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The Asymptotic Number of Graphs not Containing a Fixed Subgraph and a Problem for Hypergraphs Having No Exponent

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Abstract. Let H be a fixed graph of chromatic number r. It is shown that the number of graphs on n vertices and not containing H as a subgraph is $2^{\binom{n}{2}(1-\frac{1}{r-1}+o(1))}$. Let $h_r(n)$ denote the maximum number of edges in an r-uniform hypergraph on n vertices and in which the union of any three edges has size greater than $3r-3$. It is shown that $h_r(n) = o(n^2)$ although for every fixed $c < 2$ one has $\lim_{n\to\infty} h_r(n)/n^c = \infty$.

1. Introduction

Let H be an arbitrary graph, $|H|$ denotes the number of edges of H. Let $T_n(H)$ denote the Turán number of H, i.e., the maximum number of edges which a graph on n vertices and not containing H as a subgraph may have. Let X be an n-element set and let $X = X_1 \cup \cdots \cup X_r$ be an arbitrary partition of X. The *complete r-partite* graph $K(X_1, \ldots, X_r)$ consists of all edges connecting distinct X_i and X_j. Note that this graph contains no K_{r+1} and has chromatic number r if $X_i \neq \emptyset$, $i = 1, \ldots, r$. To maximize $|K(X_1, \ldots, X_r)|$ one chooses the X_i to have as equal sizes as possible, i.e., $\left\lfloor \frac{n}{r} \right\rfloor \leq |X_i| \leq \left\lceil \frac{n}{r} \right\rceil$. Then Turán's theorem states

Theorem 1.1. [23] $T_n(K_{r+1}) = |K(X_1, \ldots, X_r)| = \binom{n}{2}\left(1 - \frac{1}{r} + o(1)\right)$. Taking all subgraphs of $K(X_1, \ldots, X_r)$ one obtains $2^{T_n(K_{r+1})}$ distinct labeled graphs on n vertices without K_{r+1}.

Definition 1.2. Call a graph *H-free* if it contains no subgraph isomorphic to H. Let $F_n(H)$ denote the number of distinct labeled H-free graphs on n vertices. Extending earlier results of Erdös, Kleitman and Rothschild [8] Kolaitis, Prömel and Rothschild proved that the number of K_{r+1}-free graphs is asymptotic to the number of the r-partite graphs. This in particular implies

Theorem 1.3. [10]

$$F_n(K_r) = 2^{T_n(K_r)(1+o(1))}. \tag{1}$$

Let $\chi(H)$ denote the chromatic number of H. An old result of Erdös, Stone and Simonovits shows that $T_n(H)$ and $T_n(K_{\chi(H)})$ are closely related.

Theorem 1.4. [7, 9] *Set* $\chi(H) = r$, $r \geq 3$. *Then*

$$T_n(K_r) \leq T_n(H) \leq (1 + o(1))T_n(K_r). \tag{2}$$

Our first result extends (2).

Theorem 1.5. *Let* ε_0 *be an arbitrary positive number and* G *an* H*-free graph on* n *vertices. Then for* $n > n_0(\varepsilon_0, H)$ *one can remove less than* $\varepsilon_0 n^2$ *edges from* G *so that the remaining graph is* K_r*-free, where* $r = \chi(H)$.

This may be further extended in the following way: Let H_1, H_2 be two graphs. A mapping ψ: $V(H_1) \to V(H_2)$ is called a *homomorphism* if $\{x, y\} \in E(H_1)$ implies $\{\psi(x), \psi(y)\} \in E(H_2)$. Note that $\psi^{-1}(x)$ is an independent set for all $x \in V(H_2)$. Also if $\chi(H_1) = r$ then r is the smallest integer for which there exists a homomorphism ψ: $H_1 \to K_r$.

The following is a slight generalization of Theorem 1.5:

Theorem 1.5'. *Suppose that* H_2 *is a homomorphic image of* H_1, ε_0 *is an arbitrary positive real and* G *is an* H_1*-free graph with* n *vertices. Then for* $n > n_0(\varepsilon_0, H_1)$ *it is possible to remove at most* $\varepsilon_0 n^2$ *edges from* G *so that the remaining graph is* H_2*-free.* □

We do not include the proof here, it uses an argument very similar to that of the proof of Theorem 1.5. Note also that some stronger statements of the same flavor were obtained by Rödl [19]. The present proof is similar. Theorem 1.5. is shown to imply easily:

Theorem 1.6. *Suppose* $\chi(H) = r \geq 3$. *Then*

$$F_n(H) = 2^{T_n(K_r)(1+o(1))}. \tag{3}$$

Note that for $H = K_r$ (3) is much weaker than (2) and this special case was already proved in [8].

It seems likely that

$$F_n(H) = 2^{T_n(H)(1+o(1))}$$

holds for bipartite H as well. However, this is not even known for $H = C_4$, the cycle of length 4. For this case the best known upper bound $(2^{cn^{3/2}})$ is due to Kleitman and Winston [15].

Our last but probably most interesting result concerns r-uniform hypergraphs. Recall that an r-uniform hypergraph is simply a collection of distinct r-element sets, called edges. Let $g_n(v, e, r)$ denote the maximum number of edges in a r-uniform hypergraph on n vertices in which the union of any e edges has size greater than v (i.e., no v vertices span e or more edges).

Theorem 1.7. *Suppose* $r \geq 3$. *Then the following hold.*

$$g_n(3r - 3, 3, r) = o(n^2), \tag{4}$$

$$\lim_{n\to\infty} g_n(\text{[illegible]}) \text{[illegible]} = \infty \quad \text{for all [illegible]} < 2. \qquad (5)$$

Our proof of (4) is based on Szemerédi's uniformity lemma [22].

Let us mention that the special case $r = 3$ of (4) and (5) is a celebrated result of Ruzsa and Szemerédi [21]. However, the present proof is much simpler and probably more insightful. In [21] it is shown that $g_n(6,3,3) > nr_3(n)/100$ where $r_3(n)$ is the maximum size of a subset $A \subset \{1, 2, \ldots, n\}$ which contains no arithmetical progression of length 3. Thus (4) implies $r_3(n) = o(n)$ which was proved in a stronger form by Roth [20].

Let $G = (V, E)$ be a graph and $A, B \subset V$ be a pair of disjoint subsets of V. The *density of a pair* (A, B) is the fraction $d(A,B) = e(A,B)/|A||B|$ where $e(A,B)$ is the number of edges with one endpoint in A and second in B and $|A|$, $|B|$ denote the cardinalites of A and B, respectively. The *pair* (A, B) is called *ε-uniform* if for every $A' \subset A$, $B' \subset B$, $|A'| > \varepsilon|A|$, $|B'| > \varepsilon|B|$ $|d(A',B') - d(A,B)| < \varepsilon$ holds. The partition $V = C_0 \cup C_1 \cup \cdots \cup C_k$ is called ε-uniform if

i) $|C_0| < \varepsilon|V|$

ii) $|C_1| = |C_2| = \cdots = |C_k|$

iii) all but $\varepsilon\binom{k}{2}$ of the pairs (C_i, C_j) are ε-uniform, $1 \le i < j \le k$.

Uniformity Lemma [22]. *For every $\varepsilon > 0$ and positive integer ℓ, there exist positive integers $n_0(\varepsilon,\ell)$ and $m_0(\varepsilon,\ell)$ such that every graph with at least $n_0(\varepsilon,\ell)$ vertices has an ε-uniform partition into k classes, where k is an integer satisfying $\ell < k < m_0(\varepsilon,\ell)$.* □

Another simple proof of $g_n(6,3,3) = o(n^2)$ (which is also based on [22]) was independently found by E. Szemerédi.

2. Proof of Theorem 1.5.

Without loss of generality assume that $\varepsilon_0 < 1/r$ and set $\ell = \lceil 1/\varepsilon_0 \rceil$, $\varepsilon = (\varepsilon_0/6)^{|V(H)|}$ and $n_0(\varepsilon_0) > n(\varepsilon,\ell)$. Let $C_0 \cup C_1 \cup \cdots \cup C_k$ be an ε-uniform partition of $G(n)$. Consider the graph G with vertex set $\{1, 2, \ldots, k\}$ and $\{i,j\}$ joined if (C_i, C_j) is an ε-uniform pair of density at least $\varepsilon_0/3$. We prove that this graph does not contain K_r as a subgraph. This follows from the following.

Claim 2.1. *If (C_i, C_j) is $\varepsilon = (\varepsilon_0/6)^v$ uniform for every $1 \le i < j \le r$ then the graph induced on $\bigcup_{i=1}^r C_i$ contains all complete r-partite graphs on v points. (In particular, G contains H, contradicting our hypothesis.)*

Proof of Claim 2.1. As each of the pairs (C_i, C_r) $1 \le i \le r-1$ is ε-uniform we can find $(1-(r-1)\varepsilon)|C_r|$ points in C_r which are joined to at least $(\varepsilon_0/3 - \varepsilon)|C_i|$ points of C_i for each $i = 1, 2, \ldots, r-1$. Take one such point $x_1 \in C_r$ and denote by C_i' the set of all vertices of C_i which are joined to x_1 $(i = 1, 2, \ldots, r-1)$. Set $C_r' = C_r - \{x_1\}$; we have $|C_i'| \ge (\varepsilon_0/3 - \varepsilon)|C_i| > (\varepsilon_0/6)|C_i|$ for every $i = 1, 2, \ldots, r$ and hence each of the pairs (C_i', C_j') $1 \le i < j \le r$ is $(\varepsilon_0/6)^{v-1}$ uniform. Now we take x_2 from one of the sets $C_1', C_2', \ldots, C_r'$ (say C_j') and repeat the argument to construct sets $C_1^{(2)}, \ldots, C_r^{(2)}$

of size at least $(\varepsilon_0/6)^{v-2}|C_i|$, $i = 1, 2, \ldots, r$ and with the property that x_2 is joined to every point of $\bigcup_{i \neq j}^r C_i^{(2)}$. Repeating this procedure $v - 1$ times (on i-th step using that $(\varepsilon_0/6)^{v-1}(r-1) < 1$ and $\varepsilon_0/3 - (\varepsilon_0/6)^{v-1} \geq \varepsilon_0/6$) we can construct a sequence of points $x_1, x_2, \ldots, x_v$ which span a graph isomorphic to any complete r partite graph on v points. □

Now we can finish the proof of Theorem 1.5. quite easily: The number of edges not contained in pairs with density at least $\varepsilon_0/3$ is clearly at most

$$k\binom{n/k}{2} + \varepsilon_0/3\binom{k}{2}\left(\frac{n}{k}\right)^2 + \varepsilon\binom{k}{2}\left(\frac{n}{k}\right)^2 + \varepsilon n^2 < \varepsilon_0 n^2.$$

After omission of these edges we get a graph which can be mapped on G by homomorphism and hence (according to Claim 2.1.) does not contain K_r. □

3. The Proof of Theorem 1.6.

Let $\chi(H) = r$. According to Theorem 1.5. every graph on n points $n > n_0(\varepsilon)$ not containing H can be written as a union of a K_r-free graph and $\varepsilon_0 n^2$ edges. Thus the number of such graphs is according to Theorem 1.3. (here we could use also the earlier, weaker result of [8]) smaller than $(1 + o(1))2^{T_n(K_r)}\binom{\binom{n}{2}}{\varepsilon_0 n^2}$. As ε_0 can be arbitrarily small we get (3). □

4. The Proof of the First Part of Theorem 1.7.

We prove (4) in the following form: For every $\varepsilon_1 > 0$ there exists $n_1 = n_1(\varepsilon_1)$ so that if $n > n_1$ and $G = (V, E)$ is an r-uniform hypergraph with $|V| = n$ and with the property that every set of $3r - 3$ vertices spans at most two r-tuples, then $|E| \leq \varepsilon_1 n^2$. First we show that the statement holds (with n_1 replaced by n_2) if G is connected. Consider the graph $\tilde{G} = (V, F)$ defined by

$$F = \{\{x, y\}, \exists z_1, z_2, \ldots, z_{r-2}\{x, y, z_1, z_2, \ldots, z_{r-2}\} \in E\}$$

As there is no triangle with all three edges in different r-tuples (this would yield $(3r-3, 3)$ a subgraph of 3 edges on $3r - 3$ points) we infer that

i) The set of r-tuples of G = the set of r cliques of $\tilde{G}$.

Moreover, as G is connected we get that

ii) Every two r-cliques of $\tilde{G}$ intersect in at most one point (Otherwise we get an $(\ell, 2)$, $\ell \leq 2r - 2$ the vertices of which cannot be contained in any other clique since this would immediately yield $(3r-3, 3)$).

Set $H = K_{1,\ldots,1,2}$ (a complete r-partite graph with $r + 1$ points) G does not contain H for otherwise we would get (by i)) two r-tuples intersecting in $r - 1$ points which contradicts to ii). If $n_2 > n_0(\varepsilon_1)$ we get (using Theorem 1.5.) that there are $\varepsilon_1 n^2$ edges which if omitted destroy all cliques of size r. Hence by i) and ii) $|E| \leq \varepsilon_1 n^2$.

Set now $n_1 = \dfrac{1}{\varepsilon_1} n_2$ and suppose that the sizes of the vertex set of the connected components of G are $m_1, m_2, \ldots, m_p$. Let $I \subset \{1, 2, \ldots, p\}$ be the set of those i for which $m_i \geq n_2$. Then we get

$$|E| \leq \sum_{i \in I} \varepsilon_1 m_i^2 + \sum_{i \notin I} m_i^2 \leq \varepsilon_1 n^2$$

□

5. The Proof of the Lower Bound in Theorem 1.7.

For the proof of (5) we need the following statement.

Lemma 5.1. *There exists a set of positive integers $A \subset \{1, 2, \ldots, n\}$ not containing three terms of any arithmetical progression of length r and such that $|A| \geq \frac{n}{e^{c \log r \sqrt{\log n}}}$ for some absolute constant $c > 0$.*

The proof is based on the method developed by Behrend [2]. For $d \geq 2, \ell \geq 1$ we may write any a, $1 \leq a \leq n$ to the base $2rd$

$$a = a_0 + a_1(2dr) + a_2(2dr)^2 + \cdots + a_k(2dr)^k$$

Set $N(\vec{a}) = \left(\sum_{i=0}^{k} a_i^2\right)^{1/2}$, where $\vec{a} = (a_0, a_1, \ldots, a_k)$. For $s \geq 1$ set

$$A = A_{n,d,s} = \{a, 1 \leq a \leq n, 0 \leq a_i \leq d \text{ for all } i, (N(\vec{a}))^2 = s\}$$

First we prove the following.

Claim 5.2. *The set A contains no three terms of any arithmetical progression of length r.*

Proof. Suppose that A contains three distinct positive integers $a = \sum a_i(2dr)^i$, $b = \sum b_i(2dr)^i$, $c = \sum c_i(2dr)^i$ such that $r_1(b - c) = r_2(c - a)$, where r_1, r_2 are positive integers smaller than r. Then $r_2 a + r_1 b - (r_1 + r_2)c = 0$. Since $a_i, b_i, c_i \leq d$ there is no carrying in $r_2 a_i + r_1 b_i$ or $(r_1 + r_2)c_i$ for $0 \leq i \leq k$ and hence

$$r_2 a_i + r_1 b_i - (r_1 + r_2)c_i = 0 \text{ for } 0 \leq i \leq k.$$

Then

$$0 < \frac{r_2}{r_1 + r_2}(a_i - c_i)^2 + \frac{r_1}{r_1 + r_2}(b_i - c_i)^2$$

$$= \frac{r_2}{r_1 + r_2} a_i^2 + \frac{r_1}{r_1 + r_2} b_i^2 - c_i^2$$

which yields that

$$s = (N(\vec{c}))^2 < \frac{r_2}{r_1 + r_2}(N(\vec{a}))^2 + \frac{r_1}{r_1 + r_2}(N(\vec{b}))^2 = s$$

a contradiction. □

Now we finish the proof of the Lemma. For a given r and d

$$k \sim \frac{\log n}{\log(2dr)} \text{ holds.}$$

[illegible] over all s contains all sums $\sum a_i(2dr)^i \le n$, $0 \le a_i \le d$. This is approximately $n(2r)^{-k}$ elements.

Consequently for some s

$$|A_{n,d,s}| \ge \frac{n}{d^2k(2r)^k}$$

Setting $d = e^{\sqrt{\log n}}(k \sim \sqrt{\log n})$ we infer

$$|A_{n,d,s}| \ge \frac{n}{e^{c \log r \sqrt{\log n}}}$$

for some $c > 0$. □

Now we prove (5). Take r-copies $X_0, X_1, \ldots, X_{r-1}$ of $X = \{1, 2, \ldots, m\}$, where $m = \lfloor n/r \rfloor$ and consider the set $\mathscr{P}$ of all r-tuples $\{x, x + a, \ldots, x + (r-1)a\}$, where $x + ia \in X_i$ for all $i = 0, 1, \ldots, r-1$. We have clearly $|P| \ge n^{2-\varepsilon}$ for every $\varepsilon > 0$ and $n \ge n_0(\varepsilon)$. Moreover, $|P \cap P'| \le 1$ for all distinct $P, P' \in \mathscr{P}$. Suppose that there are $P_1 = \{x, x + a, \ldots, x + (r-1)a\}$, $P_2 = \{y, y + b, \ldots, y + (r-1)b\}$ and $P_3 = \{z, z + c, \ldots, z + (r-1)c\} \in \mathscr{P}$ such that $|\bigcup_{i=1}^{3} P_i| \le 3r - 3$. Then there exist i, j, k (cf. Fig. 1) such that

$$x + ia = y + ib$$

$$z + jc = x + ja$$

$$y + kb = z + kc$$

We infer that

$$(i - j)a + (k - i)b = (k - j)c$$

which contradicts to the choice of the set A. □

6. Remarks and Open Problems

The first question which comes to mind is whether Theorem 1.5. can be generalized to hypergraphs let $K_t(l, r)$ denote the t-partite complete r-graph having vertex set $X_1 \cup \cdots \cup X_t$ with $|X_i| = l$ and F, $|F| = r$ being an edge if and only if $|F \cap X_i| \le 1$ for $i = 1, \ldots, l$. That is $K_t(l, r)$ is empty for $r > t$, $K_t(1, r)$ is just $K_t(r)$, the complete r-graph on t vertices.

Problem 6.1. *Suppose H is a $K_t(l, r)$-free r-uniform hypergraph on n vertices, $t > r$. Let ε be an arbitrarily small positive real $n > n_0(\varepsilon, r, t, l)$. Is it possible to remove εn^r edges from H so that the remaining hypergraph is $K_t(r)$-free?*

A positive answer would imply that the logarithmically asymptotic number of $K_t(l, r)$-free r-uniform hypergraphs is the same as the number of those without $K_t(r)$ for $t > r$, i.e., it would extend Theorem 1.6. This number should certainly be $2^{(1+o(1))T_n(K_t(r))}$. Let us mention, however, that the determination of $T_n(K_t(r))$ appears to be a very difficult problem – it is Turán's problem (cf. [4, 5, 13] for more information).

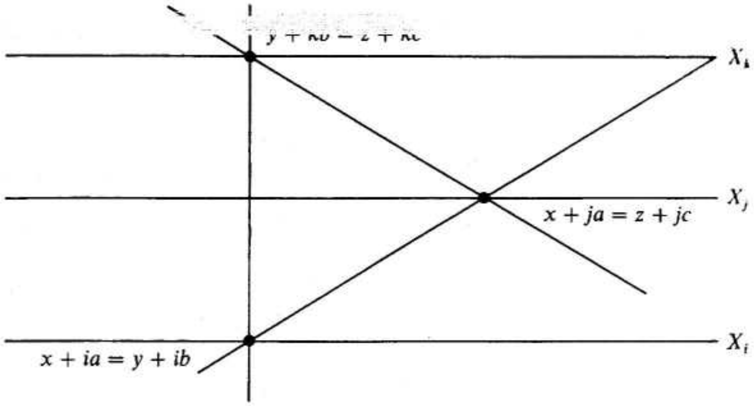

Fig. 1

Let c be a positive real and G a graph on n vertices and with at least cn^2 edges in which every edge is contained in a triangle. Szemerédi (unpublished) proved that for every integer l and $n > n_0(c, l)$ there is an edge in G which is contained in at least l triangles. This follows also easily from Theorem 1.5 choosing $r = 3$ and H the union of l triangles sharing an edge. On the other hand Alon [1] proved that the same statement does not hold for c sufficiently small and $l = \sqrt{n}$*.

The investigation of the function $g_n(v, e, r)$ goes back to Erdös [6]. Actually, the value of $g_n(3, 3, 2)$ was already determined – although in different notation – by Mantel [17] in 1907. The value is $\lfloor n^2/4 \rfloor$.

The exact and even asymptotic value of $g_n(4, 4, 2)$ is unknown. It is only known that $g_n(4, 4, 2) = \Theta(n^{3/2})$; note that $f(n) = \Theta(g(n))$ means that $c_1 < f(n)/g(n) < c_2$ holds for positive absolute constants c_1, c_2 and for n sufficiently large (cf. [10] for more problems and results concerning the $r = 2$ case).

The general problem was first considered by Brown, Erdös and Vera Sós [3].

Very little is known for $r \geq 3$. Obviously, $g_n\left(v, \binom{v}{r}, r\right) = T_n(K_v(r))$ holds, i.e., the complete determination of $g_n(v, e, r)$ would include solving Turán's problem.

Even the determination of $g_n(r + 1, 2, r)$ is difficult. It is the maximum number of r-element subsets of an n-set no two sharing $r - 1$ points. This yields the upper bound $g_n(r + 1, 2, r) \leq \binom{n}{r-1}\Big/ r$, with equality iff there exists a $S(n, r, r - 1)$ Steiner-system. Note that it is well-known that $g_n(r + 1, 2, r) \geq (1 - o(1))\binom{n}{r-1}\Big/ r$ – cf. [18] for a general asymptotic bound.

For $v = r + 1$, $e = 3$, $r \geq 3$ not even asymptotic bounds are known. It was

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* The problem of estimating $f(n, c)$ – the maximal number of triangles which must share an edge in any graph G with above properties was proposed by P. Erdös and B.L. Rothschild.

shown by Giraud [14] and by Frankl and Füredi [illegible] that

$$g_n(4,3,3) \geq \left(\frac{2}{7} - o(1)\right)\binom{n}{3}.$$

On the other hand de Caen [5] proved $g_n(4,3,3) \leq \left(\frac{1}{3} + o(1)\right)\binom{n}{3}$.

Theorem 1.7 shows that $g_n(3r-3,3,r) \neq \Theta(n^c)$ for any c. The same might hold for $g_n(\ell(r-2)+3,\ell,r)$, $\ell, r \geq 3$, in general.

Problem 6.2. *Is it true in general that for all $\ell, r \geq 3$ and $\varepsilon > 0$ $n^{2-\varepsilon} \leq g_n(\ell(r-2)+3,\ell,3) = o(n^2)$ holds for $n > n_0(\varepsilon,\ell,r)$?*

By a construction of Ruzsa [21] $g_n(7,4,3) > n^{2-\varepsilon}$ holds for all $\varepsilon > 0$, $n > n_0(\varepsilon)$. However, to prove $g_n(7,4,3) = o(n^2)$ appears to be difficult.

The proof of Theorem 1.7. implies that if a 4-uniform hypergraph on n vertices has more than εn^2 edges, $n > n_0(\varepsilon)$ then it either contains an $(11,4)$ or a $(16,6)$.

An apparently easier case is the following.

Proposition 6.3. $g_n(2+(r-2)e,e,r) = \Theta(n^2)$

Sketch of proof. The upper bound follows by noting that through given two vertices there are at most $e-1$ edges. The lower bound can be proved both by direct construction or by a random choice of cn^2 subsets of size r and then omitting all edges from every $(2+(r-2)e)$-element set containing at least e of them. □

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Remark added in proof. Problem 6.1 has been recently positively answered by P. Frankl and V. Rödl. The proof uses an extension of Szemerédi's regularity lemma to hypergraphs.