Maximal Antiramsey Graphs and the Strong Chromatic Number

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ABSTRACT

A typical problem arising in Ramsey graph theory is the following. For given graphs G and L, how few colors can be used to color the edges of G in order that no monochromatic subgraph isomorphic to L is formed? In this paper we investigate the opposite extreme. That is, we will require that in any subgraph of G isomorphic to L, *all* its edges have *different* colors. We call such a subgraph a *totally multicolored copy of L*. Of particular interest to us will be the determination of $\chi_S(n, e, L)$, defined to be the minimum number of colors needed to edge-color *some* graph $G(n, e)$ with n vertices and e edges so that all copies of L in it are totally multicolored.

It turns out that some of these questions are surprisingly deep, and are intimately related, for example, to the well-studied (but little understood) functions $r_k(n)$, defined to be the size of the largest subset of $\{1, 2, \ldots, n\}$ containing no k-term arithmetic progression, and $g(n; k, l)$, defined to be the maximum number of triples which can be formed from $\{1, 2, \ldots, n\}$ so that no two triples share a common pair, and no k elements of $\{1, 2, \ldots, n\}$ span l triples.

1. INTRODUCTION

Here we will study certain problems that combine elements of Ramsey theory, extremal set theory, and the theory of the strong chromatic number of hypergraphs. If H is a hypergraph, define the *strong chromatic number* $\chi_S(H)$ to be

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the minimum number of colors required to color the vertices in such a way that the vertices of each hyperedge of H have no repeated color. For the more familiar (ordinary) chromatic number $\chi(H)$, it is only required that the vertices of each hyperedge of H not all have the same color. The inequality $\chi_S(H) \geq \chi(H)$ holds in general, with equality when H is 2-uniform, that is, when H is a graph. When a hyperedge (or any other sort of set we happen to have in mind) is colored with no repeated color, we say the hyperedge is *totally multicolored*, which we abbreviate TMC. We use the same term for a hypergraph when all its hyperedges are TMC. We also use TMC as a verb from time to time.

Many standard problems in Ramsey theory for graphs can be thought of as relating to the ordinary chromatic number of a certain hypergraph. If G and L are graphs, define the hypergraph $H = H(G,L)$ to have vertex set equal to the edge set of G, and to have for its hyperedges the edge sets of all copies of L that are subgraphs of G. (Here, subgraph does *not* mean induced subgraph.) The usual (2-color) Ramsey arrow relation $G \to L$ is equivalent to $\chi(H(G,L)) > 2$. Here, we will be concerned with *antiramsey* results. That is, we want to know how many colors are needed to color the edges of G so that (the edges of) all copies of L in G are TMC. In other words, we want to know the value of $\chi_S(H(G,L))$; we abbreviate this expression by $\chi_S(G,L)$.

More specifically, we will study extremal antiramsey numbers. Denote a generic graph with n vertices and e edges by $G(n,e)$. Define $\chi_S(n,e,L) = \min \chi_S(G(n,e),L)$, where the minimum is taken over all graphs of type $G(n,e)$. That is, $\chi_S(n,e,L)$ is the smallest number r such that there exists a $G(n,e)$ that has an edge-coloring in r colors such that every L in G is TMC.

The function $\chi_S(n,e,L)$ has an obvious connection to extremal theory. As usual, define $ex(n,L)$ to be the greatest e such that there is a $G(n,e)$ that contains no copy of L at all. Thus, $\chi_S(n,e,L) = 1$ if $1 \leq e \leq ex(n,L)$, and $\chi_S(n,e,L) > 1$ if $e > ex(n,L)$. A problem in extremal theory that is often studied is to estimate the number of L that must occur in a graph with n vertices and e edges, which is in some sense the density of L in $G(n,e)$. Similarly, $\chi_S(n,e,L)$ could be thought of as a measure of the density of L.

In the first part of this paper we will focus our attention on the case $\chi(L) > 2$, with an emphasis on $\chi(L) = 3$. In the second part we treat the case $\chi(L) = 2$. It is well known from extremal theory that there is a fundamental dichotomy between these two cases, so this is a natural division.

We illustrate most of our results for $\chi(L) > 2$ in Table 1, which appears in the appendix, together with an explanation.

If we examine the first three rows of Table 1, we see a striking trichotomy: C_3, C_5, and all other odd cycles behave very differently. For $L = C_3$, $\chi_S(n,e,L)$ is very small and is not hard to determine; for $L = C_5$, $\chi_S(n,e,L)$ seems to behave in a complicated and poorly understood way; for the other odd cycles, $\chi_S(n,m,L)$ is very large, and good estimates are known. After preliminaries in Section 2, we deal with complete graphs and odd cycles in Sections 3–5.

In Section 6, we treat the case that L is bipartite, i.e., $\chi(L) = 2$; this case does not lead itself to representation in Table 1. In some sense, we have less to say here, partly because our questions lead immediately to well-studied difficult problems (such as the determination of the size of the largest subset of $\{1, 2, \ldots, n\}$ containing no 3-term arithmetic progression), and partly because there is probably just not as much structure for this case as there is for the case $\chi(L) > 2$. The remaining L in Table 1 are all of the form $kK_3 \cup lK_2$, that is, the disjoint union of k copies of K_3 and l copies of K_2. In Section 7, disjoint unions of graphs are studied, with an emphasis on unions of complete graphs, and with further emphasis on $kK_3 \cup lK_2$. In addition to being natural and not too hard to deal with, these graphs extend the range of behavior known to be possible. Finally, Section 8 discusses open problems.

2. SOME PRELIMINARIES

Before we come to the main results, it is necessary to introduce some more notation, and to state certain lemmas we will need later.

We begin by adopting the convention that L always denotes a graph without isolated vertices. The notation follows Harary [9], unless otherwise stated. For instance, P_k denotes a path on n vertices (not edges). However, we use $e(G)$ to denote the number of edges of G.

We will often be concerned with complete bipartite graphs and related graphs. We designate a complete bipartite graph by $K_{m,n}$ or $K(m,n)$, as convenient. We call the two maximal independent sets of such a graph its *parts* (not partite sets), and speak of the *first* and *second* parts. We frequently want to give a bipartite graph a particular coloring, called a *star coloring*. In such a coloring, we choose one of the parts of the graph and give all the edges emanating from each one of the vertices the same color, but with different colors at each of these vertices. Naturally, we can specify which part the centers of the stars are in. We also use the notation $K(m + G, n)$, where G is a graph with $p \leq m$ vertices, to represent the graph formed by identifying the vertices of G with p vertices of the first part of the $K(m,n)$. The *parts* of this graph are defined to be the same as the parts of the original $K(m,n)$.

One particular bipartite graph we will be greatly concerned with is the Turán graph $K(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$, which we will also call $T_2(n)$. Of course, $e(T_2(n)) = ex(n, K_3)$, and we also call this value $t_2(n)$. More generally, $T_k(n)$ is the extremal graph for K_{k+1}, and $t_k(n)$ is $e(T_k(n))$, which has the value $[(k-1)/k]\binom{n}{2} + O(n)$.

We now give some lemmas that will be useful later. The first is trivial, but it is convenient to have it stated explicitly.

Lemma 2.1. If $e > ex(n, L)$, then $\chi_S(n, e, L) \geq e(L)$. ∎

The remaining lemmas are all known results in extremal theory, or simple modifications thereof. In particular, the next two are implicit in [6].

Lemma 2.2. Let $L = K(k + K_2, l)$. Then if n is large enough, $ex(n, L) = t_2(n)$. ∎

This implies that $ex(n, C_k) = t_2(n)$ when $k \geq 3$ is odd and n is large. The next lemma is closely related to the preceding one. We also use this lemma to illustrate an abuse of notation that is often useful to employ. When we say αn in the following lemma, we really mean $\lfloor \alpha n \rfloor$. In other cases, it might mean $\lceil \alpha n \rceil$, as appropriate.

Lemma 2.3. Let k be fixed. Then there is an $\alpha > 0$ such that if n is large enough then for $L = K(k + K_2, \alpha n)$ we have $ex(n, L) = t_2(n)$. ∎

Our next lemma is well known in various forms, but we will prove it in the particular form we want.

Lemma 2.4. Let $0 < c < 1/2$ and $k > 0$ be given, let n be large, and let $G = G(n, cn^2)$. Then G has a subgraph with no vertex of degree less than k and with $cn^2 + O(n)$ edges.

Proof. Simply remove successively all vertices with degree less than k until none are left. No more than kn edges (actually fewer) are removed in this way, so that $cn^2 + O(n)$ edges remain. ∎

The proof of the following result uses the same idea, but with more technical care. The statement is a simplification of that of Lemma V.3.2 of [5].

Lemma 2.5 [5]. Let $1/2 > \alpha > \beta > 0$. Let n be large and let G be a $G(n, \alpha n^2)$. Then G contains a subgraph G_1 with $p > (2\beta)^{1/2} n$ vertices such that

$$\delta(G_1) > 2(\alpha - \beta)n$$

and

$$e(G_1) > e(G) - (\alpha - \beta)(n - p)(n + p + 1). \quad ∎$$

Here, as usual, $\delta(H)$ denotes the minimum degree in H. We make a final remark concerning the notation used in the statement of our results. If we say, for instance, that $e = \binom{n}{2} - \epsilon n^2$, this is to be interpreted similarly to the same statement in Table 1. That is, ϵ is any fixed constant that is small enough, and we require that $n > n_0(\epsilon)$. The interpretation is similar when c appears, except that c can be arbitrarily large. If ϵ and c are not used, then if the theorem requires $n > n_0$, this will be said explicitly. The implied constants in O-terms will depend only on ϵ or c (if present), unless otherwise stated.

3. COMPLETE GRAPHS

It is possible to give very precise results in the case of K_k, $k \geq 3$. When $k > 3$, the situation is a bit simpler than when $k = 3$, so we treat that case first. Note the relative lack of dependence on k when $k > 3$.

Theorem 3.1. Let $k > 3$, and let x be such that $t_{x-1}(n) < e \leq t_x(n)$. Then if $x < k$, $\chi_S(n, e, K_k) = 1$, and if $x \geq k$, we have $\chi_S(n, e, K_k) = \binom{x}{2}$.

Proof. The conditions on e give us that any $G(n, e)$ contains a K_x. If $x \geq k$, then any two edges in this K_x are contained in a K_k. Thus, $\chi_S(n, e, K_k) \geq \binom{x}{2}$. On the other hand, the conditions on e permit the construction of a graph $G = G(n, e)$ that has chromatic number x, namely, any subgraph of $T_x(n)$. Give G a vertex-coloring in x colors, and choose a set of $\binom{x}{2}$ colors for the edges. For each pair of color classes in this vertex-coloring, select a different one of these $\binom{x}{2}$ colors and give all the edges joining that pair of color classes this color. Any two edges with the same color have at least two nonadjacent endpoints, and so cannot be used in the same K_k. Thus, $\chi_S(G, K_k) \leq \binom{x}{2}$, and if $x < k$, we have $\chi_S(G, K_k) = 1$. ∎

Theorem 3.2. Let x be such that $t_{x-1}(n) < e \leq t_x(n)$. Then

$$\chi_S(n, e, K_3) = \begin{cases} x, & \text{if } x \text{ is odd,} \\ x - 1, & \text{if } x \text{ is even.} \end{cases}$$

Proof. Trivially, we assume $x \geq 3$. Again, the conditions on e imply that any $G(n, e)$ contains a K_x. Thus, any two adjacent edges in this K_x are contained in a K_3, but now nonadjacent edges are irrelevant. Thus, the number of colors needed to TMC all K_3 in K_x is just the edge-chromatic number of K_x, which is well known to be x if x is odd, and $x - 1$ if x is even. As in the previous proof, we have a $G = G(n, e)$ that has chromatic number x. Give G a vertex-coloring in x colors and identify each of the color classes with a vertex of K_x. Give the edges of this K_x an edge-coloring corresponding to the edge-chromatic number of K_x, and let G inherit this coloring in the obvious way. No two adjacent edges of G have the same color, so each K_3 is TMC. ∎

Note that $\chi_S(n, \binom{n}{2}, K_3) \leq n$. The only other graphs with this property are stars and matchings. Otherwise, we always have $\chi_S(n, \binom{n}{2}, L) = \binom{n}{2}$ for n large.

4. FIVE-CYCLES

The previous section gives the behavior of $C_3 = K_3$. Now we examine C_5, which is the subtlest case involving cycles. Of the four ranges of e that we are interested in, only for $e = ex(n, C_5) + 1 = t_2(n) + 1$ do we have a reasonably satisfactory answer. Indeed, a more careful analysis can be done to show that the upper bound is actually the correct answer (see [8]).

Theorem 4.1. Let n be large, and let $e = t_2(n) + 1$. Then

$$c_1 n \leq \chi_S(n, e, C_5) \leq \lfloor n/2 \rfloor + 3 .$$

Proof. Consider any graph G with n vertices and $t_2(n) + 1$ edges. By Lemma 2.3, G contains a $K(3 + K_2, c_1 n)$. Let v denote the vertex in the first part of G that is not incident to the K_2. It is easy to see that any two edges of $K(3 + K_2, c_1 n)$ incident to v belong to a C_5; thus $c_1 n \leq \chi_S(n, e, C_5)$. On the other hand, let G be the graph $K(\lceil n/2 \rceil + K_2, \lfloor n/2 \rfloor)$, which has e edges. Let u and v be the vertices of the K_2 in the first part of G. Give $G - u - v$ a star coloring, the stars having centers in the second part, requiring $\lfloor n/2 \rfloor$ colors. Choose three new colors. Give the edge $\{u, v\}$ one of these, and give the remaining edges incident to u one of the other new colors, and all the remaining edges incident to v the other. It is easy to see that every C_5 in G is TMC, and we have used $\lfloor n/2 \rfloor + 3$ colors. ∎

Theorem 4.2. Let n be large, $e = t_2(n) + x$, and let $y = \lceil (\sqrt{(8x + 1)} + 1)/2 \rceil$. Then

$$\chi_S(n, e, C_5) \leq (y + 1)\lfloor n/2 \rfloor + x .$$

Proof. The value of y assures that $x \leq \binom{y}{2}$. Let G_0 be a graph with x edges and y vertices, and let G be the graph $K(\lceil n/2 \rceil + G_0, \lfloor n/2 \rfloor)$. Color all $y\lfloor n/2 \rfloor + x$ edges incident to at least one vertex of G_0 with different colors. Now give $G - G_0$ a star coloring, with centers in the second part, using $\lfloor n/2 \rfloor$ new colors. It is straightforward that all C_5's in G are TMC. ∎

Note the special case $\chi_S(n, t_2(n) + cn, C_5) = O(n^{3/2})$, an entry in Table 1. It seems reasonable to conjecture that the upper bound in Theorem 4.2 is often sharp, at least when $x = \binom{y}{2} > 3$. (When $x = 3$, one can color all three edges of $G_0 = K_3$ the same, saving two colors, and when $x = 1$, Theorem 4.1 gives a considerably better bound.) Even if the upper bound is not very sharp, we expect that $\chi_S(n, e, C_5)$ grows moderately rapidly as e moves away from $t_2(n)$. However, the best we can prove in this direction is the following weak result:

Theorem 4.3. If $e = t_2(n) + \epsilon n^2$, then $\chi_S(n, e, C_5) > cn$ for any fixed c, once n is sufficiently large.

Proof. Let G be any graph with n vertices and e edges. We use Lemma 2.5, with $\alpha = 1/4 + \epsilon$ and $\beta = \epsilon/2$. Then by that lemma, G contains a G_1 with $p > \epsilon^{1/2} n$ vertices and with $\delta(G_1) > (1/2 + \epsilon)n$. Observe that in G_1, the endpoints of every P_4 have a point in their common neighborhood that is not in the P_4; in other words, every P_4 can be extended to a C_5. Therefore, for every C_5 in $G_1 \subset G$ to be TMC, so must every P_4. But by Theorem 6.3 below, $\chi_S(n, cn^2, P_4) > nu(n)$, where $u(n) \to \infty$ with n (although it grows quite slowly). However small ϵ is, it is fixed, and so $\epsilon^{1/2}nu(\epsilon^{1/2}n) > cn$ for any fixed c, once n is large enough. ∎

In Section 6 we mention a general result (Theorem 6.2) that applies, in particular, to C_5 when $e = (\frac{1}{2} - \epsilon)n^2$ for any fixed $\epsilon > 0$.

Theorem 4.4. If $e = (\frac{1}{2} - \epsilon)n^2$, then

$$\chi_S(n, e, C_5) = O(n^2/\log n).$$

5. ODD CYCLES WITH LENGTH AT LEAST SEVEN

Now we consider odd cycles larger than C_5. These cycles have a much simpler behavior than C_5 does. Note again that $ex(n, C_k) + 1 = t_2(n) + 1$ for all odd cycles when n is large enough.

Theorem 5.1. Let $k \geq 7$ be an odd integer. Then if n is large and $e > t_2(n)$, we have $\chi_S(n, e, C_k) \geq cn^2$.

Proof. For the moment, assume that $k \geq 9$. Consider any graph G with n vertices and at least $t_2(n) + 1$ edges. We apply Lemma 2.5, with $\alpha = 1/4$ and $\beta = 1/8$. By Lemma 2.5, G contains a graph G_1 with $p > n/2$ vertices, minimum degree $> p/4$, and with

$$\begin{aligned} e(G_1) &> t_2(n) + 1 - (n-p)(n+p+1)/8 \\ &\geq n^2/4 - (n-p)(n+p-1)/4 + (n-p)(n+p-1)/8 \\ &= p^2/4 + (n-p)/4 + (n-p)(n+p-1)/8 \\ &= p^2/4 + (n-p)(n+p+1)/8 \\ &\geq t_2(p), \end{aligned}$$

provided that $p < n$; but if $p = n$, then $e(G_1) \geq t_2(p)$ immediately. By Lemma 2.3, G_1 contains a $K(3 + K_2, c_1p)$, where we may assume that $c_1 < 1/8$. Call the two parts of this $K(3 + K_2, c_1p)$ X and Y, respectively, and let Z denote the rest of G_1. Every vertex of Y has at least $p/8 + O(1)$ edges going to Z, so that at least $p^2c_1/8 + O(p)$ edges join Y and Z. Consider the bipartite graph formed by the edges joining Y to Z. Remove all vertices from this bipartite graph with degree less than k, creating a new bipartite graph G_2 on reduced sets Y_1 and Z_1. By Lemma 2.4, this bipartite graph still has $p^2c_1/8 + O(p)$ edges.

We now claim that any two edges of G_2 are contained in some C_k. First, assume that they are adjacent. It is clear that we may extend these two edges to a P_{k-2}, where the endpoints of the path are in Y_1. Using the two adjacent vertices of X, we can extend the P_{k-2} to a C_k. Now assume that these two edges are not adjacent. Extend one of these to a P_{k-6} and the other to a (disjoint) P_3, the endpoints of each being in Y_1. Use the two adjacent vertices of X to join the P_{k-6} and P_3 into a P_{k-1} and then use the remaining vertex of X to extend this to a C_k. Therefore, for every C_k of G_1 to be TMC, all $p^2c_1/8 + O(p) \geq cn^2$ edges of G_2 must have different colors.

Now we turn to the case $k = 7$. As before, we find a graph G_1 with sets of vertices X, Y, and Z, although now we will only need the two adjacent vertices of X. Also as before, we remove all vertices of small degree in G_1, but this time we use Lemma 2.5 a second time, instead of Lemma 2.4. By Lemma 2.5, we may remove all vertices of degree $< c_2n$, where c_2 is small enough that at least c_3n vertices remain in Y_1. (For technical convenience, choose $1/c_2$ to be an integer.) Therefore, G_2 has at least $c_2c_3n^2$ edges. Again, any two adjacent edges of G_2 are in a C_7. We no longer can consider just two nonadjacent edges; instead, consider any set of $1/c_2$ such edges, and in particular, the $1/c_2$ endpoints of these edges that lie in Z_1. These each have at least c_2n edges going to Y_1, so some two of them have a common neighbor in Y_1, yielding a P_5 with endpoints in Y_1. Using the two adjacent vertices in X, this P_5 extends to a C_7.

Thus, any $1/c_2$ edges of G_2 have two that belong to a C_7. Hence, if all C_7 in G_2 are to be TMC, at least $c_2^2c_3n^2 = cn^2$ colors must be used. This completes the proof. ∎

Is it true that we can take $c = 1/8$ in Theorem 5.1? If so, this would be best possible. If may in fact be true that for all odd $k \geq 7$, if $e = t_2(n) + 1$, then $\chi_S(n, e, C_k) = (1 + o(1))(n^2/8)$.

Observe that in terms of Table 1, Theorem 5.1 covers all four ranges of e. Since $\chi_S(n, e, C_k) \leq \binom{n}{2}$ for all e, only the constant c can change for the other ranges.

We point out that for $k \geq 3$ we have

$$\chi_S(n, e, C_{2k+1}) = e - o(n^2) \quad \text{iff } e = \binom{n}{2} - o(n^2). \tag{5.1}$$

To see this, suppose on one hand that $e < \binom{n}{2}(1 - c)$ for $c > 0$. Let G be the graph $K_{\alpha n, \alpha n} \cup K_{(1-2\alpha)n}$ where $\alpha > 0$ is chosen so that $e(G) \approx e$. We can color G with $1 + \binom{(1-2\alpha)n}{2}$ colors by giving all the edges of $K_{\alpha n, \alpha n}$ the *same* color, and all the edges $K_{(1-2\alpha)n}$ *different* colors, so that all C_{2k+1}'s in G are TMC.

On the other hand, suppose $G = G(n, e)$ with $e = \binom{n}{2} - o(n^2)$. If $V(G) = \{x_1, \ldots, x_n\}$ then all but $o(n)$ of the x_k must have degree $n - o(n)$. If we c-color the edges of G and $c < (1 - \delta)e$ for $\delta > 0$, then there must be two edges e_1 and e_2 having the same color and with both endpoints of each e_i having degree $n - o(n)$. These edges can now easily be embedded into a C_{2k+1} in G, which of course is not TMC. This proves (5.1). In fact, the same argument shows that if L has two strongly independent edges (i.e., so that the four endpoints only span two edges) and $e = \binom{n}{2} - o(n^2)$, then $\chi_S(n, e, L) = e - o(n^2)$.

Of course, the situation is quite different for C_5. We can show that if $e = (1 - c)\binom{n}{2}$ then $\chi_S(n, e, C_5) < cn^2/\log n$. On the other hand, if $e > \binom{n}{2} - cn^{3/2}$ then $\chi_S(n, e, C_5) = \binom{n}{2} - o(n^2)$.

6. BIPARTITE GRAPHS

Since the Turán numbers for bipartite graphs are so small (e.g., $o(n^2)$) compared to nonbipartite graphs, it is not surprising that the general behavior of χ_S in this case is also rather different.

We begin by stating two results showing that it is important whether or not our graph L contains two *strongly independent* edges, by which we mean two disjoint edges whose four vertices span no other edges.

Theorem 6.1. Suppose L is a bipartite graph having at least two strongly independent edges, and maximum degree at least two. Then for any $\alpha > 0$, if $e > \alpha n^2$ we have

$$\chi_S(n, e, L) > \alpha' n^2 \tag{6.1}$$

for some fixed positive constant α' depending on α.

Note that "most" bipartite graphs satisfy the hypothesis of Theorem 6.1.

On the other hand, whenever L does *not* have two strongly independent edges (even when L is not necessarily bipartite) then $\chi_S(n, e, L)$ is relatively small for $e < (\frac{1}{2} - \epsilon)n^2$. A precise statement of this is the following:

Theorem 6.2. If no two edges of L are strongly independent and $e < (\frac{1}{2} - \epsilon)n^2$ for some fixed $\epsilon > 0$, then

$$\chi_S(n, e, L) = O(n^2/\log n). \tag{6.2}$$

The proofs of Theorems 6.1 and 6.2 (and their extensions) are rather lengthy and will appear in [4].

We next treat the case $L = P_k$, the path consisting of k vertices. The behavior of $\chi_S(n, e, P_k)$ depends rather strongly on whether $k = 3$, $k = 4$, or $k \geq 5$, a situation similar to the case of the odd cycles. We will treat these cases separately.

To begin with, it is easy to see that the number of colors needed to color the edges of a graph G so that all P_3's are TMC is exactly $\chi_e(G)$, the *edge-chromatic number* of G. By a well-known theorem of Vizing, this is bounded above by 1 plus the maximum degree in G. This implies that

$$\chi_S(n, e, P_3) = \frac{2e}{n} + O(1),$$

which is all we are going to say about P_3.

The case $L = P_4$ is much more substantial. Let $r_k(n)$ denote the maximum size that a subset $X \subset [n] := \{1, 2, \ldots, n\}$ can have which contains no k-term arithmetic progression.

Theorem 6.3.

$$\chi_S(n, cnr_3(n), P_4) \leq n \quad \text{for a suitable } c > 0. \tag{6.3}$$

$$\text{For any } \epsilon > 0, \quad \chi_S(n, \epsilon n^2, P_4) > cn \quad \text{for any } c \text{ if } n \text{ is sufficiently large.} \tag{6.4}$$

Proof. It was shown by Ruzsa and Szemerédi (see [10]) that for a suitable constant $c_1 > 0$, it is possible to construct a 3-uniform hypergraph $T(n)$ on $[2n]$ so that

(a) Distinct triples $T, T' \in T(n)$ have $|T \cap T'| \leq 1$.
(b) No six points in $[2n]$ span three triples in $T(n)$.
(c) $T(n)$ contains $c_1 n r_3(n)$ triples.

An easy averaging argument now implies that $[2n]$ can be partitioned into disjoint sets A, B, C with $|A| + |B| = n$, $|C| = n$ so that at least 3/16 of the triples of $T(n)$ are of the form $\{a, b, c\}$ with $a \in A, b \in B, c \in C$.

Next, we form an edge-colored bipartite graph G with vertex sets A and B which has an edge $\{a, b\}$ colored with color c iff $\{a, b, c\} \in T(n)$. It follows from (b) that every P_4 in G is TMC. Thus, (6.3) follows.

To prove (6.4), suppose $G = G(n, \epsilon n^2)$ has been cn-colored so that all P_4's are TMC, where $\epsilon > 0$ is fixed. We now reverse the preceding procedure. Namely, we first partition the vertex set of G into two sets A and B so that the bipartite subgraph G' formed by edges of the form $\{a, b\}, a \in A, b \in B$, contains at least $\frac{1}{2}\epsilon n^2$ edges; it is not hard to see that this is possible. Next, we form a hypergraph S consisting of all triples $\{a, b, c\}$ where the edge $\{a, b\}$ of G' has color c. It follows from the hypothesis that S satisfies conditions (a) and (b). However, a result from [10] implies that such a hypergraph S must satisfy $|E(S)| = o(n^2)$. This contradicts the assertion that $|E(S)| \geq \frac{1}{2}\epsilon n^2$, and the proof of (6.4) is complete. ∎

In fact, it follows from the preceding considerations that the largest value of $e = e(n)$ for which $\chi_S(n, e, P_4) \leq n$ holds satisfies

$$c_1 g(n; 6, 3) < e(n) < c_2 g(n; 6, 3)$$

where c_1 and c_2 are suitable positive constants, and $g(n; k, l)$ denotes the maximum number of triples that can be formed on $[n]$ so that no k points of $[n]$ span l triples (and, as usual, distinct triples share at most one common element).

Using a lower bound on $r_3(n)$ of Behrend [2], it follows that

$$\chi_S(n, n^2/\exp(c\sqrt{\log n}), P_4) \le n$$

for a suitable $c > 0$.

We point out here that similar considerations lead to the result

$$\chi_S(n, cg(n; 7, 4), C_4) \le n$$

for a suitable constant $c > 0$. This implies (by a remark in [10]) that

$$\chi_S(n, cr_4(n), C_4) \le n .$$

On the other hand, it is not known whether $g(n; 7, 4) = o(n^2)$, although even if this held, it is conceivable (but unlikely) that for a sufficiently small $\epsilon > 0$, we could have

$$\chi_S(n, \epsilon n^2, C_4) \le n .$$

Finally, we treat the paths P_k, $k \ge 5$. One might expect a change in behavior here since it is just these paths that have two strongly independent edges.

Theorem 6.4. For $k \ge 5$,

$$\chi_S(n, tn, P_k) = (2 + o(1))t^2, \tag{6.5}$$

for $k \le t = o(n)$ as $n \to \infty$.

Proof. Let $G = G(n, tn)$ be given, with $t = o(n)$, and for each vertex v of G, let $N_1(v)$ denote the set of edges of G that can be reached from v in at most one step, that is, edges incident with v and its neighbors. By the usual argument, we can successively delete vertices v (and their neighbors) whenever $|N_1(v)| < 2t^2$ until this is no longer possible, so that the resulting graph $G' = G'(n', t', n')$ has $t' \ge t$ and n' arbitrarily large (provided n is). By similar considerations, we can also assume that all vertices in the remaining graph also have degree at least t. Now, for any vertex v_0 in G', it is not difficult to see that if any two of the edges in $N_1(v_0)$ (restricted to G') were to have the same color, then since all vertices in G' would have degree at least $t \ge k$, a non-TMC copy of P_k would be formed. Thus, all edges in $N_1(v_0)$ must have distinct colors, which implies that at least $2t^2$ colors are needed.

On the other hand, consider the graph G_0 consisting of $n/2t$ disjoint copies of K_{2t}. Color the edges of each K_{2t} with $\binom{2t}{2}$ distinct colors, but use the same set of

$\binom{2t}{2}$ colors for each K_{2t}. Certainly, all copies of P_k (or for that matter, any connected graph) are TMC. However, G_0 has $(1 + o(1))n$ vertices and $(1 + o(1))m$ edges, and $(1 + o(1)) \cdot 2t^2$ colors are used. Thus, (6.5) follows and the theorem is proved. ∎

7. DISJOINT UNIONS OF GRAPHS

A natural class of graphs that yields a rich structure is that of disjoint unions of graphs. We use the standard notations kL and $L_1 \cup L_2$ to denote k disjoint copies of L and the disjoint union of L_1 and L_2, respectively. Our emphasis will be on unions of complete graphs, but we first study a more general case.

Theorem 7.1. If L_1 has k vertices, and if $ex(n, L_1) - ex(n, L_2) \geq kn$, then for every e,

$$\chi_S(n, e, L_1 \cup L_2) \geq \max\{\chi_S(n, e, L_1), \chi_S(n, e - kn, L_2) + e(L_1)\}.$$

Proof. Consider any $G = G(n, e)$ with $e = ex(n, L_1) + 1$. Find any copy of L_1; to TMC this L_1 requires $e(L_1)$ colors. Remove the incident edges, leaving at least $e - kn$. Suppose that only $\chi_S(n, e - kn, L_2) - 1$ new colors were available to color the remaining edges. Then some L_2 would use either a repeated color or one of the colors of the L_1. In either case, there would be an $L_1 \cup L_2$ that was not TMC. Hence, $\chi_S(n, e, L_1 \cup L_2) \geq \chi_S(n, e - kn, L_2) + e(L_1)$. On the other hand, since for every L_1 in G there is a disjoint L_2, all L_1 must be TMC. Hence, $\chi_S(n, e, L_1 \cup L_2) \geq \chi_S(n, e, L_1)$. ∎

Theorem 7.2. Let L be a $G(k, e)$ that is edge-transitive, and let $t = ex(n, L)$. Let L^* be a union of $l \geq 2$ subgraphs of L, at least one of which is L. Then if $m \geq l$,

$$\chi_S(n, t + mkn, L^*) \geq me.$$

Proof. Let G be a graph with n vertices and $t + mkn$ edges. We claim that G contains m disjoint copies of L. To see this, find an $m'L$, where m' is maximal, and remove all edges incident to this $m'L$, leaving more than $t + (m - m')kn$ edges. If $m' < m$, this remaining graph would have more than t edges, contradicting the maximality of m'. Thus, $mL \subseteq G$. If any two edges of this mL were the same color, there would be an L^* that was not TMC. Therefore, at least me colors are needed. ∎

These general theorems will be useful in the special case of disjoint unions of complete graphs, to which we now turn. It is convenient to begin by considering the upper ranges of Table 1, that is, $e = ex(n, L) + \epsilon n^2$ and $\binom{n}{2} - \epsilon n^2$.

Before beginning we mention a result that follows easily from a theorem of Moon (see [11]).

Lemma 7.1. Let $L = L_1 \cup L_2$ where $L_1 = lK_k$ and L_2 is a (possibly empty) disjoint union of graphs with chromatic number less than k. Then for n sufficiently large,

$$ex(n, L) = t_{k-1}(n - l + 1) + (l - 1)(n - l + 1) + \binom{l-1}{2}.$$

Theorem 7.3. Let L be a disjoint union of complete graphs and let x be such that $t_{x-1}(n) < e \leq t_x(n)$. Then for $n > x(x - 1)$,

$$\chi_S(n, e, L) \leq \frac{x}{2} n.$$

Proof. Let $G \subseteq T_x(n)$, the extremal graph for K_{x+1} mentioned in Section 2. $T_x(n)$ is a complete x-partite graph, with no part larger than $(n/x) + 1 < [n/(x - 1)]$. For each of the $\binom{x}{2}$ pairs of parts of G, give the bipartite graph joining them a star coloring, using different colors each time. No pair requires more than $n/(x - 1)$ colors, so that no more than $\binom{x}{2}[n/(x - 1)] = (x/2)n$ colors are needed in all to TMC all copies of L. ∎

The above theorem gives an upper bound of cn for any disjoint unions of complete graphs in all four ranges of Table 1. In the fourth range, this is sharp, as the following theorem shows:

Theorem 7.4. Let L be a disjoint union of complete graphs, the largest being K_k. If $\alpha > (k - 2)/(k - 1)$, then there are c_1 and c_2, depending only on L and α, such that, if n is large and $e = \alpha\binom{n}{2}$,

$$c_1 n \leq \chi_S(n, e, L) \leq c_2 n.$$

Proof. The upper bound comes from Theorem 7.3. The lower bound comes from Theorem 7.2. Since $ex(n, K_k) = (k - 2)/(k - 1)\binom{n}{2} + O(n)$, we have $me \geq c_1 n$ for a suitable c_1. ∎

If at least two complete graphs are smaller than the largest, cn is the right answer in all ranges.

Theorem 7.5. Let L be a disjoint union of complete graphs, where the largest is K_k and where at least two components are smaller than this. Let $e = ex(n, L) + 1$. Then there is a c such that, if n is large enough,

$$\chi_S(n, e, L) \geq cn.$$

Proof. Write L as $L_1 \cup L_2$, where $L_1 = lK_k$, and where all components of L_2 are smaller than K_k. By Lemma 7.1, $e = ex(n,L) = (k-2)/(k-1)\binom{n}{2} + O(n)$. Also by Lemma 7.1, $ex(n,L_2) \leq (k-3)/(k-2)\binom{n}{2} + O(n)$. Therefore, Theorem 7.1 gives $\chi_S(n,e,L) \geq \chi_S(n,e-lk,L_2)$, and from this, Theorem 7.4 gives $\chi_S(n,e-lk,L_2) \geq cn$ for some c, once n is large enough. ∎

By the results we have so far, we have left to consider only $L = lK_k$ and $L = lK_k \cup K_{k'}$, where $k' < k$, and only in the first three ranges of Table 1. It appears that the case $k > 3$ is different from $k = 3$; from now on we will ignore the case $k > 3$, leaving only $L = lK_3$, $l \geq 2$, and $L = lK_3 \cup K_2$, $l \geq 1$. note that the upper bound for e in the following involves $t_2(n)$, not $ex(n,L)$.

Theorem 7.6. Let L be either lK_3 or $lK_3 \cup K_2$, $l \geq 1$, and let $e \leq t_2(n) + x^2$, where $2x \leq \lceil n/2 \rceil$. Then

$$\chi_S(n,e,L) \leq 3x + 1 .$$

Proof. Let $G = K(\lceil n/2 \rceil + K_{x,x}, \lfloor n/2 \rfloor)$. We claim that $\chi_S(G,L) \leq 3x + 1$. Give the $K_{x,x}$ in the first part of G a star coloring, using x colors. Give the bipartite graph joining the vertices of this $K_{x,x}$ to the second part a star coloring, with the centers of the stars in the $K_{x,x}$, using $2x$ new colors. Finally, give all other edges one other color. We have used $3x + 1$ colors, and the only color that has an induced $K_{1,2}$ is the last one. Hence, since a graph is a disjoint union of complete graphs if it contains no induced $K_{1,2}$, only the last color could be repeated in a copy of L. But no K_3 has even one edge in the last color, so this is impossible. ∎

We note in passing that the $3x + 1$ can be improved to $3x$ when $L = lK_3$. Observe that Theorem 7.6 gives all of the upper bounds for L of the given type for the first three ranges in Table 1, except for $2K_3$ at $e = ex(n,2K_3) + 1 = t_2(n-1) + n - 1$. The next theorem takes care of this exception.

Theorem 7.7. If n is sufficiently large and $e = t_2(n-1) + n - 1$, then

$$\chi_S(n,e,2K_3) = 6 .$$

Proof. The trivial Lemma 2.1 gives $\chi_S(n,e,2K_3) \geq 6$. To get the upper bound, set $G_0 = K(\lceil (n-1)/2 \rceil + K_2, \lfloor (n-1)/2 \rfloor)$, calling the endpoints of the edge in the first part u and v. Adjoin one new vertex w to G_0 and join it to all the other vertices, calling the result G. Clearly, G has e edges. Color $\{u,v\}$ in color 1. Color all other edges incident with u and v in colors 2 and 3, respectively. Color all edges joining w to the first part of $G_0 - \{u,v\}$ in color 4, and all edges joining w to the second part of G_0 in color 5. Finally, color all remaining edges in color 6. Observe that every copy of $2K_3$ in G uses u and v in one

K_3 and w in the other. From this, it is easy to see that every copy of $2K_3$ in G has all 6 colors. ∎

Now we give a result that yields all the remaining lower bounds in Table 1 for graphs of the form under consideration.

Theorem 7.8. Let $L = lK_3 \cup K_2, l \geq 1$. Then if n is large and $e \geq \max(ex(n,L), t_2(n) + cn)$, then there is a c_1, depending only on c, such that

$$\chi_S(n,e,L) \geq c_1 n^{1/2}.$$

Proof. Let G be any $G(n,e)$, where $e \geq \max(ex(n,L), t_2(n) + cn)$. This graph contains an lK_3. Form a maximal set of vertex-disjoint triangles in G, which we will call G_0. Delete the edges of G_0 from G. In general, G will contain further triangles, all of which have some vertex in common with G_0. Successively find such triangles, deleting the edges of each as it is found, until none are left. Let G_1 denote the graph induced by the edges of all these triangles (including those in G_0); G_1 has at least cn edges, and hence was formed from at least $cn/3$ triangles. By construction, no two of these triangles have an edge in common. We claim that $\chi_S(G_1, L) \geq c_1 n^{1/2}$ for suitable c_1.

Assume that G_0 has $n^{1/2}$ triangles. Then if any two of the $3n^{1/2}$ edges have the same color, then some L will not be TMC. Hence, we can assume that G_0 has $m < n^{1/2}$ triangles. Then one of the $3m$ vertices of G_0 is contained in at least $(cn/3)/(3m) > cn^{1/2}/9$ of the triangles used in forming G_1. Let v be this vertex, let S denote the graph formed by this set of triangles at v, and let T be the triangle of G_0 that contains v; note that $T \subseteq S$. By the way G_1 was constructed, these triangles have only v in common, and no triangle of G_0, other than T, has any vertex in common with any of them. Suppose that any two independent edges in S had the same color. Then we could form an L that has these two edges, using one of them as the K_2, the other one in a triangle containing v, and as many of the triangles in $G_0 - T$ as necessary. Hence, all of the more than $cn^{1/2}/9$ independent edges of S need different colors if all copies of L are to be TMC. Therefore, the desired result is proved, with $c_1 = \min(3, c/9)$. ∎

8. OPEN QUESTIONS

We now discuss a few of the many questions raised by the preceding results. Some of these have been mentioned in earlier sections.

One of the most vexing questions is the true behavior of C_5. In the last two ranges of Table 1, the value of $\chi_S(n,e,C_5)$ is only known to within a trifle better than a factor of n, and fairly subtle methods are needed to achieve even this. There are also substantial gaps in our knowledge of the behavior of disjoint unions of complete graphs, especially when K_3 is not the largest component.

There is little doubt that the results of Section 7 could be substantially improved. In fact, it seems likely that the exact value of $\chi_S(n,e,L)$ could actually be obtained in a number of cases.

We next mention some questions concerning P_4.

(i) Is it true that

$$\chi_S(n, cnr_3(n), P_4) > c'n,$$

where $c' \to \infty$ as $c \to \infty$? If so, then by Theorem 6.2, this would be essentially best possible.

(ii) Is it true that for some $c > 0$,

$$\chi_S(n, un, P_4) < cu?$$

(iii) Is it true that for all $\epsilon > 0$,

$$\chi_S(n, \tbinom{n}{2} - n^{2-\epsilon}, P_4) > c(\epsilon)n^2?$$

It will be shown in [4] that if $e = \binom{n}{2} - n^{2-o(1)}$ then $\chi_S(n,e,P_4) = o(n^2)$. This shows that the preceding inequality, if true, would be best possible.

(iv) Observe that for $t = \epsilon n$, $\chi_S(n, tn, P_4)/t \to \infty$ as $n \to \infty$ (by Theorem 6.2). On the other hand, for t bounded, $\chi_S(n, tn, P_4)/t$ is bounded as $n \to \infty$. In fact, a result of Alon and Kahn [1] shows that for $t = (\log n)^d$, d fixed, we have $\chi_S(n, tn, P_4)/t < c(d)$. It would be of great interest to know exactly how rapidly $t = t(n)$ could grow and still have $\chi_S(n, tn, P_4)/t$ bounded. Note that by the bound of Behrend mentioned earlier, we have for suitable c, c',

$$\chi_S(n, tn, P_4) < t \exp(c\sqrt{\log t}) \quad \text{for } t < n/\exp(c'\sqrt{\log n}).$$

Returning to more general graphs, suppose L is a connected bipartite graph which is not a star. Is it then true that $\chi_S(n, \epsilon n^2, L)/n \to \infty$ as $n \to \infty$?

There are many other natural L that could be studied, but there is another way to look for new results. In Table 1, the values of $\chi_S(n,e,L)$ follow different patterns for different graphs. What patterns are possible? For instance, is it possible to have an entry of $cn^{4/3}$? Instead of starting with a graph, one might start with some pattern and look for a graph that produces that pattern. The results of Section 7 were found partly in this way. In ordinary extremal theory, the possible types of behavior can be classified according to the chromatic number of L, and this classification influences $\chi_S(n,e,L)$. Perhaps some subclassification is possible for the behavior of $\chi_S(n,e,L)$ in terms of other properties of L. We hope to return to many of these questions in a later paper.

Note. Our initial interest in this topic was motivated by a question of S. Berkowitz [3], which arose in his investigations of time/space trade-offs for

Turing machines. Specifically, he asked for bounds on the size of the largest set $S \subseteq A^3$ with $A = GF(2)^n$ so that

(i) $(\bar{x}, \bar{y}, \bar{z}) \in S \Rightarrow \bar{x} + \bar{y} + \bar{z} = \bar{0}$, where addition is componentwise modulo 2.
(ii) There do not exist $\bar{x} \neq \bar{x}', \bar{y} \neq \bar{y}', \bar{z} \neq \bar{z}'$ with

$$(\bar{x}', \bar{y}, \bar{z}) \in S,$$
$$(\bar{x}, \bar{y}', \bar{z}) \in S,$$
$$(\bar{x}, \bar{y}, \bar{z}') \in S.$$

It is easy to see in this case that S must correspond to a triple system on A^3 in which no 4 points span 2 triples and no 6 points span 3 triples. It follows from results in [10] that for a suitable $c > 0$,

$$|S| = o(n^2).$$

We do not know if $|S| > n^{2-\epsilon}$ is possible for every $\epsilon > 0$ if n is sufficiently large.

However, as noted by one of us (V.T.S.), this suggests the following attractive question:

If S is a Steiner triple system on $[n]$ then is it true that one can always select a subset S' of $n^{2-\epsilon}$ triples from S so that no 6 points span 3 triples in S'?

APPENDIX

TABLE 1. Values of $\chi_3(n, e, L)$ for Various Graphs and Ranges of e

L	$ex(n, L)$	$e = ex(n, L) + 1$	$e = ex(n, L) + cn$	$e = ex(n, L) + \epsilon n^2$	$e = \binom{n}{2} - \epsilon n$
K_k	$t_{k-1}(n)$	$\binom{k}{2}$ [3.1, 3.2]	$\binom{k}{2}$ [3.1, 3.2]	$\binom{k}{2}$ [3.1, 3.2]	c [3.1, 3.2]
C_5	$t_2(n)$	cn [4.1]	$\leq cn^{3/2}$ [4.2]	$>cn$ [4.3]	$\leq cn^2/\log n$ [4.4]
$C_7, C_9, \ldots$	$t_2(n)$	cn^2 [5.1]	cn^2 [5.1]	cn^2 [5.1]	cn^2 [5.1]
$K_3 \cup K_2$	$t_2(n)$	4 [7.6]	$cn^{1/2}$ [7.6, 7.8]	ϵn [7.6, 7.8]	cn [7.4]
$K_3 \cup kK_2$ $(k > 1)$	$t_2(n)$	cn [7.5]	cn [7.5]	cn [7.5]	cn [7.4]
$2K_3$	$t_2(n-1) + n - 1$	6 [7.7]	$\leq cn^{1/2}$ [7.6]	ϵn [7.2, 7.6]	cn [7.4]
$2K_3 \cup K_2$	$t_2(n-1) + n - 1$	$cn^{1/2}$ [7.6, 7.8]	$cn^{1/2}$ [7.6, 7.8]	ϵn [7.2, 7.6]	cn [7.4]
$2K_3 \cup kK_2$ $(k > 1)$	$t_2(n-1) + n - 1$	cn [7.5]	cn [7.5]	cn [7.5]	cn [7.4]

This table is rather schematic. We now explain how to interpret it. The reader is warned that the symbols c and ϵ are to be interpreted somewhat differently in different places, as explained below. The table has six columns. The first gives the graph L and the second gives the value of $ex(n, L)$. The function $t_k(n)$ is the Turán function $ex(n, K_{k+1})$.

The remaining four columns represent (in abbreviated fashion) estimates, bounds, or exact values for $\chi_S(n, e, L)$ in four critical ranges of e, with a reference to appropriate theorems in this paper. Often, the results in the various sections are more general than implied by the table, so for more detail, see the theorems referred to. The first column, column three, does not actually represent a range, but the particular value $ex(n, L) + 1$, the smallest e for which the entry would be greater than 1. In a few cases an exact value is given. The entry cn is to be interpreted to mean that there exist $0 < c_0 \le c_1$ such that $c_0 n \le \chi_S(n, e, L) \le c_1 n$. Note that it does *not* mean that $\chi_S(n, e, L) = cn + o(n)$ for some c. Although this is almost certainly true in all these cases, it has been proved for none of them. Entries like $cn^{1/2}$ and cn^2 are to be interpreted similarly.

Column four represents the range in which $e = ex(n, L) + cn + O(1)$, where c is any fixed positive constant. Again, an entry like cn means that in this range, $c_0 n \le \chi_S(n, e, L) \le c_1 n$, where c_0 and c_1 may (usually do) depend on c. An inequality sign in the entry, like $\le cn^{3/2}$ in the row for C_5, means that for every positive c, there is a c_1, presumably depending on c, such that in this range, $\chi_S(n, e, L) \le c_1 n^{3/2}$. In this entry, no lower bound is given. From this it may be inferred that no lower bound is known other than that in column three. Since $\chi_S(n, e, L)$ is obviously nondecreasing in e, that the entry in column three implies that $c_0 n \le \chi_S(n, e, L)$ in this range for some suitable c_0.

Column five represents the range in which $e = ex(n, L) + \epsilon n^2 + o(n^2)$, where ϵ is any sufficiently small constant. (Obviously, $\epsilon \le 1/4$, and usually it needs to be considerably smaller than this for the entry in the table to be valid.) Since $ex(n, L) = n^2/4 + o(n^2)$ for all L with chromatic number 3 [7], this could be written $n^2(1/4 + \epsilon) + o(n^2)$. The entry for C_5, $>cn$, means that there is no constant c_1 for which $\chi_S(n, e, L) \le c_1 n$ for all n and for e in such a range. When the entry is of the form cn or ϵn, it means that $\chi_S(n, e, L) \le c_1 n$ or $\epsilon_1 n$, where c_1 or ϵ_1 may depend on ϵ. However, ϵ_1 is definitely known to approach 0 as ϵ does, but this is not the case for c_1.

Finally, column six represents the range in which $e = \binom{n}{2} - \epsilon n^2 + o(n^2)$, where ϵ is any sufficiently small constant. The interpretation of each entry is similar to those in other columns. Again, the c_0 and c_1 generally depend on ϵ; they may become arbitrarily large as $\epsilon \to 0$.

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References

[1] N. Alon and J. Kahn, personal communication.

[2] F. A. Behrend, On sets of integers which contain no three in arithmetic progression, *Proc. Natl. Acad. Sci.* **23** (1946) 331–332.

[3] S. Berkowitz, personal communication.

[4] S. A. Burr, P. Erdös, P. Frankl, R. L. Graham and V. T. Sós, to appear.

[5] B. Bollobás, *Extremal Graph Theory*. Academic Press, London (1978).

[6] P. Erdös, On a theorem of Rademacher-Turán. *Illinois J. Math.* **6** (1962) 122–127.

[7] P. Erdös and M. Simonovits, A limit theorem in graph theory. *Stud. Sci. Math. Hungar.* **1** (1966) 51–57.

[8] P. Erdös and M. Simonovits, to appear.

[9] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA (1969).

[10] I. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles. *Combinatorics II*. North-Holland, Amsterdam (1978) 939–945.

[11] M. Simonovits, A method for solving extremal problems in graph theory. *Theory of Graphs*. Academic Press, New York (1968) 279–319.

[12] E. Szemerédi, Regular partitions of graphs. *Colloq. Int. C.N.R.S. No. 260, Problèmes combinatoire et théorie des graphes* (1978) 399–401.