Axiomatizability of positive algebras of binary relations

Hajnal Andréka and Szabolcs Mikulás

Abstract. We consider all positive fragments of Tarski's representable relation algebras and determine whether the equational and quasiequational theories of these fragments are finitely axiomatizable in first-order logic. We also look at extending the signature with reflexive, transitive closure and the residuals of composition.

1. Introduction

Tarski defined the class RRA of *representable relation algebras*, see [14, 20] for recent monographs, a class of algebras of binary relations with the Boolean connectives join $+$, meet $\cdot$, complement $-$ and the bottom 0 and top 1 elements, and the extra-Boolean connectives of relation composition ;, relation converse $\smile$ and the identity constant $1'$. RRA is a variety, but it is not finitely axiomatizable [22]. An important line of research in algebraic logic is to investigate fragments of RRA from the finite axiomatizability point of view. That is, we consider less expressive languages than that of RRA, typically subsignatures of RRA, and try to figure out if these versions of algebras of binary relations admit finite axiomatizations. See [27] for a somewhat outdated survey and [21] for related results.

In this paper, we mainly concentrate on *positive* similarity types Λ: the languages are restricted to subsets of $\{+, \cdot, 0, 1, ;, \smile, 1'\}$. Furthermore, we will assume that composition ; is present and that there is at least one semilattice operation (either $+$ or $\cdot$) in Λ so that an ordering $\leq$ is definable. We will systematically go through these similarity types and check if the equational and quasiequational theories are finitely axiomatizable. Since these fragments are usually not closed under homomorphic images, the problems of finitely axiomatizing the class and the variety generated by it become separate. Many of the results have been known (we will recall these from the literature), but fragments including the identity constant $1'$ attracted less attention. In particular, we will present maximal, finitely axiomatizable fragments involving the identity in Theorem 4.1 and prove this result using term graphs. We also include the proof of the non-finite axiomatizability of fragments involving join and

---

2010 *Mathematics Subject Classification*: Primary: 03G15; Secondary: 06F05.

Key words and phrases: representable relation algebras, finite axiomatizability, ordered semigroups.

Andréka's research was supported by OTKA grant Nos. 73601 and 81188.

composition, see Theorem 3.1, since this has been available only in a preprint [1]. Finally, we will look at larger similarity types that include the residuals of composition, $\backslash$ and $/$, and/or the Kleene star, *. We prove a strengthening of Theorem 3.1 where the similarity type can include any of $\backslash$, $/$ and * as well, Theorem 5.1.

2. Main definitions

First we give a formal definition of positive algebras of relations.

Definition 2.1. Let Λ be a signature such that $\Lambda \subseteq \{+, \cdot, 0, 1, ;, ^\smile, 1'\}$ and $\mathfrak{A} = (A, \Lambda)$ be an algebra. We say that $\mathfrak{A}$ is a *positive algebra of relations* if $A \subseteq \mathcal{P}(U \times U)$ for some set U, the *base* of $\mathfrak{A}$, and each operation in Λ is interpreted as follows: $+$ is union, $\cdot$ is intersection, ; is interpreted as *composition* of relations

$$x \,;\, y = \{(u, v) \in U \times U : \exists w((u, w) \in x \text{ and } (w, v) \in y)\}$$

$^\smile$ is interpreted as *converse* of relations

$$x^\smile = \{(u, v) \in U \times U : (v, u) \in x\}$$

$1'$ is the *identity* constant

$$1' = \{(u, v) \in U \times U : u = v\}$$

0 is the empty set and 1 is a subset of $U \times U$ such that 1 is the top element of $\mathfrak{A}$.

We say that an algebra $\mathfrak{A}$ is *representable* if it is isomorphic to a positive algebra of relations. We denote the class of representable Λ-algebras by $\mathsf{R}(\Lambda)$, while $\mathsf{V}(\Lambda)$ stands for the variety generated by $\mathsf{R}(\Lambda)$.

We note that $\mathsf{R}(\Lambda)$ is a quasivariety. Indeed, it is not difficult to see that it is closed under subalgebras and the formation of products, and it is closed under ultraproducts as well (since it is pseudo-axiomatizable using first-order logic, see, e.g., [6, proof of Theorem 2, pp. 141–146], and [24, sections 3–5]). Since every representable algebra $\mathfrak{A}$ is (isomorphic to) a positive algebra of relations, there is an isomorphism h from $\mathfrak{A}$ into an algebra whose elements are subsets of a relation $U \times U$ and the operations are interpreted as in the definition above. In such a case, we say that $\mathfrak{A}$ is represented on $W = \bigcup\{h(a) : a \in A\} \subseteq U \times U$. Note that, in general, $\mathsf{V}(\Lambda)$ may contain non-representable algebras, since $\mathsf{R}(\Lambda)$ may not be closed under homomorphic images.

Remark 2.2. An alternative definition of representable algebras can be given by considering positive subreducts (subalgebras of reducts) of representable relation algebras, RRA. In that case, the top element 1 must be an equivalence relation regardless of the other operations present in Λ. We feel that it is natural to be more permissive and allow a top element, say, to be irreflexive if the 1-free reduct can be represented on an irreflexive relation (see the case $\mathsf{R}(\cdot, ;)$ below).

However, the presence of certain operations in Λ forces properties on 1:

- $; \in \Lambda$ implies that 1 is transitive
- $\smile \in \Lambda$ implies that 1 is symmetric
- $1' \in \Lambda$ implies that 1 is reflexive.

Hence, if $\{;, \smile, 1'\} \subseteq \Lambda$, then 1 must be an equivalence relation and it follows that our definition coincides with subreducts of RRA.

We summarize the main results in Table 1, where

- Λ is a positive RRA-subsignature containing composition ; and at least one of the lattice operations join + or meet $\cdot$
- we do not mention the constants 0 and 1, since including them in Λ does not change the results; see below.

	$\mathsf{R}(\Lambda)$	$\mathsf{V}(\Lambda)$
$\Lambda = \{\cdot, ;\}$	Yes, [8]	Yes, [8]
$\Lambda = \{\cdot, ;, 1'\}$	No, [15]	Yes, Thm. 4.1(1)
$\Lambda \supseteq \{\cdot, ;, \smile\}$	No, [12, 17]	No, [17]
$\Lambda \supseteq \{+, ;\}$	No, [1, 2]	
$\Lambda \not\supseteq \{\cdot, ;, \smile\}$		Yes, [2, 7], Thm. 4.1

Table 1. Finite axiomatizability of positive RRA fragments

Note that the cases covered by the empty entries follow from other entries. Of the quasivarieties only the case $\mathsf{R}(\cdot, ;)$ is finitely axiomatizable, all the others are non-finitely axiomatizable. The situation concerning the varieties is simple, too: $\mathsf{V}(\Lambda)$ is finitely axiomatizable if and only if $\Lambda \not\supseteq \{\cdot, ;, \smile\}$. When do the quasivarieties and varieties coincide? We conjecture that this happens only in the case $\{\cdot, ;\}$ and that in all other cases they are different, see Problem 5.2. We will provide more explanation in the following two sections.

In Section 3 we will go through the results concerning quasivarieties $\mathsf{R}(\Lambda)$ and in Section 4 we discuss the results about varieties $\mathsf{V}(\Lambda)$. Finally, in Section 5, we extend some of the results to reflexive, transitive closure and the residuals of composition.

3. Quasivarieties

In this section, we look at the finite axiomatizability of the quasivarieties of the representable algebras. In most cases, we have non-finite axiomatizability — these proofs use ultraproduct constructions: there are non-representable algebras whose ultraproduct is representable.

Lower semilattice ordered semigroups. The class $\mathsf{R}(\cdot, ;)$ is finitely axiomatizable by equations, hence coincides with the variety $\mathsf{V}(;, \cdot)$, see [8, Corollary 7]. The axioms are

- $\cdot$ is a semilattice operation

$$a \cdot b = b \cdot a \quad a \cdot a = a \quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{SL}$$

- ; is a semigroup operation

$$(a\,;b)\,;c = a\,;(b\,;c) \tag{SG}$$

- ; is monotonic w.r.t. the ordering $\leq$ (defined in the usual way: $x \leq y$ iff $x \cdot y = x$)

$$(a \cdot b)\,;(c \cdot d) \leq a\,;c \tag{Mon}$$

This result can be proved by a step-by-step construction of a transitive and irreflexive representation, see [21].

Including 0 can be done by requiring normality

$$0\,;a = 0 = a\,;0 \tag{Nor}$$

and that 0 is the bottom element

$$0 \leq a \tag{Bot}$$

while 1 requires stating that 1 is the top element

$$a \leq 1 \tag{Top}$$

Lower semilattice ordered monoids. The class $\mathsf{R}(\cdot, ;, 1')$ is not finitely axiomatizable, [15]. The same holds if we include 0 and/or a top element 1 (whose representation is a reflexive and transitive relation).

Lower semilattice ordered involuted semigroups. For

$$\{\cdot, ;, {}^{\smile}\} \subseteq \Lambda \subseteq \{+, \cdot, 0, 1, ;, {}^{\smile}, 1'\}$$

the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable, see [12] and [17, Theorem 2.3].

Upper semilattice ordered semigroups. For

$$\{+, ;\} \subseteq \Lambda \subseteq \{+, 0, 1, ;, {}^{\smile}, 1'\}$$

the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable, [1]. Since [1] is not widely available, we will recall the proof below.

Distributive lattice ordered semigroups. For

$$\{+, \cdot, ;\} \subseteq \Lambda \subseteq \{+, \cdot, 0, 1, ;, {}^{\smile}, 1'\}$$

the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable, [2, Theorem 4].

3.1. Non-finite axiomatizability of representable upper semilattice ordered semigroups. We recall the following from [1].

Theorem 3.1. *Let* $\{+, ;\} \subseteq \Lambda \subseteq \{+, 0, 1, ;, ^\smile, 1'\}$. *The class* $\mathsf{R}(\Lambda)$ *is not finitely axiomatizable.*

Proof. For every natural number n we construct an algebra

$$\mathfrak{A}_n = (A_n, +, 0, 1, ;, ^\smile, 1')$$

such that

(1) the $\{+, ;\}$-reduct of $\mathfrak{A}_n$ is not representable
(2) any non-trivial ultraproduct over ω, $\mathfrak{A}$, of $\mathfrak{A}_n$ is representable.

We define

$$G_n = \{a, a_1', a_1'', \ldots, a_n', a_n'', b, b_1', b_1'', \ldots, b_n', b_n'', o, 1', 0\}$$

Let $(A_n, +)$ be the free upper semilattice generated freely by G_n under the defining relations:

$$\{a \leq a_i' + a_i'',\ b \leq b_i' + b_i'',\ 0 + x = x : 1 \leq i \leq n, x \in G_n\}$$

Let S denote the following set of two-element subsets of A_n:

$$S = \big\{\{a, b_1'\}\big\} \cup \big\{\{a_i', b_i''\} : 1 \leq i \leq n\big\} \cup \big\{\{a_i'', b_{i+1}'\} : 1 \leq i < n\big\} \cup \big\{\{a_n'', b\}\big\}$$

Next we define the rest of the operations on A_n as follows.

$$0 = \emptyset \qquad 1 = \sum G_n \qquad x^\smile = x$$
$$0 \,;\, x = 0 = x \,;\, 0 \qquad 1' \,;\, x = x = x \,;\, 1'$$
$$\text{if } x, y \notin \{0, 1'\}, \text{ then } x \,;\, y = \begin{cases} o & \text{if } \{x, y\} \in S \\ 1 & \text{otherwise} \end{cases}$$

(1) We define the quasiequation q_n as

$$\bigwedge_{i=1}^{n} (x \leq x_i' + x_i'' \wedge y \leq y_i' + y_i'') \rightarrow$$
$$x \,;\, y \leq x \,;\, y_1' + \sum_{i=1}^{n-1} (x_i' \,;\, y_i'' + x_i'' \,;\, y_{i+1}') + x_n' \,;\, y_n'' + x_n'' \,;\, y \tag{3.1}$$

By an induction on n one can show that q_n is valid in representable algebras. On the other hand, the evaluation ϵ given by

$$\epsilon(x) = a \quad \epsilon(x_i') = a_i' \quad \epsilon(x_i'') = a_i'' \quad \epsilon(y) = b \quad \epsilon(y_i') = b_i' \quad \epsilon(y_i'') = b_i''$$

falsifies q_n in $\mathfrak{A}_n$ (since $a \,;\, b = 1$ and each term on the right of $\leq$ in the consequent evaluates to o). Since q_n uses only the operations ; and +, it follows that already the $\{+, ;\}$-reduct of $\mathfrak{A}_n$ is not representable.

(2) We will build a representation of the ultraproduct $\mathfrak{A}$ in a step-by-step manner. First we describe $\mathfrak{A}$ to some extent using first-order properties of $\mathfrak{A}_n$. The set of *atoms* consists of all minimal, non-zero elements. An element

c is called *join-prime* if for all $d, e \in A$, $c \leq e + d$ implies $c \leq d$ or $c \leq e$. The set of atoms of $\mathfrak{A}_n$ is precisely G_n, and exactly two of them, a and b, are not join-prime. Let At denote the set of atoms of $\mathfrak{A}$, and $\bar{x}$ denote the image (under the natural embedding, note that $A_n \subseteq A_{n+1}$) of an element x of $\mathfrak{A}_n$ in $\mathfrak{A}$. Then we have the following.

- There are exactly two elements, $\bar{a}$ and $\bar{b}$, of At which are not join-prime.
- Every element of $\mathfrak{A}$ is the supremum of the atoms below it.
- Every atom is self converse: $x^{\smile} = x$. The atom $\bar{1}'$ is the identity in $\mathfrak{A}$: $x \mathbin{;} \bar{1}' = \bar{1}' \mathbin{;} x = x$.

Observing more first-order properties of $\mathfrak{A}_n$ we will arrive at the following. There is an atom $\bar{o}$, subsets A', A'' and B', B'' of At and two functions p and s such that the following hold.

- $At = \{\bar{a}, \bar{b}, \bar{1}', \bar{o}\} \cup A' \cup A'' \cup B' \cup B''$ is a disjoint union.
- The functions

  $$p\colon A' \cup B' \to A'' \cup B'' \qquad p \restriction A'\colon A' \to A'' \qquad p \restriction B'\colon B' \to B''$$
  $$s\colon \{\bar{a}\} \cup A' \cup A'' \to B' \cup B'' \cup \{\bar{b}\} \qquad s \restriction A'\colon A' \to B''$$
  $$s \restriction A''\colon A'' \to B' \smallsetminus \{s(\bar{a})\}$$

  are all bijections, $s(\bar{a}) \in B'$ and $s^{-1}(\bar{b}) \in A''$. See Figure 1, where different occurrences of α', α'', β' and β'' denote distinct elements of A', A'', B' and B'', respectively.

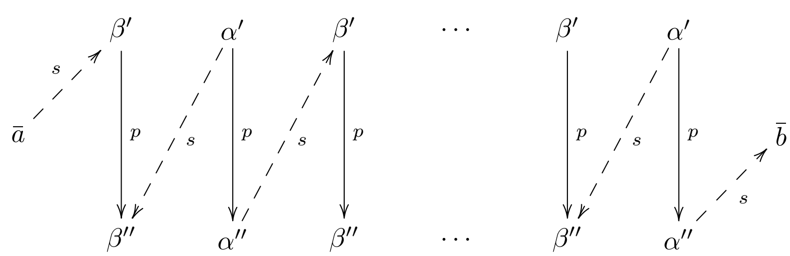

FIGURE 1. Functions p and s

- $\bar{a} \leq x + p(x)$ and $\bar{b} \leq y + p(y)$ for any $x \in A'$ and $y \in B'$. For any $x, y \in At \smallsetminus \{\bar{1}'\}$,

  $$x \mathbin{;} y = \begin{cases} \bar{o} & \text{if } y = s(x) \text{ or } x = s(y) \\ \bar{1} & \text{otherwise} \end{cases}$$

- There is no finite, undirected $\{s, p\}$-cycle in At. There is no finite, undirected $\{s, p\}$-path between $\bar{a}$ and $\bar{b}$.

We will define a labelled graph $\mathcal{G}$ whose edges are labelled by *prime filters* F of $\mathfrak{A}$ (i.e., F is upward closed and $x + y \in F$ implies $x \in F$ or $y \in F$). For any join-prime atom x, the principal filter $F(x)$ generated by x is a prime filter.

Since $\bar{a}$ and $\bar{b}$ are not join-prime, we have to "replace" them by prime filters when representing $\mathfrak{A}$. To this end we let Y_0 be the set of those atoms from which there is an undirected $\{s,p\}$-path to $\bar{b}$, and define

$$\ell(\bar{a}) = \{\bar{a}\} \cup (A'' \setminus Y_0) \cup (A' \cap Y_0) \quad \ell'(\bar{a}) = \{\bar{a}\} \cup (A' \setminus Y_0) \cup (A'' \cap Y_0)$$
$$\ell(\bar{b}) = \{\bar{b}\} \cup (B'' \setminus Y_0) \cup (B' \cap Y_0) \quad \ell'(\bar{b}) = \{\bar{b}\} \cup (B' \setminus Y_0) \cup (B'' \cap Y_0)$$

For $X \subseteq A$, let $F(X)$ denote the filter generated by X in $\mathfrak{A}$. In case $X = \{x\}$ is a singleton, we use the notation $F(x)$. For subsets X, Y of elements, we use the notation $X \,;\, Y = \{x \,;\, y : x \in X, y \in Y\}$.

Consider the subset $\mathcal{F}$ of filters of $\mathfrak{A}$:

$$\mathcal{F} = \big\{F(x), F(\{x, \bar{o}\}) : x \in At \setminus \{\bar{a}, \bar{b}\}\big\} \cup$$
$$\big\{F(Z), F(Z \cup \{\bar{o}\}) : Z \in \{\ell(\bar{a}), \ell'(\bar{a}), \ell(\bar{b}), \ell'(\bar{b})\}\big\}$$

Note that these filters are prime, $F(X)$ contains $\bar{1}'$ precisely when $\bar{1}' \in X$ and, for any element $x \in A' \cup A''$, we have $x \in F(\ell(\bar{a}))$ iff $x \in F(\ell(\bar{a}) \cup \{o\})$ iff $x \notin F(\ell'(\bar{a}) \cup \{o\})$ iff $x \notin F(\ell'(\bar{a}))$, and similarly for $y \in B' \cup B''$, $\ell(\bar{b})$ and $\ell'(\bar{b})$. Furthermore, $F(\ell(\bar{a})) \,;\, F(\ell(\bar{a})) = F(\ell(\bar{a})) \,;\, F(\ell(\bar{b})) = F(\ell(\bar{b})) \,;\, F(\ell(\bar{b})) = \{\bar{1}\}$ and $F(\ell'(\bar{a})) \,;\, F(\ell(\bar{b})) = F(\ell(\bar{a})) \,;\, F(\ell'(\bar{b})) = \{\bar{o}, \bar{1}\}$.

We are ready to define the labelled graph $\mathcal{G} = (U, E, \ell)$ where U is the set of vertices, $E \subseteq U \times U$ is the reflexive, transitive and symmetric set of edges and ℓ is a labelling function associating with every $(u,v) \in E$ a prime filter from $\mathcal{F}$.

Base step. For every $F \in \mathcal{F}$ such that $\bar{1}' \notin F$, we choose distinct points u, v such that u and v are different for distinct elements of $\mathcal{F}$. We let

$$\ell_0(u,v) = \ell_0(v,u) = F \text{ and } \ell_0(u,u) = \ell_0(v,v) = F(\bar{1}')$$

see Figure 2. We define $\mathcal{G}_0 = (U_0, E_0, \ell_0)$ as the disjoint union of these graphs.

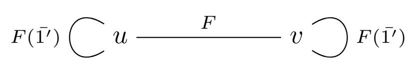

Figure 2. Base step

Note that, by the definition of $\mathcal{F}$, for any pair (x,y) of distinct atoms, there is $(u,v) \in E_0$ such that $x \in \ell_0(u,v)$ but $y \notin \ell_0(u,v)$. Observe that $F(\bar{1}') \,;\, F = F \,;\, F(\bar{1}') = F$ and $F \,;\, F = \{\bar{1}\} \subseteq F(\bar{1}')$.

Inductive step. Assume that we already defined $\mathcal{G}_k = (U_k, E_k, \ell_k)$ satisfying the following *coherence* condition of triangles $(\ell_k(u,v), \ell_k(u,w), \ell_k(w,v))$ in E_k: for every $(u,w), (w,v) \in E_k$,

$$\ell_k(u,w) \,;\, \ell_k(w,v) \subseteq \ell_k(u,v) \tag{Coh}$$

i.e., if $x \in \ell_k(u,w)$ and $y \in \ell_k(w,v)$, then $x \,;\, y \in \ell_k(u,v)$.

We assume that there is a fair scheduling function σ of all *potential defects* $(x,y,z) \in At \times At \times At$ such that $x \,;\, y \geq z$. Assume that $\sigma_k = (x,y,z)$ and

$z \in \ell_k(i,j)$ for some $(i,j) \in E_k$. If there is $w \in U_k$ such that both $x \in \ell_k(i,w)$ and $y \in \ell_k(w,j)$, then (x,y,z) is not a defect of $(i,j) \in E_k$ and we do not need to extend $\mathcal{G}_k$. Note that this includes the cases when either $x = \bar{1}'$ or $y = \bar{1}'$.

So assume otherwise: (x,y,z) is a defect of $(i,j) \in E_k$. Pick $w_{i,j}$ such that $w_{i,j} \notin U_k$. We will choose $F, G \in \mathcal{F}$ such that $x \in F$ and $y \in G$. First assume that $\bar{o} \in \ell_k(i,j)$. Consider the condition

$$\text{there is } z' \in \ell_k(i,j) \text{ such that } x\,;z' = \bar{o} \tag{3.2}$$

i.e., $s(x) \in \ell_k(i,j)$ or $s(z') = x$. If condition (3.2) holds, then we let $G = F(\ell(\bar{a}) \cup \{\bar{o}\})$ if $y = \bar{a}$, or $G = F(\ell(\bar{b}) \cup \{\bar{o}\})$ if $y = \bar{b}$, or $G = F(\{y, \bar{o}\})$ otherwise. If condition (3.2) fails, then $G = F(\ell(\bar{a}))$ or $G = F(\ell(\bar{b}))$ or $G = F(y)$ again depending on whether $y \in \{\bar{a}, \bar{b}\}$. Similarly, $F = F(\ell(\bar{a}) \cup \{\bar{o}\})$, $F = F(\ell(\bar{a}))$ or $F = F(\ell(\bar{b}) \cup \{\bar{o}\})$, $F = F(\ell(\bar{b}))$, or $F = F(\{x, \bar{o}\})$ or $F = F(x)$ depending on whether $y \in \{\bar{a}, \bar{b}\}$ and whether

$$\text{there is } z'' \in \ell_k(i,j) \text{ such that } z''\,;y = \bar{o} \tag{3.3}$$

i.e., whether $s(y) \in \ell_k(i,j)$ or $s(z'') = y$.

Next assume that $\bar{o} \notin \ell_k(i,j)$. Then $x\,;y = \bar{1}$. We have different cases again according to whether conditions (3.2) and (3.3) hold. We work out the details for the case when both conditions fail, and indicate the necessary modifications when conditions (3.2) and/or (3.3) hold in square brackets [like this]. If $\{x,y\} \cap \{\bar{a}, \bar{b}\} = \emptyset$, then we choose $F, G \in \mathcal{F}$ such that $F = F(x)$ [or $F = F(\{x, \bar{o}\})$ if condition (3.3) above holds] and $G = F(y)$ [or $G = F(\{y, \bar{o}\})$ if condition (3.2) holds]. Obviously, $\bar{o} \notin \{\bar{1}\} = F\,;G$.

Now assume that $x = \bar{a}$. We have several cases according to the value of y.
Case $y = \bar{a}$: We choose $F = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] and $G = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$].
Case $y = \bar{b}$: We let $F = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] and $G = F(\ell(\bar{b}))$ [or $F(\ell(\bar{b}) \cup \{\bar{o}\})$].
Case $y \notin \{\bar{a}, \bar{b}\} \cup B' \cup B''$: We define $F = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] and $G = F(y)$ [or $G = F(\{y, \bar{o}\})$].
Case $y \in B' \cup B''$: In this case $s^{-1}(y)$ is defined. We choose F such that $s^{-1}(y) \notin F$. In fact, $s^{-1}(y)$ is in precisely one of $F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] or $F(\ell'(\bar{a}))$ [or $F(\ell'(\bar{a}) \cup \{\bar{o}\})$], hence we can choose the one that avoids $s^{-1}(y)$. We also let $G = F(y)$ [or $G = F(\{y, \bar{o}\})$].

Note that in all cases, $\bar{o} \notin \{\bar{1}\} = F\,;G$. The case when $x = \bar{b}$ is completely analogous.

Observe that $\bar{o}$ is a "flexible" atom, i.e., $d, e \in At \smallsetminus \{\bar{1}'\}$ implies $\bar{o} \leq d\,;e$ and $d \leq \bar{1} = \bar{o}\,;e = e\,;\bar{o}$. Then it is not difficult to check that the triangle $(\ell_k(i,j), F, G)$ and all its permutations are coherent in each of the above cases.

We let

$$\begin{aligned} \ell_{k+1}(i,w_{i,j}) &= \ell_{k+1}(w_{i,j},i) = F \\ \ell_{k+1}(w_{i,j},j) &= \ell_{k+1}(j,w_{i,j}) = G \\ \ell_{k+1}(w_{i,j},w_{i,j}) &= F(\bar{1}') \\ \ell_{k+1}(q,w_{i,j}) &= \ell_{k+1}(w_{i,j},q) = \ell_{k+1}(w_{i,j},r) = \ell_{k+1}(r,w_{i,j}) = F(\bar{o}) \end{aligned}$$

for $q, r \in U_k \smallsetminus \{i,j\}$ such that $(q,i),(r,j) \in E_k$. See Figure 3. The label

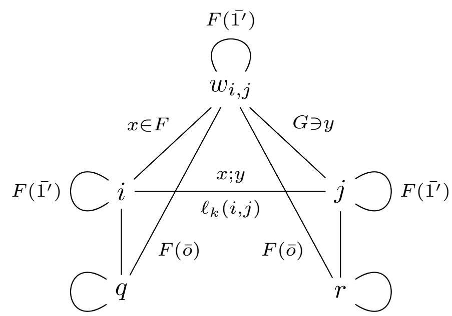

FIGURE 3. Inductive step

$\ell_{k+1}(e)$ of "old" edges $e \in E_k$ remains $\ell_k(e)$. For every $(i,j) \in E_k$ with the defect (x,y,z), we choose a different $w_{i,j}$ and define $\mathcal{G}_{k+1}$ as the union of these graphs.

Note that, for every $(u,v) \in E_{k+1}$, we have $\bar{1}' \in \ell_{k+1}(u,v)$ iff $u = v$, since the only $F \in \mathcal{F}$ such that $\bar{1}' \in F$ are $F = F(\bar{1}')$ and $F(\{\bar{1}',\bar{o}\})$. It is not hard to verify that (Coh) (with k replaced by $k+1$) holds for $\mathcal{G}_{k+1}$. We already checked this property for "primary" triangles of the form $(\ell_{k+1}(i,j), \ell_{k+1}(i,w_{i,j}), \ell_{k+1}(w_{i,j},j))$ and their permutations when we chose labels $F = \ell_{k+1}(i,w_{i,j}) = \ell_{k+1}(w_{i,j},i)$ and $G = \ell_{k+1}(w_{i,j},j) = \ell_{k+1}(j,w_{i,j})$. For all other triangles use that $\bar{o}$ is a flexible atom, whence $F' \,;G' \subseteq F(\bar{o})$ and $F(\bar{o}) \,; F' = F' \,; F(\bar{o}) = \{\bar{1}\} \subseteq G'$ for all $F', G' \in \mathcal{F}$ such that $\bar{1}' \notin F', G'$.

LIMIT STEP. For α a limit ordinal, we let $\mathcal{G}_\alpha = \bigcup_{\kappa \le \alpha} \mathcal{G}_\kappa$. In particular, $\mathcal{G} = (U,E,\ell) = \mathcal{G}_\beta$ for an ordinal β such that all potential defects have been dealt with by step β.

It is easy to check that $\mathcal{G}$ satisfies condition (Coh) (with deleting the subscript k). Furthermore, we have the following *saturation* condition: for every $(u,v) \in E$,

$$\text{if } z \in \ell(u,v) \text{ and } z \le x \,; y, \text{ then } x \in \ell(u,w) \text{ and } y \in \ell(w,v) \tag{Sat}$$

for some $w \in U$, since we assumed a fair scheduling function. We define

$$\mathsf{rep}(x) = \{(u,v) \in E : x \in \ell(u,v)\} \tag{3.4}$$

Then rep is an isomorphism from $\mathfrak{A}$ into a representable algebra with unit E. Indeed, rep is injective (see the base step), and it preserves the operations because of the following. Join $+$ is preserved because we use prime filters as labels. Composition ; is preserved by conditions (Coh) and (Sat) and by the fact that for all $c, d, e \in A$, if $c \le d \mathbin{;} e$, then there are $x, y \in At$ such that $x \le d$, $y \le e$ and $c \le x \mathbin{;} y$. Converse $\smile$ is preserved, since every element is self converse and we chose the same label for an edge and its converse. The identity constant is preserved as well, by $\bar{1}' \in \ell(u, v) = F(\bar{1}')$ iff $u = v$. The bottom element is preserved, since we used only proper filters as labels, while the top element is preserved by the fact that it is in every $F \in \mathcal{F}$. □

We will state a strengthening, Theorem 5.1, of Theorem 3.1 in the last section when we consider larger similarity types that include residuals of composition and/or Kleene star.

4. Varieties

Now we turn our attention to finite axiomatizability of the varieties $\mathsf{V}(\Lambda)$ generated by the representable algebras $\mathsf{R}(\Lambda)$. The picture radically changes, most of these varieties have finite axiomatizations. These proofs use graph-theoretic representations of terms that we will describe after recalling the main results.

Lower semilattice ordered semigroups. We have seen that $\mathsf{R}(\cdot, ;) = \mathsf{V}(\cdot, ;)$ is finitely axiomatizable by [8].

Lower semilattice ordered involuted semigroups. We have seen that for

$$\{\cdot, ;, {}^{\smile}\} \subseteq \Lambda \subseteq \{+, \cdot, 0, 1, ;, {}^{\smile}, 1'\}$$

the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable. The non-representability of the algebras used in the ultraproduct construction is witnessed by equations that fail in these algebras but are valid in $\mathsf{R}(\Lambda)$, hence in $\mathsf{V}(\Lambda)$. Since the ultraproduct is in $\mathsf{R}(\Lambda) \subseteq \mathsf{V}(\Lambda)$, we get that $\mathsf{V}(\Lambda)$ is not finitely axiomatizable, see [17] for details.

All the rest. Let Λ be such that

$$\{\cdot, ;, {}^{\smile}\} \not\subseteq \Lambda \subseteq \{+, \cdot, 0, 1, ;, {}^{\smile}, 1'\}$$

and such that it contains ; and at least one of the lattice operations $\cdot$ or $+$. Then $\mathsf{V}(\Lambda)$ is finitely axiomatizable. We will prove this result below, but mention two known cases here.

Distributive lattice ordered semigroups: The variety $\mathsf{V}(+, \cdot, ;)$ is finitely axiomatizable, since the free distributive lattice ordered semigroup, i.e., the free algebra defined by the distributive lattice axioms (DL) for $\cdot$ and $+$, semigroup axioms (SG) for ; and the additivity axiom

$$(a + b) \mathbin{;} (c + d) = a \mathbin{;} c + a \mathbin{;} d + b \mathbin{;} c + b \mathbin{;} d \tag{Add}$$

is representable, [2, Theorem 2].

Upper semilattice ordered involuted semigroups: In this case, $\mathsf{V}(+, ;, ^\smile)$ is axiomatized by the semilattice axioms (SL) for +, semigroup axioms (SG) for ;, additivity (Add), involution

$$a^{\smile\smile} = a \text{ and } (a\,;b)^\smile = b^\smile\,;a^\smile \tag{Inv}$$

and

$$a \le a\,;a^\smile\,;a \tag{DR}$$

see [7, Theorem 4].

4.1. Maximal finitely axiomatizable reducts. Next we prove finite axiomatizability of the equational theories for two signatures $\{+, \cdot, 0, 1, ;, 1'\}$ and $\{+, 0, 1, ;, ^\smile, 1'\}$, see Theorem 4.1. These varieties are maximal in the following sense: (i) we lose finite axiomatizability if we include extra operations (converse $^\smile$ or meet $\cdot$, respectively), and (ii) all their reducts generate finitely axiomatizable varieties. Finite axiomatizability for subsignatures either follows from [7], the results mentioned above, or by easy modifications (viz. by reading our proofs in such a way that we ignore everything that is not in the fragment under consideration) of the proofs below. (In [3], right above the Acknowledgements, it was claimed, mistakenly, that the equational theories of all positive subreducts are treated in [7]. In fact, all subreducts not containing $1'$ are treated in [7], but the subreducts containing $1'$ are not treated in that paper.)

Distributive lattice ordered monoids. Let $Ax(+, \cdot, ;, 1')$ be the set of the following axioms. Distributive lattice axioms (DL) for + and $\cdot$, monoid axioms for ; and $1'$: semigroup axiom (SG) for ; plus

$$a\,;1' = a = 1'\,;a \tag{Ide}$$

additivity (Add) of ;, axiom for composition below $1'$

$$(a \cdot 1')\,;(b \cdot 1') = a \cdot b \cdot 1' \tag{CbI}$$

and an axiom expressing "functionality" below $1'$

$$[(a_1 \cdot 1');c;(a_2 \cdot 1')] \cdot [(b_1 \cdot 1');d;(b_2 \cdot 1')] = (a_1 \cdot b_1 \cdot 1');(c \cdot d);(a_2 \cdot b_2 \cdot 1') \tag{FbI}$$

Upper semilattice ordered involuted monoids. Let $Ax(+, ;, ^\smile, 1')$ be the following set of axioms. Semilattice axioms (SL) for +, monoid axioms (SG) and (Ide) for ; and $1'$, additivity (Add) of ; and of $^\smile$

$$(a + b)^\smile = a^\smile + b^\smile$$

involution (Inv) and (DR), and

$$1'^\smile = 1' \tag{IC}$$

We are ready to formulate the main result of this section. Let $\vdash$ denote derivability in equational logic.

Theorem 4.1. (1) *The variety* $\mathsf{V}(+,\cdot,;,1')$ *is finitely axiomatizable:*

$$Ax(+,\cdot,;,1') \vdash \sigma = \tau \quad \textit{iff} \quad \mathsf{R}(+,\cdot,;,1') \models \sigma = \tau$$

(2) *The variety* $\mathsf{V}(+,;,^\smile,1')$ *is finitely axiomatizable:*

$$Ax(+,;,^\smile,1') \vdash \sigma = \tau \quad \textit{iff} \quad \mathsf{R}(+,;,^\smile,1') \models \sigma = \tau$$

Including 0 *requires that* 0 *is the bottom element and the normality of the operations. Including* 1 *requires that* 1 *is the top element.*

Proof. In both cases, we will show that the classes of algebraic structures defined as the +-free reducts augmented with an ordering $\leq$ such that each operation is monotonic w.r.t. $\leq$ have finitely based $\leq$-theories.

Ordered involuted monoids. Let $Ax(\leq,;,^\smile,1')$ be the set of axioms given by replacing (SL) by the axioms stating that $\leq$ is an ordering, and additivity (Add) by monotonicity of ; and $^\smile$ w.r.t. $\leq$

$$a \leq b \text{ and } c \leq d \text{ imply } a\,;c \leq b\,;d \tag{4.1}$$

$$a \leq b \text{ implies } a^\smile \leq b^\smile \tag{4.2}$$

in $Ax(+,;,^\smile,1')$.

Theorem 4.2.

$$Ax(\leq,;,^\smile,1') \vdash \sigma \leq \tau \quad \textit{iff} \quad \mathsf{R}(\leq,;,^\smile,1') \models \sigma \leq \tau$$

The same statement holds for all $\Lambda \subseteq \{\leq,;,^\smile,1'\}$ *where* $Ax(\Lambda)$ *is taken as the set of those axioms from* $Ax(\leq,;,^\smile,1')$ *which involve elements of* Λ *only.*

Lower semilattice ordered monoids. In the case of $\Lambda = \{\cdot,;,1'\}$, the ordering $\leq$ is implicit in the algebras, and its required properties (e.g., monotonicity) follow from the axioms $Ax(\cdot,;,1')$. Thus we can state the required result without explicitly mentioning the ordering $\leq$.

Let $Ax(\cdot,;,1')$ be the set of axioms given by replacing (DL) by the semilattice axioms (SL) for $\cdot$, and additivity (Add) by monotonicity (Mon) of ; in $Ax(+,\cdot,;,1')$.

Theorem 4.3. *The variety* $\mathsf{V}(\cdot,;,1')$ *is finitely axiomatizable:*

$$Ax(\cdot,;,1') \vdash \sigma = \tau \quad \textit{iff} \quad \mathsf{R}(\cdot,;,1') \models \sigma = \tau$$

The same statement holds for all $\Lambda \subseteq \{\cdot,;,1'\}$ *where* $Ax(\Lambda)$ *is taken as the set of those axioms from* $Ax(\cdot,;,1')$ *which involve elements of* Λ *only.*

Given Theorems 4.2 and 4.3, Theorem 4.1 follows from the following, see [7, Corollary 2].

Proposition 4.4. *Let* Λ *be a positive similarity type. Then* $(A,+,\Lambda) \in \mathsf{V}(+,\Lambda)$ *iff the algebraic structure* $(A,\leq,\Lambda) \in \mathsf{V}(\leq,\Lambda)$*, each operation in* Λ *is additive, and* $(A,+)$ *is a semilattice.*

Hence, if $\mathsf{V}(\leq, \Lambda)$ is axiomatized by Ax and monotonicity, then Ax together with additivity and the semilattice axioms axiomatize $\mathsf{V}(+, \Lambda)$. The key observations for proving Proposition 4.4 are

- the semilattice operation $+$ defines the ordering $\leq$
- the quasiequations (4.1) and (4.2) expressing that the operations are monotonic w.r.t. the ordering $\leq$ can be replaced by the equations stating the additivity of the operations (hence an inequality $a \leq b$ can be equivalently rewritten as $a_1 + \ldots + a_n \leq b_1 + \ldots + b_m$ where all a_i and b_j are $+$-free subterms of a and b, respectively)
- for any $+$-free terms a, b_1 and b_2, $a \leq b_1 + b_2$ is valid in $\mathsf{V}(+, \Lambda)$ iff $a \leq b_1$ or $a \leq b_2$ is valid in $\mathsf{V}(\leq, \Lambda)$.

We will prove Theorems 4.2 and 4.3 below thus completing the proof of Theorem 4.1. □

4.2. Term graphs. To prove Theorems 4.2 and 4.3 we will work with the term graphs $G(\sigma)$ of [3].

Let X be a set of variables. A *labelled graph* is a structure $G = (V, E)$ where V is a set and $E \subseteq V \times X \times V$. Given two labelled graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a *homomorphism* $h\colon G_1 \to G_2$ is a map from V_1 to V_2 that preserves labelled edges: if $(u, x, v) \in E_1$, then $(h(u), x, h(v)) \in E_2$. Given an equivalence relation θ on V, the *quotient graph* is $G/\theta = (V/\theta, E/\theta)$ where V/θ is the set of equivalence classes of V and

$$E/\theta = \{(u/\theta, x, v/\theta) : (u, x, v) \in E \text{ for some } u \in u/\theta \text{ and } v \in v/\theta\}$$

A *2-pointed graph* is a labelled graph $G = (V, E)$ with two (not necessarily distinct) distinguished vertices $\iota, o \in V$. We will call ι the *input* and o the *output* vertex of G, respectively, and denote 2-pointed graphs as $G = (V, E, \iota, o)$. In the case of 2-pointed graphs, we require that a homomorphism preserves input and output vertices as well.

Let $G_1 \oplus G_2$ denote the disjoint union of G_1 and G_2. For 2-pointed graphs $G_1 = (V_1, E_1, \iota_1, o_1)$ and $G_2 = (V_2, E_2, \iota_2, o_2)$, we define their *composition* as

$$G_1 \,;\, G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_2/\theta)$$

where θ is the smallest equivalence relation on the disjoint union $V_1 \cup V_2$ that identifies o_1 with ι_2. The *meet* of G_1 and G_2 is defined as

$$G_1 \cdot G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_1/\theta)$$

where θ is the smallest equivalence relation on $V_1 \cup V_2$ that identifies ι_1 with ι_2 and o_1 with o_2. When no confusion is likely we will identify an equivalence class u/θ with u, hence ι_i/θ with ι_i and o_i/θ with o_i for $i \in \{1, 2\}$.

We define *term graphs* as special 2-pointed graphs by induction on the complexity of terms. Let

$$G(1') = (\{\iota\}, \emptyset, \iota, \iota)$$

i.e., in this case $\iota = o$. For variable x, we let

$$G(x) = (\{\iota, o\}, \{(\iota, x, o)\}, \iota, o)$$

For terms σ and τ, we set

$$G(\sigma \cdot \tau) = G(\sigma) \cdot G(\tau) \text{ and } G(\sigma \,;\, \tau) = G(\sigma) \,;\, G(\tau)$$

while $G(\sigma^\smile)$ is defined by swapping ι and o in $G(\sigma)$.

Next we recall a characterization of validities using graph homomorphisms from [3, Theorem 1].

Theorem 4.5. *The inequality $\tau \leq \sigma$ is valid in representable algebras iff there is a homomorphism from $G(\sigma)$ to $G(\tau)$.*

The same statement holds if we replace algebras with ordered algebraic structures. The heart of the proofs of Theorems 4.3 and 4.2 will be the following observation.

> The inequality $\tau \leq \sigma$ is derivable from the axioms iff there is a homomorphism from $G(\sigma)$ to $G(\tau)$.

When no confusion is likely we will omit the axioms in front of $\vdash$ and the class of algebras in front of $\models$. We leave the easy task of checking the soundness of the axiom systems with respect to the representable classes to the reader.

Proof of Theorem 4.2. We work out the details for $\{\leq, ;, ^\smile, 1'\}$ first. Assume that we have terms τ and σ such that $\mathsf{R}(\leq, ;, ^\smile, 1') \models \tau \leq \sigma$. We have to show that $Ax(\leq, ;, ^\smile, 1') \vdash \tau \leq \sigma$.

We claim that every term ρ can be equivalently rewritten in the form $\rho_0 \,; \ldots ; \rho_{n-1}$ where every ρ_i is either a variable or $1'$ or the converse of a variable. Indeed, this claim follows from the axioms (Inv), (IC) and (SG). Furthermore, if at least one ρ_i is not $1'$, then all occurrences of $1'$ can be deleted, by using axiom (Ide).

So assume that σ and τ are either $1'$ or a composition of variables and converses of variables, say, $\sigma = \sigma_0 \,; \ldots ; \sigma_{m-1}$ and $\tau = \tau_0 \,; \ldots ; \tau_{n-1}$. We will say that σ has length m and τ has length n. By Theorem 4.5, there is a homomorphism $h \colon G(\sigma_0 \,; \ldots ; \sigma_{m-1}) \to G(\tau_0 \,; \ldots ; \tau_{n-1})$. If one of σ or τ is $1'$, then both of them equal $1'$, by the definitions of $G(1')$ and graph homomorphism. We have $\vdash \tau \leq \sigma$ in this case by the reflexivity of $\leq$. Thus we can assume that every σ_i and τ_i are different from $1'$, that is, both σ and τ are terms of the signature $\{;, ^\smile\}$. Then $\vdash \tau \leq \sigma$ essentially follows from [7, Theorem 4] with the minor change that we have $\leq$ instead of $+$. For the benefit of non-Russian speaking readers we provide a proof below (a similar argument appears in [11]).

Note that $G(\sigma_0 \,; \ldots ; \sigma_{m-1}) = G(\sigma_0) \,; \ldots ; G(\sigma_{m-1})$ and $G(\tau_0 \,; \ldots ; \tau_{n-1}) = G(\tau_0) \,; \ldots ; G(\tau_{n-1})$, hence both graphs consist of linearly ordered finite sequences of vertices $(v_0, \ldots, v_m)$ and $(u_0, \ldots, u_n)$, respectively. Since graph homomorphisms preserve the input and output nodes, we have $h(v_0) = u_0$

and $h(v_m) = u_n$. Also $m \geq n \geq 1$ in this case, and the directed edges between adjacent vertices in the graphs are labelled by variables. Since graph homomorphisms preserve labelled edges, if $h(v_i) = u_j$, then $h(v_{i+1}) = u_{j+1}$ or $h(v_{i+1}) = u_{j-1}$. In the first case, we say that at place i there is a *forward step*, and a *backward step* in the second case. Thus every homomorphism can be viewed as a sequence of forward and backward steps: $(f^{k_1}, b^{k_2}, \ldots, f^{k_r})$ for some r, where f^{k_1} represents k_1 forward steps, etc. We know that $k_1 > 0$ (every homomorphism starts with at least one forward step from the input node) and that $k_2 \leq k_1$ (we can make at most as many backward steps as the number of previous forward steps). Also note that, if (v_i, v_{i+1}) is labelled by, say, x, then (u_j, u_{j+1}) is labelled by x (in the case of a forward step) or (u_j, u_{j-1}) is labelled by x (in the case of a backward step), with an analogous statement holding if (v_{i+1}, v_i) is labelled by x. We proceed by induction on the difference $m - n$ between the length of σ and τ in establishing $\vdash \tau \leq \sigma$.

Base step $n = m$. In this case we cannot have any backward steps, and by the definition of a homomorphism, we have that τ and σ are the same sequence of the same terms. We are done, since $\leq$ is reflexive.

Inductive step $m - n > 0$. We assume as an inductive hypothesis that for every pair of terms σ' and τ' such that the difference between the length of σ' and τ' is less than $m - n$, we have $\vdash \tau' \leq \sigma'$ whenever there exists a homomorphism $h' \colon G(\sigma') \to G(\tau')$.

We claim that there is p such that

$$k_p \geq k_{p+1} \leq k_{p+2}$$

Assume indirectly that we have $k_1 \geq k_2 > k_3 \cdots > k_r$. This would mean that after k_1 forward steps we would make k_2 backward steps and then, summing up all the remaining forward and backward steps, we would end up less than k_1 steps away from the input node of τ. Hence we would not reach the output node of τ at the end of the alleged homomorphism, a contradiction.

So fix such a p and first assume that k_p is the exponent of forward steps. That is, we have $(f^{k_p}, b^{k_{p+1}}, f^{k_{p+2}})$ as part of the homomorphism, and so we have (f^k, b^k, f^k) as part of the homomorphism with $k = k_{p+1} \geq 1$, since $k_p \geq k_{p+1} \leq k_{p+2}$. Then we have a "loop" of length k of the following form: there are $i < m, j < n$ such that $h(v_i) = u_j$ and

$$h(v_{i+1}) = u_{j+1}, \ldots, h(v_{i+k}) = u_{j+k} \quad (k \text{ forward steps})$$
$$h(v_{i+k+1}) = u_{j+k-1}, \ldots, h(v_{i+2k}) = u_j \quad (k \text{ backward steps})$$
$$h(v_{i+2k+1}) = u_{j+1}, \ldots, h(v_{i+3k}) = u_{j+k} \quad (k \text{ forward steps})$$

See Figure 4 for illustrating case $k = 2$ for the inequality

$$x \mathbin{;} y^{\smile} \leq x \mathbin{;} y^{\smile} \mathbin{;} y \mathbin{;} x^{\smile} \mathbin{;} x \mathbin{;} y^{\smile}$$

The top line represents $G(x \mathbin{;} y^{\smile} \mathbin{;} y \mathbin{;} x^{\smile} \mathbin{;} x \mathbin{;} y^{\smile})$, the bottom line represents $G(x \mathbin{;} y^{\smile})$ and the dashed lines represent h. In this particular case, $\tau_0 = x$ and $\tau_1 = y^{\smile}$, that is why (u_0, u_1) is labelled by x and (u_2, u_1) is labelled by

y, and similarly (v_0, v_1) is labelled by x, (v_2, v_1) is labelled by y and (v_2, v_3) is labelled by y, etc.

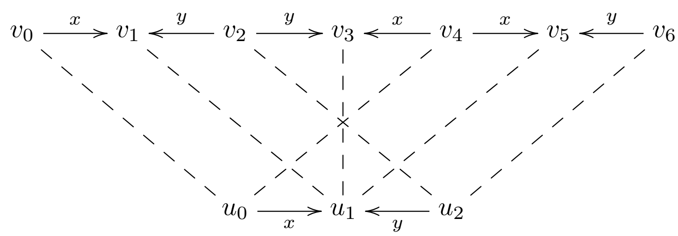

FIGURE 4. A 2-long loop in $h\colon G(x\,;y^{\smile}\,;y\,;x^{\smile}\,;x\,;y^{\smile}) \to G(x\,;y^{\smile})$

Continuing with the general proof, note that

$$\begin{array}{ccc} \sigma_i = \tau_j & \dots & \sigma_{i+k-1} = \tau_{j+k-1} \\ \sigma_{i+k} = \tau_{j+k-1}{}^{\smile} & \dots & \sigma_{i+2k-1} = \tau_j{}^{\smile} \\ \sigma_{i+2k} = \tau_j & \dots & \sigma_{i+3k-1} = \tau_{j+k-1} \end{array}$$

since homomorphisms preserve labelled edges. Hence σ has the form

$$\rho'\,;\tau_j\,;\dots;\tau_{j+k-1}\,;\tau_{j+k-1}{}^{\smile}\,;\dots;\tau_j{}^{\smile}\,;\tau_j\,;\dots;\tau_{j+k-1}\,;\rho''$$

where $\rho' = \sigma_0;\dots;\sigma_{i-1}$ and $\rho'' = \sigma_{i+3k};\dots;\sigma_{m-1}$. Let us denote $\tau_j;\dots;\tau_{j+k-1}$ by ρ, and let $\sigma' = \rho'\,;\rho\,;\rho''$. By axiom (DR) we have $\vdash \rho \leq \rho\,;\rho^{\smile}\,;\rho$, and by the second part of axiom (Inv),

$$\vdash \rho^{\smile} = (\tau_j\,;\dots;\tau_{j+k-1})^{\smile} = \tau_{j+k-1}{}^{\smile}\,;\dots;\tau_j{}^{\smile}$$

whence $\vdash \sigma = \rho';\rho;\rho^{\smile};\rho;\rho''$ and then $\vdash \sigma' \leq \sigma$ by monotonicity. It is easy to see that the restriction of h to $G(\sigma')$ is a homomorphism to $G(\tau)$. Since the length of σ' is strictly less than the length of σ, by the inductive hypothesis we have $\vdash \tau \leq \sigma'$. Hence we get $\vdash \tau \leq \sigma$ as required.

The case where k_p is the exponent of backward steps (that is, we have $(b^{k_p}, f^{k_{p+1}}, b^{k_{p+2}})$ as part of the homomorphism) is completely analogous. This finishes the proof for the full signature.

For smaller signatures an easy simplification of the above argument works. We just note that if $\leq$ is missing from the signature, then $\sigma = \tau$ is valid iff both $\sigma \leq \tau$ and $\tau \leq \sigma$ are valid (recall that $\leq$ is interpreted as the subset relation). Then the graphs of σ and τ have the same length, whence the homomorphism is the identity function. That is, σ and τ are the same terms. Then $\sigma = \tau$ is obviously derivable. □

The proof of Theorem 4.3 is more complex because of the presence of meet.

Proof of Theorem 4.3. Assume that $\mathsf{R}(\cdot, ;, 1') \models \tau \leq \sigma$. We have to prove $Ax(\cdot, ;, 1') \vdash \tau \leq \sigma$. We will proceed by induction on σ.

CASE $\sigma = 1'$: By induction on τ.

SUBCASE $\tau = x$: Just note that $\not\models x \leq 1'$.

SUBCASE $\tau = 1'$: Obviously, $\vdash 1' \leq 1'$ by (SL).

SUBCASE $\tau = \tau_1 \cdot \tau_2$: Assuming $\models \tau_1 \cdot \tau_2 \leq 1'$, by Theorem 4.5, we have a homomorphism $h \colon G(1') \to G(\tau_1 \cdot \tau_2) = G(\tau_1) \cdot G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$, $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$ and $G(\tau_1) \cdot G(\tau_2) = G = (V, E, \iota, o)$. Then $\iota = o$, since G is a homomorphic image of $G(1')$. Hence we have either $h_1 \colon G(1') \to G(\tau_1)$, whence $\models \tau_1 \leq 1'$, or $h_2 \colon G(1') \to G(\tau_2)$, i.e., $\models \tau_2 \leq 1'$. By the IH (inductive hypothesis), we have either $\vdash \tau_1 \leq 1'$ or $\vdash \tau_2 \leq 1'$. In either case, $\vdash \tau_1 \cdot \tau_2 \leq 1'$ by the semilattice axioms.

SUBCASE $\tau = \tau_1 ; \tau_2$: Assuming $\models \tau_1 ; \tau_2 \leq 1'$, by Theorem 4.5, we have a homomorphism $h \colon G(1') \to G(\tau_1 ; \tau_2) = G(\tau_1) ; G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$ and $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$. Then $\iota_1 = o_1 = \iota_2 = o_2$. Thus, for $i \in \{1, 2\}$, we have $h_i \colon G(1') \to G(\tau_i)$, whence $\models \tau_i \leq 1'$, i.e., $\vdash \tau_i \leq 1'$ by the IH. Hence $\vdash \tau_1 ; \tau_2 \leq 1'$ by the monotonicity axiom (Mon) and the monoid axiom (Ide).

CASE $\sigma = x$: By induction on τ.

SUBCASE $\tau = y$: Assuming $\models y \leq x$, we have $x = y$, whence $\vdash y \leq x$.

SUBCASE $\tau = 1'$: Just note that $\not\models 1' \leq x$.

SUBCASE $\tau = \tau_1 \cdot \tau_2$: Assuming $\models \tau_1 \cdot \tau_2 \leq x$, by Theorem 4.5, we have a homomorphism $h \colon G(x) \to G(\tau_1 \cdot \tau_2) = G(\tau_1) \cdot G(\tau_2)$. Then we have either $h_1 \colon G(x) \to G(\tau_1)$, whence $\models \tau_1 \leq x$, or $h_2 \colon G(x) \to G(\tau_2)$, i.e., $\models \tau_2 \leq x$. By the IH, we have either $\vdash \tau_1 \leq x$ or $\vdash \tau_2 \leq x$. In either case, $\vdash \tau_1 \cdot \tau_2 \leq x$ by the semilattice axioms.

SUBCASE $\tau = \tau_1 ; \tau_2$: Assuming $\models \tau_1 ; \tau_2 \leq x$, by Theorem 4.5, we have a homomorphism $h \colon G(x) \to G(\tau_1 ; \tau_2) = G(\tau_1) ; G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$ and $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$. Then $(h(\iota), x, h(o)) \in E_1 \cup E_2$. Thus we have either $h_1 \colon G(x) \to G(\tau_1)$ and $h_2 \colon G(1') \to G(\tau_2)$, whence $\models \tau_1 \leq x$ and $\models \tau_2 \leq 1'$, or $g_1 \colon G(1') \to G(\tau_1)$ and $g_2 \colon G(x) \to G(\tau_2)$, whence $\models \tau_1 \leq 1'$ and $\models \tau_2 \leq x$. In the first case, we have $\vdash \tau_1 \leq x$ and $\vdash \tau_2 \leq 1'$ by the IH and case $\sigma = 1'$. Hence $\vdash \tau_1 ; \tau_2 \leq x ; 1' = x$ by the monotonicity axiom and the monoid axioms. The other case is completely analogous.

CASE $\sigma = \sigma_1 \cdot \sigma_2$: Assuming $\models \tau \leq \sigma_1 \cdot \sigma_2$, we have $\models \tau \leq \sigma_1$ and $\models \tau \leq \sigma_2$. Then by the IH on σ, we have $\vdash \tau \leq \sigma_1$ and $\vdash \tau \leq \sigma_2$, whence $\vdash \tau \leq \sigma_1 \cdot \sigma_2$ by the semilattice axioms.

CASE $\sigma = \sigma_1 ; \sigma_2$: We want to prove $\vdash \tau \leq \sigma_1 ; \sigma_2$. In this case we will not proceed by induction along the structure of τ. The reason for this is the presence of $1'$. If $1'$ does not occur in τ, σ_1, σ_2, then the usual induction w.r.t. the construction of τ works.

MOTIVATING EXAMPLE. Let $\tau_1 = x$, $\tau_2 = (y \cdot 1') ; w ; (z \cdot 1')$ and $\sigma_1 ; \sigma_2 = y ; x ; z$. Then $\models \tau_1 \cdot \tau_2 \leq \sigma_1 ; \sigma_2$, but neither $\models \tau_1 \leq \sigma_1 ; \sigma_2$ nor $\models \tau_2 \leq \sigma_1 ; \sigma_2$. So the problem is that, because of the presence of $1'$, the subterms can affect each other.

Definition 4.6. Let τ be a term, $G(\tau) = (V, E, \iota, o)$ and $u, v \in V$. We define a term $\tau(u, v)$ which is intuitively the term corresponding to the subgraph of $G(\tau)$ between u and v. We define $\tau(u, v)$ by induction on τ. We will use the notation $G(\rho) = (V(\rho), E(\rho), \iota(\rho), o(\rho))$ for any term ρ.

Case $\tau = 1'$: In this case $V = \{\iota\} = \{o\}$, so $u = v = \iota = o$. We define $1'(\iota, \iota) = 1'$.

Case $\tau = x$: In this case $V = \{\iota, o\}$. We define $x(\iota, \iota) = x(o, o) = 1'$, $x(\iota, o) = x$, and $x(o, \iota)$ is undefined.

Case $\tau = \delta \cdot \rho$: In this case $V = V(\delta) \cup V(\rho)$ and $V(\delta) \cap V(\rho) = \{\iota, o\}$. We define $(\delta \cdot \rho)(u, v)$ by case-distinction.

Subcase $u = v \in \{\iota, o\}$: $\delta(u, u) \cdot \rho(u, u)$.
Subcase $u = \iota$ and $v = o$: $\delta \cdot \rho$.
Subcase $u = \iota$ and $v \in G(\delta) \smallsetminus G(\rho)$: $\rho(\iota, \iota) ; \delta(\iota, v)$.
Subcase $u = \iota$ and $v \in G(\rho) \smallsetminus G(\delta)$: $\delta(\iota, \iota) ; \rho(\iota, v)$.
Subcase $u \in G(\delta) \smallsetminus G(\rho)$ and $v = o$: $\delta(u, o) ; \rho(o, o)$.
Subcase $u \in G(\rho) \smallsetminus G(\delta)$ and $v = o$: $\rho(u, o) ; \delta(o, o)$.
Subcase $u, v \in G(\delta) \smallsetminus G(\rho)$ and $\delta(u, v)$ is defined: $\delta(u, v)$.
Subcase $u, v \in G(\rho) \smallsetminus G(\delta)$ and $\rho(u, v)$ is defined: $\rho(u, v)$.
All other subcases: undefined.

Case $\tau = \delta ; \rho$: In this case $V = V(\delta) \cup V(\rho)$ and $V(\delta) \cap V(\rho) = \{m\}$ for some m such that $m = o(\delta) = \iota(\rho)$. We define $(\delta ; \rho)(u, v)$ by case-distinction.

Subcase $u, v \in G(\delta) \smallsetminus G(\rho)$: $\delta(u, v)$.
Subcase $u \in G(\delta)$ and $v \in G(\rho)$: $\delta(u, m) ; \rho(m, v)$.
Subcase $u, v \in G(\rho) \smallsetminus G(\delta)$: $\rho(u, v)$.
All other subcases: undefined.

Lemma 4.7. *For any term τ we have the following: $\tau(\iota, o) = \tau$, $\tau(u, u)$ is defined for any $u \in G(\tau)$, and $\vdash \tau(u, u) \leq 1'$.*

The proof is by an easy induction. For the last case one can prove that $\models \tau(u, u) \leq 1'$, and then use our earlier case to infer $\vdash \tau(u, u) \leq 1'$. We leave the details to the reader. We will prove the following two lemmas, though.

Lemma 4.8. *Assume that $\tau(u, v)$ and $\tau(v, w)$ are defined. Then $\tau(u, w)$ is defined and $\vdash \tau(u, w) \leq \tau(u, v) ; \tau(v, w)$.*

Let $h \colon V(\tau) \to V(\sigma)$ be a map. We call h an *endfree homo* if h is a homomorphism from the "end-free" graph $(V(\tau), E(\tau))$ to the "end-free" graph $(V(\sigma), E(\sigma))$. That is, h is a homomorphism from $G(\tau)$ to $G(\sigma)$ except that it may not preserve ι and o.

Lemma 4.9. *Let $h \colon G(\tau) \to G(\sigma)$ be an endfree homo. Let $u, v \in G(\tau)$ and assume that $\tau(u, v)$ is defined. Then $\sigma(h(u), h(v))$ is defined and there is a homomorphism $g \colon G(\tau(u, v)) \to G(\sigma(h(u), h(v)))$.*

From the above lemmas we will prove our case as follows. Assume that $\models \tau \leq \sigma_1 ; \sigma_2$. Then there is a homomorphism $h \colon G(\sigma_1 ; \sigma_2) \to G(\tau)$ by

Theorem 4.5. Let $m = o(\sigma_1) = \iota(\sigma_2)$ be the point "connecting" $G(\sigma_1)$ with $G(\sigma_2)$ in $G(\sigma_1\,;\sigma_2)$. Then h is an endfree homo from $G(\sigma_1)$ to $G(\tau)$, and also from $G(\sigma_2)$ to $G(\tau)$ (by the definition of $G(\sigma_1\,;\sigma_2) = G(\sigma_1)\,;G(\sigma_2)$). Then, by Lemma 4.9, there is a homomorphism $g\colon G(\sigma_1) \to G(\tau(\iota, h(m))$ that preserves endpoints, and so $\vdash \tau(\iota, h(m)) \le \sigma_1$ by the IH. Similarly, $\vdash \tau(h(m), o) \le \sigma_2$. Thus $\vdash \tau(\iota, o) \le \sigma_1\,;\sigma_2$ by Lemma 4.8 and monotonicity. Hence it remains to prove Lemmas 4.8 and 4.9.

Proof of Lemma 4.8. We proceed by induction on τ.
Case $\tau = 1'$: In this case $\iota = u = v = w = o$ and $\tau(u, w) = \tau(u, v) = \tau(v, w) = \tau(\iota, \iota) = 1'$, so we are done by $\vdash 1' \le 1'\,;1'$.
Case $\tau = x$: In this case either $u = v = w \in \{\iota, o\}$, in which case we are done as above, or $u = v = \iota, w = o$ or $u = \iota, v = w = o$, in which cases we are done by $\vdash x \le 1'\,;x$ and $\vdash x \le x\,;1'$.
Case $\tau = \delta \cdot \rho$: We proceed by case-distinction according to the alternatives in case $\tau = \delta \cdot \rho$ in Definition 4.6. It is not hard to check that in each of the following combined situations $\tau(u, w)$ is defined.

Subcase $u = v = w = \iota$: We have to show that $\vdash \tau(\iota, \iota) \le \tau(\iota, \iota)\,;\tau(\iota, \iota)$. By Lemma 4.7, we have that $\vdash \tau(\iota, \iota) \le 1'$. Since $\vdash (x \cdot 1') \le (x \cdot 1')\,;(x \cdot 1')$ by (CbI) and (SL), we are done.

Subcase $u = v = \iota$, $w = o$: We have to show that $\vdash \tau(\iota, o) \le \tau(\iota, \iota);\tau(\iota, o)$, i.e., that $\vdash \tau \le \tau(\iota, \iota)\,;\tau$. By the IH we have that $\vdash \delta \le \delta(\iota, \iota)\,;\delta$, $\vdash \delta(\iota, \iota) \le 1'$ and $\vdash \rho \le \rho(\iota, \iota)\,;\rho$, $\vdash \rho(\iota, \iota) \le 1'$, and $\tau(\iota, \iota) = \delta(\iota, \iota) \cdot \rho(\iota, \iota)$ by definition. Then we can use axiom (FbI) to conclude this case.

Subcase $u = v = \iota$, $w \in G(\delta) \smallsetminus G(\rho)$: We have to show that $\vdash \tau(\iota, w) \le \tau(\iota, \iota);\tau(\iota, w)$. Now, we have $\tau(\iota, w) = \rho(\iota, \iota);\delta(\iota, w)$ and, by the IH, $\vdash \delta(\iota, w) \le \delta(\iota, \iota);\delta(\iota, w)$. We can use axiom (CbI) and monotonicity to conclude this case.

Subcase $u = v = w = o \neq \iota$: This is analogous to subcase $u = v = w = \iota$, we leave it to the reader.

Subcase $u = \iota$, $v \in G(\delta) \smallsetminus G(\rho)$, $w = o$: We have to show that $\vdash \tau(\iota, o) \le \tau(\iota, v);\tau(v, o)$, i.e., $\vdash \delta \cdot \rho \le \rho(\iota, \iota);\delta(\iota, v);\delta(v, o);\rho(o, o)$. We can use that by IH we have $\vdash \delta \le \delta(\iota, v)\,;\delta(v, o) \le 1'\,;\delta(\iota, v)\,;\delta(v, o)\,;1'$ and $\vdash \rho \le \rho(\iota, \iota)\,;\rho\,;\rho(o, o)$ and then axiom (FbI).

Subcase $u = \iota$, $v, w \in G(\delta) \smallsetminus G(\rho)$: We have to show that $\vdash \tau(\iota, w) \le \tau(\iota, v);\tau(v, w)$. By definition, $\tau(\iota, w) = \rho(\iota, \iota);\delta(\iota, w)$, the same for $\tau(\iota, v)$ with v in place of w, and $\tau(v, w) = \delta(v, w)$. So we have to prove $\vdash \rho(\iota, \iota)\,;\delta(\iota, w) \le \rho(\iota, \iota)\,;\delta(\iota, v)\,;\delta(v, w)$. By IH, $\vdash \delta(\iota, w) \le \delta(\iota, v)\,;\delta(v, w)$, hence the result follows.

Subcase $u \in G(\delta) \smallsetminus G(\rho)$, $v = w = o$: Proceeding as before, we have to show that $\vdash \delta(u, o)\,;\rho(o, o) \le (\delta(u, o)\,;\rho(o, o))\,;\delta(o, o)$. The same argument as in subcase $u = v = \iota$, $w \in G(\delta) \smallsetminus G(\rho)$ gives the result.

Subcase $u, v, w \in G(\delta) \smallsetminus G(\rho)$: This case is straightforward by the IH.

Subcase $u, v \in G(\delta) \smallsetminus G(\rho)$, $w = o$: We have to show that $\vdash \delta(u, o) \,;\, \rho(o, o) \leq \delta(u, v) \,;\, (\delta(v, o) \,;\, \rho(o, o))$. This is completely analogous to subcase $u = \iota$, $v, w \in G(\delta) \smallsetminus G(\rho)$.

Subcase where δ and ρ are interchanged: These cases are completely analogous with the earlier corresponding cases.

Case $\tau = \delta \,;\, \rho$: We show the most interesting alternative in case $\tau = \delta \,;\, \rho$ in Definition 4.6, viz. when $u \in G(\delta)$, $w \in G(\rho)$ and $m = v$. We have

$$\begin{aligned}
\tau(u, w) &= \delta(u, m) \,;\, \rho(m, w) && \text{by def.}\\
&\leq \delta(u, m) \,;\, \delta(m, m) \,;\, \rho(m, m) \,;\, \rho(m, w) && \text{by IH}\\
&= \delta(u, m) \,;\, \rho(m, m) \,;\, \delta(m, m) \,;\, \rho(m, w) && \text{by (CbI)}\\
&= \tau(u, m) \,;\, \tau(m, w) && \text{by def.}
\end{aligned}$$

The other alternatives easily follow from this one and the IH. This concludes the proof of Lemma 4.8. □

Proof of Lemma 4.9. We proceed by induction on τ. Let $u' = h(u)$ and $v' = h(v)$.

Case $\tau = 1'$: In this case $u = v = \iota$, hence $u' = v'$ and $\tau(u, v) = 1'$. By Lemma 4.7, $\vdash 1' \geq \sigma(u', u')$, whence $\models 1' \geq \sigma(u', u')$ by soundness. Then, by Theorem 4.5, there is $g \colon G(1') \to G(\sigma(u', u'))$ as desired.

Case $\tau = x$: In this case either $u = v \in \{\iota, o\}$ in which case $\tau(u, v) = 1'$ and we are done as above, or $u = \iota$ and $v = o$ in which case $\tau(u, v) = x = \tau$. We show that $\sigma(u', v')$ is defined and that there exists $g \colon G(\tau) \to G(\sigma(u', v'))$ by induction on σ.

Subcase $\sigma = 1'$: There is no such homomorphism h, so this case does not occur.

Subcase $\sigma = y$: In this case $y = x$, $u' = \iota$ and $v' = o$ by $h \colon G(x) \to G(y)$ and we are done.

Subcase $\sigma = \delta \cdot \rho$: In this case either $h \colon G(\tau) \to G(\delta)$ or $h \colon G(\tau) \to G(\rho)$, assume the first alternative. If $u', v' \in V(\delta) \smallsetminus V(\rho)$, then, by the IH, there is $g \colon G(\tau(u, v)) \to G(\delta(u', v')) = G(\sigma(u', v'))$ as desired. If $u' = \iota(\sigma)$ and $v' \in V(\delta) \smallsetminus V(\rho)$, then by the IH there is $g \colon G(\tau(u, v)) \to G(\delta(\iota(\sigma), v'))$. But $\delta(\iota(\sigma), v') \geq \sigma((\iota(\sigma), v'))$ by Definition 4.6 and Lemma 4.7, whence there is $f : G(\delta(\iota(\sigma), v')) \to G(\sigma(\iota(\sigma), v'))$. Hence composing the two homomorphisms we have $f \circ g \colon G(\tau(u, v)) \to G(\sigma(\iota(\sigma), v'))$. The case when $v' = o(\sigma)$ is completely analogous.

Subcase $\sigma = \delta \,;\, \rho$: If $u', v' \in V(\delta) \smallsetminus V(\rho)$, then the case follows by the IH (for δ). If $v' = o(\delta) = \iota(\rho)$, then $\sigma(u', v') = \delta(u', v') \,;\, \rho(v', v') \leq \delta(u', o(\delta))$, by Lemma 4.7. Hence the case follows as above. The case $u' = o(\delta) = \iota(\rho)$ is completely analogous.

Case $\tau = \delta \cdot \rho$: There are the following cases according to the definition of $\tau(u, v)$: (1) $u, v \in G(\delta) \cap G(\rho)$, (2) $u = \iota$ and $v \in G(\delta) \smallsetminus G(\rho)$ and the symmetric case with δ and ρ interchanged, (3) $u \in G(\delta) \smallsetminus G(\rho)$ and $v = o$ and the symmetric case with δ and ρ interchanged, (4) $u, v \in G(\delta) \setminus G(\rho)$ and the symmetric case with δ and ρ interchanged.

Subcase 1: By IH there are homomorphisms $g_1 \colon G(\delta(u,v)) \to G(\sigma(u',v'))$ and $g_2 \colon G(\rho(u,v)) \to G(\sigma(u',v'))$. Now $\tau(u,v) = \delta(u,v) \cdot \rho(u,v)$ and then, by definitions, $g_1 \cup g_2 \colon G(\tau(u,v)) \to G(\sigma(u',v'))$ is a homomorphism.

Subcase 2: By IH there are homomorphisms $g_1 \colon G(\delta(\iota,v)) \to G(\sigma(u',v'))$ and $g_2 \colon G(\rho(\iota,\iota)) \to G(\sigma(u',u'))$. Now $\tau(\iota,v) = \rho(\iota,\iota) \mathbin{;} \delta(\iota,v)$ and then, by definitions, $g_1 \cup g_2 \colon G(\tau(\iota,v)) \to G(\sigma(u',v'))$ is a homomorphism.

Subcase 3: This is completely analogous to subcase 2.

Subcase 4: This easily follows from the IH for δ (or for ρ in the symmetric case).

Case $\tau = \delta \mathbin{;} \rho$: The case when $u, v \in G(\delta) \setminus G(\rho)$ (and its symmetric version with δ and ρ interchanged) easily follows from the IH for δ (or for ρ). So assume otherwise and let $m = o(\delta) = \iota(\rho)$ and $m' = h(m)$. By the IH there are homomorphisms $g_1 \colon G(\delta(u,m)) \to G(\sigma(u',m'))$ and $g_2 \colon G(\rho(m,v)) \to G(\sigma(m',v'))$. Note that $\sigma(u',m') \mathbin{;} \sigma(m',v') \geq \sigma(u',v')$ by Lemma 4.8, whence there is a homomorphism $f \colon G(\sigma(u',m') \mathbin{;} \sigma(m',v')) \to G(\sigma(u',v'))$. Since $G(\tau(u,v)) = G(\delta(u,m);\rho(m,v))$ by definition, we get $f \circ (g_1 \cup g_2) \colon G(\tau(u,v)) \to G(\sigma(u',v'))$ as desired, finishing the proof of Lemma 4.9. □

This finishes the proof of Theorem 4.3 □

5. Further results and problems

In this section we look at alternative fragments and extended similarity types. In the previous sections we restricted ourselves to positive fragments containing composition and one of the lattice operations. We now make some comments on relaxing this restriction, without attempting completeness. For varieties $\mathsf{V}(\Lambda)$ we have already done so in Theorem 4.2.

5.1. Ordered algebraic structures.

5.1. Ordered algebraic structures. Algebraic structures with an ordering $\leq$ (interpreted as subset $\subseteq$ in representable algebras) and operations from $\{0, 1, ;, ^\smile, 1'\}$ have been investigated in the literature. We refer the interested reader to [27] about the finite axiomatizability of these classes, but we recall the relatively recent result that $\mathsf{R}(\leq, ;, 1')$ is a non-finitely axiomatizable class [13]. As far as we know the problem of finite axiomatizability of $\mathsf{R}(\leq, ;, ^\smile, 1')$ is open, and we conjecture that the answer is negative. If we look at the "inequality" theory of ordered structures (i.e., valid formulas of the form $a \leq b$), then Theorem 4.2 gives us finite axiomatizations for all positive fragments.

5.2. Other fragments.

Boolean algebras with operators. Those fragments that include negation have been investigated as well. In this case we have a Boolean algebra $(A, +, \cdot, -, 0, 1)$ equipped with extra-Boolean operations among $\{;, {}^\smile, 1'\}$, i.e., these are Boolean algebras with operators. We refer the reader to [5, Rem.1 and Fig.1] where it is indicated which of these subreducts are varieties (not all!), which of these are finitely axiomatizable, and finite axiomatizations are given where the results are positive. We note that the above paper deals with algebras of relations where the greatest element is an equivalence relation, but the results in [5] are applicable for our case (where we only require 1 to be transitive if ; is present in Λ, etc.) as well.
Without composition. For finite axiomatizability of $\mathsf{R}(\Lambda)$ and $\mathsf{V}(\Lambda)$ without composition we refer the reader to [27], generally these results are positive.

5.3. Extending the signature. One might wonder if we could improve on the finite axiomatizability results by including more operations into the signature. If we go beyond the positive subsignatures of Tarski's relation algebras and do not want the full power of Boolean algebras, then arguably the most important connectives are the residuals of composition and the Kleene star * (reflexive, transitive closure). These operations are extensively investigated in computer science and linguistics.

We recall the interpretation of the residuals in representable algebras:

$$x \setminus y = \{(u, v) \in U \times U : \forall w((w, u) \in x \text{ implies } (w, v) \in y)\} \tag{5.1}$$

$$x \,/\, y = \{(u, v) \in U \times U : \forall w((u, w) \in y \text{ implies } (v, w) \in x)\} \tag{5.2}$$

The operations $\setminus$, $/$ and * are not additive, thus Proposition 4.4 does not apply to them.
Residuated semigroups. It is shown in [4] that the variety $\mathsf{R}(\cdot, ;, \setminus, /)$ and its ordered algebraic structure version $\mathsf{R}(\leq, ;, \setminus, /)$ are finitely axiomatizable. On the other hand, neither $\mathsf{R}(+, \cdot, ;, \setminus, /)$ nor the variety $\mathsf{V}(+, \cdot, ;, \setminus, /)$ generated by it is finitely axiomatizable (even if we omit composition ;), see [16]. We note that representable residuated Boolean algebras $\mathsf{R}(+, -, ;, \setminus, /)$ have been investigated as well, see, e.g., [18].
Kleene star. We recall that the class of *relational Kleene algebras* is $\mathsf{RKA} = \mathsf{R}(+, 0, ;, {}^*, 1')$. This class is not axiomatizable in first-order logic because of the presence of *, but it is axiomatizable by infinitary quasiequations, see [23]. Its equational theory is not finitely axiomatizable, see [26] or [9] in English, and the same holds if we include converse as well: $\mathsf{V}(+, 0, ;, {}^\smile, {}^*, 1')$ is not finitely based, see [10]. It is worth noting that there are finitely axiomatizable quasivarieties that generate the same variety as RKA, see [19] and [11] for references, hence all the equations valid in RKA can be derived from a finite set of quasiequations. The same holds if we include converse as well, [10].
Residuated Kleene algebras. The class of Kleene algebras equipped with the residuals of composition are sometimes called *action algebras*, see [25].

Next we show the promised strengthening of Theorem 3.1. It follows that neither the class of relational (or representable) action algebras (with or without converse) nor its quasiequational theory is finitely axiomatizable.

Theorem 5.1. *Let* $\{;,+\} \subseteq \Lambda \subseteq \{+, 0, ;, \backslash, /, ^\smile, ^*, 1'\}$. *Neither* $\mathsf{R}(\Lambda)$ *nor the quasivariety generated by* $\mathsf{R}(\Lambda)$ *is finitely axiomatizable.*

Proof. The proof is similar to the proof of Theorem 3.1. We will work with the same algebras $\mathfrak{A}_n$, but we will equip them with the extra operations $\backslash$, $/$ and *. As before, we have that

- the $\{+,;\}$-reduct of $\mathfrak{A}_n$ is not representable, since the quasiequation q_n defined in (3.1) is valid in $\mathsf{R}(+,;)$ but fails in $\mathfrak{A}_n$.

We will show below that

- any non-trivial ultraproduct over ω, $\mathfrak{A}$, of $\mathfrak{A}_n$ is representable even in the language expanded with $\backslash$, $/$ and *.

We mention below the necessary modifications to the proof of Theorem 3.1.

For the Kleene star * note the following. In $\mathfrak{A}_n$, we have $0 ; 0 = 0$, $1' ; 1' = 1'$ and $x ; x = 1$ for every $x \in A_n \smallsetminus \{0, 1'\}$. Hence we define, in $\mathfrak{A}_n$, $0^* = 1'$, $1'^* = 1'$ and $x^* = 1$ for every $x \in A_n \smallsetminus \{0, 1'\}$, and in $\mathfrak{A}$, $\bar{0}^* = \bar{1}'$, $\bar{1}'^* = 1'$ and $\bar{x}^* = \bar{1}$ for every $x \in A \smallsetminus \{\bar{0}, \bar{1}'\}$. Then it is straightforward that the representation rep defined by the equation (3.4) preserves * as well, whence the ultraproduct $\mathfrak{A}$ of the algebras $\mathfrak{A}_n$ augmented with * is representable.

We want to define the residual $\backslash$ in the algebras $\mathfrak{A}_n$ so that in their ultraproduct $\backslash$ will be representable. To this end we define $\backslash$ so that $x \backslash y$ is the largest element z such that $x ; z \leq y$. Then the algebras $\mathfrak{A}_n$ are in fact closed under the operation $\backslash$ (since they are finite). Indeed, the extension of $x \backslash y$ is determined by

$$z \leq x \backslash y \text{ iff } x ; z \leq y$$

Note that this defines $/$ as well, since $\backslash$ and $/$ coincide in symmetric algebras (where $x ; y = y ; x$ is valid).

Looking at the multiplication table of $\mathfrak{A}_n$ we can explicitly compute the extension of $x \backslash y$ as follows. For $x \in G_n \smallsetminus \{1', o\}$, let $S(x)$ denote the unique element of G_n such that $x ; S(x) = o$, i.e., $\{x, S(x)\} \in S$. We proceed by case distinction on the form of x and y, and the value of $x \backslash y$ is determined by the first applicable case below. If $x = 1$ or $y = 1$, then

$$x \backslash 1 = 1 \qquad 1 \backslash y = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $x = 0$ or $y = 0$, then

$$0 \backslash y = 1 \qquad x \backslash 0 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $x = 1'$ or $y = 1'$, then

$$1' \setminus y = y \qquad x \setminus 1' = \begin{cases} 1 & \text{if } x = 0 \\ 1' & \text{if } x = 1' \\ 0 & \text{otherwise} \end{cases}$$

If $x = o$ or $y = o$, then

$$o \setminus y = \begin{cases} 1 & \text{if } y = 1 \\ 1' & \text{if } o \leq y < 1 \\ 0 & \text{otherwise} \end{cases} \qquad x \setminus o = \begin{cases} 1' & \text{if } x = o \\ S(x) & \text{if } x \in G_n \smallsetminus \{1', o\} \\ 0 & \text{otherwise} \end{cases}$$

When $x = c \in G_n \smallsetminus \{1', o\}$ we have

$$c \setminus y = \begin{cases} 1 & \text{if } y = 1 \\ 1' + S(c) & \text{if } c + o \leq y < 1 \\ S(c) & \text{if } o \leq y < c + o \\ 1' & \text{if } c \leq y < c + o \\ 0 & \text{otherwise} \end{cases}$$

The next case is when $x = c + 1'$ for some $c \in G_n \smallsetminus \{1', o\}$:

$$(c + 1') \setminus y = \begin{cases} 1 & \text{if } y = 1 \\ 1' + S(c) & \text{if } 1' + c + S(c) + o \leq y < 1 \\ S(c) & \text{if } S(c) + o \leq y < 1' + c + S(c) + o \\ 1' & \text{if } c + 1' \leq y < 1' + c + S(c) + o \\ 0 & \text{otherwise} \end{cases}$$

In all other cases, we have

$$x \setminus y = \begin{cases} 1' & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Thus we defined $x \setminus y$ in every $\mathfrak{A}_n$, whence the value of $\setminus$ in $\mathfrak{A}$ is determined.

Next we show that the extension of $\mathfrak{A}$ with $\setminus$ can be represented. To this end we modify the construction of the graph $\mathcal{G}$ in the proof of Theorem 3.1. We have a new type of defect: $(x, y) \in A \times A$ such that $x \setminus y \notin \ell_k(i, j)$ for some $(i, j) \in E_k$ and there is no $w \in E_k$ such that $x \in \ell_k(w, i)$ and $y \notin \ell_k(w, j)$. We assume a fair scheduling σ of all defects as in the proof of Theorem 3.1. If $\sigma_k = (x, y, z)$ is a defect of the form $z \leq x \,;\, y$, then in the $(k+1)$th step we do the construction described in the proof of Theorem 3.1. If $\sigma_k = (x, y)$ is a new type of defect of $(i, j) \in E_k$, then we do the following. We will choose prime filters $F, G \in \mathcal{F}$ such that

(1) $\bar{1}' \notin F, G$,
(2) $x \in F$ and $y \notin G$
(3) the triangle $(F, G, \ell_k(i, j))$ and all its permutations are coherent.

Provided that such filters F and G exist, we can extend the graph $\mathcal{G}_k$ with an extra node $w_{i,j}$ and define

$$\begin{aligned} \ell_{k+1}(w_{i,j}, i) = \ell_{k+1}(i, w_{i,j}) &= F \\ \ell_{k+1}(w_{i,j}, w_{i,j}) &= F(\bar{1}') \\ \ell_{k+1}(w_{i,j}, j) = \ell_{k+1}(j, w_{i,j}) &= G \end{aligned}$$

and, for every $q \in U_k \smallsetminus \{i, j\}$ with $(i, q) \in E_k$,

$$\ell_{k+1}(w_{i,j}, q) = \ell_{k+1}(q, w_{i,j}) = F(\bar{o})$$

See Figure 5 for the case when $i \neq j$. We define $\mathcal{G}_{k+1}$ as the union of these

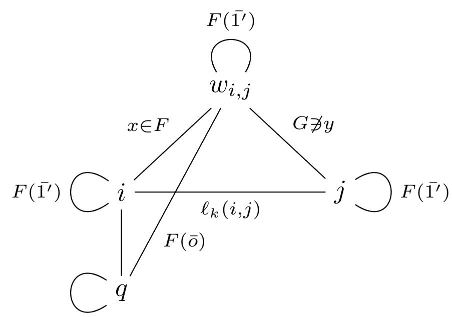

FIGURE 5. Residual when $x \setminus y \notin \ell_k(i, j)$

extensions of $\mathcal{G}_k$ for every $(i, j) \in E_k$ such that $x \setminus y$ is a defect of (i, j).

It remains to define the filters $F, G \in \mathcal{F}$ satisfying the conditions above. First consider the case when $i = j$. Since $x \setminus y \notin \ell_k(i, j) = F(\bar{1}')$, we have that $x \not\leq y$. Then we choose $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$, and also let $G = F$.

Now assume that $i \neq j$. We will proceed by case distinction according to the cases in the definition of $x \setminus y$. When $x \in \{\bar{1}, \bar{1}', \bar{0}\}$ or $y \in \{\bar{1}, \bar{1}', \bar{0}\}$, then either the choice of F and G is straightforward, or will be covered by one of the cases below.

First assume that $x = \bar{o}$. Then we can let $F = F(\bar{o})$ and G be any filter such that $y \notin G$. Using that $\bar{o} \setminus y \notin \ell_k(i, j)$ and that $\bar{o}$ is a flexible atom it is easy to show that $(F, G, \ell_k(i, j))$ satisfies the coherence condition.

The next case is when $y = \bar{o}$. We worked out the situation $x = \bar{o}$ above. Next assume that $x \in At \smallsetminus \{\bar{1}', \bar{o}\}$, whence $x \setminus \bar{o} = s(x) \notin \ell_k(i, j)$ (or the symmetric case $x \setminus \bar{o} = s^{-1}(x) \notin \ell_k(i, j)$). We have to choose F so that $s(z), s^{-1}(z) \notin F$ whenever $z \in \ell_k(i, j)$, since we have the requirements that $\bar{o} \notin G$ and $F \,;\, \ell_k(i, j) \subseteq G$. We let $F = F(\{x, \bar{o}\})$ if $x \in At \smallsetminus \{\bar{1}', \bar{o}, \bar{a}, \bar{b}\}$. If $x = \bar{a}$, then one of $F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$ or $F(\{\ell'(\bar{a}) \cup \{\bar{o}\}\})$ avoids $\{s(z), s^{-1}(z) : z \in \ell_k(i, j)\}$, because of the following. Recall that $\ell_k(i, j) \in \mathcal{F}$. If $\ell_k(i, j) = F(z)$ for some $z \in At \smallsetminus \{\bar{a}, \bar{b}\}$, then z cannot be in both $\ell(\bar{a})$ and $\ell'(\bar{a})$. If

$\ell_k(i,j) = F(\ell(\bar{a}))$ or $\ell_k(i,j) = F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$, then we can choose $F = F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$, and similarly with ℓ' instead of ℓ. Finally, if $\ell_k(i,j) = F(\ell(\bar{b}))$ or $\ell_k(i,j) = F(\{\ell(\bar{b}) \cup \{\bar{o}\}\})$, then we can let $F = F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$ (with the straightforward modifications for ℓ' instead of ℓ). Hence we can choose F with the above property. The case $x = \bar{b}$ is completely symmetric. Then G can be defined as an element of $\mathcal{F}$ with the property that $\bar{o} \notin G$ and, in case $\bar{o} \notin \ell_k(i,j)$, G avoids $\{s(z), s^{-1}(z) : z \in F\}$ (by the same reasoning as above). The case $x \notin At$ can be treated similarly by first picking an atom $x' \in At$ below x and then choosing F for x' and then G as above.

In the case when $x = c \in At \smallsetminus \{\bar{1}', \bar{o}\}$, F can be chosen as in the previous paragraph. The choice of G is a bit more intricate, since G has to avoid y (instead of $\bar{o}$) and $\{s(z), s^{-1}(z) : z \in F\}$ (in case $\bar{o} \notin \ell_k(i,j)$). One can either (i) choose an atom $d \in At \smallsetminus \{\bar{1}', \bar{a}, \bar{b}\}$ such that $d \not\leq y$ and $d \notin \{s(z), s^{-1}(z) : z \in F\}$, or (ii) show that $\bar{a} \not\leq y$ and either $\ell(\bar{a}) \cap \{s(z), s^{-1}(z) : z \in F\} = \emptyset$ or $\ell'(\bar{a}) \cap \{s(z), s^{-1}(z) : z \in F\} = \emptyset$, or (iii) show the same for $\bar{b}$ instead of $\bar{a}$. Then we can let G be the filter generated by $\bar{o}$ and d in case (i), or in case (ii) either $F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$ or $F(\{\ell'(\bar{a}) \cup \{\bar{o}\}\})$, and similarly in case (iii).

The case $x = c + 1'$ for $c \in At \smallsetminus \{\bar{1}', \bar{o}\}$ is completely analogous, since $\bar{1}'$ is not in any filter labelling an irreflexive edge. Finally, in the remaining case, one can choose an atom $c \leq x$ and use one of the above cases for defining F and G.

It is routine to show that the above choices of F and G satisfy the requirements, in particular, that $(F, G, \ell_k(i,j))$ and its permutations are coherent. Hence, using that $\bar{o}$ is a flexible atom, $\mathcal{G}_{k+1}$ satisfies the coherence condition (Coh).

We let $\mathcal{G}$ be the union of $\mathcal{G}_k$ as before. It remains to show that rep as defined by the equation (3.4) preserves $\backslash$ as well.

First assume that $(i,j) \in \mathsf{rep}(x \backslash y)$, i.e., $x \backslash y \in \ell(i,j)$. Let w be arbitrary such that $(w,i) \in \mathsf{rep}(x)$, i.e., $x \in \ell(w,i)$. Then by coherence condition (Coh), $x;(x\backslash y) \in \ell(w,j)$. Since $x;(x\backslash y) \leq y$, we have $y \in \ell(w,j)$, i.e., $\ell(w,j) \in \mathsf{rep}(y)$. Hence $(i,j) \in \mathsf{rep}(x) \backslash \mathsf{rep}(y)$ as desired.

For the other direction assume that $(i,j) \notin \mathsf{rep}(x \backslash y)$, i.e., $x \backslash y \notin \ell_k(i,j)$ for any k. Then by the construction above, we have k such that $w_{i,j} \in E_k$ and $x \in \ell_{k+1}(w_{i,j}, i)$, i.e., $(w_{i,j}, i) \in \mathsf{rep}(x)$, and $y \notin \ell_{k+1}(w_{i,j}, j)$, i.e., $(w_{i,j}, j) \notin \mathsf{rep}(y)$. Hence $(i,j) \notin \mathsf{rep}(x) \backslash \mathsf{rep}(y)$ as desired, finishing the proof of Theorem 5.1 □

5.4. Problems. We conclude with some open problems.

We have seen that neither the quasivariety nor the variety of the language $\Lambda \supseteq \{\cdot, ;, \smile\}$ is finitely axiomatizable. But we left the following open.

Problem 5.2. For which $\Lambda \supseteq \{\cdot, ;, \smile\}$ do we have $\mathsf{R}(\Lambda) = \mathsf{V}(\Lambda)$?

We noted that neither $\mathsf{R}(+, ;, \backslash, /)$ (see Theorem 5.1) nor $\mathsf{V}(+, \cdot, ;, \backslash, /)$ (see [16]) is finitely axiomatizable.

Problem 5.3. Is the variety $\mathsf{V}(+, ;, \backslash, /)$ finitely axiomatizable?

We have seen that in many cases the generated varieties are not finitely based, while in certain cases there are finitely axiomatizable quasivarieties generating these varieties (see the case of Kleene algebras). This motivates the following problem which asks for similar results in the extended similarity types.

Problem 5.4. Find finitely axiomatizable quasivarieties such that they generate the same varieties as generated by those representable (semi)lattice ordered monoids/semigroups whose signatures include the residuals of composition and/or Kleene star.

Acknowledgements. Thanks are due to the anonymous referee and I. Hodkinson for valuable comments and to R. Kontchakov for the translation from Russian.

References

[1] Andréka, H.: On the representation problem of distributive semilattice-ordered semigroups. Tech. rep., Mathematical Institute of the Hungarian Academy of Sciences (1988). Abstracted in Abstracts of the American Mathematical Society, 10(2):174, March 1989

[2] Andréka, H.: Representation of distributive lattice-ordered semigroups with binary relations. Algebra Universalis **28**, 12–25 (1991)

[3] Andréka, H., Bredikhin, D.: The equational theory of union-free algebras of relations. Algebra Universalis **33**, 516–532 (1995)

[4] Andréka, H., Mikulás, Sz.: Lambek calculus and its relational semantics: completeness and incompleteness. Journal of Logic, Language and Information **3**, 1–37 (1994)

[5] Andréka, H., Németi, I.: Axiomatization of identity-free equations valid in relation algebras. Algebra Universalis **35**, 256–264 (1996)

[6] Andréka, H., Németi, I., Sain, I.: Algebraic logic. In: D. Gabbay, F. Guenthner (eds.) Handbook of Philosophical Logic, vol. 2, second edn., pp. 133–247. Kluwer Academic Publishers, Dordrecht (2001)

[7] Bredikhin, D.: The equational theory of relation algebras with positive operations. Izv. Vyash. Uchebn. Zaved. Math. pp. 23–30 (1993). In Russian

[8] Bredikhin, D., Schein, B.: Representation of ordered semigroups and lattices by binary relations. Colloquium Mathematicum **39**, 1–12 (1978)

[9] Conway, J.: Regular Algebra and Finite Machines. Chapman and Hall, London (1971)

[10] Crvenković, S., Dolinka, I., Ésik, Z.: The variety of Kleene algebras with conversion is not finitely based. Theoretical Computer Science **230**, 235–245 (2000)

[11] Ésik, Z., Bernátsky, L.: Equational properties of Kleene algebras of relations with conversion. Theoretical Computer Science **137**, 237–251 (1995)

[12] Haiman, M.: Arguesian lattices which are not type 1. Algebra Universalis **28**, 128–137 (1991)

[13] Hirsch, R.: The class of representable ordered monoids has a recursively enumerable, universal axiomatisation but it is not finitely axiomatisable. Logic Journal of the IGPL **13**, 159–171 (2005)

[14] Hirsch, R., Hodkinson, I.: Relation Algebras by Games. Elsevier, Amsterdam (2002)

[15] Hirsch, R., Mikulás, Sz.: Representable semilattice-ordered monoids. Algebra Universalis **57**, 333–370 (2007)

[16] Hirsch, R., Mikulás, Sz.: Positive reducts of relevance logic and algebras of binary relations. Review of Symbolic Logic (to appear)

[17] Hodkinson, I., Mikulás, Sz.: Axiomatizability of reducts of algebras of relations. Algebra Universalis **43**, 127–156 (2000)

[18] Jipsen, P., Jónsson, B., Rafter, J.: Adjoining units to residuated Boolean algebras. Algebra Universalis **34**, 118–127 (1995)
[19] Kozen, D.: A completeness theorem for Kleene algebras and the algebra of regular events. Information and Computation **110**, 366–390 (1994)
[20] Maddux, R.: Relation Algebras. Elsevier, Amsterdam (2006)
[21] Mikulás, Sz.: Axiomatizability of algebras of binary relations. In: B. Löwe, B. Piwinger, T. Räsch (eds.) Classical and New Paradigms of Computation and their Complexity Hierarchies, pp. 187–205. Kluwer Academic Publishers, Dordrecht (2004)
[22] Monk, J.: On representable relation algebras. Michigan Mathematics Journal **11**, 207–210 (1964)
[23] Németi, I.: Every free algebra in the variety generated by the representable dynamic algebras is separable and representable. Theoretical Computer Science **17**, 343–347 (1982)
[24] Németi, I.: Algebraizations of quantifier logics, an introductory overview. Studia Logica **50**(3–4), 485–569 (1991)
[25] Pratt, V.: Action logic and pure induction. In: J. van Eijck (ed.) Logics in AI: European Workshop JELIA '90, pp. 97–120. Springer, Heidelberg (1990)
[26] Redko, V.: On defining relations for the algebra of regular events. Ukrain. Mat. Z. **16**, 120–126 (1964). In Russian
[27] Schein, B.: Representation of subreducts of Tarski relation algebras. In: H. Andréka, J. Monk, I. Németi (eds.) Algebraic Logic, pp. 621–635. North-Holland, Amsterdam (1991)

Hajnal Andréka

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, 13–15 Reáltanoda u., 1053 Budapest, Hungary
e-mail: andreka@renyi.hu
URL: http://www.renyi.hu/~andreka/

Szabolcs Mikulás

Department of Computer Science and Information Systems, Birkbeck, University of London, Malet Street, London WC1E 7HX, UK
e-mail: szabolcs@dcs.bbk.ac.uk
URL: http://www.dcs.bbk.ac.uk/~szabolcs/