Ramanujan and I

by

Paul Erdös

Perhaps the title "Ramanujan and the birth of Probabilistic Number Theory" would have been more appropriate and personal, but since Ramanujan's work influenced me greatly in other subjects too, I decided on this somewhat immodest title.

Perhaps I should start at the beginning and relate how I first found out about Ramanujan's existence. In March 1931 I found a simple proof of the following old and well-known theorem of Tchebychev: "Given any integer n, there is always a prime p such that $n < p < 2n$." My paper was not very well written. Kalmar rewrote my paper and said in the introduction that Ramanujan found a somewhat similar proof. In fact the two proofs were very similar; my proof had perhaps the advantage of being more arithmetical. He asked me to look it up in the Collected Works of Ramanujan which I immediately read with great interest. I very much enjoyed the beautiful obituary of Hardy in this volume [23]. I am not competent to write about much of Ramanujan's work on identities and on the τ-function since I never was good at finding identities. So I will ignore this aspect of Ramanujan's work here and many of my colleagues who are much more competent to write about it than I will do so. I will therefore write about Ramanujan's work on partitions and on prime numbers and here too I will restrict myself to the asymptotic theory.

My paper [7] on Tchebychev's theorem, which was actually my very first, appeared in 1932. One of the key lemmas was the proof that

$$\prod_{p<n} p < 4^n. \tag{1}$$

In 1939, Kalmar and I independently and almost simultaneously found a new and simple proof of (1) which comes straight from The Book! We use induction. Clearly (1) holds for $n = 2$ and 3 and we will prove that it holds for $n + 1$ by assuming that it holds for all integers $\leq n$. If $n + 1$ is even, there is nothing to prove. Thus assume $n + 1 = 2m + 1$. Observe that $\binom{2m+1}{m} < 4^m$ and that $\binom{2m+1}{m}$ is a multiple of all primes p satisfying $m + 2 \leq p \leq 2m + 1$. Now we evidently have

$$\prod_{p\leq 2m+1} p \leq \binom{2m+1}{m} \prod_{p\leq m+1} p < 4^m \prod_{p\leq m+1} p < 4^{2m+1}$$

by the induction assumption.

By more complicated arguments it can be shown that $\prod_{p<n} p < 3^n$. As is well-known, the Prime Number Theorem is equivalent to

$$\{\prod_{p\leq n} p\}^{1/n} \to e \text{ as } n \to \infty, \qquad (2)$$

but it is very doubtful if (2) can be proved by such methods.

I hope the reader will forgive me (a very old man!) for some personal reminiscenses. Denote by $\pi(n)$ the number primes not exceeding n. The Prime Number Theorem states that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$

$$(1-\varepsilon)\frac{n}{\log n} < \pi(n) < (1+\varepsilon)\frac{n}{\log n}. \qquad (3)$$

It was generally believed that for every fixed $\varepsilon > 0$, (3) can be proved by using the elementary methods of Tchebychev but that an elementary proof is not possible for every ε. In 1937 Kalmar and I found such an elementary proof. Roughly speaking our proof was based on the following Lemma: "For every $\varepsilon > 0$ there is an integer m such that for every t satisfying $m < t < m^2$ we have

$$\Big| \sum_{n \le t} \mu(n) \Big| < \varepsilon . t, \tag{4}$$

where μ is the Mobius function." It is well-known that the Prime Number Theorem is equivalent to

$$\sum_{n \le x} \mu(n) = o(x). \tag{5}$$

Thus if we know the Prime Number Theorem, then a value satisfying (4) can be found by a finite computation. But without assuming the Prime Number Theorem, we certainly cannot be sure that such an m can be found. It is perhaps an interesting fact that such a curious situation can be found in "normal" mathematics, which has nothing to do with mathematical logic!

Perhaps an explanation is needed why our paper was never published. We found our theorem in 1937, and we had a complete manuscript ready in 1938, when I arrived in the United States. At the meeting of the American Mathematical Society at Duke University I met Barkeley Rosser and I learned from him that he independently and almost simultaneously found our result and in fact he proved it also for all arithmetic progressions. Thus Kalmar and I decided not to publish our result and Rosser stated in his paper that we obtained a special case of his result by the same method. Now it so happened that Rosser's paper also was never published. This is what happened to Rosser's paper. At that time he worked almost entirely in Logic and therefore the paper was probably sent to a logician who had serious objections to some of the arguments which he perhaps did not understand completely. Thus Rosser lost interest and never published the paper. A few years ago when I told Harold Diamond of our work he thought that the result was of sufficient interest to deserve publication even now after Selberg and I had found a genuinely elementary proof of the Prime Number Theorem (using the so called fundamental inequality of Selberg.) The manuscripts of Rosser, Kalmar and myself no longer existed, but Diamond and I were able to reconstruct the proof which appeared in L'Enseignement Mathématique a few years ago [5].

I was immediately impressed when I first saw in 1932 the theorem of Hardy and Ramanujan [18] which loosely speaking states that almost all integers have about $\log\log n$ prime factors. More precisely, if $g(n)$ tends to infinity as slowly as we please then the density of integers n for which

$$|\nu(n) - \log\log n| > g(n)\sqrt{\log\log n} \tag{6}$$

is 1, where $\nu(n)$ is the number of distinct prime factors of n. The same result holds for $\Omega(n)$, the number of prime factors of n, multiple factors counted multiply. The original proof of Hardy and Ramanujan was elementary but fairly complicated and used an estimate on the number of integers $< x$ having exactly k prime factors. Landau had such a result for fixed k, and they extended it for all k.

Hardy and Ramanujan prove by induction that there are absolute constants k and c such that

$$\pi_v(x) < \frac{kx}{\log x} \frac{(\log\log x + c)^{v-1}}{(v-1)!}, \quad v = 1,2,3,\ldots,$$

where $\pi_v(x)$ denotes the number of integers $n < x$ which have v distinct prime factors. As stated above Landau had obtained for fixed v an asymptotic formula for $\pi_v(x)$ as $x \to \infty$ and it was a natural question to ask for an asymptotic formula or at least a good inequality for $\pi_v(x)$ for every v. In fact Pillai proved that

$$\pi_v(x) \gg_c \frac{x}{\log x} \frac{(\log\log x)^{v-1}}{(v-1)!} \quad \text{for } v < c.\log\log x$$

and later I showed [12] that if

$$\log\log x - c'\sqrt{\log\log x} < v < \log\log x + c'\sqrt{\log\log x} \tag{7}$$

then

$$\pi_v(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{v-1}}{(v-1)!}, \text{ as } x \to \infty \tag{8}$$

holds for every $c' > 0$; so the "critical interval" of values for v is covered.

I further conjectured that the sequence is unimodal. That is

$$\pi_1(x) < \pi_2(x) < \dots \pi_v(x) > \pi_{v+1}(x) > \pi_{v+2}(x) > \dots \tag{9}$$

holds some $v = v(x)$. I expected that the main difficulty in proving (9) will be in the critical interval (7) but it turned out to my great surprise that I was wrong. The unimodality of $\pi_v(x)$ was proved for all but the very large values of v, that is for

$$v < c''(\log x)/(\log\log x)^2$$

by Balazard*. Thus only the large values of v remain open. I first thought that the cases $v > c''(\log x)/(\log\log x)^2$ will be easy to settle but so far no one has been successful. If we put

$$f_v(x) = \sum_{a_i < x} \frac{1}{a_i},$$

*Balazard; to appear in: Séminaire de théorie des nombres de Paris 1987-88, Birkhäuser.

where the summation is extended over all the a_i which have v distinct prime factors, then I showed [12] that $f_v(x)$ is unimodal but this is much easier than (9).

In fact (8) became obsolete almost immediately. I learned from Chandrasekharan that Sathe [25] obtained by very complicated but elementary methods an asymptotic formula for $\pi_v(x)$ for $v \ll \log\log x$. Upon seeing this Selberg [26] found a much simpler proof of a stronger result by analytic methods. Later it turned out that the same method was used by Turán in his dissertation [28] which appeared only in Hungarian and was not noticed*. Kolesnik and Straus [21] and Hensley [19] further extended the range of the asymptotic formulas for $\pi_v(x)$ and currently the strongest results are in a recent paper of Hildebrand and Tenenbaum [20].

As Hardy once told me, their theorem seemed dead for nearly twenty years, but it came to life in 1934. First Turán proved [27] that

$$\sum_{n \le x} (v(n) - \log\log x)^2 < c.x \log\log x. \tag{10}$$

The proof of (10) was quite simple and immediately implied (6). Later (10) was extended by Kubilius and became the classical Turán-Kubilius inequality of Probabilistic Number Theory. Actually (10) was the well-known Tchebychev inequality but we were not aware of this because we had very little knowledge of Probability Theory.

In 1934, Turán also proved that if $f(x)$ is an irreducible polynomial, then for almost all n, $f(n)$ has about $\log\log n$ prime factors and I proved using the Brun-Titchmarsh theorem that the same holds for the integers of the form $p-1$ [8]. A couple of years later I proved [9], that the density of integers n for which

*See the paper of Alladi in this proceedings for more on this.

$\nu(n) > \log\log n$ is $1/2$. Of course (8) and the theorem of Hardy-Ramanujan immediately implies this but (8) was proved only much later and my original proof is much simpler and does not use the Prime Number Theorem. I used Brun's method and the Central Limit Theorem for the Binomial distribution. I did not at that time know the Central Limit Theorem, but in the Binomial case this was easy. At that time I could not have formulated even the special case of the Erdös-Kac theorem due to my ignorance of Probability.

All these questions were cleared up when Kac and I met in 1939 in Baltimore and Princeton. All this is described in the excellent two volume book of Elliott [6] on Probabilistic Number Theory but perhaps I can be permitted to repeat the story in my own words: "I first met Kac in Baltimore in the Winter of 1938-39. Later in March 1939, he lectured on additive number theoretic functions. Among other things he stated the following conjecture which a few hours later became the Erdös-Kac Theorem. Suppose $f(n)$ is an additive arithmetic function for which $f(p) = f(p^\alpha)$ for every α, (this is not essential and is only assumed for convenience), $|f(p)| < C$ and $\sum_p \frac{f^2(p)}{p}$ diverges to ∞. Furthermore, put

$$A(x) = \sum_{p<x} \frac{f(p)}{p} \quad \text{and} \quad B(x) = \sum_{p<x} \frac{f^2(p)}{p}.$$

Then the density of integers n for which $f(n) < A(n) + c\sqrt{B(n)}$ is

$$G(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-u^2/2} du.$$

He said he could not prove this but if we truncate $f(n)$ and put $f_k(n) = \sum_{p|n,\ p<k} f(p)$, then as $k \to \infty$, density of $d_k(c)$ of integers for which $f_k(n) < A(k) + c\sqrt{B(k)}$ approaches $G(c)$.

I was for a long time looking for a theorem like the conjecture of Kac but due to my lack of knowledge of Probability Theory I could not even formulate a theorem or conjecture like the above. But already during the lecture of Kac I realised that by Brun's method I can deduce the conjecture of Kac from his theorem. After his lecture we immediately got together. Neither of us completely understood what the other was doing, but we realised that our joint work will give the theorem and to be a little impudent and conceited, Probabilistic Number Theory was born." This collaboration is a good example to show that two brains can be better than one, since neither of us could have done the work alone. Many further theorems were proved by us and others in this subject (e.g. the Erdös-Wintner Theorem which is based on Erdös-Kac) , but I have to refer to the book of Elliott for more information. My joint papers with Kac [13] as well as with Wintner [17] appeared in the American Journal of Mathematics.

Let me state one of my favorite problems here for which our probabilistic technique does not apply. Denote by $P(n)$ the largest prime factor of n. Is it true that the density of integers for which $P(n+1) > P(n)$ is $1/2$? The reason that the probabilistic approach does not work is that $P(n)$ depends on a single prime factor and the same will hold if instead of $P(n)$ we consider $A(n) = \sum_{p_i | n} p_i$ (see my joint papers with Alladi [2], [3], for connections between $A(n)$ and $P(n)$). Pomerance and I have some weaker results than the conjecture [16], but we both feel that the problem is unattackable at present.

Note that $A(n)$ cannot have a normal order because the order of magnitude of $A(n)$ for almost all n is determined by $P(n)$ (see [2]) and $\log P(n)$ has a distribution function. In this context we point out that Elliott has shown (see [6], Ch. 15) that if $f(n)$ is additive and $f(p) > (\log p)^{1+\varepsilon}$, then f cannot have a normal order; so $A(n)$ cannot have a normal order. It should be possible to show that by neglecting a set of density zero the inequality $A(n+1) > A(n)$ will hold if and only if $P(n+1) > P(n)$.

Before I leave this subject I want to state one of my favorite theorems which was proved in 1934 and which is a strengthening of the original theorem of Hardy and Ramanujan: To every ε and $\delta > 0$ there is a $k_0(\varepsilon,\delta)$ such that the lower asymptotic density of integers n for which for every $k > k_0(\varepsilon,\delta)$

$$e^{e^{k(1-\varepsilon)}} < p_k(n) < e^{e^{k(1+\varepsilon)}}$$

is $>(1-\delta)$. Here $p_k(n)$ is the kth smallest prime factor of n, and the inequalities are considered vacuously true for integers n having fewer than k_0 prime factors. The proof of this result is not very difficult.§

Next I come to highly composite numbers. Recall that an integer n is called highly composite if for every $m < n$ we have $d(m) < d(n)$, where d is the divisor function. Ramanujan wrote a long paper [24] on this subject. Hardy rather liked this paper but perhaps not unjustly called it nice but in the backwaters of mathematics. Alaoglu and I wrote a long paper on this subject [1] sharpening and extending many of the results of Ramanujan. If we denote by $D(x)$ the number of highly composite numbers not exceeding x, then I proved that [11] there exists a $c > 0$ such that $D(x) > (\log x)^{1+c}$ for $x > x_0$. Our results were extended by J. L Nicolas, and later Nicolas and I wrote several papers on this and related topics.

Ramanujan had a very long manuscript on highly composite numbers but some of it was not published due to a paper shortage during the First World War. Nicolas has studied this unpublished manuscript of

§see my paper, "Some unconventional problems in Number Theory", Astérique, 61 (1979), p. 73-82.

Ramanujan and has written about this in an appendix to this paper. Ramanujan's paper contains many clever elementary inequalities. The reason I succeeded in obtaining $D(x) > (\log x)^{1+c}$ which is better than Ramanujan's inequality was that I could use Hoheisel's result on gaps between primes which was not available during Ramanujan's time.

Let $U_1 < U_2 < U_3 < \dots$ be the sequence of consecutive highly composite numbers. One would expect that perhaps

$$U_{k+1} - U_k < U_k/(\log U_k)^{\varepsilon}$$

but I could never prove this and in fact Nicolas does not belive that this is true. As far as I know

$$D(x) < (\log x)^{c'}$$

is not yet known. All these problems connect with deep questions on diophantine approximations and so, although these problems are not central, they are not entirely in the backwaters of mathematics!

Ramanujan in his paper on highly composite numbers obtained upper and lower bounds for $d_k(n)$, the kth iterate of $d(n)$. If we denote by $1,2,3,5,8,\dots,$ the sequence of Fibonacci numbers $f_1,f_2,f_3,\dots,$ then Katai and I proved [14] that for every $n > n_0(k,\varepsilon)$

$$d_k(n) < \exp\left(\exp\left\{\left(\frac{1}{f_k} + \varepsilon\right)\log\log n\right\}\right), \quad k > 2$$

and that for infinitely many n

$$d_k(n) > \exp\left(\exp\left\{\left(\frac{1}{f_k} - \varepsilon\right)\log\log n\right\}\right), \quad k > 2$$

which is a fairly satisfactory result. We further conjectured that

$$\sum_{n<x} d_k(n) = (c_k + o(1))x \log_{(k)} x, \quad k > 2$$

for some constant $c_k > 0$, where $\log_{(k)}(x)$ is the kth iterate of the logarithm. We could only prove this for $k \leq 4$ [15]. For $k = 2$ this was first proved by Bellman and Shapiro. Finally Katai and I proved that if $h(n)$ is the smallest integer for which $d_{h(n)}(n) = 2$, then

$$h(n) \ll \text{logloglog } n$$

for every n, but that for infinitely many n

$$h(n) > c \text{ logloglog } n, \quad \text{some } c > 0.$$

We could not obtain an asymptotic formula or even a good inequality for $\sum_{h<x} h(n)$.

Ramanujan investigates the iterates of $d(n)$ only superficially perhaps to save space. Neither he or anybody else returned to this problem until Katai and I settled it to some extent.

Now finally I have to talk about partitions. Hardy and Ramanujan (and independently Uspensky) found an asymptotic formula for $p(n)$, the number of unrestricted partitions of n. They proved that

$$p(n) \sim \frac{e^{c\sqrt{n}}}{4n\sqrt{3}}, \quad \text{where } c = \pi\sqrt{2/3}. \tag{11}$$

In fact Hardy and Ramanujan proved a good deal more; they obtained a surprisingly accurate but fairly complicated asymptotic expansion for $p(n)$ which in fact could be used to calculate $p(n)$. Later, Lehmer proved that the series of Hardy and Ramanujan diverges and Rademacher obtained a convergent series for $p(n)$. In 1942, I found an elementary but very complicated proof [10] of the first term of the asymptotic formula of Hardy and Ramanujan. My proof was based on the simple identity

$$np(n) = \sum_{k} \sum_{v} \sigma(v)p(n-kv), \qquad (12)$$

where $\sigma(v)$ is the sum of the divisors of v. I showed that (12) implies (11) by fairly complicated Tauberian arguments which show some similarity to some of the elementary proofs of the Prime Number Theorem. This was perhaps an interesting tour-de-force but no doubt the analytic proof of Hardy and Ramanujan was both simpler and more illuminating. In fact, their proof later developed into the circle method of Hardy and Littlewood which was and is one of our most powerful tools in additive number theory.

I think my most important contribution to the theory of partitions is my joint work with Lehner where we investigate the statistical theory of partitions. Using the asymptotic formula of Hardy-Ramanujan the sieve of Eratosthenes and the simplest ideas involving 'Brun's method' we obtain asymptotic formulas for the number of partitions of n where the larget summand is less than $\sqrt{n \log n} + c\sqrt{n}$.Details on this can be found in the book by Andrews [4] on the Theory of Partitions. These problems are still very much "alive" and I have some recent joint work on this with Dixmier and Nicolas and with Szalays.

Some recent work of Ivic and myself (which is not yet published and will appear in the Proceedings of the 1987 Budapest Conference on Number Theory) leads us to the following conjecture: "The number of distinct prime factors in the product $\prod_{n \leq x} p(n)$ is unbounded as $x \to \infty$." Schinzel proved this conjecture and Wirsing improved the result which will soon appear in their joint paper. In other words, they proved that the integers $p(n)$ cannot all be composed by a fixed finite set of primes. The proof is not at all trivial and I think

Ramanujan would have been pleased with this result. No doubt much more is true and presumably

$$v\Big(\prod_{n<x} p(n)\Big)/x \to \infty \quad \text{as} \quad x \to \infty$$

but at the moment this seems to be beyond our reach.

Unfortunately I never met Ramanujan. He died when I was seven years old, but it is clear from my papers that Ramanujan's ideas had a great influence on my mathematical development. I collaborated with several Indian mathematicians. S. Chowla, who is a little older than I, has co-authored many papers with me on Number Theory and I also have several joint papers with K. Alladi on number-theoretic functions. I should say a few words about my connections with Sivasankaranarayana Pillai whom I expected to meet in 1950 in Cambridge, U.S.A., at the International Congress of Mathematicians. Unfortunately he never arrived because his plane crashed near Cairo. I first heard of Pillai in connection with the following result which he proved: Let $f(n)$ denote the number of times you have to iterate Euler's function $\phi(n)$ so as to reach 2. Then, there exists constants c_1, c_2 such that

$$\frac{\log n}{\log 3} - c_1 < f(n) < \frac{\log n}{\log 2} + c_2.$$

Shapiro rediscovered these results and also proved that $f(n)$ is essentially an additive function. I always wanted to prove that $f(n)/\log n$ has a distribution function. In other words the density of integers n for which $f(n) < c.\log n$ exists for every c. I could get nowhere with this simple and attractive question and could not even decide whether there is a constant c such that for almost all n, $f(n)/\log n \to c$.

Denote by $g(x)$ the number of integers $m \leq x$ for which $\phi(n) = m$ is solvable. Pillai proved that $g(x) = o(x)$ and I proved that for every k and $\varepsilon > 0$

$$\frac{x}{\log x}(\log\log x)^k < g(x) < \frac{x}{(\log x)} \cdot (\log x)^{\varepsilon},$$

holds for sufficiently large x. Subsequently, R. R. Hall and I strengthened these inequalities and currently the best results on $g(x)$ are due to Maier and Pomerance [22]. They proved that there is an absolute constant c for which

$$g(x) = \frac{x}{\log x} e^{(c+o(1)(\log\log\log x)^2}.$$

We are very far from having a genuine asymptotic formula for $g(x)$ and it is not even clear whether such an asymptotic formula exists. I conjectured long ago that

$$\lim_{x\to\infty} \frac{g(2x)}{g(x)} = 2.$$

This is still open, but might follow from the work of Maier and Pomerance.

Pomerance, Spiro and I have a forthcoming paper on the iterations of the ϕ function but many unsolved problems remain. These problems on the iterations of arithmetic functions are certainly not central but I have to express strong disagreement with the opinion of Bombieri, a great mathematician, who said these problems are absolutely without interest.

Perhaps the most important work of Pillai was on Waring's problem, namely on the function $g(n)$, which is the smallest integer such that every integer is the sum of $g(n)$ or fewer nth-powers.

REFERENCES

1. L. Alaouglu and P. Erdös, On highly composite and similar numbers, Trans. Amer. Math. Soc. 56(1944) pp. 448-469.

2. K. Alladi and P. Erdös, On an additive artihmetic function, Pacific J. Math. 71(1977) pp. 275-294.

3. K. Alladi and P. Erdös, On the asymptotic behavior of large prime factors of integers, Pacific J. Math., 82 (1979) pp. 295-315.

4. G. Andrews, The theory of partitions, Encyclopedia of Math. and its applications, Vol. 2, Addison Wesley, Reading, Mass. (1976).

5. H. Diamond and P. Erdös, On sharp elementary prime number estimates, Enseign. Math. II Ser. 26 (1980) pp. 313-321.

6. P. D. T. A. Elliot, Probabilistic Number Theory, Vols. 1 and 2, Grundlehren 239-240, Springer Verlag, Berlin-New York (1980).

7. P. Erdös, Beweis eines Satzes von Tschebyschef, Acta Litt. Ac. Sci. Regiae Univ. Hung. Fr.-Jos Sect. Sci. Math. 5 (1932) pp. 194-198.

8. P. Erdös, On the normal number of prime factors of p-1 and some related problems concerning Euler's ϕ-function, Quart. J. Math., Oxford Ser. 6 (1935) pp. 205-23.

9. P. Erdös, Note on the number of prime divisors of integers, J. London Math. Soc., 12 (1937) pp. 308-314.

10. P. Erdös, On an elementary proof of some asymptotic formulas in the theory of partitions, Annals of Math., 43 (1942) pp. 437-450.

11. P. Erdös, On highly composite numbers, J. London Math. Soc., 19 (1944) pp. 130-133.

12. P. Erdös, Integers having exactly k prime factors, Annals of Math., 49 (1948) pp. 53-66.

13. P. Erdös and M. Kac, The Gaussian law of errors in the theory of additive number theoretic functions, Amer. J. Math., 62 (1940) pp. 738-742.

14. P. Erdös and I. Katai, On the growth of $d_k(n)$, Fibonacci Quarterly 7 (1969) pp. 267-274.

15. P. Erdös and I. Katai, On the sum $\Sigma d_4(n)$, Acta Sci. Math. Szeged, 30 (1969) pp. 313-324.

16. P. Erdös and C. Pomerance, On the largest prime factors of n and $n+1$, Aequationes Math., 17 (1978) No. 2-3, pp. 311-321.

17. P. Erdös and A. Wintner, Additive arithmetical functions and statistical independence, Amer. J. Math., 61 (1939) pp. 713-721.

18. G. H. Hardy and S. Ramanujan, On the normal number of prime factors of a number n, Quart. J. Math., Oxford, 48 (1917) pp. 76-92.

19. D. Hensley, The distribution of round numbers, Proc. London Math. Soc., 54 (1987) pp. 412-444.

20. A. Hildebrand and G. Tenenbaum, On the number of prime factors of an integer, Duke Math. Journal, 56 (1988), pp. 471-501.

21. G. Kolesnik and E. Straus, On the distribution of integers with a given number of prime factors, Acta Arith., 37 (1980) pp. 181-199.

22. H. Maier and C. Pomerance, On the number of distinct values of Euler's ϕ-function, Acta Arith. 49 (1988) pp. 263-275.

23. S. Ramanujan, Collected Papers, Chelsea, New York (1962).

24. S. Ramanujan, Highly Composite numbers, Proc. London Math. Soc. Ser 2., 14 (1915) pp. 347-409.

25. L. G. Sathe, On a problem of Hardy, J. Indian Math. Soc., 17 (1953) pp. 63-141; 18 (1954) pp. 27-81.

26. A. Selberg, Note on paper by L. G. Sathe, J. Indian Math. Soc., 18 (1954), pp. 83-87.

27. P. Turán, On a theorem of Hardy and Ramanujan, J. London Math. Soc., 9 (1934) pp. 274-276.

28. P. Turán, Az egesz szamok primosztoinak szamarol, Mat. Lapok., 41 (1934) pp. 103-130.

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APPENDIX: On Composite Numbers

By

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Highly composite numbers n are positive integers satisfying

$$d(n) > d(m) \text{ for all } m < n, \qquad (1)$$

where d is the divisor function. Srinivasa Ramanujan studied highly composite numbers in great detail and his long paper [3] is quite famous. But there was much work on highly composite numbers and related topics that Ramanujan did not publish. During his centennial in December 1987, the first published copy [2] of his Lost Notebook and other unpublished papers was released and in this impressive volume a manuscript of Ramanujan on highly composite numbers (previously unpublished) is included (pages 280-308). It is to be noted, however, that at the top of page 295 of [2] the words - "Middle of another paper" is not handwritten by Ramanujan. A short analysis of this manuscript on highly composite numbers is given in [1] p. 238-239.

The table on page 280 of [2] is not a list of highly composite numbers. This table almost coincides with the list of largely composite numbers n which satisfy the weaker inequality

$$d(n) \geq d(m) \text{ for all } m \leq n. \qquad (2)$$

Note the slight difference between (1) and (2). There are only four largely composite numbers which were omitted by Ramanujan in this table, namely, 4200, 151200, 415800, 491400. Also, as J. P. Massias has pointed out, the number 15080 in this table is not largely composite.

In this unpublished manuscript Ramanujan also has some very interesting results on $\sigma(n)$, the sum of the divisors of n. In this context we point out a result due to Robin [4] that $\sigma(n) \leq e^{\gamma}$ nloglogn for $n \geq 5041$. Here γ is Euler's constant. More precisely he showed that

$$\frac{\sigma(N)}{N\log\log N} \leq e^{\gamma}\exp\left(\frac{2(1-\sqrt{2})+c}{\sqrt{x}\log x} + O\left(\frac{1}{\sqrt{x}\log^2 x}\right)\right), \qquad (3)$$

where

$$c = \gamma + 2 - \log 4\pi.$$

In (3), N is a collossaly abundant number of parameter x and for such n we have

$$\log N = \sum_{\substack{p \leq x \\ p=\text{prime}}} \log p + O(\sqrt{x}) = x + O(\sqrt{x}\log^2 x) \qquad (4)$$

under the assumption of the Riemann Hypothesis. Using (4) we may rewrite (3) as

$$\frac{\sigma(N)}{N\log\log N} \leq e^{\gamma}\left(1 + \frac{2(1-\sqrt{2})+c}{\sqrt{\log N}\,\log\log N} + O\left(\frac{1}{\sqrt{\log N}\,(\log\log N)^2}\right)\right). \qquad (5)$$

Ramanujan wrote down a similar formula about seventy years earlier with the notation $\Sigma_{-1}(N)$ for the maximal order of $\frac{\sigma(N)}{N}$ (see [2], p. 303):

$$\overline{\lim}\,(\Sigma_{-1}(N) - e^{\gamma}\log\log N)\sqrt{\log N} \leq e^{\gamma}(2\sqrt{2} + \gamma - \log 4\pi). \qquad (6)$$

Unfortunately (5) and (6) do not agree; it seems that in formula (382) of Ramanujan ([2], p. 303) the sign of the term $2(\sqrt{2}-1)/\sqrt{\log N}$ is wrong and so the right hand side of (6) should read

$$e^{\gamma}(\gamma - \log 4\pi + 2(2-\sqrt{2})).$$

The wrong sign seems to come from Ramanujan's analysis of his formula (377) of [2]. As Ramanujan explains at the beginning of §71, p. 302 of [2], the term $(\log N)^{\frac{1}{2}-s}/\log\log N$ arises from four terms of formula (377) and in formula (379) the coefficient of this term has the wrong sign!

In the same manuscript Ramanujan has a very nice estimation of the maximal order of $\sigma(n)/n^s$ for all s, which is not at all easy to obtain. This result of Ramanujan on the maximal order of $\sigma(n)/n^s$ for $s \neq 1$ under the assumption of the Riemann Hypothesis is new (and has not yet been rediscovered!) and it will definitely be worthwhile to look into this further. I hope to do this on a later occasion.

REFERENCES

1) J. L. Nicolas, On highly composite numbers, in "Ramanujan Revisited", Proc. Centenary Conference, Univ. Illinois, Urbana, Academic Press, NY (1987), p. 215-244.

2) S. Ramanujan, "The Lost Notebook and other unpublished papers", Narosa Publishing House, New Delhi (1987), p. 280-308.

3) S. Ramanujan, Highly composite numbers, Proc. London Math. Soc., 2, 14 (1915), p. 347-409.

4) G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. pures et appl., 63 (1984), p. 187-213.

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