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## Note

# Monochromatic Sumsets

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For $S \subset N$ the sumset $P(S)$ is defined as the set of all sums $a_1 + \cdots + a_t$, $t$ arbitrary, $a_i$ distinct elements of $S$. Let $F(k)$ denote the least $n$ so that if $[n]\ (=\{1, ..., n\})$ is two colored there is a $k$-set $S$ with $P(S) \subset [n]$ and $P(S)$ monochromatic. The existence of $F(k)$ is given by Folkman's Theorem, see, e.g., [1]. Here we give a lower bound for $F(k)$.

THEOREM. $F(k) > 2^{ck^2/\lg k}$.

LEMMA. *If* $|S| = k$ *then* $|P(S)| \geqslant k(k+1)/2$.

*Proof.* Let $a_1 < \cdots < a_k$ denote the elements of $S$. The sums $a_1 + \cdots + a_j$, $1 \leqslant j \leqslant k$ and $a_1 + \cdots + a_j - a_i$, $1 \leqslant i < j \leqslant k$ have a canonical ordering and are distinct.

LEMMA. *At most* $(kn)^{\lg u} u^{2k}$ $k$*-sets* $S \subset [n]$ *have* $|P(S)| \leqslant u$.

*Proof.* Let $a_1 < \cdots < a_k$ denote the elements of $S$. Call $i$ doubling if $P(a_1, ..., a_i)$ has double the size of $P(a_1, ..., a_{i-1})$. There are at most $\lg u$ doubling $i$. Hence there are at most $k^{\lg u}$ choices for doubling positions $i$ and at most $n^{\lg u}$ choices for the values $a_i$. If $i$ is not doubling then $a_i = x - y$, where $x, y \in P(a_1, ..., a_{i-1}) \subset P(S)$ so there are at most $u^2$ choices for $a_i$.

*Proof of Theorem.* Two-color $[n]$ randomly. The expected number of $k$-sets $S$ with $P(S)$ monochromatic is then

$$\sum_{\substack{|S|=k \\ P(S) \subseteq [n]}} 2^{1-P(S)} \leqslant \sum_{u \geqslant k(k+1)/2} (kn)^{\lg u} u^{2k} 2^{-u} < 1$$

with $n < 2^{ck^2/\lg k}$, $c$ an appropriately small absolute constant.



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Attempts to remove the $\lg k$ factor in the exponent have led to an intriguing question. Define the $(r, s)$ *sumset game* as follows. Player 1 selects distinct $a_1, ..., a_r \in N$. Player 2 then selects (seeing $a_1, ..., a_r$) $a_{r+1}, ..., a_{r+s} \in N$ distinct from each other and the previous $a_i$. The payoff, to Player 1, is $|P(a_1, ..., a_{r+s})|$. Let $V(r, s)$ denote the value of this perfect information game. Can an exact formula for $V(r, s)$ be found? We conjecture $V(r, s) \geqslant cs^2 2^r$. Note $V(r, s) \leqslant \binom{s+2}{2} 2^{r-1}$ as Player 2 may select $2a_1, ..., (s+1)a_1$. Perhaps Player 1 can pick $r$ numbers sufficiently independent so that Player 2 can do no better.

*Note.* A. Taylor [2] has shown that $F(k)$ is bounded from above by a tower of threes of height $4k-3$. While not Ackermanic, this upper bound is quite far from our lower bound.

## References

1. R. L. Graham, B. L. Rothschild, and J. H. Spencer, "Ramsey Theory," Wiley, New York, 1980.
2. A. Taylor, Bounds for the disjoint union theorem, *J. Combin. Theory Ser. A* **30** (1981), 339–344.

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