EPIMORPHISMS IN CYLINDRIC ALGEBRAS AND DEFINABILITY IN FINITE VARIABLE LOGIC

H. Andréka, S. D. Comer, J. X. Madarász, I. Németi, T. Sayed Ahmed

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Abstract . The main result gives a sufficient condition for a class K of finite dimensional cylindric algebras to have the property that not every epimorphism in K is surjective. In particular, not all epimorphisms are surjective in the classes CA_n of n-dimensional cylindric algebras and the class of representable algebras in CA_n for finite $n > 1$, solving Problem 10 of [28] for finite n. By a result of Németi, this shows that the Beth-definability property fails for the finite-variable fragments of first order logic as long as the number n of variables available is > 1 and we allow models of size $\geq n + 2$, but holds if we allow only models of size $\leq n + 1$. We also use our results in the present paper to prove that several results in the literature concerning injective algebras and definability of polyadic operations in CA_n are best possible. We raise several open problems.

§0. INTRODUCTION AND THE MAIN RESULTS

In algebra, the properties of epimorphisms (in the categorial sense) being surjective and the amalgamation property in a class of algebras are well investigated, see e.g. [1] and [37]. In algebraic logic these properties turn out to be the algebraic equivalents of Beth's definability property and Craig's interpolation property, respectively (of the logic under algebraization), see [27, sec 5.6 Thm 5.6.10] for the first equivalence and [57, Thm 1.2.8] for the second equivalence. We understand Beth's and Craig's properties of abstract (or general) logics in the abstract model theoretic sense, cf. Barwise-Feferman [13, p.32, Def 1.2.4], or [9, sec 6]. The equivalence result concerning Craig's property can be traced back to Daigneault [19] in the context of polyadic algebras. Pigozzi [57] is a milestone for working out such equivalences for cylindric algebras, an alternative equational formalism of first order logic. The corresponding algebraic question (amalgamation property of cylindric algebras) was largely settled by Comer [16] (the finite dimensional case) and Pigozzi [57] (the infinite dimensional case). The equivalence between amalgamation and interpolation is studied in more general contexts in [40],[41] and [42]. The equivalence result concerning Beth's property is due to Németi [51] and applies to all algebraizable logics. That part of the result which is relevant for the present paper is fully cited and proved as Thm 5.6.10 in [27, sec 5.6]. However, unlike Pigozzi [57], Németi [51] did not settle the corresponding algebraic question i.e. whether epimorphisms are surjective in (various classes of) cylindric algebras. This appears as open Problem 10 on p.310 of [28]. In the present paper we settle this algebraic question (Problem 10 of [28]) for the finite dimensional case. The infinite dimensional case is settled by Madarász [43], [44]. Then, by the quoted Thm 5.6.10 of [27], our result will imply failure of Beth's definability property for a large variety of first order logics with finitely many variables. A precursor of the present work is the manuscript [4] which contains the first proof of the fact that Beth's definability property fails in finite variable logic. It has been improved in various ways in the meantime by the new co-authors. The results of [4] were announced in [5].

To formulate our main result we need to recall some notation.

Throughout this paper n and μ denote cardinal numbers. Unless otherwise specified, n is always finite. Concerning the classes of cylindric algebras we deal with, we follow the standard terminology of the monographs [26] and [27]. In particular, CA_n is the class of cylindric algebras of dimension n and Cs_n is the class of cylindric set algebras of dimension n. The greatest element of a Cs_n is always a *Cartesian space* i.e. a set of the form ${}^nU$ for some set U, where ${}^nU$ denotes the set of all U-termed sequences of length n. This U is called the *base* of the algebra. ${}_\mu Cs_n$ is the class of those members of Cs_n which are of base of cardinality μ. Gs_n is the class of generalized cylindric set algebras as defined in [27]. The greatest element of a Gs_n is a disjoint union of Cartesian spaces each of dimension n. If the greatest element of a Gs_n is of the form $\cup_{i\in I}{}^nU_i$, then each U_i is called a *subbase* of the algebra. ${}_\mu Gs_n$ is the class of those members of Gs_n each subbase of which has cardinality μ. For the purposes of the present paper it is enough to know that an algebra is isomorphic to a Gs_n iff it is representable, which in turn means that it is isomorphic to a subdirect product of Cs_n's. Similarly, an algebra is isomorphic to a ${}_\mu Gs_n$ iff it is isomorphic to a subdirect product of ${}_\mu Cs_n$'s. It might be useful to recall from [27] that the classes CA_n, Gs_n and ${}_\mu Gs_n$ are varieties, up to isomorphism. In the following, $n < \omega$ means that n is finite. Our main result is the following:

Proposition 1 . *For $4 \le n+2 \le \mu$ and $n < \omega$, not all epimorphisms are surjective in the following classes: CA_n, Cs_n, Gs_n, ${}_\mu Cs_n$ and ${}_\mu Gs_n$.*

The result above is a corollary to (the stronger) Theorem 6 formulated and proved in section 2. It settles Problem 10 on p.310 of [28] for the finite dimensional case. The result complements that epimorphisms are surjective in the following classes:

(1) CA_n where $n \leq 1$ (see [16]).

(2) ${}_\mu Gs_n$ where $\mu \leq n+1 < \omega$ (see [17]).

(3) Crs_α for every ordinal α where Crs_α is the class of cylindric relativized set algebras, i.e. the ones with greatest element an arbitrary set of α-ary sequences (see [53]).

(4) Bo_α (Boolean algebras with operators having the same similarity type as CA_α) and their diagonal free reducts (see [53] and [60]).

The constructions developed in this paper to prove Proposition 1 will be used to show that several results (concerning injective CA_n's in the sense of [17] and definability of polyadic operations in cylindric algebras in [8]) are best possible. This is done in Theorems 6 and 8 in this paper. We should mention that these results are quoted in [17] from the precursor [4] of the present work, but the proofs appear in print for the first time in the present paper.

Organization. The layout of this paper is as follows. Section 1 contains the algebraic investigations proving our results. In section 2 we describe the logical consequences of the algebraic investigations in section 1. In the final section we review, and in the process comment on, related results and give some historical notes.

§1. ALGEBRAIC RESULTS AND THEIR PROOFS

In this section we state and prove our results in algebraic form. We start off by recalling the notation mostly used. This includes the notation in which we deviate from the fundamental monographs [26] and [27]. Otherwise our notation is in conformity with [26] and [27].

Notation .

(i) *The full cylindric set algebra with base U and dimension n is denoted by $\mathcal{A}(n,U)$. Full here means that the universe of $\mathcal{A}(n,U)$ is $\wp({}^nU)$, the power set of ${}^nU$. The operations of $\mathcal{A}(n,U)$ are the Boolean set operations of forming union of two subsets of ${}^nU$ and forming the complement w.r.t. ${}^nU$ of a subset of ${}^nU$, together with the unary operations of i-th cylindrifications C_i and the diagonal constants D_{ij}, for each $i,j<n$ defined as follows*

$C_i(X) = \{s(i|u) : s \in X, u \in U\}$ *where $s(i|u)$ denotes the sequence we obtain from s by changing its i-th member to be u, and*

$D_{ij} = \{s \in {}^nU : s_i = s_j\}$.

(ii) *For a given CA_n, we let d_n, or even sometimes simply d, stand for the principal diagonal element that is*

$$d_n = d = \prod\{d_{kl} : k, l < n\}.$$

We let $\bar{d}$ stand for the principal co-diagonal that is

$$\bar{d} = \prod\{-d_{kl} : k, l < n, k \neq l\}.$$

(iii) *We use natural numbers in the von Neumann sense, i.e. 0 is the empty set and $n+1 = n \cup \{n\}$, hence $n+1 = \{0, 1, \ldots, n\}$. A sequence $s \in {}^nU$ is considered to be a function mapping n to U such that if $s = \langle s_0, \ldots, s_{n-1}\rangle$ then $s(i) = s_i$ for all $i \in n$.*

(iv) *Let σ be a permutation of the base U of $\mathcal{A}(n, U)$. Then σ induces an automorphism on $\mathcal{A}(n, U)$; which we denote by $\bar{\sigma}$, or sometimes also by σ, when no confusion is likely to ensue. More specifically for $X \subseteq {}^nU$, and σ a permutation of U, we let*

$$\sigma(X) = \{\sigma \circ y : y \in X\}.$$

(v) *The symmetric group on a set U is denoted by S_U. In particular, the universe of S_U is the set of all permutations of U.*

(vi) *$Aut(\mathcal{A})$ denotes the set of all automorphisms of the algebra $\mathcal{A}$. For $\mathcal{B} \subseteq \mathcal{A}$, we let $G^*(\mathcal{B}, \mathcal{A})$, or simply $G^*(\mathcal{B})$ when the big algebra is clear from context, be the subgroup of $Aut(\mathcal{A})$, fixing $\mathcal{B}$ elementwise. That is*

$$G^*(\mathcal{B}) = \{\sigma \in Aut(\mathcal{A}) : \sigma(b) = b \text{ for all } b \in B\}.$$

We often refer to $G^*(\mathcal{B})$ as the Galois group of $\mathcal{B}$.

(vii) *Id_X denotes the identity map with domain X; the subscript X will be dropped when the domain is clear. $|X|$ denotes the cardinality of X. $Rg(f)$ for a given function f denotes the range of f and $f \restriction X$ denotes the restriction of f to X. The composition of the functions f and g is defined so that the righthand-most function acts first, that is $(f \circ g)(x) = f(g(x))$ whenever $g(x) \in Rg(f)$.*

To prove Proposition 1, we need several lemmas. From now on, unless otherwise specified, it is assumed that n is a natural number > 1, U is a set, and $H \subseteq U$. For $x \in H$, we let

$$a_x = n \times \{x\} = \{\langle x : i < n\rangle\}$$

be the atom of $\mathcal{A}(n, U)$ corresponding to the function with constant value x. Let

$$\mathcal{A}(n, U, H)$$

denote the subalgebra of $\mathcal{A}(n, U)$ generated by the set $\{a_x : x \in H\}$. In particular, this subalgebra contains the element ${}^nH$ if H is finite (since n is finite, too). Let K be a class of algebras and let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism between elements of K. We say that h is an *epimorphism* in K if it has the right-cancellation property, i.e. if $f \circ h = g \circ h$ implies $g = h$ for any $\mathcal{C} \in \mathcal{K}$ and homomorphisms $f, g : \mathcal{B} \to \mathcal{C}$. Now we are ready to formulate and prove our first lemma.

Lemma 1 . *Suppose $n < \omega$, H is finite, and $\mathcal{B} \subseteq \mathcal{A}(n, U, H)$ such that ${}^nH \in \mathcal{B}$. Let $\mathcal{A}$ denote $\mathcal{A}(n, U, H)$. Then $Id_B : \mathcal{B} \to \mathcal{A}(n, U, H)$ is an epimorphism in CA_n if and only if $G^*(\mathcal{B}, \mathcal{A}) = \{Id_A\}$ (i.e. iff Id_A is the only automorphism of $\mathcal{A}$ fixing all elements of $\mathcal{B}$).*

Proof. The "only if" part is obvious. Now we prove the "if" part. Assume that $G^*(\mathcal{B}) = \{Id\}$. Let $\mathcal{C} \in CA_n$ and suppose that $f : \mathcal{A}(n, U, H) \to \mathcal{C}$ and $g : \mathcal{A}(n, U, H) \to \mathcal{C}$ are given homomorphisms that agree on $\mathcal{B}$. We now show that $f = g$. Since $\mathcal{A}(n, U, H)$ is simple (by $n < \omega$) f and g are either both equal to the zero map, or are both injective. We assume the latter, for else there is nothing more to prove. We first show that

(*) $Rg(f) = Rg(g)$.

Let h denote ${}^nH$. Observe that $\sum\{a_x : x \in H\} = h \cdot d \in B$ and thus $\sum\{f(a_x) : x \in H\} = f(h \cdot d)$, and the same for g in place of f, since H is finite. Now each atom a_x is a rectangular element [1] and is below the principal diagonal. Since f is an injective homomorphism it follows that $f(a_x)$ is also rectangular and is below the principal diagonal. By [26, 1.10.13(ii)], we have then that $f(a_x)$ is an atom in $\mathcal{C}$ and $f(a_x) \leq f(h \cdot d)$. Similarly $g(a_x)$ is an atom in $\mathcal{C}$ that is below $g(h \cdot d)$. But $h \cdot d$ is in B and f and g agree on B, hence

$$f(h \cdot d) = g(h \cdot d).$$

Thus the atoms below each coincide, i.e. we have

$$\{f(a_x) : x \in H\} = \{g(a_x) : x \in H\}.$$

Now $(*)$ readily follows since $\mathcal{A}(n, U, H)$ is generated by $\{a_x : x \in H\}$. From $(*)$ it follows that $g^{-1} \circ f$ is an automorphism of $\mathcal{A}(n, U, H)$ that is the identity

[1] We recall from [26] that an element a in a CA_n is rectangular if $c_{(\Delta)}a \cdot c_{(\Gamma)}a = c_{(\Delta \cap \Gamma)}a$ for all $\Delta, \Gamma \subseteq n$.

on $\mathcal{B}$. But $G^*(\mathcal{B}) = \{Id\}$ by assumption, so $f = g$ which shows that the identity map Id_B is an epimorphism. ■

To formulate the next two lemmas we need the notion of two elements of a cylindric algebra always belonging to the same cylinders. Namely, X, Y in a CA_n are called *cylindrically equivalent* iff $c_i X = c_i Y$ for all $i < n$. We also need the following notation. For $X \in A$ and $\mathcal{A} \in Cs_n$, we let $S_{[0,1]}X = \{s \in {}^nU : s \circ [0,1] \in X\} = \{\langle s_1, s_0, s_2, \dots\rangle : s \in X\}$. Here $[0,1]$ is the transposition on n that interchanges 0 and 1. The next Lemma is the key technical result in this paper. In fact, the element $X \in \mathcal{A}(n,U,H)$ provided by Lemma 2 will play a key role in proving all of our results. Let $\bar{h}$ denote ${}^nH \cap \bar{d}$.

Lemma 2 . *For* $4 \le n+2 \le |H| < \omega$ *with* $H \subseteq U$, *there exists* $X \in A(n,U,H)$ *for which* $X \subseteq \bar{h}$ *and* (1), (2), (3) *and* (4) *below hold:*

(1) $\{\sigma \in S_H : \bar{\sigma}(X) = X\} = \{Id_H\}$, *i.e. "$X$ is rigid".*

(2) Both X *and* $\bar{h} \smallsetminus X$ *are cylindrically equivalent to* $\bar{h}$.

(3) $|X| \ne |\bar{h} \smallsetminus X|$.

(4) Furthermore, if $H = U$ *(when* $\bar{h} = \bar{d}$, $\mathcal{A}(n,U,H) = \mathcal{A}(n,U)$, *the full set algebra with base* U*) then the following condition holds:*

$$S_{[0,1]}X \notin \{X, \bar{d} \smallsetminus X\}.$$

Proof. We start by constructing a set X that satisfies (1), (2) and (3). This will be done by induction on n. Then we prove (4). Since $\mathcal{A}(n,U,H) \cong \mathcal{A}(n,U,H')$ whenever $|H| = |H'|$, we can assume without loss of generality that H is an initial segment of the natural numbers, i.e.

$$H = m = \{0, \dots, m-1\}, \text{ where } |H| = m.$$

Now the base step of the induction is easy.

For $n = 2$ and $4 \le m < \omega$, we let

$$X = \{s \in {}^2m : s_1 = s_0 + 1(mod(m))\} \cup \{(0,2)\}.$$

Then it is not hard to check that $X \subseteq \bar{h}$ and that (1), (2), and (3) hold.

For the induction step we assume that $X \subseteq \bar{h}$ has been defined satisfying (1),(2) and (3), and we define $\bar{X} \subseteq {}^{n+1}(m+1) = \bar{H}$ (for short) which also satisfies (1), (2) and (3). First we define

$$N = \{s \in \bar{H} : m \notin Rg(s)\}$$

and for $i \leq n$, we let

$$Z_i = \{s \in \bar{H} : s_i = m\}.$$

Then the set $\{N\} \cup \{Z_i : i \leq n\}$ forms a partition of $\bar{H}$ (which will be used to separate cases). We define for each $i \in n$

$$A_i = \{s \in Z_i : s(i|s_n) \upharpoonright n \in X\}$$

and we let

$$\bar{X} = \{s \in N : s \upharpoonright n \in X\} \cup \{s \in Z_n : s \upharpoonright n \notin X\} \cup \cup_{i<n} A_i.$$

We show that $\bar{X}$ is as desired. Clearly $\bar{X} \subseteq \bar{H}$. We now consider (2), then (3), and then (1).

Proof of (2)

Suppose $i \leq n$. Clearly $c_i\bar{X} \subseteq c_i\bar{H}$. We show the reverse inclusion, namely $\bar{H} \subseteq c_i\bar{X}$. Towards this end, assume that $s \in \bar{H}$. We must show that $s \in c_i\bar{X}$. We distinguish between two cases.

Case 1. $s \in N \cup Z_n$.

Subcase 1.1. $i < n$.

If $s \in N$, then $m \notin Rg(s)$, and so $s \upharpoonright n \in \bar{h}$. Also, if $s \in Z_n$, then $s(n) = m$, and since s is one to one, it follows that $s(j) \neq m$ for all $j < n$, hence we also have $s \upharpoonright n \in \bar{h}$. Since $s \upharpoonright n \in \bar{h}$ and by induction we have $\bar{h} \subseteq c_iX$, it follows that there is an $a \in m$ such that

$$[s \upharpoonright n](i|a) = s(i|a) \upharpoonright n \in X.$$

One of two things. Either $a \neq s_n$ or $a = s_n$. In the former case we get $s(i|a) \in \bar{X}$. In the latter we have $s(i|m) \in \bar{X}$ since $s(i|m) \in Z_i$ and

$$s(i|m)(i|s_n) \upharpoonright n = s(i|a) \upharpoonright n \in X.$$

We have proved that $s \in c_i\bar{X}$. Therefore $\bar{X}$ satisfies (2).

Subcase 1.2. $i = n$.

If $s \upharpoonright n \notin X$, then by definition $s(n|m) \in \bar{X}$. Else $s \upharpoonright n \in X$. Since $m > n+1$, there exists $a \in m \smallsetminus Rg(s \upharpoonright n)$. Thus $s(n|a) \in N$, which in turn implies that $s(n|a)$ is in $\bar{X}$. In either case we get that $s \in c_n\bar{X} = c_i\bar{X}$.

Case 2. $s \in Z_k$, $k < n$ and $i \in \{k, n\}$.

Subcase 2.1. $i = k$.

Since $\bar{h} \subseteq c_kX$, there exists $a \in m$ such that $s(k|a) \upharpoonright n \in X$. Thus, $s(n|a) \in \bar{X}$ so $s \in c_n\bar{X}$. If $a \neq s_n$, then $s(k|a) \in \bar{X}$; otherwise $s \in \bar{X}$. In either case, $s \in c_k\bar{X}$.

Subcase 2.2. $i = n$.

Since $s(k|s_n) \upharpoonright n \in \bar{h} \subseteq c_iX$ there exists $a \in m$ such that $s(k|s_n)(i|a) \upharpoonright n \in X$. Thus $s(i|a) \in \bar{X}$, and so $s \in c_i\bar{X}$ as desired.

By Cases 1 and 2 above it follows that $\bar{H} \subseteq c_i\bar{X}$ hence $c_i\bar{H} = c_i\bar{X}$. The proof that $c_i\bar{H} = c_i(\bar{H} \smallsetminus \bar{X})$ is completely analogous and is therefore omitted. By this we have proved that $\bar{X}$ satisfies (2).

Proof of (3)

We now prove that $\bar{X}$ satisfies (3). For the sake of brevity, we write $Y = \bar{h} \smallsetminus X$, $\bar{Y} = \bar{H} \smallsetminus \bar{X}$, and we set for each $a \leq m$

$$X_a = \{s \in \bar{X} : s_n = a\},$$

and for $i \leq n$

$$B_i = \{s(i|m)(n|a) : s \restriction n \in X \text{ and } s_i = a\}.$$

Note that

$$|X_m| = |\{s \in \bar{H} : s \restriction n \in Y \text{ and } s_n = m\}| = |Y|.$$

For $a \in m$, we have

$$X_a = \{s \in \bar{H} : s \restriction n \in X \text{ and } a \notin Rg(s \restriction n)\} \cup \cup\{B_i : i \in n\}$$

showing that $|X_a| = |X|$. Thus, $|\bar{X}| = |Y| + m|X|$. An analogous argument shows that $|\bar{Y}| = |X| + m|Y|$, so $|\bar{X}| - |\bar{Y}| = (m-1)(|X| - |Y|)$ from which it follows that $|X| \neq |Y|$ which in turn implies that $|\bar{X}| \neq |\bar{Y}|$.

Proof of (1)

We now prove that $\bar{X}$ satisfies (1). Suppose $\sigma \in S_{m+1}$ and that $\sigma \neq Id$. We distinguish between two cases.

Case 1. $\sigma(m) = m$.

In this case $\sigma \restriction m = \tau \in S_m$, $\tau \neq Id$. Since by induction (1) holds for X, $\bar{\tau}X \neq X$ and thus $\bar{\tau}Y \neq Y$. Choose $p \in Y$ for which $\tau(p) \notin Y$ and set $f = p \cup \{(n,m)\}$. Then $f \in \bar{X}$ while $\sigma(f) = \tau(p) \cup \{(n,m)\} \notin \bar{X}$.

Case 2. $\sigma(a) = m$ for some $a \in m$.

If $\bar{\sigma}\bar{X} = \bar{X}$ then $\bar{\sigma}X_a = X_m$. But $|\sigma X_a| = |X| \neq |Y| = |X_m|$ by (3), and so $\bar{\sigma}\bar{X} = \bar{X}$. This completes the proof of the induction step, hence $\bar{X}$ satisfies $(1), (2)$ and (3).

Now we prove (4). We distinguish between the case $n = 2$ and the case $n > 2$. Let $n = 2$. Let X be as defined above in the base step of the induction. Then we have $(1,2) \in X$ but $(2,1) \notin X$. This shows that $S_{[0,1]}X \neq X$. Also $(1,3) \notin X$ and $(3,1) \notin X$, hence $S_{[0,1]}X \neq \bar{d} \smallsetminus X$. Now let $2 < n$. Let $p = \{(0,1),(1,2),(3,0)\} \cup \{(i, m+i-2) : 3 \leq i < n\}$. Then $p \in {}^nm$. For $2 \leq \beta \leq n$, we denote X and N, constructed above in the induction step for dimension β by X_β and N_β, respectively. It is not hard to see that by the construction of X_β, we get

$$(\forall 2 \leq \beta < n)(p \restriction \beta + 1 \in X_{\beta+1} \text{ iff } p \restriction \beta \in X_\beta).$$

In particular, $p \in X_n$ iff $p \upharpoonright 2 \in X_2$. It follows thus that $p \in X_n$ but $S_{[0,1]}p \notin X_n$. Thus $S_{[0,1]}X_n \neq X_n$. Now let $q = \{(0,1),(1,3),(3,0)\} \cup \{(i, m+i-2) : 3 \leq i < n\}$. Then it is easy to see that $q \notin X_n$ and $S_{[0,1]}q \notin X_n$. Thus $S_{[0,1]}X_n \neq \bar{d} \smallsetminus X_n$. By this the proof of Lemma 2 is complete. ■

We let Bo_n stand for the class of Boolean algebras with operators with same similarity type as CA_n. Let $\mathcal{D} \in Bo_n$. Then $Bl\mathcal{D}$ denotes the Boolean reduct of $\mathcal{D}$ which is of course a Boolean algebra. Recall that an atom of a Boolean algebra is a minimal non-zero element. $At(\mathcal{D})$ denotes the set of all atoms of $Bl\mathcal{D}$. For an algebra $\mathcal{A}$ and $X \subseteq A$, $Sg^{\mathcal{A}}X$ or simply SgX when $\mathcal{A}$ is clear from context, denotes the subalgebra of $\mathcal{A}$ generated by X.

Lemma 3 . *Suppose $\mathcal{A}$ is a finite subalgebra of $\mathcal{C} \in Bo_n$. Let W be an atom of $\mathcal{A}$ and let Y be a partition of W into finitely many elements, each of which is cylindrically equivalent to W. Let $\mathcal{B} = Sg^{\mathcal{C}}(A \cup Y)$, i.e. $\mathcal{B}$ is the subalgebra of $\mathcal{C}$ generated by $A \cup Y$. Then (i) and (ii) below hold:*

(i) $Y \subseteq At(\mathcal{B})$.

(ii) $\mathcal{B} = Sg^{Bl\mathcal{C}}(A \cup Y)$, *i.e $\mathcal{B}$ coincides with the Boolean subalgebra of $\mathcal{C}$ generated by $A \cup Y$.*

Proof. For the sake of brevity, let $\mathcal{D} = Sg^{Bl\mathcal{C}}(A \cup Y)$. We first show that $D = B$. Clearly $D \subseteq B$, since B is closed under the Boolean operations. Since D, by definition, is closed under the Boolean operations and contains all the diagonal elements, to show that $B \subseteq D$ it remains to show that D is closed under cylindrifications. Towards this end, let

$$Z = (At(\mathcal{A}) \smallsetminus \{W\}) \cup Y.$$

Then $D = Sg^{BlC}Z$ because $W = \sum Y$ and because A is generated as a Boolean algebra by its atoms. Therefore

(iii) $Y \subseteq At(\mathcal{D})$ and

(iv) Every element of D is a sum of a subset of Z.

Now for each $z \in Z$, $c_i z \in A$ because $A \subseteq C$ and for each $y \in Y$, we have $c_i y = c_i W \in A$. Since c_i is additive (iv) implies $c_i b \in A$ for all $b \in D$. It follows that $D = B$ which proves (ii). Now (i) readily follows from (iii). ■

Our final lemma before proving Proposition 1 shows that a certain subalgebra $\mathcal{A}_0$ of $\mathcal{A}(n, U, H)$ is finite (even when U is infinite).

Lemma 4 . *For $4 \leq n+2 \leq |H| < \omega$ and $H \subseteq U$, the subalgebra of $\mathcal{A}(n, U, H)$ generated by ${}^nH$ is finite.*

Proof. Let $G = \{\sigma \in S_U : \sigma(u) \in H \text{ for all } u \in H\}$. Let us call a subset a of ${}^nU$ G-stable if $\bar{\sigma}(a) = a$ for all $\sigma \in G$. It is easy to see that G-stable elements generate G-stable ones. Since ${}^nH$ is clearly G-stable, it suffices to show that there are only finitely many G-stable elements. Now each atom of the Boolean algebra whose universe is the set of all G-stable elements has the form f^G for some $f \in {}^nU$ where $f^G = \{\sigma \circ f : \sigma \in G\}$ is the G-orbit of f under the action of G on ${}^nU$. For a function $f : A \to B$, the *kernel* of f is defined as $ker(f) = \{(a, b) \in A \times A : f(a) = f(b)\}$. It is easy to see that $f^G = \{s \in {}^nU : ker(s) = ker(f) \text{ and } [s_i \in H \text{ iff } f_i \in H]\}$. Let

$$\Pi_n = \{(\pi, \lambda) : \pi \text{ is a partition of } n \text{ and } \lambda : n/\pi \to \{0, 1\}\}.$$

Here n/π is the set of all blocks of the partition. Then of course Π_n is finite. Moreover we have

$$|\{f^G : f \in {}^nU\}| = |\Pi_n|,$$

because f^G corresponds to the pair (π, λ) where $\pi = ker f$ and $\lambda(i/\pi) = 0$ if $f(i) \in H$ and 1 otherwise. The Lemma follows. ■

Lemma 5 . *Suppose* $4 \leq n + 2 \leq |H| < \omega$, $H \subseteq U$ *and assume that* $X \subseteq {}^nH \cap \bar{d}$ *satisfies (1)-(2) of Lemma 2. Then* (*i*) *and* (*ii*) *below hold.*

(*i*) *X is an atom in the subalgebra of $\mathcal{A}(n, U, H)$ generated by $\{X\}$.*

(*ii*) *Suppose $\mathcal{B} \subseteq \mathcal{A}(n, U, H)$ is such that X is an atom in $\mathcal{B}$. Then $Id_B : \mathcal{B} \to \mathcal{A}(n, U, H)$ is a non-surjective epimorphism in CA_n.*

Proof. For the sake of brevity let $\mathcal{C} = \mathcal{A}(n, U, H)$, let $\bar{h} = {}^nH \cap \bar{d}$ and let $\mathcal{A}_0$ be the subalgebra of $\mathcal{C}$ generated by ${}^nH$. Then $\mathcal{A}_0$ is finite and $\bar{h}$ is an atom of $\mathcal{A}_0$. Let $\mathcal{B}$ be the subalgebra of $\mathcal{C}$ generated by $\{X\}$. Then $\bar{h} \in \mathcal{B}$ because $\bar{h} = c_0\bar{h} \cap c_1\bar{h} \cap -d_{01} = c_0X \cap c_1X \cap -d_{01}$ by property (2) of X. Hence

$$\mathcal{B} = Sg^{\mathcal{C}}\{X\} = Sg^{\mathcal{C}}[\{X, \bar{h} \smallsetminus X\} \cup A_0],$$

and so X is an atom of $\mathcal{B}$ by Lemma 3 (*i*). This proves (*i*) of Lemma 5.

To prove (*ii*), assume the hypotheses. To prove that Id_B is an epimorphism we apply Lemma 1. Let s be an arbitrary element of $G^*(B)$, the Galois group of B, i.e. s an automorphism of $\mathcal{C}$ that fixes $\mathcal{B}$ elementwise. We shall prove that $s = Id_C$ by which we will be done. Now s fixes ${}^nH \cap d = \sum\{a_x : x \in H\}$ because the latter is in $\mathcal{B}$, thus it permutes the set $\{a_x : x \in H\}$. Therefore there is a permutation of H, σ say, such that $\bar{\sigma}(a_x) = s(a_x)$ for all $x \in H$. But $\{a_x : x \in H\}$ generates $\mathcal{C}$ and so $\bar{\sigma} = s$. By property (1) of X (i.e. by X being "rigid"), we get $\sigma = Id_H$ because s fixes $X \in B$. Thus $s = \bar{Id}_H = \bar{Id}_C$ and $G^*(\mathcal{B}) = \{Id_C\}$. It follows from Lemma 1 that Id_B is an epimorphism in CA_n from $\mathcal{B}$ to $\mathcal{C}$. To see that this map is not surjective it suffices to show that $\mathcal{B}$

is a proper subalgebra of $\mathcal{C}$. By property (2) of X we get that $|X| \geq 2$. But it is easy to see that $\{f\}$ is in $\mathcal{C}$ for every $f \in {}^nH$. Since $X \subseteq {}^nH$ we get that X is not an atom of $\mathcal{C}$. On the other hand, by our assumptions X is an atom of $\mathcal{B}$. Thus $B \neq C$ as desired. By this the proof of Lemma 5 is complete. ■

Now we are ready to prove our main algebraic proposition stated in the introduction. For a class K, SK and IK denote the classes of all subalgebras and of all isomorphic copies of members of K, respectively. We prove something stronger than Proposition 1, namely:

Theorem 5 . *Let $1 < n < \omega$ and let U be a set, $\mu = |U|$. Let $K \subseteq CA_n$ be such that $SK = K$ and $\mathcal{A}(n, U, H) \in K$, for some $H \subseteq U$ with $n + 2 \leq |H| < \omega$. Then not all epimorphisms in K are surjective. In particular, not all epimorphisms are surjective in CA_n, Cs_n, ${}_\mu Cs_n$, Gs_n and ${}_\mu Gs_n$.*

Proof. Theorem 5 immediately follows from Lemma 5. ■

It is proved in [17] and [8] that n-dimensional cylindric set algebras of base $\leq n + 1$ have rather nice properties, e.g. the full Cs_n with base $\mu \leq n + 1$ is homogeneous and is $I_\mu Gs_n$-injective, and the substitution-operations are term definable in Cs_n's with base $\leq n+1$. The construction in the proof of Lemma 2 can be used to show that all these nice properties get lost if the base is bigger than $n + 1$. Thus $n + 1$ is a kind of "turning point" for these properties.

Now we turn to showing that 2.8, 2.9, 3.7, 3.8, 3.10, 5.2(2) and 5.5 of [17] cannot be improved. In the proof we apply the ideas used above to the special case where $H = U$, $|U| = \mu$. For $4 \leq n + 2 \leq \mu < \omega$ and $H = U$ observe that $\mathcal{A}(n, U, H)$ is the full set algebra $\mathcal{A}(n, U)$. It is easily seen (cf. the proof of Lemma 5) that every $s \in Aut(\mathcal{A}(n, U))$ is induced by a permutation on U, i.e. has the form $s = \bar{\sigma}$ for some $\sigma \in S_U$ the symmetric group on U. In what follows, for undefined terminology the reader is referred to [17].

We recall that for $\mathcal{B} \subseteq \mathcal{A}(n, U)$, $G^*(\mathcal{B})$ denotes the Galois group of $\mathcal{B}$. For $\rho \in {}^nU$, we let $\rho^{G(B)}$ denote the orbit of ρ under the action of $G^*(\mathcal{B})$ on ${}^nU$, and we let X^B_ρ denote the atom of $\mathcal{B}$ that contains ρ.

Theorem 6 . *Suppose $4 \leq n + 2 \leq \mu < \omega$ and $|U| = \mu$. Then*

(i) $\mathcal{A}(n, U)$ is neither homogeneous nor $I_\mu Gs_n$-injective.

(ii) $I_\mu Gs_n$ does not have enough injectives.

(iii) There exists $\mathcal{B} \subseteq A(n, U)$ for which $G^*(\mathcal{B}) = \{Id\}$, $\mathcal{B} \neq \mathcal{A}(n, U)_{G(B)}$ and $X^B_\rho \neq \rho^{G(\mathcal{B})}$ for all $\rho \in \bar{d}$.

Proof. Choose $\mathcal{B} \subseteq \mathcal{A}(n, U)$ similarly as in the proof of Lemma 5, i.e. let $X \subseteq \bar{d}$ satisfy conditions (1)−(3) of Lemma 2 and let $\mathcal{B}$ be the subalgebra of $\mathcal{A}(n, U)$

generated by X. Let $f \in Aut(\mathcal{B})$ be the automorphism that interchanges the atoms X and $Y = \bar{d} \smallsetminus X$ of $\mathcal{B}$. Such an automorphism exists since $\mathcal{B}$ is generated by its atoms. By (3) of Lemma 2 we have $|X| \neq |Y|$ and so f cannot be induced by a permutation of U. This shows that $\mathcal{A}(n, U)$ is not homogenous since f does not extend to an automorphism of $\mathcal{A}(n, U)$. Since $\mathcal{A}(n, U)$ is simple, a similar argument shows that $\mathcal{A}(n, U)$ is not $I_\mu Gs_n$ injective and cannot be embedded in one. (i) and (ii) follow. The first two properties of $\mathcal{B}$ in (iii) follow from the proof of Lemma 5. For $\rho \in \bar{d}$ we have $\rho^{G(B)} = \{\rho\}$ because $G(\mathcal{B}) = \{Id\}$ and X^B_ρ is either X or $\bar{d} \smallsetminus X$. Thus $\rho^{G(B)} \neq X^B_\rho$ and the proof of Theorem 6 is now complete. ∎

Theorem 2.5 of [17] states that all subalgebras of a full cylindric set algebra of dimension n and with base $\leq n+1$ are one-generated. We do not know whether Theorem 2.5 of [17] can be extended to $\mu = n+2$, or is [17, Thm.5.2] also best possible. In more detail:

Open question 1 . *Are all subalgebras of $\mathcal{A}(n, n+2)$ one-generated if $5 \leq n < \omega$?*

On the background of this problem: By using the lemmas in this paper, it is not difficult to show that if $\bar{d}$ of $\mathcal{A}(n, U)$ can be partitioned into three cylindrically equivalent subsets, then these three subsets generate a subalgebra which is not one-generated. It is proved in [56] that $\bar{d}(n, n+2) = \{s \in {}^n(n+2) : (\forall i < j < n) s_i \neq s_j\}$ can be partitioned into three cylindrically equivalent subsets if and only if $1 < n < 5$. This shows that there are subalgebras of $\mathcal{A}(n, n+2)$ which are not one-generated if $1 < n < 5$. We do not know whether all subalgebras of $\mathcal{A}(5, 7)$ are one-generated or not. Related information can be found in [28, I.4.8(p.65), Problem I.2(p.127), Problem 8 (p.311)], see also [27, Problem 3.3(p.103)]. We note that the technique of using cylindrically equivalent subsets of $\bar{d}$, first used in [7], proved to be rather fruitful in all kinds of later investigations, cf., e.g., [3], [15], [31], [39, p.38], [43], [68], [69].

Our next result formulated as Theorem 8 below concerns definability of substitutions in cylindric algebras. It shows that Theorem 1 of [8] to be quoted below is best possible. Before formulating our result, we review some needed notation and terminology. Let n be arbitrary. Let $V \subseteq {}^nU$. Let $X \subseteq V$. Let $i, j \in n$. Then

$$S^V_{[i,j]}X = \{f \in V : f \circ [i, j] \in X\}.$$

The superscript V is omitted when no confusion is likely to ensue. $S_{[i,j]}$ is called a substitution operation corresponding to the transposition $[i, j]$ on n, or simply a substitution. Quasipolyadic (generalized) set algebras of dimension n are (generalized) cylindric set algebras of dimension n expanded with the substitution operations $S_{[i,j]}$ for every $i, j \in n$ and $RPEA_n$ stands for the class of all quasipolyadic generalized set algebras of dimension n. On the other hand, $QPEA_n$ stands for the class of (abstract) quasipolyadic equality algebras as defined in [27]. We adopt the equivalent formalism of $QPEA_n$ as defined in [63]. We recall from [63] that $QPEA_n$ are expansions of CA_n with unary (substitution) operations p_{ij} for $i, j \in n$. The interpretation of the abstract operation p_{ij} in set algebras is the concrete operation $S_{[i,j]}$. For $\mathcal{A} \in QPEA_n$, the cylindric algebra $Rd_{ca}\mathcal{A}$ denotes the cylindric reduct of $\mathcal{A}$ obtained by discarding the p_{ij}'s. Before formulating Theorem 8 we need a lemma which roughly says that any unary operation f defined on full generalized cylindric set algebras satisfying the polyadic axioms of the (abstract) substitution p_{ij} is the "genuine" substitution. [2] More precisely:

Lemma 7 . *Let $n > 1$. Let $\mathcal{A} \in QPEA_n$. Assume that $Rd_{ca}\mathcal{A}$ is a full Gs_n. Then $p^{\mathcal{A}}_{ij}X = S_{[i,j]}X$, for every $i, j \in n$.*

Proof. We will prove more. Namely, let n be an arbitrary ordinal (not necessarily finite). Let Gws_n be the class of generalized weak set algebras in the sense of [27, Def. 3.1.2]. Let $\mathcal{A} \in Gws_n$ be such that every $x \in A$ is a (possibly infinite) union of rectangular elements of $\mathcal{A}$. Note that $Gws_n = Gs_n$ when n is finite. Furthermore, it is easy to see that every full Gs_n when $n < \omega$ satisfies the above conditions. Assume that $\mathcal{A} = Rd_{ca}\mathcal{C}$ for some $\mathcal{C} \in QPEA_n$. We first show that

(**) $p^{\mathcal{C}}_{ij}x = \{f \in 1^A : f \circ [i, j] \in x\}$

for all $i, j \in n$ and all $x \in A$. (This is equivalent to showing that $\mathcal{C}$ is a representable $QPEA_n$.) In what follows we use the axiomatization $(P0) - (P11)$ of polyadic algebras in [27, 5.4.3] restricted to the similarity type of $QPEA_n$ involving—besides the cylindric operations—only the substitution operations p_{ij} and s^j_i for $i < j < n$. We note that the unary operations s^j_i are term definable in CA_n (and $QPEA_n$) by $s^j_i x = c_i(x \cdot d_{ij})$. Now let $i < j < n$. Let x be rectangular. Then $x = c_i x \cap c_j x$. Then we have by $(P8), (P9)$ and $(P10)$

$$p_{ij}x = p_{ij}(c_i x \cdot c_j x) = p_{ij}c_i x \cdot p_{ij}c_j x = s^j_i c_i x \cdot s^i_j c_j x.$$

Next we show that when $x \in A$ is rectangular, we have

$$s^j_i c_i x \cap s^i_j c_j x = \{f \in 1^A : f \circ [i, j] \in X\},$$

by which we will be done. The inclusion $\geq$ always holds. Assume that $f \in s^j_i c_i x \cap s^i_j c_j x$. Then $f \in c_j(d_{ij} \cdot c_i x)$, hence $f(j|fi) \in c_i x$, thus $f(j|fi, i|fj) = f \circ [i, j] \in c_i x$. Similarly $f \in s^i_j c_j x$ implies $f \circ [i, j] \in c_j x$. Thus $f \circ [i, j] \in$

[2] We note that it can well happen that $\mathcal{A}$ is a $QPEA_n$ such $Rd_{ca}\mathcal{A}$ is representable while $\mathcal{A}$ is not a representable $QPEA_n$, i.e. p_{ij} remains abstract in any representation of $\mathcal{A}$.

$c_ix \cap c_jx = x$. We have seen that (**) holds for every rectangular $x \in A$. Next we show that (**) holds for every element of A. Let $R = \{x \in A : x \text{ is rectangular }\}$. Let $y \in A$ be arbitrary. Then $y = \sum\{x \in R : x \leq y\}$ and $p_{ij}y = \sum\{x \in R : x \leq p_{ij}y\}$ by our assumption on A and since p_{ij} preserves sums. It is not hard to show that $p_{ij}x$ is rectangular for all $x \in R$. Thus $p_{ij}y = \bigcup\{p_{ij}x : x \in R, x \leq y\}$. Hence (**) holds. By this the proof is complete. ■

We are ready to formulate and prove our final theorem in this section. We let Ws_α be the class of α-dimensional weak cylindric set algebras as defined in [27, Def.3.1.2]. We recall from [27] that $Cs_n = Ws_n$ when $n < \omega$.

Theorem 8 . *Substitutions are term definable neither in ${}_\mu Cs_n$ nor in Ws_α for $2 \leq n < \omega$, $\mu \geq n+2$ and $\alpha \geq 2$. In more detail, there exists a $\mathcal{C} \in {}_\mu Cs_n$, such that no term function $f \in {}^CC$ would satisfy the polyadic axioms for $S_{[0,1]}$; and the same for Ws_α in place of ${}_\mu Cs_n$.*

Proof. We start with the case of Cs's. Let $1 < n < \omega$. Let everything be as in the hypothesis of Lemma 2(4) with $H = U$, $\mu = |U| = n+2$. For brevity let $\mathcal{C} = \mathcal{A}(n, U)$ be the full Cs_n with base U. Assume $f \in {}^AA$ satisfies the axioms for p_{01}. Then by Lemma 7 we have $f = S_{[0,1]}$. We now show that the algebra $\mathcal{B} = Sg^{\mathcal{C}}X$ constructed in Theorem 5 is not closed under the substitution operation $S_{[0,1]}$. By Lemma 2(2), applied to the special case when $H = U$, we have X, $\bar{d} \smallsetminus X$, and $\bar{d}$ are cylindrically equivalent. By Lemma 3 (i) we have that $X, \bar{d} \smallsetminus X$ are atoms of B. By the definition of $S_{[0,1]}$ we have that $|S_{[0,1]}X| = |X|$, thus $S_{[0,1]}X \neq \bar{d}$. Then $S_{[0,1]}X \notin \mathcal{B}$ since by the above and Lemma 3(4) we have that $S_{[0,1]}X \notin \{0, \bar{d}, X, \bar{d} \smallsetminus X\}$.

Now we consider the case of Ws's. We assume that $n \geq \omega$. Here we use a construction of Németi in [52]. Let $\mathcal{A}$ be the weak set algebra constructed in [52, Statement 1]. Let $\mathcal{C}$ be the full weak set algebra having the same unit as $\mathcal{A}$. Then by the proof of Lemma 7 we have that in C any function satisfying the polyadic axioms for p_{01} is $S_{[0,1]}$. But $\mathcal{A} \subseteq \mathcal{C}$ is a cylindric subalgebra of $\mathcal{C}$ that is not closed under $S_{[0,1]}$ as shown in [52, Statement 2]. ■

Theorem 8 complements the result that subsititutions are term definable in CA_n when $n \leq 1$ and in ${}_{<\mu}Gs_n$ for every n and every $\mu \leq n+1$, a result of Andréka and Németi, cf. [8, Theorem 1]. This result was preceded by the classical result of Comer and Henkin addressing the case when $\mu < n$, see [27, Theorem 3.2.53]. In this case ${}_{<\mu}Gs_n$ coincides with the class of the so-called cylindric algebras of positive characteristic [26].

§2. LOGICAL CONSEQUENCES

The logical consequences concern the (Beth definability property for) finite-variable fragments of first order logic. Both the finite-variable fragments and

Beth Definability property are quite well investigated. Historical notes on both of these are given at the end of the paper, in section 3 (7)-(9).

The *n-variable fragment* L_n of first order logic (FOL) is the usual FOL where we use only the first n variables $\{v_0, \ldots, v_{n-1}\}$; and for simplicity we do not allow constant or function symbols and we use only n-place relation symbols. Otherwise, the formulas, models, validity are the usual. Let M be a class of cardinal numbers. Then n-variable (fragment of first order) logic *with models of size in* M is denoted as ${}_M L_n$, this is the same as L_n except that we use only models of size μ where $\mu \in M$. With this notation, L_n is the same as ${}_{Card} L_n$ where $Card$ denotes the class of all cardinal numbers.

The atomic formulas of L_n are not "independent" of each other, because of the presence of substitution of variables in the atomic formulas $R(v_{i_1} \ldots v_{i_n})$. In usual FOL, all atomic formulas $R(v_{i_1} \ldots v_{i_n})$ are semantically equivalent to formulas built up from $R(v_0 \ldots v_{n-1})$ and $v_i = v_j$ for some i, j. This allows one to concentrate on the so-called *substitution-free fragment* L_n^- of FOL, which is that part of FOL which uses only atomic formulas of the form $R(v_0 \ldots v_{n-1})$ (and of course $v_i = v_j$) where n is the rank of R.[3] Though FOL is equivalent with its substitution-free fragment, we loose this equivalence in the n-variable logic if we do not restrict the size of the models, this is what Corollary 1 below states. Let ${}_M L_n^-$ denote n-variable first order substitution-free logic with models of size in M. We call two languages *equivalent* if there is a translation function between their sets of formulas which preserves validity and semantical consequence.

Corollary 1 . *Let $2 \leq n < \omega$. Then n-variable fragment is equivalent with n-variable substitution-free fragment iff we use only models of size $< n+2$, i.e. for a class M of cardinal numbers we have*

$${}_M L_n \text{ is equivalent with } {}_M L_n^- \quad \text{iff} \quad M \subseteq n+2.$$

Proof. For $M \subseteq n+2$ the statement follows from [8, Thm.1] which states that substitutions are term definable in ${}_\mu Cs_n$ where $\mu \leq n+1$. For $M \not\subseteq n+2$ the statement follows from Theorem 8 in section 1 which states in a strong form that substitutions are not term definable in ${}_\mu Cs_n$ if $\mu \geq n+2$. ■

The name Beth Definability Theorem is a generic title for assertions of the form "A logic has the Beth definability property". What Beth himself proved is that first order logic has the Beth definability property. The Beth Definability Theorem is one of the cornerstones of first order logic. Indeed, the Beth Definability Theorem together with the so-called Downward Löwenheim Skolem Theorem characterizes first order logic. This, in turn, is known as

[3] These are called "restricted" formulas in [27, sec. 4.3].

Lindström Theorem. The Beth Definability Theorem (for first order logic) relates two notions of definability, implicit definability and explicit definability. A set Σ of formulas *implicitly defines* a relation symbol P if for any structure of the symbols in $\Sigma \smallsetminus \{P\}$ this structure has at most one expansion that is a model of Σ. On the other hand, Σ *defines* P *explicitly* if there is a formula built up of symbols distinct from P that turns out to be equivalent to P in any model of Σ. It is straightforward to see that explicit definability implies implicit definability. The converse which is nothing more than the Beth Definability Theorem is true for first order logic. But when we restrict our attention to finitely many variables (and do not restrict the sizes of the models) we loose this nice property of first order logic. This was first proved by Németi in [51] as announced in [5].

We now turn to formulating our main results. We start by writing out the notions of implicit and explicit definitions in more detail. Let $L = \langle Fm, Mod\rangle$ be a fragment of FOL where Fm denotes the set of formulas of L, and Mod denotes the class of all models of L. Let Fm_n and Fm_n^- denote the sets of formulas of L_n and L_n^- respectively, and let Mod_n denote the class of all first order models with only n-place relations. With this notation, $L_n = \langle Fm_n, Mod_n\rangle$, and ${}_ML_n^- = \langle Fm_n^-, \{\mathcal{A} \in Mod_n : |A| \in M\}\rangle$.

Definition 1 . *Let $L = \langle Fm, Mod\rangle$ be a fragment of FOL. Let $\Sigma \subseteq Fm$, $k < \omega$, P and P' be relation symbols of rank $k+1$ such that P' does not occur in Σ. Then $\Sigma[P/P']$ denotes the set of formulas obtained from Σ by replacing every occurrence of P by P'. Now, we recall from [18, p.87] definitions (i) and (ii) below.*

(i) Σ defines P implicitly iff

$$\Sigma \cup \Sigma[P/P'] \models \forall v_0 \dots v_k[P(v_0 \dots v_k) \longleftrightarrow P'(v_0 \dots v_k)].$$

(ii) Σ defines P explicitly iff there is a $\phi(v_0 \dots v_k) \in Fm$ such that P does not occur in ϕ and

$$\Sigma \models \forall v_0 \dots v_k[\phi(v_0 \cdots v_k) \longleftrightarrow P(v_0 \cdots v_k)].$$

(iii) A logic has the semantic Beth definability property (cf. [13, p.32, Def.1.2.4]) if implicit definability of P implies explicit definability of P (for every Σ and P as above).

The following is an improvement of the result that L_n fails to have the Beth definability property announced in [5].

Corollary 2 . *Let $1 < n < \omega$. First order logic with n variables has the semantic Beth definability property iff we restrict the models to be of size $\leq n+1$. In more detail,* *(i)* *n-variable first order logic with models of size $\le n+1$ as well as n-variable substitution-free logic with models of size $\le n+1$ have the semantic Beth definability property.*

(ii) *n-variable substitution-free logic with models of size in M where $M \not\subseteq n+2$ does not have the semantic Beth definability property. In more detail, there are Σ and P as in Definition 1, such that Σ defines P implicitly but not explicitly.*

(iii) *n-variable logic with models of size in M where $M \not\subseteq 2n+1$ does not have the semantic Beth definability property.*

Proof. Corollary 2 follows from algebraic results via using Theorem 5.6.10 in the monograph [27]. Let t be the similarity type of CA_n's, let X denote the set of our relation symbols, and let $K = I_M Gs_n = I\{\mathcal{A} \in Gs_n :$ all subbases of $\mathcal{A}$ have cardinality $\in M\}$. Using the notation of [27, p.259] then it is easy to see that ${}_M L_n^-$ is equivalent with the logic denoted in [27, p.259] as $\langle \mathcal{F}r_X, K\rangle$, and K is closed under taking subalgebras and forming direct products. Now, [27, Thm.5.6.10] states that the semantical Beth definability property holds for $\langle \mathcal{F}r_X, K\rangle$ iff all almost-onto epimorphisms in K are surjective. Here, a homomorphism $h : \mathcal{A} \to \mathcal{B}$ is called *almost-onto* iff $\mathcal{B}$ is generated (as an algebra) by the range of h together with a single element of $\mathcal{B}$. Epimorphisms are surjective in ${}_\mu Gs_n$ for $\mu \le n+1$ (cf. [17]), hence the semantical Beth definability property holds for ${}_M L_n^-$ if $M \subseteq n+2$, by [27, Thm.5.6.10]. On the other hand, let $n+2 \le \mu < \omega$, and take a homomorphism $f : \mathcal{B} \to \mathcal{A}(n, U, H)$ from Lemma 5 with $|U| = \mu$. Lemma 5 (ii) states that f is an epimorphism which is not surjective. We have that f is almost-onto because the algebra $\mathcal{A}(n, U, H)$ is generated by the single element $\{\langle h_i, h_{i+1}, u_1, \ldots, u_{n-2}\rangle : i+1 < |H|, u_1, \ldots, u_{n-2} \in H\}$ where $h : |H| \to H$ is any bijection. Hence ${}_M L_n^-$ does not have the semantical Beth definability property if $M \not\subseteq n+2$.

To prove Corollary 2 (iii), we can use the construction given in [43]. We recall the construction. Let $1 < n < \omega$, let $U_0, U_1, \ldots, U_{n-1}$ be disjoint sets such that $|U_0| \ge 3$ and $|U_i| = 2$ for $1 \le i < n$. Let $U = \bigcup\{U_i : i < n\}$, $T = U_0 \times U_1 \times \cdots \times U_{n-1}$, let $q \in T$ and $a \in U_0$ be arbitrary and define

$$X = \{s \in T : s_0 = a \text{ and } |\{0 < i < n : s_i \ne q\}| \text{ is odd}\},$$

$$Y = \{s \in T : s_0 \ne a \text{ and } |\{0 < i < n : s_i \ne q\}| \text{ is even}\}.$$

Let $\mathcal{B}$ be the $RPEA_n$ with base U generated by $\{X\}$, and let $\mathcal{A}$ be the $RPEA_n$ with base U generated by $\{X \cup Y\}$. It is proved in [43] that the inclusion homomorphism $Id : \mathcal{B} \to \mathcal{A}$ is a non-surjective $RPEA_n$ epimorphism. ∎

Open question 2 . *Let $2 < n < \omega$ and $n+2 \le \mu < 2n+1$. Does n-variable* *logic with models of size μ have the semantic Beth definability property? In algebraic form this question is the following. Are the epimorphisms surjective in ${}_\mu RPEA_n$ if $2 < n < \omega$ and $n+2 \leq \mu \leq 2n$?*

To formulate Corollary 3 which is the *proof theoretic* consequence of our algebraic proposition in section 0, we will use the syntactical derivability relation $\vdash_{n,r}$, or briefly $\vdash_r$, of first order logic with n variables as defined in [27, p.157]. Roughly, $\vdash_r$ is obtained by restricting the usual Hilbert-style axioms and proof rules of first order logic to the formulas of L_n^- (i.e. *only n variables* and substitution-free formulas can be used in proofs). As proved in [27], $\vdash_r$ is not complete. In fact, for every $m > n > 2$, there is a formula built up of n variables and one relation symbol, ϕ say, such that ϕ can be proved using $m+1$ variables but cannot be proved using m variables [29]. Thus $\vdash_r$ is different from the semantical consequence relation $\models$. A survey of properties of this provability relation can be found in [9, Def.65, pp.223-228]. The syntactical version of the Beth definability property is:

Definition 2 . *Let L, Σ, P, P' and k be as in Definition 1.*

(i) Σ defines P implicitly via $\vdash_r$ iff

$$\Sigma \cup \Sigma[P/P'] \vdash_r \forall v_0 \dots v_k [P(v_0 \dots v_k) \longleftrightarrow P'(v_0 \dots v_k)].$$

(ii) Σ defines P explicitly via $\vdash_r$ iff there is a $\phi(v_0 \cdots v_k) \in Fm$ such that P does not occur in ϕ and

$$\Sigma \vdash_r \forall v_0 \cdots v_k [\phi(v_0 \cdots v_k) \longleftrightarrow P(v_0 \cdots v_k)].$$

(iii) The provability relation $\vdash_r$ has the syntactical Beth definability property if implicit definability via $\vdash_r$ implies explicit definability via $\vdash_r$ for every Σ as above.

Corollary 3 . *Let $1 < n < \omega$. Then the provability relation $\vdash_{n,r}$ of first order logic fails to have the syntactic Beth definability property. In more detail, there are Σ and P as in Definition 2, such that Σ defines P implicitly via $\vdash_{n,r}$ but Σ does not define P explicitly via $\vdash_{n,r}$.*

Corollary 3 follows from our algebraic proposition by Thm 5.6.10 and the first ten pages of sec 4.3 ("Connections between logic and $CA's$") of the monograph [27], in the spirit of the proof of Corollary 2. In more detail, Theorems 4.3.25 and 4.3.28(i) of [27] state that CA_n and $\vdash_{n,r}$ correspond to each other in such a way that [27, Thm.5.6.10] becomes applicable.

In fact a stronger statement follows from our theorems in section 1. Namely, there is an implicit definition which is already valid in a very weak version of first order logic (corresponding to CA_n) for which there is no explicit definition which would be valid *semantically* on standard first order models. (A result of this spirit is proved for certain modal logics in [49].) In more detail:

Corollary 4 . *Let $1 < n < \omega$. There are Σ and P as in Definition 1, such that Σ defines P implicitly via $\vdash_{n,r}$ but Σ does not define P explicitly via $\models$; in particular Σ does not define P explicitly via $\vdash_{m,r}$ for any $m \geq n$.*

§3. Related results and some historical notes

After the first version of this paper was completed, several results were obtained by various people related to the subject matter of the paper. Such results address surjectiveness of epimorphisms, (strong) amalgamation, and (strong) embedding properties in the sense of [57] in classes of algebras frequently studied in algebraic logic, and Beth definability properties in finite variable fragments. Some of these answer problems posed by Pigozzi in his landmark paper [57]. We now briefly review those related results.

(1) In [64] Sayed Ahmed proves that for $1 < n < \omega$ and $\mu < n$, the class $I_{<\mu}Gs_n$ (of CA_n's of positive characteristic) has the strong amalgamation property strengthening Comer's result quoted in the introduction, for it is known that in the case of varieties strong amalgamation implies that epimorphisms are surjective. He also proves that cylindric algebras of positive characteristic of *any* dimension has the strong amalgamation property answering a question of Pigozzi in [57].

(2) ${}_\infty Cs_\alpha$ stands for the class of cylindric set algebras of dimension α with infinite base. I.e. ${}_\infty Cs_\alpha = \bigcup\{{}_\mu Cs_\alpha : \mu \geq \omega\}$. Similarly for ${}_\infty Gs_\alpha$. For a class K of algebras we write ES holds in K if epimorphisms are surjective in K. For $\alpha \geq \omega$, Madarász [44] proves the infinite analogue of Proposition 1 herein, namely that ES fails in Gs_α, ${}_\infty Gs_\alpha$ and ${}_\infty Cs_\alpha$. It follows that these classes fail to have the strong amalgamation property. Madarász also proves that the classes of the so-called diagonal cylindric algebras in the sense of [57] and semisimple algebras of infinite dimension fail to have ES. Madarász also proves that such classes fail to have the strong amalgamation property even if the strong amalgam is sought in the bigger class of representable algebras. Sayed Ahmed proves that such classes together with the class of infinite dimensional representable cylindric algebras have the strong embedding property [44]. In contrast, the classes of algebras addressed in our Proposition 1 do not have even the embedding property, cf. [16] and [48].

(3) Madarász and Simon prove that CA_ω does not have the embedding property [44], [70]. However, if we add the so-called merry go round identities to CA_ω, the resulting class has the embedding property [64]. On

the connections between the merry go round identities and cylindric and quasipolyadic algebras see also [21], [22].

(4) Df_α, QPA_α and $QPEA_\alpha$ stand for the classes of diagonal free reducts of cylindric algebras, quasipolyadic algebras and quasipolyadic equality algebras of dimension α, respectively. Sain [60] proves that ES fails for QPA_n and $QPEA_n$ when $1 < n < \omega$, together with their concrete versions namely the representable ones, by adapting the proof of Theorem 5 herein (which was available in [4]). The infinite analogue is proved by Madarász [44]. Sain [60] proves that ES fails in Df_α for $\alpha > 1$. The $\alpha < \omega$ case can also be obtained from the present proof of Proposition 1 by adapting the proof of Lemma 3 on p.313 of [16]. In [16] Comer proves that CA_n for $n > 1$ does not have the embedding property. Marx [48] contains a stronger version of Comer's quoted Lemma in one direction, showing that the embedding property fails in finite dimensional algebras having the same similarity type as CA_n under rather mild conditions, namely that the first two cylindrifications commute one way. We do not know whether commutativity of the first two cylindrifications kills ES. Sayed Ahmed [67] proves that Df_n and RDf_n have the strong embedding property for any n. Comer [16] proves that the amalgamation property fails for Df_n for $n > 1$.

(5) We note that by the results of this paper and of [43], almost all of the questions concerning ES in varieties of cylindric algebras are solved. However, a few remains open and some of these are given in the survey paper [44].

(6) Concerning relation algebras, let RA and RRA stand for the classes of relation algebras, and representable relation algebras. ${}_\infty RRA$—the relation algebraic version of ${}_\infty Gs_\alpha$—consists of subdirect products of RA's representable such that the greatest element is of the form $U \times U$ for some infinite set U. Németi [54] proves that ES fails in RA, RRA and ${}_\infty RRA$. He also proved that the amalgamation property fails and the embedding property holds in ${}_\infty RRA$ complementing a result of McKenzie [46, p.116]. Now let QRA stand for the class of Q-relation algebras as defined in e.g. [71]. Sain [60] proves that ES holds for QRA by proving that QRA has the strong amalgamation property. [4] Marx [50] shows that a weak form of associativity in algebras having the same similarity type as RA's forms a borderline; in the sense that any K containing RRA in which $K \models (x;1);1 \leq x;1$ fails to have the embedding property. Here

[4]The cylindric version of QRA's, is the class of the so-called *directed* CA_3's. This class is introduced by Németi and is investigated in e.g. [58]. The class of directed CA_3's has the strong amalgamation is proved by Sayed Ahmed and Sági [59].

; stands for the binary operation abstracting the concrete operation of composition of binary relations. In particular, the class of semiassociative RA's, or SA for short, fails to have the embedding property. This is independently proved by Madarász [44]. The CA analogue of Marx's results is the result of Marx quoted in the previous item, since associativity in RA-like algebras of relations corresponds to commutativity in CA-like algebras of relations. In contrast, Németi [54] and Marx [50] prove that the class of weakly associative RA's, or WA's for short, has the strong amalgamation property. The reader is referred to [50] for definitions of SA and WA. These classes were originally introduced by Maddux.

(7) Johnson [36] proves that the class of polyadic algebras of infinite dimension has the strong amalgamation property. Sayed Ahmed [65] proves that various reducts thereof like the classes of algebras investigated in [62] also have the strong amalgamation property, complementing a result of Madarász in [44]. The latter states that ES fails for those reducts of polyadic algebras of infinite dimension for which the substitutions available are indexed by surjective transformations. The main result in [65] shows that surjectiveness here is necessary.

(8) Let $\omega > n > 1$. Let L_n denote first order logic restricted to the first n variables. Bíró [15] uses the construction developed herein in Lemma 2 to show that Vaught's Theorem on the existence of prime models fails for L_n. Another construction showing that Vaught's Theorem fails for L_n is due to Andréka [2]. This was used by Sayed Ahmed to show that the Henkin-Orey omitting types Theorem fails for L_n, cf. [12], [66] and [11]. L_n was first systematically studied in Henkin [25], more results on L_n are surveyed in [27, sec.4.3], [30], [9, sec.7, pp.220-231,237].

(9) Beth definability property (BDP) can be traced back to Padoa's method for showing definability of primitive notions in a language. E. W. Beth [14] proved that first order logic (FOL) has the BDP, he was motivated by applications of logic in science. Classical propositional logic, intuitionistic propositional logic, the minimal modal logic K and all normal extensions of the modal logic K4 have the BDP, see [45]. Failure of BDP for L_n was first announced in [5], proved in [51] via the methods of algebraic logic. Sain [60] proves that BDP fails for L_n without equality. Gurevich [23] shows that FOL with only finite models fails BDP. Finite model theory is intimately connected to finite variable logics, and there is a strong connection between definability properties and complexity issues in computer science. This is the main issue of descriptive complexity theory, see e.g. [20]. E.g., BDP fails for FOL with finite models, finitely many variables, but infinite conjunctions ($L^{\omega}_{\infty\omega}$), see Kolaitis [38],

Hodkinson [30], [20], [24].

(10) We should mention that for the highly relevant guarded and loosely guarded fragments of first order logic the Beth definability property holds even for the finite-variable case [34] [35].

(11) An important variant of BDP is the weak BDP, see e.g. [13, pp.73-76,689-716]. An implicit definition $\Sigma(R)$ is called strong if in all models it has exactly one solution R. The *weak BDP* requires explicit definitions for strong implicit definitions only. Sain [60],[61], Hoogland [32],[33] and [9, chap.6] contain algebraic characterizations for the weak BDP. The question whether the distinguished kinds of CA_n's have these properties remains open. For example, is Cs_ω^{reg} contained in a full reflective proper subcategory K of IGs_ω? More on this can be found in [9, Def.56–Thm.59 on pp.212-214, and pp.228-229], [33], [61]. Even weak BDP fails for L_n, a result of Németi, Simon and Hodkinson, see [55] for $n = 3$ and [30] for large n (i.e., for all $n \geq N$ for some $N \geq 5$). Also, weak BDP often fails in finite model theory, see [20], [24], [30], [38]. There is a version of the Beth property, called "weak local BDP" which holds for L_n, for arbitrary n, see [9, p.228, below Thm.70].

The following question concerning cylindric algebra for both finite and infinite dimensions is (to the best of our knowledge) still open:

Open question 3 . *Let $\alpha \geq 2$. Is every Gs_α-epimorphism a CA_α-epimorphism?*

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Andréka, H., Madarász, J. X., and Németi, I.
Rényi Institute of Mathematics, Budapest, Hungary
andreka@renyi.hu, madarasz@renyi.hu, nemeti@renyi.hu

Comer, S. D.
Mathematics and Computer Science Department, The Citadel, Charleston, USA, steve.comer@citadel.edu

Sayed Ahmed, T.
Department of Mathematics, Faculty of Science, Cairo University, Cairo, Egypt
rutahmed@rusys.eg.net