of the Mal'cev's problem of the existence of an associative (semigroup) ring which is not embeddable into any skew-field, but its multiplicative semigroup is embeddable into a group.

For more informations, see also (Book and Otto 1993).

G.8 Relational Algebras

by Hajnal Andréka, Judit X. Madarász, and István Németi in Budapest, Hungary

Boolean algebras (BA's for short) can be regarded as algebras of unary relations; i.e., the elements of a BA, say $\mathcal{B}$, are unary relations and the operations of $\mathcal{B}$ are the natural operations on unary relations. The purpose of relational algebra is to expand the natural algebras of unary relations (i.e., BA's) to natural algebras of relations of higher ranks, i.e., of relations in general. What will be the elements of the new algebras? The elements of BA's can be visualized as sets of *points*. Then, the elements of the new algebras will be *sets of sequences* (the reason for this is that the elements of relations are sequences independently of whether our relations are binary, ternary or n-ary).

The simplest case is when we concentrate on binary relations. For a set U, $\mathcal{P}(U)$ denotes its power set (the set of all subsets of U) while $\mathcal{P}(U)$ denotes the BA $(\mathcal{P}(U); \cup, \cap, -)$ with universe $\mathcal{P}(U)$. The ***full relation algebra*** over the set U is defined to be the algebra

$$\mathcal{R}e(U) = (\mathcal{P}(U \times U), \circ, ^{-1}, \mathrm{Id}_U)$$

where "$\circ$" is the usual composition of two relations, R^{-1} is the usual converse (or inverse) of the relation R and $\mathrm{Id} = \mathrm{Id}_U$ is the identity relation on U. The class RRA of ***representable relation algebras*** is defined as

$$\mathsf{RRA} = \mathbf{SP}\{\mathcal{R}e(U) \mid U \text{ is a set}\}$$

where $\mathbf{S}$ and $\mathbf{P}$ are the operators on classes of algebras corresponding to taking isomorphic copies of subalgebras and direct products, respectively.

G.8.1 Theorem (Tarski) RRA *is a discriminator variety. The equational theory of* RRA *is recursively enumerable but not decidable.*

Before discussing RRA's further, let us look at algebras of relations of higher ranks (e.g., ternary, n-ary relations). The natural algebras are called cylindric algebras. In the following, n denotes a natural number. The ***full cylindric algebra*** of n-ary relations over a set U is defined as

$$\mathcal{R}el_n(U) = \big(\mathcal{P}(U^n), c_0, \ldots, c_{n-1}, \mathrm{Id}\,\big)$$

where Id is the n-ary identity relation $\mathrm{Id}_{n,U} = \{\,(a, \ldots, a) \mid a \in U\}$ and c_i is a unary operation for each $i < n$ defined by $c_i(R) = c_i^{(U)}(R) = \{\,(b_0, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_{n-1}) \mid (b_0, \ldots, b_{n-1}) \in R \text{ and } a \in U\}$, for any $i < n$ and $R \subseteq {}^nU$. We will omit the superscript U. Let $R \subseteq U^n$ be a relation. Then the relation $c_i(R)$ is called the smallest ***i-cylinder*** containing R. Choosing $n = 3$ and U the real numbers, we obtain the greatest element $U \times U \times U$ of our algebra as the usual Cartesian space, and i-cylinders appear as *cylinders* parallel to the i-th axis. Let $n = 2$ and $R \subseteq U \times U$. Then $c_0(R) = U \times \mathrm{Rg}(R)$ and $c_1(R) = \mathrm{Dom}(R) \times U$. This example shows that the operations c_i are natural ones (on relations). The class RCA_n of n-ary ***representable cylindric algebras*** is defined as

$$\mathsf{RCA}_n = \mathbf{S}\,\mathbf{P}\{\,\mathcal{R}el_n(U) \mid U \text{ is a set }\,\}.$$

G.8.2 Theorem (Tarski) RCA_n *is a discriminator variety. The equational theory of* RCA_n *is recursively enumerable, and if* $n > 2$ *then undecidable.*

To have all finitary relations over U in a single algebra, we need to extend cylindric algebras to α-ary relations with α an arbitrary ordinal. For this, we need to replace our single (α-ary) identity relation Id with $\alpha \times \alpha$ many identity relations $\mathrm{Id}_{ij} = \{\, q \in {}^{\alpha}U \mid q_i = q_j\,\}$, for $i, j \in \alpha$. Throughout, α is an arbitrary (possibly finite) ordinal. Now, we define the full algebra of α-ary relations as

$$\mathcal{R}el_\alpha(U) = \big(\mathcal{P}(U^\alpha), c_i, \mathrm{Id}_{ij} \mid i, j < \alpha\,\big),$$

where $c_i(R)$ and Id_{ij} are defined as above. Thus, besides the Boolean operations, $\mathcal{R}el_\alpha(U)$ has α many unary operations c_i (one for each $i < \alpha$) and $\alpha \times \alpha$ many constants Id_{ij}. Now, for $\alpha < \omega$ we have two versions for $\mathcal{R}el_\alpha(U)$ but they are polynomially equivalent.. Indeed, if e.g., $\alpha = 3$ then $\mathrm{Id}_{1,2} = c_0(\mathrm{Id})$ while $\mathrm{Id} = \mathrm{Id}_{01} \cap \mathrm{Id}_{12}$, $\mathsf{RCA}_\alpha = \mathbf{S}\,\mathbf{P}\{\,\mathcal{R}el_\alpha(U) \mid U$ is a set$\,\}$.

G.8.3 Theorem (Tarski) RCA_α *is an arithmetical variety. The equational theory of* RCA_α *is recursively enumerable, but it is undecidable if* $\alpha > 2$.

So far, the greatest elements of our algebras were Cartesian spaces, i.e., of the form U^α (both in the cases of RRA's and RCA's). However, this restriction is not always convenient (cf. e.g., Andréka et al. (1998), van Benthem (1996), Monk (2000), Henkin et al. (1981)). Removing this restriction motivates the definition of cylindric-relativized set algebras. Let $V \subseteq U^\alpha$ be an arbitrary α-ary relation. Then the ***algebra of subrelations*** of V is defined as

$$\mathcal{R}el(V) = \big(\mathcal{P}(V), c_i^V, \mathrm{Id}_{ij}^V \mid i, j < \alpha\big)$$

where $c_i^V(R) = V \cap c_i(R)$ and $Id_{ij}^V = V \cap \mathrm{Id}_{ij}$. The class of α-ary ***cylindric-relativized set algebras*** is defined as

$$\mathsf{Crs}_\alpha = \mathrm{S}\{\,\mathcal{R}el(V) \mid V \subseteq U^\alpha \text{ for some set } U\,\}.$$

The finite algebra part of the next theorem is the result of a joint work with Hajnal Andréka and Ian Hodkinson.

G.8.4 Theorem (Németi) *Let $\alpha \neq 1$. Then Crs_α is an arithmetical variety. The equational theory of Crs_α is decidable. A finite Crs_n is isomorphic to one with finite greatest element.*

It is natural to ask whether any one of the distinguished kinds RRA, RCA_α, Crs_α of algebras of relations is axiomatizable by a finite set of equations. If $\alpha \geq \omega$, then having a finite set of axioms is impossible because there are infinitely many basic operations, but we still could hope for a finite scheme of equations like the scheme $c_i\,\mathrm{Id}_{ij} = 1$, for all $i, j < \alpha$.

G.8.5 Theorem (Monk, Monk, Németi, Jónsson, Andréka)
Assume $\alpha > 2$. None of the varieties RRA, RCA_α, Crs_α *is axiomatizable by a finite scheme of equations. None of* RRA *or* RCA_α *is axiomatizable by a scheme Σ of universally quantified formulas such that Σ involves only finitely many variables.*

The negative result above motivates the definition of the ***finitely axiomatizable approximations*** RA and CA_α of RRA and RCA_α. The axioms for RA are **(R1)** – **(R3)** below.

(R1) The Boolean axioms; and the operations $\circ, ^{-1}$ are "$\cup$"-distributive, i.e., they commute with the Boolean join "$\cup$".

(R2) $\circ, ^{-1}, \mathrm{Id}$ form an involuted monoid , where an ***involuted monoid*** is a monoid with an extra unary operation $^{-1}$ satisfying the two equations $x^{-1-1} = x^{-1}$, $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.

(R3) $x^{-1} \circ -(x \circ y) \leq -y$.

The axioms for CA_α are **(E1)** – **(E5)** below.

(E1) The Boolean axioms.

(E2) The c_i's are commuting complemented $\cup$-distributive closure operations (e.g., $c_i c_j x = c_j c_i x$, $c_i - c_i x = -c_i x$ etc).

(E3) $\mathrm{Id}_{ii} = 1$ and $\mathrm{Id}_{ij} = \mathrm{Id}_{ji}$ (i.e., notational trivialities)

(E4) $\mathrm{Id}_{ik} = c_j(\mathrm{Id}_{ij} \cap \mathrm{Id}_{jk})$ if $j \notin \{i, k\}$.

(E5) $x \leq \mathrm{Id}_{ij} \Rightarrow c_i(x) \cap \mathrm{Id}_{ij} = x$.

Clearly, $\mathsf{RA} \supseteq \mathsf{RRA}$ and $\mathsf{CA}_\alpha \supseteq \mathsf{RCA}_\alpha$. CA_1's are also called ***monadic algebras***. Both approximations RA and CA_α were introduced by Tarski. In some sense, RA is close to RRA and CA_α is close to RCA_α. However, it is hard to make it precise what we mean by *close* here. It is possible to introduce natural properties such that

$$\mathsf{RA} \cap \text{“property”} \subsetneq \mathsf{RRA} \quad \text{and} \quad \mathsf{CA}_\alpha \cap \text{“property”} \subsetneq \mathsf{RCA}_\alpha.$$

However, one can replace RA by a bigger class RA^- and CA_α with CA_α^- such that all the above style representation theorems remain true. Using the well established connections between logic and algebraic logic, one can argue that the axioms for RA and CA_α are optimal in some sense. E.g., the CA_α axioms correspond to one of the usual axiomatizations of first order logic and most of the equations separating RCA_α from CA_α would look strange to the logician as a possible extra axiom (unless he is trying to axiomatize the finite-variable fragments L_n of first order logic).

CA_α's correspond to first order logic L_α with equality. If we *algebraize* the same logic, but without equality, we obtain ***substitution-cylindrification algebras*** which are obtained from CA_α by throwing away the constants Id_{ij} and replacing them with the term-functions $s^i_j(x) = c_i(\mathrm{Id}_{ij} \cap x)$. ***Quasi-polyadic algebras*** (of Halmos) are almost the same as these, cf. Henkin et al. (1985) for both kinds of algebras.

Connections with logic are in Tarski and Givant (1987), Andréka et al. (2001), Henkin et al. (1985), Németi (1991) except for new kinds of recent applications of Crs_α-theory to the finite-variable fragments, finite model theory, the bounded fragments, and the guarded fragment, cf. e.g., Hoogland and Marx (2001), Andréka et al. (1998), van Benthem (1996). Cf. also Craig (1974), Henkin et al. (1971, 1985).

The negative result Theorem G.8.5 gave rise to the Finitization Problem which asks whether we could define our algebras of relations in such a way that they would form a finitely axiomatizable variety. There has been extensive research work on this problem recently, cf. e.g., Németi and Sain (2000) for further references.

Algebras of relations have been extensively applied in computer science, AI, linguistics and other areas, cf. e.g., Bergman et al. (1990), Marx et al. (1996), van Benthem (1996).

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G.9 Partial Algebras

by Peter Burmeister in Darmstadt, Germany

Introduction

Quite often (e.g., in Computer Science, but also for the multiplicative inverse in fields) the operations in an "algebra" are not everywhere defined. Moreover, quite often constructions for "*total algebras*" make use of ***partial algebras*** and some general construction principles like *universal solutions*. For such "*partial algebras*" , a highly developed theory has been worked out. This *theory of partial algebras* lies in between those of *total algebras* and *relational systems* (see Section G.8). From *total algebras* it inherits in particular the concepts of *terms*, *direct products* (with the structure defined componentwise whenever possible in *all* components), *closed subsets* (and in connection with them *(closed) subalgebras* as the partial algebras obtained by restricting the structure to closed subsets, and the concept of *generation*). From *relational systems* it inherits the wealth of possible concepts, since partial algebras can model relational systems (cf. Burmeister 1986, 13.4.2). Moreover, *many-sorted (partial) algebras* can easily be considered as partial algebras on the disjoint union of the carriers of the different sorts, and their homomorphisms then have just to be compatible with the canonical homomorphisms into the set of sorts with the specification of the many-sorted (partial) operations as fundamental partial operations (cf. Burmeister 1986). Here we can only introduce some of the basic concepts and applications of a language for partial algebras.

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