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Problems and results on extremal problems in number theory, geometry, and combinatorics

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During my long life I wrote many papers about solved and unsolved problems. I will start with number theory.

1. Perhaps my first serious conjecture which goes back to 1931 or 32 states as follows:

Let $1 < a_1 < a_2 < \dots < a_n$ be a sequence of integers. Assume that all the sums

$$\sum_{i=1}^{n} \varepsilon_i a_i, \quad \varepsilon_i = 0 \text{ or } 1,$$

are distinct. Is it then true that there is an absolute constant c for which

$$a_n > 2^{n+c} \ ? \qquad (1)$$

I immediately proved by a simple counting argument that

$$a_n >_c \exp \log 2 \left(n + \frac{\log n}{\log 2} + c\right)$$

and in 1954 using the second moment method Leo Moser and I proved that

$$a_n > \exp \log 2 \left(n + \frac{\log n}{2 \log 2} + c\right)$$

which is the current record. Conway and Guy proved that for large n $a_n < 2^{n-2}$ is possible and it has been conjectured that $a_n < 2^{n-3}$ is not possible.

I offer 500 Dollars for a proof or disproof of (1).

Our proof with L. Moser is appeared in [1].

2. A few days ago Sárközy and I asked the following question: Let $a_1 < a_2 < \ldots < a_n$ be such that the set of $2^n - 1$ integers

$$\sum_{i=1}^{n} \varepsilon_i a_i, \quad \varepsilon_i = 0 \text{ or } 1, \ \varepsilon_i \text{ not all } 0,$$

does not contain an arithmetic progression of three terms. Is it then true that

$$\min a_n = 3^{n-1} \ ? \qquad (2)$$

The conjecture (2) is perhaps too optimistic, but we are convinced that

$a_n > 3^{n-c}$ holds but we can only prove $a_n > 3^n n^{-c}$. A paper of ours on this and related problems will appear soon.

3. Let $a_1 < a_2 < \ldots < a_t \leq n$ be a sequence of integers. Assumme that $a_i \nmid a_u + a_v$, $i < u$, $i < v$, i.e. that no a divides the sum of two larger a's. Is it then true that

$$\max t = [\tfrac{n+2}{3}] + 1 \ ?$$

Sárközy and I conjectured this 20 years ago and it is annoying that we could not settle this problem (see [2]).

4. Now to a really serious problem which has important consequences. Let $W(n)$ be the smallest integer for which if we divide the integers $\leq W(n)$ into two classes at least one class contains an arithmetic progression of n terms. Van der Waerden in his classical paper proved that $W(n)$ exists but he has only a very poor upper bound for $W(n)$, his bound increased as fast as the Ackermann function, only very recently Shelah obtained a primitive recursive upper bound for $W(n)$. This was certainly a sensational triumph but Shelah's bound is probably still too high. It would be very nice to prove

$$W(n)^{1/n} \longrightarrow \infty$$

but we do not even know $W(n)/(2^n) \longrightarrow \infty$.

More than 50 years ago Turán and I asked:

Let $r_k(n)$ be the smallest integer for which if $a_1 < a_2 < \ldots < a_t \leq n$, $t = r_k(n)$ then the a's contain an arithmetic progression of k terms. $r_k(n) < \frac{n}{2}$ implies $W(k) < n$, i.e. Van der Waerden's theorem. We conjectured

$$r_k(n)/n \longrightarrow 0. \tag{3}$$

I offered 1000 dollars for a proof or disproof of (3) and in 1972 Szemerédi proved (3). His proof is a master-piece of combinatorial reasoning and his method (i.e. his regularity Lemma) can be used in many other problems. A few years later Fürstenberg proved (3) by methods of ergodic theory. His methods will no doubt be used in many other parts of combinatorial number theory.

Here is my 3000 dollar conjecture:

Let $\sum 1/a_n = \infty$, is it then true that the a's contain arbitrarily long arithmetic progressions ? If it is true this would imply that there are arbitrarily long arithmetic progressions among the primes.

Probably there are arbitrarily long arithmetic progressions among consecutive primes but this question is completely unattackable at present and can certainly not be approached by combinatorial methods (see [3, 4]).

Now I discuss some problems in <u>geometry.</u>

Let $x_1, \dots, x_n$ be n distinct points in the plane. Denote by $d(x_i,x_j)$ the distance between x_i and x_j. Let

$$\min d(x_i,x_j) = d, \quad \max d(x_i,x_j) = D.$$

D is the diameter of $x_1, \dots, x_n$. Denote by $A(x_1,\dots,x_n)$ the number of distinct distances determined by the points $x_1, \dots, x_n$, and by $R(x_1,\dots,x_n)$ the number of times the same distance can occur. In other words, $R(x_1,\dots,x_n)$ is the largest integer t for wich $d(x_i,x_j)$ takes the same value.

1. In 1946 I conjectured

$$\min A(x_1,\dots,x_n) > c_1\, n/\sqrt{\log n} \tag{4}$$

and

$$\max R(x_1,\dots,x_n) < n^{1+c_2/\log\log n} \tag{5}$$

In (4) and (5) the minimum and the maximum is taken over all possible choices of $x_1, \dots, x_n$, respectively.
The lattice points show that (4) and (5) if true are best possible.

I offer 500 dollars for a proof or disproof of each of the conjectures (4) or (5). I am afraid, there are easier ways of earning 1000 dollars than deciding these conjectures. Partial results have been proved by L. Moser, Beck and Spencer, Szemerédi and Trotter, Fan Chang, Graham and others. I believe the current record is $n^{4/5}$ in (4) and $n^{5/4}$ in (5). See also [5-8].

2. Denote by $F(n)$ the maximum number of times the diameter D can occur. It is easy to see that $F(n) = n$. I believe this was first observed by Erika Pannwitz. Trivially the minimum distance can occur fewer than $3n$ times. Denote this number by $f(n)$. Harborth determined $f(n)$ exactly.
Pach and I conjectured

$$f(n)\, F(n) < n^2. \tag{6}$$

(6) if true is best possible, the regular polygon of odd number of vertices shows this. We proved

$$f(n) + F(n) < 3n - c\sqrt{n}$$

but did not determine the exact value of $\max f(n)+F(n)$, the method of Harborth with some care will perhaps give this. Reutter posed the problem of determination of $f(n)$ in Elemente der Mathematik about 1964 and stated the formula for $f(n)$. Harborth proved it also in Elemente der Mathematik.

3. Let $x_1, \dots, x_n$ be a convex n-gon. Denote by $h(n)$ the maximum number of times the same distance can occur. In 1958 Leo Moser and I conjectured that $h(n) < cn$ holds for some absolute constant c. This conjecture is still open and I offer 100 dollars for a proof or disproof.

Moser and I observed that $h(n) > \frac{5}{3}n$ and last year Peter Hajnal proved $h(n) \geq \frac{9}{5}n$ which was improved by Edelsbrunner to $h(n) \geq 2n - 7$ which is as far as I know the current record. Füredi very recently proved $h(n) < cn \log n$, which is the best upper bound.

Perhaps the following stronger result holds: There is an r (perhaps $r = 4$) so that there is an x_i which has no r other vertices equidistant from it. I once conjectured this with $r = 3$, but this was disproved by Danzer. By the way Pach believes that our conjecture with Moser is wrong and $h(n)/n$ can tend to infinity very slowly like the inverse Ackermann function, but both Füredi and I believe $h(n) < cn$. Füredis paper will appear soon in J. Combin. Theory Ser. A.

4. Assume that

$$d(x_i,x_j) \geq 1,\ 1 \leq i < j \leq n, \text{ and } |d(x_i,x_j) - d(x_k,x_l)| \geq 1. \tag{7}$$

In other words if two distances differ, they differ by at least one. I conjecture that if (7) holds then

$$D(x_1,\dots,x_n) > cn \tag{8}$$

and perhaps for $n > n_o$

$$D(x_1,\dots,x_n) = n - 1. \tag{9}$$

In other words (7) implies that for $n > n_o$ the diameter is minimal if the points are on a line. If (4) holds then (7) implies

$$D(x_1,\dots,x_n) > \frac{cn}{\sqrt{\log n}}$$

Kanold proved in 1981 that (7) implies $D(x_1,\dots,x_n) > cn^{3/4}$. I posed $D(x_1,\dots,x_n) > cn^{2/3}$ in Elemente der Mathematik 1981, Kanolds proof appeared soon afterwards. Makai has some new inequalities for small values of n.

5. Finally a simple problem which has perhaps been neglected.

Let $x_1, \dots, x_n$ be n points no four on a line. It is easy to see that one can find a subsequence $x_{i_1}, \dots, x_{i_k}$, $k > c\sqrt{n}$, no three of them are on a line. The proof is quite simple. Is it true that $k > c\sqrt{n}$ can be improved ? I can not even prove that $k > \epsilon n$ does not hold for every $x_1, \dots, x_n$. Perhaps I overlook a trivial point. Füredi just tells me that Rödl and Phelps proved in a different context that $k > \sqrt{n \log n}$. In fact they proved the following beautiful theorem.

Let $|S| = n$, $A_i \subset S$, $|A_i| = 3$, $|A_{i_1} \cap A_{i_2}| \leq 1$, then there is a subset $S_1 \subset S$, $|S_1| > \sqrt{n \log n}$, and S_1 contains none of the A_i's.

Now finally I discuss some extremal problems in <u>combinatorics.</u>

1. In my old paper with Ko and Rado many problems were stated. All but one of them has been solved. Here is the one which is still open:

Let $|S| = 4n$, denote by $f(n)$ the largest integer for which there is a family $A_i \subset S$, $|A_i| = 2n$, $1 \leq i \leq f(n)$ for which $|A_i \cap A_j| \geq 2$. We conjectured that

$$f(n) = [\binom{4n}{2n} - \binom{2n}{n}^2]/ 2. \tag{10}$$

(10) if true is best possible. To see this consider all subsets of size $2n$ of the integers $1 \leq x \leq 4n$ which contain at least $n + 1$ integers not exceeding $2n$.

More general conjectures have been stated by Peter Frankl and Cooper. Many papers have appeared on the Erdös-Ko-Rado theorem, here I only refer to three of them [9 - 11].

2. Let H be a graph, $T(n;H)$ the Turàn number of H is the largest integer for which there is a $G(n;T(n;H))$ (in other words a graph of n vertices and $T(n;H)$ edges) which does not contain H as a subgraph. Turán determined $T(n;H)$ if H is a complete graph. The exact value of $T(n;H)$ is known only for very few graphs. Rényi, V. T.-Sós and I proved that

$$T(n;C_4) = (\tfrac{1}{2} + o(1))n^{3/2}, \tag{11}$$

but the exact value of $T(n;C_4)$ is known only if $n = p^2 + p + 1$ where p is a power of a prime. Füredi proved

$$T(p^2 + p + 1, C_4) = \tfrac{1}{2}p^3 + p^2 + \tfrac{p}{2}.$$

I published many papers on extremal graph theory, here I just state a theorem and two conjectures of Simonovits and myself.

Let H be the edges of a cube, H is a regular bipartite graph of 8 vertices and 12 edges. We proved

$$T(n;H) < cn^{8/5}$$

probably

$$T(n;H) > cn^{8/5},$$

but we could not even prove that $T(n;H) / n^{3/2} \longrightarrow \infty$.

We conjectured that if H is a bipartite graph which has degree 3 then

$$T(n;H) < cn^{3/2}. \qquad (12)$$

On the other hand if all vertices of H have degree ≥ 3 then

$$T(n;H) > n^{3/2 + \epsilon}. \qquad (13)$$

Both, (12) and (13), are rather doubtful. In fact let H have the vertices x ; y_1,y_2,y_3,y_4; z_1,z_2,z_3,z_4,z_5,z_6. x is joined to y_1,y_2,y_3,y_4, each z is joined to two y's distinct z's to distinct pairs. We could not prove

$$T(n;H) < cn^{3/2}.$$

I recommend to the interested reader the excellent book of Bolobas [12] and the very nice paper of Simonovits [13].

To end the paper I would like to state one of our oldest problems. In 1931 E. Klein (Mrs. Szekeres) observed that from any 5 points in the plane, no three of which are on a line, one can always find four which form the vertices of a convex quadrilateral. Then she asked:

Let $f(n)$ be the smallest integer for which if $f(n)$ points are in the plane (no three on a line) one can always select n of them which form a convex n-gon. It is not clear at all that $f(n)$ exists. Szekeres and I proved

$$2^{n-2}+ 1 < f(n) < \binom{2n-4}{n-2}.$$

Szekeres conjectured $f(n) = 2^{n-2}+ 1$. This was proved by Turán and E. Makai for $n = 5$. It is not known yet whether from 17 points one can always find a convex hexagon.

About 10 years ago I asked:

Let $F(n)$ be the smallest integer for which if $F(n)$ points are given in the plane (no three on a line) can one always find n of them which form a convex n-gon whose interior contains none of the other points. $F(4) = 5$ is trivial and Harborth proved $F(5) = 10$. Harborth conjectured that for $n > 7$ $F(n)$ does not exist and this was indeed shown by Horton. It is not known yet if $F(6)$ exists.

Finally many decades ago Richard Guy and I observed that if $h(n)$ is the largest integer for which every set of n points (no three on a line) contains at least $h(n)$ convex quadrilaterals, then

$$\lim_{n\to\infty} \frac{h(n)}{n^4} = c$$

exists. The value of this limit is not known yet, perhaps $c = 1/69$? The exact determination of $h(n)$ will perhaps be difficult. See [14], where my papers with Szekeres [15,16] are reprinted.

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