{R}(\mathbf{Ax\ syt_0}).$$

We note that Thm.4.7.11 below points in the direction that denoting $\mathbf{R}(\mathbf{sym})$ by say $\mathbf{R}(\mathbf{Ax\ symm})$ seems to be justified. (However we do not do this here.) I.e. $\mathbf{R}(\mathbf{sym})$ could be considered as the Reichenbachized version of **Ax(symm)** since under some assumptions the latter is the same as $\mathbf{Ax(eqspace)}+\mathbf{Ax(syt_0)}$, cf. Thm.4.7.11.

The axioms $\mathbf{R}(\mathbf{Ax\ syt_0})$, $\mathbf{R}(\mathbf{Ax\ eqsp})$, $\mathbf{R}^+(\mathbf{Ax\ eqsp})$ and $\mathbf{R}(\mathbf{sym})$ are all testable by thought-experiments. The following proposition states that they are simultaneity-stable.

**PROPOSITION 4.7.6** *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }))$. *Then (i)–(iv) below hold.*

**(i)** $\mathfrak{M} \models \mathbf{R}(\mathbf{Ax\ syt_0}) \quad \Leftrightarrow \quad \mathbf{Asim}(\mathfrak{M}) \models \mathbf{R}(\mathbf{Ax\ syt_0})$.

**(ii)** $\mathfrak{M} \models \mathbf{R}(\mathbf{Ax\ eqsp}) \quad \Leftrightarrow \quad \mathbf{Asim}(\mathfrak{M}) \models \mathbf{R}(\mathbf{Ax\ eqsp})$.

**(iii)** $\mathfrak{M} \models \mathbf{R}^+(\mathbf{Ax\ eqsp}) \quad \Leftrightarrow \quad \mathbf{Asim}(\mathfrak{M}) \models \mathbf{R}^+(\mathbf{Ax\ eqsp})$.

**(iv)** $\mathfrak{M} \models \mathbf{R}(\mathbf{sym}) \quad \Leftrightarrow \quad \mathbf{Asim}(\mathfrak{M}) \models \mathbf{R}(\mathbf{sym})$.

**Proof:** The proof is easy. We omit it. ∎

By Thm.4.3.11 and Lemma 3.1.6 we have the following.

**FACT 4.7.7** *In models of* $\mathbf{Bax}^-$*, every* $\mathsf{f}_{mk}$ *can be obtained as a composition of an affine transformation and a map* $\widetilde{\varphi}$ *induced by a field automorphism* $\varphi$.

$\lhd$

The next theorem says that assuming $\mathbf{Bax}^-$ (and some auxiliary axioms), $\mathbf{R}(\mathbf{Ax\ syt_0})$ implies that the $\mathsf{f}_{mk}$'s are affine transformations, i.e. there are no field automorphisms involved in the $\mathsf{f}_{mk}$'s. As we already said in Remark 3.1.5 we do not find it extremely nice (or inspiring) to state that the $\mathsf{f}_{mk}$'s are affine as an axiom, but we are happy to have it as a theorem.

**THEOREM 4.7.8**

$$\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{R}(\mathbf{Ax\ syt_0}) \models (\forall m,k)\mathsf{f}_{mk} \in \mathit{Aftr}.$$

**On the proof:** The proof is similar to the proof of Prop.3.9.50(i) (p.392) saying that

$$\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathbf{syt_0}) \models (\forall m,k)\mathsf{f}_{mk} \in \mathit{Aftr}.$$ ∎

The next theorem says that $\mathbf{R}(\mathbf{Ax\ syt_0})$ *excludes FTL observers*, even in dimension 2, under assuming $\mathbf{Bax}^-$ and some auxiliary axioms. As a contrast, it is important to note here that $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\mathbf{syt_0})$ allows FTL observers in 2 dimensions, cf. Thm.2.8.2 on p.127, Thm.2.8.9 on p.132 and Thm.3.9.8(iii) on p.352.

**THEOREM 4.7.9**
$\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{R}(\mathbf{Ax\ syt_0}) \models$ "$\not\exists$ **FTL observers**".

**On the proof:** Assume $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}_t) + \mathbf{R}(\mathbf{Ax\ syt_0})$. Assume $m, k \in \mathit{Obs}$ and that $k$ moves FTL as seen by $m$. By $\mathbf{Bax}^- + \mathbf{Ax}(\mathit{Triv}_t)$ we can assume that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. Let $p := \mathit{view}_m(\mathsf{f}_{km}(1_t))$. Then $\overline{p\mathsf{f}_{km}(1_t)} = \mathit{tr}_m(ph)$, for some $ph \in \mathit{Ph}$. Let such a $ph$ be fixed. See Figure 208. It is easy to check that the "causal direction" of $ph$ is the opposite for $k$ as for $m$. In Figure 208 for $k$ it is this ↘ while for $m$ is this ↖. By this and by applying $\mathbf{R}(\mathbf{Ax\ syt_0})$ for $m$ and $k$ one concludes that $1_t \in \mathit{tr}_m(ph) \cap \bar{t}$, i.e. that $p = 1_t$ (since by $\mathbf{Ax}(\sqrt{\ })$, $\mathsf{f}_{mk}$ is order preserving). Let $m' \in \mathit{Obs}$ such that $m'$ does not move FTL as seen by $m$. Then applying $\mathbf{R}(\mathbf{Ax\ syt_0})$ for $m'$ and $k$ one concludes that $\mathit{tr}_m(ph) \cap \mathit{tr}_m(m') =$

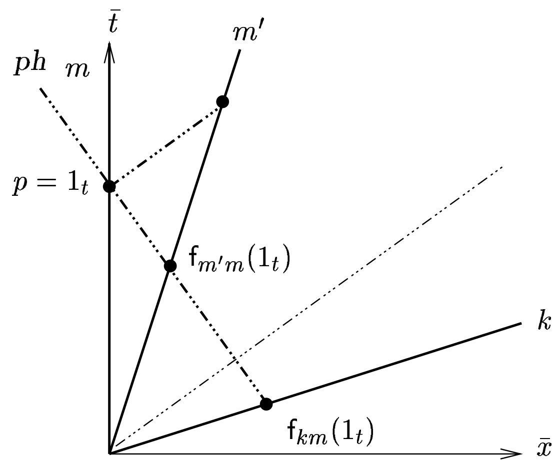



Figure 208:

$\{\mathsf{f}_{m'm}(1_t)\}$. But then $m$ and $m'$ see each other's clocks by "*view*" differently, and this contradicts $\mathbf{R}(\mathbf{Ax\ syt_0})$. ∎

Intuitively, the next theorem says that in **Bax** *the "Reichenbachian" symmetry axioms are equivalent with their old versions,* if $n > 2$ is assumed. For $n = 2$ this is not so because by Thm.4.7.9, $\mathbf{R}(\mathbf{Ax\ syt_0})$ (under assuming $\mathbf{Bax}^-$ and some auxiliary axioms) excludes FTL observers, while $\mathbf{Ax(syt_0)}$ does not exclude them.

**THEOREM 4.7.10** *Assume $n > 2$. Then (i)–(iv) below hold.*

**(i)** $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{R}(\mathbf{Ax\ syt_0}) \leftrightarrow \mathbf{Ax(syt_0)}$.

**(ii)** $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{R}(\mathbf{Ax\ eqsp}) \leftrightarrow \mathbf{Ax(eqspace)}$.

**(iii)** $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) \models \mathbf{R}^+(\mathbf{Ax\ eqsp}) \leftrightarrow \mathbf{Ax(eqspace)}$.

**(iv)** $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) \models \mathbf{R}(\mathbf{sym}) \leftrightarrow \mathbf{Ax(symm)}$.

**On the proof:** Item (i) can be proved by using median observers as follows. Throughout the proof we will tacitly use that both $\mathbf{Ax(syt_0)}$ and $\mathbf{R}(\mathbf{Ax\ syt_0})$, under assuming $\mathbf{Bax}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax}(\mathit{Triv})$, imply that the $\mathsf{f}_{mk}$'s are affine transformations by Prop.3.9.50(i) (p.392) and Thm.4.7.8. Assume $m, k \in Obs$ such that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. Let $h$ be a median observer for $m$ and $k$. Such an $h$ exists.[508] Now, it is

[508] The existence of such an $h$ can be proved as follows. By changing the lengths of meter-rods each model of $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ })$ can be transformed to a model of $\mathbf{Newbasax}+\mathbf{Ax}(\sqrt{\ })$. Since models of **Newbasax** are disjoint unions of models of **Basax** in the new $\mathbf{Newbasax}+\mathbf{Ax}(\sqrt{\ })$

easy to see that in the world-view of $h$ the simultaneity of $m$ and the simultaneity of $k$ are $\bar{t}$-symmetric, formally $\mathsf{f}_{mh}[S]$ and $\mathsf{f}_{kh}[S]$ are $\bar{t}$-symmetric.[509] By this one can prove that $\mathbf{Ax(syt_0)}$ holds for $m$ and $k$ iff $h$ sees the clocks of $m$ and $k$ slowing down with the same rate, cf. the proof of Lemma 3.9.52 on p.395. But since the light-cone is also $\bar{t}$-symmetric, it is easy to see that $\mathbf{R}(\mathbf{Ax\ syt_0})$ holds for $m$ and $k$ iff $h$ sees the clocks of $m$ and $k$ slowing down with the same rate. Thus $\mathbf{Ax(syt_0)}$ holds for $m$ and $k$ iff $\mathbf{R}(\mathbf{Ax\ syt_0})$ holds for $m$ and $k$. The rest of the proof of (i) goes by using $\mathbf{Ax}(\mathit{Triv})$.

In item (iii) the direction $\mathbf{R}^+(\mathbf{Ax\ eqsp}) \to \mathbf{Ax(eqspace)}$ can be proved by using that in **Bax** if two clocks do not get out of synchronism then they are orthogonal to movement, formally for every $p, q \in {}^nF$ with $p \neq q$

$$(p_t = q_t \quad \wedge \quad \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \quad \Rightarrow \quad (\overline{pq} \perp_e tr_m(k) \quad \wedge \quad \overline{\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(q)} \perp_e tr_k(m)).$$

The direction $\mathbf{Ax(eqspace)} \to \mathbf{R}^+(\mathbf{Ax\ eqsp})$ can be proved by using that in **Bax** if two clocks are orthogonal to movement then they do not get out of synchronism and remain orthogonal to movement, formally for every $p, q \in {}^nF$ with $p \neq q$

$$(p_t = q_t \quad \wedge \quad \overline{pq} \perp_e tr_m(k)) \quad \Rightarrow \quad (\mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t \quad \wedge \quad \overline{\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(q)} \perp_e tr_k(m)).$$

Item (ii) is a corollary of Prop.4.7.4 and item (iii).

Item (iv) follows from items (i), (ii) and Thm.4.7.11 below. ∎

Thm.4.7.11 below says that $\mathbf{Ax(syt_0)}$+$\mathbf{Ax(eqspace)}$ is equivalent with $\mathbf{Ax(symm)}$, assuming **Bax** and some auxiliary axioms. We note that, assuming **Basax** (and some auxiliary axioms), both $\mathbf{Ax(syt_0)}$ and $\mathbf{Ax(eqspace)}$ are equivalent with $\mathbf{Ax(symm)}$, cf. Prop.3.9.47 (p.391) and Thm.2.8.16 (p.137). For similar equivalence theorems we refer the reader to §3.9.

**THEOREM 4.7.11** *Assume $n > 2$. Then*

$$\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) \models \mathbf{Ax(symm)} \leftrightarrow (\mathbf{Ax(syt_0)} + \mathbf{Ax(eqspace)})$$

**Proof:** The theorem can be proved by using the "median observer" proof methods from §3.9. We note that the proof of "direction →" uses Prop.3.9.37. The proof will be filled in later. ∎

In Thm.4.7.10 above we have seen that, for $n > 2$, $\mathbf{Ax(syt_0)}$ and $\mathbf{R}(\mathbf{Ax\ syt_0})$ are equivalent. For $n = 2$ these two axioms are not equivalent, since $\mathbf{R}(\mathbf{Ax\ syt_0})$

model there is a median observer $h$ for $m$ and $k$ by Thm.3.8.25 on p.306. It can be easily checked that this $h$ will be a median observer for $m$ and $k$ in the original $\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ })$ model too.

[509] This can be seen by using the $\mathbf{Newbasax}$+$\mathbf{Ax}(\sqrt{\ })$ model mentioned in footnote 508.

excludes FTL observers,[510] while $\mathbf{Ax(syt_0)}$ does not. The next theorem *describes how FTL observers see each other's clocks via photons* in 2-dimensional $\mathbf{Basax}+\mathbf{Ax(syt_0)}$ models. (The way they see each other, according to our next theorem, is only slightly different from what $\mathbf{R}(\mathbf{Ax\ syt_0})$ says, namely at one place "$p$" is replaced by "$-p$".)[511]

**THEOREM 4.7.12** *Assume $n$=2. Assume* $\mathbf{Basax}+\mathbf{Ax(syt_0)}+\mathbf{Ax}(\surd\ )$. *Assume $m, k \in Obs$ and that $k$ moves FTL as seen by $m$ and that $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. Then*

$$(\forall p \in \bar{t}\,)\ |view_m(\mathsf{f}_{km}(p))| = |view_k(\mathsf{f}_{mk}(-p))|;$$

*see Figure 209.*

**Proof:** The proof will be filled in later. ∎

The following is a corollary of Prop.4.7.6 and Thm.4.7.10. Roughly speaking, it says that in a relativization of a **Bax** model, the "Reichenbachian version" of a symmetry axiom holds iff in the original **Bax** model the original symmetry axiom holds.

**COROLLARY 4.7.13** Assume $n > 2$. Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax}(\surd\ ))$. Assume that $\mathfrak{M}^+$ is an art-sim version of $\mathfrak{M}$. Then (i)–(iv) below hold.

**(i)**

$$\mathfrak{M}^+ \models \mathbf{R}^+(\mathbf{Ax\ eqsp}) \ \Leftrightarrow\ \mathfrak{M} \models \mathbf{Ax(eqspace)}.$$

**(ii)** Assume $\mathfrak{M}^+ \models \mathbf{Ax}(\mathit{Triv})$ or $\mathfrak{M} \models \mathbf{Ax}(\mathit{Triv})$. Then

$$\mathfrak{M}^+ \models \mathbf{R}(\mathbf{Ax\ eqsp}) \ \Leftrightarrow\ \mathfrak{M} \models \mathbf{Ax(eqspace)}.$$

**(iii)** Assume $\mathfrak{M}^+ \models \mathbf{Ax}(\mathit{Triv})$ or $\mathfrak{M} \models \mathbf{Ax}(\mathit{Triv})$. Then

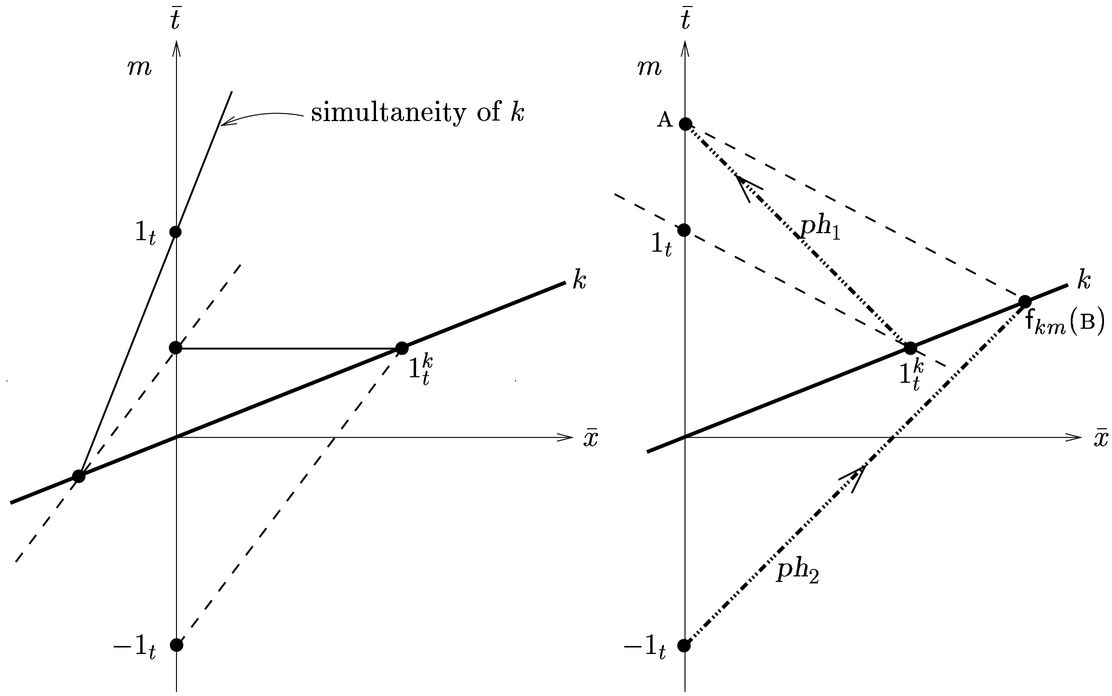
$$\mathfrak{M}^+ \models \mathbf{R}(\mathbf{Ax\ syt_0}) \ \Leftrightarrow\ \mathfrak{M} \models \mathbf{Ax(syt_0)}.$$

**(iv)** Assume ($\mathfrak{M}^+ \models \mathbf{Ax}(\mathit{Triv})$ or $\mathfrak{M} \models \mathbf{Ax}(\mathit{Triv})$) and ($\mathfrak{M}^+ \models \mathbf{Ax}(\|)$ or $\mathfrak{M} \models \mathbf{Ax}(\|)$). Then

$$\mathfrak{M}^+ \models \mathbf{R}(\mathbf{sym}) \ \Leftrightarrow\ \mathfrak{M} \models \mathbf{Ax(symm)}.$$

[510] Cf. Thm.4.7.9.

[511] Of course the conclusion of the next theorem has to be different from that of $\mathbf{R}(\mathbf{Ax\ syt_0})$ by the just quoted Thm.4.7.9.




$1_t^k := \mathsf{f}_{km}(1_t)$

By $\mathbf{Ax(syt_0)}$, $|\mathsf{f}_{km}(1_t)_t| = |\mathsf{f}_{mk}(1_t)_t|$.

$\text{A} := view_m(\mathsf{f}_{km}(1_t))$, $\text{B} := view_k(\mathsf{f}_{mk}(-1_t))$

$|\text{A}| = |\text{B}|$ by Thm.4.7.12.

Figure 209: Illustration for Thm.4.7.12.

**Proof:** The corollary follows from Prop.4.7.6, Thm.4.7.10 and Prop.4.7.14 below. ∎

The next proposition says the following. Consider a relativization $\mathfrak{M} \mapsto \mathfrak{M}/P$. If this represents a move from **Reich**(**Bax**) to **Bax** (i.e. if $\mathfrak{M}/P \models$ **Bax**) then truth of **Ax**(*Triv*) and of **Ax**(||) are preserved. Further, this becomes false if "the move" is in the other direction (i.e. from **Bax** to **Reich**(**Bax**)).

**PROPOSITION 4.7.14** *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }))$ *and that* $\mathfrak{M}^+$ *is an art-sim version of* $\mathfrak{M}$. *Then*

| | | | | |
|---|---|---|---|---|
| **(i)** | $\mathfrak{M}^+ \models \mathbf{Ax}(\mathit{Triv})$ | $\Rightarrow$ | $\mathfrak{M} \models \mathbf{Ax}(\mathit{Triv})$ | *and* |
| **(ii)** | $\mathfrak{M}^+ \models \mathbf{Ax}(\|\|)$ | $\Rightarrow$ | $\mathfrak{M} \models \mathbf{Ax}(\|\|)$. | *But* |
| **(iii)** | $\mathfrak{M} \models \mathbf{Ax}(\mathit{Triv})$ | $\not\Rightarrow$ | $\mathfrak{M}^+ \models \mathbf{Ax}(\mathit{Triv})$ | *and* |
| **(iv)** | $\mathfrak{M} \models \mathbf{Ax}(\|\|)$ | $\not\Rightarrow$ | $\mathfrak{M}^+ \models \mathbf{Ax}(\|\|)$. | |

**Proof:** The proof will be filled in later. ∎

Thm.4.7.15 below says that $\mathbf{R}(\mathbf{Ax\ syt_0})$ implies the twin paradox, under assuming $\mathbf{Bax}^- + (\mathsf{c}_m(d) < \infty)$ and some auxiliary axioms. In connection with Thm.4.7.15 below we note that **Ax(TwP)** is simultaneity-stable.[512]

**THEOREM 4.7.15**

$$\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + (\mathsf{c}_m(d) < \infty) + \mathbf{R}(\mathbf{Ax\ syt_0}) \quad \models \quad \mathbf{Ax(TwP)}.$$

*Therefore*

$$\mathbf{Reich}_0(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) + (\mathsf{c}_m(d) < \infty) + \mathbf{R}(\mathbf{Ax\ syt_0}) \quad \models \quad \mathbf{Ax(TwP)}.$$

**On the proof:** Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + (\mathsf{c}_m(d) < \infty) + \mathbf{R}(\mathbf{Ax\ syt_0}))$. Let $m, k, k_1 \in$ *Obs* such that they satisfy the hypothesis of **Ax(TwP)**. Then there is a plane $P$ such that $\bar{t} \cup tr_m(k) \cup tr_m(k_1) \subseteq P$. There is a 2-dimensional model $\mathfrak{M}^*$ which is obtained from $\mathfrak{M}$ by restricting $w_m$ to plane $P$ such that

$$\mathfrak{M}^* \models \mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(\mathit{Triv}) + (\mathsf{c}_m(d) < \infty) + \mathbf{R}(\mathbf{Ax\ syt_0}).$$

[512]Moreover, **Ax(TwP)** is testable by thought experiments which property is reflected by the fact that it is expressible in the language of the geometry $\mathfrak{G}_{\mathfrak{M}}$ associated to $\mathfrak{M}$ in §6. We will see in §6 that the geometric structure $\mathfrak{G}_{\mathfrak{M}}$ associated to $\mathfrak{M}$ can be considered as a *reduct* of the model $\mathfrak{M}$. Therefore *some* of the formulas of $\mathfrak{M}$ are expressible in the language of $\mathfrak{G}_{\mathfrak{M}}$ while others might be not such. We have reasons to believe that the ones expressible in the language of $\mathfrak{G}_{\mathfrak{M}}$ are closer to testability than the others. This is even more so in the language of the geometry $\mathfrak{G}^R_{\mathfrak{M}}$ (cf. §6). Cf. also footnote 481 on p.581.

The reader is invited to construct such an $\mathfrak{M}^*$. Let such an $\mathfrak{M}^*$ be fixed. Since $\mathfrak{M}^*$ is 2-dimensional and since $\mathbf{Bax}^-(2) + \mathbf{Ax}(\surd\,) \models \mathbf{Reich}(\mathbf{Bax})$, we have that $\mathfrak{M}^* \models \mathbf{Reich}(\mathbf{Bax})$. Thus $\mathfrak{M}^*$ is an art-sim version of some $\mathfrak{M}^+ \in \mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax}(\surd\,))$ by Thm.4.5.13. But then by Corollary 4.7.13, $\mathfrak{M}^+ \models \mathbf{Ax(syt_0)}$. Since $\mathbf{Bax} + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\surd\,) + c_m(p,d) < \infty \models \mathbf{Ax(TwP)}$ by Thm.4.2.9 on p.461, we have $\mathfrak{M}^+ \models \mathbf{Ax(TwP)}$. By this, $\mathfrak{M}^* \models \mathbf{Ax(TwP)}$, since $\mathbf{Ax(TwP)}$ is simultaneity-stable. But then $\mathbf{Ax(TwP)}$ holds for $m, k, k_1$ in $\mathfrak{M}$, too. ∎

**Questions for future research 4.7.16**

- **(i)** Is $\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax(TwP)} \models \mathbf{R}(\mathbf{Ax\ syt_0})$ true?
- **(ii)** What kinds of consequences are derivable from $\mathbf{Ax(TwP)}$ if we assume $\mathbf{Reich}(\mathbf{Basax})$ ?

◁

We note that the question in (i) above, by Corollary 4.7.13(i) and Thm.4.5.13 (under assuming $\mathbf{Ax}(\mathit{Triv})$) is equivalent with the following question.

**QUESTION 4.7.17** *Is* $\mathbf{Basax} + \mathbf{Ax}(\surd\,) + \mathbf{Ax(TwP)} \models \mathbf{Ax(syt_0)}$ *true?*

◁

The following theorem says that our “Reichenbachian” symmetry principles are not too strong. Namely they are true in a very broad class of “Reichenbachian” models (i.e. models obtained by artificial simultaneities).

**THEOREM 4.7.18** *Assume $n > 2$. Then (i)–(iii) below hold.*

| | | | |
|---|---|---|---|
| **(i)** | $\mathbf{Asim}(\mathbf{Bax} + \mathbf{Ax(syt_0)})$ | $\models$ | $\mathbf{R}(\mathbf{Ax\ syt_0})$. |
| **(ii)** | $\mathbf{Asim}(\mathbf{Bax} + \mathbf{Ax(eqspace)})$ | $\models$ | $\mathbf{R}(\mathbf{Ax\ eqsp})$. |
| **(iii)** | $\mathbf{Asim}(\mathbf{Bax} + \mathbf{Ax(symm)} + \mathbf{Ax}(\|\|))$ | $\models$ | $\mathbf{R}(\mathbf{sym})$. |

**On the proof:** Let $n > 2$. Item (i) follows by Corollary 4.7.13(i) since it can be easily seen that every model of $\mathbf{Bax}+\mathbf{Ax(syt_0)}$ is a submodel of a $\mathbf{Bax} + \mathbf{Ax(syt_0)} + \mathbf{Ax}(\mathit{Triv})$ model, and every sub-model of an $\mathbf{R}(\mathbf{Ax\ syt_0})$ model is an $\mathbf{R}(\mathbf{Ax\ syt_0})$ model. Similarly, items (ii) and (iii) follow by Corollary 4.7.13(ii),(iii). ∎

The following is a corollary of Thm.4.7.18(iii). It says that $\mathbf{R}(\mathbf{sym})$ *does not blur* the distinction between $\mathbf{Reich}(\mathbf{Basax})$ and $\mathbf{Basax}$. (This information was already implicit in some of the previous theorems.)

**COROLLARY 4.7.19**

$$\mathbf{Reich(Basax)} + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{R(sym)} \not\models \mathbf{Bax}.$$

**On the proof:** By Thm.4.7.18 it remains to check that there are non-trivial relativizations of models of $\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(symm)} + \mathbf{Ax}(\|)$ validating $\mathbf{Ax}(\mathit{Triv})$ and $\mathbf{Ax}(\|)$. This can be easily checked, but for completeness we include some of the details.

Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(symm)} + \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv}))$. Let $P : Obs \longrightarrow \mathcal{P}({}^nF)$ be a function such that 1–3 below hold.

1. $(\forall m \in Obs)\ P_m$ is an $m$-space-like hyper-plane containing $\bar{0}$.

2. $(\exists m \in Obs)\ P_m \neq S$.

3. $(\forall m, k \in Obs)\ (\mathsf{f}_{mk} \in \mathit{Triv} \ \Rightarrow\ P_k = \mathsf{f}_{mk}[P_m])$.

Then $\mathfrak{M}/P \models \mathbf{Reich(Basax)} + \mathbf{R(sym)}$ by Thm.4.5.13 and Thm.4.7.18(iii). By condition 3 on $P$ one can check that $\mathfrak{M}/P \models \mathbf{Ax}(\|) + \mathbf{Ax}(\mathit{Triv})$. By condition 2, we have $\mathfrak{M}/P \not\models \mathbf{Bax}$. ∎

The following theorem says that $\mathbf{Ax(syt_0)}$ does not hold in any non-trivial relativization of a $\mathbf{Bax}+\mathbf{Ax(syt_0)}$ model. Therefore $\mathbf{Ax(syt_0)}$ is very far from being simultaneity-stable.

**THEOREM 4.7.20** *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(syt_0)})$. *Assume* $\mathfrak{M}^+$ *is an art-sim version of* $\mathfrak{M}$ *such that* $\mathfrak{M}^+ \neq \mathfrak{M}$. *Then* $\mathfrak{M}^+ \not\models \mathbf{Ax(syt_0)}$.

**On the proof:** Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Bax} + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax(syt_0)})$. Let $\mathfrak{M}^+$ be a non-trivial relativization of $\mathfrak{M}$. For simplicity we will assume that $n = 2$. The proof to be given for $n = 2$ can be easily generalized to arbitrary $n$. By $w^+$, $tr^+$ we denote the world-view and trace in $\mathfrak{M}^+$, while in $\mathfrak{M}$ they are denoted as $w$, $tr$. For every $m \in Obs$ let $\bar{x}_m$ be the artificial simultaneity of $m$ in $\mathfrak{M}$, i.e. $\bar{x}_m \stackrel{\text{def}}{=} (w^+_m \circ w_m)[\bar{x}]$. Since $\mathfrak{M}^+ \neq \mathfrak{M}$ there is $h \in Obs$ such that $\bar{x}_h \neq \bar{x}$. Let this $h$ be fixed. Let $m, k \in Obs$ be such that $h$ is a median observer for $m$ and $k$ in $\mathfrak{M}$. Without loss of generality we can assume that $w_m(\bar{0}) = w_k(\bar{0}) = w_h(\bar{0})$ and that $h$ sees the clocks of $m$ and $k$ ticking forward. Since we assumed $\mathfrak{M} \models \mathbf{Bax}+\mathbf{Ax}(\sqrt{\ })+\mathbf{Ax(syt_0)}$, $h$ sees the clocks of $m$ and $k$ slowing down with the same rate in $\mathfrak{M}$ by Lemma 3.9.52 on p.395. This means that the time unit vectors of $m$ and $k$ in the world-view $w_h$ of $h$ are $\bar{t}$-symmetric, i.e. letting $1^k_t := \mathsf{f}_{kh}(1_t)$ and $1^m_t := \mathsf{f}_{mh}(1_t)$ we have that $1^k_t$ and $1^m_t$ are $\bar{t}$-symmetric. We will prove that if $\mathbf{Ax(syt_0)}$ holds for the observer pairs $\langle h, m\rangle$ and

$\langle h,k\rangle$ in $\mathfrak{M}^+$ then $\mathbf{Ax(syt_0)}$ cannot hold for $\langle m,k\rangle$ in $\mathfrak{M}^+$, and this will prove that $\mathfrak{M}^+ \not\models \mathbf{Ax(syt_0)}$. By Prop.3.9.50(i) the $\mathsf{f}_{mk}$'s are affine transformations both in $\mathfrak{M}$ and $\mathfrak{M}^+$. Throughout the rest of the proof the reader is asked to consult Figure 210. Assume that $\mathbf{Ax(syt_0)}$ holds for $\langle h,m\rangle$ and $\langle h,k\rangle$ in $\mathfrak{M}^+$. Let $\mathrm{M},\mathrm{M}' \in tr_h^+(m)$,

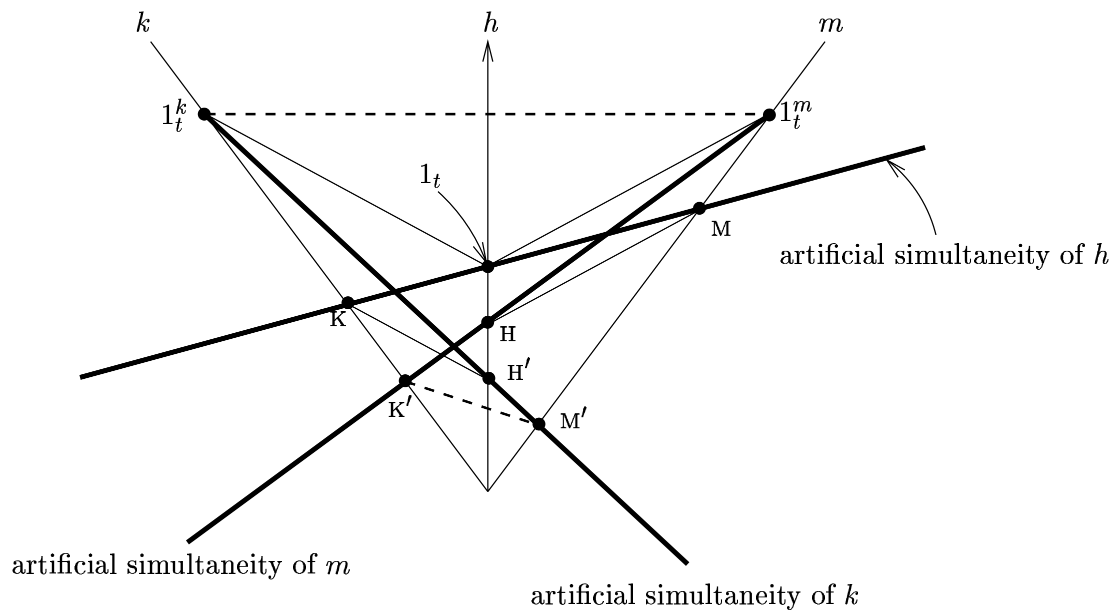



Figure 210: Illustration for the proof of Thm.4.7.20.

$\mathrm{K},\mathrm{K}' \in tr_h^+(k)$, $\mathrm{H},\mathrm{H}' \in \bar{t}$ such that in $\mathfrak{M}^+$ events $w_h(1_t)$, $w_h(\mathrm{M})$ and $w_h(\mathrm{K})$ are simultaneous for $h$, events $w_h(1_t^m)$, $w_h(\mathrm{H})$ and $w_h(\mathrm{K}')$ are simultaneous for $m$ and events $w_h(1_t^k)$, $w_h(\mathrm{H}')$ and $w_h(\mathrm{M}')$ are simultaneous for $k$. Since for $\langle h,m\rangle$ and $\langle h,k\rangle$ $\mathbf{Ax(syt_0)}$ holds in $\mathfrak{M}^+$ and in $\mathfrak{M}^+$ all the $\mathsf{f}_{mk}$'s are affine, we have that

$$\overline{1_t 1_t^m} \parallel \overline{\mathrm{HM}} \quad \text{and} \quad \overline{1_t 1_t^k} \parallel \overline{\mathrm{H}'\mathrm{K}}.$$

By this, by $\bar{x}_h \neq \bar{x}$ and by $\sigma_{\bar{t}}(1_t^m) = 1_t^k$ it can be easily seen that

$$\overline{1_t^m 1_t^k} \not\parallel \overline{\mathrm{M}'\mathrm{K}'}.$$

But this implies that $\mathbf{Ax(syt_0)}$ does not hold for $\langle m,k\rangle$ in $\mathfrak{M}^+$. ∎

Thm.4.7.21 below is in contrast with Corollary 4.7.19 above. It says that $\mathbf{Ax(syt_0)}$+ $\mathbf{R}(\mathbf{Ax\ syt_0})$ blurs the distinction between **Reich**(**Bax**) and **Bax**.

**THEOREM 4.7.21**

$$\mathbf{Reich}_0(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax(syt_0)} + \mathbf{R}(\mathbf{Ax\ syt_0}) \models \mathbf{Bax}.$$

**On the proof:** Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}_0(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax(syt_0)} + \mathbf{R}(\mathbf{Ax\ syt_0}))$. Assume $\mathfrak{M} \not\models \mathbf{Bax}$. Then $\mathfrak{M}$ has a 2-dimensional slice $\mathfrak{M}^*$ such that $\mathfrak{M}^* \not\models \mathbf{Bax}$, but $\mathfrak{M}^* \models \mathbf{Reich}_0(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax(syt_0)} + \mathbf{R}(\mathbf{Ax\ syt_0})$. Since $\mathfrak{M}^*$ is 2-dimensional, we have that $\mathfrak{M}^* \models \mathbf{Reich}(\mathbf{Bax})$. Thus, by Thm.4.5.13 there is $\mathfrak{M}^+ \in \mathsf{Mod}(\mathbf{Bax})$ such that $\mathfrak{M}^*$ is an art-sim version of $\mathfrak{M}^+$. Let this $\mathfrak{M}^+$ be fixed. By Corollary 4.7.13, $\mathfrak{M}^+ \models \mathbf{Ax(syt_0)}$. But then, by $\mathfrak{M}^* \models \mathbf{Ax(syt_0)}$ and Thm.4.7.20 we have that $\mathfrak{M}^+ = \mathfrak{M}^*$. Hence $\mathfrak{M}^* \in \mathsf{Mod}(\mathbf{Bax})$. This contradicts $\mathfrak{M}^* \not\models \mathbf{Bax}$. ∎

In connection with Thm.4.7.21 above we note that by the following item **Ax(syt)** does not blur the distinction between **Reich**(**Basax**) and **Basax**; this is *despite* of the fact that **Ax(syt)** is *not adequate* for **Reich**(**Basax**). I.e. though **Ax(syt)** is not Reichenbach-adequate, it is not so much "inadequate" as **Ax(symm)** is (cf. Thm.4.7.31(i)).

**Remark 4.7.22** $\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax(syt)} \not\models \mathbf{Bax}$.

◁

**Question for future research 4.7.23** Does **Ax(eqspace)** blur the distinction between **Reich**(**Basax**) and **Basax**? I.e. is

$$\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax(eqspace)} \models \mathbf{Basax}$$

true?

◁

Item (i) of the next theorem says that the temporal symmetry principle $\mathbf{R}(\mathbf{Ax\ syt_0})$ is equivalent with the spatial symmetry principle $\mathbf{R}(\mathbf{Ax\ eqsp})$, if we assume $\mathbf{Reich}(\mathbf{Flxbasax}) + (\mathsf{c}_m(d) < \infty) + \mathbf{Ax}(\mathit{Triv})$. Item (ii) is the non-Reichenbachian version of item (i), we include it only for completeness. For similar equivalence theorems we refer the reader to §3.9.

**THEOREM 4.7.24** *Assume $n > 2$. Then (i) and (ii) below hold.*

**(i)** $\mathbf{Reich}(\mathbf{Flxbasax}) + (\mathsf{c}_m(d) < \infty) + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{R}(\mathbf{Ax\ syt_0}) \leftrightarrow \mathbf{R}(\mathbf{Ax\ eqsp})$.

**(ii)** $\mathbf{Flxbasax} + \mathbf{Ax}(\sqrt{\ }) + (c < \infty) + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{Ax(syt_0)} \leftrightarrow \mathbf{Ax(eqspace)}$.

**Proof:** The proof of item (ii) will be filled in later. Item (i) follows from item (ii) by Thm.4.5.13 and Corollary 4.7.13. ∎

The following is a corollary of Thm.4.7.15 and Thm.4.7.24.

**COROLLARY 4.7.25** Assume $n > 2$. Then

$$\mathbf{Reich}(\mathbf{Flxbasax}) + (\mathsf{c}_m(d) < \infty) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{R}(\mathbf{Ax\ eqsp}) \models \mathbf{Ax(TwP)}.$$

**Proof:** The corollary is an immediate corollary of Thm's 4.7.15 and 4.7.24. ∎

The following is a corollary of Thm.4.7.21 and Thm.4.7.24.

**COROLLARY 4.7.26** Assume $n > 2$. Then

$$\mathbf{Reich}(\mathbf{Basax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\mathbf{syt_0}) + \mathbf{R}(\mathbf{Ax\ eqsp}) \models \mathbf{Basax}.$$

**Proof:** The corollary follows from Thm's 4.7.21 and 4.7.24. ∎

The following theorem says that **R**(**sym**) blurs the distinction between **Reich**(**Bax**) and **Reich**(**Flxbasax**).[513]

**THEOREM 4.7.27** *Assume $n > 2$. Then*

$$\mathbf{Reich}(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{R}(\mathbf{sym}) \models (m \overset{\odot}{\to} k \to c_{m,2} = c_{k,2}),$$

*where for every $m \in Obs$, $c_{m,2}$ denotes the two-way speed of light for $m$. Therefore*

$$\mathbf{Reich}(\mathbf{Bax}) + \mathbf{Ax}(\mathit{Triv}) + \mathbf{Ax}(\|) + \mathbf{Ax6} + \mathbf{R}(\mathbf{sym}) \models \mathbf{Reich}(\mathbf{Flxbasax}).$$

**Proof:** The theorem follows from Cor.4.7.13, Prop.3.9.37 (p.386) and Thm.4.5.13 (p.576). ∎

Based on the results stated so far, we consider[514] **R**(**sym**) as the symmetry principle adequate for **Reich**(**Basax**). Therefore, we define the *symmetric version of* **Reich**(**Basax**) to be the theory **Reich**(**Basax**) + **R**(**sym**).

Recall that for any theory *Th* we defined $\mathbf{Reich}'(Th)$ as the set of simultaneity-stable consequences of $Th + \mathbf{Ax}(\sqrt{\ })$.

---

[513]Therefore, in **Reich**(**Bax**), what **R**(**sym**) expresses is really a kind of *symmetry*, namely it says that for *everyone* the two-way speed of light is the same.

[514]in harmony with the spirit of §4.2

**PROPOSITION 4.7.28** *Let $n > 2$.*

$$\mathbf{Reich}'(\mathbf{Basax} + \mathbf{Ax(symm)}) \supseteq \mathbf{Reich}(\mathbf{Basax}) + \mathbf{R}(\mathbf{sym}).$$

**Proof.** $\mathbf{Reich}(\mathbf{Basax}) \subseteq \mathbf{Reich}'(\mathbf{Basax} + \mathbf{Ax(symm)})$ because the former theory consists of symultaneity-stable formulas which are all consequences of $\mathbf{Basax} + \mathbf{Ax(symm)}$ by Thm.4.5.13, Prop.4.7.6, Thm.4.7.18, and because $\mathbf{Basax} + \mathbf{Ax(symm)} \models \mathbf{Ax(\|)}$ for $n > 2$. ∎

**QUESTION 4.7.29** *Let $n > 2$. Is the following true?*

$$\mathbf{Reich}'(\mathbf{Basax} + \mathbf{Ax(symm)}) \mathrel{=\!\mid\!\models} \mathbf{Reich}(\mathbf{Basax}) + \mathbf{R}(\mathbf{sym})?$$

◁

### A new symmetry principle for our non-Reichenbachian theories

We devote this last part of the section to proving that **Ax(symm)** is too strong for **Reich**(**Basax**). We introduce a very natural and simple symmetry principle **Ax(sy)**. It says that "I see you moving with the same speed as you see me moving". Then we will show that **Ax(sy)** is a part of **Ax(symm)** which already is too strong for **Reich**(**Basax**).

**Ax(sy)** $(\forall m, k \in Obs)\ v_m(k) = v_k(m)$.

We note that **Ax(sy)** is almost true in **Basax** in the following sense.

**FACT 4.7.30** **Basax** + (*the* $\mathsf{f}_{mk}$*'s are affine*) $\models$ **Ax(sy)**, *cf. Thm.2.8.6.*

◁

We will see that this very innocent looking[515] axiom already blurs the distinction between **Reich**(**Basax**) and **Basax**.

Besides discussing **Ax(sy)**, Thm.4.7.31 below also says that **Ax(symm)** blurs the distinction between **Reich**(**Basax**) and **Basax**, as we promised already a few times way above.

[515] and sometimes very useful, too

**THEOREM 4.7.31** *Items (i) and (ii) below hold.*

**(i)** $\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax}(\|) + \mathbf{Ax(symm)} \models \mathbf{Basax}$.

**(ii)** $\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)} \models \mathbf{Basax}$.

For the proof of Thm.4.7.31 we will need some lemmas. Therefore the proof comes after the lemmas.

**LEMMA 4.7.32** $\mathbf{Bax}^- + \mathbf{Ax}(\|) + \mathbf{Ax(symm)} \models \mathbf{Ax(sy)}$.
*Therefore* $\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax}(\|) + \mathbf{Ax(symm)} \models \mathbf{Ax(sy)}$.

**Proof:** Assume $\mathbf{Bax}^- + \mathbf{Ax}(\|) + \mathbf{Ax(symm)}$. Let $m, k \in Obs$. Let $m', k' \in Obs$ such that $tr_m(m') = tr_k(k') = \bar{t}$ and $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$. Such $m', k'$ exist by $\mathbf{Ax(symm)}$. $v_m(k) = v_{k'}(m')$ holds by $\mathsf{f}_{mk} = \mathsf{f}_{k'm'}$. By $\mathbf{Ax}(\|)$ and $tr_m(m') = tr_k(k') = \bar{t}$, we have $v_{k'}(m') = v_k(m)$. Hence $v_m(k) = v_k(m)$. ∎

For formulating our next lemma we need to define a new symmetry principle $\mathbf{Ax(sy_0)}$.

$$\mathbf{Ax(sy_0)}\ \ (\forall m, k \in Obs)\Big[tr_m(k) \cap \bar{t} \neq \emptyset \ \Rightarrow$$
$$(\exists k' \in Obs)[tr_k(k') = \bar{t} \ \ \wedge \ \ tr_m(k') = tr_{k'}(m)\Big].$$

The lemma below says that assuming $\mathbf{Bax}^-$ and some auxiliary axioms $\mathbf{Ax(sy_0)}$ and $\mathbf{Ax(sy)}$ are equivalent.

**LEMMA 4.7.33**

**(i)** $\mathbf{Bax}^- + \mathbf{Ax}(\|) \models \mathbf{Ax(sy_0)} \to \mathbf{Ax(sy)}$. *Therefore*
$\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax}(\|) \models \mathbf{Ax(sy_0)} \to \mathbf{Ax(sy)}$.

**(ii)** $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t) \models \mathbf{Ax(sy)} \to \mathbf{Ax(sy_0)}$. *Therefore*
$\mathbf{Reich}_0(\mathbf{Bax}) + \mathbf{Ax}(Triv_t) \models \mathbf{Ax(sy)} \to \mathbf{Ax(sy_0)}$.

**Proof:**
*Proof of (i):* Assume $\mathbf{Bax}^- + \mathbf{Ax}(\|) + \mathbf{Ax(sy_0)}$. Let $m, k \in Obs$. Let $k' \in Obs$ such that $tr_m(k') = tr_{k'}(m)$, $tr_m(k') \cap \bar{t} \neq \emptyset$ and $tr_k(k') \parallel \bar{t}$. Such a $k'$ exists by $\mathbf{Ax(sy_0)}$. $v_m(k') = v_{k'}(m)$ by $tr_m(k') = tr_{k'}(m)$. By $\mathbf{Ax}(\|)$, we have $v_{k'}(m) = v_k(m)$ and obviously $v_m(k) = v_m(k')$. Hence $v_m(k) = v_k(m)$.
*Proof of (ii):* Assume $\mathbf{Bax}^- + \mathbf{Ax}(\sqrt{\ }) + \mathbf{Ax}(Triv_t) + \mathbf{Ax(sy)}$. Let $m, k \in Obs$ such that $tr_m(k) \cap \bar{t} \neq \emptyset$. $v_m(k) = v_k(m)$ holds by $\mathbf{Ax(sy)}$. By $v_m(k) = v_k(m)$ and $v_m(k) \cap \bar{t} \neq \emptyset$, we have that there is $f \in Triv$ such that $f[\bar{t}] = \bar{t}$ and $f[tr_k(m)] = tr_m(k)$. Let such an $f$ be fixed. By $\mathbf{Ax}(Triv_t)$ there is $k'$ such that $\mathsf{f}_{k'k} = f$. For this $k'$, $tr_k(k') = \bar{t}$ and $tr_m(k') = tr_{k'}(m)$ hold. ∎

**LEMMA 4.7.34** *Assume* $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)})$. *Then* $\mathfrak{M} \subseteq \mathfrak{N}$ *for some* $\mathfrak{N} \in \mathsf{Mod}(\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)} + \mathbf{Ax(}Triv_t\mathbf{)})$.

**Proof:** We omit the proof. ∎

The following two lemmas use the set $PT$ of photon-preserving affine transformations. The set $PT$ is defined in Def.3.6.2 on p.265.

**LEMMA 4.7.35** *Assume* $f \in PT$. *Then*

$$ang^2(f[\bar{t}]) = ang^2(f^{-1}[\bar{t}]).$$

**Proof:** The lemma follows from Thm.2.8.6 on p.129 and Thm.3.6.16 on p.273. ∎

**LEMMA 4.7.36** *Assume* $f, g \in PT(2, \mathfrak{F})$. *Assume* $\sqrt{ang^2(f[\bar{t}])}$ *and* $\sqrt{ang^2(g[\bar{t}])}$ *are rational numbers. Then* $\sqrt{ang^2((f \circ g)[\bar{t}])}$ *is a rational number, too.*

**Proof:** The proof will be filled in later. ∎

The following lemma is the 2-dimensional version of our Thm.4.7.31(ii).

**LEMMA 4.7.37** *Assume* $n = 2$. *Then* $\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)} \models \mathbf{Basax}$.

**Proof:** Assume $n = 2$. Assume $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)})$. Let $\mathfrak{N} \in \mathsf{Mod}(\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)} + \mathbf{Ax(}Triv_t\mathbf{)})$ such that $\mathfrak{M} \subseteq \mathfrak{N}$. Such an $\mathfrak{N}$ exists by Lemma 4.7.34. Clearly

$$\mathfrak{N} \models \mathbf{Basax} \quad \Rightarrow \quad \mathfrak{M} \models \mathbf{Basax}.$$

Thus to prove the lemma it is enough to prove that $\mathfrak{N} \models \mathbf{Basax}$. Since $\mathfrak{N}$ is 2-dimensional, $\mathfrak{N} \models \mathbf{Reich}(\mathbf{Basax})$. Then by Thm.4.5.13, $\mathfrak{N}$ is an art-sim version of some $\mathfrak{N}^+ \in \mathsf{Mod}(\mathbf{Basax})$. Let this $\mathfrak{N}^+$ be fixed.

By $w^+, tr^+, v^+, \mathsf{f}^+$ we denote the "world-view", the "trace", the "speed" and the "world-view transformation" in $\mathfrak{N}^+$, while in $\mathfrak{N}$ they are denoted as $w, tr, v, \mathsf{f}$. For every $m \in Obs$ let $\bar{x}_m$ denote the artificial simultaneity of $m$ in $\mathfrak{N}^+$, i.e.

$$\bar{x}_m \stackrel{\text{def}}{=} (w_m \circ (w^+_m)^{-1})[\bar{x}].$$

Clearly $(\forall m \in Obs)\ \bar{0} \in \bar{x}_m \in \mathsf{Eucl}$. Now, if $(\forall m \in Obs)\ \bar{x}_m = \bar{x}$ then $\mathfrak{N} = \mathfrak{N}^+$ and $\mathfrak{N} \models \mathbf{Basax}$. To prove $(\forall m \in Obs)\ \bar{x}_m = \bar{x}$, let $m \in Obs$ be fixed. By Lemma 4.7.33(ii),

$$\mathfrak{N} \models \mathbf{Ax(sy_0)}.$$

Let $k, h \in Obs$ such that 1–5 below hold. (1–4 are understood in $\mathfrak{N}$, while 5 is understood in $\mathfrak{N}^+$). See Figure 211.

1. $k$ moves forward in direction $1_x$ and slower than light as seen by $m$.
2. $h$ moves forward in direction $-1_x$ and slower than light as seen by $m$.
3. $tr_m(k) = tr_k(m)$ and $tr_m(h) = tr_h(m)$.
4. $\bar{0} \in tr_m(k) \cap tr_m(h)$.
5. $\sqrt{v^+_m(k)}$ and $\sqrt{v^+_m(h)}$ are rational numbers.

Such $k$ and $h$ exist by $\mathfrak{M} \models \mathbf{Ax(sy_0)}$.

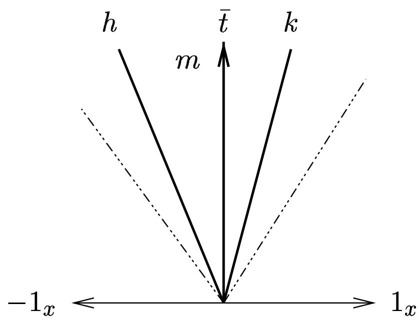



Figure 211:

We claim that

$$(\star) \qquad \bar{x}_m = \bar{x}_k = \bar{x}_h$$

holds because of the following. First we prove that

$$v^+_m(k) = v^+_k(m) \quad \text{and} \quad v^+_m(h) = v^+_h(m).$$

Since $\mathfrak{M}^+ \models \mathbf{Basax}$, $\mathsf{f}^+_{mk} = \widetilde{\varphi} \circ f$, for some $\varphi \in Aut(\mathbf{F})$ and $f \in PT$ by Prop.3.6.5 (p.267). Let this $f$ and $\varphi$ be fixed. Since $\sqrt{v^+_m(k)}$ is rational we have that $\widetilde{\varphi}[tr^+_m(k)] = tr^+_m(k)$. By Lemma 4.7.35, we have that $ang^2(f^{-1}[\bar{t}]) = ang^2(f[\bar{t}])$. So, $v^+_m(k) = ang^2(tr^+_m(k)) = ang^2(\widetilde{\varphi}[tr^+_m(k)]) = ang^2((\mathsf{f}^+_{km} \circ \widetilde{\varphi})[\bar{t}]) = ang^2(f^{-1}[\bar{t}]) = ang^2(f[\bar{t}]) = ang^2((\widetilde{\varphi} \circ f)[\bar{t}]) = ang^2(\mathsf{f}^+_{mk}[\bar{t}]) = ang^2(tr^+_k(m)) = v^+_k(m)$. By the above computation $v^+_m(k) = v^+_k(m)$ holds. Analogously $v^+_m(h) = v^+_h(m)$.
Now, $v^+_m(k) = v^+_k(m)$, $tr_m(k) = tr_k(m)$ and $\mathfrak{M}^+ \in \mathbf{Asim}(\mathfrak{M})$ imply that $tr^+_m(k) = tr^+_k(m)$ and $\bar{x}_k = \bar{x}_m$. Analogously $\bar{x}_h = \bar{x}_m$. Hence $(\star)$ above holds.

Claim **4.7.38** $\sqrt{v^+_k(h)}$ is a rational number.

*Proof:* Let us recall that $\mathsf{f}^+_{m\underset{\sim}{k}} = \widetilde{\varphi} \circ f$, $\varphi \in Aut(\mathbf{F})$ and $f \in PT$. By Prop.3.6.5 and $\mathfrak{N}^+ \models$ **Basax**, $\mathsf{f}^+_{hm} = \widetilde{\psi} \circ g$, for some $g \in PT$ and $\psi \in Aut(\mathbf{F})$. Let this $g$ and $\psi$ be fixed. In the following computation we will use that $\widetilde{\varphi}[tr^+_m(h)] = tr^+_m(h)$. This is so because $\sqrt{v^+_m(h)}$ is a rational number. So, $v^+_k(h) = ang^2(tr^+_k(h)) = ang^2(\mathsf{f}^+_{mk}[tr^+_m(h)]) = ang^2((\widetilde{\varphi} \circ f)[tr^+_m(h)]) = ang^2(f[tr^+_m(h)]) = ang^2((\mathsf{f}^+_{hm} \circ f)[\bar{t}]) = ang^2((\widetilde{\psi} \circ g \circ f)[\bar{t}]) = ang^2((g \circ f)[\bar{t}])$. By the above computation

$$v^+_k(h) = ang^2((g \circ f)[\bar{t}])$$

Further, $f[\bar{t}] = (\widetilde{\varphi} \circ f)[\bar{t}] = \mathsf{f}^+_{mk}[\bar{t}] = tr^+_m(k)$. Similarly $g[\bar{t}] = tr^+_h(m)$. Therefore $\sqrt{ang^2(f[\bar{t}])}$ and $\sqrt{ang^2(g[\bar{t}])}$ are rational numbers since $\sqrt{v^+_m(k)}$ and $\sqrt{v^+_m(h)}$ are rational numbers and $v^+_m(h) = v^+_h(m)$. This and $v^+_k(h) = ang^2((g \circ f)[\bar{t}])$, by Lemma 4.7.36, imply that $\sqrt{v^+_k(h)}$ is a rational number.

QED (Claim 4.7.38)

Since $v^+_k(h)$ is a rational number (by Claim 4.7.38), analogously to the proof of $v^+_m(k) = v^+_k(m)$ one can prove that

$$v^+_k(h) = v^+_h(k).$$

Now, by 1–4 above it is easy to check that

($\star\star$) $k$ sees $h$ moving forward in direction $1_x$ (both in $\mathfrak{N}^+$ and $\mathfrak{N}$) and
$h$ sees $k$ moving forward in direction $-1_x$ (both in $\mathfrak{N}^+$ and $\mathfrak{N}$).

By $\mathfrak{N} \models$ **Ax(sy)**,

$$v_k(h) = v_h(k).$$

Finally, ($\star$), ($\star\star$), $v^+_k(h) = v^+_h(k)$ and $v_k(h) = v_h(k)$ imply $\bar{x}_m = \bar{x}$, cf. Figure 212. ∎

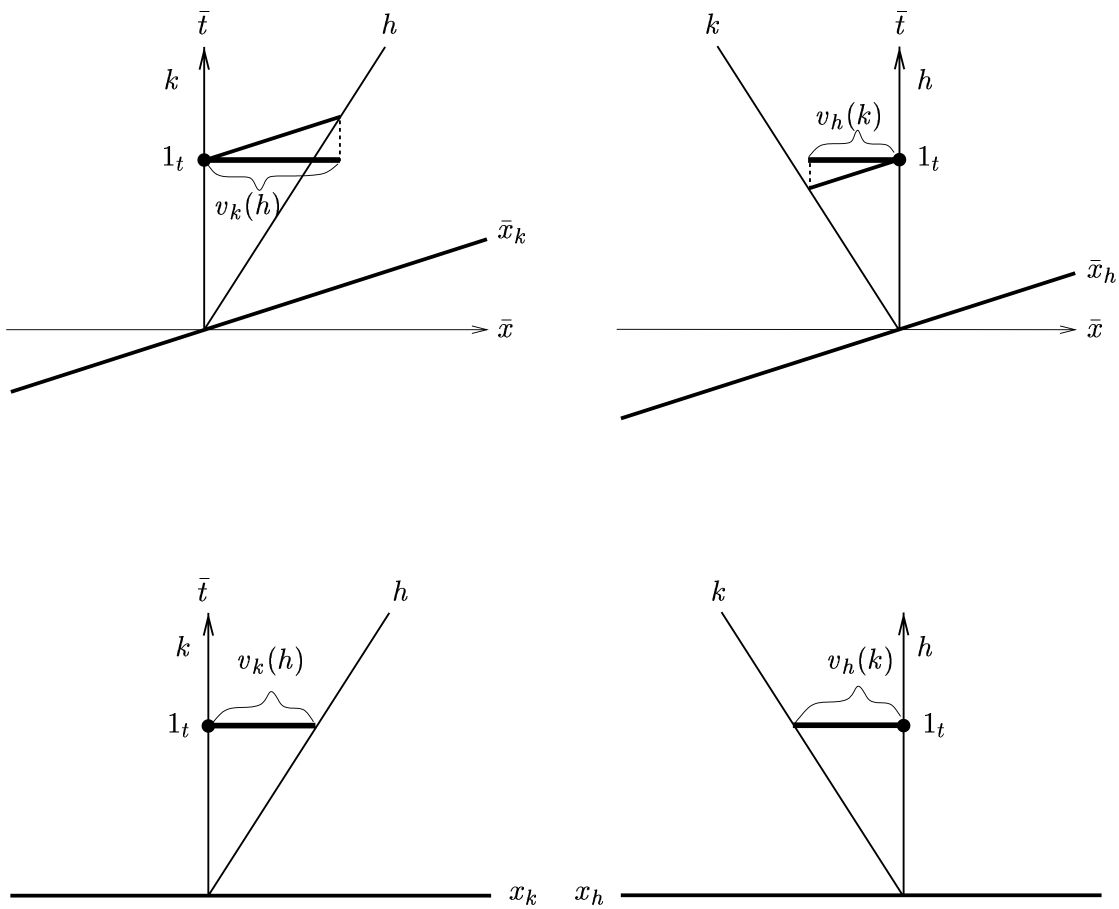
**Proof of Thm.4.7.31:** By Lemma 4.7.32, we have (ii) ⇒ (i). Thus it is enough to prove (ii). Let $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)})$. Assume that $\mathfrak{M} \not\models$ **Basax**. We will derive a contradiction from these assumptions.

Since $\mathfrak{M} \not\models$ **Basax** and $\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Basax})$ there is $m \in Obs$, a plane $P$ and $ph \in Ph$ such that

$$\bar{t} \subseteq P \quad \wedge \quad tr_m(ph) \subseteq P \quad \wedge \quad v_m(ph) \neq 1.$$

Let such $m, P, ph$ be fixed.

We will construct a 2-dimensional $\mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)}$ model, call it $\mathfrak{M}^*$, such that the world-views of $m$ in $\mathfrak{M}$ and $\mathfrak{M}^*$ will be basically the same. But then



$\bar{t}$
$h$
$k$
$1_t$
$v_k(h)$
$\bar{x}_k$
$\bar{x}$
$k$
$\bar{t}$
$h$
$v_h(k)$
$1_t$
$\bar{x}_h$
$\bar{x}$
$\bar{t}$
$h$
$k$
$v_k(h)$
$1_t$
$x_k$
$k$
$h$
$v_h(k)$
$1_t$
$x_h$


Figure 212:

by Lemma 4.7.37, $\mathfrak{M}^* \models \mathbf{Basax}$ will hold, and this will contradict $v_m(ph) \neq 1$. Formally: Let

$$\mathfrak{M}^* = \langle (B^*;\ Obs^*, Ph^*, Ib^*), \mathfrak{F}, G;\ \in, W^* \rangle$$

be a 2-dimensional model defined as follows. Let

$$\begin{aligned} Obs^* &\stackrel{\text{def}}{=} \{ k \in Obs \ : \ tr_m(k) \subseteq P \}, \\ Ph^* &\stackrel{\text{def}}{=} \{ ph' \in Ph \ : \ tr_m(ph') \subseteq P \}, \\ B^* &\stackrel{\text{def}}{=} Ib^* \stackrel{\text{def}}{=} Ph^* \cup Obs^*. \end{aligned}$$

For every $k \in Obs^*$ let $\varrho_k \in Triv \cap Linb$ be such that $\varrho_k[\mathsf{Plane}(\bar{t}, \bar{x})] = \mathsf{f}_{mk}[P]$. Since for every $k \in Obs^*$ $\bar{t} = \mathsf{f}_{mk}[tr_m(k)] \subseteq \mathsf{f}_{mk}[P]$ such $\varrho_k$'s exist. Each $\varrho_k$ induces a function $\varrho'_k : {}^2F \longrightarrow \mathsf{f}_{mk}[P]$ the natural way, i.e. $(\forall q \in {}^2F)\ \varrho'_k(q) \stackrel{\text{def}}{=} \varrho_k(\langle q_0, q_1, 0, \ldots, 0\rangle)$. Let

$$W^* \stackrel{\text{def}}{=} \left\{ \langle k, q, b\rangle \in Obs^* \times {}^2F \times B^* \ : \ \langle k, \varrho'_k(q), b\rangle \in W \right\}.$$

By the above $\mathfrak{M}^*$ has been defined. Now, by

$$\mathfrak{M} \models \mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)},$$

it is easy to check that

$$\mathfrak{M}^* \models \mathbf{Reich}_0(\mathbf{Basax}) + \mathbf{Ax(sy)}.$$

Now, by Lemma 4.7.37 (since $\mathfrak{M}^*$ is 2-dimensional) we get

$$\mathfrak{M}^* \models \mathbf{Basax}.$$

Obviously $m \in Obs^*$, $ph \in Ph^*$ and $v_m(ph)$ in $\mathfrak{M}$ is the same as $v_m(ph)$ in $\mathfrak{M}^*$. But then $v_m(ph) = 1$ since $\mathfrak{M}^* \models \mathbf{Basax}$, and this contradicts $v_m(ph) \neq 1$. ∎

## 4.8 The paradigmatic effects and our lattice of relativity theories

In the present section, we will map the lattice $\mathbf{Basax} \geq \ldots \mathbf{Bax}^{--}$ of our theories represented in Figure 180 on p.552 from the point of view of the paradigmatic effects of relativity introduced in §2. In this section we always assume $n > 2$ and $\mathbf{Ax}(\sqrt{\ })$, for simplicity, without mentioning this.

Below, we will summarize the various paradigmatic effects in the form of statements (E1) – (E7). Then we will look at the various theories like **Bax**, **Reich(Bax)**, $\mathbf{Bax}^{-}$ etc. and will ask ourselves which ones of the effects (E1) – (E7) are provable in the theory in question. At the end of this section we represent the status of the paradigmatic effects in the lattice of our theories, see Figure 223 on p.653.

Usually these statements (Ei) will have two free variables $m, k$. This will enable us to state theorems like $Th \models$ (Ei) which automatically means $Th \models (\forall m, k \in Obs)$(Ei); and also to say that in some model $\mathfrak{M}$ of $Th$, for some choices of $m, k$ (Ei) holds, etc. Thus, availability of the free variables $m, k$ will give us a certain amount of flexibility for stating our theorems.

First we state the three most basic paradigmatic effects (E1)–(E3). These correspond to statements (I)–(III) on p.90 in section 2.5; and also to formulas **(clock)**, **(meter)**, **(asynch)** on p.436. In their formulations, we will add the conditions $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ and $tr_k(m) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ for convenience only; this way the formalized versions of the effects will be simpler.

(E1) Relative motion makes clocks slow down. Formally:

$$[v_m(k) \neq 0 \ \wedge \ \mathsf{f}_{mk}(\bar{0}) = \bar{0}] \Rightarrow [k\text{'s clocks are slow as seen by } m].$$

Here "$k$'s clocks are slow as seen by $m$" is formalized as $|\mathsf{f}_{km}(1_t)_t| > 1$, see Figure 213, and also cf. Thm.2.5.2(iii) (p.92).

(E2) Relative motion makes spaceships (or meter-rods) in direction of movement shrink. Formally:

$$[v_m(k) \neq 0 \ \wedge \ \mathsf{f}_{mk}(\bar{0}) = \bar{0} \ \wedge$$
$$k \text{ thinks that the direction of movement is parallel with } \bar{x}] \Rightarrow$$
$$[\text{the } x\text{-meter-rod of } k \text{ shrinks as seen by } m^{516}].$$

[516] i.e. is shorter than $1_x$

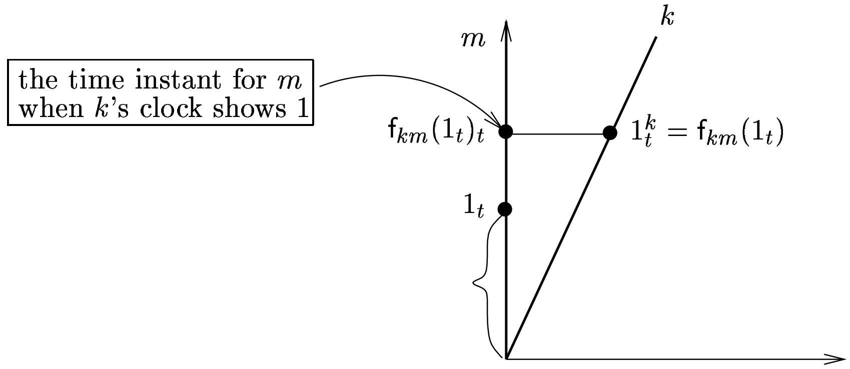



Figure 213: $m$ thinks that $k$'s clocks are slow.

Here, "$k$ thinks that the direction of movement is parallel to $\bar{x}$" is formalized as "$tr_k(m) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$". Then the $x$-meter rod represents the meter-rod in direction of movement. Further, "the $x$-meter rod of $k$ shrinks as seen by $m$" is formalized as[517] "$1_x \in tr_k(k') \parallel \bar{t} \Rightarrow \|tr_m(k')(0)\| < 1$". Intuitively, $k'$ is an observer representing the nose of $k$'s spaceship, while $k$ itself represents the rear of the ship. Then $tr_m(k')$ is the life-line of the nose of the spaceship of $k$ as seen by $m$, and so $\|tr_m(k')(0)\|$ represents the <u>*length*</u> of $k$'s spaceship as seen by $m$. See Figure 214, and cf. Thm.2.5.9 (p.100), Fig.38 (p.101), and Remark 2.5.11.

(E3) Moving clocks get <u>*out of synchronism*</u>. Formally:

$$v_m(k) \neq 0 \Rightarrow \text{(there are events } e_1, e_2 \text{ which are seen by both } m \text{ and } k \text{ and}$$

$$\text{which are simultaneous for } m \text{ but not for } k).$$

Intuitively, (E3) intends to say that the clocks in the nose and the rear of the spaceship of $m$ are seen by $k$ as being out of synchronism. Cf. Thm.2.5.5 and Figure 35 on p.96.

We also call the next two effects basic paradigmatic effects.

[517] Here we use our earlier convention that $tr_m(k)$ can be regarded as a function $tr_m(k) : F \to S$.

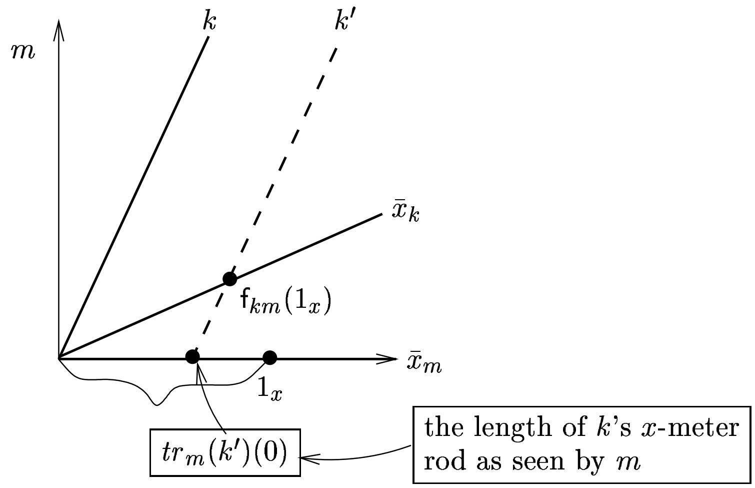



Figure 214: $m$ thinks that $k$'s $x$-meter rods are short.

(E4) Relative motion causes *either clocks slow down or meter-rods shrink*. In more detail: $m$ sees that either $k$'s clocks slow down, or else $k$'s meter-rods in the direction of movement shrink. Formally:

$$(\text{E1}) \vee (\text{E2})$$

This effect was studied in §2.6.

The next basic paradigmatic effect states that "moving spaceships get fat". It is a natural version of (E2) saying that moving spaceships shrink, and, implicitly, it was also studied in §2.5 and in §2.6 (and explicitly in §4.7). E.g. Thm.2.8.8(i) says that moving spaceships shrink under assuming **Basax** + **Ax(symm)**. The variant we will formulate in (E5) below remains true in theories much weaker than **Basax** + **Ax(symm)**. The conclusions of Thm.s 2.8.8, 2.8.12 imply that moving ships get "*fat*", that is their length becomes short relative to their width. This new formulation saying "fat" instead of "shrink" is insensitive to what the *units of measurement* used by the various observers are (while "shrink" is sensitive to what the units of mesurements are, and that is why **Ax(symm)** is needed in §2.8 to prove "shrink"). See Figure 215.

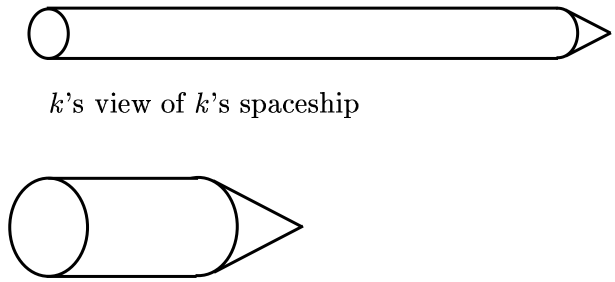



Figure 215: The length of $k$'s spaceship gets short relative to its width as seen by $m$.

(E5) Moving spaceships get distorted, intuitively, they get <u>*"fat"*</u> in the following sense.

$$[v_m(k) \neq 0 \ \wedge \ \mathsf{f}_{mk}(\bar{0}) = \bar{0} \ \wedge \ tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})] \Rightarrow$$

[the length of $k$'s spaceship gets short relative to its width as seen by $m$].

Here, the <u>*length*</u> of $k$'s spaceship is formalized as in (E2), i.e. it is represented by $\|tr_m(k')(0)\|$ where $k' \in Obs$ is such that $1_x \in tr_k(k') \parallel \bar{t}$. On the other hand, the <u>*width*</u> of $k$'s spaceship is represented as the distance of the straight line parallel with the movement of $k$, as follows. Let $P \subseteq {}^nF$ be a plane such that $1_y \in P \parallel \mathsf{Plane}(\bar{t}, \bar{x})$.[518] Now the width of $k$'s spaceship as seen by $m$ is $\mathsf{Eudist}(\mathsf{f}_{km}[P], \mathsf{f}_{km}[\mathsf{Plane}(\bar{t}, \bar{x})])$. See Figure 216. The reason why we had to be more careful with the formulation of "width" (as seen by $m$) than with that of "length" was explained in section 4.7, cf. p.608, in the definition of $\mathbf{R}(\mathbf{Ax\ eqsp})$. See also Figures 203–205 on pp.610–611.

Now we formalize "the length of $k$'s spaceship gets short relative to its width as seen by $m$" as follows:

($\star$) Assume $1_x \in tr_k(k') \parallel \bar{t}$ for $k' \in Obs$, and $1_y \in P \parallel \mathsf{Plane}(\bar{t}, \bar{x})$ where $P \subseteq {}^nF$ is a plane. Then

$$\|tr_m(k')(0)\| < \mathsf{Eudist}(\mathsf{f}_{km}[P], \mathsf{f}_{km}[\mathsf{Plane}(\bar{t}, \bar{x})]).$$

We note that in **Bax**, the conclusion part of ($\star$) above is equivalent with the simpler statement

[518] $P$ is the "life-line" of the straight line $\ell \subseteq S$ where $1_y \in \ell \parallel \bar{x}$.

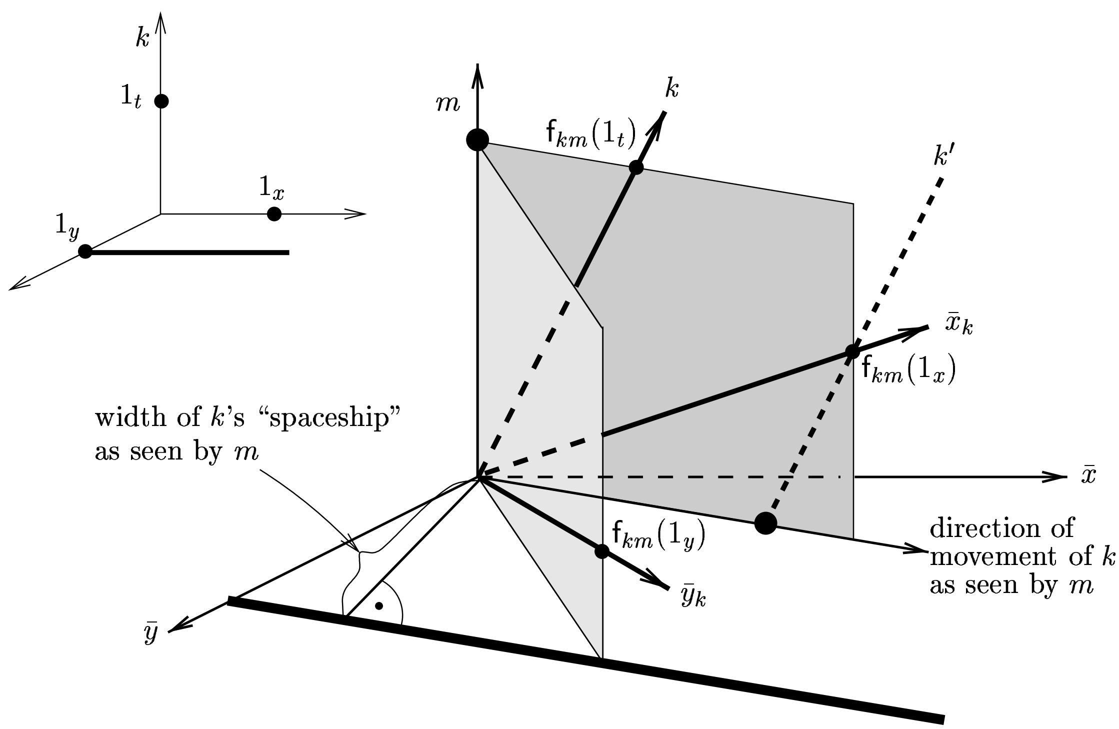



Figure 216: The width of $k$'s spaceship as seen by $m$.

(⋆⋆)

$$\|tr_m(k')(0)\| < \|\mathsf{f}_{km}(1_y)\|.$$

However, as explained in §4.7, in **Reich**(**Basax**) these statements are no longer equivalent.

We will use three kinds of *variants* of the basic effects. Let i $\in \{1, 2, 4, 5\}$. We will investigate the following variants of (Ei).

The disjunction-version:

$$\text{(Ei)}(or) \stackrel{\text{def}}{=} \text{(Ei)}(m, k) \vee \text{(Ei)}(k, m).$$

E.g. (E1)($or$) states that relative motion makes at least one of the involved clocks slow down. We will use these versions to make them insensitive to the choice of the units of measurements. E.g., we have seen that **Basax** $\not\models$ (E1) because in a model, an observer can choose one of his units of measurement arbitrarily, but **Basax** $\models$ (E1)($or$) is true. Cf. §2.6.

The "for fast-enough ovserver" version: (Ei)($fast$) states that (Ei) holds if $k$ is moving fast enough relative to $m$:

$$\text{(Ei)}(fast) \quad \text{is} \quad (\forall m)(\forall d \in \mathsf{directions})(\exists v \in F)(\forall k)$$

[$k$ moves in direction $d$ with a speed greater than $v$ as seen by $m$][519] $\Rightarrow$ (Ei)].

The "distortion gets arbitrarily large" version: We have seen such versions in §2.5. Here we state the "large-version" $(E5)^+$ of (E5). This is a natural variant of (E5) which is both stronger and weaker, namely it is stronger because we can make the distortion arbitrarily large, and it is weaker because we need to choose a *fast enough* spaceship.

[519] I.e. $\vec{v}_m(k) \parallel d$ and $v_m(k) > v$.

(E5)$^+$ The distortion effect in (E5) can get arbitrarily large, assuming we increase the speed of $k$ sufficiently. Formally:

$$(\forall m)(\forall d \in \mathsf{directions})(\forall \lambda \in {}^+\mathrm{Rationals})(\exists v_0 < \mathsf{c}_m(d))(\forall k, k')$$
$$\Big([v_m(k) \geq v_0 \ \wedge \ \vec{v}_m(k) \parallel d \ \wedge \ \mathsf{f}_{mk}(\bar{0}) = \bar{0} \ \wedge \ tr_k(m) \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \ \wedge$$
$$1_x \in tr_k(k') \parallel \bar{t} \ \wedge \ 1_y \in P \parallel \mathsf{Plane}(\bar{t}, \bar{x})] \Rightarrow$$
$$\|tr_m(k')(0)\| < \lambda \cdot \mathsf{Eudist}(\mathsf{f}_{km}[P], \mathsf{f}_{km}[\mathsf{Plane}(\bar{t}, \bar{x})])\Big).$$

Whenever we prove $Th \models$ (E5), also $Th \models$ (E5)$^+$ will be true; and also for suitable versions (E1)$^+$, (E2)$^+$. To save space, in this section we will not deal with these "large-versions".

We will also systematically investigate the following two *additional paradigmatic effects*:

(E6) There are *no faster than light* (FTL) *observers*. Formally,

$$v_m(k) < c_m(\bar{0}, \vec{v}_m(k)) \ \text{ if } \ [\bar{0} \in tr_m(k) \text{ and } v_m(k) \neq 0, \infty].^{520}$$

The formal version states that "$k$ moves slower than the speed of light, as seen by $m$".

We note here that we decided to concentrate on the $n > 2$ case in this section (for simplicity). If we wanted to deal with the case $n = 2$ also, then in (E6) we should add "$n > 2$" to the hypothesis part, and in (E1) – (E5) we should include $v_m(k) \neq \infty \ \wedge \ v_m(k) < c_m(\bar{0}, \vec{v}_m(k))$ into the hypothesis parts.

(E7) is the Existential *Twin Paradox* **Ax(∃TwP)**, cf. Def.4.2.8 on p.460. We note that in investigating any one of our theories like e.g. **Basax**, instead of (E7), one uses e.g. the paradigmatic effect "**Ax(symm)** → (E7)". By this we mean to say that the twin paradox in itself is almost never provable, because to prove it, one needs some symmetry principle like **Ax(symm)**, or **Ax(symm$_0$)** , or **Ax(syx)**. Therefore, when using (E7) as a paradigmatic effect potentially true in one or another member of our lattice of theories, we will use it in the form

(a symmetry principle) → (E7).

For example, **Basax** $\not\models$ (E7) but **Basax** $\models$ (**Ax(symm)** → (E7)). We will choose the symmetry principle in accordance with §4.2.

[520] Compare the definition of $STL$ on p.460.

Let us turn to "mapping" our lattice of theories by using (E1)–(E7) i.e. to discussing the question which one of (E1)–(E7) is true in which one of our theories. We note that exploring this issue is strongly related to exploring the "lego" character (i.e. "modularity") of our theories, and also to the idea of addressing the "why" type questions as they were discussed in the introduction of the present work.[521]

Let us start with the top (**Basax**) of our hierarchy.

<u>(1) **Basax** and **Newbasax**:</u> All the paradigmatic effects occur here in full form. We have to use (E1)(*or*), (E2)(*or*) in place of (E1), (E2) if we do not want to assume a symmetry principle. (The reason is that the symmetry principle **Ax(symm)** "fixes" the units of measurements, and if we want to allow all possible units, we can use a "disjunction-version" of the paradigmatic effect.)

**THEOREM 4.8.1**

(i) $\mathbf{Basax} + \mathbf{Ax(symm)} \models \{(\mathrm{E1}), (\mathrm{E2})\}$.

(ii) $\mathbf{Newbasax} \models \{(\mathrm{E1})(or), (\mathrm{E2})(or), (\mathrm{E3}), (\mathrm{E4}), (\mathrm{E5}), (\mathrm{E6})\}$.

(iii) $\mathbf{Newbasax} \models \mathbf{Ax(symm)} \to (\mathrm{E7})$.

On the **proof**: All these have been proved (explicitly or implicitly) in previous parts of this material (mainly in §2). ∎

The assumption **Ax(symm)** can be considerably weakened in the (E7), i.e. twin paradox, part. The axioms **Ax(syx)**, **Ax(eqspace)**, **Ax(eqm)**, are introduced in the present material. (cf. §6 for **Ax(syx)**).

**THEOREM 4.8.2**

(i) $\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{Ax(syx)} \to (\mathrm{E7})$.

(ii) $\mathbf{Newbasax} \models \mathbf{Ax(eqspace)} \to (\mathrm{E7})$.

(iii) $\mathbf{Newbasax} + \mathbf{Ax}(\mathit{Triv}) \models \mathbf{Ax(eqm)} \to (\mathrm{E7})$.

[521] C.f. also p.550.

Idea of **proof:** **Newbasax** + **Ax(syx)** + **Ax(***Triv***)** $\models$ **Ax(symm)** and similarly for "equimeasure" and "eqspace". ∎

In investigating **Basax** and **Newbasax**, Theorem 4.8.2 above plays the role of an interesting curiosity, but it will become essential in investigating weak systems like **Reich**(**Basax**), where **Ax(symm)** is considered as a too strong symmetry principle (to fit the philosophy of the Reichenbachian versions of our theories). Cf. §4.2 and 4.7.

(2) Newtonian Kinematics: It is natural to expect all our paradigmatic effects to fail in **NewtK**, since they distinguish the pre-relativistic world view (say of Newton) from the various relativistic world views (e.g. of Einstein or of Reichenbach). Cf. e.g. Corollary 4.1.13 in §4.1. The only exception is the no-FTL principle (E6), which simply does not make sense in the more "autenthical" no-photon version $\mathbf{NewtK^n}$.

**THEOREM 4.8.3** **NewtK** $\models \{\neg(\mathrm{E1}), \ldots, \neg(\mathrm{E7})\} \setminus \{\neg(\mathrm{E6})\}$.

That is, effects (E1) etc. are not only not provable in **NewtK**, but they can be proved to be false there. This follows from Thm.4.1.12.

Since all of our theories *Th* below **Newbasax** admit **NewtK** as a special case, by Thm.4.8.3 above none of our theories will prove any[522] of the paradigmatic effects for trivial reasons. Therefore, to avoid triviality, we introduce the following notation. Let *Th* be one of our theories. Then

$$Th^{\oplus} \stackrel{\text{def}}{=} Th + c_m(p,d) < \infty.$$

(Recall that implicitly $m, p, d$ are universally quantified here.) Now we can safely ask ourselves for which one of our theories *Th* is $Th^{\oplus} \models (\mathrm{Ei})$ true for some i $\in \{1, \ldots, 7\}$.

(3) **Flxbasax** and **Bax**: We will find that all paradigmatic effects, except for (E4), hold for **Bax**. Effect (E4) becomes true just in the transition **Bax** $\mapsto$ **Flxbasax**.

**THEOREM 4.8.4**

(i) $\mathbf{Flxbasax}^{\oplus} \models (\mathrm{E1}) \vee (\mathrm{E2})$.

(ii) $\mathbf{Bax}^{\oplus} \not\models (\mathrm{E1}) \vee (\mathrm{E2})$.

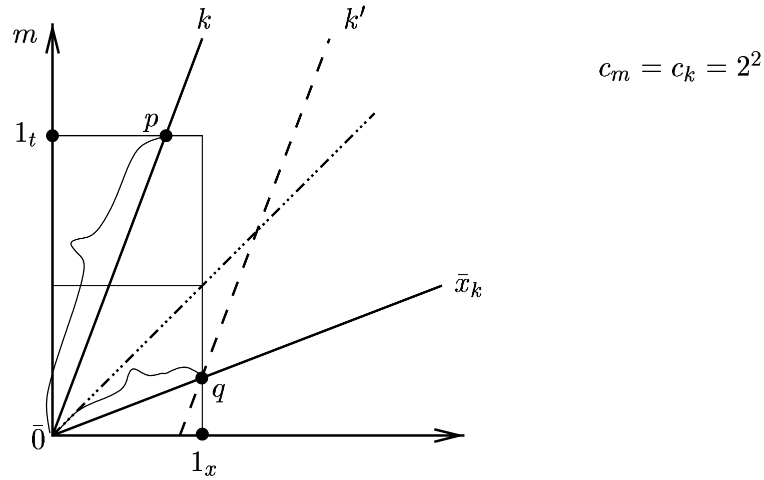



Figure 217: Illustration of proof of Thm.4.8.4(i).

**Proof.** The proof of (i) is represented in Figure 217.

In the figure we represented how observer $m$ sees the coordinate system of observer $k$, in the $\mathsf{Plane}(\bar{t}, \bar{x})$. In the picture[523] $c_m = 2^2$, and we want to show that if $m$ does not think that $k$'s clocks are slow, then $m$ will think that $k$'s spaceship is short. "$k$'s clocks are not slow according to $m$" means that $k$'s $1_t$, i.e. $\mathsf{f}_{km}(1_t)$, lies between $\bar{0}$ and $p$, where $p$ is the point on $k$'s life-line which is simultaneous with $1_t$ according to $m$. Since we are in a **Flxbasax**-model, the speed of light $c_k$ in $k$'s world-view is the same as that in $m$'s world-view, i.e. $c_k = c_m = 2^2$. Therefore $1_x$ of $k$, i.e. $\mathsf{f}_{km}(1_x)$, has to lie between $\bar{0}$ and $q$ (for $q$ see the figure). But already if $q$ would be $1_x$ of $k$, $m$ would see $k$'s spaceship shorter than 1, as illustrated in the figure.

The proof of (ii) is similar, and is represented in Figure 218. ∎

**THEOREM 4.8.5**

(i) $\mathbf{Bax}^{\oplus} \models \{(\mathrm{E1})(or), (\mathrm{E2})(or), (\mathrm{E3}), (\mathrm{E5}), (\mathrm{E6})\}$.

(ii) $\mathbf{Bax}^{\oplus} \models \mathbf{Ax(symm)} \rightarrow (\mathrm{E7})$.

(iii) $\mathbf{Bax}^{\oplus} \models \mathbf{Ax(syt_0)} \rightarrow (\mathrm{E7})$.

[522]Well, in principle (E6) could be proved in these theories without adding the condition $c_m(p,d) < \infty$, because **NewtK** $\models$ (E6). But this is not the case by Thm.4.3.25 and Thm.4.4.14.

[523]Recall that, actually, $c_m$ is the square of usual speed of light, cf. Def.2.2.2(ii) on p.46.

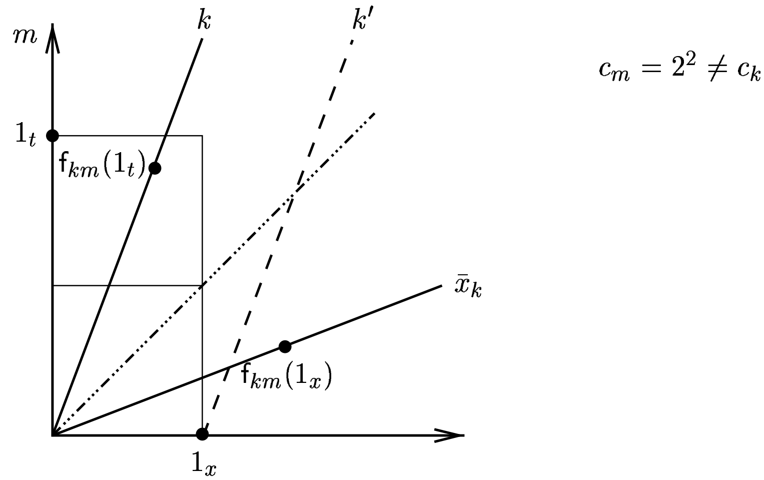



Figure 218: A model of **Bax** in which (E4) fails for $m, k$. Since $c_k \neq c_m$ is allowed, $\mathsf{f}_{km}(1_t)$ does not prescribe the place of $\mathsf{f}_{km}(1_x)$.

(iv) $\mathbf{Bax}^{\oplus} \models \mathbf{Ax(syt_0)} \rightarrow \mathbf{Ax(TwP)}$.

On the **proof**: One can obtain a proof for Thm.4.8.5 by using earlier theorems, and by using the fact that every model of **Bax** can be obtained from a model of **Basax** by (possibly) changing the time-units of each observer, or by changing the space-units of each observer. See Figure 75 and pp. 233–243. See also footnote 354 on p.432. ∎

Thm.4.8.5 above seems to suggest that from, say, the "philosophical" point of view, **Bax** is a very strong relativity theory, since it proves the same but one paradigmatic effects (as far as we looked) as **Basax** does. From Thm.4.8.5 above we infer that in establishing the paradigmatic effects, it was not important to assume that the speed of light is the same for all observers, it is enough to assume that $c_m(p, d_1) = c_m(p, d_2)$ for all $m, p, d_1, d_2$. Moreover, we conjecture that for the paradigmatic effects to hold it is enough to use that the speed of light is the same in direction $d$ and $-d$, for all $d$ (i.e. that $c_m(p, d) = c_m(p, -d)$), which is a weak form of isotropy. (Cf. the problem on p.593.)

Let us turn to our Reichenbach-style theories:

(4) Reichenbachian theories **Reich**(**Basax**), ..., **Reich**(**Bax**). We will find that the synchronism effect (E3) completely disappears here (as was to be expected), but the other effects remain true in their "fast-enough observer" forms.

The next theorem says that the "synchronism effect" (E3) fails in Reichenbachian theories just as strongly as in Newtonian Kinematics.

**THEOREM 4.8.6** **Reich**(**Basax**) + **R**(**sym**) $\not\models$ (E3), *i.e. moving clocks need not get out of synchronism. Moreover, there is a model* $\mathfrak{M} \models$ **Reich**(**Basax**) + **R**(**sym**) *such that in* $\mathfrak{M}$*, every observer thinks that the other observer's clocks are not out of synchronism. I.e.* $\mathfrak{M} \models \neg$(E3).

**Proof.** To prove the theorem, we have to construct a model of **Reich**(**Basax**) + **R**(**sym**) in which all observers agree as far as simultaneity is concerned. In fact, each model $\mathfrak{M}$ has such a simultaneity-version: we simply choose $\langle P_m : m \in Obs \rangle$ such that $\mathsf{f}_{mk}[S] = S$ will be true for all $m, k \in Obs$. E.g., fix $k \in Obs$, and define $P_m \stackrel{\text{def}}{=} \mathsf{f}_{km}[S]$ for all $m \in Obs$. See Figure 219.

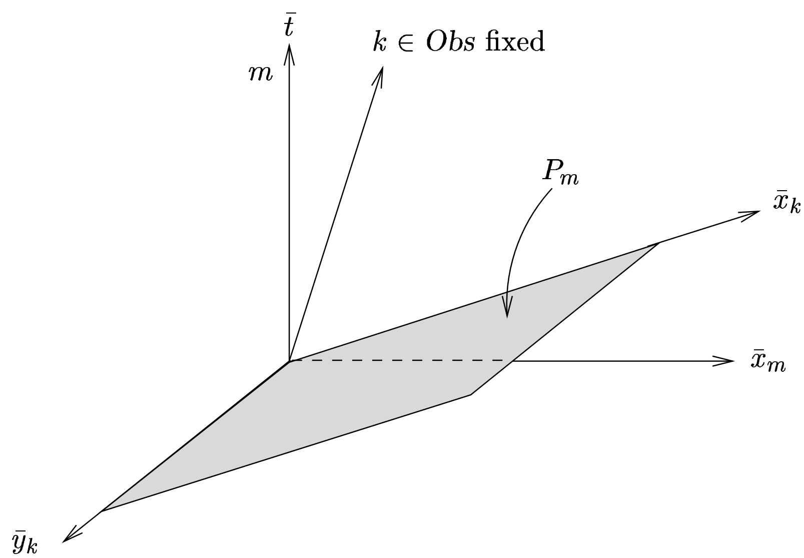



Figure 219: Illustration for the proof of Thm.4.8.6. An art-sim version in which all observers agree in simultaneity matters.

Now if we choose $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Basax} + \mathbf{Ax(sym)} + \mathbf{Ax(}\mathit{Triv}\mathbf{)} + \mathbf{Ax(}\|\mathbf{)})$, then $\mathfrak{M}/P \models$ **Reich**(**Basax**) + **R**(**sym**) by Thm.s 4.5.8, 4.7.6(iv), 4.7.10(iv).

We note that the models we constructed in the present proof are similar to models of **NewtK** in that $\mathsf{f}_{mk}[S] = S$ in them. However, the analogy stops here. E.g., $\mathsf{f}_{mk}$ is not the identity function on $S$. ∎

The above theorem implies that the Reichenbachian relativity theories are drastically different from the usual ones (in some respects). On the other hand, this result was to be expected, since Reichenbachian relativity theories consider simultaneities as matters of convention only.

Next we show that the other two basic paradigmatic effects also fail in Reichenbachian theories in their original form. However, we will see that they do not fail in such a strong way as (E3) does: clocks of fast-enough observers slow down, and spaceships of fast-enough observers shrink.

**THEOREM 4.8.7** $\mathbf{Reich}(\mathbf{Basax}) + \mathbf{R}(\mathbf{sym}) \not\models$ (E1)(*or*) $\vee$ (E2)(*or*). *Moreover, there are a model* $\mathfrak{M}$ *of* $\mathbf{Reich}(\mathbf{Basax}) + \mathbf{R}(\mathbf{sym})$ *and observers* $m, k$ *in* $\mathfrak{M}$ *moving relative to each other in pre-standard configuration such that both* $k$ *and* $m$ *think that the other's clocks are on time, and the other's x-meter rods are precise. Thus* (E1)(*or*) $\vee$ (E2)(*or*) *fails for* $m, k$.

**Proof.** The proof is illustrated in Figure 220.

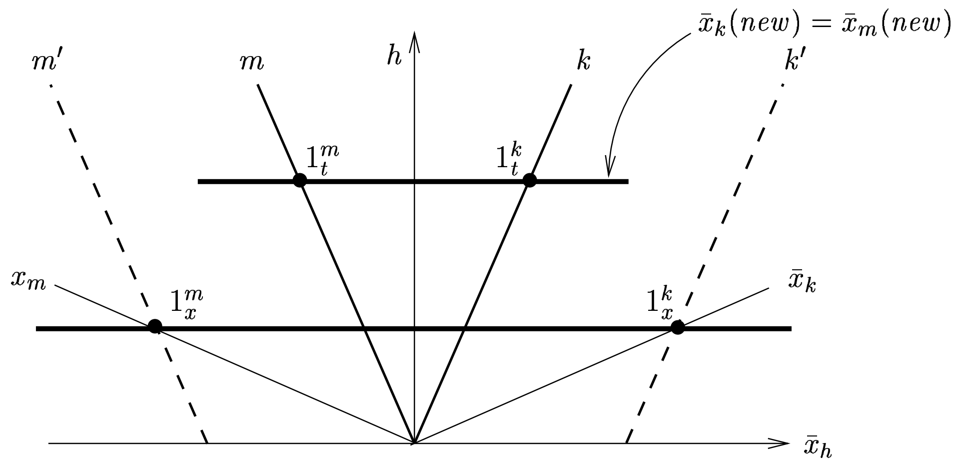



Figure 220: The new simultaneities of $m$ and $k$ are the simultaneity of their median observer $h$.

In the figure, we are in a model of $\mathbf{Basax} + \mathbf{Ax(symm)}$. $m, k, h$ are observers, and $h$ is the median observer for $m$ and $k$. $m'$ and $k'$ represent the "noses" of $m$'s and $k$'s spaceships, respectively. If we take the artificial simultaneities for $m$ and $k$ to be $\mathsf{f}_{hm}[S]$ and $\mathsf{f}_{hk}[S]$ respectively, then in $\mathfrak{M}/P$, $m, k$ will be as in the statement of the theorem. ∎

Next we state theorems to the effect that those versions of the basic paradigmatic effects which we state for "fast enough" observers (i.e. (E1), (E2), (E4), (E5)), hold in Reichenbachian theories.

**THEOREM 4.8.8**

(i) $\mathbf{Reich}(\mathbf{Bax})^{\oplus} \models \text{(E5)}(\mathit{fast})$.

(ii) $\mathbf{Reich}(\mathbf{Bax})^{\oplus} \not\models \text{(E4)}(\mathit{fast})$.

(iii) $\mathbf{Reich}(\mathbf{Flxbasax})^{\oplus} \models \text{(E4)}(\mathit{fast})$.

**Proof.** (iii): To prove $\mathbf{Reich}(\mathbf{Flxbasax}) \models \text{(E4)}(fast)$, we will refine the proof of $\mathbf{Flxbasax} \models \text{(E4)}$, which was illustrated in Figure 217. Assume that $\mathfrak{N}$ is a model of $\mathbf{Reich}(\mathbf{Flxbasax})$. Then $\mathfrak{N}$ is a relativized version of a model $\mathfrak{M}$ of $\mathbf{Flxbasax}^{\oplus}$. For simplicity, in the drawing we assume that $\mathfrak{M} \models \mathbf{Basax}$, i.e. that the speed of light in the model $\mathfrak{M}$ is 1. Let $m \in Obs^{\mathfrak{N}}$ be arbitrary. Figure 221 represents $m$'s world-view in $\mathfrak{M}$. $\bar{x}_m(new)$ is the new simultaneity of $m$ (in $\mathfrak{N}$). Let everything be as in the figure. Particularly, $\overline{1_t q}$ is parallel with $\bar{x}_m(new)$, and $\overline{\bar{0}p}$ is parallel with $\overline{1_x q}$.

To any $r$ between $p$ and $q$ define $r'$ as follows: Let $r''$ be the symmetric image (w.r.t. the line $\overline{\bar{0}q}$) of $r$ in between $1_x$ and $q$, and let $r'$ be on $\bar{x}_m(new)$ be such that $\overline{r', r''}$ is parallel with $\overline{\bar{0}r}$. See Figure 221. We can see that as $r$ approaches $q$, so will $r'$ approach $\bar{0}$, in a monotonic way. Thus there is $r$ such that $r'$ is in between $\bar{0}$ and $1_x(new)$. Then we can choose $v$ as the velocity of an observer whose life-line is $\overline{\bar{0}r}$, because of the following. For simplicity, let $k \in Obs$ be such that $tr_m(k) = \overline{\bar{0}r}$. If $m$ sees that $k$'s clocks are not slow, then $k$'s time unit is between $\bar{0}$ and $r$, and thus $k$'s x-unit is between $\bar{0}$ and $r''$, in the original model $\mathfrak{M} \models \mathbf{Basax}$. Thus the life-line of $k' \in Obs$ which represents the nose of $k$'s spaceship as seen by $m$ in $\mathfrak{M}$, will be parallel with $\overline{\bar{0}r}$, and will intersect $\bar{x}_m(new)$ between $\bar{0}$ and $r'$. In the relativized model $\mathfrak{N}$, still $k'$ will represent the nose of $k$'s spaceship. This finishes the proof of (iii).

We omit the proofs of (i) and (ii). ∎

We have seen that $\text{(E5)}(fast)$ is true in $\mathbf{Reich}(\mathbf{Bax})$. We can prove a version of (E5) to hold already in $\mathbf{Reich}_0(\mathbf{Bax})_\partial$. Notice that we do not know whether $\mathbf{Reich}_0(\mathbf{Bax})$ is the same as $\mathbf{Reich}(\mathbf{Bax})$. Thus $\mathbf{Reich}_0(\mathbf{Bax})_\partial$ may be situated in

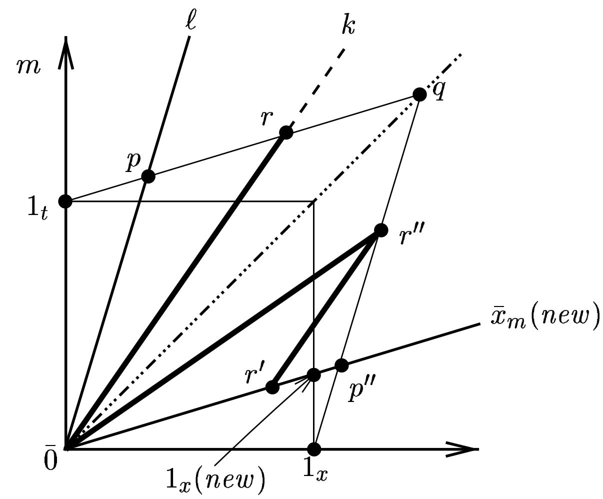



Figure 221:

our lattice of theories like in Figure 222. Thus, if $\mathbf{Reich}_0(\mathbf{Bax})_\partial \neq \mathbf{Reich}(\mathbf{Bax})_\partial$, then this is a new place in our lattice where a paradigmatic effect appears. Apart from this, we include the following theorem also because its proof contains interesting ideas.

The version of (E5) which we prove to hold in $\mathbf{Reich}_0(\mathbf{Bax})_\partial$ is obtained from the "distortion gets arbitrarily large" version $(E5)^+$ of (E5) by replacing the quantifier "$(\forall d \in \mathsf{directions})$" in it with "$(\exists d \in \mathsf{directions})$". We denote the so obtained version by $(E5)^{+\exists}$. We claim that $(E5)^{+\exists}$ is a "fully blown" relativistic effect, almost as good[524] as (E2) is. It says that for any observer $m$ there is a direction $d$ such that if you move fast enough in direction $d$, then your spaceship will be as distorted as you want.

**THEOREM 4.8.9**

$$\mathbf{Reich}_0(\mathbf{Bax})^{\oplus}_{\partial} \models (E5)^{+\exists},$$

*i.e. there are directions such that spaceships moving fast enough in that direction get distorted (and this effect increases with speed).*

The **proof** can be found in [20]. ∎

[524] or as counter-intuitive

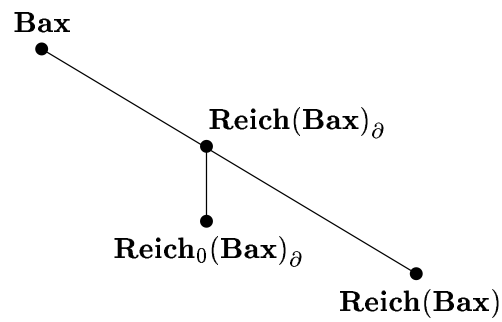



Figure 222:

The next theorem is an analogon of theorems in section 2.8. It states that if we assume some symmetry axiom, then the "clocks slow down" and the "meter rods shrink" effects hold for fast-moving observers.

**THEOREM 4.8.10**

(i) **Reich**(**Bax**) + **R**(**Ax syt**$_\mathbf{0}$) + **Ax(***Triv***)** $\models$ (E1)($fast$).

(ii) **Reich**(**Bax**) + $\mathbf{R}^+$(**Ax eqsp**) $\models$ (E2)($fast$).

(iii) *can be replaced with* **R**(**Ax eqsp**) + **Ax(***Triv***)**.

**Proof.** We omit the proof. ∎

By the above theorem, beginning with **Reich**(**Bax**) we do have some non-negligible basic paradigmatic effects.

We have already seen (in §4.3, §4.7) that the other two paradigmatics effects, (E6) and (E7) hold in $\mathbf{Bax}^-$, so they hold in the Reichenbachian theories, too. We want to point out that the twin-paradox effect holds under assuming the symmetry principle **R**(**Ax syt**$_\mathbf{0}$), which is an adequate symmetry principle for the Reichenbachian theories. Cf. Thm.'s 4.3.24, 4.7.15.

<u>(5) Weak theories $\mathbf{Bax}^-$, **Bax(P1)**, $\mathbf{Bax}^{--}$.</u> We have seen in section 4.3 that the basic paradigmatic effects fail to hold in $\mathbf{Bax}^{-\oplus}$ just as much as in Newtonian Kinematics, because $\mathbf{Bax}^{-\oplus}$ has models in which the world-view transformations

are all Galilean. See Thm.4.3.21, and the text preceding it. We also saw that the twin paradox effect (E7) fails in $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(symm)}$, cf. Thm.4.3.22. But there are some paradigmatic effects which hold in $\mathbf{Bax}^{-}$ already: Thm.4.3.24 states that there are no FTL observers in models of $\mathbf{Bax}^{-\oplus}$, i.e. (E6) holds in $\mathbf{Bax}^{-}$. Also, Thm.4.7.15 states that under a special symmetry principle, the twin paradox holds in $\mathbf{Bax}^{-}$. This shows that the assumption that in each direction there is a photon moving forwards is a rather strong assumption. We summarize the status of the paradigmatic effects in $\mathbf{Bax}^{-}$, as far as we know them, in the following theorem.

**THEOREM 4.8.11**

(i) *The basic paradigmatic effects fail in* $\mathbf{Bax}^{-\oplus}$ *in a strong sense, namely there are models* $\mathfrak{M}$ *of* $\mathbf{Bax}^{-\oplus}$ *such that*

$$\mathfrak{M} \models \{\neg(\mathrm{E1}), \neg(\mathrm{E2}), \neg(\mathrm{E3}), \neg(\mathrm{E4}), \neg(\mathrm{E5})\}.$$

(ii) $\mathbf{Bax}^{-\oplus} \models (\mathrm{E6})$*, i.e. there are no FTL observers in models of* $\mathbf{Bax}^{-\oplus}$*.*

(iii) $\mathbf{Bax}^{-\oplus} \not\models \mathbf{Ax(symm)} \rightarrow (\mathrm{E7})$*, but*

(iv) $\mathbf{Bax}^{-\oplus} + \mathbf{Ax(}\mathit{Triv}\mathbf{)} \models \mathbf{R}(\mathbf{Ax\ syt_0}) \rightarrow \mathbf{Ax(TwP)}$.

Summing up, with the exception of (E6), (E7), $\mathbf{Bax}^{-\oplus}$ can prove no one of our paradigmatic effects. So, $\mathbf{Bax}^{-}$ is much weaker than **Bax** or **Reich**(**Bax**), and one could be tempted to say that (from the paradigmatic point of view), **Bax** and **Reich**(**Bax**) are "relativity theories", while $\mathbf{Bax}^{-}$ is not a "relativity theory".[525] However, adding innocent looking axioms like

$$(\forall m, k, m')(\exists k')(\mathsf{f}_{mk} = \mathsf{f}_{m'k'})$$

to $\mathbf{Bax}^{-}$ may change its behaviour in the direction of proving more relativistic effects. The kinds of axioms we are having in mind are found in Tőke [259].

We now turn to the theory **Bax**(**P1**). We proved the following in §4.4 as Thm.4.4.14.

**THEOREM 4.8.12** *None of the paradigmatic effects* (E1) – (**Ax(symm)** $\Rightarrow$ (E7)) *discussed in this section are provable from* $\mathbf{Bax}(\mathbf{P1})^{\oplus}$*, for any n.*

[525]in the sense that practically none of the paradigmatic effects distinguishing relativity can be proved in it

**Question for future research 4.8.13** Use the results and conjectures in the present section to obtain "philosophically substantial" answers to the "why" type questions[526] involving one or more of (E1) – (E7). We note that one still has to do some intellectual work to obtain satisfactory answers in this direction.[527]

We sum up what we got in this section in Figure 223. In the figure we gave names "slow", "shrink", "async", "fat", "noftl", "twin" to effects (E1), (E2), (E3), (E5), (E6), (E7) respectively. If we indicate an effect (Ei) beside a theory *Th* in the lattice, then we mean to say that *Th* is the "turning-point" for effect (Ei), i.e. *Th* is the first place in the lattice where (Ei) appears. In more detail, this means that effect (Ei) is not yet present in the theories weaker (i.e. below) *Th*, while (Ei) is present in all theories stronger (i.e. above) *Th*. E.g. effect (E4) ("slow or shrink") is true in **Flxbasax**, **Newbasax**, ..., while it is not true in **Bax**, **Reich**(**Bax**), etc.

One can see in the figure that "noftl", "twin" appear first in $\mathbf{Bax}^-$, "fat" appears (in some form) in **Reich**(**Bax**), "async", "slow", "shrink" appear in **Bax**. These three theories seem to be main turning points in the lattice. We see that in $\mathbf{Bax}^-$ there are practically no relativistic effects[528] while in **Bax** practically all the relativistic effects hold. One could interpret this by saying that $\mathbf{Bax}^-$ is not a relativity theory, while **Bax** is a fully grown typical relativity theory. One could say that in **Reich**(**Bax**) half of the relativistic effects hold.

**Reich**(**Bax**) is a turning-point from the point of view of basic paradigmatic effects, it is the first place where non-negligible basic effects ("spaceships get fat for fast observers") occur. Thus, one could say that the transition from "non-relativity theories" to "relativity theories" is somewhere in between $\mathbf{Bax}^-$ and **Reich**(**Bax**). However, already in $\mathbf{Bax}^-$ we have paradigmatic effects ("there are no faster than light observers", "symmetry principle implies the twin paradox"). Two other turning points can be observed in Figure 223. Effect (E3) ("clocks get out of synchronism") appears first in **Bax**, and with it also appear effects (E1), (E2) ("clocks slow down", "spaceships shrink"). This shows that **Bax** is quite a strong relativity theory. We can see from the transition $\mathbf{Bax} \mapsto \mathbf{Flxbasax}$ that effect (E4) ("clocks slow down or spaceships shrink") is sensitive to whether the speed of light is the same for all observers, while the other effects are not sensitive to this, they only require that the speed of light be the same in all direction, as far as one observer is concerned.

[526] cf. the introduction
[527] Cf. p.550.
[528] no basic ones

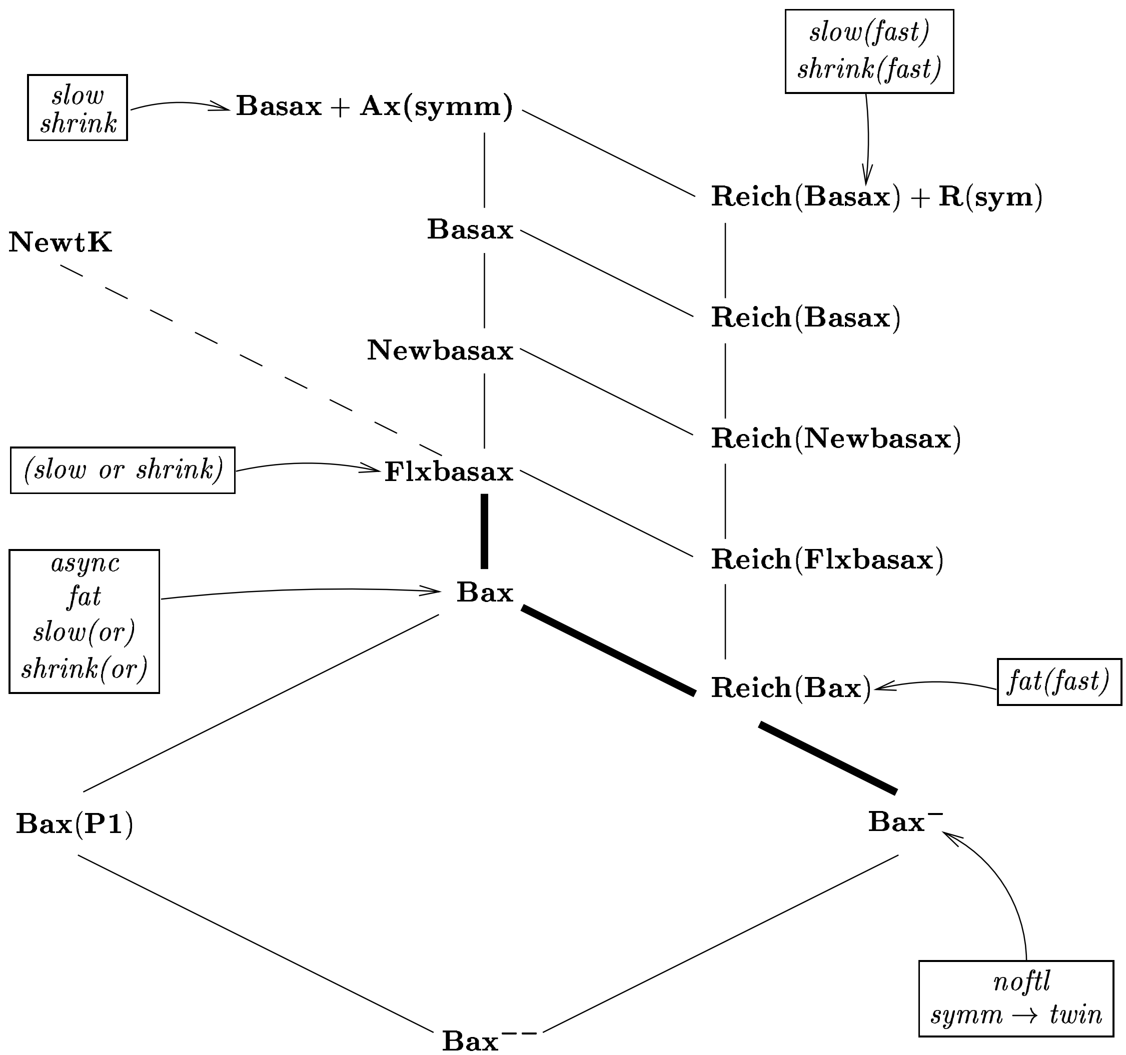



Figure 223: Paradigmatic effects appear at the indicated places in our lattice of relativity theories. (We assume **Ax**($\sqrt{\ }$) and $n > 2$).

## 4.9 Partial world-view versions of our theories

In the previous parts of this section we studied several weaker, more general versions of our original theories **Basax** and **Newbasax**, in which we refined the central assumption (**AxE**, $\mathbf{AxE_0}$, $\mathbf{AxE_R}$ etc) concerning the way observers see photons. Other axioms also had to be relaxed in order to be compatible with the new axiom of photons, and to preserve the distinction between the genearalized weak axiom systems. In this section we shall relax our systems in a different way: we shall drop our earlier assumption that every observer $m$ associates some events to each point of his coordinate system ${}^nF$. In other words, we no longer assume that any observer can "see" (potential) observers and photons even at large distance.

Slightly more formally, we will have to replace our assumption $Dom(w_m^-) = {}^nF$ with the weaker one $Dom(w_m^-) \subseteq {}^nF$, where $Dom(w_m^-)$ is the set of those points where $m$ can see some body. (Recall that in footnote 198 on p.188 we introduced the notation $w_m^-$ to replace $w_m$ so that we could say more conveniently that $Dom(w_m^-) \neq {}^nF$ without having a contradiction with our conventions made in Chapter 2; cf. Fig.3, p.34). Of course, we will need some *new* axioms on what $Dom(w_m^-)$ may be like. The first step in this direction was made when we introduced **Newbasax**, where $w_m^{-1}(Rng(w_k)) \subsetneq {}^nF$ is possible. Cf. pp. 187–193 (§3.3).

(We emphasize that "seeing" in this sense is not the same thing as observing by means of photons. We only refer to the ability of assigning coordinates to events. As we shall discuss (p.660), even if the domain of coordinatized events is bounded in the coordinate space for some observer, there might be events the observer cannot send photons to, or receive photons from.)

In the process of *generalizing our theories towards general relativity* we shall have to make this step anyway. We will need $Dom(w_m^-) \subsetneq {}^nF$ already for accelerated observers[529] (cf. e.g. the accelerated observers section of Andréka-Madarász-Németi-Sági [24] or the accelerated observers part of Misner et. al. [196] or [127]).

First, we redefine the notions of a world-view, a world-view transformation and speed so that they become suitable for observers with partial coordinate spaces. We need to do this because in our earlier concepts $Dom(w_m) = {}^nF$ (even thought $m$ may see $\emptyset$ at certain points), $Dom(\mathsf{f}_m) = {}^nF$, and $v_m$ was only defined for bodies whose traces are in $\mathsf{Eucl}$ (i.e. they are complete lines, not only subsets of lines).

**Definition 4.9.1** Let $\mathfrak{M}$ be a frame model and $m \in Obs$.

**(i)** $w_m^- \stackrel{\text{def}}{=} \{\langle p, e\rangle \in w_m \,:\, e \neq \emptyset\}$.

[529] In the Euclidean sense.

Intuitively, $w^-_m$ is the "true" or "real" world-view of $m$ in the sense that if $m$ sees nothing, *not even potential* bodies at $p \in {}^nF$ then the "real" world-view $w^-_m$ of $m$ is <u>not</u> defined at point $p$.

**(ii)** $\mathsf{f}^-_{mk} :\stackrel{\text{def}}{=} (w^-_m) \circ (w^-_k)^{-1}$.

Thus $\langle p, q\rangle \in \mathsf{f}^-_{mk}$ means that $m$ and $k$ see the same, *nonempty* event at $p$ and $q$, respectively.

Notice that the only difference between $\mathsf{f}_{mk}$ and $\mathsf{f}^-_{mk}$ is that for $p \in {}^nF$, if $p \notin Dom(\mathsf{f}^-_{mk})$ then $p$ is $\mathsf{f}_{mk}$-related to *all* those $q$'s for which $q \notin Dom(w^-_k)$. So $\mathsf{f}_{mk} = \mathsf{f}^-_{mk} \cup \{-Dom(w^-_m)\} \times \{-Dom(w^-_k)\}$. I.e. the "blind spots" of $m$ are $\mathsf{f}_{mk}$-related with those of $k$.

But there is an *essential difference.* Namely the following: In **Newbasax**, $\mathsf{f}^-_{mk}$ is everywhere defined i.e. **Newbasax** $\models [\,(m \stackrel{\odot}{\to} k) \;\to\; (\mathsf{f}^-_{mk} : {}^nF \longrightarrow {}^nF)\,]$. However, in our partial domain theories (e.g. **Loc**(**Bax**$^-$) introduced in Def. 4.9.3 on p.4.9.3) $\mathsf{f}^-_{mk}$ is a *partial function* only (like in general relativity). We shall see (cf. Prop. 4.9.4):

$$\mathbf{Loc}(\mathbf{Bax}^-) \models \mathsf{f}^-_{mk} : {}^nF \stackrel{\circ}{\longrightarrow} {}^nF.$$

This is not helped if we try to work with $\mathsf{f}_{mk}$ instead of $\mathsf{f}^-_{mk}$; namely **Loc**(**Basax**) $\not\models$ ($\mathsf{f}_{mk}$ is a function), although **Loc**(**Basax**) will be a quite strong partial world-view theory. Therefore, this is a point where we will have to be careful when generalizing our "old" definitions, proofs etc.

We note that $\quad \mathsf{f}^-_{mk} = \{\, \langle p, q\rangle \in {}^nF \times {}^nF \;:\; w^-_m(p) = w^-_k(q)\,\}$.

**(iii)** Let $m \in Obs$ and $b \in B$ and $a \in F$. Let us recall from §2 that by definition

$$v_m(b) = a \quad\Longleftrightarrow\quad [\, tr_m(b) \in \mathsf{Eucl} \;\text{ and }\; ang^2(tr_m(b)) = a\,].$$

In this sub-section we will have to refine the definition of $v_m(b)$. The new definition of $v_m(b)$ is the following.

$$v_m(b) = a \quad\stackrel{\text{def}}{\Longleftrightarrow}\quad [\,|tr_m(b)| > 1 \;\wedge\; (\exists \ell \in \mathsf{Eucl})(tr_m(b) \subseteq \ell \;\wedge\; ang^2(\ell) = a)\,].$$ [530]

$\lhd$

[530]Note, that if $|tr_m(b)| \le 1$ or $(\nexists \ell \in \mathsf{Eucl})\; tr_m(b) \subseteq \ell$ then $v_m(b)$ is <u>undefined</u>. Also note that $v_m(b) = \infty$ is possible.

Next, we would like to *refine* some of our theories *Th* (from e.g. **Basax** to $\mathbf{Bax^-}$) so that the new version $\mathbf{Loc}(Th)$ should permit $Dom(w_m^-) \subsetneq {}^nF$, and even that $Dom(w_m^-)$ be a bounded subset of ${}^nF$ (i.e. that $Dom(w_m^-)$ fits into a finite $n$-dimensional ball in ${}^nF$). This direction of generalization might be called "orthogonal" to our way of weakening the assumption concerning the speed of light earlier in this section. By this we mean that it is possible to turn any of our earlier theories to a partial version.

Recalling our relativity theories discussed so far, we think that, by and large, their axioms can be classified as follows:

1. **Ax1** and **Ax2** (saying that $G = \mathsf{Eucl}$ and $Ph \cup Obs \subseteq Ib$), which were always assumed, merely fix the framework we work in within special relativity. These axioms will have to be assumed in the partial domain theories, too.

2. A large part of the axioms characterize the world-view of any given observer. They tell us where the observer in question can see himself/herself, where and how it can see other observers and photons etc. For example, in the case of **Basax**, **Ax3**, **Ax4**, **Ax5** and **AxE** belong to this category. If one reviews our weaker theories, one can check that the weakened versions of **Ax3**, ..., **AxE** play a similar role in those theories.

3. There are axioms that establish or characterize the relationship between the world-views of any two observers. To this category belong **Ax6**, $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$ and all of our symmetry axioms.

Although this is not supposed to be a clear-cut classification, we feel that it is informative, and even the less fundamental axioms (i.e. our auxiliary axioms) have a well-defined place in it (mostly Class 3).

When generalizing our full-domain relativity theories to the case of partial world-views we shall have to concentrate on Class 2. The reason is that in our earlier theories the Class 2 axioms together with Class 1 (**Ax1** and **Ax2**) imply $Dom(w_m^-) = {}^nF$ or at least $\bar{t} \subseteq Dom(w_m^-)$. We shall have no reason to change Class 1, at least as long as we deal with special relativity, because **Ax1** and **Ax2** do not describe what observers can see. But allowing $Dom(w_m^-) \subsetneq {}^nF$ is an essential step towards general relativity. Some changes to Class 3 will be necessary but straightforward, e.g. we shall have to replace $\mathsf{f}_{mk}$ with $\mathsf{f}_{mk}^-$ in $\mathbf{Ax6_{01}}$.

**CONVENTION 4.9.2** For any full-domain theory *Th* we shall write $\mathbf{Loc}(Th)$ for the new, partial-domain version of *Th*.

We claim that there is a natural way to generalize the Class 2 axioms of any full-domain relativity theory *Th*. We shall not give a fully specified algorithm for this generalization, but we shall describe the procedure informally and demonstrate its application in the switch from $\mathbf{Bax^-}$ to $\mathbf{Loc(Bax^-)}$.

Given the Class 2 axioms of some theory *Th* (plus Class 1), one can "imagine" what the world-view of some arbitrary observer $m$ looks like is some model. By this we do not mean a full specification of the models of *Th* as we had e.g. for **Basax**, **Newbasax**, **Flxbasax** and **Bax**; one only has to consider what is required immediately by the axioms. The following pictures depict how one can do this in the case of our earlier theories: Fig.178 (p.548), Fig.179 (p.549), Fig.159 (p.507), Fig.11 (p.53). We proceed further by changing the Class 2 axioms so that they should settle the same requirements as previously, but relativized to a domain $Dom(w_m^-)$. We shall see examples below for $\mathbf{Bax^-}$ and for several of our most distinguished relativity theories. If necessary, one has to adjust the Class 3 axioms a little, too. Additionally, one might have to introduce new axioms to prescribe some properties of $Dom(w_m^-)$; but, as we shall see, some properties will already follow from the modified axioms we will have at this stage. But one has to check carefully whether the above process of *relativization* has or has not opened the way for too exotic models. Should this be the case, then one has to fill in the gap by new postulates.

Let us recall that $\mathbf{Bax^-}$ (cf. Def.4.3.7 on p.479) was one of our weakest theories, and it was the common root of both our relativistic and Newtonian theories. It consists of the following axioms:

$$\{\,\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Obs}}, \mathbf{Ax5_{Ph}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxP1}, \mathbf{AxE_{01}}\,\}.$$

We shall concentrate on obtaining $\mathbf{Loc(Bax^-)}$ first. We note in advance that later one can obtain the partial-world-views versions of other relativity theories by changing e.g. **AxP1** and $\mathbf{AxE_{01}}$ to the stronger form known e.g. from **Flxbasax** or **Newbasax**. (Some other axioms might need to be strengthened, too. We shall return to this issue way below.) Alternatively, one can proceed by the same method we shall use to arrive at $\mathbf{Loc(Bax^-)}$.

From our present point of view, **Ax1**, **Ax2**, $\mathbf{Ax6_{00}}$, **AxP1**, $\mathbf{AxE_{01}}$ are *harmless*. They do not "pump up" the size of $Dom(w_m^-)$ to be big (or unbounded). So, let us turn our attention to $\mathbf{Ax3_0}$, **Ax4**, $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax5_{Ph}}$, $\mathbf{Ax6_{01}}$. Thus $\mathbf{Loc(Bax^-)}$ will be obtained from $\mathbf{Bax^-}$ by replacing $\mathbf{Ax3_0}$, **Ax4**, $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax5_{Ph}}$, $\mathbf{Ax6_{01}}$ with their partial versions $\mathbf{Ax3_0^{par}}$, $\mathbf{Ax4^{par}}$, $\mathbf{Ax5_{Obs}^{par}}$, $\mathbf{Ax5_{Ph}^{par}}$, $\mathbf{Ax6_{01}^{par}}$. The latter axioms are introduced below.

**Definition 4.9.3**

$$\mathbf{Loc}(\mathbf{Bax}^-) \;=\; \{\,\mathbf{Ax(}\text{Frame}\mathbf{)}, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}_\mathbf{0}^{\text{par}}, \mathbf{Ax4}^{\text{par}}, \mathbf{Ax5}_\mathbf{Obs}^{\text{par}}, \mathbf{Ax5}_\mathbf{Ph}^{\text{par}}, \mathbf{Ax6}_\mathbf{00}, \mathbf{Ax6}_\mathbf{01}^{\text{par}}, \mathbf{AxP1}, \mathbf{AxE}_\mathbf{01}\,\},$$

where $\mathbf{Ax1}$, $\mathbf{Ax2}$, $\mathbf{Ax6}_\mathbf{00}$, $\mathbf{AxP1}$, $\mathbf{AxE}_\mathbf{01}$ are like in $\mathbf{Bax}^-$, and the rest of the axioms are defined as follows.

$\mathbf{Ax(}$Frame$\mathbf{)}$ $Obs \neq \emptyset$.

Actually, this axiom has always been assumed. We are planning to update the text of this study accordingly. Thus $\mathbf{Ax(}$Frame$\mathbf{)}$ is not new at all.

$\mathbf{Ax3}_\mathbf{0}^{\text{par}}$ $(\forall h \in Ib)\Big((\exists \ell \in G)[\,tr_m(h) \;=\; \ell \cap Dom(w_m^-) \quad \text{or} \quad tr_m(h) = \emptyset\,] \quad \text{and} \quad (\exists m \in Obs)\; tr_m(h) \neq \emptyset\,\Big).$

That is, $\mathbf{Ax3}_\mathbf{0}^{\text{par}}$ still asserts that the traces of inertial bodies are subsets of straight lines, but it does not require that these lines themselves be complete straight lines, but it does require that the traces of inertial bodies be straight lines "within the domain of $w_m^-$" (or empty).

$\mathbf{Ax4}^{\text{par}}$ $(\forall m \in Obs) \qquad tr_m(m) \;=\; \bar{t} \cap Dom(w_m^-) \;\neq\; \emptyset.$

Intuitively, $m$ can see himself only on the $\bar{t}$-axis. But there may be points on the $\bar{t}$-axis where $m$ can see nothing.[531] We changed $\mathbf{Ax4}$ because it forced $Dom(w_m^-)$ to be unbounded.

$\mathbf{Ax5}_\mathbf{Obs}^{\text{par}}$ Intuitively, let us fix an observer $m$, a direction $d$, and a point $p \in Dom(w_m^-)$. We shall speak about things moving forwards in direction $d$ through point $p$ as seen by $m$ (without mentioning all these data). Assume there is a photon moving in direction $d$. Then there is a photon in the given direction which is limiting in the following sense: For all speeds slower than this limiting photon, there is an observer moving with this speed.

Formally: $(\forall m \in Obs)(\forall p \in Dom(w_m^-))(\forall d \in \mathsf{directions})$
$\Big(\Big[(\exists ph \in Ph)(p \in tr_m(ph) \;\wedge\; (ph \text{ is moving forwards in } d \text{ as seen by } m))\Big] \;\Rightarrow$
$\Big[(\exists ph \in Ph)\Big(p \in tr_m(ph) \;\wedge\; (ph \text{ is moving forwards in } d \text{ as seen by } m) \;\wedge$
$(\forall \lambda \in F)(0 \leq \lambda < v_m(ph) \;\Rightarrow\; (\exists k \in Obs)(p \in tr_m(k) \;\wedge\; v_m(k) = \lambda \;\wedge$
$(k \text{ is moving forwards in direction } d \text{ as seen by } m))\Big)\Big]\Big).$

[531]Intuitively, such a point may be [a point at "space" zero but] *sometime* after the "Big Crunch". Or for an observer falling into a Schwarzchild black hole the point (measured by his own clock i.e. his proper time) on his life-line where his life-line intersects the singularity.

**$\mathbf{Ax5^{par}_{Ph}}$** Intuitively, from any point $p \in Dom(w^-_m)$ in any direction there is a photon moving forwards in that direction. Formally,

$$(\forall m \in Obs)(\forall p \in Dom(w^-_m))(\forall d \in \mathsf{directions})(\exists ph \in Ph)$$
$$[\, p \in tr_m(ph) \;\land\; (ph \text{ is moving forwards in direction } d \text{ as seen by } m)].$$

**$\mathbf{Ax6^{par}_{01}}$** $(\forall m, k \in Obs)\, Dom(\mathsf{f}^-_{mk}) \in Open.$

I.e., we replaced $\mathsf{f}_{mk}$ with $\mathsf{f}^-_{mk}$ in this axiom.

◁

Recall that $\mathbf{Bax}^{-\oplus} = \mathbf{Bax}^- +$ "the speed of photons in not $\infty$". $\mathbf{Bax}^{-\oplus}$ was our weakest theory from which some genuine relativistic effects follow. Similarly,

$$\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) = \mathbf{Loc}(\mathbf{Bax}^-) + \text{"the speed of photons is not } \infty\text{"}$$

can be considered as our weakest theory of special relativity. ◁

A potential problem with this theory (as well as the other straightforward partial domain theories derived below from their "full domain" ancestors) is that it might be too *permissive* about what $Dom(w^-_m)$ can be like. We shall see that a large number of interesting propositions and theorems can be proven without making any further restrictions on $Dom(w^-_m)$. We introduce the following axioms as optional postulates, which might be used or omitted according to our need. To our knowledge they are fully compatible with the standard treatment of relativity, including the general theory.[532]

We obtain $\mathbf{Ax6^{par}_{01}} \models (Dom(w^-_m)$ is an *open* subset of ${}^nF)$ if we take $m = k$ in the axiom; thus we do not introduce a distinct postulate for this. At the formulation of $\mathbf{Ax6_{01}}$ (p.191), we already discussed how such a statement can be translated into our first-order frame language, hence we do not discuss that problem here again.

As a contrast to saying that a definable set like $Dom(w^-_m)$ is open, it is harder to say in our first-order frame language that $Dom(w^-_m)$ is *connected*. Therefore we shall postulate a slightly stronger property, namely *star-connectedness*.

**Ax(star)** $Dom(w^-_m)$ is a *star-connected* subset of ${}^nF$. Formally,

$$(\exists c \in Dom(w^-_m))(\forall p \in Dom(w^-_m))\, [p, c] \subseteq Dom(w^-_m),$$

where $[p, c]$ is the segment connecting $p$ and $c$, i.e. $[p, c] = \{\, q \in \overline{pc} : \mathsf{Betw}(p, q, c)\,\}$.

[532] In this subsection we do not intend to introduce too many such axioms which will have to be dropped or totally transformed for general relativity.

Alternatively, instead of star-connectedness of $Dom(w_m^-)$ one can postulate the following:

**Ax(clock-conn)** $(\exists k \in Obs)\Big( [\, tr_m(k)$ is connected in the usual sense according to $k$'s clock $]$[533] $\wedge \quad (\forall p \in Dom(w_m^-))(\exists q \in tr_m(k))\,[p,q] \subseteq Dom(w_m^-)\Big)$.

Someone might claim that it is not realistic to let observers coordinatize events that they cannot see by photons. Again others may think that it is a precondition to coordinatizing events to introduce (conventional) simultaneities, and this requires us to potentially send and receive photons to/from the events. The following two axioms formalize these two possible restrictions.

**Ax(photon)** $(\forall p \in Dom(w_m^-))(\exists ph \in Ph)\,(p \in tr_m(ph) \;\wedge\; tr_m(ph) \cap \bar{t} \neq \emptyset)$.

In simple terms, **Ax(photon)** says that an observer can only coordinatize those events from which it can receive a photon, or to which it can send a photon.

**Ax(simult)** $(\forall p \in Dom(w_m^-))(\exists ph_0, ph_1 \in Ph)\,(p \in tr_m(ph_0) \cap tr_m(ph_1) \;\wedge$ $tr_m(ph_0) \cap \bar{t} \neq \emptyset \neq tr_m(ph_1) \cap \bar{t} \;\wedge\; (\forall d \in \mathsf{directions})$ $[ph_0$ moves forwards in direction $d \;\Rightarrow ph_1$ moves backwards in direction $d])$.

That is, **Ax(simult)** says that any observer should be able to send *and* receive photons to/from the events to which it assigns coordinates.

Both of these axioms are optional. Those who feel that e.g. Occam's razor requires these restrictions, might assume either. Other might prefer to analyze simpler and weaker axiom systems, and choose to omit these axioms. We feel that they do not influence the models of our partial world-view theories essentially, although we did not check this conjecture.

On the other hand, when we enter the realm of general relativity, **Ax(photon)** and **Ax(simult)** (or suitably adjusted versions of these) will gain more significance. Imagine an observer $m$ who is located outside the event horizon of a black hole. Then some will argue that $m$ should be unable to coordinatize events within the horizon because $m$ cannot receive photons from the events within the horizon. Similarly, thos who "live" within the horizon, cannot coordinatize events that are outside. Further, those observers who fall through the horizon can assign coordinates to

[533] $tr_m(k)$ is connected $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall p, q \in \bar{t})[\, w_k^-(p), w_k^-(q) \in Dom(w_m^-) \;\Rightarrow\; (\forall r \in \bar{t})\big(\mathsf{Betw}(p,r,q) \;\Rightarrow$ $w_k(r) \in Dom(w_m^-)\,)] \;\wedge\; \mathsf{f}_{km}^- \restriction \bar{t}$ is continuous (on its domain).

both domain. Those who have this opinion will propose **Ax(simult)**. But we feel that **Ax(simult)** will be optional even in our general relativity theories.

The following axiom excludes certain exotic models. Recall that $\overset{\odot}{\to}$ used to be an equivalence relation in (almost all of) our earlier full-domain theories. We doubt that the same applies in the partial-world-views case.

**Ax(mut)** $m \overset{\odot}{\to} k \Rightarrow k \overset{\odot}{\to} m$.

The next axiom postulates the continuity of $\mathsf{f}^-_{mk}$. We deem this axiom harmless, and it is usually assumed both in the special and the general theories of relativity.

**Ax(continuity)** $\mathsf{f}^-_{mk}$ is continuous. Formally,

$$(\forall m, k \in Obs)\big(\text{ for every open ball } S \subseteq Dom(\mathsf{f}^-_{mk}) \text{ we have } \mathsf{f}^-_{mk}[S] \text{ is also open}\big).$$

Our next candidate axiom says that $\mathsf{f}^-_{mk}$ is a partial collineation on ${}^nF$. Though we had this as a theorem in **Bax$^-$**, we *conjecture* that it is not provable in **Loc**(**Bax$^-$**) (not even in stronger partial domain relativity theores introduced way below).

**Ax(pcoll)** $\mathsf{f}^-_{mk} : {}^nF \overset{\circ}{\longrightarrow} {}^nF$ is a partial collineation on ${}^nF$, i.e. for any $\ell \in G$, $\mathsf{f}^-_{mk}[\ell] \subseteq \ell_1 \in G$ for some $\ell_1$. In other words, assume $p, q, r \in Dom(\mathsf{f}^-_{mk})$ are collinear. Then $\mathsf{f}^-_{mk}(p), \mathsf{f}^-_{mk}(q), \mathsf{f}^-_{mk}(r)$ are collinear, too.

◁

Let us turn to the issue of how the other relativity theories can be turned partial. The general schema of defining **Loc**($Th$) from $Th$ is that we replace **Ax3$_0$**, **Ax4**, **Ax5**, **Ax6$_{01}$** (or its variants) with their partial-domain versions by the method described on p. 657.

**Basax** and **Newbasax** contain the **Ax5**. This needs to be replaced by its relativization to $Dom(w^-_m)$.

**Ax5$^{\text{par}}$** $(\forall m \in Obs)(\forall \ell \in G)$

$$\Big(\,[\,(ang^2(\ell) < 1 \ \wedge\ \ell \cap Dom(w^-_m) \neq \emptyset) \ \Rightarrow\ (\exists k \in Obs)\, tr_m(k) = \ell \cap Dom(w^-_m)\,]$$
and
$$[\,(ang^2(\ell) = 1 \ \wedge\ \ell \cap Dom(w^-_m) \neq \emptyset) \ \Rightarrow\ (\exists ph \in Ph)\, tr_m(ph) = \ell \cap Dom(w^-_m)]\Big).$$

Now we can formulate our first approximation of $\mathbf{Loc}(\mathbf{Newbasax})$.[534] $\mathbf{Ax3_0^{par}}$, $\mathbf{Ax4^{par}}$, $\mathbf{Ax6_{01}^{par}}$ can be reused.

$\mathbf{Loc}(\mathbf{Newbasax}) \stackrel{\text{def}}{=}$

$$\{\, \mathbf{Ax(}\text{Frame}\mathbf{)}, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0^{par}}, \mathbf{Ax4^{par}}, \mathbf{Ax5^{par}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}^{par}}, \mathbf{AxE_0} \,\}\,.$$

$\triangleleft$

**Basax** differs from **Newbasax** in that in **Basax** every observer has access to the same events. Similarly

$$\mathbf{Loc}(\mathbf{Basax}) \stackrel{\text{def}}{=} \mathbf{Loc}(\mathbf{Newbasax}) \setminus \{\, \mathbf{Ax6_{00}} \,\} + \mathbf{Ax6}$$

We shall see that $\mathbf{Loc}(\mathbf{Basax})$ still has partial models. We conjecture that keeping $\mathbf{Ax6_{01}^{par}}$ is essential to have $\mathbf{Loc}(\mathbf{Basax}) > \mathbf{Loc}(\mathbf{Newbasax})$. The reader is invited to check whether $\mathbf{Ax6_{01}^{par}}$ is independent from the rest of $\mathbf{Loc}(\mathbf{Basax})$. On the other hand, $\mathbf{Ax6_{00}}$ has been omitted because $\mathbf{Ax6}$ is a stronger assumption. $\triangleleft$

In a completely similar spirit, $\mathbf{Ax(5^{Obs})}^{\text{par}}$ and $\mathbf{Ax(5^{Ph})}^{\text{par}}$ are obtained by replacing $\ell$ with $\ell \cap Dom(w_m^-)$ in the atomic formulas of the pattern "$\ell = tr_m(\ldots)$". $\mathbf{Loc}(\mathbf{Bax})$ is defined analogously.

$\mathbf{Ax(5^{Obs})}^{\text{par}}$ $(\forall \ell \in G)\,[\ell \cap Dom(w_m^-) \neq \emptyset \;\Rightarrow$
$$\Big((\exists ph \in Ph)(m \overset{\odot}{\to} ph \;\wedge\; v_m(ph) > ang^2(\ell)) \;\Rightarrow\; (\exists k \in Obs)tr_m(k) \subseteq \ell\Big)\Big].$$

$\mathbf{Ax(5^{Ph})}^{\text{par}}$ $(\forall \ell \in G)\,[\ell \cap Dom(w_m^-) \neq \emptyset \;\Rightarrow$
$$\Big((\exists ph \in Ph)(m \overset{\odot}{\to} ph \;\wedge\; v_m(ph) = ang^2(\ell)) \;\Rightarrow\; (\exists ph_1 \in Ph)tr_m(ph_1) \subseteq \ell\Big)\Big].$$

$$\begin{aligned}\mathbf{Loc}(\mathbf{Bax}) \;\stackrel{\text{def}}{=}\; & \{\, \mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0^{par}}, \mathbf{Ax4^{par}}, \mathbf{Ax(5^{Obs})}^{\text{par}}, \mathbf{Ax(5^{Ph})}^{\text{par}}, \\ & \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}^{par}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}} \,\}\end{aligned}$$

And finally,

$$\mathbf{Loc}(\mathbf{Flxbasax}) \stackrel{\text{def}}{=} \mathbf{Loc}(\mathbf{Bax}) + \mathbf{AxE_{02}}.$$ [535]

$\triangleleft$

[534] By a first approximation we mean that we might have to add some auxiliary axioms to $\mathbf{Loc}(\mathbf{Newbasax})$ in order to exclude some particularly exotic models. In other words, some auxiliary axioms might be needed to fill in the gap that might result from relativizing our earlier postulates on **Newbasax** world-views to partial domains.

[535] Note that by the convention below the definition of $\mathbf{AxE_0}$ $v_m(ph)$ is defined even if $0 \neq tr_m(ph)$ is finite, *hence* $\mathbf{AxE_{00}}$, $\mathbf{AxE_{01}}$ or $\mathbf{AxE_{02}}$ do *not* force the life-lines of photons to be *complete* lines.

Now, for $Th \in \{\mathbf{Loc}(\mathbf{Bax}^-), \mathbf{Loc}(\mathbf{Bax}^{-\oplus}), \mathbf{Loc}(\mathbf{Bax}), \mathbf{Loc}(\mathbf{Flxbasax}), \mathbf{Loc}(\mathbf{Newbasax}), \mathbf{Loc}(\mathbf{Basax})\}$ we have a much more flexible version $\mathbf{Loc}(Th)$[536] of $Th$ which is much more suitable for generalizations in the direction of general relativity. I.e. **Loc**(**Newbasax**), **Loc**(**Flxbasax**), etc. are much *closer to* the spirit of general relativity than the original **Newbasax** etc. were. This list is not complete. To generalize the less frequently used theories for the partial world-view paradigm is left to the reader.

Let us examine the most basic properties of our partial-domain theories.[537]

**PROPOSITION 4.9.4** *Assume* $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$. *Then the following items hold.*

**(i)** $(\forall b \in Ib)\, tr_m(b) \subseteq \ell$ *for some* $\ell \in G$. *I.e. the traces of inertial bodies are (parts of) straight lines.*

**(ii)** $(\forall m \in Obs)\, Dom(w_m^-) \neq \emptyset$. *Actually,* $\mathbf{Ax4}^{\mathrm{par}} \models Dom(w_m^-) \neq \emptyset$. *I.e. there are no "blind" observers.*

**(iii)** $Obs \cap Ph = \emptyset$.

**(iv)** $(\forall p \in Dom(w_m^-))\, (p \in tr_m(k) \cap tr_m(ph) \Rightarrow v_m(k) \neq v_m(ph))$. *I.e. no observer can travel together with a photon, even locally.*

**(v)** $w_m^-$ *is injective.*

**(vi)** $\mathsf{f}_{mk}^-$ *is a bijection between* $Dom(\mathsf{f}_{mk}^-)$ *and* $Dom(\mathsf{f}_{km}^-) = Rng(\mathsf{f}_{mk}^-)$.

**(vii)** $\mathsf{f}_{mm}^- = \mathrm{Id} \restriction Dom(w_m^-)$, $(\mathsf{f}_{mk}^-)^{-1} = \mathsf{f}_{km}^-$ *and* $\mathsf{f}_{mk}^- \restriction D = \mathsf{f}_{mh}^- \circ \mathsf{f}_{hk}^-$, *where* $D = \{x \in {}^nF : w_m^-(x) \in Rng(w_h^-) \cap Rng(w_k^-)\}$.

**(viii)** $w_m^- \restriction Dom(\mathsf{f}_{mk}^-) = \mathsf{f}_{mk}^- \circ w_k^-$.

**(ix)** $\mathsf{f}_{mk}^-[tr_m(b)] \subseteq tr_k(b)$.

*Assume* **Loc**(**Bax**). *Then the following two items hold.*

[536] Although **Loc**(**Newbasax**) is only one of a series of partial domain relativity theories, it can be considered as a kind of standard: it does not share the non-classical permissiveness of $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$, **Loc**(**Bax**) etc. about the speed of light, and it does not make the unreasonable assumption **Ax6**, unlike **Loc**(**Basax**).

[537] Cf. Prop. 2.3.3 for the corresponding properties of **Basax**. Most of the statements of Prop. 2.3.3 were preserved by the weak full-domain systems analysed later in this opus.

**(x)** $\big[ang^2(\ell) < c_m \ \wedge \ \ell \cap Dom(\mathsf{f}^-_{mk}) \neq \emptyset\big] \Rightarrow (\exists \ell' \in \mathsf{Eucl})\mathsf{f}^-_{mk}[\ell \cap Dom(\mathsf{f}^-_{mk})] = \ell' \cap Rng(\mathsf{f}^-_{mk})$.

*That is, $\mathsf{f}^-_{mk}$ maps "slow lines" (to be more exact, those points of slow lines which are seen both by $m$ and $k$) into straight lines, and the images are full in the sense that no point of $Rng(\mathsf{f}^-_{mk})$ could be added to it. Notice that* **Loc(Bax)** *allows us to speak about slow lines, i.e. lines that are slower than the photons in a given world-view.*

**(xi)** $\big[ang^2(\ell) = c_m \ \wedge \ \ell \cap Dom(\mathsf{f}^-_{mk}) \neq \emptyset\big] \Rightarrow (\exists \ell' \in \mathsf{Eucl}) ang^2(\ell') = c_k \ \wedge \ \mathsf{f}^-_{mk}[\ell \cap Dom(\mathsf{f}^-_{mk})] = \ell' \cap Rng(\mathsf{f}^-_{mk})$.

*I.e. $\mathsf{f}^-_{mk}$ maps photon lines into photon lines in a sense similar to the previous item.*

**Proof:** The proofs are similar to those of Prop. 2.3.3. We omit them. ∎

Next, we introduce two different weakened versions of **Ax(Bw)**. Without **Ax(Bw)**$^{\text{par}}$ one can have quite exotic models, as we shall see below. Recall that **Ax(Bw)** was only rarely used, because the world-view transformations were bijective collineations even in **Bax**$^-$, and **Ax**$(\sqrt{\ })$ was enough to ensure that they preserve betweenness. However, this is not the case in the partial domain theories anymore. We are not even sure if lines are preserved locally in **Loc(Newbasax)** without **Ax(Bw)**$^{\text{par}}$ (although slow and photon lines are preserved by Prop. 4.9.4(x)-(xi)).

**Ax(Bw)**$^{\text{par}}$ Intuitively, if $m \overset{\odot}{\to} k$ then $\mathsf{f}^-_{mk}$ is locally betweenness preserving; formally:

$$(\forall m, k \in Obs)(\forall p \in Dom(\mathsf{f}^-_{mk}))(\exists \varepsilon \in {}^+F)[\,\mathsf{f}^-_{mk} \restriction S(p,\varepsilon) \text{ is } \mathsf{Betw} \text{ preserving}\,].$$

**Ax(syBw)**$^{\text{par}}$ $(\forall m, k \in Obs)(\forall p \in Dom(\mathsf{f}^-_{mk}))(\exists \varepsilon \in {}^+F)$
$\big[(\forall q, r, s \in S(p,\varepsilon))\mathsf{Betw}(q,r,s) \Leftrightarrow \mathsf{Betw}(\mathsf{f}^-_{mk}(q), \mathsf{f}^-_{mk}(r), \mathsf{f}^-_{mk}(s))\big]$.

In other words, **Ax(syBw)**$^{\text{par}}$ is the "symmetric version" of **Ax(Bw)**$^{\text{par}}$. It is straightforward to check that **Ax(syBw)**$^{\text{par}}$ > **Ax(Bw)**$^{\text{par}}$.

### 4.9.1 Characterizing $\mathsf{f}^-_{mk}$ in models of Loc(Bax)

The key step in the analysis of models of **Basax**, **Newbasax**, **Flxbasax**, **Bax** etc. was a theorem stating the the world view transformations are bijective collineations defined on ${}^nF$. Clearly, we have to characterize the world-view transformations differently for partial domain theories. We shall prove that $\mathsf{f}^-_{mk}$ *locally* preserves parallelism and planes (assuming $\mathbf{Ax(Bw)}^{\mathrm{par}}$). Further, we shall show a theorem analogous to Thm. 3.1.4, which characterized the world-view transformations in terms of affine transformations and maps generated by field automorphisms.

**PROPOSITION 4.9.5** *The world-view transformations preserve planes locally. I.e. there is a neighbourhood around every point in their domain such that they preserve the intersections of planes with this neighbourhood. Formally,*
$\mathbf{Loc(Bax)} + \mathbf{Ax(Bw)}^{\mathrm{par}} \models (\forall p \in Dom(\mathsf{f}^-_{mk}))(\exists \varepsilon \in {}^+F)\ \big(S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk}) \wedge$
$(\forall P \in Plane)\ \big[P \cap S(p,\varepsilon) \neq \emptyset\ \Rightarrow\ \mathsf{f}^-_{mk}[P \cap S(p,\varepsilon)]$ *is part of a plane*$\big]\big)$.

**Proof:** Actually, this is a corollary of Lemma 4.9.16 below. ∎

**PROPOSITION 4.9.6** *In models of* $\mathbf{Loc(Bax)} + \mathbf{Ax(Bw)}^{\mathrm{par}} + \mathbf{Ax}(\sqrt{\ })$*,* $\mathsf{f}^-_{mk}$*'s preserve parallelism locally. Formally:*
$\mathbf{Loc(Bax)} + \mathbf{Ax(Bw)}^{\mathrm{par}} \models (\forall p \in Dom(\mathsf{f}^-_{mk}))(\forall \delta \in F^+)(\exists \varepsilon \in {}^+F)$
$\big(S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk})\ \wedge\ (\forall \ell_0, \ell_1 \in G)\,[\ell_0 \cap S(p,\varepsilon) \neq \emptyset \neq \ell_1 \cap S(p,\varepsilon)\ \wedge\ \ell_0 \parallel \ell_1 \wedge$
$\mathrm{ang}^2(\ell_0) \notin (c_m - \delta, c_m + \delta)\ \Rightarrow \mathsf{f}^-_{mk}[\ell_0 \cap S(p,\varepsilon)] \parallel \mathsf{f}^-_{mk}[\ell_1 \cap S(p,\varepsilon)]]\big)$.

For the proof of Prop. 4.9.6 we shall need the following two lemmas. We shall include their proof after the proof of the proposition.

**LEMMA 4.9.7** *Assume* $\mathbf{Ax}(\sqrt{\ })$*. Let us fix some* $c \in F^+$*. Let us call the line* $\ell$ *c-line if* $\mathrm{ang}^2(\ell) = c$*, slow line if* $\mathrm{ang}^2(\ell) < c$ *and fast line if* $\mathrm{ang}^2(\ell) > c$*. Then the planes of* ${}^nF$ *can be classified as follows:*

1. *Planes that contain to c-lines. Each line of such a plane is fast.*
2. *Planes that contain fast lines and c-lines, but no slow lines. Then each pair of c-lines within such a plane are parallel.*[538]
3. *Planes that contain slow lines, as well as c-lines and fast lines. Through each point of such a plane there are two c-lines within the plane, and the c-lines of the plane belong to two equivalence classes of* $\parallel$*.*

[538]Such planes are known as Robb-planes in the literature, assuming that $c$ is the speed of light.

**LEMMA 4.9.8** *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Let* $c \in F^+$ *and* $S = S(p, \varepsilon)$ *be fixed. Let* $0 < \delta < c$. *Let* $P$ *be a plane that contains a line* $\ell$ *with* $ang^2(\ell) = \delta$. *Then there is* $S_0 = S(p, \gamma)$ *such that*

$$(\forall \ell_1, \ell_2 \in G)\,(\ell_1 \cap S_0 \neq \emptyset \neq \ell_2 \cap S_0 \;\land\; \ell_1 \cup \ell_2 \subseteq P \;\land\; ang^2(\ell_1) = ang^2(\ell_2) = c \;\land$$

$$\ell_1 \not\parallel \ell_2 \;\Rightarrow\; (\forall q \in \ell_1 \cap \ell_2) q \in S)\,.$$

*Informally, if* $\ell_1$ *and* $\ell_2$ *are non-parallel c-lines that intersect* $S_0$, *then they meet within* $S$.

**Proof of Prop. 4.9.6:** Assume $\mathbf{Loc(Bax)} + \mathbf{Ax(Bw)}^{\rm par} + \mathbf{Ax}(\sqrt{\ })$. Let $p \in Dom(\mathsf{f}^-_{mk})$ for some $m, k \in Obs$ and let $S = S(p, \varepsilon)$ be such that $S \subseteq Dom(\mathsf{f}^-_{mk})$, and $\mathsf{f}^-_{mk} \restriction S$ preserves $\mathsf{Betw}$. Such an $S$ exists by $\mathbf{Ax6}^{\rm par}_{\mathbf{01}}$ and $\mathbf{Ax(Bw)}^{\rm par}$.

We are defining a neighbourhood $S_0 \subseteq S$ such that $\mathsf{f}^-_{mk}$ will preserve $\parallel$ within $S_0$. Let $S' = S(p, \gamma)$ be the neighbourhood of $p$ with the property of Lemma 4.9.8, i.e. each plane $P$ that contains a line $\ell$ with $ang^2(\ell) = \delta$, each pair of non-parallel lines $\ell_a, \ell_b$ with $ang^2(\ell_a) = ang^2(\ell_b) = c_m$, $S' \cap \ell_a \neq \emptyset \neq S' \cap \ell_b$, we have $\emptyset \neq \ell_a \cap \ell_b \subseteq S$. Let $S_0 = S(p, \frac{\gamma}{2})$.

Let $\ell_1, \ell_2 \in G$ be such that $\ell_1 \cap S_0 \neq \emptyset \neq \ell_2 \cap S_0$ and $\ell_1 \parallel \ell_2$. We have to show that $\ell_1$ and $\ell_2$ are mapped into a pair of parallel lines.

<u>Case 1:</u> If $\ell_1, \ell_2$ are photon-lines (i.e. $ang^2(\ell_1) = ang^2(\ell_2) = c_m$), then we have a relatively easy task. For the proof of this case the reader is asked to consult Figure 224. We shall prove Case 1 for $S$ instead of $S_0$, i.e. we shall assume only that $\ell_1 \cap S \neq \emptyset \neq \ell_2 \cap S$.

As $\ell_1 \parallel \ell_2$, they determine a plane $P$. By Prop. 4.9.5, $\mathsf{f}^-_{mk}$ takes $P$ into a plane $P'$. Figure 224 shows both $P$ in $m$'s world-view and $P'$ in $k$'s world-view. By Prop. 4.9.4(xi), $\mathsf{f}^-_{mk}$ takes $\ell_1$ and $\ell_2$ into lines that have the same axis with $\bar{t}$. Formally,

$$(\exists \ell'_1, \ell'_2 \in G)\left[ang^2(\ell'_1) = ang^2(\ell'_2) = c_k \;\land\; \mathsf{f}^-_{mk}[\ell_1] \subseteq \ell'_1 \;\land\; \mathsf{f}^-_{mk}[\ell_2] \subseteq \ell'_2\right].$$

Of course, we still have to show $\ell'_1 \parallel \ell'_2$. By Lemma 4.9.7, through each $q \in P$ there are at most two photon-lines (i.e. lines with angle $c_m$) within the plane and the photon-lines of $P$ belong to at most two equivalence classes of $\parallel$.

Now, let $\ell_3 \in G$ be such that $ang^2(\ell_3) = c_m$, and $\ell_3$ intersects both $\ell_1$ and $\ell_2$ within $S_0$. As $\mathsf{f}^-_{mk}$ is injective, $\ell_1 \cap \ell_3, \ell_2 \cap \ell_2 \in S_0$ and $\ell_3 \subseteq P$, the line $\ell'_3$ containing the image of $\ell_3$ has to cross both $\ell'_1$ and $\ell'_2$ in $P'$. Consequently, $\ell'_1 \not\parallel \ell'_3 \not\parallel \ell'_3$. As the photon-lines of $P'$ belong to at most two equivalence classes of $\parallel$, we have $\ell'_1 \parallel \ell'_2$.

<u>Case 2:</u> Assume $P$ contains slow lines, i.e. lines with an angle $< c_m$. Let us choose $a \in \ell_1 \cap S_0$ and $c \in \ell_2 \cap S_0$ arbitrarily. From this point the reader is asked to follow Figure 225. Let $\ell_a, \ell_b, \ell_c, \ell_d \in G$ be chosen so that $a \in \ell_a \cap \ell_b$, $c \in \ell_c \cap \ell_d$,

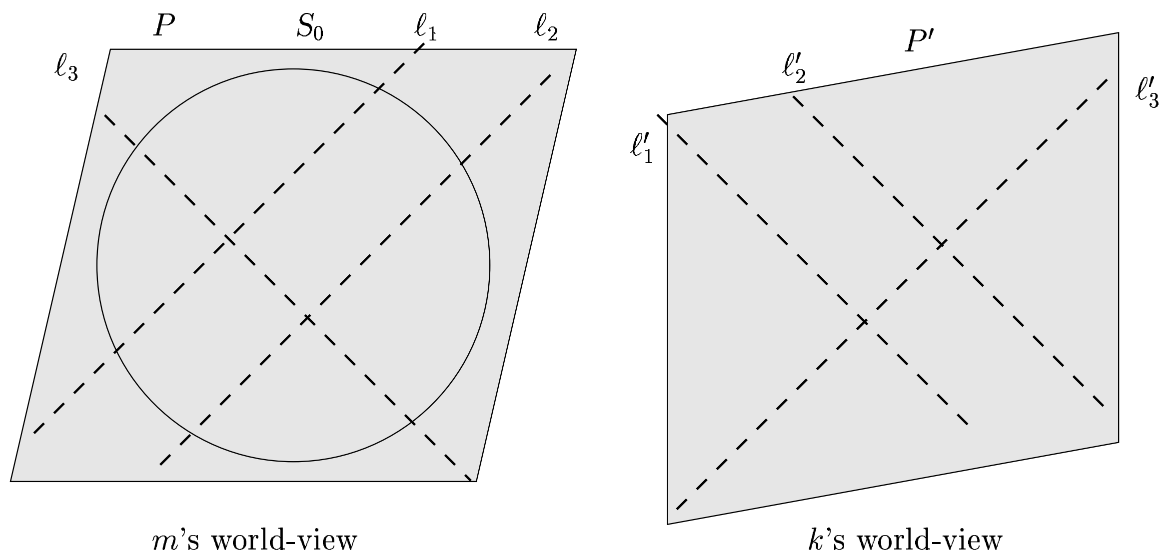



Figure 224: The idea of the proof of Prop. 4.9.6, Case 1. Dashed lines represent potential photon traces, i.e. lines with an angle $c_m$ to $\bar{t}$ in $m$'s world-view and $c_k$ in $k$'s world-view.

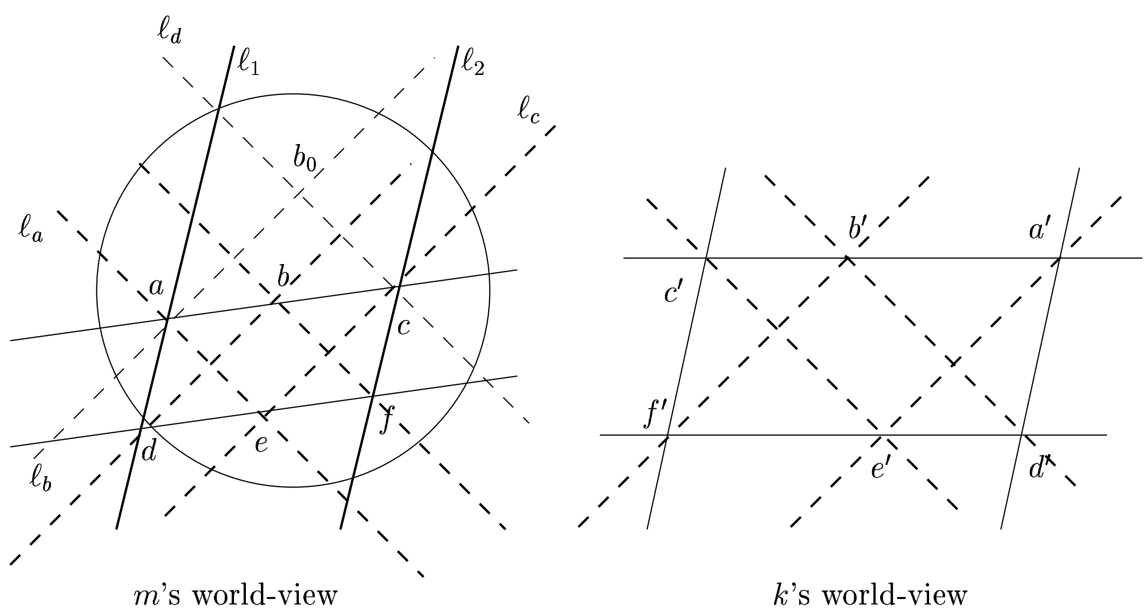



Figure 225: Illustration for Prop. 4.9.6, proof of Case 2. Again, dashed lines represent potential photon traces.

$ang^2(\ell_a) = ang^2(\ell_b) = ang^2(\ell_c) = ang^2(\ell_d) = c_m$, and these lines are pairwise distinct and fall within $P$. By Lemma 4.9.7 such lines exist. Let $e$ be the intersection of $\ell_a$ and $\ell_c$, and $b_0$ be the intersection of $\ell_b$ and $\ell_d$. Next, let us translate the $ab_0c$ triangle in a direction parallel to $\ell_1$ (and $\ell_2$), so that $b_0$ is mapped onto some $b \in \overline{ac}$. Let $d, f \in P$ be the points where $a$ and $c$ are mapped to, respectively. It is easy to check that $b, d, f \in S$, i.e. they fall within the sphere where $\mathsf{f}^-_{mk}$ is defined and preserves $\mathsf{Betw}$. We recommend the reader to have a look at Figure 225 again. We note that $\overline{ad} \parallel \overline{cf}$ (because $\ell_1 \parallel \ell_2$), $\overline{ae} \parallel \overline{bf}$ (because they are parallel photon-lines), and $\overline{db} \parallel \overline{ec}$.

Now let us switch to $k$'s world-view. We have seen at Case 1 that parallel photon-lines are mapped into parallel photon-lines within $S$, therefore, denoting the $\mathsf{f}^-_{mk}$-image by a prime,

$$\overline{a'e'} \parallel \overline{b'f'} \quad \text{and} \quad \overline{d'b'} \parallel \overline{e'c'}$$

Since $\mathsf{f}^-_{mk} \restriction S$, we have $\mathsf{Betw}(a', b', c')$ and $\mathsf{Betw}(d', e', f')$. Then we can apply the Pascal-Pappus theorem,[539] and obtain $\overline{a'd'} \parallel \overline{c'f'}$. That is, $\ell'_1 \parallel \ell'_2$.

<u>General case:</u> For the general case (when $P$ contains no slow lines)[540], we can proceed as follows. One takes $\ell_0 \in G$ such that $\ell_0 \parallel \ell_1 \parallel \ell_2$, $\ell_0 \cap S_0 \neq \emptyset$, and $\mathsf{Plane}(\ell_0, \ell_1)$ and $\mathsf{Plane}(\ell_0, \ell_2)$ contain two photon-lines through each of their points. We shall prove below that such an $\ell_0$ always exists. Then one can apply the proof of Case 2 to show that $\mathsf{f}^-_{mk}[\ell_0] \parallel \ell'_1$ and $\mathsf{f}^-_{mk}[\ell_0] \parallel \ell'_1$. Then $\ell'_1 \parallel \ell'_2$ by the transitivity of $\parallel$.

Why does such an $\ell_0$ exist? First, it is easy to check that the scenarios that do not belong to Case 1 or Case 2 above are such that $ang^2(\ell_1) = ang^2(\ell_2) > c_m$. Then let us take a plane $P_0$ such that $P_0$ intersects both $\ell_1$ and $\ell_2$ within $S_0$, $\ell_1 \perp P_0 \perp \ell_2$. Such a $P_0$ clearly exists, e.g. the plane containing $p$ and perpendicular to $\ell_1$ can be checked to be such. Let us have a look at $P_0$, see Figure 226.

As $ang^2(\ell_1) = ang^2(\ell_2) > c_m$, $P_0$ must also contain slow lines, and, consequently, for each $q \in P_0$ there are two distinct photon-lines in $P_0$ going through $q$. We shall skip the easy proof of this step. Let $p_1 \stackrel{\text{def}}{=} \ell_1 \cap P_0$ and $p_2 \stackrel{\text{def}}{=} \ell_2 \cap P_0$. Let $\ell_{ph_1}, \ell_{ph_2} \in G$ be such that $p_1 \in \ell_{ph_1}$, $p_2 \in \ell_{ph_2}$, $\ell_{ph_1} \cup \ell_{ph_2} \subseteq P_0$, and $\ell_{ph_1} \nparallel \ell_{ph_2}$. There are two pairs of such $\ell_{ph_1}$ and $\ell_{ph_2}$, and it can be checked by Lemma 4.9.8 that for one of the two choices $\ell_{ph_1}$ and $\ell_{ph_2}$ intersect within $S_0$. Let us assume that $\ell_{ph_1}$ and $\ell_{ph_2}$ are chosen so.

Now, $\ell_{ph_1}$ and $\ell_{ph_2}$ divide $P_0 \cap S_0$ into four domains; and one of them is such that its points are connected with both $p_1$ and $p_2$ via slow lines. Let us choose $p_0 \in P_0 \cap S_0$ from this quarter. We only have to take $\ell_0$ to be the line containing $p_0$

[539] The Pascal-Pappus theorem can be applied in the $\langle {}^nF, \mathsf{Betw}\rangle$ geometry by Fact 6.6.25.

[540] Cf. Class 2 and 3 of Lemma 4.9.7.

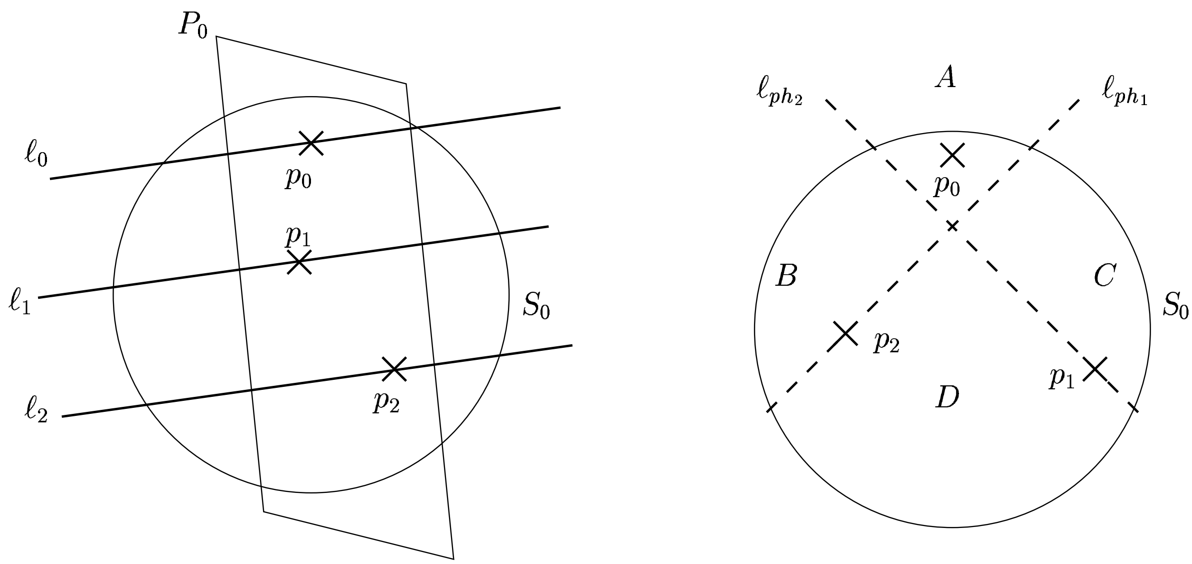



Figure 226: Illustration to the proof of the general case of Prop. 4.9.6. The second picture focusses on the plane $P_0 \perp \ell_1$. $\ell_{ph_1}$ and $\ell_{ph_2}$ divide $P_0 \cap S_0$ into four domains. If $q \in A$, then both $\overline{p_1 q}$ and $\overline{p_2 q}$ are slow lines.

and parallel with $\ell_1$. ∎

**Proof of Lemma 4.9.7:** Let $P$ be an arbitrary plane. It is enough to show the following items:

(a) If there is a slow line or $c$-line in $P$, then there are fast lines in $P$, too.

(b) If there is a slow line in $P$, then through each $p \in P$ there are two distinct $c$-lines, and the $c$-lines of $P$ belong to two equivalence classes of $\parallel$.

(c) If there is no slow line in $P$, then

$$(\forall \ell_1, \ell_2 \in G)\,[(\ell_1 \cup \ell_2 \subseteq P \ \wedge\ \ell_1, \ell_2 \text{ are } c\text{-lines}) \ \Rightarrow\ \ell_1 \parallel \ell_2]\,.$$

To show (a), we first notice that if there is a slow line $\ell_s \subseteq P$, then through each $p \in P$ there is an $\ell_p \in G$ such that $\ell_p \parallel \ell_s$ and $p \in \ell_p \subseteq P$. Let $\ell_1, \ell_2 \in G$ be distinct, parallel, slow, and $\ell_1 \cup \ell_2 \subseteq P$. Let $p_1 \in \ell_1 \subseteq \mathit{Space}$, and $p_2 \in \ell_2 \subseteq \mathit{Space}$. Then $\overline{p_1 p_2} \subseteq P$ and $\mathit{ang}^2(\overline{p_1 p_2}) = \infty$, i.e. $\overline{p_1 p_2}$ is fast.

For the proof of (c) the reader is asked to consult Figure 232. Suppose there is no slow line in $P$, and $\ell_1, \ell_2$ are distinct and non-parallel $c$-lines in $P$. Let $p \in \ell_1 \cap \ell_2$. Let $H$ be a horizontal hyperplane, i.e. $H \parallel \mathit{Space}$, such that $p \notin H$. Let $q \in \ell_1 \cap H$ and $r \in \ell_2 \cap H$. Let $s \in H$ be the projection of $p$ to $H$, i.e. $\mathit{space}(s) = \mathit{space}(p)$, and $s_0 = q_0 = r_0$.

Now, by the definition of $ang^2()$, we have

$$\|q - s\| = c\|p - s\| = \|r - s\|. \tag{297}$$

Let $w \in {}^nF$ be such that $\mathsf{Betw}(q, w, r)$. Such a $w$ exists by $\ell_1 \nparallel \ell_2$ and $p \in \ell_1 \cap \ell_2$.

It is easy to show $\|w - s\| < \|q - s\|$. We only have to consider $m \in [qr]$ for which $\overline{sm} \perp \overline{qr}$. Such an $m$ exists because $qrs$ is an isosceles triangle by (297). If $w = m$, then $\|w - s\| < \|q - s\|$ follows by Pythagoras' theorem. If $w \neq m$, then either $\mathsf{Betw}(q, w, m)$ or $\mathsf{Betw}(r, w, m)$. In the former case one has to apply Pythagoras' theorem to $qsm$ and $qwm$, yielding

$$\|w - s\| = \|s - m\| + \|w - m\| < \|s - m\| + \|q - m\| = \|q - s\|,$$

and similarly for the case $\mathsf{Betw}(r, w, m)$.

But if $\|w - s\| < \|q - s\|$, then by (297),

$$\|w - s\| < c\|p - s\|,$$

and hence $ang^2(\overline{wp}) < c$, contrary to the supposition that there is no slow line in $P$.

To show (b), consider a slow line $\ell_s \subseteq P$ and $p \in \ell_s$. Let $H$ be the hyperplane such that $H \parallel \mathit{Space}$, and $t \stackrel{\text{def}}{=} p + 1_t \in H$. See Figure 231. Let $\ell \in G$ be such that $\ell \subseteq P \cap H$. Such an $\ell$ exists because $P$ cannot be parallel with $H$ (as $P$ contains a slow line while $H$ does not). Let $s \in \ell_s \cap H$. As $ang^2(\ell_s) < c$, $\|s - t\| < c$. Then by $\mathbf{Ax}(\sqrt{\ })$ there are $p_1, p_2 \in \ell$ such that $\|p_1 - p\| = \|p_2 - p\| = c$ and $p_1 \neq p_2$. Then $\overline{p_1 p}, \overline{p_2 p} \subseteq P$ are distinct $c$-lines going through $p$.

Suppose there is a $c$-line $\ell' \subseteq P$ such that $p \in \ell'$. Then $\ell'$ has to intersect $H$ in some $p'$. Clearly, $p'$ must be on the $(n-1)$-dimensional sphere (or, if $n = 3$, a circle) $S$ with a radius $c$ around $t$ within $H$. But as $|S \cap \ell| \leq 2$, either $p' = p_1$ (and hence $\ell' = \overline{p_1 p}$) or $p' = p_2$ (and $\ell' = \overline{p_2 p}$).

We still have to show that the $c$-lines of $P$ belong to two equivalence classes of $\parallel$. We have already seen that they belong to at least two classes.

Let us fix $p \in P$. Let $\ell_1, \ell_2 \in G$ be such that $p \in \ell_1 \cap \ell_2$, $\ell_1 \cup \ell_2 \subseteq P$, and $\ell_1, \ell_2$ are distinct $c$-lines. See Figure **??**. Let $q \in P$ and $\ell_3, \ell_4$ be distinct $c$-lines through $q$ such that $\ell_3 \cup \ell_4 \subseteq P$. We have to show

$$(\ell_1 \parallel \ell_3 \ \wedge\ \ell_2 \parallel \ell_4) \ \vee\ (\ell_1 \parallel \ell_4 \ \wedge\ \ell_2 \parallel \ell_3).$$

Assume e.g. $\ell_2 \not\parallel \ell_3$. Let $r \in \ell_2 \cap \ell_3$. Let $\ell_p$ and $\ell_r$ be the bisector of the angle between $\ell_1$ and $\ell_2$, and $\ell_3$ and $\ell_2$, respectively. Let $\ell_p, \ell_r$ be such lines that $\ell_p \parallel \bar{t} \parallel \ell_r$, $p \in \ell_p$ and $r \in \ell_r$. As $\ell_1, \ell_2, \ell_3$ are $c$-lines, $angle(\ell_1, \ell_p) = angle(\ell_2, \ell_p) = angle(\ell_3, \ell_r) = angle(\ell_2, \ell_r)$. Then, as $\ell_1 \not\parallel \ell_2$ and $\ell_2 \neq \ell_3$, we have $\ell_1 \parallel \ell_3$. By an analogous argument, $\ell_2 \parallel \ell_4$. And if $\ell_2 \parallel \ell_3$, then a similar argument yields $\ell_1 \parallel \ell_4$. ∎

**Proof of Lemma 4.9.8:** Filled in later. ∎

**PROPOSITION 4.9.9** *The world-view transformations can be characterized as compositions of a translation, a relatively "harmless" map (to be specified below), and an affine transformation.*[541] *Formally,*
**Loc(Bax)+Ax(Bw)**$^{\mathrm{par}}$**+Ax(continuity)+Ax**$(\sqrt{\ })$ $\models (\forall p \in Dom(\mathsf{f}^-_{mk}))(\exists \varepsilon \in {}^+F)$

$$\left[S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk}) \ \land \ \mathsf{f}^-_{mk} \restriction S(p,\varepsilon) = \tau_{-p} \circ \widetilde{\varphi} \circ A \restriction S(p,\varepsilon)\right],$$

*for some $A \in Aftr$ and $\varphi : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ with the following properties:*

*(i)* $\varphi$ *is injective,*

*(ii)* $\varphi$ *preserves addition,*

*(iii)* $\varphi$ *preserves order,*

*(iv)* $\varphi$ *has the following property:*

$$(\forall p, q, r \in S(p,\varepsilon)) \left(pq = r\frac{\varepsilon}{2} \ \Rightarrow \ \varphi(p)\varphi(q) = \varphi(r)\frac{\varepsilon}{2}\right).$$

**Proof:** Assume **Loc(Bax) + Ax(Bw)**$^{\mathrm{par}}$ **+ Ax(continuity) + Ax**$(\sqrt{\ })$. Let $m, k \in Obs$ be such that $m \overset{\odot}{\to} k$. Let $p \in Dom(\mathsf{f}^-_{mk})$. Let $S = S(p,\varepsilon)$ be such that $S \subseteq Dom(\mathsf{f}^-_{mk})$, and $\mathsf{f}^-_{mk} \restriction S$ preserves both $\mathsf{Betw}$ and $\parallel$ for lines $\{\ell : ang^2(\ell) \notin (\frac{c_m}{2}, \frac{3c_m}{2})\}$. Such and $S$ exists by $\mathbf{Ax6^{par}_{01}}$, **Ax(Bw)**$^{\mathrm{par}}$ and Prop. 4.9.6.

**Claim 4.9.10** $\mathsf{f}^-_{mk}$ takes the $\frac{\varepsilon}{2}$-long unit vectors around $p$ to linearly independent vectors. Formally, the vectors of

$$E \overset{\mathrm{def}}{=} \{\mathsf{f}^-_{mk}(p + \frac{\varepsilon}{2}e_0) - \mathsf{f}^-_{mk}(p), \ldots, \{\mathsf{f}^-_{mk}(p + \frac{\varepsilon}{2}e_{n-1}) - \mathsf{f}^-_{mk}(p)\}$$

are linearly independent. [542]

[541] See Thm. 3.1.4 for the stronger form of this statement that applies in the full-domain theory **Basax + Ax**$(\sqrt{\ })$. By Thm. 3.4.40 we have the same in **Bax + Ax**$(\sqrt{\ })$, and **Bax + Ax**$(\sqrt{\ })$ $\models$ **Ax(Bw)**. Therefore Prop. 4.9.9 provides a characterization of $\mathsf{f}^-_{mk}$'s in a strinctly corresponding case.

[542] Actually, for Claim 4.9.10 we do not need that $\mathsf{f}^-_{mk}$ preserves $\parallel$.

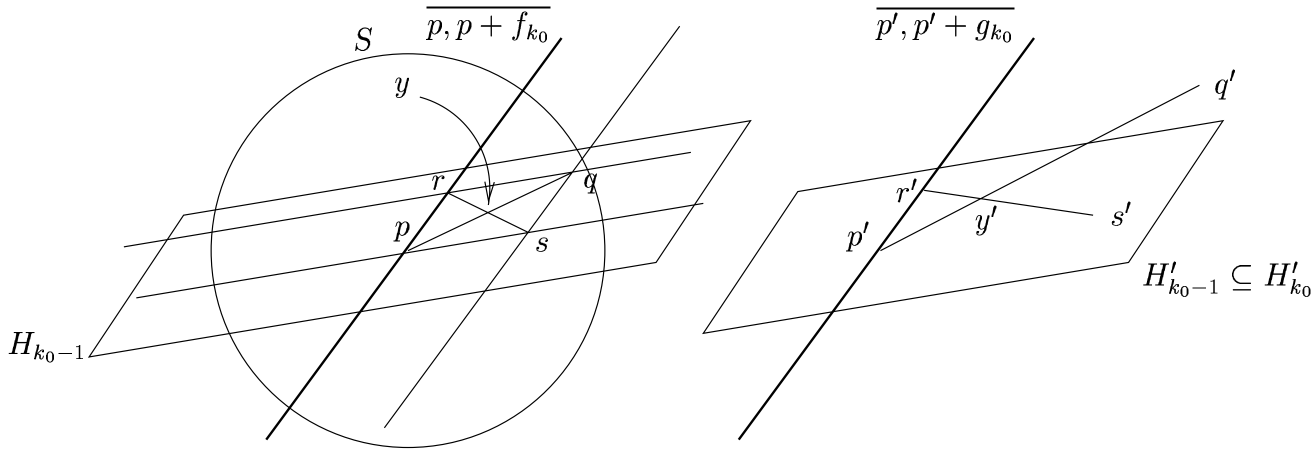



Figure 227: Idea of the proof of Claim 4.9.10. Notice that we did not use that $\mathsf{f}^-_{mk}$ preserves $\parallel$, only that it preserves $\mathsf{Betw}$.

**Proof of Claim 4.9.10:** Suppose the members of $E$ are not linearly independent. Then $\{\mathsf{f}^-_{mk}(p), \mathsf{f}^-_{mk}(p) + q : q \in E\}$ generate at at most $(n-1)$-dimensional hyperplane $H$. We are going to show that $\mathsf{f}^-_{mk}$ has to map $S$ into $H$ by the supposition. This will contradict **Ax(continuity)**, by which $\mathsf{f}^-_{mk}[S]$ must be an open subset of ${}^nF$.

Let $f_i \stackrel{\text{def}}{=} \frac{\varepsilon}{2}e_i$, $g_i \stackrel{\text{def}}{=} \mathsf{f}^-_{mk}(p + \frac{\varepsilon}{2}e_0) - \mathsf{f}^-_{mk}(p)$ for $i \in n$.[543] Let $q \in S$ be arbitrary. For convenience, let $p' \stackrel{\text{def}}{=} \mathsf{f}^-_{mk}(p)$. Then

$$q = p + x_0 f_0 + \ldots + x_{n-1} f_{n-1}$$

for some $x_0, \ldots, x_{n-1} \in F$. We shall proceed by induction. Let $H_i$ by the hyperplane generated by $\{p, p+f_0, \ldots, p+f_i\}$ for $i \in n$. Then e.g. $H_0 = \overline{p, p + f_0}$ and $H_{n-1} = {}^nF$, as $\{\frac{\varepsilon}{2}e_i : i \in n\}$ are linearly independent. Similarly, let $H'_i$ be the hyperplane generated by $\{p', p' + g_0, \ldots, p' + g_i\}$.

Assume $q \in H_0 \cap S$. Then $\mathsf{f}^-_{mk}(q) \in H$ because both $p$ and $p + f_0$ are mapped into $H$ and $\mathsf{f}^-_{mk}$ preverses collinearity via $\mathsf{Betw}$.

Assume $\mathsf{f}^-_{mk}$ takes $H_k$ into $H$ for each $k \in k_0$. Let $q \in H_{k_0} \cap S$. Then

$$q = p + (r - p) + (s - p),$$

for some $s \in H_{k_0-1} \cap S$ and $r \in \overline{p, p + f_{k_0}}$. See Figure 227. It is well-known from geometry that $prqs$ is a parallelogram, and that the diagonals of a parallelogram bisect

[543] That is, $E = \{g_i : i \in n\}$.

each other. Thus, letting $y \stackrel{\text{def}}{=} \frac{p+q}{2} = \frac{r+s}{2}$, we have $\mathsf{Betw}(p, y, q)$ and $\mathsf{Betw}(r, y, s)$. But then $\mathsf{Betw}(\mathsf{f}^-_{mk}(r), \mathsf{f}^-_{mk}(y), \mathsf{f}^-_{mk}(s))$ and $\mathsf{Betw}(\mathsf{f}^-_{mk}(p), \mathsf{f}^-_{mk}(y), \mathsf{f}^-_{mk}(q))$. Then, by the former, $y$ is mapped into $H_{k_0}$ and, by the latter, $q$ is taken into $H_{k_0}$ as well.

Thus $S = S \cap H_{n-1}$ is taken into $H$ and $\mathsf{f}^-_{mk}[S]$ cannot be an open subset of ${}^nF$, contrary to **Ax(continuity)**. (Claim 4.9.10) ∎

Let us continue with the proof of Prop. 4.9.9. Let $A \in \mathit{Aftr}$ be such that $A$ takes $\bar{0}, f_0, \ldots, f_{n-1}$ to $p', p'+g_0, \ldots, p'+g_{n-1}$, respectively. Such an $A$ exists and is unique, and is a bijection of ${}^nF$ by Claim 4.9.10. We have to show that $\tau_p \circ \mathsf{f}^-_{mk} \circ A^{-1} = \widetilde{\varphi}$ with $\varphi$ having properties (i) to (iv) above.

Let $g \stackrel{\text{def}}{=} \tau_p \circ \mathsf{f}^-_{mk} \circ A^{-1}$. $g$ is defined of $S = S(p, \varepsilon)$, is injective and preserves both $\mathsf{Betw}$ and parallelism, because each of $\tau_p$, $\mathsf{f}^-_{mk}$, $A^{-1}$ were such. Further, $g$ leaves $\bar{0}, f_0, \ldots, f_{n-1}$ and hence the axes $\bar{x}_0 \cap S, \ldots, \bar{x}_{n-1} \cap S$ fixed. We are going to code $+, \cdot, \leq$ by parallel lines and $\mathsf{Betw}$ to show that $g$ is a map generated by some $\varphi$ with properties (i) to (iv).[544]

First, let us see how $g$ behaves on $\bar{t}$. Within this proof let us use the temporary notation

$$x^{(j)} \stackrel{\text{def}}{=} \langle \underbrace{0, \ldots, 0}_{(j-1)\text{-times}}, x, \ldots, 0 \rangle.$$

Let $z = x + y$ for some $x, y, z \in (-\varepsilon, \varepsilon)$. Then there is an $\ell \in G$ such that $\ell \parallel \bar{t}$, $\ell \neq \bar{t}$, and there are $a, b \in \ell \cap S$ such that

$$\overline{\bar{0}a} \parallel \overline{x^{(0)}b}, \text{ and}$$
$$\overline{y^{(0)}a} \parallel \overline{bz^{(0)}}$$

See Figure 228. We do not restrict generality with assuming $ang^2(\overline{\bar{0}a})$, $ang^2(\overline{x^{(0)}b})$, $ang^2(\overline{y^{(0)}a})$, $ang^2(\overline{bz^{(0)}}) < \frac{c_m}{2}$, as $\ell$ can be chosen "close enough" to $\bar{t}$.

As $g$ leaves $\bar{0}$ and $f_0$ fixed, it takes $\bar{t} \cap S$ into $\bar{t}$. As $g$ is injective and preserves parallelism,

$$g[\ell \cap S] \not\subseteq \bar{t} \text{ and } g[\ell] \parallel \bar{t},$$
$$\overline{g(\bar{0})g(a)} \parallel \overline{g(x^{(0)})g(b)},$$
$$\overline{g(y^{(0)})g(a)} \parallel \overline{g(b)g(z^{(0)})}.$$

[544] The procedure followed below to code $+, \cdot, \leq$ by parallel lines and $\mathsf{Betw}$ is very close to the method we shall use in the geometry chapter (§6) below to define a field $\mathfrak{F}_{oe}$ from the structure $\langle Mn, Bw \rangle$ where, roughly speaking, $Mn$ will be the points of the observer-independent geometry, corresponsing to ${}^nF$ in our case, and $Bw$ is the betweenness in $Mn$, corresponding to $\mathsf{Betw}$ in ${}^nF$.

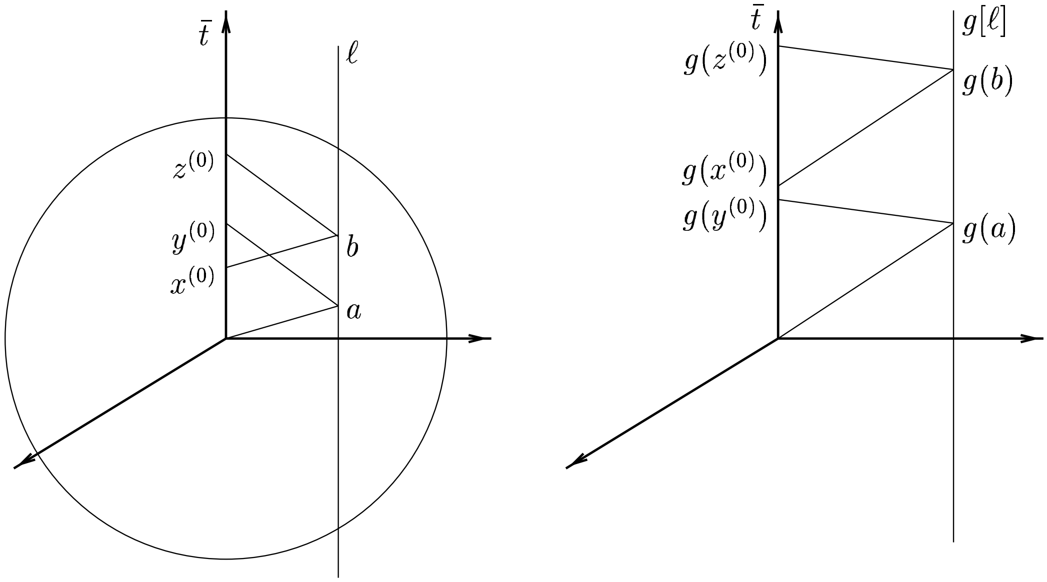



Figure 228: Illustration for the proof of Prop. 4.9.9. $g$ preserves addition on $\bar{t}$.

Consequently, $g(x^{(0)}), g(y^{(0)}), g(z^{(0)}) \in \bar{t}$ and $g(z) = g(x) + g(y)$. In other words, $g$ preserves addition on $\bar{t} \cap S$.

It is easy to check that $g$ preserves order ($\leq$) on $\bar{t}$, too. It is straightforward that

$$x \geq y \iff \left(\mathsf{Betw}(x^{(0)} - y^{(0)}, f_0, \bar{0}) \ \lor \ x^{(0)} - y^{(0)} = f_0 \ \lor \ \mathsf{Betw}(f_0, x^{(0)} - y^{(0)}, \bar{0})\right).$$

Since $g$ preserves $\mathsf{Betw}$, we have $g(x^{(0)})_t \geq g(y^{(0)})_t$, and hence $x \geq y$.

Let us turn to the problem of multiplication. Unfortunately, we cannot fix multiplication by parallel lines, because $1_t \notin S$ is possible (and it is also possible that no point with a rational $t$-coordinate falls within $S$). But we can define the multiplication "as if" the unit were e.g. $\frac{\varepsilon}{2}$. This "as if" is responsible to the strange substitute for multiplication in property (iv).

Let $\frac{\varepsilon}{2}z = xy$ for some $x, y, z \in (-\varepsilon, \varepsilon)$. Then there is an $\ell \in G$ such that $\bar{0} \in \ell$, $\ell \neq \bar{t}$, and there are $a, b \in \ell \cap S$ with the following properties:

(298) $$a \neq \bar{0} \neq b,$$

(299) $$\overline{az^{(0)}} \parallel \overline{by^{(0)}}, \text{ and}$$

(300) $$\overline{ax^{(0)}} \parallel \overline{bf_0}.$$

See Figure 229. We do not restrict

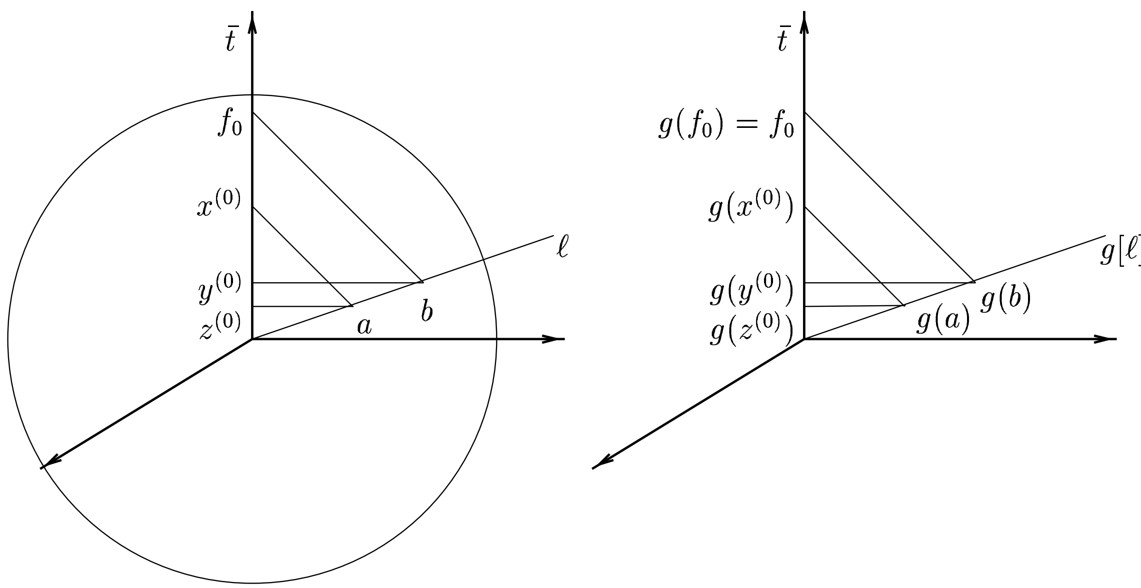



Figure 229: Illustration to the proof of Prop. 4.9.9, property (iv). The relation $R(x, y, z) \stackrel{\text{def}}{\Longleftrightarrow} xy = z\frac{\varepsilon}{2}$ can be coded by parallel lines, which are preserved by $g$.

generality with assuming $ang^2(\overline{az^{(0)}}), ang^2(\overline{by^{(0)}}), ang^2(\overline{ax^{(0)}}), ang^2(\overline{bf_0}) < \frac{c_m}{2}$, as $\ell$ can be chosen "close enough" to $\bar{t}$.

As $g$ preserves $\bar{t}, \bar{0}, f_0$, Betw and $\|$ on $S$ and is injective, it maps the scenario of Figure 229 to a scenario where the images of $x^{(0)}$, $y^{(0)}$, $z^{(0)}$, $a$, $b$ have properties corresponding to (298)-(300), that is:

$$\begin{aligned} g[\ell \cap S] &\not\subseteq \bar{t} \quad \text{and} \\ g(a) &\neq \bar{0} = g(\bar{0}) \neq g(b), \\ \overline{g(a)g(z^{(0)})} &\parallel \overline{g(b)g(y^{(0)})}, \text{ and} \\ \overline{g(a)g(x^{(0)})} &\parallel \overline{g(b)g(f_0)} = \overline{g(b)f_0}. \end{aligned}$$

One can check that this implies $\frac{\varepsilon}{2}g(z) = g(x)g(y)$.

Up to this point we have seen that $g$ acts of $\bar{t}$ just like it should, i.e. $g \restriction \bar{t} = \widetilde{\varphi} \restriction \bar{t}$ for some $\varphi : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ with properties (i) to (iv). By exactly the same argument $g$ acts this way on all of the axes $\bar{x}_0, \ldots, \bar{x}_{n-1}$. That is,

$$(\forall q \in S)\,((\exists i \in n) q \in x_i \;\Rightarrow\; g(q) = \langle \varphi_0(q_0), \ldots \varphi_{n-1}(q_{n-1}) \rangle)$$

for some $\varphi_0, \ldots, \varphi_{n-1}$ with properties (i) to (iv). We still have to show $\varphi_j = \varphi_k$ for any $j, k \in n$, and that $g$ acts the same way of the whole of $S$.

Let $x \in (-\varepsilon, \varepsilon)$. Let $j \neq k$, $j, k \in n$. Then $\overline{f_j f_k} \parallel \overline{x^{(j)} x^{(k)}}$. Since $g$ preserves parallelism and $f_j, f_k$, we have

$$\overline{f_j f_k} \parallel \overline{g(x^{(j)})g(x^{(k)})}.$$

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But $g(x^{(j)}) = \widetilde{\varphi}_j^{(j)}(x)$, and $g(x^{(k)}) = \widetilde{\varphi}_k^{(k)}(x)$. Hence $\varphi_j(x) = \varphi_k(x)$, i.e. $\varphi_j = \varphi_k$. ∎

**Remark 4.9.11** **(i)** It can be checked easily that the choice of $\frac{\varepsilon}{2}$ below is not essential. It could be substituted by any number from $(0, \varepsilon)$. If the interval $(0, \varepsilon)$ contains a rational number (i.e. if the sphere $S(p, \varepsilon)$ is not infinitely small), item (iv) can be turned to saying that $\varphi$ preserves multiplication.

**(ii)** It follows from the proof of Prop. 4.9.9 that for any $p \in Dom(\mathsf{f}^-_{mk})$, $\mathsf{f}^-_{mk}$ can be extended from the neighbourhood $S(p, \varepsilon)$ where it preserves parallelism of lines slower than $c_m$ to a betweenness-preserving bijective collineation on ${}^nF$ iff a function $\varphi$ with the properties (i) to (iv) above (see in Prop. 4.9.9) can be extended to an order-preserving field automorphism.

### 4.9.2 Some partial models

A simple but not exhaustive "algorithm" for building models for partial world-view relativity theories is similar to the technique we used in e.g. §§2.4, 3.2, 3.5, 3.6 for building models of **Basax**. We shall first deal with **Loc**(**Basax**). This method serves as a simple tool for testing questionable hypotheses (about partial theories); we shall apply it way below e.g. to prove that **Loc**($\mathbf{Bax}^{-\oplus}$) has models where $\mathsf{f}^-_{mk}$'s do not preserve parallelism. By and large, one can proceed according to the following pattern.

1. Let us fix an arbitrary "original" observer, say $m$. Define the domain of $m$, i.e. fix $Dom(w^-_m)$. Make sure that $Dom(w^-_m)$ is an open subset of ${}^nF$. Observer $m$ can be identified with $Dom(w^-_m) \cap \bar{t}$, in order to obey $\mathbf{Ax4}^{\mathrm{par}}$.

2. Let us inhabit the world-view of $m$ by observers and photons so that $\mathbf{Ax1}$, $\mathbf{Ax2}$, $\mathbf{Ax3}^{\mathrm{par}}_{\mathbf{0}}$, $\mathbf{Ax5}^{\mathrm{par}}$, $\mathbf{AxE}$ are obeyed in $m$'s world-view.

3. In Step 2 we obtained $Obs$ an $Ph$ so that $(\forall k \in Obs) m \overset{\odot}{\to} k$ and $(\forall ph \in Ph) m \overset{\odot}{\to} ph$. Define $\mathsf{f}^-_{mk}$ for each $k \neq m$. Do this in a conservative way; by this we mean that one should not take the risk that the axioms become invalid in other observers' world-views. For example, if $Dom(w^-_m)$ contains $p$ and $q$ such that $p - q$ has rational coordinates, $\mathsf{f}^-_{mk}$ should be a composition of a map generated by a field automorphism and a rhombus transformation. Let $Dom(\mathsf{f}^-_{mk}) \overset{\text{def}}{=} Dom(w^-_m)$. This will make sure that **Ax6** holds.

4. Let $w^-_k \overset{\text{def}}{=} \mathsf{f}^-_{mk} \circ w^-_m$. One has to check the axioms in any observer's world-view.

To obtain models of **Loc**(**Newbasax**), **Loc**(**Flxbasax**), **Loc**(**Bax**) etc. one can use the common techniques we used so far: use windows, change the speed of light, either the same way for every observer or differently for each etc.

First, we shall show that **Loc**(**Basax**) has a model in which the world-views are bounded. Let $\mathfrak{F}$ be an Euclidean ordered field, and $G \overset{\text{def}}{=} \mathsf{Eucl}$ (**Ax1** is fulfilled). Further,

$$\begin{aligned} Obs &\overset{\text{def}}{=} \{\ell \in \mathsf{SlowEucl} : \ell \cap S(\bar{0}, 2) \neq \emptyset\} \times Triv_t, \\ Ph &\overset{\text{def}}{=} \{\ell \in \mathsf{PhtEucl} : \ell \cap S(\bar{0}, 2) \neq \emptyset\}, \\ Ib &\overset{\text{def}}{=} Obs \cup Ph. \end{aligned}$$

For any $k \in Obs$ let $k = \langle \ell_k, g_k \rangle$. Let $m = \langle \bar{t}, \mathrm{Id} \rangle$ be fixed.

$$\begin{aligned} Dom(w_m^-) &\stackrel{\text{def}}{=} S(\bar{0}, 2), \\ w_m^-(p) &\stackrel{\text{def}}{=} \{ph \in Ph : p \in ph\} \cup \{k \in Obs : p \in \ell_k\}. \end{aligned}$$

By this choice of $w_m^-$ **Ax2**, $\mathbf{Ax3_0^{par}}$, $\mathbf{Ax4^{par}}$, $\mathbf{Ax5^{par}}$, **AxE** are true in $m$'s world-view.

Next, let $\sigma : Obs \to {}^nF \times {}^nF \times Poi$ be such that if $\sigma(k) = \langle o_k, t_k, poi_k \rangle$ then $o_k, t_k \in \ell_k \cap S(\bar{0}, 2)$, $o_k \neq t_k$, and $poi_k$ takes $\bar{0}$ and $1_t$ to $o_k$ and $t_k$, respectively. Such a $\sigma$ exists, and it will be a parameter of the model. Now, having $k = \langle \ell_k, g_k \rangle$,

$$\mathsf{f}_{mk}^- \stackrel{\text{def}}{=} g_k \circ poi_k.$$

It is easy to check that $\mathsf{f}_{mk}^-$ is injective, thus **Ax6** is fulfilled. $\mathsf{f}_{mk}^-$ takes $\bar{t}$ to $tr_m(k)$, thus $tr_k(k) \subseteq \bar{t}$ and therefore $\mathbf{Ax4^{par}}$ is fulfilled. $\mathbf{Ax3_0^{par}}$ holds because $\mathsf{f}_{mk}^-$ is affine. **AxE** holds because both $g_k$ and $poi_k$ take photon lines to photon lines. $Dom(\mathsf{f}_{mk}^-) = Dom(w_m^-)$ is open, and $Rng(\mathsf{f}_{mk}^-)$ is open because $\mathsf{f}_{mk}^-$ is affine. Now, as $\mathsf{f}_{k_1k_2}^- = \mathsf{f}_{k_1m}^- \circ \mathsf{f}_{mk_2}^-$ is also affine, $\mathbf{Ax6_{01}^{par}}$ holds, too. $\mathbf{Ax5^{par}}$ can also be checked easily by $\mathsf{f}_{mk}^-$ being affine. In this model each observer's domain is bounded because it is an image of the sphere $S(\bar{0}, 2)$ by an affine transformation. $\lhd$

We are going to use the just outlined method to show that $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$[545] permits models in which light-cones are tilted. This phenomenon is not characteristic in special relativity, but it is common in general relativity. We do not think that by weakening $\mathbf{Bax}^{-\oplus}$ to $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$ we have entered the realm of general relativity; clearly, a couple of other changes will be necessary, too. But we find this result interesting because it demonstrates how weakening a theory from a full domain version to a partial one can induce a *qualitative change* in the range of models (in contrast to the less striking case when the observers of $\mathbf{Loc}(Th)$ can see fractions of a world which looks more or less like the world in models of $Th$). At the same time, we shall see that $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$ does not imply that $\mathsf{f}_{mk}^-$'s preserve parallelism, even locally.

Moreover, the following model construction demonstrates the fundamental difference between two ways of formalizing the "there are no FTL observer" statement. Roughly, one can give a "local" and a "global" form. The global form will fail in the following model, but the local form will still be valid. By contrast, in $\mathsf{Mod}(\mathbf{Bax}^-)$ the global and the local form will be equivalent. We shall return to this issue in §4.9.3 (p.4.9.3) below.

[545] One could easily add $\mathbf{Ax}(\sqrt{\ })$, **Ax(star)**, **Ax(clock-conn)**, **Ax(photon)**, **Ax(simult)**, **Ax(mut)**, **Ax(continuity)**, **Ax(pcoll)**, $\mathbf{Ax(syBw)}^{\mathrm{par}}$.

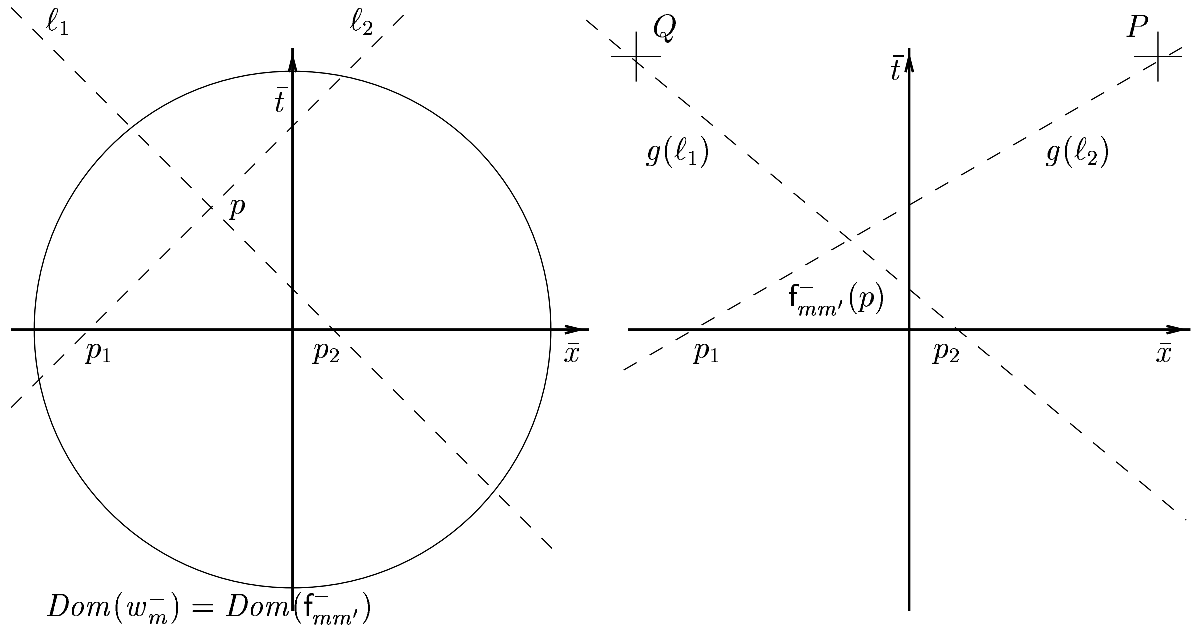


Figure 230: To the definition of $g$ and $\mathsf{f}_{mm'}^-$ in the definition of the non-$\|$-preserving model of $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$.

Let us start with a two-dimensional model ($n = 2$) $\mathfrak{M}$ of $\mathbf{Loc}(\mathbf{Basax})$ obtained above. Let us select an observer $m$ whose domain contains $\bar{0}$, e.g. the "original" observer $\langle \bar{t}, \mathrm{Id} \rangle$ we used for building the model. We are going to add a new observer $m'$ to $\mathfrak{M}$ so that $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$ is still obeyed in the new model $\mathfrak{N}$ and the world-view of $m'$ contains *tilted light-cones*, and at the same time, $\mathsf{f}_{mm'}^-$ does not preserve parallelism.

First we shall define an auxiliary function $g : \mathsf{PhtEucl} \to \mathsf{Eucl}$, which declares how $\mathsf{f}_{mm'}^-$ will act on photon lines. Next we shall define $\mathsf{f}_{mm'}^-$ in terms of $g$.

For the definition of $g$ the reader is asked to have a look at Figure 230. Let us work in $m$'s world-view. We know that $Dom(w_m^-)$ is bounded. Now let us fix two points, $P$ and $Q$, with the following properties. $P$ and $Q$ are one another's mirror images to $\bar{t}$, and $|P_x|, |P_t|$ are outside of the bound for $Dom(w_m^-)$ (i.e. if $Dom(w_m^-) \subseteq S(\bar{0}, K)$ for some $K \in F^+$, then $|P_x| > K$ and $|P_t| > K$). This choice is certainly possible. Let e.g. $P_x < 0$ and $Q_x > 0$.

Let $\ell \in \mathsf{PhtEucl}$. If $\ell$ moves forwards in direction $+1 \in \mathsf{directions}$, then let $g[\ell]$ be the line going through $Q$ and crossing $\bar{x}$ is $\ell \cap \bar{x}$. (In other words, the intersection point of $\ell$ and $\bar{x}$ is fixed, while $g[\ell]$ is expected to go through $Q$.) If $\ell$ moves backwards in direction $+1 \in \mathsf{directions}$, then $P$ should be used instead of $Q$. See Figure 230. Thus $g$ is defined on $\mathsf{PhtEucl}$ and it can be checked that if $\ell \cap Dom(w^-_m) \neq \emptyset$, then

$(\star)$ ($\ell$ moves forwards in direction $+1$) $\iff$ ($g[\ell]$ moves forwards in direction $+1$),

$(\star\star)$ $ang^2(g[\ell]) > 0$.

Now, let us define $\mathsf{f}^-_{mm'}$ (and thereby $w^-_{m'}$) the following way. If $p \in Dom(w^-_m)$, then there are $\ell_0, \ell_1 \in \mathsf{PhtEucl}$ such that $p \in \ell_0 \cap \ell_1$, and $\ell_0$ moves forwards in direction $+1$, while $\ell_1$ moves backwards in direction $+1$. Further, $\ell_0$ and $\ell_1$ are unique. See Figure 230. By $(\star)$ and $(\star\star)$, $g[\ell_0]$ and $g[\ell_1]$ are intersecting lines. Let $\mathsf{f}^-_{mm'}(p)$ be their intersection, i.e. $\mathsf{f}^-_{mm'}(p) \in g[\ell_0] \cap g[\ell_1]$. Thus $\mathsf{f}^-_{mm'}$ is a function defined on $Dom(w^-_m)$.

**Claim 4.9.12** $\mathsf{f}^-_{mm'}$ has the following properties.

(i) $\mathsf{f}^-_{mm'}$ is bijective,

(ii) $\mathsf{f}^-_{mm'}$ is continuous,

(iii) $\mathsf{f}^-_{mm'}$ preserves $\mathsf{Betw}$.[546]

Let us postpone the proof of Claim 4.9.12 for while, and proceed with the proof of $\mathfrak{N} \models \mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\text{par}}$, where $\mathfrak{N}$ is defined below.

Let $\mathsf{f}^-_{m'm} \stackrel{\text{def}}{=} (\mathsf{f}^-_{mm'})^{-1}$. If $\mathfrak{M} = \langle \mathfrak{F}; G; B, Ib, Obs, Ph, W\rangle$, then

$$\begin{aligned} \mathfrak{N} &\stackrel{\text{def}}{=} \langle \mathfrak{F}; G; B \cup \{m'\}, Ib \cup \{m'\}, Obs \cup \{m'\}, Ph, W'\rangle \quad \text{where} \\ W' &\stackrel{\text{def}}{=} W \cup \{\langle k,p,m'\rangle : \langle k,p,m\rangle \in W\} \cup \{\langle m',p,b\rangle : b \in (\mathsf{f}^-_{m'm} \circ w^-_m)(p)\}.^{547} \end{aligned}$$

Clearly, $\mathfrak{N} \models \{\mathbf{Ax(}\text{Frame}\mathbf{)}, \mathbf{Ax1}, \mathbf{Ax2}\}$.

$\mathfrak{N} \models \mathbf{Ax3}_\mathbf{0}^{\text{par}}$ follows by Claim 4.9.12(iii) and $tr_m(m') \neq \emptyset$.

$\mathbf{Ax4}^{\text{par}}$ holds in $\mathfrak{N}$ because by the above construction of $\mathsf{f}^-_{mm'}$, $\mathsf{f}^-_{mm'}[\bar{t}] \subseteq \bar{t}$, as $P$ and $Q$ are each other's mirror images to $\bar{t}$.

$\mathbf{AxE_{01}}$ holds because $|P_x| = |Q_x| > K$.

$\mathbf{Ax6_{00}}$ holds because $tr_k(m') = tr_k(m)$, and $Rng(w^-_m) = Rng(w^-_{m'})$.

$\mathbf{Ax6_{01}^{\text{par}}}$ holds because $Dom(\mathsf{f}^-_{km'}) = Dom(\mathsf{f}^-_{km}) \in Open$, $Dom(\mathsf{f}^-_{m'k}) = Rng(\mathsf{f}^-_{km'}) = Rng(\mathsf{f}^-_{km} \circ \mathsf{f}^-_{mm'})$, and $\mathsf{f}^-_{mm'}$ is continuous by Claim 4.9.12(ii).

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[546] Actually, $\mathsf{Betw}(x,y,z) \iff \mathsf{Betw}(\mathsf{f}^-_{mm'}(x), \mathsf{f}^-_{mm'}(y), \mathsf{f}^-_{mm'}(z))$ holds, too.

[547] The last item can also be written as $\{\langle m',p,b\rangle : \langle m, \mathsf{f}^-_{m'm}(p), b\rangle\}$. Intuitively, $m'$ can see the same events $m$ can see, but moved by the function $\mathsf{f}^-_{mm'}(p)$.

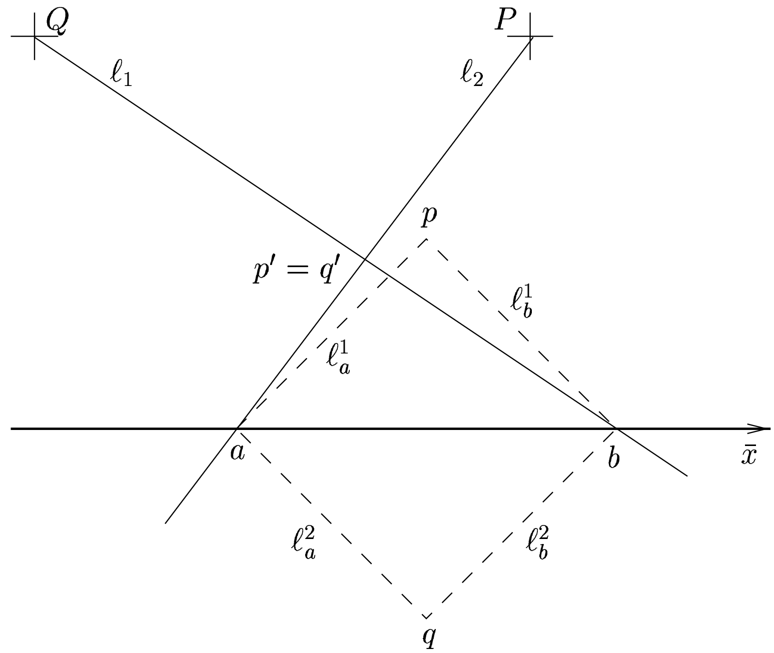


Figure 231: Illustration to the proof of Claim 4.9.12(i).

**AxP1** and $\mathbf{Ax5^{par}_{Ph}}$ hold by the construction of $g$ (and $\mathsf{f}^-_{mm'}$). In more detail, if $p \in Rng(\mathsf{f}^-_{mm'}) = Dom(w^-_{m'})$ and $ph \in Ph$ goes through $p$ in the $+1$ direction, then $tr_{m'}(ph) \subseteq \overline{pQ}$, and if $ph$ goes through $p$ in the $-1$ direction, then $tr_{m'}(ph) \subseteq \overline{pP}$. Cf. $(\star)$ above.

$\mathbf{Ax5^{par}_{Obs}}$ follows by Claim 4.9.12(ii)-(iii) and **AxP1**.

$\mathbf{Ax(Bw)^{par}}$ holds by $\mathfrak{M} \models \mathbf{Ax(Bw)^{par}}$ and Claim 4.9.12(iii). $\lhd$

**Proof of Claim 4.9.12(i):** Assume $\mathsf{f}^-_{mm'}(p) = \mathsf{f}^-_{mm'}(q) = r$ for some $p, q \in Dom(\mathsf{f}^-_{mm'})$. If $r_t = 0$, then $p = q = r$ as $\mathsf{f}^-_{mm'} \restriction \bar{x} = \mathrm{Id} \restriction \bar{x}$. Assume then e.g. $r_t > 0$. Throughout the proof of item (i) the reader is asked to consult Figure 231.

By the construction of $g$ (and $\mathsf{f}^-_{mm'}$) above, there are two distinct $\ell_1, \ell_2 \in \mathsf{Eucl}$ such that $r \in \ell_1 \cap \ell_2$, and for $a \in \ell_1 \cap \bar{x}$ and $b \in \ell_2 \cap \bar{x}$, $\{a, b\} \subseteq Rng(\mathsf{f}^-_{mm'})$. Clearly, $a$ and $b$ are distinct and unique, and $\mathsf{f}^-_{mm'}$ leaves them fixed.

Let us turn to the world-view of $m$. By the definition of $\mathsf{f}^-_{mm'}$, $p$ and $q$ must be the intersections of some $\ell_a, \ell_b \in \mathsf{PhtEucl}$ such that $a \in \ell_a$ and $b \in \ell_b$. As $c_m = 1$ and $n = 2$, there two such pair of lines, say $\langle \ell^1_a, \ell^1_b \rangle$ and $\langle \ell^2_a, \ell^2_b \rangle$. It is easy to check that only one of these pairs intersect in a point above $\bar{x}$. As $\mathsf{f}^-_{mm'}$ maps the points above $\bar{x}$ to points above $\bar{x}$ (this is easy to check), we have $p = q$. ∎

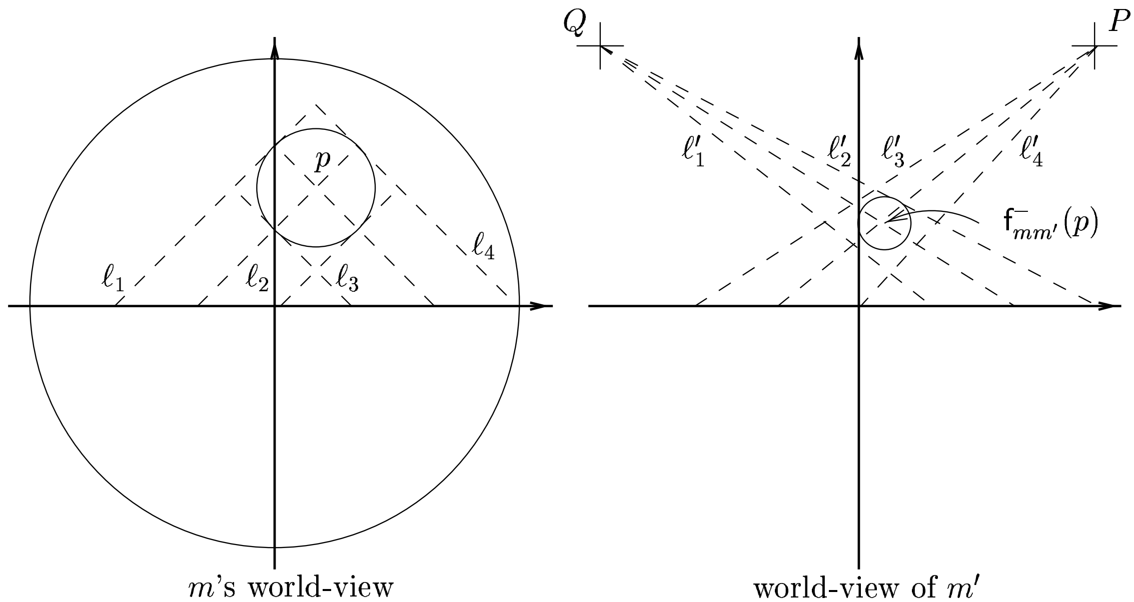


Figure 232: Illustration to the proof of Claim 4.9.12(ii).

**Proof of Claim 4.9.12(ii):** The reader is asked to consult Figure 232. It is enough to show that $\mathsf{f}^-_{mm'}$ takes each "small enough" open sphere of $Dom(\mathsf{f}^-_{mm'})$ to an open set.[548]

Let $S = S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mm'}) = Dom(w^-_m)$. Let $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathsf{PhtEucl}$ be the lines that "project" $S$ to $\bar{x}$. We are not going to formulate this, but follow Figure 232. Now, if $\varepsilon$ is so small that $\ell_1 \cap \ell_4 \subseteq Dom(\mathsf{f}^-_{mm'})$, then the poligon enclosed by $\ell_1, \ell_2, \ell_3, \ell_4$ is mapped to a poligon within $Rng(\mathsf{f}^-_{mm'})$ by the construction of $\mathsf{f}^-_{mm'}$ above. As $p$ will be an inner point of this poligon, there will be $S' = S(\mathsf{f}^-_{mm'}(p), \gamma)$ such that $S' \subseteq Rng(\mathsf{f}^-_{mm'}) = Dom(w^-_{m'})$.

It is easy to check that $\varepsilon$ can be chosen "small enough" in the above sense. For example, if $S_0 = S(p,\delta) \subseteq Dom(w^-_m)$, then $\varepsilon = \frac{\delta}{2}$ (actually, $\frac{\delta}{\sqrt{2}}$ if $\mathfrak{F}$ is Euclidean) is suitable. ∎

**Proof of Claim 4.9.12(iii):** Assume $\mathsf{Betw}(p,q,r)$ for some $p,q,r \in Dom(\mathsf{f}^-_{mm'})$. The case $\overline{pr} \parallel \bar{x}$ is trivial. So let us assume $\overline{pr} \nparallel \bar{x}$. The reader is asked to consult Figure 233 for the elements of the proof.

[548] Because each open set is the union of "small enough" spheres, and the union of open sets (the $\mathsf{f}^-_{mm'}$-images of these spheres) is open.

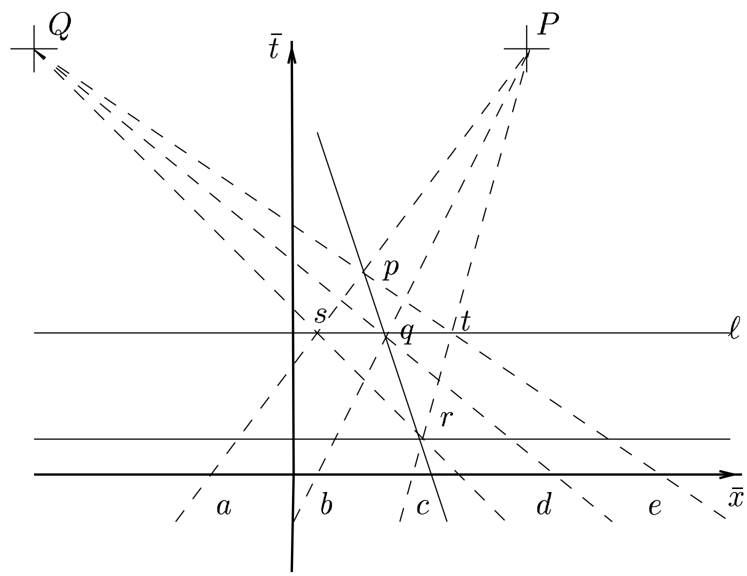


Figure 233: Illustration to the proof that in the definition of the non-$\|$-preserving model of $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$ is a collineation. (Part of Claim 4.9.12(iii). The picture shows the world-view of the new observer $m'$.

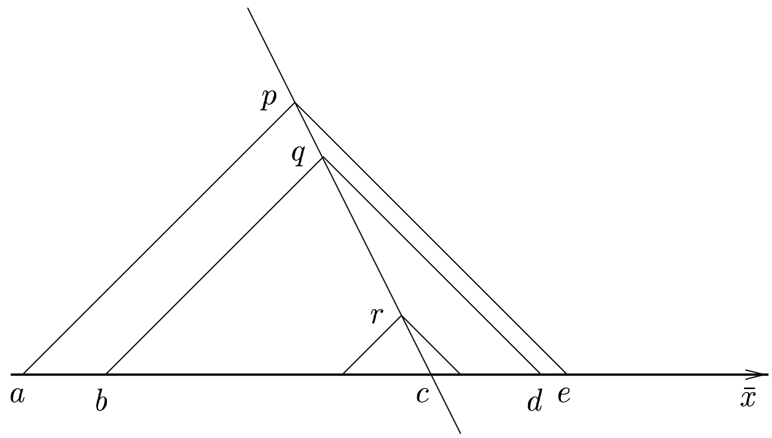


Figure 234: To the proof of Claim 4.9.12(iii). The picture shows the world-view of the old observer $m$.

Let $p' \stackrel{\text{def}}{=} \mathsf{f}^{-}_{mm'}(p)$, $q' \stackrel{\text{def}}{=} \mathsf{f}^{-}_{mm'}(q)$ and $r' \stackrel{\text{def}}{=} \mathsf{f}^{-}_{mm'}(r)$. Let $\ell \parallel \bar{x}$ be such that $q \in \ell$. Let $s \in \ell \cap \overline{pP}$, $t \in \ell \cap \overline{pQ}$, $a \in \bar{x} \cap \overline{pP}$, $b \in \bar{x} \cap \overline{qP}$, $c \in \bar{x} \cap \overline{pq}$, $d \in \bar{x} \cap \overline{qQ}$, $e \in \bar{x} \cap \overline{pQ}$, like in Figure 233.

As the $abP$ triangle is similar to $sq'P$, and $deQ$ is similar to $qtQ$, we have

$$\frac{|a-b|}{|s-q'|} = \frac{Q_t}{q'_t} \quad \text{and} \quad \frac{|d-e|}{|q'-t|} = \frac{Q_t}{q'_t},$$

and hence

(301) $$\frac{|a-b|}{|s-q'|} = \frac{|d-e|}{|q'-t|}.$$

On the other hand, as $p, q$ and $c$ were collinear, in the world-view of $m$ we have

$$\frac{|a-b|}{|a-c|} = \frac{|p-q|}{|p-c|} \quad \text{and} \quad \frac{|d-e|}{|c-e|} = \frac{|p-q|}{|p-c|},$$

and hence

(302) $$\frac{|a-c|}{|c-e|} = \frac{|a-b|}{|d-e|}.$$

See Figure 234.

From (301) and (302) one obtains

(303) $$\frac{|a-c|}{|c-e|} = \frac{|s-q'|}{|q'-t|}.$$

As $p'st$ and $p'ae$ are similar triangles and their sides are parallel, (303) implies $\overline{p'q'} \parallel \overline{p'c}$. But then $\overline{p'q'} = \overline{p'c}$. By exactly the same argument, $\overline{p'r'} = \overline{p'c}$. Thus $p', q', r'$ are collinear. Checking $\mathsf{Betw}(p', q', r')$ is easy and is left to the reader. ∎

### 4.9.3 Faster than light observers

When discussing the possibility of faster than light travel in models of a partial domain theory, one has to face a novel problem which did not come to light with full domain theories. This problem is the difference between the *local* and the *global* formulation of the FTL-relationship between two observers. Recall that by Thm. 4.3.17 (p.488) all light-cones are alike in $\mathsf{Mod}(\mathbf{Bax}^-)$. I.e., the speed of light in direction $d \in \mathsf{directions}$ does not depend on the space-time location: $c_m(p, d) = c_m(q, d)$ forall $p, q \in {}^nF$. However, as we saw in the $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$ model on pp. 678-685, in certain weak local relativity theories this is not the case anymore. Thus the question arises naturally: To which photon should an observer, say $m$, compare the trace of some observer, say $k$, to judge whether it travels faster than light? It seems to us that there are two intuitively convincing (not arbitrary) options:

1. Observer $k$ travels faster than light for $m$ at some point $p \in Dom(w_m^-)$ if $k$'s trace for $m$ is not contained "inside" of the light-cone around $p$ in some neighbourhood of $p$. In this case we shall say that $k$ travels FTL for $m$ *locally* at $p$. See Figure **??**(a). We shall give the formula of this relationship in Def. 4.9.13(i) below. Notice that the possibility that $k$ is FTL for $m$ at some $p \in tr_m(k)$ while it is not FTL (it is STL) for $m$ at some other $q \in tr_m(k)$ cannot be excluded beforehand.

2. Observer $m$ can compare the speed of $k$ to some light-cone on its own path ($tr_m(m)$). See Figure **??**(b). Notice that one cannot exclude *a priori* the possibility that the light-cones on $m$'s trace (as seen by himself) are different. Therefore, this concept needs one more parameter; namely, the time when $m$ measures the speed of light. In this case it might be possible that at time $\tau_0 \in F$ $m$ thinks that $k$ travels FTL while at some other $\tau_1 \neq\in F$ $m$ thinks that $k$ travels STL. The formal definition comes as Def. 4.9.13(ii) below. This option will be called the *global* version of the FTL-relationship.

The above distinction (between the local and the global FTL perspective) will gain more significance when we will be able to discuss general relativity. Imagine, for example, that you are approaching the event horizon of a Schwarzschild black hole. It is known from the standard books on relativity that in such a case you will observe the world outside of the black hole "speed up" as a whole. Eventually, you will be able to observe the whole history of the outer universe before crossing the event horizon. But this involves that the photons of the outside world have to speed up too! It is possible (although at this point this is nothing but a fairy tale) that the speed of ordinary observers (e.g. spaceships, planets etc.) of the outer universe will exceed the speed of light in your neighbourhood. In other words, they will travel FTL from a *global* perspective (option 2 above). On the other hand, they need not travel FTL *locally*, i.e. the observers of the world outside of the horizon of the black hole might still be slower than the photons in *their* neighbourhood.

On the other hand, the distinction between the local and the global FTL relationship will gain a cosmological significance when investigating inflationary universe scenarios. In that case some observer $m$ will experience that distant parts of the universe escape with speeds greater than the speed of light in the proximity of $m$'s life-line, while still slower than the *local* speed of light in the receding areas.

Although we shall not be able to discuss the phenomena indicated above adequately in the realm of localized (or partialized) theories of special relativity, we can explore these notions here due to the flexibility of our theory $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$. Let us turn to the formal definitions.

**Definition 4.9.13** **(i)**

$$\text{LocFTL}(m,k,p) \overset{\text{def}}{\iff} (\exists ph \in Ph)\,[p \in tr_m(k) \cap tr_m(ph) \wedge m \text{ can see } k \text{ and } ph \text{ move forwards in the same direction } \wedge\ v_m(k) > v_m(ph)],$$

$$\exists\text{LocFTL}(m,k)l \overset{\text{def}}{\iff} (\exists p \in Dom(w_m^-))\text{LocFTL}(m,k,p),$$

$$\forall\text{LocFTL}(m,k) \overset{\text{def}}{\iff} m \overset{\odot}{\to} k \ \wedge\ (\forall p \in tr_m(k))\text{LocFTL}(m,k,p).$$

**(ii)** $$\text{GlobFTL}(m,k,p) \overset{\text{def}}{\iff} (\exists ph \in Ph)\,[p \in tr_m(m) \cap tr_m(ph) \wedge m \text{ can see } k \text{ and } ph \text{ move forwards in the same direction } \wedge\ v_m(k) > v_m(ph)],$$

$$\exists\text{GlobFTL}(m,k) \overset{\text{def}}{\iff} (\exists p \in Dom(w_m^-))\text{GlobFTL}(m,k,p),$$

$$\forall\text{GlobFTL}(m,k) \overset{\text{def}}{\iff} (\forall p \in tr_m(m))\text{GlobFTL}(m,k,p).$$

By definition, $\text{LocFTL}(m,k,p)\ \vee\ \text{GlobFTL}(m,k,p) \Rightarrow m \overset{\odot}{\to} k$. It is easy to check that

$$\begin{aligned} \forall\text{LocFTL}(m,k) &\Rightarrow \exists\text{LocFTL}(m,k) \\ \forall\text{GlobFTL}(m,k) &\Rightarrow \exists\text{GlobFTL}(m,k) \end{aligned}$$

$\triangleleft$

**THEOREM 4.9.14** *Assume $n > 2$. Then*

$$\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(syBw)}^{\mathrm{par}} \models \neg(\exists m, k \in Obs)\exists \mathrm{LocFTL}(m,k).$$

As a contrast to Thm. 4.9.14, we would like to point out that an analogous conclusion with $\exists$GlobFTL in the place of $\exists$LocFTL does *not* follow. This simply follows by the model construction on pp. 678-685 above, where light-cones are tilted and $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(syBw)}^{\mathrm{par}}$ holds. Although we worked in $n = 2$, that model construction can be generalized to higher dimensions, too. This step is left to the reader.

**FACT 4.9.15**

$$\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(syBw)}^{\mathrm{par}} \not\models \neg(\exists m, k \in Obs)\exists \mathrm{GlobFTL}(m,k).$$

To prove Thm. 4.9.14 we shall need the following lemma.

**LEMMA 4.9.16** **(i)** *Assume that $f$ is a* Betw*-preserving function on a sphere $S = S(p,\varepsilon)$. If $H$ is a $k < n$ dimensional hyperplane such that $H \cap S \neq \emptyset$, then there is an $H'$ $k' \leq k$ dimensional hyperplane such that $f[H \cap S] \subseteq H'$.*

**(ii)** *Assume further that $f$ is continuous in the sense of* **Ax(continuity)** *for $\mathsf{f}^-_{mk}$. That is,*

$$(\textit{for every open sphere } S_0 \subseteq Dom(f)) \;\; f[S_0] \textit{ is an open subset of } {}^nF.$$

*Then $f[S \cap H]$ generates a $k$-dimensional hyperplane. In other words, there is no $k_0 < k$ dimensional hyperplane $H_0$ such that $f[S \cap H] \subseteq H_0$.*

We shall return to the proof of this lemma after demonstrating no-FTL theorem.

**Proof of Thm. 4.9.14:** Assume $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(syBw)}^{\mathrm{par}}$ and let $m, k \in Obs$ be such that $m$ FTL $k$. Then there are $p \in {}^nF$ and $ph_1 \in Ph$ such that $tr_m(k) \cap tr_m(ph_1) = \{p\}$, $m$ can see $k$ move forwards in some $d \in$ directions, and $v_m(k) > v_m(ph_1)$. Let $d, p, ph_1$ be fixed. First, we claim the following:

**Claim 4.9.17** There is an observer $k'$ such that $v_m(k') = \infty$.

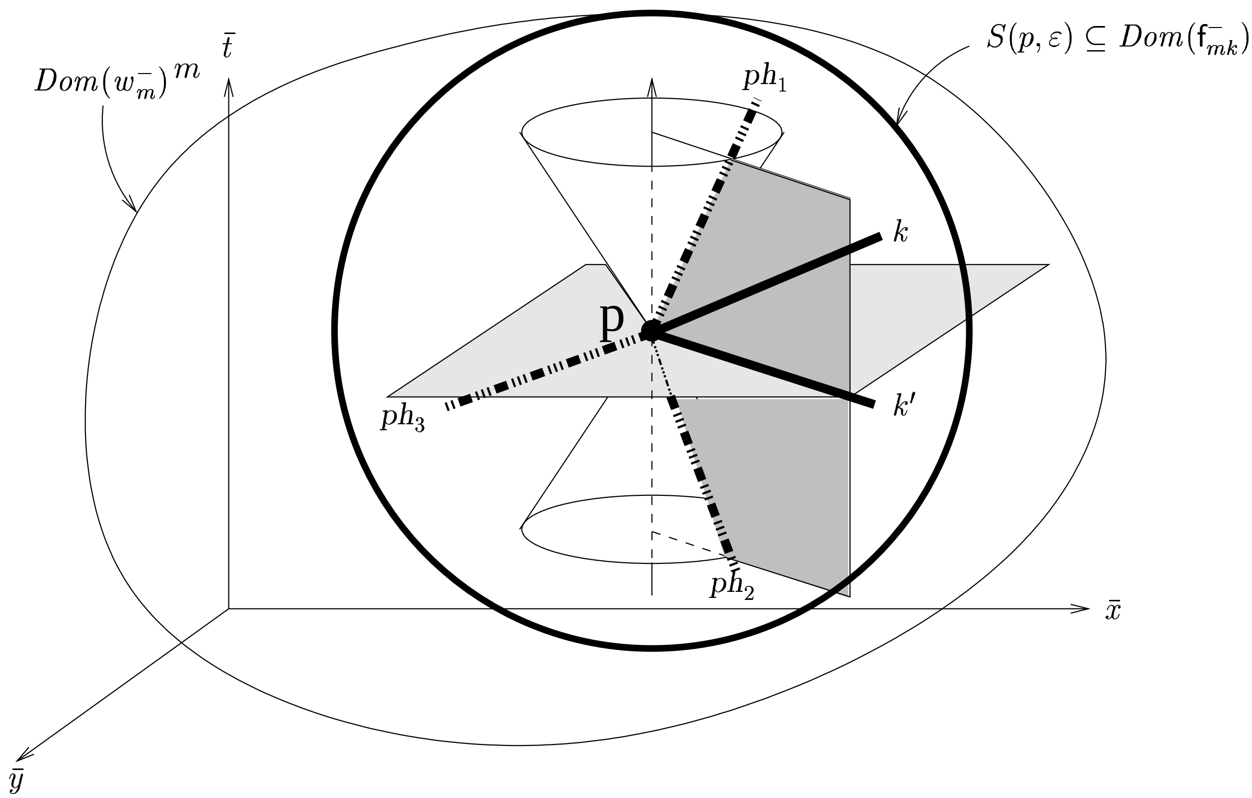


Figure 235: Illustration of the main idea of the proof of Thm. 4.9.14, Claim 4.9.17.

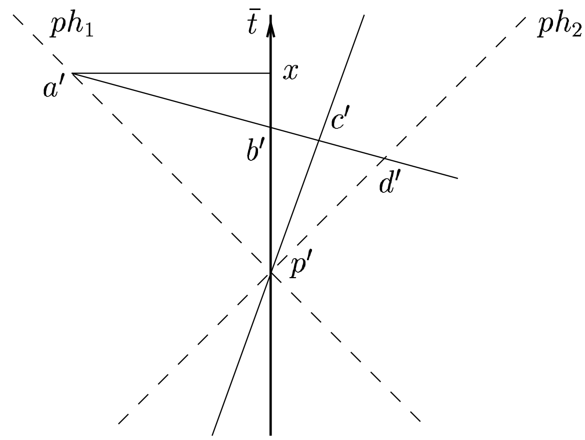


Figure 236: The world-view of observer $k$ in the proof of Thm. 4.9.14. The plane $P'$ is represented, which contains $\mathsf{f}^-_{mk}[P \cap S_0]$.

**Proof of Claim 4.9.17:** We can assume $v_m(k) \neq \infty$, otherwise we are ready. By $\mathbf{Ax5^{par}_{Ph}}$, there is a $ph_2 \in Ph$ such that $p \in tr_m(ph_2)$ and $m$ sees $ph_2$ move forwards in direction $-d$. See Figure 235. Let $ph_2$ be fixed. Then $m$ sees $ph_2$ move backwards in direction $d$. Let $\ell_0 \in G$ be a line with $ang^2(\ell_0) = \infty$ going through $p$ in direction $d$. Then there is a $P \in Plane$ such that $\bar{t} \parallel P$, and $tr_m(k) \cup tr_m(ph_1) \cup tr_m(ph_2) \cup \ell_0 \subseteq P$. (This can be checked by the definition of "$m$ can see $b$ move forwards/backwards in direction $d$".)

Let $S = S(p, \varepsilon)$ be such that $S \subseteq Dom(\mathsf{f}^-_{mk})$ and $\mathsf{f}^-_{mk} \restriction S$ preserves betweenness. Such an $S$ exists by $\mathbf{Ax6^{par}_{01}}$ and $\mathbf{Ax(Bw)}^{\mathrm{par}}$. Let $\ell_1 \in G$ be such that $\ell_1 \parallel \bar{t}$ and $\ell_1$ intersects each of $tr_m(k), tr_m(ph_1), tr_m(ph_2), \ell_0$ within $S$ (in $b$, $a$, $d$, $c$, respectively), but not in $p$. Such an $\ell_1$ exists by $v_m(ph_1), v_m(ph_2) > 0$ (by $\mathbf{AxE_{01}}$), $v_m(k) > 0$ (by $m$ FTL $k$), $ang^2(\ell_0) = \infty > 0$. We have $\mathsf{Betw}(a, b, c)$ by $v_m(ph_1) > v_m(k)$ and the definition of speed. Further, $\mathsf{Betw}(b, c, d)$ by the fact that $m$ can see $k$ move forwards and $ph_2$ move backwards, respectively, in the same direction $d$.

Let us turn to the world-view of $k$. See Figure 236. As $S \subseteq Dom(\mathsf{f}^-_{mk})$, $k$ can see the events that $m$ can see at points $p, a, b, c, d$ at some $p', a', b', c', d' \in Rng(\mathsf{f}^-_{mk})$, respectively. As $\mathsf{f}^-_{mk}$ is injective (cf. Prop 4.9.4(v)), $a', p', d'$ are pairwise distinct and $a', p', d'$ are not collinear. Thus $a', p'$ and $d'$ determine a plane $P'$. We have $b', c' \in P'$ because $\mathsf{f}^-_{mk} \restriction S$ preserves betweenness, and $\mathsf{Betw}(a, b, d)$, $\mathsf{Betw}(a, c, d)$. Further, $b' \in \bar{t}$, because $k \in w_m(b) = w_k(b')$ and $\mathbf{Ax4}^{\mathrm{par}}$.

We would like to show $ang^2(\overline{p'c'}) < v_k(ph_2)$, because we would like to find an observer on $\overline{p'c'}$. First, let us show that $\overline{p'c'} \cap \overline{b'd'} = \{c'\}$. To begin with, $c' \in \overline{p'c'} \cap \overline{b'd'}$

by $\mathsf{Betw}(b', c', d')$. Now suppose $\overline{p'c'} = \overline{b'd'}$. Then $\{p', b'\} \subseteq \bar{t}$, and $b' \neq p'$, and hence $c' \in \overline{p'c'} = \overline{b'd'} \subseteq \bar{t}$. By $\mathbf{Ax3_0^{par}}$, $tr_k(ph_2)$ is a part of a line. But then $tr_k(ph_2) \subseteq \bar{t}$, because $ph_2 \in w_k(p') \cap w_k(d')$, $p', d' \in \bar{t}$, $p' \neq d'$. But $v_k(ph_2) = 0$ contradicts $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$.

Let us proceed with $\overline{p'c'} \cap \overline{b'd'} = \{c'\}$. Let $x \in \bar{t}$ be such that $x_t = d'_t$. If $x = b'$, we are ready, because $b'_t = c'_t = d'_t$ and $\mathsf{Betw}(b', c', d')$ imply $ang^2(\overline{p'c'}) < v_m(ph_2)$ by definition. So let us turn to the case $x \neq b'$. Consider the $b'xd'$ triangle and the line $\overline{p'c'}$. $\overline{p'c'}$ intersects the $[b'd']$ side at an inner point. Then by Pasch's axiom[549] we have that $x \in \overline{p'c'}$, or $\overline{p'c'}$ intersects $[b'x]$ or $[xd']$. In the first two cases ($x \in \overline{p'c'}$ or $\overline{p'c'}$ crosses $[b'x]$), we have that $\overline{p'c'} \subseteq \bar{t}$, by the fact that $\overline{p'c'}$ intersects $\bar{t}$ twice, and the intersection points are distinct because $x = p'$ would imply $v_k(ph_2) = 0$, contrary to $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$. On the other hand, if $\overline{p'c'}$ intersects $\overline{xb'}$ between $x$ and $b'$, then $ang^2(\overline{p'c'}) < v_k(ph_2)$ by the definition of speed.

We have shown $ang^2(\overline{p'c'}) < v_k(ph_2)$. Then by $\mathbf{Ax5_{Obs}^{par}}$, there is $k' \in Obs$ such that $v_k(k') = ang^2(\overline{p'c'})$ and $k$ can see $k'$ move forwards in the same direction, say $e$, in which it can see $ph_2$ move forwards. It can be checked that $tr_k(k') = \overline{p'c'}$.

Returning to the world-view of $m$, $m \overset{\odot}{\rightarrow} k'$ follows by $k' \in w_k(p') = w_m(p)$. Moreover, $k' \in w_k(c') = w_m(c)$ and $c' \in Dom(\mathsf{f}^-_{km})$, thus $tr_k(k') \subseteq \ell_0$. Finally, $v_m(k') = ang^2(\ell_0) = \infty$. (Claim 4.9.17) ∎

Let us return to the proof of Thm. 4.9.14. Let $H$ be the $n-1$ dimensional hyperplane containing $p$ and parallel with Space. If $k'$ is the observer with infinite speed whose existence is warranted by Claim 4.9.17, then $tr_m(k') \subseteq H$.

Let us turn to the world-view of $k'$. See Figure 237. Let $S_0 = S(p, \delta)$ be such that $S_0 \subseteq S$ (recall that $\mathsf{f}^-_{mk} \restriction S$ preserves $\mathsf{Betw}$), $S_0 \subseteq Dom(\mathsf{f}^-_{mk'})$, and $\mathsf{f}^-_{mk'} \restriction S_0$ preserves betweenness. By Lemma 4.9.16, $H \cap S_0$ is taken into some at most $n-1$ dimensional hyperplane $H'$ or ${}^nF$. We have $\bar{t} \subseteq H'$ because $tr_m(k') \subseteq H$. As $H'$ is an at most $n-1$ dimensional hyperplane, there must be a direction, say $e \in \mathsf{directions}$, such that the lines that cross $\bar{t}$ and point in the direction $e$ are not contained in $H'$. (This is easy to check.) Let us fix such an $e$. By $\mathbf{Ax5_{Obs}^{par}}$, there is a limiting photon $ph$ going through $p'' \overset{\text{def}}{=} \mathsf{f}^-_{mk'}(p)$ moving forwards in direction $e$ for $k'$. By $\mathbf{AxE_{01}}$ $v_{k'}(ph) > 0$. Then there must be a $h \in Obs$ such that $tr_{k'}(h)$ goes through $p''$ and moves forwards in direction $e$ for $k'$, and $v_{k'}(h) > 0$. Let $h$ with these properties be fixed. Because of the way $e$ was chosen $tr_{k'}(h)$ is not contained in $H'$; actually,

$$tr_{k'}(h) \cap H' = \{p''\}. \tag{304}$$

Moreover, let us fix a $ph_0 \in Ph$ such that $p'' \in tr_{k'}(ph_0) \subseteq H'$. It is not a trivial question whether such a $ph_0$ exists, and this is the point where we shall need $n > 2$,

[549] The Pasch's axiom holds in $\langle {}^nF, \mathsf{Betw} \rangle$ by Fact 6.6.28.

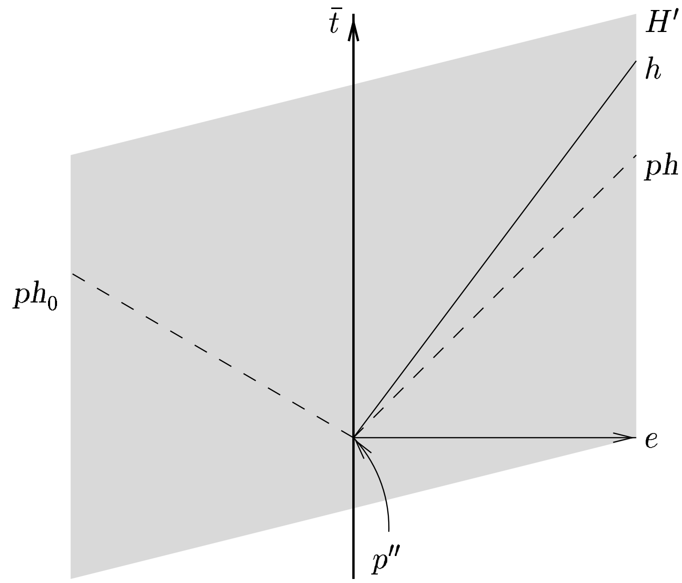


Figure 237: To the proof of Thm. 4.9.14. The picture shows the world-view of $k'$. $H'$ is the hyperplane that contains $\mathsf{f}^{-}_{mk'}[H \cap S_0]$.

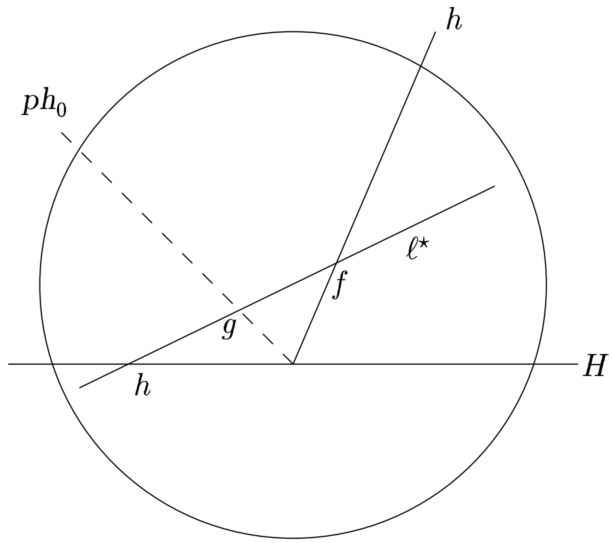


Figure 238: Illustration for the final step of the proof of Thm. 4.9.14. The picture focusses on the plane $P_0 \stackrel{\text{def}}{=} \mathsf{Plane}(tr_m(k), tr_m(ph_0))$.

and that we assumed the "bidirectional" version $\mathbf{Ax(syBw)}^{\text{par}}$ of the assumption of local betweenness-preserving. If $n > 2$ then $H$ is at least two-dimensional. We have to show that $H'$ is also at least two-dimensional, i.e. that $H'$ is not contained within a line. Suppose it is. As $H$ is a least two-dimensional, there are $a, b, c \in H \cap S_0$ which are not collinear. But $\mathsf{f}^-_{mk'}(a)$, $\mathsf{f}^-_{mk'}(b)$ and $\mathsf{f}^-_{mk'}(c)$ are collinear by the supposition. This contradicts $\mathbf{Ax(syBw)}^{\text{par}}$.

Let us return to $m$'s world-view. See Figure 235 again. We have $ph_0, h \in w^-_m(p) = w^-_{k'}(p'')$. $tr_m(ph_0)$ is not contained in $H$, because $v_m(ph_0) < \infty$ by $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$. Both $ph_0$ are $h$ are present "everywhere" in $S_0$, in the sense that if $tr_m(ph_0) \subseteq \ell_{ph_0}$ for some $\ell_{ph_0} \in G$ (which must be the case by $\mathbf{Ax3_0^{\text{par}}}$), then $\ell_{ph_0} \cap S_0 \subseteq tr_m(ph_0)$; and similarly for $h$. Consider the plane $P_0$ that contains both $tr_m(ph_0)$ and $tr_m(h)$. It is easy to show that there is a line $\ell^\star$ that intersect $tr_m(h)$, $tr_m(ph_0)$ and $H$ within $S_0$ in $f, g, h$, respectively, that are pairwise distinct and also distinct from $p$; see Figure 238. Since $f, g, h \in S_0$ and $\mathsf{f}^-_{mk'} \restriction S_0$ preserves collinearity (via betweenness), $f' \stackrel{\text{def}}{=} \mathsf{f}^-_{mk'}(f)$, $g' \stackrel{\text{def}}{=} \mathsf{f}^-_{mk'}(g)$ and $h' \stackrel{\text{def}}{=} \mathsf{f}^-_{mk'}(h)$ are collinear, too. Further, $g', h' \in H'$ as both $H$ and $tr_m(ph_0)$ are mapped into $H'$. But then $f' \in H'$, too. As $f' \neq p''$ by $\mathsf{f}^-_{mk'}$ being injective, we have $tr_m(h) \subseteq H'$. Thus we have reached a contradiction with (304). ∎

**Proof of Lemma 4.9.16(i):** We shall proceed by induction. If $k = 1$, the proposition follows by $f$ being Betw-preserving.

Assume that the lemma holds for $k$. For the present induction argument the reader is asked to study Figure 227 again. Let $H$ be a $k+1 < n$ dimensional hyperplane. I.e. $H$ is a translation of a $(k+1)$-dimensional subspace of ${}^nF$. Then there is a $k$-dimensional hyperplane $H_0$ and some $\ell \in G$ such that $\ell \cap H_0 = \{p\} \in S$ and

$$(\forall x \in H)(\exists q \in \ell)(\exists r \in H_0)\ x = (q - p) + (r - p) + p.$$

See Figure 227. Let $y \stackrel{\text{def}}{=} \frac{p+x}{2}$. It is straightforward to check $\mathsf{Betw}(q, y, r)$ and $\mathsf{Betw}(p, y, x)$.

Now, $f$ takes $H_0$ to some at most $k$-dimensional hyperspace $H_0'$ and $\ell$ to a line $\ell'$. Let $H'$ be generated by $H_0'$ and $\ell'$. Clearly, $f(p'), f(r') \in H_0'$ and $f(p'), f(q') \in \ell'$. As $f$ preserves Betw, we have $\mathsf{Betw}(f(q), f(y), f(r))$ and $\mathsf{Betw}(f(p), f(y), f(x))$. The former implies $f(y) \in H'$, and then the latter implies $f(x) \in H'$.

**Proof of Lemma 4.9.16(ii):** If $H$ is a $k$-dimensional hyperplane, then there are $p, f_0, f_1, \ldots, f_{k-1} \in H \cap S$ and $f_k, \ldots, f_{n-1} \in S \setminus H$ such that the members of

$$E = \{f_j - p : j \in n\}$$

are linearly independent. By an argument similar to the proof of Claim 4.9.10 one can check that the members of

$$\{f(f_j) - f(p) : j \in n\}$$

are linearly independent, too. Then $\{f(f_j) - f(p) : j \in k\}$ must be independent, too. But then $f[H \cap S]$ generates a $(k-1)$-dimensional hyperplane. ∎

**Conjecture 4.9.18** *We conjecture that our no FTL results generalize to* $\mathbf{Loc}(\mathbf{Bax}^{-\oplus})$*, assuming $n > 2$ of course. (Some extra auxiliary assumptions might be needed like e.g. a partial-domain version of* $\mathbf{Ax}(\|)$ *or of some of our other auxiliary axioms.)*

**Question for future research 4.9.19** It would be interresting to investigate whether Lemma 4.9.16(ii) can be demonstrated without using continuity. If necessary the symmetric version of Betw-preserving can be assumed, i.e.

$$(\forall p, q, r \in S)\,(\mathsf{Betw}(p, q, r) \ \Leftrightarrow\ \mathsf{Betw}(f(p), f(q), f(r)))\,.$$

This question is purely of mathematical interest, because the cost of of assuming **Ax(continuity)** in relativity is not really high and we shall do it whenever necessary.

### 4.9.4 Symmetry principles suitable for partial world-view theories

To prove "nice" or "classical" theorems in our partial world-view theories, we shall need some kind of symmetry axioms. Just like in the earlier cases (cf. §§2.8, 3.9), these will say that "the way you can observe my clocks, meter rods, etc. being affected by my movement is the same way I can see your corresponding properties being affected by your movement." We have to update our symmetry axioms because they usually assumed that $\mathsf{f}^-_{mk}$ is defined everywhere (or, alternatively, that $\mathsf{f}_{mk}$ is a function defined on ${}^nF$), or at least on the whole of $\bar{t}$. For purely technical reason it seems that $\mathbf{Ax(syt_0)}$ is the easier symmetry axiom to generalize. (The reason is that $\mathbf{Ax(syt_0)}$ does not work with the notion of "one of $m$'s brothers", whose generalization needs some careful analysis.)

$\mathbf{Ax(syt_0)}^{\mathrm{par}}$ $(m \overset{\odot}{\to} k \ \wedge\ k \overset{\odot}{\to} m) \Rightarrow (\forall p \in Dom(\mathsf{f}^-_{mk}) \cap \bar{t})(\forall q \in Dom(\mathsf{f}^-_{km}) \cap \bar{t})$
$(\exists \varepsilon \in {}^+F)\ \big(S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk})\ \wedge\ S(q,\varepsilon) \subseteq Dom(\mathsf{f}^-_{km})\ \wedge\ (\forall \delta \in F)$
$\big[|\delta| < \varepsilon\ \Rightarrow\ |\mathsf{f}^-_{mk}(p+\varepsilon 1_t)_t - \mathsf{f}^-_{mk}(p)_t| = |\mathsf{f}^-_{km}(q+\varepsilon 1_t)_t - \mathsf{f}^-_{km}(q)_t|\big]\big)$.

Informally, $\mathbf{Ax(syt_0)}^{\mathrm{par}}$ says that I can see your clock slow down in exactly the same proportion you can see my clock slow down. $\mathbf{Ax(syt_0)}^{\mathrm{par}}$ says nothing in the case the two observers cannot see one another's clock.

Now we can define the *generalized*, partial-domain version of **Specrel**.

$$\mathbf{Loc(Specrel)} \overset{\mathrm{def}}{=} \mathbf{Loc(Newbasax)} + \mathbf{Ax(syt_0)}^{\mathrm{par}} + \mathbf{Ax(Bw)}^{\mathrm{par}} + \mathbf{Ax(continuity)} + \mathbf{Ax}(\sqrt{\ }).$$

◁

The following theorem is the key theorem that opens the way to the generalization of most of our earlier results. Basically, it says that the $\mathsf{f}^-_{mk}$'s are the same sort of transformations (Lorentz transformations) locally, as our earlier $\mathsf{f}_{mk}$'s were globally.

**THEOREM 4.9.20**

**(i)** $\mathbf{Loc(Specrel)} \models (\forall p \in tr_m(k) \cap \bar{t})(\exists \varepsilon \in {}^+F)$
$\big[S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk})\ \wedge\ \mathsf{f}^-_{mk} \restriction S(p,\varepsilon) = A \restriction S(p,\varepsilon)\big]$, *for some* $A \in Aftr$.

**(ii)** *Moreover, the previous statement can be strengthened to assert* $\mathsf{f}^-_{mk} \restriction S(p,\varepsilon) = poi \restriction S(p,\varepsilon)$ *for some* $poi \in Poi$.

*Notice that this theorem describes only the case when the two observers meet.*

**Proof of item(i):** Assume **Loc**(**Specrel**). Let $m, k \in Obs$ and $p \in tr_m(k) \cap \bar{t}$. By $\mathbf{Ax6_{00}}$, $p \in Dom(\mathsf{f}^-_{mk})$. Let $q \stackrel{\text{def}}{=} \mathsf{f}^-_{mk}(p)$. Then $q \in \bar{t}$ by $\mathbf{Ax4}^{\text{par}}$. By Prop. 4.9.9 there is $S_1 = S(p, \varepsilon_1)$ such that

$$\mathsf{f}^-_{mk} \upharpoonright S_1 = (\tau_{-p} \circ \widetilde{\varphi} \circ A) \upharpoonright S_1,$$

for a translation $\tau_{-p}$ (by the vector $-p$), $A \in Aftr$ and $\varphi$ having properties (i) to (iv) described at Prop. 4.9.9. By $\mathbf{Ax(syt_0)}^{\text{par}}$ there is a $S_2 = S_2(p, \varepsilon_2)$ such that if $|\delta| < \varepsilon_2$, then

$$|\mathsf{f}^-_{mk}(p + \delta 1_t)_t - \mathsf{f}^-_{mk}(p)_t| = |\mathsf{f}^-_{km}(q + \delta 1_t)_t - \mathsf{f}^-_{km}(q)_t|. \tag{305}$$

Let $S = S(p, \varepsilon) \stackrel{\text{def}}{=} S_1 \cap S_2$. If $|\delta| < \varepsilon$, then (305) holds.

It is easy to check that $A = B \circ \tau_q$ for some linear bijection $B$. Then (305) implies

$$|B(\widetilde{\varphi}(\delta 1_t))_t| = |B^{-1}(\widetilde{\varphi}^{-1}(\delta 1_t))_t|.$$

Then by $B$'s linearity,

$$|\varphi(\delta) B(1_t)_t| = |\varphi^{-1}(\delta) B^{-1}(1_t)_t|.$$

Then, for some $\alpha, \beta \in F^+$,

$$\varphi(\delta)\alpha = \varphi^{-1}(\delta)\beta. \tag{306}$$

Let us apply property (iv) to $x, y = x\frac{\varepsilon}{2}, z = x^2$ for some $x \in (-\varepsilon, \varepsilon)$. We obtain

$$\varphi(x)\varphi(x\frac{\varepsilon}{2}) = \varphi(x^2)\frac{\varepsilon}{2} \quad \text{and} \quad \varphi^{-1}(x)\varphi^{-1}(x\frac{\varepsilon}{2}) = \varphi^{-1}(x^2)\frac{\varepsilon}{2}. \tag{307}$$

By (306) and (307) one obtains

$$\varphi(x^2)\frac{\varepsilon}{2} = \varphi^{-1}(x)\varphi^{-1}(x\frac{\varepsilon}{2})\frac{\beta}{\alpha}. \tag{308}$$

On the other hand, applying (307) first to $\varphi(x^2)$, and then (306), one gets

$$\varphi(x^2)\frac{\varepsilon}{2} = \varphi^{-1}(x)\varphi^{-1}(x\frac{\varepsilon}{2})\left(\frac{\beta}{\alpha}\right)^2. \tag{309}$$

By (308) and (309), $\alpha = \beta$. Then $\varphi(\delta) = \varphi^{-1}(\delta)$ and $\varphi^2(\delta) = \delta$.

Now suppose $\varphi \neq \mathrm{Id} \upharpoonright (-\varepsilon, \varepsilon)$. E.g. let $a < \varphi(a)$ for some $a \in (-\varepsilon, \varepsilon)$. Then by $\varphi$ preserving $<$ and $\varphi^2 = \mathrm{Id} \upharpoonright (-\varepsilon, \varepsilon)$,

$$\varphi(a) < \varphi^2(a) = a.$$

We have reached a contradiction, therefore $\varphi = \mathrm{Id} \restriction (-\varepsilon, \varepsilon)$. ∎

**Proof of item(ii):** ∎

Theorem 4.9.20 seems to imply that $\mathbf{Ax(syt_0)}^{\mathrm{par}}$ is an adequate symmetry axiom for **Loc**(**Newbasax**). It seems likely that it is adequate for **Loc**(**Flxbasax**) and **Loc**(**Bax**), too. Nevertheless let us turn to formulating the partial domain version $\mathbf{Ax(symm_0)}^{\mathrm{par}}$ of our symmetry axiom **Ax(symm)**.Recall that

$$\mathbf{Ax(symm)} = \mathbf{Ax(symm_0)} + \mathbf{Ax(eqtime)}.$$

$\mathbf{Ax(symm_0)}^{\mathrm{par}}$ will say $(\forall m, k \in Obs)(\forall p \in Dom(\mathsf{f}^-_{mk}))(\exists \varepsilon \in {}^+F)[\, S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk}) \ \wedge \ (\exists m', k' \in Obs)(Dom(w^-_{m'}) = Dom(w^-_m) \ \wedge \ Dom(w^-_{k'}) = Dom(w^-_k) \ \wedge \ tr_m(m') \cup tr_k(k') \subseteq \bar{t} \ \wedge \ \mathsf{f}^-_{mk} \restriction S(p,\varepsilon) = \mathsf{f}^-_{k'm'} \restriction S(p,\varepsilon)\,)\,)\,]$.

A stronger possible version of $\mathbf{Ax(symm_0)}^{\mathrm{par}}$ would say

$\mathbf{Ax(symm_0)}^{\mathrm{par+}}$ $(\forall m, k \in Obs)(\forall p)(\forall \varepsilon \in {}^+F)$
$[\, S(p,\varepsilon) \subseteq Dom(\mathsf{f}^-_{mk}) \ \Rightarrow$ here would come the rest of $\mathbf{Ax(symm_0)}^{\mathrm{par}}$.

A shorter, helpful, equivalent form of $\mathbf{Ax(symm_0)}^{\mathrm{par}}$ is the following.

$\mathbf{Ax(symm_0)}^{\mathrm{par'}}$ $(\forall m, k \in Obs)(\exists m', k' \in Obs)[\, Do^-(m) = Do^-(m') \ \wedge \ Do^-(k) = Do^-(k') \ \wedge \ tr_m(m') \cup tr_k(k') \subseteq \bar{t}$ and $(\exists$ affine transformation $h : {}^nF \longrightarrow {}^nF)\ \mathsf{f}^-_{mk} \subseteq h \supseteq \mathsf{f}^-_{m'k'}\,]$.

The *intuitive* meaning of the partial-domain versions $\mathbf{Ax(symm_0)}^{\mathrm{par'}}$, $\mathbf{Ax(symm_0)}^{\mathrm{par}}$ of $\mathbf{Ax(symm_0)}$ remains the same as that of $\mathbf{Ax(symm_0)}$[550], *but* here we have to circumnavigate the difficulty that we have no reason to assume that the domain $Do^-(m)$ of $m$ is the same as (or is similar to) that of $Do^-(k)$ of $k$. Hence we cannot assume that $Dom(\mathsf{f}^-_{mk}) = Dom(\mathsf{f}^-_{k'm'})$. So, we want to say that $\mathsf{f}^-_{mk}$ does the same kind of "distortions" to "space-time" as $\mathsf{f}_{k'm'}$ does (unit-vectors shrink, grow etc) without claiming that their domains agree.

**Warning 4.9.21**

**(i)** At this point we did not carefully check whether our philosophy (in the new partial-domain situation) permits something like $Dom(\mathsf{f}^-_{mk}) = Dom(\mathsf{f}^-_{k'm'})$. Perhaps it does, perhaps it does not. We leave it to future research to find out. Until then, to be on the safe side, we do not assume it.

[550] As I see you being affected by your motion so does a brother of yours see a brother of me.

**(ii)** We did not check carefully whether in the new, partial-domain situation it is still something like $\mathbf{Ax(symm_0)}^{\mathrm{par}}$ which expresses that observers $m$ and $k$ are equivalent (in the spirit of the Einsteinean-Machian philosophy of equivalence of observers[551]).

Let us turn to formulating the partial-domain versions of **Ax(eqtime)**. (This is the second part of **Ax(symm)**.)

$$\mathbf{Ax(eqtime)}^{\mathrm{par}} \quad (\forall m, m' \in Obs)\,[\, tr_m(m') \subseteq \bar{t} \supseteq tr_{m'}(m) \;\Rightarrow\; (\forall p, q \in \bar{t} \cap Dom(\mathsf{f}^{-}_{mm'}))\; |p-q| = |\mathsf{f}^{-}_{mm'}(p) - \mathsf{f}^{-}_{mm'}(q)|\,].$$

$$\mathbf{Ax(symm)}^{\mathrm{par}} \stackrel{\mathrm{def}}{=} \mathbf{Ax(symm_0)}^{\mathrm{par}} + \mathbf{Ax(eqtime)}^{\mathrm{par}}.$$

◁

**Question for future research 4.9.22** Is

$$\mathbf{Loc}(\mathbf{Newbasax}) + \text{some auxiliary axioms} \models \mathbf{Ax(symm_0)}^{\mathrm{par}} \leftrightarrow \mathbf{Ax(symm_0)}^{\mathrm{par}\prime}$$

true?

◁

**Future research task 4.9.23**

1. Find other symmetry principles (cf. e.g. § 2.8, 3.9) which are more adequate to the new, partial-domain generalizations **Loc**(**Newbasax**), **Loc**(**Flxbasax**), **Loc**($\mathbf{Bax}^-$) of our relativity theories.

2. Investigate whether $\mathbf{Ax(symm)}^{\mathrm{par}}$ is adequate to the philosophy of the new theories **Loc**(**Newbasax**), **Loc**(**Flxbasax**) etc.

The symmetric versions of our new, partial-domain version relativity theories are **Loc**(**Newbasax**) + $\mathbf{Ax(symm)}^{\mathrm{par}}$, **Loc**(**Flxbasax**) + $\mathbf{Ax(symm)}^{\mathrm{par}}$, **Loc**(**Bax**) + $\mathbf{Ax(symm)}^{\mathrm{par}}$ etc. The new theories are **Loc**(**Newbasax**), ..., **Loc**(**Bax**), **Loc**(**Reich**(**Newbasax**)), **Loc**(**Reich**(**Bax**)), etc. or analogously with $\mathbf{Ax(syt_0)}^{\mathrm{par}}$.

[551] which, in our opinion intends to be a consequence of Occam's razor i.e. trying to keep our model of the world as simple as possible if there is no reason for doing otherwise

### 4.9.5 Paradigmatic effects

We are going to demonstrate, roughly speaking, that the paradigmatic effects of relativity (see §§2.5, 4.8) can be reformulated in such a way that they remain valid in the partial domain version of our theories, in the cases their "ancestor" was true in the original theory. For simplicity, we shall only work with those effects that follow by a symmetry axiom (it is usually irrelevant which symmetry axiom one chooses; in the partial domain case we shall work with $\mathbf{Ax(syt_0)}^{\mathrm{par}}$), a relativity theory, and some auxiliary axioms. To generalize our theorems below to the case without symmetry might be a future research task.

First, we are reformulating the paradigmatic effects of §4.8 to incorporate the possibility of partial domains. Unlike in §4.8, we shall not use the convenience assumptions $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$ (or $\mathsf{f}^-_{mk}(\bar{0}) = \bar{0}$) and $tr_m(k) \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$, because $Dom(\mathsf{f}^-_{mk}) \subsetneq {}^nF$ might make such forms of the paradigmatic effects vacuous for many observers.

(E1)$^{\mathrm{par}}$ $(0 < v_m(k) < c_m \ \wedge \ v_m(k) \neq \infty) \Rightarrow (\forall p \in tr_m(k) \cap \bar{t})(\exists \varepsilon \in F^+)$
$(\forall \delta \in (-\varepsilon, \varepsilon)) |\mathsf{f}^-_{mk}(p + \delta 1_t)_t - \mathsf{f}^-_{mk}(p)_t| > \delta.$

That is, if $m$ can see $k$ move with a non-zero speed but slower than the speed of light, and $m$ and $k$ meet (at some point $p$), then $m$ can see $k$'s clock slow down in the neighbourhood of their meeting point. Formula (E1)$^{\mathrm{par}}$ does not describe the case the two observers do not meet. If they meet, $tr_m(k) \cap \bar{t}$ is non-empty. Cf. the original formula (E1) on p.635.

(E2)$^{\mathrm{par}}$ $(\forall p \in tr_m(m))(\exists \varepsilon \in F^+) \big(S(p, \varepsilon) \subseteq Dom(\mathsf{f}^-_{mk}) \wedge$
$[\emptyset \neq tr_m(k) \cap S(p, \varepsilon) \parallel tr_m(k_1) \cap S(p, \varepsilon) \neq \emptyset \ \wedge \ p \in tr_m(k) \ \wedge$
$tr_m(k_1) \subseteq \mathsf{Plane}(\bar{t}, tr_m(k)) \ \wedge \ 0 < v_m(k) < c_m \ \wedge \ v_m(k) \neq \infty] \Rightarrow$
$\big[(\forall q \in tr_m(k) \cap S(p, \varepsilon))(\forall r \in tr_m(k_1) \cap S(p, \varepsilon))(\forall s \in tr_k(k_1) \cap Rng(\mathsf{f}^-_{mk}))$
$(q_0 = r_0 \ \wedge \ s_0 = \mathsf{f}^-_{mk}(q)_0 \ \wedge \ \mathsf{f}^-_{km}(s) \in S(p, \varepsilon)) \ \Rightarrow \ |q - r| < |s - \mathsf{f}^-_{mk}(q)|\big]\big).$

Informally, the length of those spaceships $m$ meets and can see with a non-zero speed but slower than the speed of light shrink in the direction of movement, at least as long as they are not too distant from their meeting point. The front of the spaceship is $k_1$, the rear if $k$. The spaceship's length is $|q - r|$ for $m$ and $|s - \mathsf{f}^-_{mk}(q)|$ for $k$, who "travels" on the spaceship. See Figure **??**. Cf. also (E2) on p.635.

For completeness, we are also introducing (E4)$^{\mathrm{par}}$, although we shall not use it in this subsection, because (E4) is nothing but a weaker version of (E1) and (E2) suitable for theories without a symmetry axiom.

$$\text{(E4)}^{\mathrm{par}} \overset{\mathrm{def}}{=} \text{(E1)}^{\mathrm{par}} \ \vee \ \text{(E2)}^{\mathrm{par}}.$$

Notice that $m$ is a free variable of both $(\mathrm{E1})^{\mathrm{par}}$ and $(\mathrm{E2})^{\mathrm{par}}$.

**THEOREM 4.9.24 (Clocks slow down and spaceships shrink)**
**Loc(Specrel)** $\models \{(\mathrm{E1})^{\mathrm{par}}, (\mathrm{E2})^{\mathrm{par}}\}$.[552]

**Proof:** ∎

On a semi-formal level, $(\mathrm{E3})^{\mathrm{par}}$ will be the same as (E3), i.e. we have no reason to change essentially the original form of (E3) as it was given on p.636. However, if we want to turn the conclusion of (E3) into a precise formula, we have to take care of $Dom(\mathsf{f}^-_{mk})$, that is:

$(\mathrm{E3})^{\mathrm{par}}$ $0 < v_m(k) < c_m \ \wedge \ v_m(k) \neq \infty \ \Rightarrow$
$\big[(\exists p, q \in Dom(\mathsf{f}^-_{mk}))(p_0 = q_0 \ \wedge \ \mathsf{f}^-_{mk}(p)_0 \neq \mathsf{f}^-_{mk}(q)_0)\big]$.

Moreover, in §2.5 we saw that this effect can be formulated somewhat more specifically, saying that the clocks that are separated by a vector that falls in the direction of movement *must* get out of synchronism, while while those that are separated by a direction orthogonal to it *must not* get out of synchronism. We are going to give the appropriate formulae for both of these cases.

$(\mathrm{E3})^{\mathrm{par}_{\parallel}}$ $0 < v_m(k) < c_m \ \wedge \ v_m(k) \neq \infty \ \wedge \ tr_m(k) \cap \bar{t} \neq \emptyset \ \Rightarrow$
$\big[(\exists p, q \in Dom(\mathsf{f}^-_{mk}))(p_0 = q_0 \ \wedge \ p - q \in \mathsf{Plane}(\bar{t}, tr_m(k)) \ \Rightarrow \ \mathsf{f}^-_{mk}(p)_0 \neq \mathsf{f}^-_{mk}(q)_0\big]$.
I.e. clocks falling in the direction of movement are not synchronous for $k$ if they were synchronous for $m$.

$(\mathrm{E3})^{\mathrm{par}_{\perp}}$ $0 < v_m(k) < c_m \ \wedge \ v_m(k) \neq \infty \ \wedge \ tr_m(k) \cap \bar{t} \neq \emptyset \ \Rightarrow$
$\big[(\exists p, q \in Dom(\mathsf{f}^-_{mk}))(p_0 = q_0 \ \wedge \ p - q \perp \mathsf{Plane}(\bar{t}, tr_m(k)) \ \Rightarrow \ \mathsf{f}^-_{mk}(p)_0 = \mathsf{f}^-_{mk}(q)_0\big]$.
I.e. clock separated by a vector orthogonal to the direction of movement are not affected by the movement. If they were synchronous, they remain so.

**THEOREM 4.9.25 (Clocks get out of synchronism)**
**Loc(Specrel)** $\models \{(\mathrm{E3})^{\mathrm{par}}, (\mathrm{E3})^{\mathrm{par}_{\parallel}}, (\mathrm{E3})^{\mathrm{par}_{\perp}}\}$.[553]

**Proof:** ∎

Effect (E6) said that faster than light observers do not exist. Actually, we have already seen a theorem (Thm. 4.9.14 above) saying this. Let us recall what was stated there.

[552] Cf. Thms 4.8.1, 2.5.2, 2.5.9.
[553] Cf. Thms 4.8.1(ii), 2.5.5, 2.5.6.

(E6)$^{\rm par}$ $\neg(\exists m, k \in Obs)\, m$ FTL $k$,

where $FTL$ comes from Def. 4.9.13.

**Remark 4.9.26** **Loc(Specrel)** $\models$ (E6)$^{\rm par}$ by Thm. 4.9.14.

The last one of our paradigmatic effects, (E7) by the notation of the previous subsection, was the existential form of the twin paradox. As the partiality of the world-view of observers may cause difficulties, we are going to reformulate the (E7) below.

(E7)$^{\rm par}$ $(\exists m, k_1, k_2 \in Obs)(\exists p, q, r \in Dom(w^-_{mk}))$
$(p, r \in tr_m(m) \wedge p, q \in tr_m(k_1) \wedge q, r \in tr_m(k_2) \wedge m \text{ STL } k_1 \wedge m \text{ STL } k_2 \wedge$
$|r - q| > |\mathsf{f}^-_{mk}(p) - \mathsf{f}^-_{mk}(q)| + |\mathsf{f}^-_{mk}(q) - \mathsf{f}^-_{mk}(r)|)$.

**THEOREM 4.9.27 (Existential twin paradox)** **Loc(Specrel)** $\models$ (E7)$^{\rm par}$.

**Proof:** ∎

**Future research task 4.9.28** Try to push through the investigations, discussions and results in the present work (Chapters 2–6) to the **Loc(Newbasax)**, **Loc(Flxbasax)** etc. like we did for the versions of our theories at the end of Chapter 4. I.e. try to push through the present results to the partial-domain versions of our theories. Compare the results with the old ones. E.g. what do the geometries (cf. Chap.6) look like? Are there duality theories between the geometries and the observer oriented $\mathfrak{M}$-type models (cf. §6.6 in Chap.6).

How does the Reichenbachian-version contra Einsteinian version comparison elaborated in Chapter 4 (§4.5) look like in the new partial-domains "paradigm"?

* * *

**Remark 4.9.29** The generalization $Th \mapsto \mathbf{Loc}(Th)$ of our relativity theories is analogous with what is called relativization in Algebraic Logic, cf. e.g. Németi [206], Németi [?], , Henkin et al. [130], Németi [?], Németi [204], Marx [187], Mikulás [194], Marx-Venema [189]. In algebraic logic, relativization made the structures under investigation more flexible, and it had an effect of "decomposing" the large "square" structures to many overlapping little "mosaics" cf. the so called mosaic method in the above quoted works e.g. in Németi [?]. It is interesting to observe that in the present relativity setting the same idea of relativization induces similar effects, e.g.

ez a relativizálós cikk a Marx-Masuch-Pólos [188] kötetből

Csirmaz kötet logic collocquium-ban levő Németi cikk!

Marx-Masuch-Pólos kötetben!

our relativistic geometry will consist instead of big "square" windows (like in the case of **Newbasax**, etc.) many small perhaps overlapping non-square windows in analogy with the mosaics in the algebraic logic case.

It would be interesting to investigate the connections further, since in Algebraic Logic the "mosaics and relativization" proved extremely useful. E.g. Venema [271] emphasizes the advantages of the relativized non-square approach over the square one. These advantages were further exploited in Andréka-Benthem-Németi [31].

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**Future research task 4.9.30**

1. Recall the process of Reichenbachization $Th \mapsto \mathbf{Reich}(Th)$ of relativity theories $Th$ from §4.5 p. 553. Put Reichenbachization

$$Th \quad \mapsto \quad \mathbf{Reich}(Th)$$

in analogy with localization

$$Th \quad \mapsto \quad \mathbf{Loc}(Th)$$

Execute all the things we did with $Th \mapsto \mathbf{Reich}(Th)$ for the new $Th \mapsto \mathbf{Loc}(Th)$.

2. Try to characterize the $\mathsf{f}_{mk}$ transformations in the style we did in Fig.15 on p.63. E.g. are the $\mathsf{f}_{mk}$'s like in Figures 15, 16 on pp. 63–64? Of course we mean this figures now restricted to the domain of observer e.g. to $Dom(w_m^-)$. In more detail take Fig.16 restrict the picture to $Dom(w_m^-)$ and contemplate wether the $\mathsf{f}_{mk}$'s look like this in $\mathbf{Loc}(Th)$.

3. Try to push through what we did in §4.7 with, now instead of $\mathbf{Reich}(Th)$ for $\mathbf{Loc}(Th)$.

4. How much of the axiom $\mathbf{Ax(Bw)}^{\mathrm{par}}$ (of betweenness preserving) is needed in the no FTL proof for e.g. $\mathbf{Loc}$(**Newbasax**)?

5. Is collinearity of the $\mathsf{f}_{mk}$'s sufficient in the no FTL proof instead of $\mathbf{Ax(Bw)}^{\mathrm{par}}$?

6. Can collinearity of the $\mathsf{f}_{mk}$'s be proved from the rest of the axioms of $\mathbf{Loc}(Th)$ perhaps under some extra mild assumptions like we proved in the case of **Basax** etc cf. Thm.4.3.13?

### 4.9.6 Alternative, more flexible partial domain theories

As we have seen above, our localized theories $\mathbf{Loc}(\mathbf{Bax}^-),\ldots,\mathbf{Loc}(\mathbf{Basax})$ of special relativity are quite flexible. We have seen that $\mathbf{Loc}(\mathbf{Bax}^{-\oplus}) + \mathbf{Ax(Bw)}^{\mathrm{par}}$ has models which presursor, is some sense, the "paradigmatic" phenomena of general relativity. (Although we know there is still a long way to go.) However, $\mathbf{Loc}(\mathit{Th})$ (for $\mathit{Th} \in \{\mathbf{Bax}^-,\ldots,\mathbf{Basax}\}$) is still unflexible in one important sense if one has the switch to general relativity in mind. Namely, consider the scenario of Figure **??**. Let us have an object with a long history, e.g. a spaceship, a planet or a star. It is natural to attempt—and it is a common practice in general relativity—to cover the life-line of the object by many finite, overlapping "local" coordinatizations (i.e. observers) $m_1, m_2, m_3$ etc. Consider e.g. observer $m_2$. $m_2$ can see the traces of $m_1$ and $m_3$ overlap with its own trace. However, by $\mathbf{Ax3}_\mathbf{0}^{\mathrm{par}}$ $m_1$ and $m_2$ cannot simply "disappear" at some points within $Dom(w^-_{m_2})$: if $m_2 \overset{\odot}{\to} m_1$ and $m_2 \overset{\odot}{\to} m_3$, then $m_1$ and $m_2$ must be present in the whole history of the object within the entire $Dom(w^-_{m_2})$, i.e. in $tr_{m_2}(m_2)$. (The reader should not be disturbed by the contingent fact that Figure **??** does not show the traces as straight lines.) But if $m_1$ and $m_3$ are present in the whole of $tr_{m_2}(m_2)$, then by $\mathbf{Ax6_{00}}$ $m_1$ and $m_3$ must also see (coordinatize) the events of $w^-_{m_2}[tr_{m_2}(m_2)]$. Thus the world-view of $m_1, m_2, m_3$ etc. are "pumped up" to cover the whole of the object's life-line. We have basically the same effect as by $\mathbf{Ax4}$.

The solution we propose is to weaken $\mathbf{Ax6_{00}}$ in our localized relativity theories. Our purpose is twofold: (i) to experiment with more flexible theories (where observers' world-views can overlap in a less restricted way), (ii) to approach the spirit of the usual treatments of the theory of general relativity. The following definitions indicate a possible move in this direction.

**Definition 4.9.31** [observer brothers]

$$m \text{ is a brother of } k \overset{\text{def}}{\Longleftrightarrow} (\forall m_1 \in Obs)\, tr_{m_1}(m) = tr_{m_1}(k).$$

$\mathbf{Ax6}_{\mathbf{00}}^{\mathrm{par}}$ $tr_m(k) \subseteq \bigcup\{tr_h(k) : \ h \text{ is a brother of } k\ \}$.

Informally, if $m$ can see some event $e \ni k$, then some brother $h$ of $k$ can see $e$.

**Definition 4.9.32** $Loc_1(\mathit{Th}) \overset{\text{def}}{=} \mathbf{Loc}(\mathit{Th}) \setminus \{\mathbf{Ax6_{00}}\} \cup \{\mathbf{Ax6}_{\mathbf{00}}^{\mathrm{par}}\}$, for any partial domain theory obtained by the operator $\mathbf{Loc}$.

**Future research task 4.9.33**

**(i)** Investigate the models of $Loc_1(Th)$ for $Th \in \{\mathbf{Bax}^-, \dots, \mathbf{Basax}\}$ similary to the way we investigated $\mathbf{Loc}(Th)$ above. Are FTL observers allowed? Can the $\mathsf{f}^-_{mk}$ transformations be characterized in a streamlined way? Are the paradigmatic effects true?

**(ii)** Check if $Loc_1(Th)$ really serves its purpose. That is, build a model in which each observer coordinatizes only a finite domain, while the union of their views makes up some characteristic infinite special relativistic universe, in some sense.