NEARLY DISJOINT COVERING SYSTEMS

Marc A. Berger[1]
Paul Erdös[2]
Alexander Felzenbaum[1]
Aviezri S. Fraenkel[1]

[1]Faculty of Mathematical Sciences
The Weizmann Institute of Science
Rehovot 76100, ISRAEL.

[2]Mathematical Institute
The Hungarian Academy of Sciences
Budapest, HUNGARY.

ABSTRACT

We pose the problem of the existence of incongruent covering systems of residue sets, where two residue sets intersect if and only if their moduli are relatively prime. We show how such covering systems give rise to nearly disjoint cell covers of a lattice box, and thereby obtain a partial solution. In particular, we show that the number of primes dividing the ℓ.c.m. of the moduli of the residue sets of such an *incongruent* covering system must be at least five.

NOTATION

$\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_+$ the non-negative integers and $\mathbb{Q}$ the rationals. For $a, b \in \mathbb{Z}$, $< a, b >$ denotes the integer interval

$$< a, b > := \{a, a+1, \cdots, b\} .$$

(If $a > b$ this is the empty set.) An empty product is defined to be 1. The complement of the set S is denoted $\bar{S}$. $S_1 \subseteq S_2$ denotes that S_1 is a subset of S_2, and $S_1 \subset S_2$ denotes that S_1 is a strict subset of S_2 (i.e. $S_1 \subseteq S_2$ but $S_1 \neq S_2$). If $\mathcal{F}$ is a non-empty family of non-empty sets, the *derived family* $\mathcal{F}^{(1)} \supseteq \mathcal{F}$ is the family of all non-empty intersections of these sets.

This research was supported by grant No. 85-00368 from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

INTRODUCTION

A *residue set* is a set $R \subseteq \mathbb{Z}$ of the form

$$R = \{k \in \mathbb{Z} : k \equiv a(mod\ n)\} \quad ,$$

for some $a \in \mathbb{Z}$, $n \in \mathbb{N}$. This set is denoted $a(n)$. We refer to n as the *modulus* of R, and two residue sets are *congruent* if they have the same modulus. If $\mathcal{R}$ is a family of residue sets

$$\mathcal{R} = \{a_i(n_i) : i \in < 1, t >\} \ ,$$

the modulus $n = n_k$ is *division maximal*, or simply *divmax*, if it is maximal relative to division among the moduli of the sets of $\mathcal{R}$. That is,

$$n \Big| n_i \implies n = n_i \qquad (i \in < 1, t >) \ .$$

If the sets in $\mathcal{R}$ cover $\mathbb{Z}$, then $\mathcal{R}$ is a *covering system*. If the moduli n_i are all distinct then $\mathcal{R}$ is *incongruent*.

Let $\mathcal{R}$ be a family of residue sets which cover $\mathbb{Z}$ and have the property that they are *nearly disjoint* in the sense that no two distinct residue sets of $\mathcal{R}$ intersect, unless their moduli are relatively prime. Is it necessary that some of the residue sets of $\mathcal{R}$ be congruent, or can $\mathcal{R}$ be incongruent? We know of course from the Newman-Znám result that if the residue sets of $\mathcal{R}$ are pairwise disjoint, then necessarily the divmax moduli from $\mathcal{R}$ must occur repeatedly. We show that under a certain condition described below, a similar result holds for these nearly disjoint covering systems.

Recall the Bell numbers b_n, $n \geq 1$, which count the number of distinct partitions into subsets of $< 1, n >$ ($b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 15$, $b_5 = 52$, etc.). These numbers satisfy the recurrence

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_k \qquad (n \geq 0) \ ,$$

where $b_0 = 1$.

Our main result is the following

THEOREM: *Let $\mathcal{R} = \{a_i(n_i) : 1 \le i \le t\}$ be a nearly disjoint family of residue sets which cover $\mathbb{Z}$, $0(1) \notin \mathcal{R}$, and let $p_1 < \cdots < p_\ell$ be the prime divisors of the ℓ.c.m. of the moduli n_i. Assume*

$$\min_{0 \le t \le k-4} \Big(\prod_{j=1}^{t} p_j\Big) b_{k-t} < 2 + \sum_{j=1}^{k} (p_j - 2) \quad \textit{for every} \quad k \in < 5, \ell > . \tag{1}$$

Then $n_i = n_j$ for some $i \neq j$.

The repetition of a modulus in $\mathcal{R}$ derives from the repetition of a divmax modulus in the associated system $\mathcal{R}^{(1)}$. The precise statement appears below in §3.

Special cases of the Theorem

(i) If $\ell \le 4$ then (1) is satisfied vacuously – hence there is no nearly disjoint incongruent covering system, the ℓ.c.m. of whose moduli is divisible by at most 4 primes.

(ii) There is no nearly disjoint incongruent covering system, the ℓ.c.m. of whose moduli has precisely 5 distinct prime divisors, if *any* of the following conditions hold:

(a) One of the primes is ≥ 29;

(b) $p_1 \ge 7$;

(c) $p_2 \ge 11$;

(d) $p_1 = 2,\ p_2 \ge 7$.

Similar other conditions can be given.

§1. LATTICE GEOMETRY

Let $n \in \mathbb{N}$ and let $S \subseteq \mathbb{Z}^n$. For $i \in < 1, n >$ define the $i^{\underline{th}}$ *projection* $\pi_i(S)$ of S by

$$\pi_i(S) := \{y_i : \boldsymbol{y} = (y_1, \cdots, y_n) \in S\} .$$

For $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{Z}^n$ with $x_i \ge 2$ $(i \in < 1, n >)$ define the n-dimensional *lattice box*, or simply *box*

$$B = B(n; \boldsymbol{x}) := \{\boldsymbol{y} = (y_1, \cdots, y_n) : 0 \leq y_i < x_i \qquad (i \in < 1, n >)\}$$
$$= \pi_1(B) \times \cdots \times \pi_n(B) \qquad ,$$

where $\pi_i(B) = < 0, x_i - 1 >$ $(i \in < 1, n >)$. If $x_1 = \cdots = x_n = d$ then $B(n; \boldsymbol{x})$ is the *d-cube*, or simply *cube* $Q(n; d)$.

Given a box $B = B(n; \boldsymbol{x})$ let $I \subseteq < 1, n >$ and for any $i \notin I$ let u_i be any fixed integer in $\pi_i(B)$. An $I-$*cell*, or simply *cell* of B is a set of the form

$$C := \{\boldsymbol{y} = (y_1, \cdots, y_n) : 0 \leq y_i < x_i \text{ for } i \in I,\ y_i = u_i \text{ for } i \notin I\}$$

$$= \pi_1(C) \times \cdots \times \pi_n(C) \qquad ,$$

where $\pi_i(C) = \pi_i(B)$ for all $i \in I$, and $\pi_i(C) = \{u_i\}$ for all $i \notin I$. The set I is the *index* of C, denoted

$$I = Index(C) \qquad .$$

Two cells of B are *parallel* if they have the same index. The *dimension* of a cell C is

$$dim(C) :=\mid Index(C) \mid \qquad .$$

LEMMA I: *Let C_1, C_2 be cells of an n-dimensional box B.*

(i) If $C_1 \cap C_2 \neq \emptyset$ then $C_1 \cap C_2$ is a cell of B with

$$Index(C_1 \cap C_2) = Index(C_1) \cap Index(C_2) \quad .$$

(ii) If

$$Index(C_1) \cup Index(C_2) = < 1, n >$$

then $C_1 \cap C_2 \neq \emptyset$.

PROOF: Both parts of this Lemma follow from the observation that

$$C_1 \cap C_2 = (\pi_1(C_1) \cap \pi_1(C_2)) \times \cdots \times (\pi_n(C_1) \cap \pi_n(C_2)) \quad .$$

Let $\mathcal{C}$ be a family of cells of a box B. A point $\boldsymbol{y} \in B$ is *isolated* (with respect to $\mathcal{C}$) if for any other point $\boldsymbol{z} \in B$ there exists a cell in $\mathcal{C}$ which contains $\boldsymbol{y}$ but not $\boldsymbol{z}$. Equivalently $\boldsymbol{y} \in B$ is isolated if

$$\cap(C : C \in \mathcal{C},\ \boldsymbol{y} \in C) = \{\boldsymbol{y}\} \ ;$$

or equivalently if

$$\cap(Index(C) : C \in \mathcal{C},\ \boldsymbol{y} \in C) = \emptyset \ .$$

Denote the isolated points of B with respect to $\mathcal{C}$ by $Isol(B;\mathcal{C})$, or simply $Isol(B)$. The family $\mathcal{C}$ is *nearly disjoint* if whenever C_1, C_2 are distinct cells of $\mathcal{C}$ with $C_1 \cap C_2 \neq \emptyset$, then

$$Index(C_1) \cup Index(C_2) = <1, n> \ .$$

LEMMA II: *Let $\mathcal{C}$ be a nearly disjoint family of cells of an n-dimensional box B.*

(i) If $C_0, \cdots, C_t$ are distinct members of $\mathcal{C}$ with $\cap_{i=0}^{t} C_i \neq \emptyset$, then

$$Index(C_0) \supseteq \overline{Index(\cap_{i=1}^{t} C_i)} \ .$$

(ii) Suppose $B \notin \mathcal{C}$. Then to each isolated point $\boldsymbol{y} \in B$ corresponds a unique sub-family $\mathcal{C}' \subseteq \mathcal{C}$ for which

$$\cap(C : C \in \mathcal{C}') = \{\boldsymbol{y}\} \ .$$

PROOF: (i) By the nearly disjointness of $\mathcal{C}$

$$Index(C_0) \cup Index(C_i) = <1, n> \quad (i \in <1, t>) \ .$$

Thus by Lemma I(i)

$$\begin{aligned} Index(C_0) \cup Index(\bigcap_{i=1}^{t} C_i) \ &= \ Index(C_0) \cup \left[\bigcap_{i=1}^{t} Index(C_i)\right] \\ &= \ \bigcap_{i=1}^{t} [Index(C_0) \cup Index(C_i)] \ = \ <1, n> \ . \end{aligned}$$

(ii) Let $\mathcal{C}' \subseteq \mathcal{C}$ be such that

$$\cap(C : C \in \mathcal{C}') = \{\boldsymbol{y}\} \quad ,$$

and let C be any cell in $\mathcal{C}$ which contains $\boldsymbol{y}$. If $C \notin \mathcal{C}'$ then it would follow from part (i) above that $C = B$.

∎

LEMMA III: *Let $\mathcal{C}$ be a nearly disjoint family of cells of an n-dimensional box B, and let $\mathcal{C}'$ be any sub-family of $\mathcal{C}$ with*

$$C_* := \cap(C : C \in \mathcal{C}') \neq \emptyset \quad .$$

Then

$$\mathcal{C}_* := \{C \cap C_* : C \in \mathcal{C},\ C \cap C_* \neq \emptyset\}$$

is also a nearly disjoint family of cells of C_*, and

$$Isol(C_*;\mathcal{C}_*) = Isol(B;\mathcal{C}) \cap C_* \quad .$$

Furthermore if $D_1 \cap C_*$, $D_2 \cap C_*$ are distinct parallel cells of $\mathcal{C}_*$ $(D_1, D_2 \in \mathcal{C})$, then D_1, D_2 are distinct parallel cells of $\mathcal{C}$.

PROOF: To see that $\mathcal{C}_*$ is nearly disjoint observe that if $(C_1 \cap C_*) \cap (C_2 \cap C_*) \neq \emptyset$, $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, then $C_1 \cap C_2 \neq \emptyset$; and so

$$Index(C_1) \cup Index(C_2) = < 1, n > \quad .$$

Thus by Lemma I(i)

$$Index(C_1 \cap C_*) \cup Index(C_2 \cap C_*)$$

$$= \ [Index(C_1) \cup Index(C_2)] \cap Index(C_*) \ = \ Index(C_*) \quad .$$

Next, regarding the isolated points, observe that for any $\boldsymbol{y} \in C_*$

$$\cap(C : C \in \mathcal{C},\ \boldsymbol{y} \in C) = \cap(C : C \in \mathcal{C}_*,\ \boldsymbol{y} \in C) \quad .$$

Finally observe that if $D \in \mathcal{C} \setminus \mathcal{C}'$, $D \cap C_* \neq \emptyset$, then by Lemma II(i) $Index(D) \supseteq \overline{Index(C_*)}$. Now if $D_1 \cap C_*$, $D_2 \cap C_*$ are distinct parallel cells in $\mathcal{C}_*$

then $D_1, D_2 \in \mathcal{C} \setminus \mathcal{C}'$. Since $Index(D_1)$ and $Index(D_2)$ each contain $\overline{Index(C_*)}$, and since $Index(D_1 \cap C_*) = Index(D_2 \cap C_*)$, it follows from Lemma I(i) that $Index(D_1) = Index(D_2)$.

∎

LEMMA IV: *Let* $I_k,\ I'_k\ (k \in <1,t>)$ *be subsets of* $<1,n>$ *satisfying*

$$\bigcap_{k=1}^{t} I_k = \bigcap_{k=1}^{t} I'_k = \emptyset \ ,$$

$$I_j \cup I_k = I_j \cup I'_k = <1,n> \qquad (j \neq k) \ .$$

Then $I_k = I'_k \qquad (k \in <1,t>)$.

PROOF:

$$I_k \supseteq \bigcup_{j \neq k} \overline{I'_j} = \overline{\bigcap_{j \neq k} I'_j} = I'_k \ .$$

∎

PROPOSITION V: *Let $\mathcal{C}$ be a nearly disjoint family of cells of an n-dimensional box B. If*

$$|Isol(B) \cap D| > b_k$$

for some $k \in <1,n>$ and some k-dimensional cell D of B, then there are two points $\boldsymbol{y}, \boldsymbol{z} \in Isol(B) \cap D$ with the following property. Each cell of $\mathcal{C}$ containing $\boldsymbol{y}$ is parallel to a corresponding cell of $\mathcal{C}$ containing $\boldsymbol{z}$, and vice versa.

PROOF: Without loss of generality we may assume that $B \notin \mathcal{C}$, since $\mathcal{C} \setminus \{B\}$ is nearly disjoint and keeps the same points isolated. The proof proceeds by induction on $n = dim(B)$. The case $n = 1$ is easy.

Let $\mathcal{C}' = \{C \in \mathcal{C} : C \supseteq D\}$. If $\mathcal{C}' \neq \emptyset$ then we may apply Lemma III and consider $\mathcal{C}_*$ instead of $\mathcal{C}$, thereby reducing the dimension from $dim(B)$ to $dim(C_*)$. (Observe that $C_* \neq B$ since we assumed that $B \notin \mathcal{C}$.) In this case the induction hypothesis applies.

Otherwise if $\mathcal{C}' = \emptyset$ then, on account of the nearly disjointness of $\mathcal{C}$, for any isolated point $\boldsymbol{y} \in D$ the family

$$\Pi_{\boldsymbol{y}} := \{Index(D) \setminus Index(C) : C \in \mathcal{C},\ \boldsymbol{y} \in C\}$$

forms a partition of $Index(D)$. Since there are only a total of b_k partitions of $Index(D)$, it follows that there must be two isolated points $\boldsymbol{y}, \boldsymbol{z} \in D$ with

$$\Pi_{\boldsymbol{y}} = \Pi_{\boldsymbol{z}} := \{S_1, \cdots, S_t\} \quad .$$

Let C_j and C'_j be the (unique) cells of $\mathcal{C}$ containing $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively, for which

$$Index(C_j \cap D) \; = \; Index(C'_j \cap D) \; = \; Index(D) \setminus S_j \quad (j \in < 1, t >) \quad .$$

Then for $i \neq j$, since $S_i \cap S_j = \emptyset$,

$$Index(C_i \cap D) \cup Index(C'_j \cap D) \; = \; Index(D) \quad ;$$

and so by Lemma I(ii) $C_i \cap C'_j \neq \emptyset$. Thus by the nearly disjointness of $\mathcal{C}$, the sets

$$I_j = Index(C_j), \quad I'_j = Index(C'_j)$$

satisfy the hypotheses of Lemma IV. It follows from this Lemma that C_j and C'_j are parallel $\quad (j \in < 1, t >)$.

∎

COROLLARY VI: *Let $\mathcal{C}$ be a nearly disjoint family of cells of the box $B(n; \boldsymbol{x})$. If*

$$|Isol(B) \cap D| > \min_{0 \leq t \leq k-1} (\prod_{i=1}^{t} v_i) b_{k-t}$$

for some $k \in < 1, n >$ and some k-dimensional cell D of B, where $v_1 \leq v_2 \leq \cdots \leq v_k$ is a consecutive ordering of $(x_i : i \in Index(D))$, then there are two points $\boldsymbol{y}, \boldsymbol{z} \in Isol(B) \cap D$ with the following property. Each cell of $\mathcal{C}$ containing $\boldsymbol{y}$ is parallel to a corresponding cell of $\mathcal{C}$ containing $\boldsymbol{z}$, and vice versa.

PROOF: D can be partitioned into $\Pi_{i=1}^{t} v_i$ cells of dimension $k - t$.

∎

§2. CYCLIC GROUP ALGEBRA

Let $G := \mathbb{Z}/m\mathbb{Z}$ be the additive cyclic group $< 0, m-1 >$ modulo m, and let m have the prime factorization

$$m = \prod_{j=1}^{\ell} p_j^{\alpha_j} \ (p_1 < \cdots < p_\ell) \ .$$

Let B be the box

$$B := Q(\alpha_1; p_1) \times \ \cdots \ \times Q(\alpha_\ell; p_\ell) \ .$$

Observe that B is the box $B(n; \boldsymbol{x})$ where $n = \sum_{j=1}^{\ell} \alpha_j$ and

$$x_k = p_j \ \text{ for } \ \sum_{i<j} \alpha_i < k \leq \sum_{i \leq j} \alpha_i \ .$$

Recall the mapping $\Phi : G \to B$ defined in [1]. Given $u \in G$ and $j \in < 1, \ell >$ let

$$\Phi^{(j)}(u) := \boldsymbol{x}^{(j)} = (x_1^{(j)}, \cdots, x_{\alpha_j}^{(j)}) \in Q(\alpha_j; p_j)$$

be the α_j-tuple of p_j-ary coefficients for $u(mod\ p_j^{\alpha_j})$. That is

$$\Phi^{(j)}(u) = \boldsymbol{x}^{(j)} \iff u(mod\ p_j^{\alpha_j}) = \sum_{i=1}^{\alpha_j} x_i^{(j)} p_j^{\alpha_j - i} \ .$$

Then set

$$\Phi(u) := (\Phi^{(1)}(u), \cdots, \Phi^{(\ell)}(u)) \in B \ .$$

The following result, proved in [1], describes an important property of Φ.

LEMMA VII: *Φ is bijective, and if K is a coset of G, say*

$$|K| = \prod_{j=1}^{\ell} p_j^{\beta_j} \qquad (\beta_j \in < 0, \alpha_j >; \ \ j \in < 1, \ell >)$$

then $C = \Phi(K)$ is a cell of B with index

$$Index(C) = \bigcup_{j=1}^{\ell} (\sum_{i<j} \alpha_i + \{1, \cdots, \beta_j\}) \ .$$

Two cosets $K_1, K_2 \subseteq G$ are *congruent* if $|K_1| = |K_2|$. Let $\mathcal{K}$ be a family of cosets of $G = \mathbb{Z}/m\mathbb{Z}$. An element $u \in G$ is *isolated* (with respect to $\mathcal{K}$) if for any other element $v \in G$ there exists a coset in $\mathcal{K}$ which contains u but not v. Denote the isolated points of G with respect to $\mathcal{K}$ by $Isol(G; \mathcal{K})$, or simply $Isol(G)$. The family $\mathcal{K}$ is *nearly disjoint* if whenever K_1, K_2 are distinct cosets of $\mathcal{K}$ with $K_1 \cap K_2 \neq \emptyset$ then

$$\ell.c.m.(|K_1|,|K_2|) = m \quad .$$

The *covering function* $f = f_{\mathcal{K}} : G \to \mathbb{Z}_+$ is defined by

$$f(u) := |\{K \in \mathcal{K} : u \in K\}| = \text{the number of cosets in } \mathcal{K} \text{ which contain } u \quad .$$

PROPOSITION VIII: *Let $\mathcal{K}$ be a nearly disjoint family of cosets of $G := \mathbb{Z}/m\mathbb{Z}$ which cover G, and suppose $G \notin \mathcal{K}$. Then*

$$\sum_{u \in Isol(G)} (-1)^{f(u)} \omega^u = 0 \quad ,$$

where ω is a primitive $m^{\underline{th}}$ root of unity, and $f = f_{\mathcal{K}}$ is the covering function.

PROOF: It follows from Lemma VII that

$$\ell.c.m.(|K_1|,|K_2|) = m \iff Index(\Phi(K_1)) \cup Index(\Phi(K_2)) = <1, n> \quad ,$$

where n is the dimension of $B := \Phi(G)$. Then $\mathcal{C} := \Phi(\mathcal{K})$ is a nearly disjoint family of cells of the box B which cover B, and

$$Isol(B) = \Phi(Isol(G)) \quad .$$

Thus by Lemma II(ii) to every isolated point $u \in G$ there corresponds a unique sub-family $\mathcal{K}' \subseteq \mathcal{K}$ with

$$\cap(K : K \in \mathcal{K}') = \{u\} \quad .$$

For any coset $K \subseteq G$ with $|K| > 1$ we have

$$\sum_{u \in K} \omega^u = 0 \quad .$$

Since $\mathcal{K}$ covers G we can use the inclusion-exclusion principle now to write

$$\sum_{u \in Isol(G)} (-1)^{f(u)} \omega^u = -\sum_{u \in G} \omega^u + \sum_{\substack{K \in \mathcal{K} \\ |K|>1}} \sum_{u \in K} \omega^u - \sum_{\substack{K_1,K_2 \in \mathcal{K} \\ |K_1 \cap K_2|>1}} \sum_{u \in K_1 \cap K_2} \omega^u$$

$$+ \sum_{\substack{K_1,K_2,K_3 \in \mathcal{K} \\ |K_1 \cap K_2 \cap K_3|>1}} \sum_{u \in K_1 \cap K_2 \cap K_3} \omega^u \pm \cdots = 0 \quad .$$

∎

The next result is from Conway and Jones [3, Thm. 5].

LEMMA IX: *Let $U \subseteq \mathbb{Z}_+$ and $\{q_u : u \in U\} \subseteq \mathbb{Q}$ be such that $\sum_{u\in U} q_u\omega^u = 0$, where ω is a primitive $m^{\underline{th}}$ root of unity. Suppose that $0 \in U$ and that no proper subsum $\sum_{u\in U'} q_u\omega^u$ equals zero, $\emptyset \subset U' \subset U$. Then*

$$|U| \ \geq \ 2 + \sum_{p|r}(p-2) \ \ ,$$

where

$$r := \frac{m}{g.c.d.(u : u \in U)}$$

and the sum here is over the distinct prime divisors of r.

§3. PROOF OF THEOREM

Let $\mathcal{C}$ be a family of cells of a box B. The index $I = Index(C)$, $C \in \mathcal{C}$, is *subset minimal*, or simply *submin*, if it is minimal with respect to set inclusion among the indices of the cells of $\mathcal{C}$. That is,

$$C' \in \mathcal{C},\ Index(C') \subseteq I \implies Index(C') = I \ .$$

Similarly let $\mathcal{K}$ be a family of cosets of a cyclic group G. The order $n = |K|$, $K \in \mathcal{K}$, is *division minimal*, or simply *divmin*, if it is minimal with respect to division among the orders of the cosets of $\mathcal{K}$. That is,

$$K' \in \mathcal{K},\ |K'|\,\Big|\,n \implies |K'| = n \ \ .$$

Observe by Lemma VII that $n = |K|$ is divmin in $\mathcal{K}$ if and only if
$I = Index(\Phi(K))$ is submin in $\Phi(\mathcal{K})$.

THEOREM X: *Let $\mathcal{K}$ be a nearly disjoint family of cosets of $\mathbb{Z}/m\mathbb{Z}$ which cover G, and suppose $G \notin \mathcal{K}$. Let $p_1 < \cdots < p_\ell$ be the prime divisors of m. Assume*

$$\mu_k := \min_{0\leq t\leq k-1} (\prod_{j=1}^{t} p_j) b_{k-t} \ < \ 2 + \sum_{j=1}^{k}(p_j - 2) \qquad (k \in <5, \ell>) \ . \tag{2}$$

Let $n = |K_1|$ be divmin for $\mathcal{K}^{(1)}$. Then there exist two distinct congruent cosets

$K, K' \in \mathcal{K}^{(1)}$ of order n such that each coset of $\mathcal{K}$ containing K is congruent to a corresponding coset of $\mathcal{K}$ containing K', and vice versa.

PROOF: First observe that (2) automatically holds for $k \leq 4$, so that in fact the assumption of the Theorem is equivalent to

$$\mu_k \ < \ 2 + \sum_{j=1}^{k}(p_j - 2) \qquad (k \in < 1, \ell >) \ . \tag{3}$$

Let $\mathcal{C}$ be a nearly disjoint family of cells of an n-dimensional box B which cover B. Let $I = Index(C_1)$, $C_1 \in \mathcal{C}^{(1)}$, be submin in $\mathcal{C}^{(1)}$. In particular $Index(C) \supseteq I$ for any $C \in \mathcal{C}$. Define the cell

$$C_* := \{\boldsymbol{y} = (y_1, \cdots, y_n) \in B : y_i = 0 \ \text{ for } \ i \in I\} \ .$$

Observe that $Index(C_*) = \bar{I}$. Now $\mathcal{C}$ induces a nearly disjoint family of cells

$$\mathcal{C}_* := \{C \cap C_* : C \in \mathcal{C}, \ C \cap C_* \neq \emptyset\}$$

which cover C_*. Furthermore there is a one-to-one correspondence between isolated points $\boldsymbol{y} \in C^*$ with respect to $\mathcal{C}_*$ and cells in $\mathcal{C}^{(1)}$ parallel to C_1. Indeed if

$$C_* \cap (C \in \mathcal{C}, \ \boldsymbol{y} \in C) = \{\boldsymbol{y}\}$$

then

$$J = Index(\cap(C \in \mathcal{C}, \ \boldsymbol{y} \in C)) \subseteq I \ \ ,$$

and since I is submin, $J = I$. Additionally if $D_1, D_2 \in \mathcal{C}$ are such that $D_1 \cap C_*$ and $D_2 \cap C_*$ are parallel, then since $Index(D_1)$ and $Index(D_2)$ each contain I, it follows from Lemma I(i) that in fact D_1 and D_2 are parallel. Thus if we establish that C_* has two isolated points $\boldsymbol{y}$ and $\boldsymbol{z}$ with respect to $\mathcal{C}_*$, for which the cells of $\mathcal{C}_*$ containing them correspond and are parallel one to another, then it will follow that $\mathcal{C}^{(1)}$ contains two I-cells with this same property relative to $\mathcal{C}$.

In our case let $\mathcal{C} := \Phi(\mathcal{K})$ be the family of cells of the box $B := \Phi(G)$ which correspond to the cosets of $\mathcal{K}$. Then $I_1 = Index(\Phi(K_1))$ is submin in $\mathcal{C}^{(1)} = \Phi(\mathcal{K}^{(1)})$. By Lemma VII, restricting to the cell C_* defined above corresponds to restricting to the quotient $G/S_1 \cong \mathbb{Z}/m_1\mathbb{Z}$, where S_1 is the subgroup congruent to K_1 and $m_1 = m/|K_1|$. Thus by restricting to the cyclic group $G_1 = G/S_1$ we may assume that K_1 is a singleton. In other words it suffices to prove our Theorem here for the special case where K_1 is an isolated singleton. Furthermore by shifting the cosets in $\mathcal{K}$ all by a fixed amount we may even assume that $K_1 = \{0\}$. So let us make that assumption now!

Next, as in [2], let S_* be the subgroup of G with

$$|S_*| = \prod_{j=1}^{\ell} p_j \quad .$$

Then $\mathcal{K}$ induces a nearly disjoint family of cosets of S_*,

$$\mathcal{K}_* := \{K \cap S_* : K \in \mathcal{K},\ K \cap S_* \neq \emptyset\}$$

which cover S_* . Let $m = |G|$ have the prime factorization

$$m = \prod_{j=1}^{\ell} p_j^{\alpha_j} \quad .$$

Suppose the element $u \in S_*$ is isolated with respect to $\mathcal{K}_*$; i.e.

$$S_* \cap (K : K \in \mathcal{K},\ u \in K) = \{u\} \quad .$$

Then we claim that on account of the nearly disjointness of $\mathcal{K}$, the cosets $K \in \mathcal{K}$ which contain u have the following special property:

$$(P) \qquad\qquad \text{If} \quad p_j \,\Big|\, |K| \ \text{ then } \ p_j^{\alpha_j} \,\Big|\, |K| \ .$$

In other words if $|K \cap S_*| = \Pi_{j \in J} p_j$ for some $J \subseteq < 1, \ell >$ then $|K| = \Pi_{j \in J} p_j^{\alpha_j}$. In particular if two cosets $L_1 \cap S_*,\ L_2 \cap S_* \in \mathcal{K}_*$ containing isolated points of S_* are congruent, then in fact the cosets $L_1, L_2 \in \mathcal{K}$ are themselves congruent.

To see why (P) holds suppose $K \in \mathcal{K}$ contains the isolated point $u \in S_*$. If $p_j \nmid |K \cap S_*|$ then $p_j \nmid |K|$, and so, by the nearly disjointness, any other coset L of $\mathcal{K}$ containing u must be such that $p_j^{\alpha_j} \,\Big|\, |L|$.

The upshot of this is that it suffices now to show that there are two isolated points of S_* with respect to $\mathcal{K}_*$, for which the cosets of $\mathcal{K}$ containing them correspond and are parallel one to another. In other words it suffices to prove our Theorem here for numbers $m = \Pi_{j=1}^{\ell} p_j$ which are square-free. So let us make that assumption now!

In summary, then, it suffices to prove our Theorem for the special case where

(i) $m = \Pi_{j=1}^{\ell} p_j$ is square-free,

(ii) $0 \in G$ is an isolated point.

The rest is quick! According to Proposition VIII

$$\sum_{u \in Isol(G)} (-1)^{f(u)} \omega^u = 0 \quad ,$$

where ω is a primitive $m^{\underline{th}}$ root of unity, and $f = f_{\mathcal{K}}$ is the covering function. Let $\sum_{u \in U} (-1)^{f(u)} \omega^u$ be a minimal subsum which also equals zero, $0 \in U \subseteq Isol(G)$. This polynomial then satisfies the hypotheses of Lemma IX.

Suppose the conclusion of our Theorem were false. If

$$r := \frac{m}{g.c.d.(u : u \in U)} = \prod_{j \in J} p_j, \quad |J| = k$$

then the isolated points $\{\Phi(u) : u \in U\}$ all lie in a k-dimensional cell of

$$B := \Phi(G) = B(\ell; (p_1, \cdots, p_\ell)) \quad .$$

Thus by Corollary VI

$$|U| \leq \mu_k \quad .$$

On the other hand by Lemma IX

$$|U| \ \geq 2 + \sum_{j \in J} (p_j - 2) \ \geq \ 2 + \sum_{j=1}^{k} (p_j - 2) \quad .$$

Regardless of what $k \in < 1, \ell >$ is, though, this conflicts with (3).

∎

REMARK: Observe that the minimum in the expression for μ_k in (2) can be (slightly) simplified to

$$\mu_k = \min_{0\le t\le k-4}\left(\prod_{j=1}^{t} p_j\right) b_{k-t}$$

(with $k-1$ replaced by $k-4$). This is because

$$p_t b_{k-t} > b_{k-t+1} \text{ for } k \ge 5\,,\ \ t \in \langle k-3, k-1\rangle\ .$$

We use this observation in the statement of Theorem XI below and the Theorem in the Introduction.

From Theorem X follows the Theorem in the Introduction. In fact we can say something about *which* moduli are necessarily repeated.

THEOREM XI: *Let $\mathcal{R}$ be a nearly disjoint family of residue sets which cover $\mathbb{Z}$, $\mathbb{Z} \notin \mathcal{R}$, and let $p_1 < \ldots < p_\ell$ be the prime divisors of the l.c.m. of the moduli of the sets of $\mathcal{R}$. Assume*

$$\min_{0\le t\le k-4}\left(\prod_{j=1}^{t} p_j\right) b_{k-t} < 2 + \sum_{j=1}^{k}(p_j-2) \ \ \textit{for every}\ \ k \in \langle 5, \ell\rangle\ .$$

Let n be any divmax modulus of $\mathcal{R}^{(1)}$. Then there exist two distinct congruent sets $R, R' \in \mathcal{R}^{(1)}$ of modulus n such that each set of $\mathcal{R}$ containing R is congruent to a corresponding set of $\mathcal{R}$ containing R', and vice versa.

The conclusion here means that we can label all the sets $\{R_1, \ldots, R_s\}$ of $\mathcal{R}$ which contain R, and all the sets $\{R'_1, \ldots, R'_s\}$ of $\mathcal{R}$ which contain R' so that R_i is congruent to R'_i $(i \in \langle 1, s\rangle)$. In particular it will follow that these two families consist of the same number, s, of sets. It may be that $R_i = R'_i$ for some i, but this cannot be the case for all $i \in \langle 1, s\rangle$, since R and R' are distinct.

For example consider the nearly disjoint covering system

$$\mathcal{R} = \{0(2), 0(3), 1(4), 3(4)\}\ .$$

Then

$$\mathcal{R}^{(1)} = \{0(2), 0(3), 0(6), 1(4), 3(4), 3(12), 9(12)\}\ ,$$

and the divmax modulus $n = 12$ is repeated: $R = 3(12)$, $R' = 9(12)$. There are $s = 2$ sets of $\mathcal{R}$ containing R: $R_1 = 0(3)$, $R_2 = 3(4)$. Likewise there are 2 sets of

$\mathcal{R}$ containing R': $R_1' = 0(3)$, $R_2' = 1(4)$. The sets R_1, R_1' are congruent (in fact equal); and the sets R_2, R_2' are also congruent. In particular $\mathcal{R}$ contains the two distinct congruent sets R_2 and R_2'.

PROOF: Let $\mathcal{R} = \{a_i(n_i) : i \in \langle 1, t\rangle\}$, and set $m = \ell.c.m.(n_1, \ldots, n_t)$. Then

$$\mathcal{R} \cap G := \{a_i(n_i) \cap G : i \in \langle 1, t\rangle\}$$

is a nearly disjoint family of cosets of $G := \mathbb{Z}/m\mathbb{Z}$. Furthermore $G \notin \mathcal{R} \cap G$. Observe that

$$(\mathcal{R} \cap G)^{(1)} = \mathcal{R}^{(1)} \cap G ;$$

and that if $n|m$ then

$$|a(n) \cap G| = m/n ,$$

implying that residue sets of $\mathcal{R}^{(1)}$ with divmax moduli corresponding to cosets of $(\mathcal{R} \cap G)^{(1)}$ with divmin order. Apply Theorem X, then, with $\mathcal{K} = \mathcal{R} \cap G$ to arrive at the desired conclusion.

∎

REFERENCES

1. Berger, M.A., Felzenbaum, A. and Fraenkel, A.S., A non-analytic proof of the Newman-Znám result for disjoint covering systems, *Combinatorica* 6 (1986), 235-243.

2. Berger, M.A., Felzenbaum, A. and Fraenkel, A.S., Irreducible disjoint covering systems (with an application to Boolean algebra), (to appear).

3. Conway, J.H. and Jones, A.J., Trigonometric diaphantine equations (on vanishing sums of roots of unity), *Acta Arith.* 30 (1976), 229-240.