A non representable infinite dimensional quasi-polyadic equality algebra with a representable cylindric reduct

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Abstract

We construct an infinite dimensional quasi-polyadic equality algebra $\mathfrak{A}$ such that its cylindric reduct is representable, while $\mathfrak{A}$ itself is not representable. [1]

The most well known generic examples of algebraizations of first order logic are Tarski's cylindric algebras (CA) and Halmos' polyadic equality algebras (PEA). The theory of cylindric algebras is well developed in the treatise [10], [11], [12] and is still an active part of research in algebraic logic. The generic examples of CA's are set algebras. More precisely, let α be an ordinal. Let U be a set. Then we define for $i, j \in \alpha$ and $X \subseteq {}^{\alpha}U$:

$$\mathsf{c}_i X = \{s \in {}^{\alpha}U : \exists t \in X, s(j) = t(j) \text{ for all } i \neq j\},$$

$$\mathsf{d}_{ij} = \{s \in {}^{\alpha}U : s(i) = s(j)\}.$$

For a set X, let $\mathfrak{B}(X) = \langle \wp(X), \cup, \cap, \sim, \emptyset, X\rangle$ be the full Boolean set algebra with universe $\wp(X)$. A cylindric set algebra of dimension α is a subalgebra of an algebra of the form

$$\langle \mathfrak{B}({}^{\alpha}U), \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{i,j<\alpha}.$$

The class of representable cylindric algebras of dimension α, or RCA_α for short, is the class of subdirect products of set algebras of dimension α. These

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are the standard models, so to speak, of cylindric algebras. The operations c_i of cylindrifications can be generalized and new operations called substitutions can be introduced on set algebras as follows: For $\Gamma \subseteq \alpha$, $\tau \in {}^{\alpha}\alpha$ and $X \subseteq {}^{\alpha}U$ set:

$$\mathsf{c}_{(\Gamma)}X = \{s \in {}^{\alpha}U : \exists t \in X, s(j) = t(j) \text{ for all } j \notin \Gamma\},$$

$$\mathsf{S}_{\tau}X = \{s \in {}^{\alpha}U : s \circ \tau \in X\}.$$

A full polyadic equality set algebra of dimension α is an algebra of the form

$$\langle \mathfrak{B}({}^{\alpha}U), \mathsf{c}_{(\Gamma)}, \mathsf{S}_{\tau}, \mathsf{d}_{ij}\rangle_{\Gamma\subseteq\alpha, i,j\in\alpha, \tau\in{}^{\alpha}\alpha},$$

and the class of representable polyadic algebras consists of all subdirect products of these. The class of polyadic equality algebras of a given dimension is obtained from set algebras (of the same dimension) by a process of an abstraction. An axiomatization of polyadic equality algebras is given in [10] 5.3.1. which we now recall.

Definition 1 . *Let α be an ordinal. By a polyadic equality algebra of dimension α, or a PEA_α for short, we understand an algebra of the form*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}_{\tau}, \mathsf{d}_{ij}\rangle_{i,j\in\alpha, \Gamma\subseteq\alpha, \tau\in{}^{\alpha}\alpha}$$

where $\mathsf{c}_{(\Gamma)}$ ($\Gamma \subseteq \alpha$) and s_{τ} ($\tau \in {}^{\alpha}\alpha$) are unary operations on A, $\mathsf{d}_{ij} \in A$ ($i, j \in \alpha$), such that postulates 1-15 below hold for $x, y \in A$, $\tau, \sigma \in {}^{\alpha}\alpha$, $\Gamma, \Delta \subseteq \alpha$ and all $i, j \in \alpha$.

1. *$\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra*
2. $\mathsf{c}_{(\Gamma)}0 = 0$
3. $x \leq \mathsf{c}_{(\Gamma)}x$
4. $\mathsf{c}_{(\Gamma)}(x \cdot \mathsf{c}_{(\Gamma)}y) = \mathsf{c}_{(\Gamma)}x \cdot \mathsf{c}_{(\Gamma)}y$
5. $\mathsf{c}_{(0)}x = x$
6. $\mathsf{c}_{(\Gamma)}\mathsf{c}_{(\Delta)}x = \mathsf{c}_{(\Gamma\cup\Delta)}x$
7. $\mathsf{s}_{Id}x = x$
8. $\mathsf{s}_{\sigma\circ\tau} = \mathsf{s}_{\sigma} \circ \mathsf{s}_{\tau}$
9. $\mathsf{s}_{\sigma}(x + y) = \mathsf{s}_{\sigma}x + \mathsf{s}_{\sigma}y$
10. $\mathsf{s}_{\sigma}(-x) = -\mathsf{s}_{\sigma}x$
11. *if $\sigma \restriction (\alpha \sim \Gamma) = \tau \restriction (\alpha \sim \Gamma)$, then* $\mathsf{s}_{\sigma}\mathsf{c}_{(\Gamma)}x = \mathsf{s}_{\tau}\mathsf{c}_{(\Gamma)}x$

12. *if* $\tau^{-1}\Gamma = \Delta$ *and* $\tau \restriction \Delta$ *is one to one, then* $\mathsf{c}_{(\Gamma)}\mathsf{s}_\tau x = \mathsf{s}_\tau \mathsf{c}_{(\Delta)} x$

13. $\mathsf{d}_{ii} = 1$

14. $x \cdot \mathsf{d}_{ij} \leq \mathsf{s}_{[i|j]} x$

15. $\mathsf{s}_\tau \mathsf{d}_{ij} = \mathsf{d}_{\tau(i)\tau(j)}$

Polyadic equality algebras are proper expansions of cylindric algebras. That is if $\mathfrak{A} \in PEA_\alpha$ then the cylindric reduct of $\mathfrak{A}$ obtained by discarding the operations not in the similarity type of CA_α is a CA_α. In polyadic equality algebras all substitution operations are available. That is if $\mathfrak{A} \in PEA_\alpha$ and $\tau \in {}^\alpha\alpha$ is a transformation on α, then $\mathsf{s}_\tau^{\mathfrak{A}}$ is a unary (substitution) operation on $\mathfrak{A}$, that happens to be, among other things, a Boolean endomorphism of $\mathfrak{A}$. Morever (generalized) cylindrifications $\mathsf{c}_{(\Gamma)}$ are defined for every $\Gamma \subseteq \alpha$. Quasi-polyadic equality algebras of dimension α (QEA_α) on the other hand, are obtained from PEA_α by restricting the similarity type and axiomatization of the latter to finite cylindrifications and substitutions corresponding to finite transformations only. A finite transformation is one for which $\{i \in \alpha : \tau(i) \neq i\}$ is finite. For the finite dimensional case, polyadic equality algebras coincide of course with quasi-polyadic equality algebras, but in the infinite dimensional case, the distinction is highly significant. For example PEA_ω has uncountably many operations while QEA_ω has countably many operations. Furthermore, the equational theory of $RPEA_\omega$ is extremely complex [15].

The class of locally finite dimensional quasi-polyadic algebras was introduced by Halmos [6], but the class of quasi-polyadic algebras without the restriction of local finiteness has not been studied except relatively recently [8], [7]. Quasi-polyadic equality algebras are still expansions of cylindric algebras. Quasi-polyadic equality set algebras are defined as the PEA case, and so is the class $RQEA_\alpha$ of representable algebras of dimension α (by restricting the signature to the appropriate similarity type). Though $RPEA_\alpha$ is not a variety for infinite α, $RQEA_\alpha$ is a variety. The theories of QEA_α and CA_α for infinite α are quite close. How close, is a question that remains to be settled, and indeed our main result does shed light on this rather vague question. Quoting Henkin Monk and Tarski in [11] p. 266-267: "Quasi-polyadic algebras: These are like polyadic algebras, except that s_τ is allowed only for finite transformations, and $\mathsf{c}_{(\Gamma)}$ only for finite Γ. Their theory has not been much developed, but they form an interesting stage between cylindric and polyadic algebras."

If $\mathfrak{A} \in QEA_\alpha$ then its cylindric reduct, in symbols $\mathfrak{Rd}_{ca}\mathfrak{A}$, is in CA_α. Further, if $\mathfrak{A} \in RQEA_\alpha$ then $\mathfrak{Rd}_{ca}\mathfrak{A} \in RCA_\alpha$. Subsititutions corresponding to finite transformations are *not* term definable in cylindric algebras (except in some very special cases, like for instance dimension complemented algebras). That is the theories of cylindric algebras and quasi - polyadic algebras are essentially distinct. The inter-connections between the two theories have been recently studied by many authors, to mention a few references in this connection see [17], [14], [10], [16], with similarities and differences illuminating both theories.

It is known that the class RCA_α of representable cylindric algebras for $\alpha > 2$ is not axiomatizable by a set of universal formulas containing finitely many variables [1], same for $RQEA_\alpha$ [13], [16]. (A proof of the latter result for the infinite dimensional case is only sketched in [16], and it seems to us that there are some serious gaps in this sketch). A striking result of Andréka [1] is that for finite $\alpha > 2$ the class $RQEA_\alpha$ is not finitely axiomatizable over RCA_α. The analogous result for infinite ordinals is unknown. In this paper, we address the infinite dimensional case. We do not recover Andréka's result in its strongest form, but we prove a necessary condition for the class $RQEA_\omega$ to be non-finitely axiomatizable over RCA_ω. We will show that there is an $\mathfrak{A} \in QEA_\omega$ such that its cylindric reduct $\mathfrak{Rd}_{ca}\mathfrak{A}$ is representable, while $\mathfrak{A}$ itself is not representable. This means that the finitely many polyadic axiom schemas do not define $RQEA_\omega$ over RCA_ω. (In principal, there could be another finite schema that defines the quasi-polyadic operations). Our construction is based on an unpublished construction of Andréka and Németi [3] proving the same result for finite $\alpha > 3$. (This construction can be extended to $\alpha = 3$, see [5].) This latter result is surpassed by Andréka's result mentioned above. In our treatment of cylindric algebras and quasi-polyadic equality algebras we follow [10], [11]. Thus quasi-polyadic equality algebras of dimension α are polyadic algebras in the sense of Definition 1 when cylindrifications are restricted to finite subsets and substitutions considered are only those corresponding to finite transformations. We shall prove:

Theorem 2 . *There exists an $\mathfrak{A} \in QEA_\omega$ such that $\mathfrak{Rd}_{ca}\mathfrak{A} \in RCA_\omega$, but $\mathfrak{A}$ is not representable*

Proving the analogous result for polyadic equality algebras is easy since for any PEA_ω its cylindric reduct is representable and there are easy examples of non representable PEA_ω's. But for QEA_ω the proof is much more intricate. Our example will be constructed from a *weak* set algebra. A cylindric weak set algebra is an algebra whose unit is a weak space, i.e. a set of the form ${}^\alpha U^{(p)} = \{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$ where p is a fixed sequence in ${}^\alpha U$. The operations of a weak set algebra with unit V are the Boolean operations of union, intersection and complementation with respect to V, and cylindrifications and diagonal elements are defined like in set algebras but relativized to V. We shall need to characterize abstractly (countable) quasi-polyadic equality *weak* set algebras where we require that the algebra is also closed under finite substitutions. This was done for cylindric algebras by Andréka, Németi and Thompson [2]. It turns out that, in the countable case, weak set algebras coincide with the class of weakly subdirect indecomposable algebras for both CA's and QEA's. This follows from the facts that subdirect indecomposability and its weak version are defined for general algebras via congruences, congruences correspond to ideals, and that for $\mathfrak{A} \in QEA_\alpha$, I is a quasi-polyadic ideal of $\mathfrak{A}$ if and only if it is a cylindric ideal of $\mathfrak{Rd}_{ca}\mathfrak{A}$. This ultimately makes the abstract characterization of weak set algebras for countable quasi-polyadic algebras coincide with that of (countable) cylindric algebras. Now let $WQEAs_\alpha$ denote the class of quasi-polyadic equality set algebras. Then we have $RQEA_\alpha = \mathbf{SP}WQEAs_\alpha$. Here $\mathbf{SP}$ denotes the operation of forming subdirect products. This is proved exactly like the cylindric case. Next, we give the definition of subdirect indecomposability and its weak version relative to congruences in general algebras.

Definition 3 .

(i) *An algebra* $\mathfrak{A}$ *is weakly subdirectly indecomposable if* $|A| \geq 2$ *and if the formulas* $R, S \in Co\mathfrak{A}$ *and* $R \cap S = Id \upharpoonright A$ *always imply that* $R = Id \upharpoonright A$ *or* $S = Id \upharpoonright A$.

(ii) *An algebra* $\mathfrak{A}$ *is subdirectly indecomposable if* $|A| \geq 2$ *and if for every system* R *of relations satisfying* $R \in {}^I Co\mathfrak{A}$ *and* $\bigcap_{i \in I} R_i = Id \upharpoonright A$, *there is an* i *such that* R_i *coincides with the identity relation.*

We shall need to specify ideals in quasi-polyadic equality algebras. Ideals are congruence classes containing the least element. From now on α will denote an infinite ordinal and FT_α denotes the set of finite transformations on α. $x \subseteq_\omega y$ denotes that x is a finite subset of y and $Sb_\omega\alpha$ denotes the set of all x such that $x \subseteq_\omega \alpha$.

Definition 4 . *Let* $\mathfrak{A} \in QEA_\alpha$. *A subset* I *of* $\mathfrak{A}$ *is an ideal if the following conditions are satisfied:*

(i) $0 \in I$,

(ii) *If* $x, y \in I$, *then* $x + y \in I$,

(iii) *If* $x \in I$ *and* $y \leq x$ *then* $y \in I$,

(iv) *For all* $\Gamma \subseteq_\omega \alpha$ *and* $\tau \in FT_\alpha$ *if* $x \in I$ *then* $\mathsf{c}_{(\Gamma)}x$ *and* $\mathsf{s}_\tau x \in I$.

If $X \subseteq \mathfrak{A} \in QEA_\alpha$, then $\mathfrak{Ig}^{\mathfrak{A}}X$ is the ideal generated by X.

Lemma 5 . *Let* $\mathfrak{A} \in QEA_\alpha$ *and* $X \in A$. *Then* $\mathfrak{Ig}^{\mathfrak{A}}X = \{y \in A : y \leq \mathsf{c}_{(\Gamma)}(x_0 + \ldots x_{k-1})\}$: *for some* $x \in {}^kX$, *and* $\Gamma \subseteq_\omega \alpha\}$.

Proof. Let H denote the set of elements on the right hand side. It is easy to check $H \subseteq \mathfrak{Ig}^{\mathfrak{A}}X$. Conversely, assume that $y \in H$, $\Gamma \subseteq \omega$. It is clear that $\mathsf{c}_{(\Gamma)}y \in H$. H is closed under substitutions, since for any finite transformation τ, any $x \in A$ there exists finite $\Gamma \subseteq \omega$ such that $\mathsf{s}_\tau x \leq \mathsf{c}_{(\Gamma)}x$. Now let $z, y \in H$. Assume that $z \leq \mathsf{c}_{(\Gamma)}(x_0 + \ldots x_{k-1})$ and $y \leq \mathsf{c}_{(\Delta)}(y_0 + \ldots y_{l-1})$, then

$$z + y \leq \mathsf{c}_{(\Gamma\cup\Delta)}(x_0 + \ldots x_{k-1} + y_0 \ldots + y_{l-1}).$$

The Lemma is proved. ■

It follows from [11] 2.3.8 that if $\mathfrak{A} \in QEA_\alpha$ and I is a cylindric ideal of $\mathfrak{Rd}_{ca}\mathfrak{A}$ then I is an ideal of A. Therefore $\mathfrak{A}$ is (weakly) subdirectly indecomposable if and only if $\mathfrak{Rd}_{ca}\mathfrak{A}$ is (weakly) subdirectly indecomposable. Now we prove the analogue of a result of Thompson for quasi-polyadic equality algebras. The proof is the same as that given by Andréka, Németi and Thompson in [2] theorem 3, but for the sake of completeness (and because the proof is short) we include the proof adapted to the quasi-polyadic equality (present) case. $\mathbf{I}K$ denotes the class of all isomorphic images of algebras in K. We now have:

Lemma 6 . *Let $\mathfrak{A} \in RQEA_\alpha$ be countable. Then (i) and (ii) are equivalent*

(i) $\mathfrak{A} \in \mathbf{I}WQEAs_\alpha$

(ii) $\mathfrak{A}$ *is weakly subdirectly indecomposable.*

Proof. We shall only need that $(ii) \implies (i)$. So assume that that $\mathfrak{A}$ is weakly subdirectly indecomposable quasi-polyadic algebra of dimension α. Then by [11] 2.4.46 which works for quasi-polyadic algebras, we have that

$$(*) \quad (\forall x, y \in A \sim \{0\})(\exists \Delta \subseteq_\omega \alpha) x \cdot \mathsf{c}_{(\Delta)}y \neq 0.$$

Let $a : \omega \to A \sim \{0\}$ be any enumeration of $A \sim \{0\}$. We define $\Gamma : (A \sim \{0\}) \to Sb_\omega\alpha$ step by step, so that

$$(**) \qquad b_n = \prod\{\mathsf{c}_{(\Gamma a_m)}a_m : m < n\} \neq 0 \text{ for all } n \in \omega, n \neq 0.$$

Let $\Gamma(a_0) = 0$. Let $n \in \omega$, $n > 0$, and assume that $\Gamma(a_m)$ has been defined for each $m < n$ such that $b_n \neq 0$ holds. By $(*)$, there is a $\Delta \subseteq_\omega \alpha$ such that $b_n \cdot \mathsf{c}_{(\Delta)}a_n \neq 0$. Set $\Gamma(a_n) = \Delta$. Then clearly $b_{n+1} = b_n \cdot \mathsf{c}_{(\Delta)}a_n \neq 0$. Since $A \sim \{0\} = \{a_n : n \in \omega\}$, the function Γ is defined. By $(**)$ $\Gamma : \mathfrak{A} \to Sb_\omega\alpha$ satisfies

$$(***) \quad (\forall A_0 \subseteq_\omega (A \sim \{0\})) \prod\{\mathsf{c}_{(\Gamma a)}a : a \in A_0\} \neq 0.$$

Then there is a maximal proper ideal of $Bl\mathfrak{A}$ such that $m \supseteq \{-\mathsf{c}_{(\Gamma a)}a : a \in A\}$. Let $\mathfrak{Cm}\mathfrak{A}$ be the canonical embedding algebra of $\mathfrak{A}$. $\mathfrak{Cm}\mathfrak{A}$ is defined like the CA

case [10] definition 2.7.3. In particular, it has domain $\wp(M)$, which we denote by $Em\mathfrak{A}$, where M is the set of maximal Boolean ideals of $\mathfrak{A}$. Substitutions are defined on $Em(A)$ as follows:

$$\mathsf{s}_\tau X = \bigcup_{I\in X}\{J \in M : J \subseteq \mathsf{s}_\tau I\}.$$

Let $z = \{m\}$. Then $z \in Em\mathfrak{A}$ and $0 \neq z \leq em(\mathsf{c}_{(\Gamma a)}a)$ for all $a \in A$. Here em is the map that embeds $\mathfrak{A}$ into $\mathfrak{Cm}\mathfrak{A}$; $em(x) = \{I \in M : x \notin M\}$. Let $I = \{y \in Em\mathfrak{A} : (\forall\Gamma \subseteq_\omega \alpha)\mathsf{c}_{(\Gamma)}y \cdot z = 0\}$. Then I is an ideal of $\mathfrak{Cm}\mathfrak{A}$ and $I \cap em(A) = \{0\}$. Let $\mathfrak{B} = \mathfrak{Cm}\mathfrak{A}/I$. Then $\mathfrak{B} \in RQEA_\alpha$ and $\mathfrak{A}$ is embeddable in $\mathfrak{B}$. Here we are using that if $\mathfrak{A} \in RQEA_\alpha$, then so is $\mathfrak{Cm}\mathfrak{A}$. The proof of this is identical to the CA case. Also $\mathfrak{B}$ is subdirectly indecomposable by [10] 2.4.44. By [11] 3.1.86 $\mathfrak{A}$ is isomorphic to a weak set algebra. Though 2.4.44 in [10] and 3.1.86 in [11] are formulated for CA's they are true for QEA's. ■

The following corollary which we shall need is now immediate:

Corollary 7 . *Let $\mathfrak{A} \in RQEA_\alpha$ be countable such that $\mathfrak{Rd}_{ca}\mathfrak{A}$ is weakly subdirectly indecomposable (equivalently isomorphic to a cylindric weak set algebra). Then $\mathfrak{A} \in \mathbf{I}WQEAs_\alpha$.*

From the proof of Theorem 2, we have

Corollary 8 . *There exists a countable $\mathfrak{A} \in QEA_\omega$ that is weakly subdirectly irreducible but not representable.*

Proof of Theorem 2

Let $U = \mathbb{N}$. Let $Z \in {}^\omega\wp(\mathbb{N})$ be defined by $Z_0 = Z_1 = 3 = \{0, 1, 2\}$ and $Z_i = \{2i - 1, 2i\}$ for $i > 1$. Let $p : \omega \to \omega$ be defined by $p(i) = 2i$. Let $V = {}^\omega U^{(p)} = \{s \in {}^\omega U : |\{i \in \omega : s_i \neq 2i\}| < \omega\}$. We will work inside the weak set algebra with universe $\wp(V)$ and cylindrifications and diagonal elements for $i, j < \omega$ defined for $X \subseteq V$ by:

$$\mathsf{c}_i X = \{s \in V : \exists t \in X, t(j) = s(j) \quad \forall j \neq i\}$$

and

$$\mathsf{d}_{ij} = \{s \in V : s_i = s_j\}.$$

Let

$$\mathsf{P}Z = \{s \in V : (\forall i \in \omega)s_i \in Z_i\}.$$

Let

$$t = \{s \in {}^{\omega\sim 2}U : |\{i \in \omega \sim 2 : s_i \neq 2i\}| < \omega, (\forall i > 2)s_i \in Z_i\}.$$

Let

$$X = \{s \in t : |\{i \in \omega \sim 2 : s(i) \neq 2i\}| \text{ is even } \},$$
$$Y = \{s \in t : |\{i \in \omega \sim 2 : s(i) \neq 2i\}| \text{ is odd } \},$$
$$R = \{(u,v) : u \in 3, v = u + 1 (mod 3)\},$$
$$B = \{(u,v) : u \in 3, v = u + 2 (mod 3)\},$$

and

$$a = \{s \in \mathsf{P}Z : (s \restriction 2 \in R \text{ and } s \restriction \omega \sim 2 \in X) \text{ or } (s \restriction 2 \in B \text{ and } s \restriction \omega \sim 2 \in Y)\}.$$

Let $Eq(\omega)$ be the set of all equivalence relations on ω. For $E \in Eq(\omega)$, let $e(E) = \{s \in V : kers = E\}$. Note that $e(E)$ may be empty. Let

$$d = \mathsf{P}Z \cap \mathsf{d}_{01}.$$

$\pi(\omega) = \{\tau \in FT_\omega : \tau \text{ is a bijection } \}$. For $\tau \in FT_\omega$ and $X \subseteq V$, recall that the substitution (unary) operation S_τ is defined by

$$\mathsf{S}_\tau X = \{s \in V : s \circ \tau \in X\}.$$

Let

$$P' = \{\mathsf{S}_\tau a : \tau \in \pi(\omega)\}, \quad P = P' \cup \{\mathsf{S}_\delta d : \delta \in \pi(\omega)\}.$$

More concisely,

$$P = \{\mathsf{S}_\tau x : \tau \in \pi(\omega), \quad x \in \{a, d\}\}.$$

For $W \in {}^\omega RgZ^{(Z)}$, let

$$\mathsf{P}W = \{s \in V : (\forall i \in \omega) s_i \in W_i\}.$$

Let

$$T = \{\mathsf{P}W \cdot e(E) : W \in {}^\omega RgZ^{(Z)}, (\forall \delta \in \pi(\omega)) W \neq Z \circ \delta, E \in Eq(\omega)\},$$
$$At = P \cup T,$$

and

$$A = \{\bigcup X : X \subseteq At\}.$$

Claim 1 . *A is a subuniverse of the full cylindric weak set algebra*

$$\langle \wp(V), +, \cdot, -, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j \in \omega}.$$

Furthermore $\mathfrak{A}$ *is atomic and* $At\mathfrak{A} = At \sim \{0\}$.

Notice that the Boolean operations of the algebra are denoted by $+$, $\cdot$, $-$ standing for Boolean join (union), Boolean meet (intersection) and complementation, respectively.

Proof of Claim 1. Let $b = \mathsf{P}Z \sim \mathsf{d}_{01}$. Then

(1) $a \cdot \mathsf{S}_{[0,1]}a = 0$, $a + \mathsf{S}_{[0,1]}a = b$, $(\forall i \in \omega)\mathsf{c}_i a = \mathsf{c}_i\mathsf{S}_{[0,1]}a = \mathsf{c}_i b$.

It is not difficult to check that (1) holds. One can check first $B = \mathsf{S}_{[0,1]}R$, $B \cdot R = 0, B + R = {}^2 3 - \mathsf{d}_{01}, (\forall i \in 2)\mathsf{c}_i R = \mathsf{c}_i B = \mathsf{c}_i {}^2 3, X \cdot Y = 0, X \cup Y = t$, and $(\forall i \in \omega \sim 2)\mathsf{c}_i X = \mathsf{c}_i Y = \mathsf{c}_i t$. From (1) we immediately get

(2) $\mathsf{P}Z = a + \mathsf{S}_{[0,1]}a + d$. For, $\mathsf{P}Z = \mathsf{P}Z \sim \mathsf{d}_{01} + \mathsf{P}Z \cdot \mathsf{d}_{01} = b + d$.

(3) $\mathsf{S}_\delta \mathsf{P}W = \mathsf{P}(W \circ \delta^{-1})$ for every $\delta \in \pi(\omega)$ and $W \in {}^\omega(RgZ)^{(Z)}$.

Indeed, we have $s \in \mathsf{S}_\delta\mathsf{P}W$ iff $s \circ \delta \in \mathsf{P}W$ iff $s \circ \delta_i \in W_i \quad \forall i \in \omega$ iff $s_j \in W_{\delta_j^{-1}} \quad \forall j \in \omega$ iff $s \in \mathsf{P}(W \circ \delta^{-1})$.

(4) $\mathsf{P}W \in A$ for every $W \in {}^\omega(RgZ)^{(Z)}$.

Assume $W = Z \circ \delta^{-1}$ for some $\delta \in \pi(\omega)$. Then $\mathsf{P}W = \mathsf{S}_\delta\mathsf{P}Z = \mathsf{S}_\delta a + \mathsf{S}_{\delta \circ [0,1]}a + \mathsf{S}_\delta d \in A$ by (3) and (2). Assume $W \neq Z \circ \delta, \quad \forall \delta \in \pi(\omega)$. Then by $V = \sum\{e(E) : E \in Eq(\omega)\}$ we have $\mathsf{P}W = \sum\{\mathsf{P}W \cdot e(E) : E \in Eq(\omega)\} \in A$.

(5) $(\forall x, y \in At)(x \neq y \Rightarrow x \cdot y = 0)$ and $V = \sum At$.

If $E \neq E', E, E' \in Eq(\omega)$ then $e(E) \cap e(E') = 0$ and if $W \neq W'$, $W, W' \in {}^\omega RgZ^{(Z)}$ then $\mathsf{P}W \cap \mathsf{P}W' = 0$. Thus the elements of T are disjoint from each other and from the elements of P since $(\forall x \in P)(\exists \delta \in \pi(\omega))x \subseteq \mathsf{S}_\delta\mathsf{P}Z = \mathsf{P}(Z \circ \delta^{-1})$ by (3). Let $\delta, \delta' \in \pi(\omega)$. Clearly $\mathsf{S}_\delta a \cdot \mathsf{S}_{\delta'} d = 0$ since $\mathsf{S}'_\delta a \subseteq \prod\{-\mathsf{d}_{ij} : i < j < \omega\}$ while $\mathsf{S}_{\delta'} d \subseteq \mathsf{d}_{\delta' 0 \delta' 1}$. Let $y \in \{a, d\}$ and assume $\delta' \neq \delta$. If $\delta' \neq \delta \circ [0,1]$ then [2] $Z \circ \delta'^{-1} \neq Z \circ \delta^{-1}$ hence $\mathsf{P}(Z \circ \delta^{-1}) \cap \mathsf{P}(Z \circ \delta'^{-1}) = 0$, thus $\mathsf{S}_\delta y \cdot \mathsf{S}_{\delta'} y = 0$ since $\mathsf{S}_\sigma y \subseteq \mathsf{S}_\sigma \mathsf{P}Z = \mathsf{P}(Z \circ \sigma^{-1}) \quad \forall \sigma \in \pi(\omega)$ by (3). If $\delta' = \delta \circ [0,1]$ then $\mathsf{S}_\delta a \cdot \mathsf{S}_{\delta'} a = \mathsf{S}_\delta(a \cdot \mathsf{S}_{[0,1]}a) = 0$ by (1) and $\mathsf{S}_\delta d = \mathsf{S}_\delta \mathsf{S}_{[0,1]} d = \mathsf{S}_{\delta'} d$. Thus all the elements of At are disjoint from each other. By $U = \bigcup RgZ$ we have $V = \sum\{\mathsf{P}W : W \in {}^\omega RgZ^{(Z)}\} \subseteq \sum At$ by (4). Thus $V = \sum At$.

(6) A is closed under the Boolean operations.

For, (6) is an immediate corollary of (5) and the definition of A.

[2] For, we show $\delta \neq \delta'$ and $Z \circ \delta^{-1} = Z \circ \delta'^{-1}$ imply $\delta = \delta \circ [0,1]$. Let $k \in \omega \sim 2$ and $j = \delta k$. Then $\delta_j^{-1} = k \notin 2$, hence $Z\delta_j^{-1} + Z\delta_j'^{-1}$ implies $k = \delta_j^{-1} = \delta_j'^{-1}$, i.e., $\delta, k = j$. We have seen $\delta \restriction \omega \sim 2 \subseteq \delta'$. By this and by $\delta \neq \delta'$ we have $\delta 0 = \delta' 1$ and $\delta 1 = \delta' 0$. Thus $\delta = \delta' \circ [0,1]$.

(7) Let $\mathcal{M}$ denote the minimal subalgebra of $\wp(V)$, i.e., $\mathcal{M} = \mathfrak{Sg}^{(\wp(V))}0$. Then $\mathcal{M} \subseteq A$.

Let $i < j < \omega$. Then $\mathsf{d}_{ij} = \sum\{e(E) : (i,j) \in E, E \in Eq(\omega)\} = \sum(\{\mathsf{P}W \cdot e(E) : W \in {}^{\omega}RgZ^{(Z)},\ W \neq Z \circ \delta \quad \forall \delta \in \pi(\omega),\ (i,j) \in E \in Eq(\omega)\} \bigcup \{\mathsf{S}_\delta d : \delta \in \pi(\omega), \{\delta 0, \delta 1\} = \{i,j\}\}) \in A$. Let $k < \omega$. Then $\mathsf{c}_{(k)}\bar{d}(k \times k) \in \{0, V\} \subseteq A$ by (5). Thus by [10] [2.2.24], and (6) we have $\mathcal{M} = \mathfrak{Sg}^{(\wp(V))}\{\mathsf{d}_{ij} : i < j < \omega\} \subseteq A$.

(8) $\mathsf{P}W \in A$ for every $W \in {}^{\omega}(RgZ \cup \{U\})^{(Z)}$.

Let $\Im = \{i \in \omega : W_i \neq U\}$. Then $\mathsf{P}W = \sum\{\mathsf{P}W' : W' \in {}^{\omega}RgZ, W' \restriction \Im \subseteq W\}$ by $U = \bigcup RgZ$. Thus $\mathsf{P}W \in A$ by (4).

(9) $\mathsf{S}_\tau \mathsf{P}W \in A$ for every $W \in {}^{\omega}RgZ^{(Z)}$.

For if $\mathsf{S}_\tau \mathsf{P}W = 0$, then we are done. Assume that $\mathsf{S}_\tau \mathsf{P}W \neq 0$. Let $z \in \mathsf{S}_\tau \mathsf{P}W$ be arbitrary. Let $\eta \in {}^{\omega}\omega$ such that $z_i \in Z_{\eta(i)}$. Such an η exists by $U = \bigcup RgZ$. Now we have

$$(*) \quad (\forall i \in \omega) W_i = W_{\eta\tau(i)}.$$

since $(\forall i \in \omega) z\tau(i) \in W_i \cap W_{\eta\tau(i)}$ by $z \in \mathsf{S}_\tau \mathsf{P}W$ and by the definition of η, hence $W_i = W_{\eta\tau(i)}$ since the elements of RgZ are disjoint from each other. Let $sup\tau = \{i \in \omega : \tau(i) \neq i\}$. Let $W' \in {}^{\alpha}RgZ^{(Z)}$ be defined by $(\forall i \in \omega \sim sup\tau) W'_i = W_i$ and for all $(\forall i \in sup\tau) W'_i = W_{\eta(i)}$. Then

$$\mathsf{S}_\tau \mathsf{P}W = \{s \in V : (\forall i \in \omega) s\tau(i) \in W_i\} = \mathsf{P}W'.$$

(10) $\mathsf{S}_\tau x \in A$ for every $x \in A$.

It is enough to show (10) for $x \in At$ since S_τ is additive. If $\tau, \delta \in \pi(\omega)$ then $\mathsf{S}_\tau \mathsf{S}_\delta a = \mathsf{S}_{\tau\circ\delta} a \in P \subseteq A$ since $\tau \circ \delta \in \pi(\omega)$. If $\tau \in FT_\omega \sim \pi(\omega)$ then $\mathsf{S}_\tau \mathsf{S}_\delta a = 0 \in A$. Note that $\mathsf{S}_\delta d = P(Z \circ \delta^{-1}) \cdot \mathsf{d}_{\delta 0, \delta 1}$. By (9) and the above, to finish the proof (10), it is enough to show $\mathsf{S}_\tau g \in A$ for all g of the form $\mathsf{P}W \cdot e(E)$ since S_τ is a Boolean homomorphism. Let $g = \mathsf{P}W \cdot e(E)$. Then by

$$e(E) = \prod\{\mathsf{d}_{ij} : (i,j) \in E\} \cdot \prod\{-\mathsf{d}_{ij} : (i,j) \notin E\},$$

there exists a finite $K \subseteq \{\mathsf{d}_{ij} : i < j < \omega\} \cup \{-\mathsf{d}_{ij} : i < j < \omega\}$ such that $g = \mathsf{P}W \cdot \prod K$. Here we are using that there exists $n \in \omega$ such that $\mathsf{P}W \subseteq -\mathsf{d}_{ij}$ for all $n \leq i < j$ since the elements of RgZ are disjoint from each other and $W \in {}^{\omega}RgZ^{(Z)}$. The rest follows from (9), the fact that S_τ is a Boolean homomorphism and that $\mathsf{S}_\tau \mathsf{d}_{ij} = \mathsf{d}_{\tau i \tau j} \in A$.

(11) $\mathsf{c}_i x \in A$ for every $x \in A$ and $i \in \omega$.

It is enough to show (11) for $x \in At$ since c_i is additive. Now $\mathsf{c}_i \mathsf{S}_\delta a = \mathsf{S}_\delta \mathsf{c}_{\delta_i} a$. Indeed let $j = \delta_i$. Then $\mathsf{c}_j a = \mathsf{c}_j b = \mathsf{P}Z(j/U) \cdot \gamma$ where $\gamma = 1$ if $j \in 2$ and $\gamma = -\mathsf{d}_{01}$ if $j \in \omega \sim 2$. Thus $\mathsf{c}_i \mathsf{S}_\delta a \in A$ by (10), (8) and (7). Let $x \in At \sim P'$. Then $x = \mathsf{P}W \cdot \prod K$ for some $W \in {}^{\omega}(RgZ \cup \{U\})^{(Z)}$ and $K \subseteq_\omega \{\mathsf{d}_{ij} : i < j < \omega\} \cup \{-\mathsf{d}_{ij} : i < j < \omega\}$. Assume $\mathsf{P}W \cdot \prod K \neq 0$. We will show $\mathsf{c}_i(\mathsf{P}W \cdot \prod K) = \mathsf{c}_i \mathsf{P}W \cdot \mathsf{c}_i \prod K$. Let $\Gamma = \{j \in \omega : \mathsf{d}_{ij} \in K\}$ and $\Omega = \{j \in \omega : -\mathsf{d}_{ij} \in K\}$. It is enough to show $\mathsf{c}_i \mathsf{P}W \cdot \mathsf{c}_i \prod K \subseteq \mathsf{c}_i(\mathsf{P}W \cdot \prod K)$. Let $s \in \mathsf{c}_i \mathsf{P}W \cdot \mathsf{c}_i \prod K$. Assume $\Gamma \neq 0$. Let $j \in \Gamma$. Then $s(i/s_j) \in \mathsf{P}W \cdot \prod K$ since $W_i = W_j$ by $\mathsf{P}W \cdot \prod K \neq 0$. Assume $\Gamma = 0$. Let $\Delta = \{j \in \Omega : W_j = W_i\}$. Then $|\Delta| < |W_i|$ by $\mathsf{P}W \cdot \prod K \neq 0$. Let $u \in W_i \sim \{s_i : j \in \Delta\}$. Then $s(i/u) \in \mathsf{P}W \cdot \prod K$. Thus $\mathsf{c}_i(\mathsf{P}W \cdot \prod K) = \mathsf{c}_i \mathsf{P}W \cdot \mathsf{c}_i \prod K = \mathsf{P}W(i/U) \cdot \mathsf{c}_i \prod K \in A$ by (8) and (7).

By (6), (7) and (11) we have proved $A \in Su\wp(V)$. (A is a subuniverse of $\wp(V)$). By (5) then we have $At\mathfrak{A} = At \sim \{0\}$.

The construction of $\mathfrak{B} \in QEA_\omega$:

Let $\tau, \delta \in FT_\omega$. We say that "$\tau, \delta$ transpose" iff $(\delta 0 - \delta 1).(\tau\delta 0 - \tau\delta 1)$ is negative.

Now we first define $\mathsf{s}_\sigma : At \to A$ for every $\sigma \in FT_\omega$.

$$\mathsf{s}_\sigma(\mathsf{S}_\delta a) = \begin{cases} \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a & \text{if} \quad \text{“}\sigma, \delta \quad \text{transpose”} \\ \mathsf{S}_{\sigma\circ\delta} a & \text{otherwise} \end{cases}$$

$$\mathsf{s}_\sigma(x) = \mathsf{S}_\sigma x \quad \text{if} \quad x \in At \sim P'.$$

Then we set:

$$\mathsf{s}_\sigma(\sum X) = \sum\{\mathsf{s}_\sigma(x) : x \in X\} \quad \text{for} \quad X \subseteq At.$$

We shall first prove that $\mathsf{s}_\sigma : A \to A$.

(12) From the definition of s_σ we immediately get $\mathsf{s}_\sigma \mathsf{S}_\delta a \in \{\mathsf{S}_{\sigma\circ\delta} a, \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a\}$ for $\delta \in \pi(\omega)$.

(13) $\mathsf{s}_\sigma x = \mathsf{S}_\sigma x$ for $\sigma \in FT_\omega \sim \pi(\omega)$ and $x \in At$.

If $\tau \in FT_\omega \sim \pi(\omega)$ then $\mathsf{S}_\tau a = 0$, hence $\mathsf{s}_\sigma \mathsf{S}_\delta a = 0 = \mathsf{S}_\sigma \mathsf{S}_\delta a$ by (12). For $x \in At \sim P'$ we have $\mathsf{s}_\sigma x = \mathsf{S}_\sigma x$ by definition.

(14) $\mathsf{s}_\sigma : At \to At$, is a bijection for $\sigma \in \pi(\omega)$

By (12) we have $\mathsf{s}_\sigma : P' \to P'$. Assume $\delta \neq \delta'$, $\delta, \delta' \in \pi(\omega)$. If $\delta \neq \delta' \circ [0,1]$ then $\{\sigma \circ \delta, \sigma \circ \delta \circ [0,1]\} \cap \{\sigma \circ \delta', \sigma \circ \delta' \circ [0,1]\} = 0$, hence $\mathsf{s}_\sigma \mathsf{S}_\delta a \neq \mathsf{s}_\sigma \mathsf{S}_{\delta'} a$. Assume $\delta = \delta' \circ [0,1]$. In this case "$\sigma, \delta$ transpose" iff "σ, δ' transpose", hence $\mathsf{s}_\sigma \mathsf{S}_\delta a = \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a \neq \mathsf{S}_{\sigma\circ\delta'\circ[0,1]} a = \mathsf{s}_\sigma \mathsf{S}_{\delta'} a$, by [3] $[\tau \neq \tau' \Rightarrow \mathsf{S}_\tau a \neq \mathsf{S}_{\tau'} a] \quad \forall \tau, \tau' \in \pi(\omega)$. We have seen that $\mathsf{s}_\sigma : P' \to P'$. Let $\tau = \sigma^{-1} \circ \delta$. Define $\tau' = \tau \circ [0,1]$ if "σ, τ transpose", $\tau' = \tau$ otherwise. Then "σ, τ transpose" iff "σ, τ' transpose", hence $\mathsf{s}_\sigma \mathsf{S}_{\tau'} a = \mathsf{S}_{\sigma\circ\tau} a = \mathsf{S}_\delta a$. Thus $\mathsf{s}_\sigma : P' \to P'$ is onto. By $\mathsf{s}_\sigma \mathsf{S}_\delta d = \mathsf{S}_\sigma \mathsf{S}_\delta d$ then we have $\mathsf{s}_\sigma : (P \sim P') \to (P \sim P')$. Next we show $\mathsf{s}_\sigma : T \to T$ is a bijection. Let $E \in Eq(\omega)$. Define $E(\tau) = \{(\tau i, \tau j) : (i,j) \in E\}$ for any $\tau \in FT_\omega$. Then it is not difficult to check that by $\sigma \in \pi(\omega)$ we have $E(\sigma) \in Eq(\omega)$ and $(ker(s \circ \sigma) = E$ iff $Kers = E(\sigma))$. Thus $\mathsf{S}_\sigma e(E) = e(E(\sigma))$. Now $\mathsf{s}_\sigma(\mathsf{P}W \cdot e(E)) = \mathsf{S}_\sigma(\mathsf{P}W \cdot e(E)) = \mathsf{S}_\sigma \mathsf{P}W \cdot \mathsf{S}_\sigma e(E) = \mathsf{P}(W \circ \sigma^{-1}) \cdot e(E(\sigma)) \in T$ if $W \neq Z \circ \delta$ for any $\delta \in \pi(\omega)$. If $W \neq W'$ or $E \neq E'$ then $W \circ \sigma^{-1} \neq W' \circ \sigma^{-1}$ or $E(\sigma) \neq E'(\sigma)$, thus $\mathsf{s}_\sigma : T \to T$ is one to one. The fact that $\mathsf{s}_\sigma(\mathsf{P}(W \circ \sigma^{-1}) \cdot e(E(\sigma^{-1}))) = \mathsf{P}W \cdot e(E)$ shows that $\mathsf{s}_\sigma : T \to T$ is onto.

Now we have proved that $\mathsf{s}_\sigma : A \to A$. Define

$$\mathfrak{B} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{s}_\tau, \mathsf{d}_{ij} \rangle_{i,j\in\omega, \tau\in FT_\omega}.$$

Claim 2 . $\mathfrak{B} \in QEA_\omega$

We shall proceed via several steps.

(15) s_τ is a Boolean homomorphism on $\mathfrak{A}$, for any $\tau \in FT_\omega$.

If $\tau \in \pi(\omega)$ then (15) follows from (14) and from the definition of s_σ. If $\tau \in FT_\omega \sim \pi(\omega)$ then (15) follows from (13).

(16) $\mathsf{s}_\tau \mathsf{s}_\sigma \mathsf{S}_\delta a = \mathsf{s}_{\tau\circ\sigma} \mathsf{S}_\delta a$ for $\delta \in \pi(\omega)$, $\tau, \sigma \in FT_\omega$.

Assume $\delta 0 < \delta 1$.
Case 1: $\tau\sigma\delta 0 < \tau\sigma\delta 1$. Then $\mathsf{s}_{\tau\circ\sigma} \mathsf{S}_\delta a = \mathsf{S}_{\tau\circ\sigma\circ\delta} a$ since "$\tau \circ \sigma, \delta$ do not transpose ". If $\sigma\delta 0 < \sigma\delta 1$ then "σ, δ do not transpose " and "$\tau, \sigma \circ \delta$ do not transpose ", hence $\mathsf{s}_\sigma \mathsf{S}_\delta a = \mathsf{S}_{\sigma\circ\delta} a$ and $\mathsf{s}_\tau \mathsf{S}_{\sigma\circ\delta} a = \mathsf{S}_{\tau\circ\sigma\circ\delta} a$ and we are done. Similarly, if $\sigma\delta 0 > \sigma\delta 1$ then $\mathsf{s}_\sigma \mathsf{S}_\delta a = \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a$ and

[3]This follows from the proof of (5).

$\mathsf{s}_\tau \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a = \mathsf{S}_{\tau\circ(\sigma\circ\delta\circ[0,1])\circ[0,1]} a = \mathsf{S}_{\tau\circ\sigma\circ\delta} a$ and we are done.

Case 2: $\tau\sigma\delta 0 > \tau\sigma\delta 1$. Then $\mathsf{s}_{\tau\circ\sigma}\mathsf{S}_\delta a = \mathsf{S}_{\tau\circ\sigma\circ\delta\circ[0,1]} a$. If $\sigma\delta 0 < \sigma\delta 1$ then $\mathsf{s}_\sigma \mathsf{S}_\delta a = \mathsf{S}_{\sigma\circ\delta} a$ and $\mathsf{s}_\tau \mathsf{S}_{\sigma\circ\delta} a = \mathsf{S}_{\tau\circ\sigma\circ\delta\circ[0,1]} a$ and we are done. If $\sigma\delta 0 > \sigma\delta 1$ then $\mathsf{s}_\sigma \mathsf{S}_\delta a = \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a$ and $\mathsf{s}_\tau \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a = \mathsf{S}_{\tau\circ\sigma\circ\delta\circ[0,1]} a$.

The case $\delta 0 > \delta 1$ is completely analogous, hence we omit it.

(17) $\mathsf{s}_\sigma(\mathsf{S}_\delta a + \mathsf{S}_{\sigma\circ[0,1]} a) = \mathsf{S}_\sigma(\mathsf{S}_\delta a + \mathsf{S}_{\sigma\circ[0,1]} a)$.

"σ, δ transpose " iff "$\sigma, \delta\circ[0,1]$ transpose ". Hence $\{\mathsf{s}_\sigma \mathsf{S}_\delta a, \mathsf{s}_\sigma \mathsf{S}_{\delta\circ[0,1]} a\} = \{\mathsf{S}_{\sigma\circ\delta} a, \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a\}$ by the definition of s_σ.

(18) $\mathsf{s}_\tau \mathsf{s}_\sigma x = \mathsf{s}_{\tau\circ\sigma} x$ for every $\tau, \sigma \in FT_\omega$ and $x \in A$.

It is enough to show (18) for $x \in At$. For $x \in P'$, (18) is true by (16). Let $x \in At \sim P'$. Then $\mathsf{s}_\sigma x = \mathsf{S}_\sigma x$ by definition. Now $\mathsf{S}_\sigma x = \mathsf{P}W \cdot \prod K$ for some $W \in {}^\omega(RgZ \cup \{U\})^{(Z)}$ and $K \subseteq_\omega \{\mathsf{d}_{ij} : i < j < \omega\} \cup \{-\mathsf{d}_{ij} : i < j < \omega\}$, by the proof of (10). Assume $\mathsf{S}_\delta a \subseteq \mathsf{S}_\sigma x$ for some $\delta \in \pi(\omega)$. We will show that then $\mathsf{S}_{\delta\circ[0,1]} a \subseteq \mathsf{S}_\sigma x$, too. $\mathsf{S}_\delta a \cup \mathsf{S}_{\delta\circ[0,1]} a \subseteq \mathsf{P}Z \circ \delta^{-1} \cdot e(Id_\omega)$, thus $\mathsf{S}_\delta a \le \mathsf{S}_\sigma x$ implies $[\mathsf{P}Z \circ \delta^{-1} \cdot e(Id_\omega)] \cap \mathsf{P}W \cdot \prod K \neq 0$. But then $\mathsf{P}Z \circ \delta^{-1} \cdot e(Id_\omega) \subseteq \mathsf{P}W \cdot \prod K$, thus $\mathsf{S}_{\delta\circ[0,1]} a \subseteq \mathsf{S}_\sigma x$, too. Thus $\mathsf{s}_\tau \mathsf{s}_\sigma x = \mathsf{S}_\tau \mathsf{S}_\sigma x = \mathsf{S}_{\tau\circ\sigma} x = \mathsf{s}_{\tau\circ\sigma} x$ by (17) and by the definition of $\mathsf{s}_\tau, \mathsf{s}_\sigma, \mathsf{s}_{\tau\circ\sigma}$.

(19) $\mathsf{c}_{(\Gamma)} \mathsf{S}_\delta a = \mathsf{c}_{(\Gamma)}(\mathsf{S}_\delta a + \mathsf{S}_{\delta\circ[0,1]} a)$ if $\Gamma \subseteq_\omega \omega$, $\Gamma \neq 0$.

Let $i \in \omega$ be arbitrary. Then $\mathsf{c}_i a = \mathsf{c}_i(a + \mathsf{S}_{[0,1]} a)$ holds by (1). Thus $\mathsf{c}_i \mathsf{S}_\delta a = \mathsf{S}_\delta \mathsf{c}_{\delta_i} a = \mathsf{S}_\delta \mathsf{c}_{\delta_i}(a + \mathsf{S}_{[0,1]} a) = \mathsf{c}_i(\mathsf{S}_\delta a + \mathsf{S}_{\delta\circ[0,1]} a)$.

(20) $\mathsf{s}_\sigma \mathsf{c}_{(\Gamma)} x = \mathsf{S}_\sigma \mathsf{c}_{(\Gamma)} x$ for every $x \in A$ if $\Gamma \subseteq_\omega \omega, \Gamma \neq 0$.

Let $x \in A$ be arbitrary. Then $\mathsf{c}_{(\Gamma)} x = \sum X$ for some $X \subseteq At$, by (11). Assume $\mathsf{S}_\delta a \in X$. Then $\mathsf{c}_{(\Gamma)} \mathsf{S}_\delta a \subseteq \mathsf{c}_{(\Gamma)} x$, hence $\mathsf{S}_{\delta\circ[0,1]} a \subseteq \mathsf{c}_{(\Gamma)} x$ by (19). Therefore $\mathsf{S}_{\delta\circ[0,1]} a \in X$, too. Now $\mathsf{s}_\sigma \mathsf{c}_{(\Gamma)} x = \sum\{\mathsf{s}_\sigma y : y \in X\}$ and (17) finish the proof of (20).

(21) $\sigma \restriction (\omega \sim \Gamma) = \tau \restriction (\omega \sim \Gamma) \Rightarrow \mathsf{s}_\sigma \mathsf{c}_{(\Gamma)} x = \mathsf{s}_\tau \mathsf{c}_{(\Gamma)} x$ for every $x \in A$, $\sigma, \tau \in FT_\omega$, and $\Gamma \subseteq_\omega \omega$.

(21) follows from (20).

(22) $\mathsf{c}_{(\Gamma)}\mathsf{s}_\sigma x = \mathsf{c}_{(\Gamma)}\mathsf{S}_\sigma x$, for every $x \in A$, $\sigma \in FT_\omega$, if $\Gamma \subseteq_\omega \omega, \Gamma \neq 0$.

It is enough to check (22) for $x \in P'$. Let $\delta \in \pi(\omega)$. Then $\mathsf{s}_\sigma \mathsf{S}_\delta a \in \{\mathsf{S}_{\sigma\circ\delta} a, \mathsf{S}_{\sigma\circ\delta\circ[0,1]} a\}$ by definition of s_σ and $\mathsf{c}_{(\Gamma)}\mathsf{S}_{\sigma\circ\delta\circ[0,1]} a = \mathsf{c}_{(\Gamma)}\mathsf{S}_{\sigma\circ\delta} a = \mathsf{c}_{(\Gamma)}\mathsf{S}_\sigma \mathsf{S}_\delta a$ by (19).

(23) $\tau \restriction (\tau^{-1}\Gamma)$ is one - one then $\mathsf{c}_{(\Gamma)}\mathsf{s}_\tau x = \mathsf{s}_\tau \mathsf{c}_{(\Delta)} x$ where $\Delta = \tau^{-1}\Gamma$.

If $\Gamma = 0$ then $\Delta = 0$ and we are done. If $\Gamma \neq 0$ and $\Delta = 0$ then $\pi \notin \pi(\omega)$ hence we are done by (13). Assume $\Gamma \neq 0, \Delta \neq 0$. Then we are done by (22) and (20).

Now we are ready to show $\mathfrak{B} \in QPEA_\omega$. We have to show that $(1-15)$ in definition 1 are satisfied in $\mathfrak{B}$. $(1-6)+13$ are satisfied since $\mathfrak{Rd}_{ca}\mathfrak{B} \in Ws_\omega$. 7 holds because "$Id_\omega, \delta$ don't transpose" $\forall \delta \in FT_\omega$. $8, 11, 12$ hold by (18), (22), (23) respectively. $9-10$ are satisfied by (15). 14 holds by (13) and 15 holds since $\mathsf{s}_\tau \mathsf{d}_{ij} = \mathsf{S}_\tau \mathsf{d}_{ij}$ by definition of s_τ.

We finally show:

Claim 3 . $\mathfrak{B} \notin RPEA_\omega$.

Proof. Assume $\mathfrak{B} \in RPEA_\omega$. Then by Corollary 7 $\mathfrak{B}$ is isomorphic to some weak set algebra $\mathfrak{C}$ since $\mathfrak{Rd}_{ca}\mathfrak{B}$ is weakly subdirectly indecomposable. Let U' be the base of $\mathfrak{C}$. The unit of $\mathfrak{C}$ is of the form ${}^\alpha U'^{(p)}$ for some sequence p. Let $h : \mathfrak{B} \twoheadrightarrow \mathfrak{C}$ be an isomorphism. Let $x = Z_0 \times U \times U \times U \times Z_5 \times Z_6 \ldots$. That is $x = \{s \in V : s_0 \in Z_0 : (\forall i > 4)(s_i \in Z_i)\}$. Then $x \in A$ by (8), and $\mathsf{c}_i x = x$ for $i \in \{1,2,3\}$. So $\mathsf{c}_i h(x) = h(x)$ for $i \in \{1,2,3\}$, thus $h(x) = Z' \times U' \times U' \times U' \times \ldots$. for some $Z' \subseteq U'$. Let $\bar{x} = \prod\{\mathsf{s}_{[0,i]} x : i \in 4\}$. Then $\bar{x} = Z_0 \times Z_0 \times Z_0 \times Z_0 \times Z_5 \times Z_6 \ldots$. For a relation R, recall that $\bar{d}(R) = \prod_{(i,j)\in R \sim Id} -\mathsf{d}_{ij}$. Then we have $\bar{x} \cdot \bar{d}(3\times 3) \neq 0$ and $\bar{x} \cdot \bar{d}(4 \times 4) = 0$ imply the same for $h(x)$, therefore $|Z'| = 3$.

Let $b' = h(b), a' = h(a), g = \mathsf{S}_{[0,1]} a, g' = h(g)$. Then $b \leq x \cdot \mathsf{s}_{[0,1]} x - \mathsf{d}_{01}$ hence $b' \subseteq h(x) \cdot \mathsf{S}_{[0,1]} h(x) - \mathsf{d}_{01}$, thus

$$\forall s \in b' \quad (s_0, s_1) \in {}^2Z' \sim \mathsf{d}_{01} \quad \text{and} \quad |Z'| = 3. \qquad (\star)$$

In $\mathfrak{A}$ we have $a + g = b \neq 0, a \cdot g = 0, \mathsf{s}_{[0,1]} a = a, \mathsf{s}_{[0,1]} g = g$ and $\mathsf{c}_i a = \mathsf{c}_i g = \mathsf{c}_i b \quad \forall i \in 2$.

Therefore

$$(*) \quad a' + g' = b' \neq 0, a' \cdot g' = 0$$

$$(**) \quad \mathsf{S}_{[0,1]}a' = a', \mathsf{S}_{[0,1]}g' = g' \quad \text{and}$$
$$(***) \quad \mathsf{c}_i a' = \mathsf{c}_i g' = \mathsf{c}_i b' \quad \forall i \in 2$$

Let $q \in b'$ be arbitrary. q^{01}_{uv} is the function q' that agrees with q everywhere except that $q'(0) = u$ and $q'(1) = v$. Define

$$\bar{a} = \{(u, v) : q^{01}_{uv} \in a'\}$$

and

$$\bar{g} = \{(u, v) : q^{01}_{uv} \in g'\}.$$

Then by $(*) - (\star)$ we have

$$(*)' \quad \bar{a} + \bar{g} = {}^2Z' \sim \mathsf{d}_{01}, \bar{a} \cdot \bar{g} = 0,$$

$$(**)' \quad \mathsf{S}_{[0,1]}\bar{a} = \bar{a}, \mathsf{S}_{[0,1]}\bar{g} = \bar{g} \quad \text{and}$$
$$(***)' \quad \mathsf{c}_0\bar{a} = \mathsf{c}_0\bar{g} = \mathsf{c}_0{}^2Z'.$$

We show that $(*)' - (***)'$ together with $|Z'| = 3$ is impossible. By $(***)'$ we have $Rg\bar{a} = Rg\bar{g} = Z'$, hence $|\bar{a}| \geq 3$ and $|\bar{g}| \geq 3$. By $(*')$ we have then $|\bar{a}| = |\bar{g}| = 3$ by $\bar{a} \cdot \bar{g} = 0$ and $|{}^2Z'_1 \sim \mathsf{d}_{01}| = 6$. But by $(**)'$ and $\bar{a} \leq -\mathsf{d}_{01}$ we have $|\bar{a}| \geq 4$, contradiction. ■

Claims 1-3 prove Theorem 2.

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