Algebras of relations of various ranks, some current trends and applications

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Abstract. Here the emphasis is on the main pillars of Tarskian structuralist approach to logic: relation algebras, cylindric algebras, polyadic algebras, and Boolean algebras with operators. We also tried to highlight the recent renaissance of these areas and their fusion with new trends related to logic, like the guarded fragment or dynamic logic. Tarskian algebraic logic is far too broad and too fruitful and prolific by now to be covered in a short paper like this. Therefore the overview part of the paper is rather incomplete, we had to omit important directions as well as important results. Hopefully, this incompleteness will be alleviated by the accompanying paper of Tarek Sayed Ahmed [81].

The *structuralist approach* to a branch of learning aims for separating out the really essential things in the phenomena being studied abstracting from the accidental wrappings or details (called in computer science "syntactic sugar"). As a result of this, eventually one associates to the original phenomena (or systems) being studied streamlined elegant mathematical structures. These streamlined structures can be algebras in the sense of universal algebra, or other kinds of elegant well understood mathematical structures like e.g. space-time geometries in the case of relativity theory. In the present case they will be algebras, but as we said, this is not essential to the approach, they could be geometric metric spaces, topologies, etc.

Along the above structuralist lines, one arrives at a kind of *duality theory* between the theory describing the original systems (or phenomena) and the corresponding streamlined mathematical structures. The important point in calling this a duality is the requirement that from the streamlined abstract mathematical structures one should be able to reconstruct the originals up to a certain equivalence determined by what we consider essential in our first step mentioned above. So far, two expressions showed up which we will use in the above outlined sense. These are "structuralist approach to something", and "duality theory" (in this structuralist spirit).

Algebraic logic can be regarded as a structuralist approach to the subject matter of logic. In other words, algebraic logic sets up a duality theory between the world of logics and the world of classes of algebras.

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The algebraic counterpart of classical propositional logic is the world of Boolean algebras (BA's for short). BA theory was immensely successful in helping and improving, clarifying classical propositional logic, see e.g. [32]. What happens then if we want to extend the algebraization of classical propositional logic yielding BA's to first-order logic?

Well, BA's are algebras of unary relations. That is, the elements of a BA $\mathfrak{B}$ are unary relations and the operations of $\mathfrak{B}$ are the natural operations on unary relations e.g. intersection, complementation. The problem of extending this approach to predicate logics boils down to the problem of expanding the natural algebras of unary relations to natural algebras of relations of higher ranks, i.e. of relations in general, more precisely, algebras of not necessarily binary relations e.g. algebras of ternary relations. The reason for this is, roughly speaking, the fact that the basic building blocks of predicate logics are predicates, and the meanings of predicates can be relations of arbitrary ranks. Indeed, already in the middle of the last century, when De Morgan wanted to generalize algebras of propositional logic in the direction of what we would call today predicate logic, he turned to algebras of binary relations.[1] That was probably the beginning of the quest for algebras of relations in general. Returning to this quest, the new algebras will, of course, have more operations than BA's, since between relations in general there are more kinds of connections than between unary relations (e.g. one relation might be the converse, sometimes called inverse, of the other).

The *framework* for the quest for the natural algebras of relations *is universal algebra*. The reason for this is that universal algebra is the field which investigates classes of algebras in general, their interconnections, their fundamental properties. Therefore universal algebra can provide us for our search with a "map and a compass" to orient ourselves. There is a further good reason for using universal algebra. Namely, universal algebra is not only a unifying framework, but it also contains powerful theories. E.g. if we know in advance some general properties of the kinds of algebras we are going to investigate, then universal algebra can reward us with a powerful machinery for doing these investigations. Among the special classes of algebras concerning which universal algebra has powerful theories are the so called *discriminator varieties* and the *arithmetical varieties*. At the same time, algebras originating from logic turn out to fall in one of these two categories, in most cases.

Let us return to our task of moving from BA's of unary relations to expanded BA's of relations in general. What are the elements of a BA? They are sets of

[1] De Morgan illustrated the need for expanding the algebras of unary relations (i.e. BA's) to algebras of relations in general (the topic of the present paper) by saying that the scholastics, after two millennia of Aristotelian tradition, were still unable to prove that if a horse is an animal, then a horse's tail is an animal's tail. ("v_0 is a tail of v_1" is a binary relation.)

"points". What will be the elements of the expanded new algebras? One thing about them seems to be certain, they will be sets of sequences. Why? Because relations in general are *sets of sequences*. These sequences may be just pairs if the relation is binary, they may be triples if the relation is ternary, or they may be longer — or more general kinds of sequences. (There is another consideration pointing in the direction of sequences. Namely, the semantics of quantifier logics is defined via satisfaction of formulas in models, which in turn is defined via evaluations of variables, and these evaluations are sequences. The meaning of a formula in a model is the *set of* those *sequences* which satisfy the formula in that model. Thus we arrive again at sets of sequences.) So, one thing is clear at this point, namely that the elements of our expanded BA's of relations will be sets of sequences. Indeed, this applies to practically all known algebraizations of predicate logics or quantifier logics.

Summing up: the key paradigm for moving from BA's to richer versions of algebraic logic is replacing sets of points with sets of sequences as the elements of our algebras[2]; e.g. in the case of relation algebras, our algebras are complex algebras of oriented graphs i.e. their elements are sets of "arrows". This is where the name "*arrow logic*" comes from, cf. [16], [62]. In the more general case of longer sequences, our algebras are like complex algebras of lists in the sense of the programming language LISP, so we can regard a sequence or a list as a generalization of an arrow, say like a longer arrow which has points in the middle, too.

1 Algebras of binary relations

Let us start concentrating on the simplest nontrivial case, namely that of the algebras of binary relations. Actually, these algebras will be strong enough to be called a truly first-order (as opposed to propositional) algebraic logic, namely [29], [87, §5.3] show that the logic captured by binary relation algebras is strong enough to serve as a vehicle for set theory and hence for ordinary metamathematics.

A *binary relation algebra* (BRA for short) is an algebra whose *elements* are binary relations and whose *operations* are the operations of taking the union $R \cup S$ of two relations R, S, taking the complement $-R$ of R, taking the relation algebraic composition $R \circ S = \{\langle a, b\rangle : \exists c(aRc \text{ and } cSb)\}$ of two relations R, S

[2] When explaining the so-called "holy grail of modern physics" (grand unification theories) in his video "The Elegant Universe", Brian Greene said that the key idea was replacing point-like building blocks by string-like blocks (actually: superstrings), in their paradigm-shift. The motivation he gave was that strings have more "degrees of freedom" than points do (strings can wriggle while points cannot). The same can be said about switching from points to sequences which we are describing here (sequences can wriggle, too). So perhaps the two paradigm shifts (the one described here and the one in cosmology) might turn out to have something in common.

and taking the inverse $R^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}$ of the binary relation R. That is,

Definition 1. *A BRA is an algebra*

$\mathfrak{A} = \langle \mathcal{A}; \cup, -, \circ, ^{-1}\rangle$ *where*

(i) *$\mathcal{A}$ is a nonempty set of binary relations,*
(ii) *$\langle \mathcal{A}; \cup, -\rangle$ is a Boolean set algebra, i.e. $R, S \in \mathcal{A} \Rightarrow R \cup S, -R \in \mathcal{A}$, and*
(iii) *$\mathcal{A}$ is closed under taking relation algebraic composition and inverses, i.e. $R, S \in \mathcal{A} \Rightarrow R \circ S, R^{-1} \in \mathcal{A}$.*

BRA *denotes the class of all algebras* isomorphic to *BRA's.* □

Item (ii) in the definition of a BRA above implies that $\mathcal{A}$ has a biggest element, called the *unit* of $\mathcal{A}$. Let $V = \bigcup \mathcal{A}$ be this unit. Then (iii) above implies that $V \circ V \subseteq V$ and $V^{-1} \subseteq V$, hence V is an equivalence relation. Thus the unit of a BRA is always an equivalence relation.

Let us return to universal algebra as a unifying framework. If $\mathfrak{A}$ is a BRA as above, then the similarity type or signature of $\mathfrak{A}$ consists of the function symbols $\cup, -, \circ, ^{-1}$ where the first two are Boolean join and complementation.[3] Homomorphisms, equations etc. are defined accordingly; e.g. homomorphisms should preserve all four operations, and $(x \cup y) \circ z = (x \circ z) \cup (y \circ z)$, $(x \cup y)^{-1} = x^{-1} \cup y^{-1}$ are typical equations.[4] We will often denote by 0 and 1 the Boolean zero and unit, i.e. the smallest and biggest elements of the Boolean algebra.

Notice that a BRA is a BA enriched with further operations $\circ, ^{-1}$, and these extra-Boolean operations all distribute over $\cup$. Such algebras are called *Boolean algebras with operators* (BAO's for short). A BAO $\langle \mathcal{B}; \cup, -, f_i : i \in I\rangle$ is called *normal* if the Boolean zero 0 is a zero-element for all the extra-Boolean operations f_i, i.e. if $f_i(x_1, \ldots, x_n) = 0$ whenever one of the arguments $x_1, \ldots, x_n$ is 0. The literature of BAO's is quite extensive, see e.g. [46], [47] to mention two papers.

Having a fresh look at our BRA's with an *abstract algebraic* eye, we notice that they should be very familiar from the abstract algebraic literature. Namely, a BRA $\mathfrak{A} = \langle \mathcal{A}; \cup, -, \circ, ^{-1}\rangle$ consists of two well known algebraic structures,

(R1) a *Boolean algebra* $\langle \mathcal{A}; \cup, -\rangle$ and

[3] In the literature various other symbols are used for these operations, perhaps the most often used symbols are $+, -, ;, ^{\smile}$. We hope it will not lead to confusion that we use the same symbols for the operations and the operation symbols denoting these operations.
[4] These two equations express that composition and inverse distribute over $\cup$.

(R2) an *involuted semigroup* $\langle \mathcal{A}; \circ, ^{-1} \rangle$

sharing the same universe. Further, the connection between these two structures is described by saying that

(R3) $\mathfrak{A} = \langle \mathcal{A}; \cup, -, \circ, ^{-1} \rangle$ is a *normal BAO.*

These properties are all expressible with equations.

The above (R1)-(R3) define a nice equational class V containing BRA and is a reasonable starting point for an axiomatic study of the algebras of relations. Postulates like above already appear in De Morgan [23], and since then investigations of relation algebras have been carried on for almost 130 years.

The question naturally arises whether V equals BRA. It will turn out that BRA indeed can be defined by a set E of equations, but this set E of equations cannot be chosen to be finite. Thus V is bigger than BRA since V is defined by a finite set of equations.

To illuminate the use of universal algebraic methods in algebraic logic, first we recall some standard material from Universal Algebra.

Subdirect products of algebras, and subdirectly irreducible algebras are defined in practically every textbook on Universal Algebra. An algebra $\mathfrak{A}$ is *subdirectly irreducible* if it is not (isomorphic to) a subdirect product of algebras all different from $\mathfrak{A}$ or the one-element algebra. A *subdirect product* is a subalgebra of a product of algebras, having a special "economy" property. For our purposes, this "economy" property will not be needed. By Birkhoff's classical theorem, every algebra is a subdirect product of some subdirectly irreducible ones. Therefore, the subdirectly irreducible algebras are often regarded as the basic building blocks of all the other algebras. In particular, when studying an algebra $\mathfrak{A}$, it is often enough to study its subdirectly irreducible building blocks.

For example, the unique (up to isomorphism) subdirectly irreducible BA is the 2-element one. It is not hard to check that the subdirectly irreducible BRA's are exactly the isomorphic copies of the BRA's whose unit element as an equivalence relation has exactly one equivalence block, i.e. whose unit element is a Descarte space $U \times U$.

Definition 2. *A class* K *of algebras is called a* discriminator class *iff conditions (i),(ii) below hold.*

(i) There is a term $\tau(x, y, z, u)$ *in the language of* K *such that in every subdirectly irreducible member of* K *we have*

$$\tau(x, y, z, u) = \begin{cases} z, & \text{if } x = y, \\ u, & \text{otherwise.} \end{cases}$$

(ii) Every algebra in K *is a subdirect product of subdirectly irreducible members of* K*.*

The term τ *in (i) above is called a* discriminator term.

A class K *of algebras is called a* variety *if there is a set* E *of equations such that* K *consists of exactly those algebras in which this set* E *of equations is valid. By a* discriminator variety *we understand a discriminator class which is a variety.*

Warning: Sometimes instead of the 4-ary τ*, the ternary discriminator term* $t(x, y, z) = \tau(x, y, z, x)$ *is used. They are interdefinable, since* $\tau(x, y, z, u) = t(t(x, y, u), t(x, y, z), z)$*. Therefore, it does not matter which one is used.* $\square$

Having a discriminator term provides us with powerful methods, e.g. the discriminator term can be used to code up any universal formula into an equation in the subdirectly irreducible members of K.

Theorem 1. (Tarski) BRA *is a discriminator variety. In particular, there is a set* E *of equations defining the class* BRA*, i.e. such that an algebra* $\mathfrak{A}$ *belongs to* BRA *iff the set* E *of equations is valid in* $\mathfrak{A}$*.*

We outline a proof for Theorem 1 which relies on universal algebraic methods. This method is widely usable in algebraic logic.

Outline of a proof for Theorem 1: Let Si denote the class of all subdirectly irreducible members of BRA.

(a) As we mentioned above, the elements of Si are (up to isomorphism) exactly those BRA's whose unit element is of form $U \times U$.

(b) BRA is a discriminator class. Let $c(x) = 1 \circ x \circ 1$ where 1 denotes the Boolean unit. By (a) then, in every member of Si, $c(0) = 0$ and $c(x) = 1$ whenever x is nonzero. From this "switching term" c and the Boolean operations it is not difficult to write up a discriminator term for Si, as follows. Let $x \oplus y$ denote the *symmetric difference* of x and y, i.e. $x \oplus y = [x \cap -y] \cup [y \cap -x]$. Then ($x = y$ iff $c(x \oplus y) = 0$) and ($x \neq y$ iff $c(x \oplus y) = 1$) for any x, y, in members of Si. Let $\tau(x, y, z, u) = [z \cap -c(x \oplus y)] \cup [u \cap c(x \oplus y)]$. Then τ is a discriminator term for Si, and hence for BRA, too.

(c) It is not hard to show, e.g. by considering BRA's as many-sorted structures and collecting the first-order formulas that define this class, that Si is closed under taking ultraproducts. This is done e.g. in [14, section 1, Lemma 5].

(d) Direct products of BRA's correspond to "disjoint union of units" just as in the case of BA's, i.e. if $\mathfrak{A}, \mathfrak{B}$ are BRA's with unit elements V, W respectively, then the direct product $\mathfrak{A} \times \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is (isomorphic to) a subalgebra of a BRA with unit element the disjoint union of V and W. The same applies to more than two algebras.

If K is a class of algebras, then $\mathbf{H}\mathsf{K}, \mathbf{S}\mathsf{K}, \mathbf{P}\mathsf{K}, \mathbf{I}\mathsf{K}, \mathbf{Up}\mathsf{K}$ denote the classes of all homomorphic images, all subalgebras, all direct products, all isomorphic copies and all ultraproducts of members of K, respectively. The fundamental theorem of discriminator varieties states that $\mathbf{HSP}\mathsf{K} = \mathbf{ISPUp}\mathsf{K}$ whenever K is a discriminator class.

(e) From (c), (d) and (a) we get that $\mathsf{BRA} = \mathbf{ISPUp}\mathsf{Si}$. From the fundamental theorem of discriminator varieties and (b) then we get $\mathsf{BRA} = \mathbf{HSP}\mathsf{Si}$.

(f) The fundamental theorem of varieties, Birkhoff's variety theorem, states that a class V of algebras is a variety iff $\mathsf{V} = \mathbf{HSP}\mathsf{L}$ for some class L of algebras. Thus BRA is a variety by (e).

BRA is a discriminator variety by (b) and (f). This finishes the proof of Theorem 1. $\square$

We note that Theorem 1 implies that the homomorphic image of a BRA is again a BRA. We do not know a simple direct proof for this.

Theorem 1 indicates that BRA is indeed a promising start for developing a nice algebraization of (at least a part of) first-order logic, or to put it more plainly, for developing an algebraic theory of relations. After Theorem 1, the question comes up naturally if we can strengthen the postulates (R1)-(R3) defining V to obtain a finite set E of equations describing the variety BRA. The answer is

Theorem 2. (Monk [66]) BRA *is not finitely axiomatizable, i.e. the set E of equations in Theorem 1 cannot be chosen to be finite. However, E can be chosen to be decidable.*

Theorem 2 was strengthened in many ways. Maddux [56] proved that the set of equations containing only one variable and valid in BRA is not finitely axiomatizable, either. Jónsson [45] proved that no set E of equations containing only finitely many variables can axiomatize BRA, i.e. if E is as in Theorem 1 then E has to contain infinitely many variables. Andréka [4] proved that besides the necessity of using infinitely many variables in E, one has to use all the operation symbols $\cup, -, \circ$ also: for any natural number k there have to be infinitely many equations e in E such that e contains more than k distinct variables and all the operation symbols $\cup, -, \circ$. This latter result does not extend to the operation symbol $^{-1}$ because, as is proved in [12], converse $^{-1}$ is finitely axiomatizable over the other operations in BRA. Venema [89] shows that not all elements of E can have the shape of the so-called Sahlqvist equations. Hodkinson-Venema [39] shows that E cannot consist of so-called canonical equations only (though BRA is a so-called canonical variety).

One can obtain a decidable infinite set E of equations characterizing the variety BRA, by Monk [67]. Lyndon [51] outlines another recipe for obtaining a different such E which may work for BRA. However, the structures of these E's are rather involved. Hirsch-Hodkinson [37], [38] contain other recursively enumerable E's for Thm.1. In this connection, we note that the following is still an important open problem of algebraic logic:

Problem 1. Find <u>*simple*</u>, mathematically transparent, decidable sets E of equations characterizing BRA. (A solution for this problem has to be considerably simpler than, or at least markedly different from, the E's discussed above.)

The efforts of trying to get rid of Theorem 2 became known as *finitizability investigations*. Why would we want to get rid of Theorem 2? Algebras of unary relations, i.e. BA's, admit a nice finite axiomatization. Theorem 2 says that the same is not possible for BRA's, binary relation algebras. Since we are in the middle of the search for the right notion of algebras of (not only unary) relations, the question naturally comes up whether Theorem 2 was perhaps only a consequence of an unfortunate choice of the basic operations $\circ$ and $^{-1}$ of BRA's.

Apparently, it is not so easy to get rid of Theorem 2 by moving to reducts of BRA, e.g. by omitting operations from BRA's. Axiomatizability of reducts of BRA's is a well investigated area even today. As a sample result we mention that if we have $\cup$ and $\circ$ among the operations of the reduct, then the class remains non-finitely axiomatizable (Andréka [3]). However, if we keep only $\cap$ and $\circ$, then the so obtained reduct is a finitely axiomatized variety (Bredikhin and Schein [22]).

One can get very far in doing algebraic logic (for quantifier or predicate logics) via BRA's. If we want to investigate nonclassical quantifier logics, we can replace the Boolean reduct $\langle \mathcal{A}; \cup, - \rangle$ of $\langle \mathcal{A}; \cup, -, \circ, ^{-1} \rangle \in \mathsf{BRA}$ with the algebras corresponding to the propositional version of the nonclassical logic in question (e.g. Heyting algebras). By Andréka's above quoted result, if instead of $\langle \mathcal{A}; \cup, - \rangle$ we use (an expansion of, like Heyting algebras) a distributive lattice, then Theorem 2 carries over.

It is a second central open problem[5] of algebraic logic to find out whether we can get rid of Theorem 2 by moving in the direction opposite to taking reducts. This would mean to expand BRA's by new, "set-theoretically defined" operations on relations such that the new class would become finitely axiomatizable. This quest is motivated by the fact that there are many known examples for a non-finitely axiomatizable class K of structures such that an expansion K^+ of K is finitely axiomatizable. As an example, here we mention two results of Bredikhin [20], [21].: If we take as basic operations $\circ, ^{-1}$, we get a nonfinite-axiomatizable but axiomatizable class, just as BRA is. However, if we add the unary operation dom of taking the domain $\mathsf{dom}(R) = \{\langle a, a\rangle : \exists b\langle a, b\rangle \in R\}$ of a relation R, and add the binary relation $\subseteq$ of inclusion between relations, we get a finitely axiomatizable class.

The following problem was raised in various forms by Jónsson, Henkin, Monk, Tarski–Givant [87, lines 9–11 of p. 62, the first sentence of the paragraph preceding 3.5.(ix), and the first page of §3.5 (p. 56)].

Problem 2. Can one add to the operations of BRA finitely many new set-theoretically defined operations $f_1, \dots, f_n$ on relations such that the class $\{\langle \mathfrak{A}, f_1, \dots, f_n\rangle : \mathfrak{A}$ is a BRA$\}$ would generate a finitely axiomatizable variety.

The finitization problem is discussed in detail in [70], [81]. About the finitization problem there is a large number of new positive solutions, e.g. [50], [77], [78], [79], [83], [84], [85].

The natural logical counterpart of BRA's is classical first-order logic restricted to three individual variables v_0, v_1, v_2 and without equality ([29],[87]). As shown in [87, §5.3], [29] this system is an adequate framework for building up set theory and hence metamathematics in it. One can illustrate most of the main results, ideas and problems of algebraic logic by using only BRA's.

BRA's also play an important rôle in theoretical computer science (cf. e.g. the references given in [70]). Jónsson [44] calls a finitely axiomatizable variety approximating BRA *Program Specification Algebras*.

If we want to algebraize first-order logic with equality, we have to add an extra constant Id, representing equality, to the operations. RRA denotes the class

[5] This is not really a single problem; it is rather a large circle of problems motivated (and "punctuated") by many deep results. In a sense, it is a rich theory built around a deep, open, almost philosophical question.

of so obtained algebras. RRA abbreviates representable relation algebras. RRA has been investigated more thoroughly than BRA; actually, Theorems 1,2 above were proved first for RRA. For historical notes on Theorem 1 for RRA see [87, Thm.8.3(v) on p. 240].

2 Algebras of more general relations

By this point we might have developed some vague picture of how algebras of binary relations are introduced, investigated etc. One might even sense that they give rise to a smooth, elegant, exciting and powerful theory. However, our original intention was to develop algebras of relations in general, which should surely incorporate not only binary but also ternary, and in general n-ary relations.

Let us see how to generalize our BRA's to relations of higher ranks. Let us fix n to be a natural number. Defining composition of n-ary relations for $n > 2$ is complicated (however, not impossible). Instead, we will single out simple basic operations on n-ary relations and then show that composition, and all kinds of other operations on ternary etc. relations are expressible via these basic simple operations.

From now on for a while, let n be a fixed natural number. Let U be any set. Then ${}^nU$ denotes the n-th Cartesian power of U, i.e. ${}^nU$ is the set of all U-termed sequences of length n, i.e. ${}^nU = U \times \cdots \times U$ (n times). An n-ary relation over U is then just a subset of ${}^nU$. The following are natural operations on n-ary relations.

Let $1 \leq i \leq n$ and $R \subseteq {}^nU$. The i-th *projection* or *cylindrification* of the relation R is defined as

$$\mathsf{c}_i^U(R) = \{\langle a_1, \ldots, a_{i-1}, u, a_{i+1}, \ldots, a_n\rangle : \langle a_1, \ldots, a_n\rangle \in R \text{ and } u \in U\}.$$

The *n-ary identity relation* over U is defined to be

$$\mathsf{Id}_{n,U} = \{\langle u, \ldots, u\rangle : u \in U\} \subseteq {}^nU.$$

Intuitively, i-th cylindrifying a relation R is "deleting the information content of the i-th argument", and $\mathsf{Id}_{n,U}$ can be used to transfer "information" between the arguments of a relation.

The relation $c_i(R)$ is called the smallest i-cylinder containing R. Using $n = 3$ and U the real numbers, we obtain the greatest element $U \times U \times U$ of our algebra as the usual Cartesian space, and i-cylinders appear as "cylinders" parallel with the i-th axis. Let $n = 2$ and $R \subseteq U \times U$. Then $c_0(R) = U \times Rg(R)$ and $c_1(R) = Dom(R) \times U$. This example shows that the operations c_i are natural ones (on relations). See Figures 1,2.

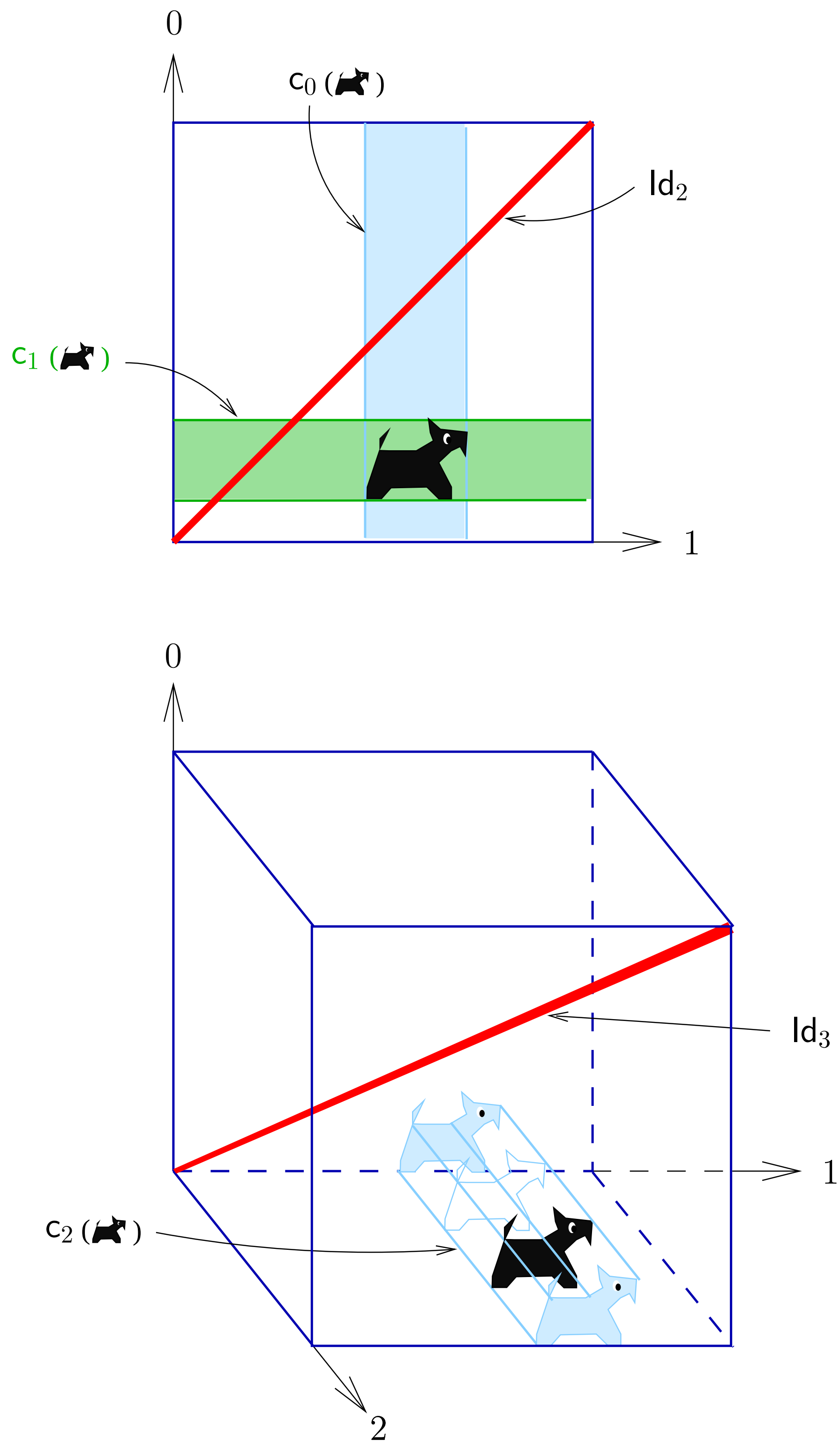

Fig. 1. Drawing cylindrifications in the case of $n = 3$.

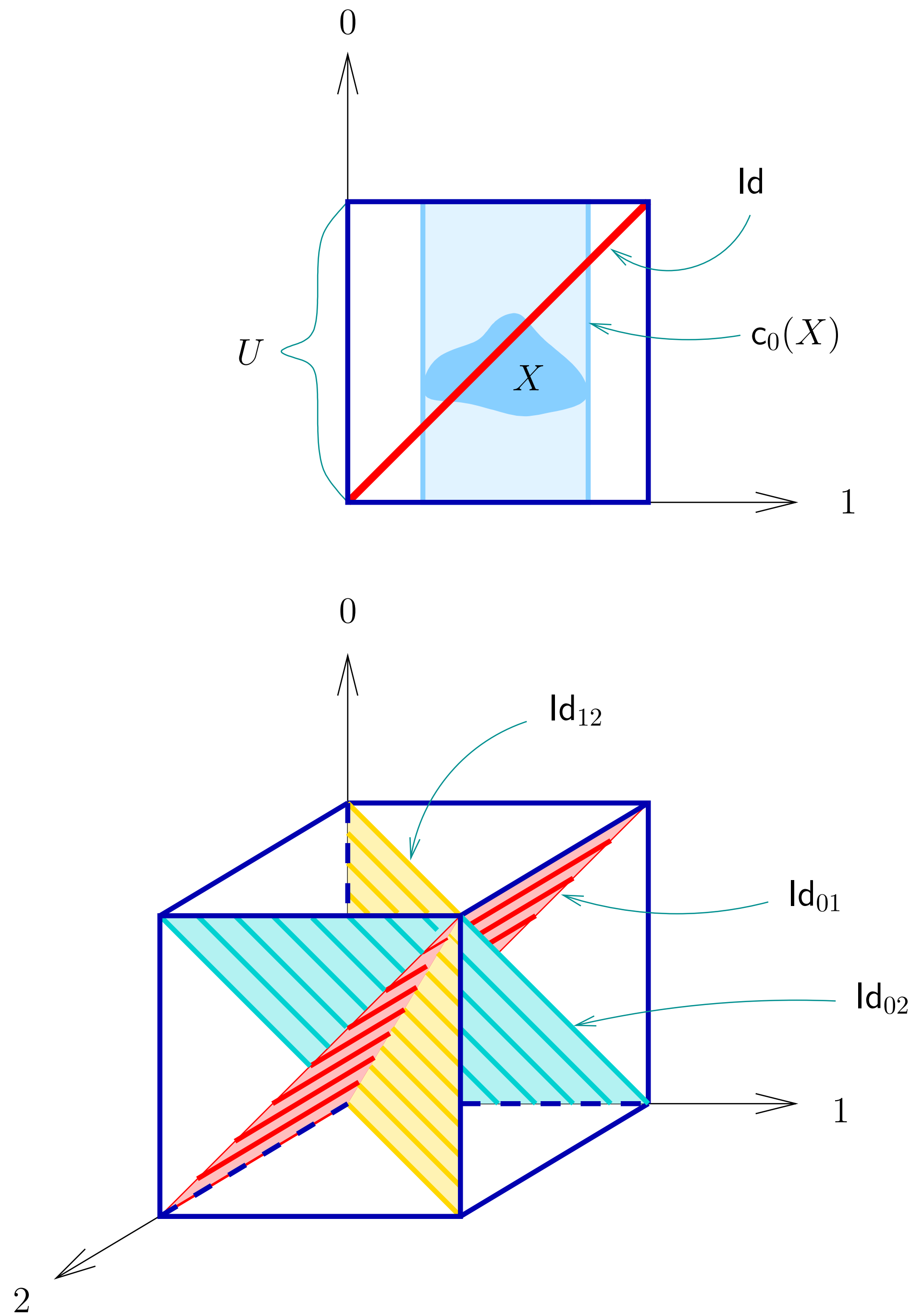

Fig. 2. Drawing the diagonal constants in the case of $n = 3$.

Definition 3. *A* cylindric algebra of n-ary relations over the set U *is an algebra*

$$\langle \mathcal{A}; \cup, -, \mathsf{c}_1^U, \ldots, \mathsf{c}_n^U, \mathsf{Id}_{n,U} \rangle \quad \textit{where}$$

(i) $\mathcal{A}$ *is a set of* n*-ary relations on* U *such that*
(ii) $\mathcal{A}$ *is closed under the specified operations, i.e.*
$\mathsf{Id}_{n,U} \in \mathcal{A}$ *and* $R, S \in \mathcal{A} \Rightarrow R \cup S, -R, \mathsf{c}_1^U(R), \ldots, \mathsf{c}_n^U(R) \in \mathcal{A}$. □

Note that $\langle \mathcal{A}; \cup, - \rangle$ is just the usual Boolean set algebra with universe $\mathcal{A}$.

A *cylindric algebra of relations*, a Cs in short, is a cylindric algebra of n-ary relations over some set U. Looking at Cs's with an *abstract algebraic* eye, we notice that Cs's are normal BAO's since the extra-Boolean operations, the cylindrifications c_i, distribute over Boolean join and are "normal", i.e.

$$\mathsf{c}_i(R \cup S) = \mathsf{c}_i(R) \cup \mathsf{c}_i(S) \quad \text{and} \quad \mathsf{c}_i(0) = 0.$$

The cylindrifications have some extra good properties, namely in addition to being "normal operators", they are also "*closure* operators", i.e.

$$\mathsf{c}_i \mathsf{c}_i(R) = \mathsf{c}_i(R),$$

moreover they are so-called *complemented* closure operators and they *commute* with each other, i.e.

$$\mathsf{c}_i(-\mathsf{c}_i(R)) = -\mathsf{c}_i(R), \quad \text{and} \quad \mathsf{c}_i \mathsf{c}_j(R) = \mathsf{c}_j \mathsf{c}_i(R).$$

There are similar natural equations concerning the constant $\mathsf{Id}_{n,U}$. These state that $\mathsf{Id}_{n,U}$ codes up a "Leibnizian equivalence relation".: For $1 \leq i \leq j \leq n$ define

$$\mathsf{Id}_{ij} = \mathsf{c}_1 \ldots \mathsf{c}_{i-1} \mathsf{c}_{i+1} \ldots \mathsf{c}_{j-1} \mathsf{c}_{j+1} \ldots \mathsf{c}_n \mathsf{Id}_{n,U}.$$

Then $\mathsf{Id}_{ij} = \{s \in {}^nU : s_i = s_j\}$. Now the "Leibnizian equivalence axioms" are

$$\mathsf{Id}_{ii} = 1, \quad \mathsf{Id}_{ij} = \mathsf{Id}_{ji}, \quad \mathsf{Id}_{ik} = c_j(\mathsf{Id}_{ij} \cap \mathsf{Id}_{jk}) \text{ if } j \notin \{i, k\}, \quad \text{and}$$
$$x \leq \mathsf{Id}_{ij} \Rightarrow c_i(x) \cap \mathsf{Id}_{ij} = x.$$

Thus, a Cs is a

(C1) a normal BAO with commuting complemented closure operators c_i and
(C2) the constant element $\mathsf{Id}_{n,U}$ satisfies the Leibnizian equivalence axioms.

The above (C1), (C2) define a nice variety called the variety $\mathsf{CA_n}$ of *cylindric algebras*. (The subscript n in $\mathsf{CA_n}$ refers to the rank n of our relations, fixed earlier.) Advantages of cylindric algebras of relations over BRA's are the following. (1) Cs's contain only unary extra-Boolean operations, and unary operations are much easier to handle in *algebraic* investigations than binary ones. (2) The cylindrifications and the diagonal constant have very clear *geometrical* meanings, they are easy to draw, cf. Figures 1,2. On the other hand, it is not so easy to draw the geometrical meaning of relation composition of binary relations. (3) Finally, the *logical* "content" of the Cs operations are very straightforward (cf. e.g. [81]), while it is not so transparent how relation composition codes up quantification.

We believe that algebra, geometry, and logic form a coherent unit which lies at the very heart of algebraic logic. These three subjects - algebra, geometry, logic - often show up together and give three complementing faces of the same thing. This "trinity" is very visible in cylindric algebra theory, but it is also very visible in e.g. space-time theory, too, where all these subjects play important roles, cf. e.g. [2], [53], [55].

Let $\mathsf{RCA_n}$ denote the class of all isomorphic copies of subdirect products of Cs's. By using the notation introduced in the proof-idea for Theorem 1, $\mathsf{RCA_n} = \mathbf{ISP}\mathsf{Cs_n}$, where $\mathsf{Cs_n}$ denotes the class of all cylindric algebras of n-ary relations. Algebras in $\mathsf{RCA_n}$ are isomorphic copies of natural algebras of relations with greatest element a disjoint union of Cartesian spaces. One can see this by using step (b) in the proof-idea for Theorem 1.

Theorem 3. (Tarski, Monk [67]) $\mathsf{RCA_n}$ *is a discriminator variety that can be axiomatized with a decidable set E of equations but if $n > 2$, then cannot be axiomatized with a finite set E of equations.*

Strengthened versions of Theorem 3, similar to those for Theorem 2, are known, cf. e.g. [4]. This algebra of n-ary relations can fruitfully be extended to infinite ordinals n, where we get a similar landscape (however, as expected, where new kinds of phenomena make the landscape more interesting). We do not follow up that direction here; we only note the following.

The class RCA_ω of infinite-dimensional representable cylindric algebras is the natural generalization of $\mathsf{RCA_n}$ where n goes to the least infinite ordinal ω. If we want to treat all finitary relations in one kind of algebras, then we need RCA_ω (comprising all the $\mathsf{RCA_n}$'s). It is important to emphasize that the logical counterpart of RCA_ω is the *schema* version of first-order logic, cf. [69]. More concretely, the formulas of this logic are the first-order formula schemas instead of being only the first-order formulas themselves. So this schema version of $L_{\omega\omega}$

can be seen as a generalization, or extension, of usual first-order logic $L_{\omega\omega}$. This formula-schema version of $L_{\omega\omega}$ was called rank-free version of first-order logic in [14, pp. 232-234]. For more on the schema-version of $L_{\omega\omega}$ we refer to [69], [76, §3.7, especially pp. 361-362], [63, p. 167] and to [82, the last 8 lines on p. 668].

3 Decidability, non-square approach

A class K of algebras is called *decidable* iff its equational theory $\mathsf{Eq}(\mathsf{K}) = \{\mathsf{e} : \mathsf{K} \models \mathsf{e},\ \mathsf{e}$ is an equation$\}$ is decidable. In algebraic logic, the question of decidability of equational theories is an important one. Besides asking whether, say, BRA (or $\mathsf{RCA_n}$, or whichever algebras of relations we are contemplating) is decidable, it is also important to ask which varieties contained in BRA or containing BRA are decidable. Recent works in this direction include [6], [41], [49], [65], [71], [86].

Most of the classes studied so far are undecidable, e.g. the variety V defined by (R1)-(R3) in section 1 is already undecidable, moreover, it remains such if we add equations valid in BRA. On the other hand, if we omit the axiom of "$\circ$" distributing over join, then the class becomes decidable (a result of V. Gyuris). The variety of BAO's (of any fixed similarity type) is also decidable and has many decidable subvarieties, cf. e.g. [41].

Some of the reducts of BRA are decidable, while others are not, e.g. omitting complementation from BRA makes it decidable, cf. [11]. Further, let $\mathsf{c}_0(x) = 1 \circ x$. Then e.g. the reduct containing the Booleans, $^{-1}$, Id, and c_0 only is decidable (a version of this is proved in [33], a simpler proof is available from the authors). Interestingly, most of what are called relativized versions of algebras of relations turn out to be decidable (even if we add new operators not available in BRA), cf. e.g. [61], [65], [71]. These "relativized versions" are based on the following idea.

When looking for a natural concept of a (subdirectly irreducible) algebra of relations, we need not necessarily require the greatest element of an algebra of relations to be a Cartesian square (i.e. of form $U \times U$, or of form ${}^nU$ for some n). Of course, this greatest element has to be a relation (since all elements are supposed to be relations), but perhaps not necessarily of the form $U \times U$. This approach is called the "*non–square approach*" by Yde Venema (cf. Venema [88]), which might be a more suggestive name than the traditional "relativized version" used e.g. in [34]. The non-square approach still yields algebras consisting of real relations, so it is a reasonable and healthy part of the quest for natural algebras of relations outlined in §2. A surprising result in Marx [61] says that all Boolean Algebras with Operators (BAO's) are isomorphic to such non-square algebras of relations (if there are only finitely many operations in our algebras).[6] This seems to point in the direction that the unifying power and circle of applicability of algebras of relations is considerably greater than we thought (which in turn was already much greater than what one would naturally expect).

So far, the greatest elements of our algebras were Cartesian spaces, i.e. of the form ${}^nU$ (both in the case of BRA's and Cs's). Removing this restriction motivates the definition of cylindric-relativized set algebras.

Definition 4. *Let α be an arbitrary (perhaps infinite) ordinal. Let $V \subseteq {}^\alpha U$ be an arbitrary α-ary relation. Then by an* algebra of subrelations of V *we understand a structure*

$$\langle \mathcal{A}; \cup, -, \mathsf{c}_i^V, \mathsf{Id}_{ij}^V : i, j < \alpha \rangle \quad \textit{where}$$

(i) *$\langle \mathcal{A}; \cup, - \rangle$ is a Boolean set algebra with greatest element V,*
(ii) *the operations c_i^V and Id_{ij}^V are defined by*
$$\mathsf{c}_i^V(R) = V \cap \mathsf{c}_i^U(R) \quad \textit{and} \quad \mathsf{Id}_{ij}^V = V \cap \mathsf{Id}_{ij}^U, \quad \textit{for all } i, j < \alpha \textit{ and } R \subseteq {}^nU,$$
(iii) *$\mathcal{A}$ is closed under the operations c_i^V and Id_{ij}^V, i.e.*
$$i, j < \alpha \textit{ and } R \in \mathcal{A} \quad \Rightarrow \quad \mathsf{Id}_{ij}^V, \mathsf{c}_i^V(R) \in \mathcal{A}.$$

A cylindric relativized algebra of α-ary relations*, a Crs_α in short, is an algebra of subrelations of an α-ary relation, and*

Crs_α *is the class of such algebras (up to isomorphism).* □

Clearly, Crs's are normal BAO's, but it may happen that c_i^V is not a closure operator.

Theorem 4. (Németi) *Let $\alpha \neq 1$. Then Crs_α is an arithmetical variety which is non-finitely axiomatizable when $\alpha > 2$. The equational theory of Crs_α is decidable.*

Though Crs_α can still not be defined with finitely many equations, in contrast with BRA and $\mathsf{RCA_n}$, it can be defined by a set E of equations which contains only three variables, a result of Resek [75]. Moreover, $\mathsf{CA_n} \cap \mathsf{Crs_n}$ is a nice finitely axiomatizable variety of n-ary relations. Actually, adding the two so-called Merry-Go-Round equations to the equations defining $\mathsf{CA_n}$ we get an axiomatization for $\mathsf{CA_n} \cap \mathsf{Crs_n}$, for a proof see Andréka-Thompson [15]. In some sense, one could say

[6] This representation becomes especially natural if any two extra-Boolean operators are so called conjugates of each other (this does not restrict the ranks of these operators).

that $\mathsf{CA_n} \cap \mathsf{Crs_n}$ is the real algebra of n-ary relations, and it was only an accident that Tarski omitted the so-called Merry-Go-Round equations from the definition of a cylindric algebra. (The Merry-Go-Round equations are due to Henkin, we do not recall these simple and elegant equations here.) A sample literature on cylindric relativized set algebras is [35], [57], [68], [71], [75], [83].

The theory of Crs's and its methods proved to be very fruitful in applications. The Amsterdam school used Crs's very fruitfully in connection with arrow logic, dynamic logic, and well behaved fragments of first-order logic. The definition of the bounded and guarded fragments of first-order logic, which proved to be so successful, grew out from Crs-theory, see Andréka-Benthem-Németi [5].

$\mathsf{RCA_n}$ is the algebraic counterpart of the n-variable fragment of first-order logic, while $\mathsf{CA_n} \cap \mathsf{Crs_n}$ is the algebraic counterpart of the bounded and guarded fragments of first-order logic.

The guarded fragment is basically the ideal logic of bounded quantifiers. The basic idea is that instead of unrestricted quantifiers like $\forall x \varphi(x)$ one allows only bounded quantifiers like $(\forall x \in y)\varphi(x, y)$. Intuitively, in the bounded or guarded fragments one allows only statements like "all dogs are black" instead of uncontrollably general statements like "all things are black". In our opinion, the so restricted logic is quite natural. Actually, there was a period in the history of logic where the only quantifiers allowed were guarded ones.

$\mathsf{CA_n}$ and $\mathsf{Crs_n}$ turned out to be very useful in giving a new, wider semantics for first-order logic, and this new semantics made possible a fine-structure analysis of first-order logic, see [71], [72].

More detailed and more concrete connections between algebraic logic and logic are in [14], [33], [70] except for new kinds of recent applications of $\mathsf{Crs_n}$-theory to the finite-variable fragments, finite model theory, the bounded fragments, and the guarded fragment. On the latter see [28], [38], [62].

4 New trends, applications

The above seems to illustrate that history does not repeat itself. Namely, the algebraization of propositional logic turned out to be an immensely powerful engine for driving propositional logic in its original form in a very straightforward way. Were we naive, we would ask the question: how does Tarskian algebraic logic (CA's, RA's, Crs's) simplify the theory of $L_{\omega\omega}$, i.e. first-order predicate logic, in its classical form. This turns out to be the wrong question. The right question is: what new do the results and tools of Tarskian algebraic logic tell us about the subject matter of logic and its application areas. Indeed, in the category theoretic version of algebraic logic this has always been the way of formulating the question, e.g. in the FOLDS approach of Michael Makkai [59], the definition of first-order logic is modified according to the patterns emerging naturally from the ways topos theory behaves. Analogously, an application of the negative behavior of RCA's contrasted with the positive behavior of Crs's is the introduction of the guarded fragment of first-order logic in [5] yielding many positive results about the guarded fragment of logic, e.g. [30], [31].

When obtaining the strong negative results on non-finitizability of $\mathsf{RCA_n}$ by Monk and then by Jónsson, Maddux, Andréka, Venema, Hodkinson-Venema, a pessimist could have said that Tarskian algebraic logic failed, and this is the end of the story. This would be analogous to Morley's attitude to the outcome of the Michelson-Morley experiment saying that it represents a failure. The right interpretation of the same experiment e.g. by Einstein was that it is an immense success, and that it provides us with much more profound insights into the nature of the world than was ever hoped for by Michelson and Morley, the designers of the experiment. Analogously, when looking at the strong negative results concerning finitizability of the square version of Tarskian algebraic logic, we should ask ourselves what do these results teach us about the subject matter of logic and its applications, related areas. Then we would immediately conclude several useful insights, e.g. quantifier logic is much more complex than propositional logic, the formula schema version of quantifier logic is drastically different from the formula version, the schema version being inherently non-finitizable, and there seems to be no simple way of extending Gödel's completeness theorem from the formula version to the schema version of this logic. As a contrast, in propositional logic the schema version and the formula version practically coincide. Further, the finite variable fragments of $L_{\omega\omega}$ are inherently non-finitely axiomatizable. However, this can be helped by switching over to studying the guarded version of first-order logic. Namely, the finite variable fragments of the guarded version behave well.

The celebrated Resek-Thompson theorem about finitizability of $\mathsf{CA}_\alpha \cap \mathsf{Crs}_\alpha$ shows that if we are willing to loosen up the squareness of the standard model theory of $L_{\omega\omega}$, then we can obtain nice streamlined finite axiomatizations even for the formula schema version of this logic. Further, all these investigations lead us to defining very nice and illuminating new logics which are useful for understanding various kinds of phenomena studied in connection with applying logic. Examples are e.g. in the semantics of natural language discourse an example of which is the so called dynamic semantics paradigm for linguistics or in other words, dynamic logic of natural language semantics, cf. Benthem [16]. More general examples for such positive applications are summarized in the book on logical dynamics by van Benthem [17]. Further rich sources of examples of positive applications of Tarskian algebraic logic are Gabbay-Kurucz-Wolter-Zakharyaschev [28] and Rybakov [76]. Rybakov completely independently of algebraic logic, investigated the schema properties of first-order logic motivated by essential considerations arising from the very nature of logic as such. He found a very rich collection of algebraic logic results immediately applicable to the questions he investigated.

These kinds of insights have led to a renaissance of Tarskian algebraic logic centered about e.g. Amsterdam, Budapest, London, Stanford (in alphabetical order), e.g. [1], [5], [9], [13], [17], [19], [28], [36] - [40], [49], [52], [61] - [65], [72], [79] - [86]. The new philosophy is summarized as follows. Instead of trying to use Tarskian algebraic logic as a device designed for studying the rigidly fixed formalism known as $L_{\omega\omega}$ and then applying $L_{\omega\omega}$ to the "world of logic", we sidestep $L_{\omega\omega}$ and ask ourselves what Tarskian algebraic logic and its variations can tell us about the world of logic in general (and its applications). There are natural ideas about the world of logic which do not show up at all in the standard framework of $L_{\omega\omega}$, and which come up naturally in Tarskian algebraic logic. Such an idea is illustrated by the category theory motivated approach to logic called Institutions theory putting an emphasis on the category of theory morphisms and theories. This structure does show up quite naturally in Tarskian algebraic logic but not in $L_{\omega\omega}$. This is one of the reasons why the search for an algebraic logic version of the Lindström theorems (about $L_{\omega\omega}$) remains an open problem though it was published by Craig in 1988. (We note that $L_{\omega\omega}$ and its treatment in Tarskian algebraic logic are discussed in detail in e.g. [33, Part II], and in [14, section 7].)

A further rich application area of Tarskian algebraic logic is the direction known as substructural logics and Lambek calculus, cf. [9], [13], [16], [62].

Actually, the presently suggested strategy for algebraic logic (to sidestep the standardized forms of $L_{\omega\omega}$) has been fruitfully followed by the category theoretic version of algebraic logic, e.g. by topos theory. This attitude of category theoretic algebraic logic made it very attractive for many application areas beginning with theoretical computer science and ending with the efforts for generalizing general relativity to the direction of quantum gravity, cf. e.g. works of Jeremy Butterfield and Lee Smolin (cosmo-logic). An example for the relativistic applications of topos theory is to obtain via quantum logic a quantum-first-order logic and then a quantum set theory. But if this can be done by toposes, then it also can be done by the orthomodular-lattice valued cylindric algebras. Namely, probabilistic cylindric algebras have been studied by Ferenczi and others, cf. [25]-[27], [73], [74]. Following the same recipe one can replace Boolean valued cylindric algebras by orthomodular lattice valued cylindric algebras. Then this can be used as indicated above.

Almost all application areas of logic can benefit from Tarskian algebraic logic because the latter provides us with a more refined, more sensitive picture of the landscape of the subject matter of logic. A relatively new application area of logic is the logical analysis of relativity theory, cf. [7], [8], [55]. The structuralist spirit of algebraic logic outlined at the beginning of the present paper has already been applied in this area in the duality between the streamlined relativistic geometries and the observational models of relativity theory, cf. [53, §4.].

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