How to Define an Irregular Graph

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I have lived in Michigan nearly all my life, having been brought up in Sault Ste. Marie, educated at Michigan State University, studying at the University of Michigan, and employed by Western Michigan University. I've enjoyed graph theory from the day I heard my eventual thesis advisor E. A. Nordhaus speak on it. I've had eleven doctoral students, the most recent being Ortrud Oellermann, and the two of us are now very pleased to have Erdös number 1. I enjoy musicals and many sports.

The misfortune of birth overtook me on March 26, 1913. My parents were both mathematicians, and I learned a great deal from them. I got my Ph.D. at the University of Budapest in 1934. I spent the years 1938-1948 in Manchester, England, and I have been in the United States since 1954. I travel constantly around the world: Hungary, Israel, the U.S.A., England, etc. My subjects are number theory, combinatorics, set theory and probability.

I was born in Vryheid, South Africa, and have been interested in mathematics since childhood. After receiving my predoctoral degrees from the University of Natal in Durban, I came to the United States to study graph theory at Western Michigan University, where I received my Ph.D. in 1986, and where I am now on the faculty. I enjoy playing the piano, violin and recorder. I speak German, Afrikaans and Zulu, in addition to English.

There are several curious real-life puzzles and problems that can be translated into the language of graph theory, where they are often found to have interesting solutions. One such problem is to show that in any group of $n \geq 2$ people, there are always at least two people who have the same number of acquaintances in the group. A graph G with n vertices can be constructed to model this situation by associating each vertex of G with one person in the group, and joining two vertices by an edge if and only if the corresponding people are acquainted. In the resulting graph, the *degree* of a vertex v is the number of vertices adjacent to v. The original problem may now be rephrased in terms of graphs as follows: Show that in any graph G with $n \geq 2$ vertices, there are at least two vertices having the same degree.

To solve this problem, notice first that the degree of a vertex in a graph G with n vertices is one of the numbers $0, 1, 2, \ldots, n-1$. If all the degrees of the vertices of G were distinct, then each of these n numbers has to occur exactly once as the degree of a vertex in G. However, if v is the vertex of G that has degree $n-1$, then v is adjacent to every other vertex of G, including the vertex having degree 0, which is impossible.

If all people in the group had the same number of acquaintances, then the corresponding graph would be called *regular*. Later in this article, we will look at some properties of regular graphs and investigate graphs which are, in some sense, opposite to regular graphs.

We now introduce a few definitions that will be useful in what follows. A graph with n vertices is called *complete*, and is denoted by K_n, if every two vertices are adjacent. The five smallest complete graphs are shown in Figure 1.

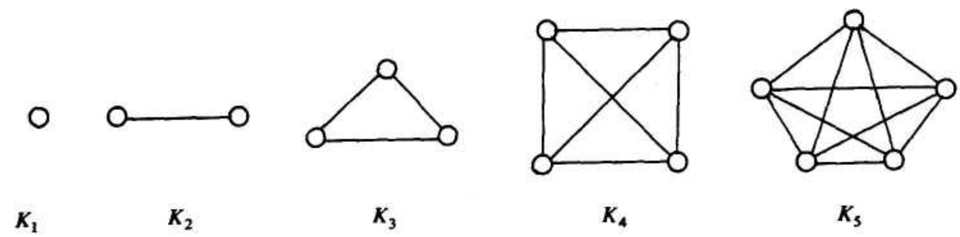

Figure 1. Five complete graphs.

The *degree set* $\mathscr{D}(G)$ of a graph G is the set of degrees of the vertices of G. For example, $\mathscr{D}(K_n) = \{n-1\}$, while $\mathscr{D}(G_1) = \{1, 2, 3, 4\}$ and $\mathscr{D}(G_2) = \{0, 1, 2, 3\}$ for the graphs G_1 and G_2 of Figure 2.

A *path* between vertices u and v is a sequence of distinct vertices, consecutive ones of which are adjacent, beginning with u or v and ending with the other, together with the resulting edges. A graph G is *connected* if there is a path between every two vertices of G. The graph G_1 of Figure 2 is connected, while G_2 is not connected (disconnected) since there is no path in G_2 between v_3 and any vertex $v \in \{v_1, v_2, v_4, v_5\}$.

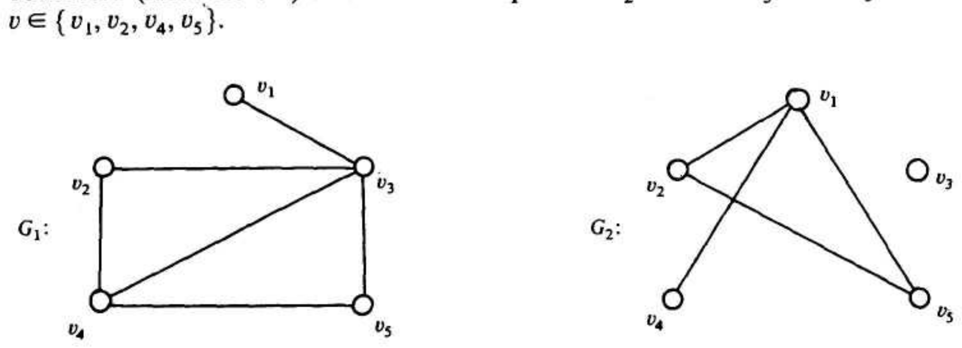

Figure 2. Complementary graphs.

The *complement* $\bar{G}$ of a graph G is that graph having the same vertices as G, with vertices u and v being adjacent in $\bar{G}$ if and only if u and v are not adjacent in G. Notice that $G_2 = \bar{G}_1$ in Figure 2.

As mentioned previously, a graph G is *regular* if all of its vertices have the same degree. (Equivalently, a graph G is regular if $\mathscr{D}(G)$ consists of a single element.) If this common degree is r, then G is *r-regular*. For an r-regular graph with n vertices, we have $0 \leq r \leq n-1$.

The sum of the degrees of the vertices of a graph is always even since this sum counts each edge twice. So, if G is an r-regular graph with n vertices, then rn is even. On the other hand, if either r or n is an even integer with $0 \leq r \leq n-1$, then there is an r-regular graph with n vertices. A 4-regular graph with 7 vertices is shown in Figure 3. The reader may find it interesting to construct some r-regular graphs of order n.

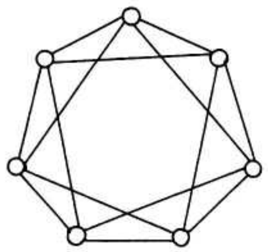

Figure 3. A 4-regular graph with 7 vertices.

Although it may seem a bit rare for a graph to be regular, these graphs occur with surprising 'regularity.' In 1936, in the first book ever written on graph theory, Denes König [5] showed that for every graph H, there exists an r-regular graph G, where r is the largest degree among the vertices of H, such that H is an *induced subgraph* of G (i.e., G is obtained by adding vertices to H in such a way that every edge of G is either an edge of H or is incident with at least one vertex of G which is not in H). For example, consider the graph H with degree set $\mathscr{D}(H) = \{2, 3, 4\}$ in Figure 4. If we take two copies of H and join corresponding vertices in the two copies whose degrees are less than 4, then we produce the graph F with $\mathscr{D}(F) = \{3, 4\}$. Repeating this process with F gives a 4-regular graph G. This procedure can then be used to prove König's result.

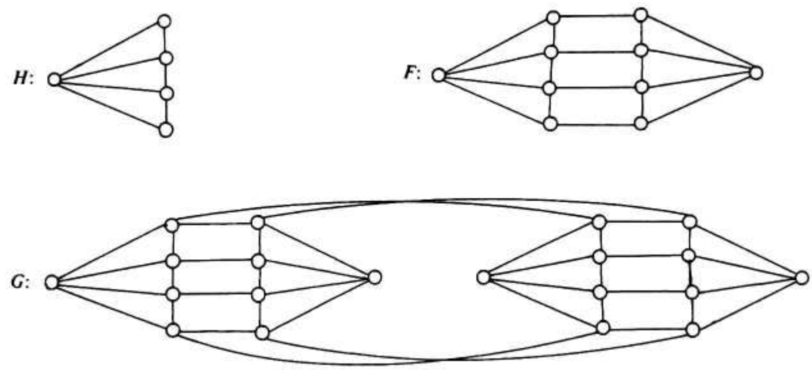

Figure 4. Constructing regular graphs.

The Search for "Irregularity." The results mentioned above constitute only a very small sample of what's known about regular graphs. We now propose a question and, at the same time, illustrate how one might go about introducing a new concept that can become a topic for future research. The question is:

Which class of graphs is opposite to the regular graphs?

Or, perhaps, we may rephrase this as:

How should one define an irregular graph?

Here, of course we are taking "irregular" as an antonym for "regular"—it is *not* meant as a synonym for "nonregular." In research, the goal is not only to come up with a definition that seems natural but to arrive at a class of graphs with interesting, and perhaps even some surprising, properties.

Probably the most obvious candidate for the definition of an irregular graph is a graph whose vertices have distinct degrees. However, we have already seen that no graph (with at least two vertices) has this property. We might vary this definition a little and define a graph G with n vertices to be irregular if the vertices of G have distinct degrees, except for one pair (i.e., G is irregular if $|\mathscr{D}(G)| = n - 1$). We know such types of graphs exist since each of the complementary graphs, G_1 and G_2, of Figure 2 satisfies this definition. However, this is not an ideal way to define an irregular graph, for it was proved in [2] that there is only one connected graph G with $n \geq 2$ vertices such that $|\mathscr{D}(G)| = n - 1$, and that the only other graph with n vertices and having a degree set of cardinality $n - 1$ is $\overline{G}$. The reader might enjoy constructing these graphs G and determining which degree is repeated. Further, such graphs have been completely determined, and so defining these graphs to be irregular does not produce any new problems of interest to study.

In order to consider another approach to define an irregular graph, we introduce the idea of distance. The *length* of a path in a graph is the number of edges in it. For distinct vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the smallest length of all paths in G between u and v. For example, in the graph G_1 of Figure 2, we have $d(v_1, v_3) = 1$ and $d(v_1, v_2) = 2$. Note that in any graph, $d(u, v) = 1$ if and only if u and v are adjacent. This, and the idea of distance degree regular graphs defined by Bloom, Kennedy, and Quintas [3], suggests another way to look at regular graphs and, possibly, to define irregular graphs.

A graph G is regular if the same number of vertices are at distance 1 from each vertex of G. For appropriate integers k, we could define a graph G to be *distance-k regular* if every vertex of G has the same number of vertices at distance k from it. Thus, distance-1 regular graphs and regular graphs are the same thing. The graph G of Figure 5 is distance-2 regular (though it's not regular) since there are exactly two vertices at distance 2 from each of its vertices.

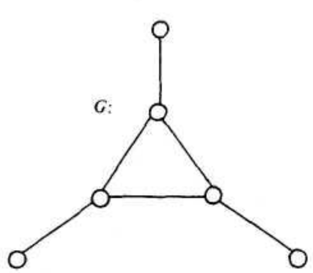

Figure 5. A distance-2 regular, nonregular graph.

The preceding discussion suggests defining a connected graph G with $n \geq 2$ vertices to be *distance-k irregular* (k, a positive integer) if for every two distinct vertices u and v of G, the number of vertices at distance k from u is different from the number of vertices at distance k from v. We have already noted that no graph is distance-1 irregular. For $k \geq 2$, no graph is distance-k irregular either. To see this, suppose that G is a distance-k irregular graph with n vertices. Then G must contain a vertex u having no vertices at distance k from it, and a vertex v such that $n-1$ vertices are at distance k from v. But then all other vertices of G, including u, are at distance k from v, and a contradiction is produced.

We have suggested some possible definitions of an irregular graph, but for various reasons, all attempts to define these graphs have been unsatisfactory. Now, however, we will describe a definition with some potential.

Let G be a regular graph. Then all the neighbors of v (the vertices adjacent to v) have degree r if G is r-regular. This means that every regular graph is *locally regular*. This suggests yet another way to define an irregular graph. First, define a graph G to be *locally irregular* if for each vertex v of G the neighbors of v have distinct degrees. These graphs *do* exist. Some locally irregular graphs are shown in Figure 6.

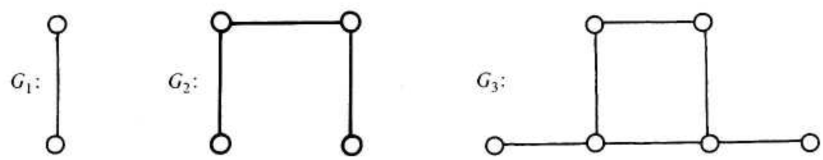

Figure 6. Three locally irregular graphs.

Locally irregular graphs that are also connected have been defined and studied recently in [1], and have been referred to as *highly irregular graphs*. There are surprisingly many highly irregular graphs, as was pointed out in [1].

Let's look at a few properties of these graphs. Let G be a highly irregular graph whose largest degree is d. If v is a vertex of degree d in G, then (since all the neighbors of v have different degrees) the degrees of the neighbors of v must be $1, 2, \ldots, d$. Suppose u is the neighbor of v that has degree d. Then u too must have d neighbors with different degrees. Further, since G is highly irregular, u and v cannot have a common neighbor w since u and v would then be two neighbors of w with the same degree. Therefore, if G has n vertices, then $n \geq 2d$. Thus, the largest degree in a highly irregular graph is at most half the number of vertices.

From a highly irregular graph G, we can construct another highly irregular graph H having $2n$ vertices by taking two copies of G and joining a vertex of degree d in one copy to a vertex of degree d in the other. In [1], it was proved that there exist highly irregular graphs with n vertices for every positive integer n except 3, 5, and 7. There are only two connected graphs with three vertices, and neither of these is highly irregular since each vertex of degree 2 has two neighbors with the same degree.

Let's see next why there is no highly irregular graph with 5 vertices. Suppose that there is such a graph G. By our earlier observation about largest degrees, the maximum degree d must satisfy $d \leq 5/2$, so $d \leq 2$. Since G is connected, we cannot have $d \in \{0, 1\}$. Hence, $d = 2$. Let u be a vertex of degree 2 in G. Then one neighbor w of u must have degree 1, and the other neighbor v of u must have degree 2. Similarly, v is adjacent to u and to a vertex of degree 1. But this implies that G has only four vertices, which is a contradiction.

We leave it as an interesting exercise to verify that there is no highly irregular graph with 7 vertices.

As mentioned earlier, König proved that if H is a graph whose largest degree is r, then there is an r-regular graph G containing H as an induced subgraph. In [1], it was shown that for every graph H there is a highly irregular graph G containing H as an induced subgraph. To illustrate the proof of this analogue, consider the non-highly irregular graph H in Figure 7. Taking two copies H_1 and H_2 (as indicated in Figure 7), we join each pair v_i and v_i' $(i = 1, 2, 3)$ but not v_4 and v_4'. Further, each vertex v_i of H_1 is joined to the vertices of H_2 which correspond to the vertices of H_1 that are not adjacent to v_i. The construction of the desired highly irregular graph G is completed by adding four new vertices u_1, u_2, u_1', u_2', and joining u_1 to v_1; u_1' to v_1'; u_2 to v_1 and v_2; and u_2' to v_1' and v_2'.

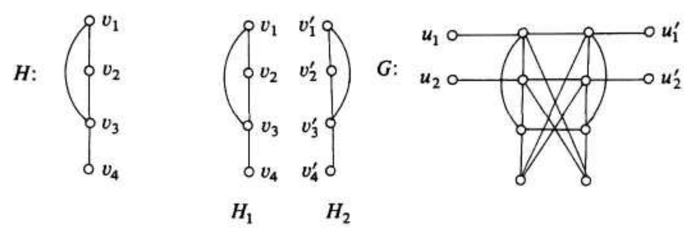

Figure 7. Constructing a highly irregular graph.

The class of highly irregular graphs seems to be sufficiently numerous and diverse so as to be an appropriate answer to our question posed in the title. However, before we conclude, let us mention one other alternative. The concept of the degree of a vertex in a graph G can be generalized in the following way: For a given graph F, the *F-degree* of a vertex v in G is the number of subgraphs of G, isomorphic to F, to which v belongs. Consequently, the ordinary degree of a vertex v is the K_2-degree of v. (The complete graph K_2 was shown in Figure 1.)

A graph G is *F-regular* of degree r if every vertex of G has F-degree r. The graph G of Figure 9 is K_3-regular of degree 3, but it is not regular. Let P_3 denote the graph that is a path with three vertices (see Figure 9). It was shown in [4] that a graph G is P_3-regular of degree at least 2 if and only if G is regular.

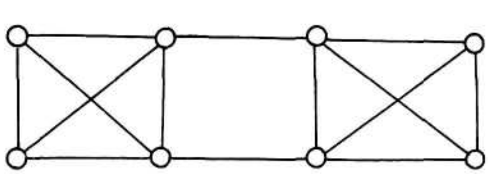

Figure 8. A k_3-regular graph.

The concept just introduced suggests yet another type of irregular graph. For a graph F, let a graph G be called *F-irregular* if the vertices of G have distinct F-degrees. Since we have seen earlier that no graph with at least two vertices is distance-1 irregular, no such graph is F-irregular for $F = K_2$. However, this is not the case for *all* choices of F; for example, the graph G of Figure 9 is P_3-irregular. (Each vertex of this graph is labeled with its P_3-degree.)

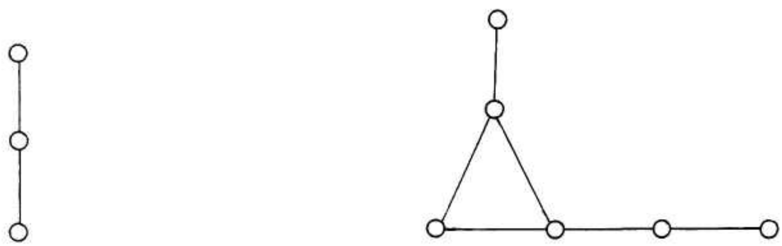

Figure 9. A P_3-irregular graph G.

P. Erdös, L. Székely, and W. T. Trotter have shown the existence of infinitely many K_3-irregular graphs, while it was shown in [4] that K_n-irregular graphs exist for every $n \geq 3$. In fact, we believe that F-irregular graphs exist for every connected graph F with at least three vertices. We noted earlier that there is no regular, P_3-irregular graph. This suggests a question, whose answer is unknown to us: *Do regular, K_3-irregular graphs exist*?

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The statistician is no longer an alchemist expected to produce gold from any worthless material offered him. He is more like a chemist capable of assaying exactly how much of value it contains, and capable also of extracting this amount, and no more. In these circumstances, it would be foolish to commend a statistician because his results are precise or to reprove because they are not. If he is competent in his craft, the value of the result follows solely from the value of the material given him. It contains so much information and no more. His job is only to produce what it contains.

R. A. Fisher