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Some Complete Bipartite Graph — Tree Ramsey Numbers

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Dedicated to the memory of G. A. Dirac

We investigate $r(K_{a,a}, T)$ for $a = 2$ and $a = 3$, where T is an arbitrary tree of order n. For $a = 2$, this Ramsey number is completely determined by $r(K_{2,2}, K_{1,m})$ where $m = \Delta(T)$. For $a = 3$, we do not find such an "exact" result, but we do show that $r(K_{3,3}, T) \leq \max\{n + \lceil cn^{1/3} \rceil, r(K_{3,3}, K_{1,m})\}$. Except for the choice of c this result is best possible.

1 Introduction

Let T denote a tree of order n and maximum degree $\Delta(T) = m$. The *Ramsey number* $r(K_{a,a}, T)$ is the smallest integer p such that in every two-coloring of the edges of K_p there is either a monochromatic $K_{a,a}$ or

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else a monochromatic copy of T. By tradition, we shall let the colors be R (red) and B (blue) with the resulting edge-induced subgraphs denoted $\langle R\rangle$ and $\langle B\rangle$, respectively. It is well-known that the computation of $r(K_{a,a},T)$ is quite easy in some cases (e.g., for $T = P_n$, a path of order n) but difficult in general. In particular, the "star" case $T = K_{1,n-1}$ is known to be complicated. In this paper, we show that for $a = 2$ the problem of computing $r(K_{2,2},T)$, i.e. $r(C_4,T)$, reduces to that of computing $r(C_4,K_{1,m})$.

The following terminology will be used. An *end-vertex* is a vertex of degree one. An *end-edge* is an edge which is incident with an end-vertex. A *suspended path* in a graph G is a path $(x_0,x_1,\ldots,x_k)$ in which $x_1,\ldots,x_{k-1}$ have degree two in G.

2 C_4-tree Ramsey numbers

Let T be a tree of order n and maximum degree $\Delta(T) = m$. In this section, we shall prove that

$$r(C_4,T) = \max\{4, n+1, r(C_4,K_{1,m})\}.$$

Thus, $r(C_r,T)$ is easily determined if $f(m) = r(C_4,K_{1,m})$ is known. In [3], Parsons proved that if q is prime power, then $f(q^2) = q^2+q+1$ and $f(q^2+1) = q^2+q+2$. A table of $f(m)$ for small values of m is shown below.

$m =$	1	2	3	4	5	6	7	8	9	10
$f(m) =$	4	4	6	7	8	9	11	12	13	14

We start with the following bit of graph-theoretic folklore, which however trivial, is the starting point for many results in extremal graph theory and Ramsey theory involving trees.

Basic Lemma. *If $\delta(G) \geq n-1$ then G contains every tree of order n.*

Lemma 1.1. *If F is any forest with q edges, then $r(C_4,F) \leq 2(q+1)$.*

Proof. For the case in which F is a tree, this follows easily by a double application of the Basic Lemma. For the general case, it follows by induction on the number of components. □

Except for some very special cases, the bound of Lemma 1.1 can be improved.

Lemma 1.2. *If F is a forest with q edges and two or more components, then $r(C_4, F) \leq 2q - 1$ unless $F \cong qK_2$ or $F \cong (q-2)K_2 \cup K_{1,2}$.*

The next two results are well known. Lemma 1.3 is proved by Parsons in [3].

Lemma 1.3. *For all $m \geq 2$, $r(C_4, K_{1,m}) \leq m + \lceil\sqrt{m}\rceil + 1$.*

Lemma 1.4. *For all $n \geq 3$, $r(C_4, P_n) = n + 1$.*

To prove Lemma 1.4, just consider a maximal length path in $\langle B \rangle$. Its two end-vertices are joined to all of the remaining vertices in $\langle R \rangle$. Now we prove the main result of this section.

Theorem 1. *If T is a tree of order n and maximum degree $\Delta(T) = m$, then $r(C_4, T) = \max\{4, n+1, r(C_4, K_{1,m})\}$.*

Proof. It is plain that $p = \max\{4, n+1, r(C_4K_{1,m})\}$ is a lower bound for $r(C_4, T)$. Let (R, B) be any two-coloring of the edges of K_p in which $\langle R \rangle \not\supseteq C_4$. We claim that there is then an embedding of T into $\langle B \rangle$. For $n \leq 6$, the verification is straightforward. There are 13 trees to consider. These trees and the corresponding values of $r(C_4, T)$ are shown in Figure 1 overleaf. Our proof that the result holds for all $n \geq 7$ will be by induction. Since the case in which T is a path has been settled, we may assume that $m \geq 3$. To exhibit an embedding of T into $\langle B \rangle$, there are two basic strategies.

(a) *Embed a maximum degree vertex first.*

Let x be a maximum degree vertex of T and choose $\sigma(x)$ to be a maximum degree vertex in $\langle B \rangle$. Thus, $\sigma(x)$ has degree $k \geq m$ in $\langle B \rangle$. Let N_R and N_B denote the neighborhoods of $\sigma(x)$ in $\langle R \rangle$ and $\langle B \rangle$, respectively. We wish to extend σ to an embedding of T into $\langle B \rangle$. There are several cases in which such a strategy will be successful.

(i) $k \leq 2(m-1)$.

Rewriting the inequality, we find $k \geq 2(k - m + 1)$. From Lemma 1.1, we see that within the subgraph of $\langle B \rangle$ spanned by N_B there is any desired forest with $k - m$ edges. Let T' be a tree obtained from T by deleting (one at a time) end-vertices other than those which are adjacent to x until there are precisely $k + 1$ vertices left. Then $F' = T' - x$ has k vertices and $k - m$ edges. Thus, we are able to extend σ so that it provides an embedding of T' into $\langle B \rangle$, and σ maps $V(F')$ onto N_B. We now seek to

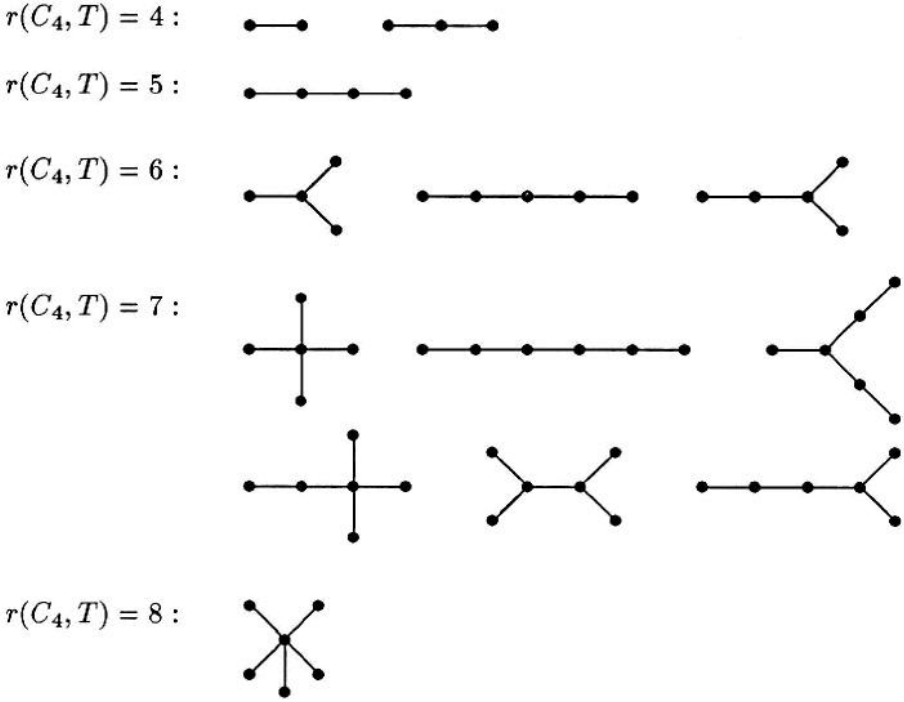

Figure 1. $r(C_4, T)$ for trees of order at most six.

extend σ to an embedding of T into $\langle B\rangle$. Suppose that such an extension fails to exist. Then since $p \geq n+1$, there must exist a vertex $y \in V(T)$, $y \neq x$, such that $\sigma(y)$ is adjacent in $\langle R\rangle$ to two vertices of N_R. In other words, $\langle R\rangle$ contains a C_4, a contradiction.

Henceforth we assume that $k \geq 2m-1$.

(ii) *T has a vertex of degree $m \geq 3$ which is an "end-star."*

Suppose that x is adjacent to $m-1$ vertices of degree 1. Then $T-x$ consists of a tree T' and $m-1$ isolated vertices. Choose $\sigma(x)$ to be a vertex of degree $k \geq 2m-1$ in $\langle B\rangle$ and attempt to extend the embedding as in case (i). In particular, let T'' be a tree obtained from T' by successively deleting end-vertices (but keeping all vertices which are adjacent to x) until there are precisely $k-m+1$ vertices left. Now we want to extend σ so that it provides an embedding of T'' into the subgraph of $\langle B\rangle$ spanned by N_B. If this is possible, then σ can be extended to an embedding of T into $\langle B\rangle$. For, otherwise, there is a vertex other than $\sigma(x)$ which is adjacent in $\langle R\rangle$ to two vertices of N_R, so $\langle R\rangle$ contains a C_4. Note that since T'' is a subgraph of T, we have $\Delta(T'') \leq m$. Now for $m = 3$, the desired embedding of T'' follows immediately from the induction hypothesis if $k \geq r(C_4, K_{1,3}) = 6$. If $k = 5$, then T'' is the tree of order three and $r(C_4, T'') = 4 < k$. For $m \geq 4$, use of the induction hypothesis shows that it suffices to verify that $r(C_4, K_{1,m}) \leq 2m - 1$. This is easily done; $r(C_4, K_{1,4}) = 7$, $r(C_4, K_{1,5}) = 8$ and $m + \lceil\sqrt{m}\rceil + 1 \leq 2m - 1$ for $m \geq 6$ by an elementary calculation.

Henceforth we assume that $F = T - x$ has two or more nontrivial components.

(iii) $F \cong mK_2$ *or* $F \cong (m-2)K_2 \cup K_{1,2}$.

Consider first the case where $F \cong mK_2$. Then $r(C_4, F) = 2m + 1$ for $m \geq 2$ and we wish to prove that $r(C_4, T) = 2m + 2$ for $m \geq 3$. If $k = 2m - 1$, then the subgraph of $\langle B \rangle$ spanned by N_B contains $(m-1)K_2$. The remaining vertex of N_B is adjacent to one of the two vertices in N_R and this gives the desired graph. If $k = 2m$, then the subgraph of $\langle B \rangle$ spanned by $N_R \cup N_B$ contains mK_2 and again we find the desired graph. The case where $F \cong (m-2)K_2 \cup K_{1,2}$ is similar. In view of Lemma 1.2, we may now assume that $r(C_4, F) \leq 2q - 1$, when F is of size q.

(iv) $2m \geq n - 1$.

Since F has $n - m - 1$ edges, then $r(C_r, F) \leq 2(n - m - 1) - 1 \leq 2m - 1 \leq k$. In this case, the subgraph of $\langle B \rangle$ spanned by N_B contains F so we have an embedding of T into $\langle B \rangle$.

Now suppose that strategy (a) fails. Then $F = T - x$ has at least two non-trivial components and $2\Delta(T) < n - 1$. The first condition implies that T has (at least) three independent end-edges. This observation gives rise to the second strategy.

(b) *Complete the embedding of* T *using a matching.*

Let T' denote the tree obtained from T by deleting the three independent end-edges. Then T' is a tree of order $n - 3$ and maximum degree $\Delta(T') \leq m < \lfloor \frac{n}{2} \rfloor$.

We now claim that $r(C_4, T') = n - 2$. This follows from the induction hypothesis if $r(C_4, K_{1,m}) \leq n - 2$ when $m < \lfloor \frac{n}{2} \rfloor$. This is surely the case for $n = 7$, 8, and 9, and it follows for $n \geq 10$ by an elementary calculation using the result of Lemma 1.3, $r(C_4, K_{1,m}) \leq m + \lceil\sqrt{m}\rceil + 1$.

Observe that the Basic Lemma yields the simple fact that $r(K_{1,2}, T) = n+1$. Thus, we may assume the existence of a $K_{1,2}$ in $\langle R \rangle$. Since $r(C_4, T') = n - 2$, we may assume a T' in $\langle B \rangle$ which is vertex disjoint from the $K_{1,2}$ in $\langle R \rangle$. Let $Y = \{y_1, y_2, y_3, y_4\}$ denote the set of four vertices which are disjoint from the T' which is embedded in $\langle B \rangle$. In view of the way T' was constructed, there is a set of three vertices $X = \{x_1, x_2, x_3\}$ for which a matching of X into Y in $\langle B\rangle$ would yield an embedding of T. Since the subgraph of $\langle R\rangle$ spanned by Y contains a $K_{1,2}$, every vertex of X is adjacent in $\langle B\rangle$ to at least one vertex of Y. Moreover, Hall's condition cannot fail for any other subset of X since this would also imply a C_4 in $\langle R\rangle$. Hence, there is a matching of X into Y and so an embedding of T into $\langle B\rangle$.

Thus, in all cases either strategy (a) or strategy (b) is successful and the proof is complete. □

The preceding theorem reduces the problem of computing $r(C_4, T)$ to one of computing $r(C_4, K_{1,m})$. Outside of special values of m (e.g., $m = q^2$ or $m = q^2 + 1$), little is known concerning exact values of $f(m)$ for large m. The following result gives a modest improvement in our knowledge of the asymptotic behavior of this sequence of Ramsey numbers.

Theorem 2. *Let p_k denote the kth prime. If*

$$p_{k+1} - p_k < p_k^{\alpha} \qquad (*)$$

for all sufficiently large k, then

$$r(C_4, K_{1,n}) > n + \lfloor n^{\frac{1}{2}} - 6n^{\frac{\alpha}{2}} \rfloor$$

for all sufficiently large n.

Remark. At present, the best known value of α for which $(*)$ holds is less than $11/20$, but improvements in this value are being obtained rapidly. In any case, by Lemma 1.3 and Theorem 2, $r(C_4, K_{1,n})$ is determined to within $6n^{\frac{11}{40}}$.

Proof of Theorem 2. Let p be the smallest prime which exceeds $n^{\frac{1}{2}}$. In [1] and [2], it is shown that there exists a graph G_0 of order $N = p^2 + p + 1$ which contains no C_4 and in which the degree of each vertex is p or $p + 1$. Set $m = \lfloor n^{\frac{1}{2}} - 6n^{\frac{\alpha}{2}} \rfloor$. We wish to establish the existence of a graph of order $n + m$ which contains no C_4 and in which the degree of each vertex is at least m. To this end, we randomly delete $d = N - (n + m)$ vertices from G_0 to obtain a random graph G. A given vertex x which is of degree p in G_0 will be described as *bad* if it is not deleted and has degree $< m$ in G. Let B_x denote the event: "x is bad." Thus, the random graph G will belong to B_x if for some $k > p - m$, k vertices are deleted from the p vertices in the neighborhood of x and the remaining $d - k$ deleted vertices do not include x. It follows that

$$P(B_x) = \sum \binom{p}{k} \binom{N-p-1}{d-k} \Big/ \binom{N}{d},$$

where the sum extends over all $k > p - m$. By elementary analysis,

$$P(B_x) < \sum \{epd/k(N-d)\}^k,$$

where $e = 2.71828\ldots$. By our choice of p and m, all terms in the sum have $k > p - m > 6n^{\frac{\alpha}{2}}$. Also, $p < n^{\frac{1}{2}} + n^{\frac{\alpha}{2}}$ and $N - d = n + m > n$. In the worst case, p (and therefore d) are as large as possible while k and $N - d$ are as small as possible. Even in this case, $epd/k(N-d) < \frac{2e}{6} + 0(1)$. It follows that $nP(B_x) \to 0$ as $n \to \infty$. Clearly, the same analysis holds if x has degree $p + 1$. Consequently, there is a positive probability that all vertices are good when n is sufficiently large. Thus our random graph construction is successful. □

3 $K_{3,3}$-tree Ramsey numbers

The following result shows that the Ramsey number $r(K_{3,3}, T)$ is determined by $m = \Delta(T)$ if m is quite close to n. Otherwise, our result yields only an upper bound.

Theorem 3. *There exists a constant c such that for every tree of order n and maximum degree $\Delta(T) = m$,*

$$r(K_{3,3}, T) \leq \max\{n + \lceil cn^{\frac{1}{3}} \rceil, r(K_{3,3}, K_{1,m})\}.$$

To set the stage for the proof of this result, we first recall the simple argument by which one may obtain an upper bound for $r(K_{a,b}, K_{1,n})$. Namely, by the pigeonhole principle, if

$$p\binom{p-n}{a} > (b-1)\binom{p}{a},$$

then $p > r(K_{a,b}, K_{1,n})$. The argument of Theorem 1 can be extended to show that there is a c such that $r(K_{2,b}, T) \leq \max\{n + c, r(K_{2,b}, K_{1,m})\}$ for all trees of order n and maximum degree m. Assembling these facts, we have the following result.

Lemma 3.1. *Let T be an arbitrary tree of order n. Then, for all sufficiently large n,*

(i) $r(K_{2,3}, T) < n + 2n^{\frac{1}{2}}$,

(ii) $r(K_{2,4}, T) < n + 3n^{\frac{1}{2}}$,

(iii) $r(K_{3,3}, K_{1,n}) < n + 3n^{\frac{2}{3}}$.

Remark. The constants chosen here are rather generous, so the given inequalities hold for even quite small values of n.

Proof of Theorem 3. The proof will be by induction. We shall argue that with $c = 20$ there is an integer n_0 so that for all $n > n_0$, the truth of the theorem for all $k < n$ implies its truth for n. Having done so, we need only adjust (if necessary) the value of c so the result holds for all n. In what follows there is always the tacit assumption that n is sufficiently large. Set $p = \max\{n + \lceil 20n^{\frac{1}{3}} \rceil, r(K_{3,3}, K_{1,m})\}$ and assume that (R, B) is a two-coloring of the edges of K_p in which $\langle R \rangle \not\supseteq K_{3,3}$. There are four cases to consider.

(i) $m > \frac{3n}{4}$.

Use the first strategy of Theorem 1; if x is a maximum degree vertex in T, then make $\sigma(x)$ a maximum degree vertex of $\langle B \rangle$. Thus $\sigma(x)$ has degree $k \geq m$ in $\langle B \rangle$ and, by Lemma 3.1, we may also assume that $k > n - 3n^{\frac{2}{3}}$. Certainly, the number of vertices to which $\sigma(x)$ is adjacent in $\langle R \rangle$ does not exceed $3n^{\frac{2}{3}} + 20n^{\frac{1}{3}} < 4n^{\frac{2}{3}}$. As in the proof of Theorem 1, let N_R and N_B denote the neighborhoods of $\sigma(x)$ in $\langle R \rangle$ and $\langle B \rangle$, respectively. Describe a vertex in N_B as *powerless* if it is adjacent to at least $d = \lceil 10n^{\frac{1}{3}} \rceil$ vertices of N_R in $\langle R \rangle$. If $|N_R| = s$ and there are t powerless vertices, then

$$t\binom{d}{3} \leq \binom{s}{3},$$

for, otherwise, there is a $K_{3,3}$ in $\langle R \rangle$ with two powerless vertices and $\sigma(x)$ on one side and three vertices from N_R on the other. It follows that $t < \frac{n}{10}$. This number is much less than then number of isolated vertices in $F = T - x$. The remaining vertices in N_B will be called *powerful.* Note that there are at least $\frac{4n}{5}$ powerful vertices. This is more than enough to ensure that there is an embedding of the non-trivial components of F into the subgraph of $\langle B \rangle$ spanned by the powerful vertices. (Here we use the crude bound $r(K_{3,3}, F) \leq 3(q + 1)$ for a forest with q edges and note that our F has at most $\frac{n}{4}$ edges.) We now face the problem that there may not be enough vertices left over in N_B to account for all the isolated vertices of F. To find an embedding of T into $\langle B \rangle$ we need to take appropriate "small" trees (altogether involving at most $3n^{\frac{2}{3}}$ vertices) rooted at powerful vertices and find an embedding of an isomorphic tree where all vertices except the root reside in N_R. Let $\sigma(y)$ denote a powerful vertex which is the root of such a tree. The existence of such an embedding is ensured by the fact that the vertex in question is powerful. As such, it is adjacent in $\langle B \rangle$ to at least

$10n^{\frac{1}{3}}$ vertices in excess of what may be needed to complete its tree. Let Y denote the set of vertices in N_R to which $\sigma(y)$ is adjacent in $\langle B\rangle$. The subgraph of $\langle R\rangle$ spanned by Y cannot contain a $K_{2,3}$, for then $\langle R\rangle$ would contain a $K_{3,3}$. Suppose that within the subgraph of $\langle B\rangle$ spanned by Y we need to find a forest F' with r vertices. Success is ensured since $r < 3n^{\frac{2}{3}}$ and

$$|Y| > r + 10n^{\frac{1}{3}} > r + 2r^{\frac{1}{2}} > r(K_{2,3}, F').$$

We now assume that $\Delta(T) < \frac{3n}{4}$.

(ii) *T has six independent end-edges.*

Since $m = \Delta(T) < \frac{3n}{4}$, we have

$$p = n + \lceil 20n^{\frac{1}{3}} \rceil > r(K_{2,4}, T).$$

It follows that $\langle R\rangle$ contains a $K_{2,4}$. Let T' denote the tree obtained from T by deleting six independent end-edges. Then $r(K_{3,3}, T') \leq p - 6$, so we find a $K_{2,4}$ in $\langle R\rangle$ and a T' in $\langle B\rangle$ which are vertex disjoint. Let Y denote the set of all vertices disjoint from the embedded T'. Thus $|Y| = p - n + 6 \geq 14$. In view of the construction of T', there is a set $X = \{x_1, x_2, \ldots, x_6\}$ for which a matching of X into Y in $\langle B\rangle$ would yield an embedding of T. Since the subgraph of $\langle R\rangle$ spanned by Y contains $K_{2,4}$, every vertex of X is adjacent in $\langle B\rangle$ to at least two vertices of Y. Hence, failure of Hall's condition would yield a $K_{r,s}$ in $\langle R\rangle$ with $r \geq 3$ and $s \geq |Y| - (r-1) > 3$. Consequently, Hall's condition must be satisfied and there is an embedding of T into $\langle B\rangle$.

(iii) *T has a suspended path of length at least six.*

Let T' be a tree obtained from T by reducing the length of an appropriate suspended path by one. By induction, we find an embedding of T' into $\langle B\rangle$. It is not difficult to show that if there are as many as eleven vertices external to the embedded T', then the absence of a $K_{3,3}$ in $\langle R\rangle$ implies that there is an embedding of T into $\langle B\rangle$. In fact, there are more than $\lceil 20n^{\frac{1}{3}} \rceil$ additional vertices, so this case causes no problem.

(iv) *The residue.*

By now, our tree is quite special. It has at most five vertices of degree three or more and it has no suspended path of length six. As usual, let x denote a maximum degree vertex of T. In view of the observations just made, we can be sure that x is adjacent to about $\frac{n}{5}$ or more vertices of degree one. Furthermore, the forest $F = T - x$ has at most four nontrivial components. Now we return to the first strategy and let $\sigma(x)$ by a maximum degree vertex in $\langle B\rangle$.

Note that almost all vertices of F are either isolated or have degree one. (The number of exceptions is limited by the fact that at most five vertices have degree three or more and the fact that the suspended paths which join these vertices are of length at most five.) Suppose that $\sigma(x)$ has degree k in $\langle B\rangle$. Delete end-vertices from F to obtain a forest F' of order k. Now delete all isolated vertices from F' to obtain F''. We claim that there is an embedding of F'' into the subgraph of $\langle B\rangle$ induced by the set of powerful vertices in N_B. This follows immediately from the induction hypothesis, since F'' is of order at most $k - \frac{n}{5}$ and has no vertex of degree $> \frac{n}{2}$, whereas there are at least $k - \frac{n}{10}$ powerful vertices. Now we just need to extend the embedding into N_R to obtain an embedding of T into $\langle B\rangle$. This only involves choosing the appropriate vertices within N_R to play the role of end-vertices in T and (at the same time) possibly releasing some vertices within N_B so they can play the role of end-vertices which are adjacent to x. But there is no problem, because each powerful vertex is adjacent in $\langle R\rangle$ to at most $10n^{\frac{1}{3}}$ vertices of N_R and the whole graph contains $20n^{\frac{1}{3}}$ vertices in addition to the number needed for T. This completes the proof. □

Except for the value of the constant c, this result is best possible. Let $p \equiv 3 \pmod 4$ be prime and consider a tree T of order $n = p^3 - p + 2$ which has exactly two vertices of degree greater than one. Of necessity, these two vertices are adjacent. We claim that $r(K_{3,3}, T) \geq p^3 + 1$. To see this, we use the example of W. G. Brown [1]. This is a graph of order p^3 which is regular of degree $p^2 - p$ and contains no $K_{3,3}$. This graph has the property that for any two non-adjacent vertices, there are exactly $p - 1$ vertices to which the two are commonly adjacent. It follows that the complement of this graph does not contain T since there are always $p - 1$ vertices which cannot be used in any attempted embedding. Now, if the degrees of the two "high degree" vertices of T are as balanced as possible, we ensure that $r(K_{3,3}, K_{1,m}) < n + \lceil cn^{\frac{1}{3}}\rceil$, at least when n is large. Thus, in this example the result $r(K_{3,3}, T) \leq n + \lceil cn^{\frac{1}{3}}\rceil$ cannot be improved except for the choice of c.

4 Open questions

There is still much more which could be known concerning the asymptotic behavior of $f(n) = r(C_4, K_{1,n})$. For example, it is conjectured that for every constant c there are infinitely many n for which every graph of order n and minimum degree $\geq \sqrt{n} - c$ contains a C_4. That is, infinitely often $f(m) < m + \sqrt{m} - c'$. One of the authors (E.P.) offers \$100 for a proof or disproof. Clearly, infinitely often $f(n+1) = f(n) + 1$ and infinitely often $f(n+1) > f(n) + 1$. Is it true that $f(n+1) = f(n)$ holds i.o. but that the density of these n is 0 and is it true that $f(n+1) \leq f(n) + 2$ for all n?

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