A marriage of groups and Boolean algebras.
In memory of Steven R. Givant

Andréka, H. and Németi, I.

Relation Algebras

Already in his 1941 article, Tarski remarked that the theory of relation algebras seemed to be a kind of union of the theories of Boolean algebras and of groups. The penultimate theorem we shall discuss provides an explanation of this connection.

RA = Groups + BA

GROUPS

Group: (A, •, 1', $^{-1}$)

• binary operation

a•b in A, for all a,b

Invertible monoid

Brandt groupoid: (A, •, I, $^{-1}$)

• partial binary operation

a•b in A, for some a,b

Invertible monoid

Polygroupoid: (A, •, I, $^{-1}$)

• many-valued binary operation

a•b subset of A, for all a,b

Invertible monoid

Group: (A, •, 1', $^{-1}$)

- binary operation

a•b in A, for all a,b

Invertible monoid

- associative: a(bc)=(ab)c

1' is identity: a1' = 1'a =a

$^{-1}$ is inverse: $aa^{-1} = a^{-1}a = 1'$

Brandt groupoid: (A, •, I, $^{-1}$)

- partial binary operation

a•b in A, for some a,b

Invertible monoid

- associative: a(bc)=(ab)c if ab, bc exist

e in I is identity: ae = a, ea =a, if exist

$^{-1}$ is inverse: $aa^{-1}a=a$, and $aa^{-1}a$ exists

Polygroupoid: (A, •, I, $^{-1}$)

- multivalued binary operation

a•b subset of A, for all a,b

Invertible monoid

- associative: a(bc)=(ab)c

I is set of identities: aI = Ia =a

$^{-1}$ is inverse: multivalued version

Polygroupoid: (A, •, I, $^{-1}$)

• multivalued binary operation

a•b subset of A, for all a,b

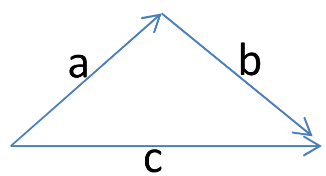
Invertible monoid

• associative: a(bc)=(ab)c complex multiplication

I is identity: aI=Ia=a

$^{-1}$ is inverse:

a in bc iff b in ac^{-1} iff c in b^{-1}a .



Complex multiplication:

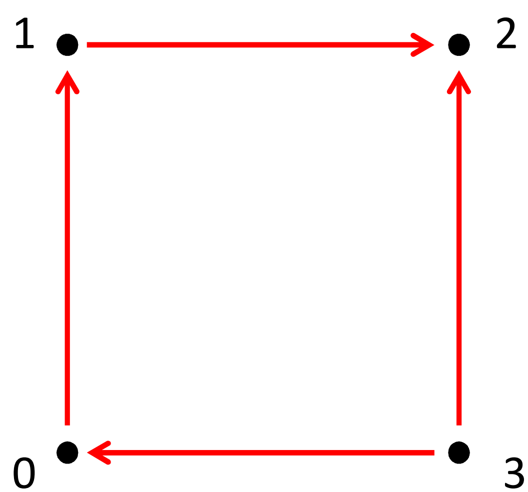
XY = unionof { ab : a in X, b in Y}

GROUPS

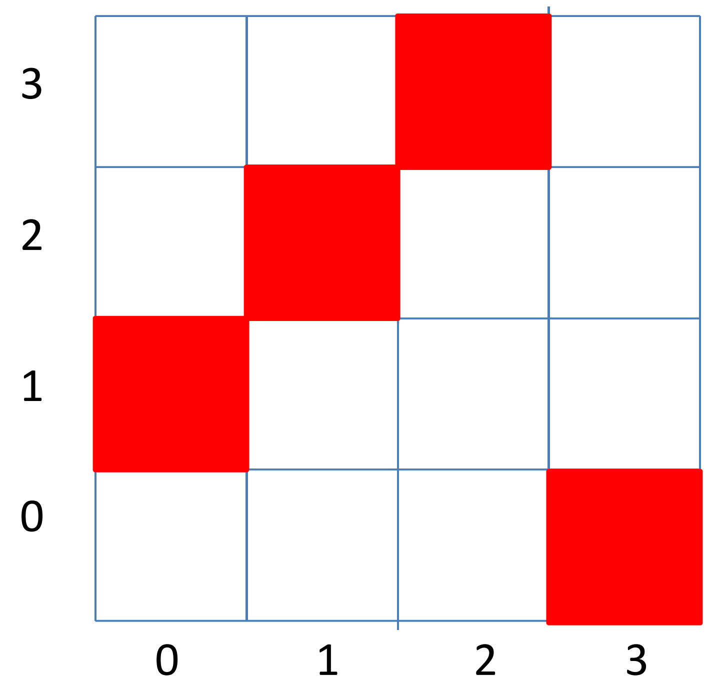
Cayley representation

Group: (A, •, 1′, $^{-1}$)
• binary operation

Cayley representation:
A is set of permutations on a set
Composition, identity map, inverse



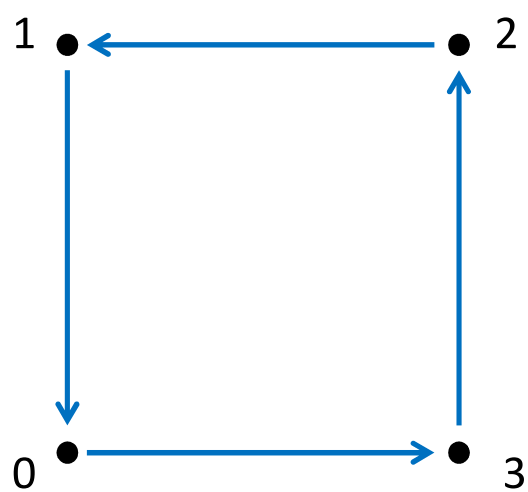
Suc = { (0,1), (1,2), (2,3), (3,0) }



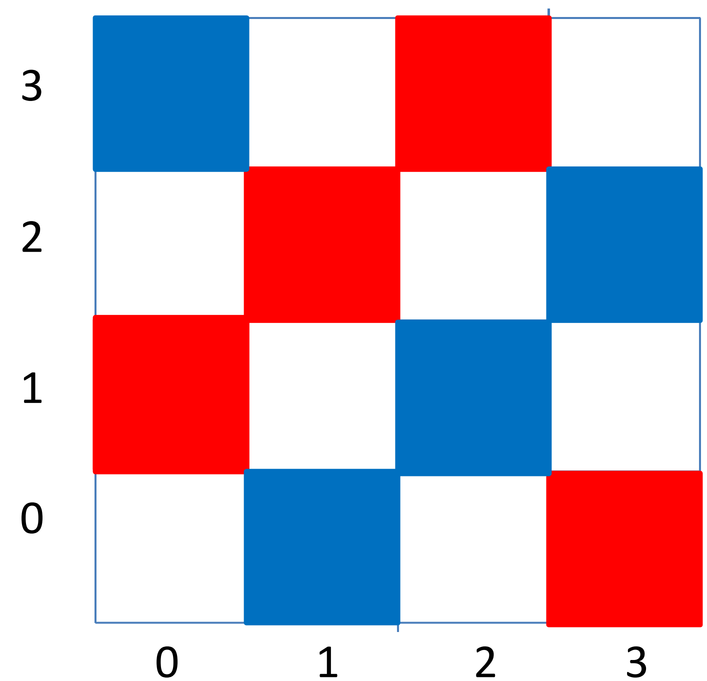
Inverse

Group: (A, •, 1', $^{-1}$)
• binary operation

Cayley representation:
A is set of permutations on a set
Composition, identity map, inverse



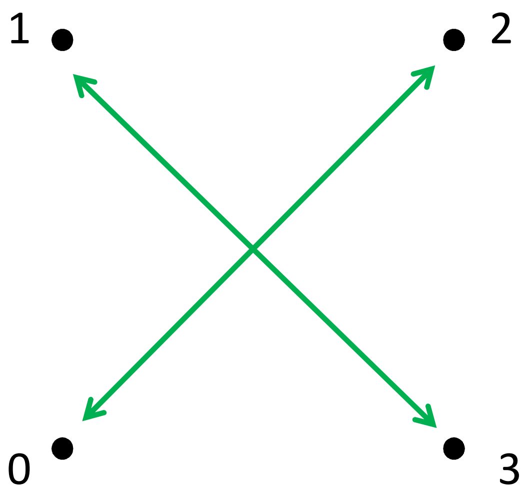
Suc = { (0,1), (1,2), (2,3), (3,0) }
Suc^{-1} = { (1,0), (2,1), (3,2), (0,3) }



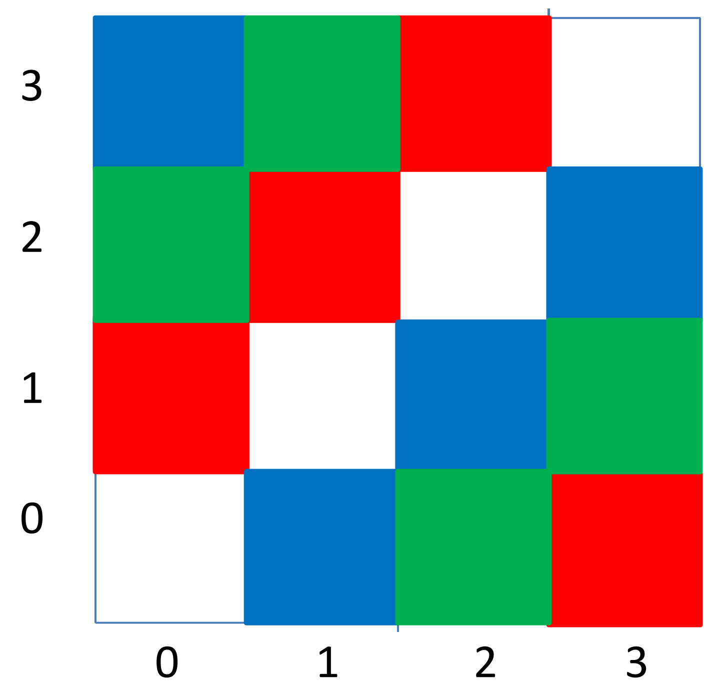
Composition, identity

Group: (A, •, 1′, $^{-1}$)
• binary operation

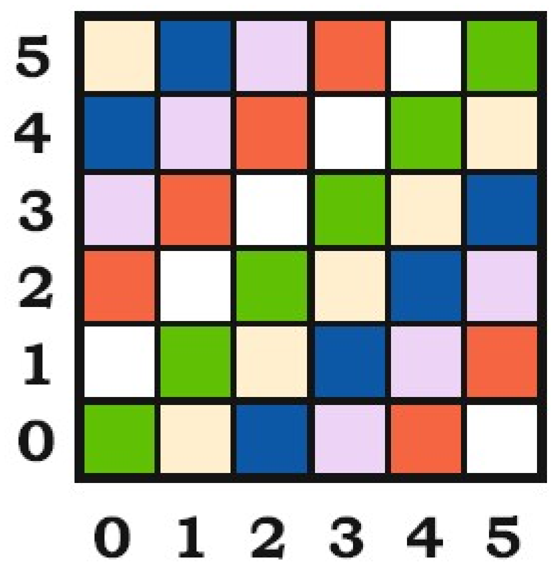
Cayley representation:
A is set of permutations on a set
Composition, identity map, inverse



Suc = { (0,1), (1,2), (2,3), (3,0) }
Suc^{-1} = { (1,0), (2,1), (3,2), (0,3) }
Suc^2 = { (0,2), (1,3), (2,0), (3,1) }



Cayley representation of Z_6

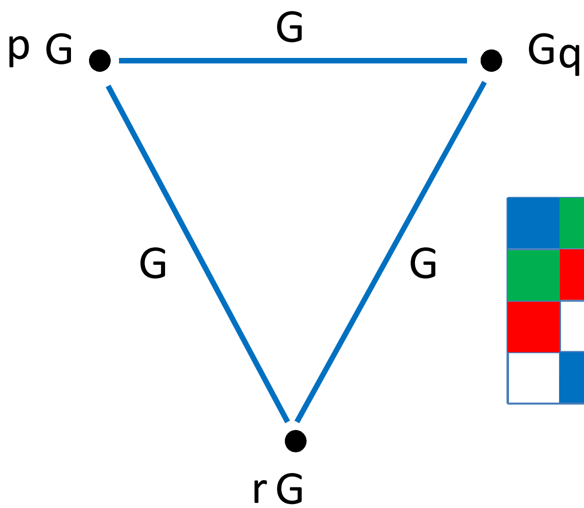


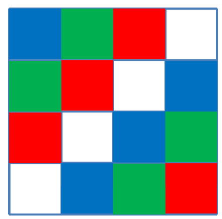
BRANDT GROUPOIDS

Brandt groupoid structure

Brandt groupoid: (A, •, I, $^{-1}$)

- partial binary operation





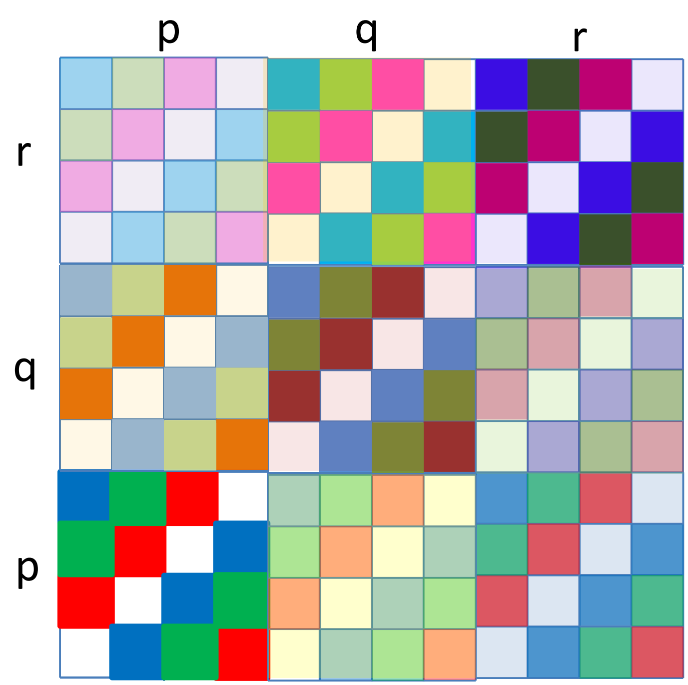
A = { (i,g,j) : i,j in I and g in G }

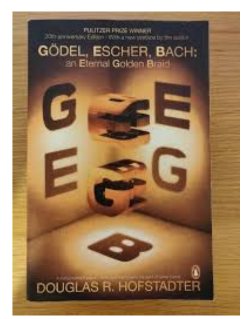
(i,g,j) •(j,h,k) = (i, gh, k)

multiplication in G

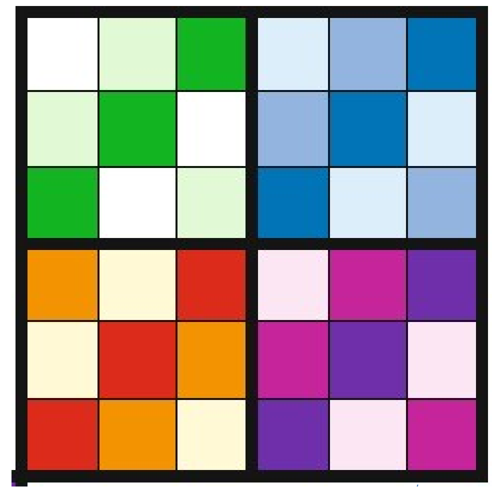
Structure:

Copies of a group on the full graph on I

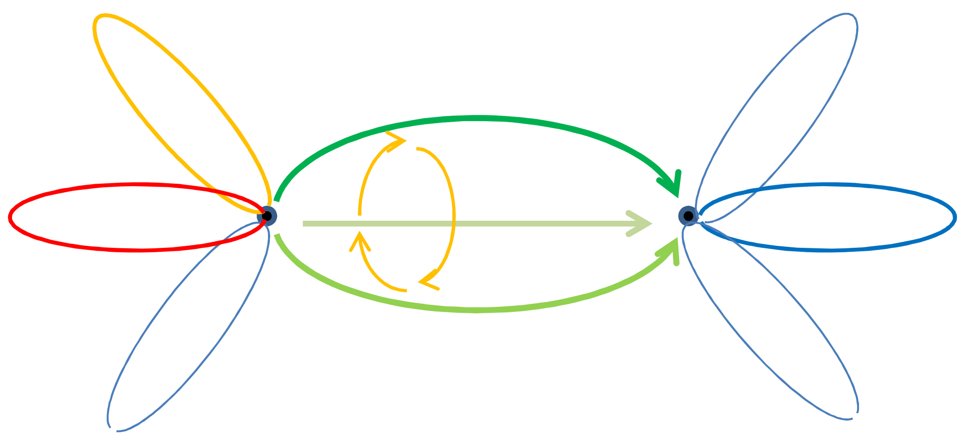




Brandt groupoid with I={p,q} and G=Z_3



Category



POLYGROUPOIDS

Polygroupoid structure

Polygroupoid: (A, •, I, $^{-1}$)

• many-valued binary operation

Structure:

Theorem (Comer, 1983)
Polygroupoids are exactly atom-structures of atomic relation algebras.

RA = SCm PG.

Representation of a polygroupoid:
Elements of A with binary relations
• as composition of binary relations
I as identity relation
$^{-1}$ as converse of a relation

Complete representations of RA: determined by polygroupoid
Incomplete representations of RA: determined by BA structure
Subject of second part of the talk

STORY

- Representation theorem of Jónsson and Tarski 1952
- Discovery of Roger Maddux 1991
- Idea of Steven Givant 1991
- Vision of Steve Comer 1983

Loop polygroupoids

a is a loop if there is x in I such that xax=a.

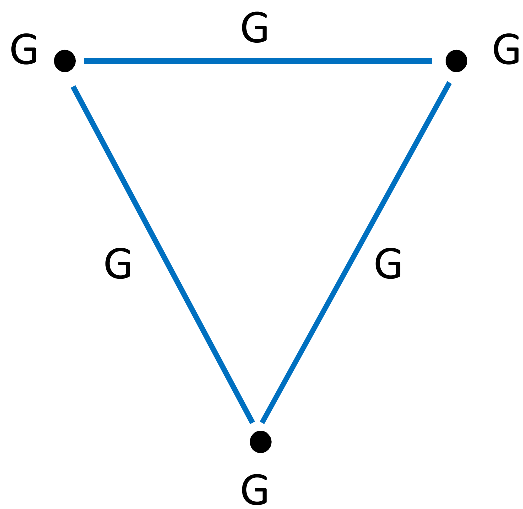
A polygroupoid is a loop-polygroupoid iff the product on loops is a partial function.
LPG

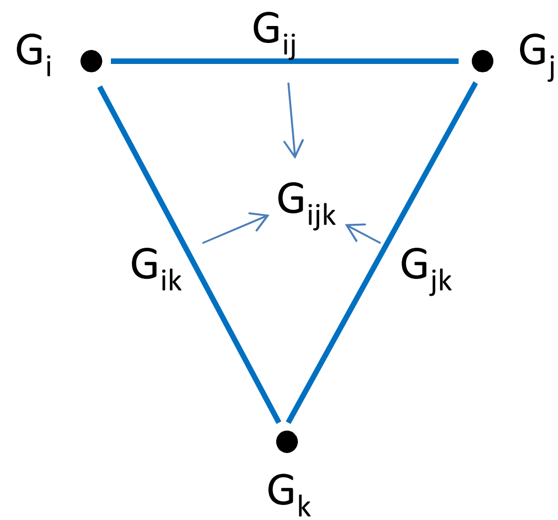
A relation algebra is measurable iff the identity is the sum of atoms, and for each subidentity atom x the square x;1;x is the supremum of functional elements.
MRA

The structure of LPGs is very similar to BGs:

Groups on the vertices, but different groups possible,
Factor groups on the edges.
Plus a common factor group in the middle of each triangle.

LOOP POLYGROUPOIDS





A = { (i,g,j) : i,j in I and g in G }

(i,g,j) • (j,h,k) = (i, gh, k)

multiplication in G

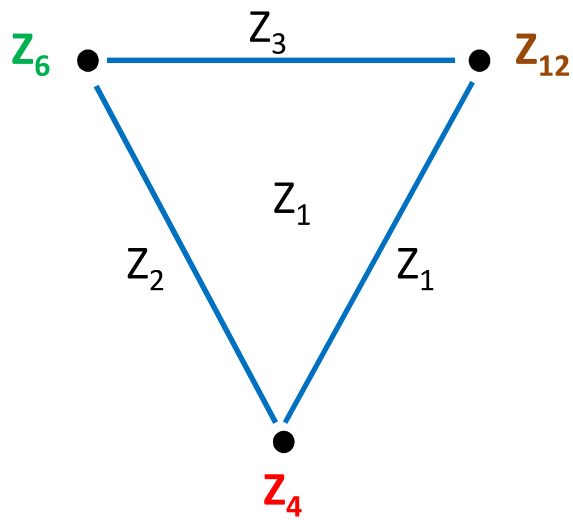
A = { (i,g,j) : i,j in I and g in G_{ij} }

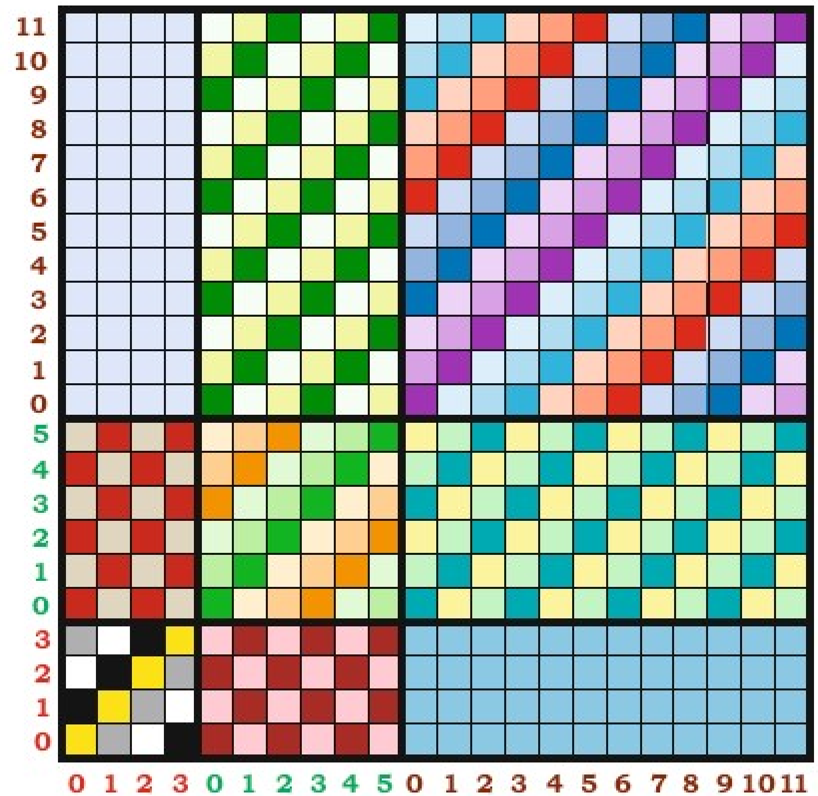
(i,g,j) • (j,h,k) = { (i,q,k) : $\pi q = \pi g \cdot \pi h$ }

multiplication in G_{ijk}

LOOP POLYGROUPOIDS

REPRESENTABLE EXAMPLES





Multicategory (on blackboard)

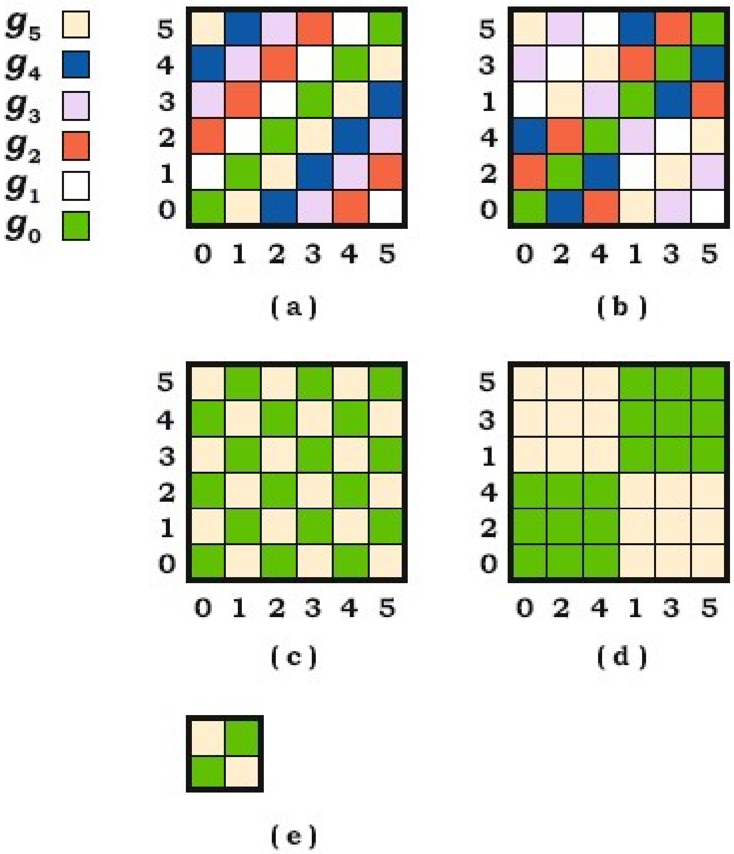
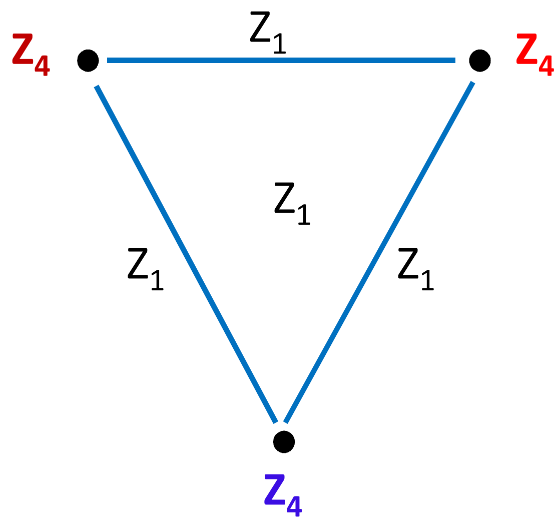
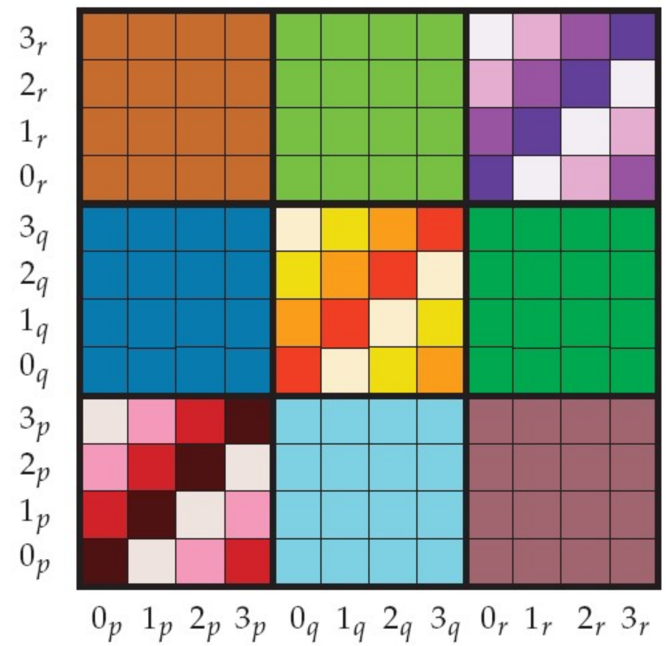


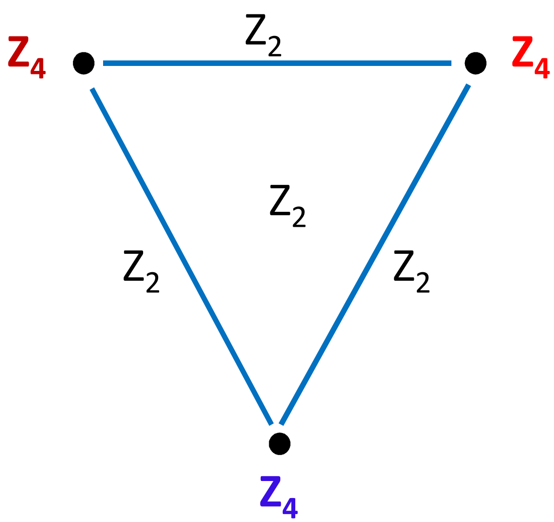
Fig. 10.3 (a) The Cayley representation of the group complex algebra $\mathfrak{C}$. (b) The same representation with a reordered base set to show that the representation of ι is an equivalence relation on G with equivalence classes $\{0,2,4\}$ and $\{1,3,5\}$. (c) The induced representation of $\mathfrak{C}/\iota$. (d) The induced representation with a reordered base set. (e) The contracted representation of $\mathfrak{C}/\iota$.

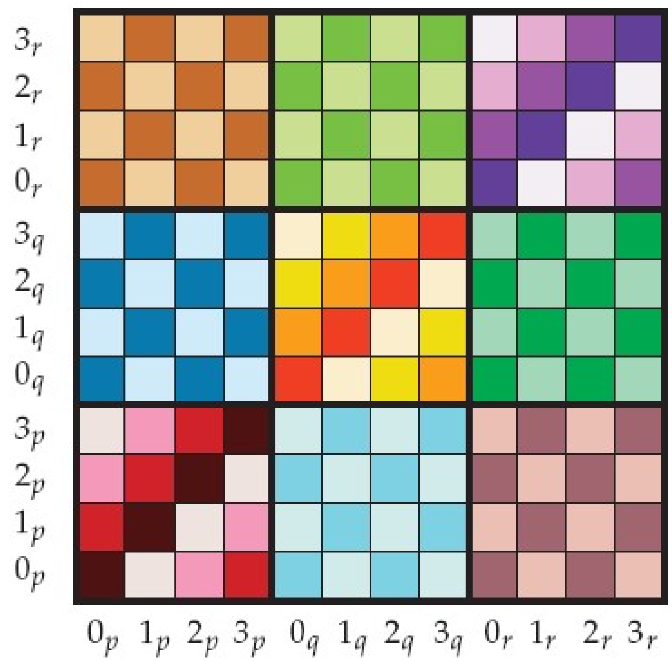
Loop polygroupoid with I={p,q,r} and G_x=Z4 and G_{xy}=Z1



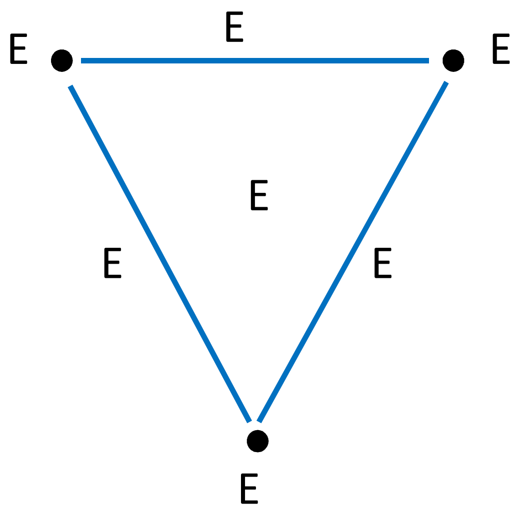


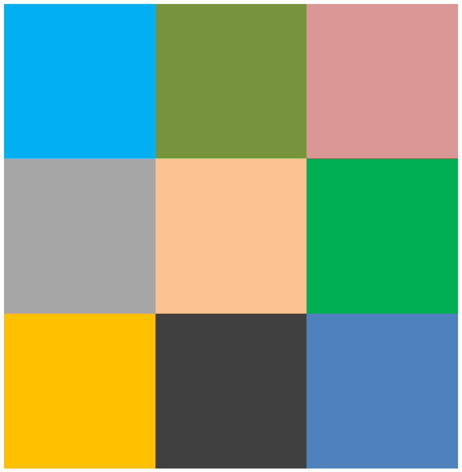
Loop polygroupoid with I={p,q,r} and G_x=Z4 and G_{xy}=Z2





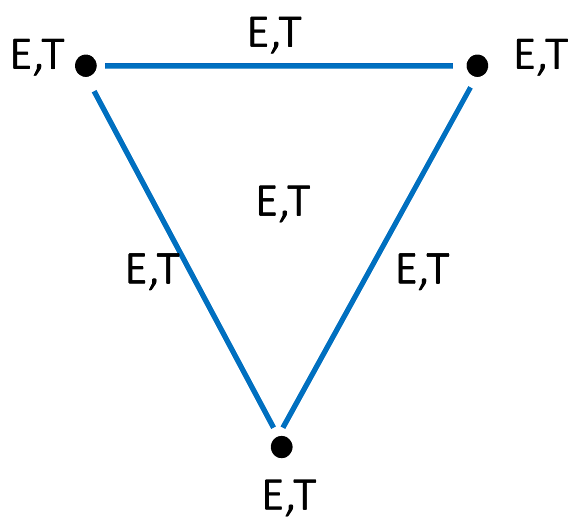
Point dense RA: (A, •, I, $^{-1}$)
all the groups are one-element

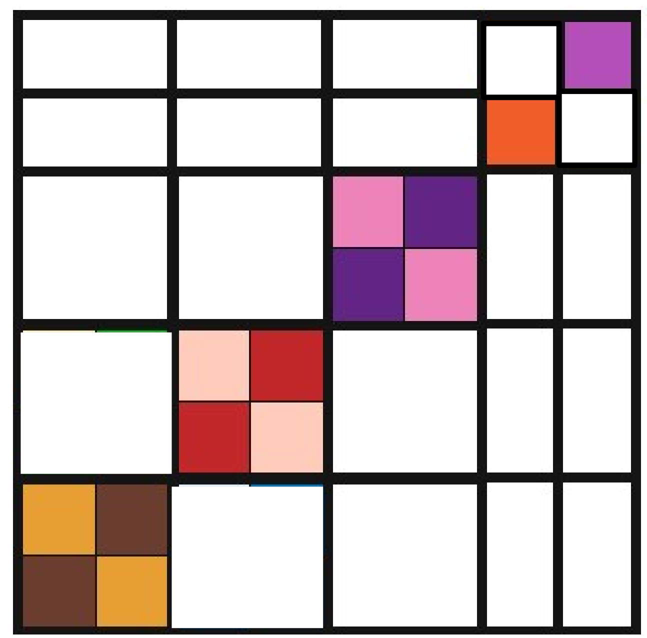




Loop polygroupoid with I={p,q,r} and $G_x=Z_1$

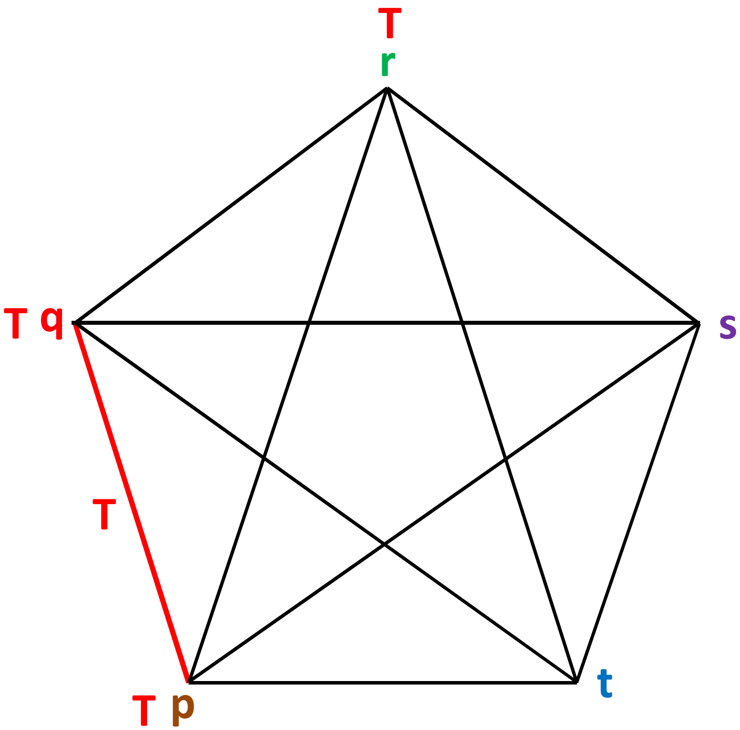
Pair dense RA: (A, •, I, $^{-1}$)
all the groups are one- or two-element

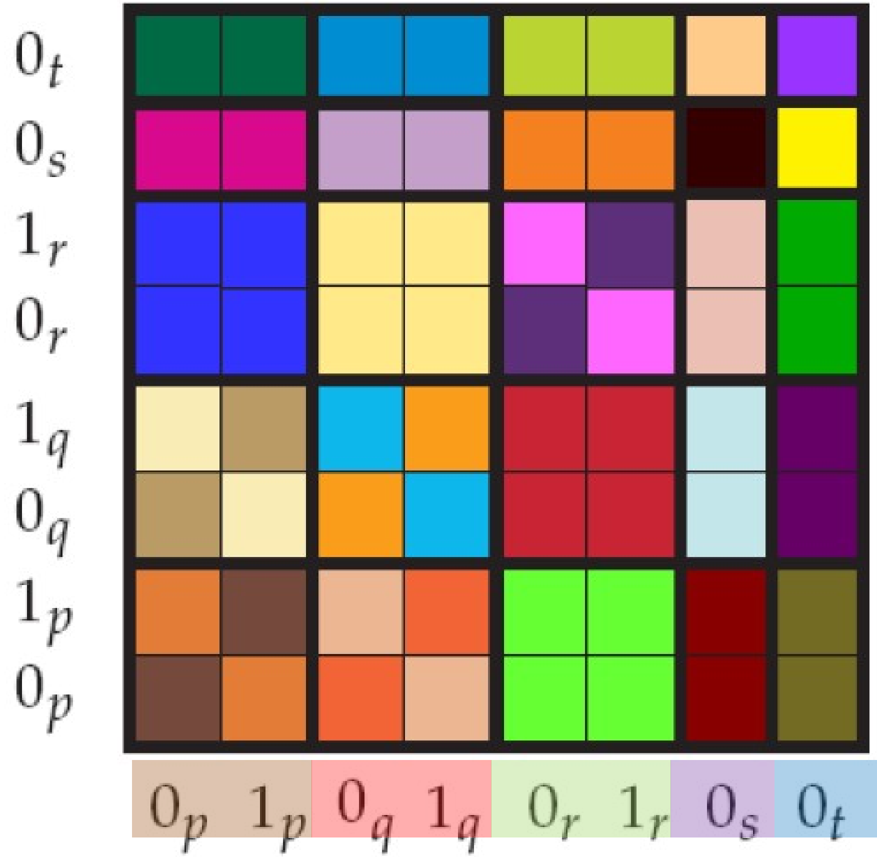




Loop polygroupoid with I={p,q,r,s,t} and $G_x=Z_2$ or $G_x=Z_1$

Pair dense RA: (A, •, I, $^{-1}$)
all the groups are one- or two-element



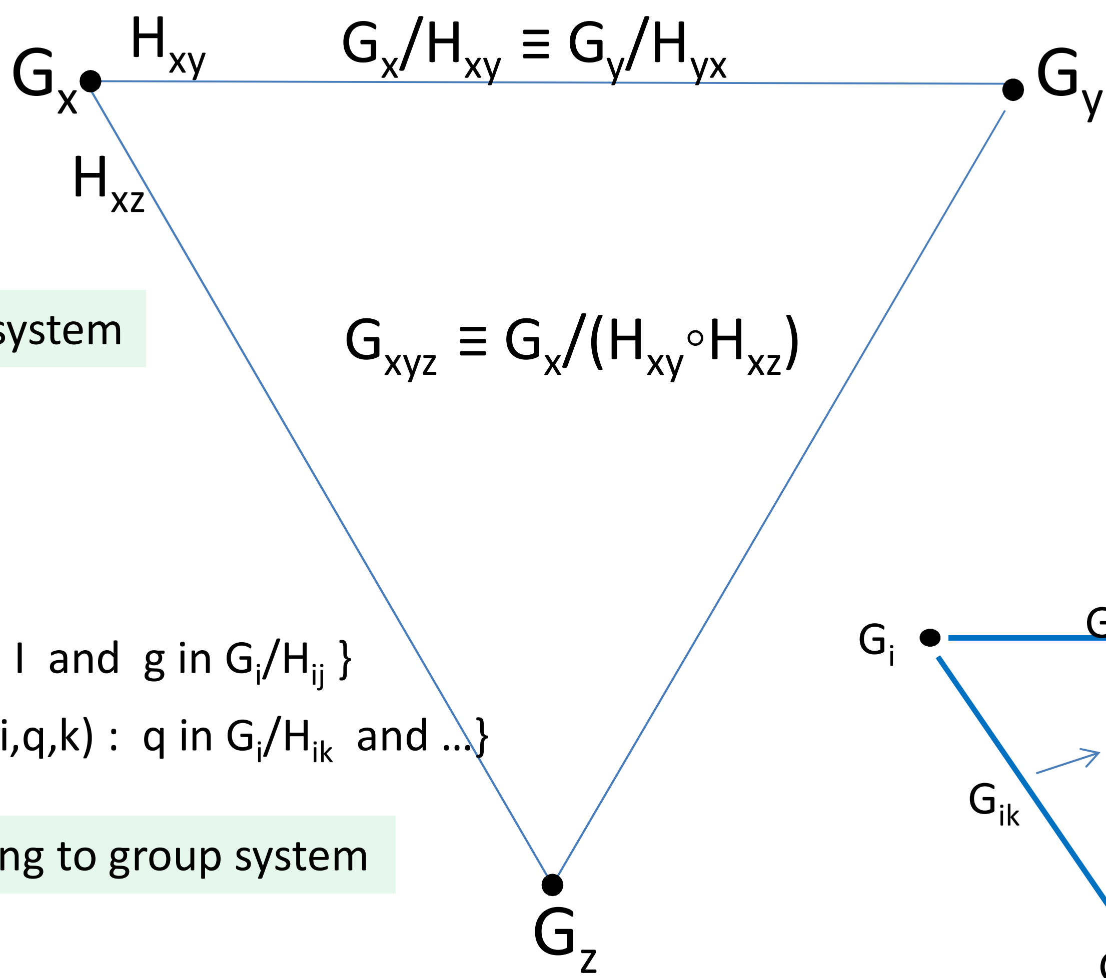


Loop polygroupoid with I={p,q,r,s,t} and $G_p=G_q=G_r=G_{pq}=Z_2$, all the others are Z_1

LPG structure: Group system

G_{xy} is a common factor group of G_x and G_y

⇓

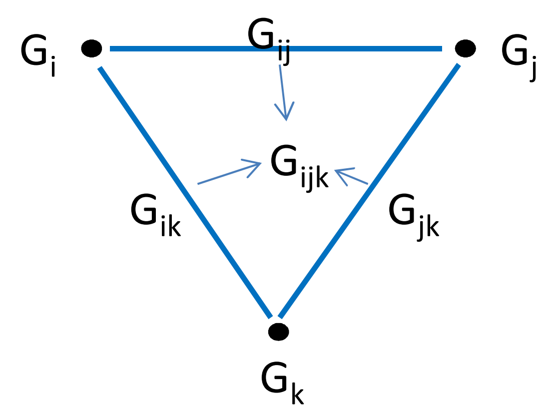


Group system

A = { (i,g,j) : i,j in I and g in G_i/H_{ij} }

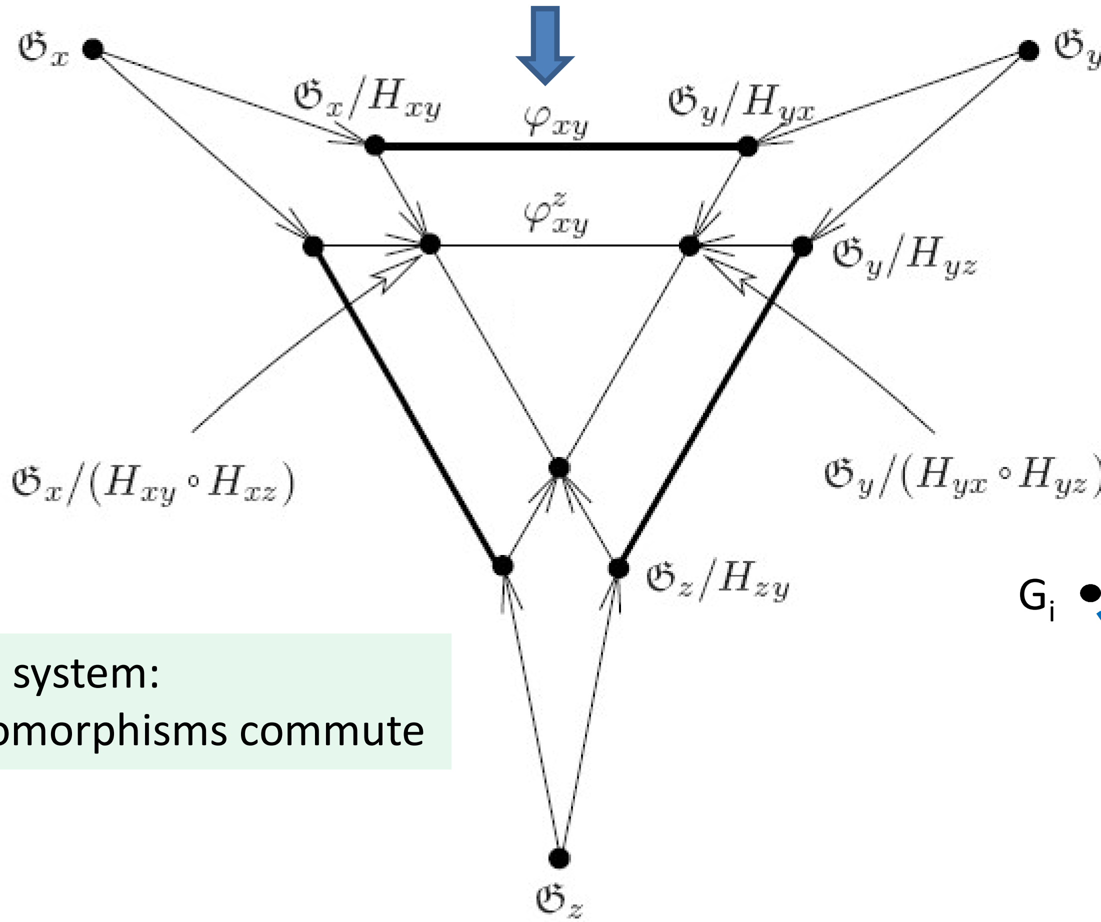
(i,g,j) •(j,h,k) = { (i,q,k) : q in G_i/H_{ik} and …}

structure belonging to group system

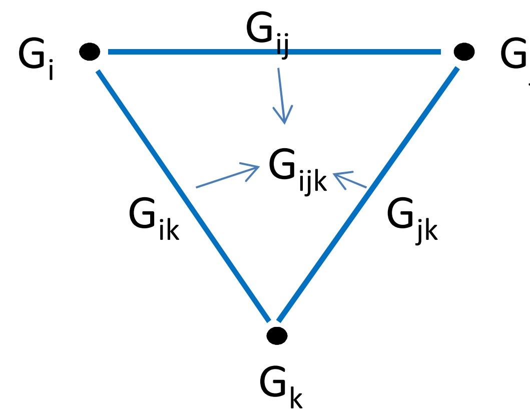


Loop polygroupoid structure

G_{xy} is a common factor group of G_x and G_y



Group system:
the isomorphisms commute



Theorem (G): Representable LPGs are exactly the structures belonging to group systems.

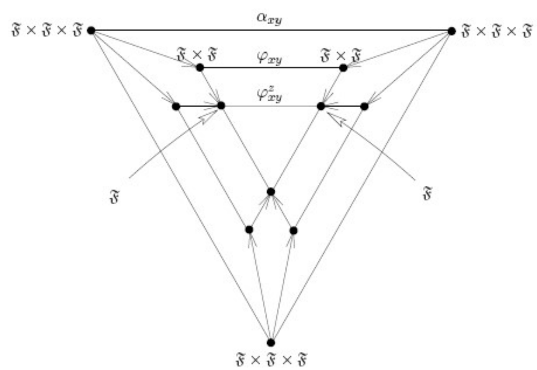
Problem: are all LPGs representable?

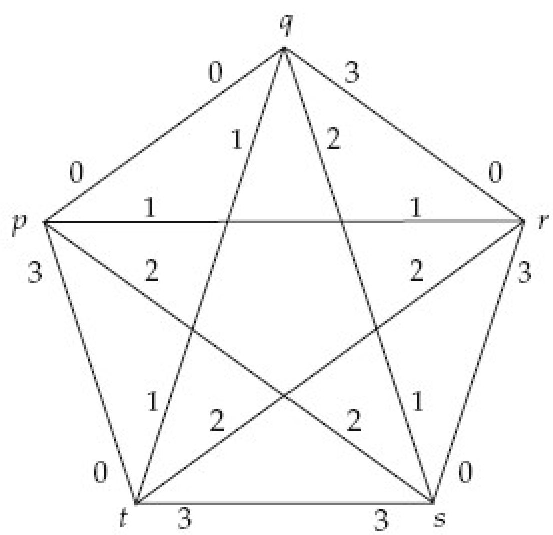
Partial results (AG):
LPGs with I having less than 5 elements are representable.
LPGs with all groups direct products of at most two finite cyclic groups are representable.
LPGs with less than “three levels” are representable.

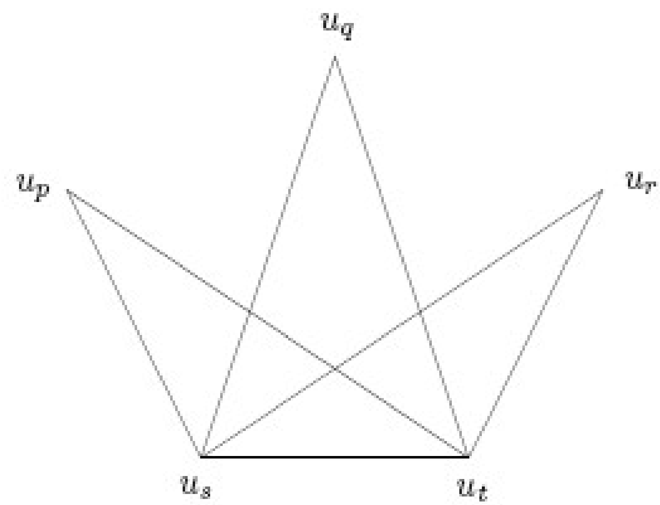
Surprise (AG):
There is a nonrepresentable LPG with I having 5 elements,
the groups on the vertices Z2xZ2xZ2, the groups on the edges Z2xZ2, and
the group in the middle Z2.

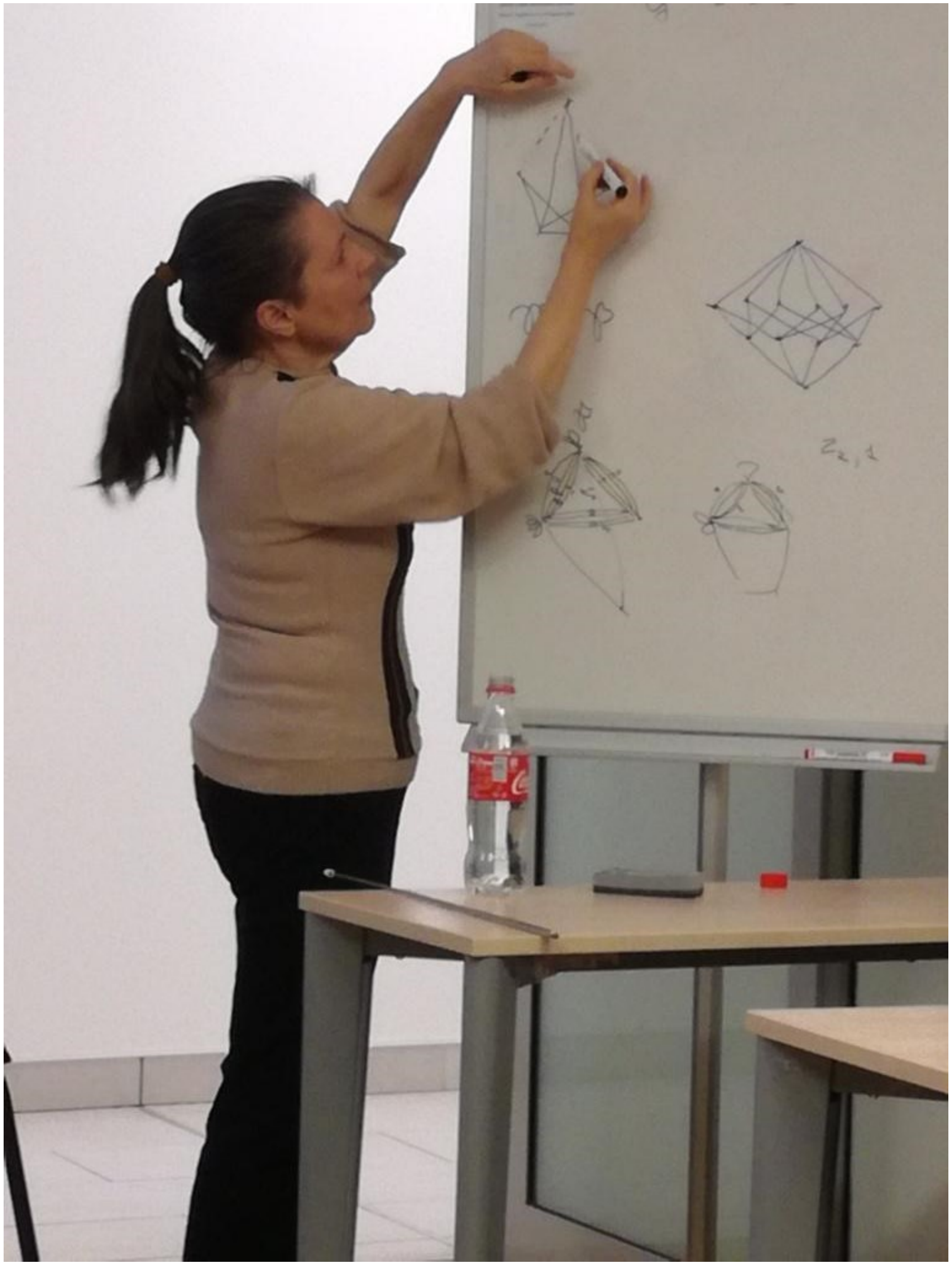
NONREPRESENTABLE LPG

On blackboard





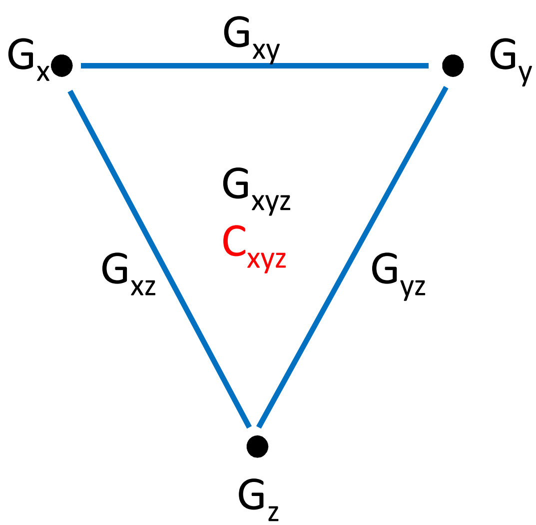




Loop polygroupoid structure in general

Loop polygroupoid: $(A, \bullet, I, ^{-1})$

- $\bullet$ partial binary operation on loops



$A = \{ (x,g,y) : x,y \text{ in } I \text{ and } g \text{ in } G_{xy} \}$

$(x,g,y) \bullet (y,h,z) = \{ (x, ghC_{xyz}, z) \}$

element of factor group G_{xyz} of Gx

C_{xyz} is called the shift in the triangle xyz

Conditions on next slide

Structure:

Groups on the vertices, factor groups on the edges of the full graph on I, a group with a shift in the middle of each triangle

Representation Theorem for LPG (AG): LPGs are exactly the structures belonging to shifted group systems.

(i) φ_{xx} *is the identity function on* $G_x/\{e_x\}$*, where* e_x *is the identity element of* G_x.

(ii) φ_{yx} *is the inverse of* φ_{xy}*. In particular,* $K_{xy} = H_{yx}$.

(iii) $\varphi_{xy}[H_{xz}/H_{xy}] = H_{yz}/H_{yx}$.

Assume that (iii) holds. Define $\varphi^z_{xy}(g/(H_{xy} \circ H_{xz})) = \varphi_{xy}(g/H_{xy}) \circ H_{yz}$.

(iv) $\varphi^z_{xy} \mid \varphi^x_{yz} = \tau(C_{xyz}) \mid \varphi^y_{xz}$.

(v) $C_{xyy} = H_{xy}$.

(vi) $\varphi_{xz}[C_{xyz}] = C^{-1}_{zyx}$.

(vii) $\varphi_{xy}[C_{xyz}] = C^{-1}_{zyx}$.

(viii) $C_{xyz} \circ C_{xzw} = \varphi_{yx}[C_{yzw} \circ H_{yx}] \circ C_{xyw}$.

Open Problems

OProblem1.
Are these all the nonrepresentable LPGs?

OProblem2.
Can each measurable RA be embedded into an atomic measurable RA?

OProblem3.
Are all representable measurable RAs completely representable?

The same for other structures, general systems theory