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EXTREMAL CLIQUE COVERINGS OF COMPLEMENTARY GRAPHS

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Let $cc(G)$ (*resp.* $cp(G)$) be the least number of complete subgraphs needed to cover (*resp. partition*) the edges of a graph G. We present bounds on $\max\{cc(G)+cc(\bar{G})\}$, $\max\{cp(G)+cp(\bar{G})\}$, $\max\{cc(G)cc(\bar{G})\}$ and $\max\{cp(G)cp(\bar{G})\}$ where the maxima are taken over all graphs G on n vertices and $\bar{G}$ is the complement of G in K_n. Several related open problems are also given.

Introduction

Let G be a graph on n vertices and let $\bar{G}$ be its complement in K_n, the complete graph on n vertices. If f is a real valued function defined on graphs, what are the extreme values of $f(G)+f(\bar{G})$ and $f(G)\,f(\bar{G})$? E. A. Nordhaus and J. W. Gaddum (see e.g. [5]) considered those questions when the function is the chromatic number. D. Taylor, R. D. Dutton and R. C. Brigham [5] studied the questions for several other functions. One of those is the *clique covering number*. That is $cc(G)$, the least number of complete subgraphs (*cliques*) of G necessary to cover the edge set of G. We continue their investigation. We also consider the questions for another function the *clique partition number*. That is $cp(G)$, the least number of cliques needed to partition the edge set of G.

In Theorem 1, we establish the right inequality of $\lfloor n^2/4\rfloor+2\leq\max\{cc(G)+cc(\bar{G})\}\leq(n^2/4)(1+o(1))$ where the maximum is taken over all graphs G on n vertices. The bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ assumes the lower bound.

In Theorem 2 we modify the proof of Theorem 1 to show that $\max\{cc(G)cc(\bar{G})\}\leq(n^4/256)(1+o(1))$, where the maximum is taken over all n-vertex graphs G. D. Taylor et al. [5, Theorem 5] gave an example of a graph F for which $cc(F)cc(\bar{F})=n^2(n+8)^2/256$. The graph F is obtained from two copies A_1 and A_2 of $K_{n/4}$ and two copies A_2 and A_3 of $\bar{K}_{n/4}$ by joining each vertex of A_i to each vertex of A_{i+1} ($i=1, 2$ and 3). When n is not divisible by 4 the construction can be modified to yield a similar graph. Hence Theorem 2 establishes the conjecture made in [5], that $\max\{cc(G)cc(\bar{G})\}\sim n^4/256$ where the maximum is taken over all n-vertex graphs G.

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Somewhat weaker results for the clique partition number are obtained in Theorems 3, 4 and 5. They imply

$$\frac{7}{25}n^2+O(n) \leqq \max\{cp(G)+cp(\bar{G})\} \leqq \frac{13}{30}n^2+O(n) \quad \text{and}$$

$$\frac{29}{2000}n^4+O(n^3) \leqq \max\{cp(G)cp(\bar{G})\} \leqq \frac{169}{3600}n^4+O(n^3)$$

where the maxima are taken over all n-vertex graphs G.

We state several related open problems at the end of the paper.

Results

Theorem 1. *For some $d>0$ and all graphs G on n vertices, $cc(G)+cc(\bar{G})<$ $<(n^2/4)(1+d/\log n)$.*

Proof. Suppose $4^c \leqq n/4c^3$. From a sequence $\mathcal{S}=\{K^1, K^2, \ldots, K^l\}$ of cliques K^j in K_n by choosing K^i to be a clique in G or in $\bar{G}$ which covers at least c edges uncovered by $K^1 \cup K^2 \cup \ldots \cup K^{i-1}$. The process halts when such a selection is no longer possible. Now $l \leqq n^2/c$. If a vertex has fewer than n/c incident edges in G or in $\bar{G}$, augment $\mathcal{S}$ by adding these edges separately, and continue repeating this step until there are no such vertices remaining. At most $2n^2/c$ new cliques have been added to $\mathcal{S}$. Let H_1 (or H_2) denote the subgraph of K_n induced by the set of edges of G (respectively $\bar{G}$) not contained in the union of the cliques in $\mathcal{S}$, and put $H=H_1 \cup H_2$. Let T denote the set of vertices of H with degree at least n/c in both H_1 and H_2, and let U and V denote the sets of vertices in $K_n - T$ with degree at least n/c in H_1 and H_2 respectively. Note that vertices in U and V have degree 0 in H_2 and H_1 respectively.

In [2] it is shown that $cc(D) \leqq k^2/4$ for all k-vertex graphs D. Therefore the edges of H with both ends in U or both ends in V can be covered by at most $|U|^2/4$ or $|V|^2/4$ cliques respectively. We further augment $\mathcal{S}$ by these cliques, which adds at most $n^2/4$ cliques to $\mathcal{S}$.

We next show that $|T| \leqq n/c$. Assume $|T|>n/c$. Then at least $n^2/2c^2$ edges of H_1 have at least one end in T. It follows that some set E of at least $n/2c^2$ such edges are all incident with some vertex p. Let $T'=\{v \in T: pv \in E\}$. Then $|T'| \geqq n/2c^2$, so at least $n^2/4c^3$ edges of H_2 have at least one end in T'. Then a set F of $n/4c^3$ or more such edges are all incident with some vertex q. Let $T''=\{v \in T': qv \in F\}$. Then $|T''| \geqq n/4c^3$. By the bound for Ramsey's Theorem given for example in [1, Theorem 7.5], G of $\bar{G}$ contains a clique K with c vertices in T''. Therefore the clique spanned by K and p (or K and q) covers c edges of H_1 (respectively H_2) incident with p (respectively q). But this contradicts the definition of $\mathcal{S}$. Thus $|T| \leqq n/c$ as claimed. Hence we can further augment the cliques in $\mathcal{S}$ by adding all edges of H incident with vertices in T as separate cliques. There are at most n^2/c such edges.

The cliques in $\mathcal{S}$ now form a clique covering of G and a clique covering of $\bar{G}$, and $|\mathcal{S}| \leqq n^2/4+4n^2/c$. For large n we can take $3c>\log n$, which gives the theorem. ∎

Theorem 2. *For some $d>0$ and all graphs G on n vertices, $cc(G)\cdot cc(\bar{G})<$ $<n^4(1+d/\log n)/256$.*

Proof. In the proof of Theorem 1, we obtained a clique covering of G using at most $4n^2/c+|U|^2/4$ cliques, and a clique covering of $\bar{G}$ using at most $4n^2/c+|V|^2/4$ cliques, where $4^c\leqq n/4c^3$ and $|U|+|V|\leqq n$. Hence $cc(G)cc(\bar{G})\leqq(4n^2/c+a^2/4)(4n^2/c+b^2/4)$ where each of these factors is at most $n^2/2$, and $a+b\leqq n$. This product is at most $4n^4/c+a^2b^2/16$, which is maximised when $a=b=n/2$. Hence $cc(G)cc(\bar{G})\leqq$ $\leqq 4n^4/c+n^4/256$. Taking $3c>\log n$ as in Theorem 1, we obtain the result. ∎

Corollary. *For each graph G on n vertices* $\min(cc(G), cc(\bar{G}))\leqq n^2/16(1+o(1))$. ∎

If G_1 and G_2 are vertex-disjoint graphs, then $G_1\vee G_2$ is the graph formed from the union of G_1 with G_2 by adding edges joining each vertex of G_1 to each vertex of G_2.

Lemma 1. [3, Theorem 3]. *Let $G=A\vee\bar{K}_q$. If A has p vertices and e edges, and the edge-chromatic number $\chi'(A)$ of A is at most q, then $cp(H)=pq-e$.* ∎

We note that $\chi'(K_m)=m$ or $m-1$ according as m is odd or even. Therefore for all $m\geqq 1$,

$$(1)\quad cp(K_m\vee\bar{K}_m)=m^2-\binom{m}{2}$$

and

$$(2)\quad cp(K_{m+r}\vee\bar{K}_{2m})=2m(m+r)-\binom{m+r}{2}\quad\text{when}\quad 0\leqq r\leqq m.$$

Let A and B be replicas of K_m and let H_m be the graph diagrammed in Figure 1. There, as in all figures below, a double line joining two graphs G_1 and G_2 indicates that every vertex in G_1 is adjacent to every vertex in G_2.

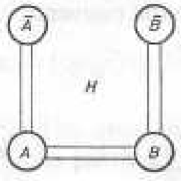

Fig. 1

Lemma 2. *For all $m\geqq 1$, $cp(H_m)\geqq\dfrac{7}{4}m^2+m$.*

Proof. Let $\mathscr{C}$ be a clique partition of $H=H_m$ of least cardinality (so that $|\mathscr{C}|=$ $=cp(H)$). Denote the subfamily $\{K^1, K^2, \ldots, K^\sigma\}$, consisting of those cliques in $\mathscr{C}$ with vertices in both graphs A and B, by $\mathscr{S}$. From subgraphs A' and B' of A and B by deleting the edges of all cliques in $\mathscr{S}$ from A and B respectively. Let d_i and e_i be the number of vertices of K^i in A and B respectively. Denote the clique partitions of $\bar{A}\vee A'$ and $\bar{B}\vee B'$ induced by $\mathscr{C}-\mathscr{S}$ by $\mathscr{C}_A$ and $\mathscr{C}_B$ respectively. Thus $cp(H)=$ $=|\mathscr{C}_A|+|\mathscr{C}_B|+\sigma$. But

$$|\mathscr{C}_A|\geqq cp(\bar{A}\vee A')=m^2-\binom{m}{2}+\sum_{i=1}^{\sigma}\binom{d_i}{2}$$

by Lemma 1. Similar statements for B imply that

$$cp(H) \geqq m^2+m+\sigma+\sum_{i=1}^{\sigma}\binom{d_i}{2}+\binom{e_i}{2}.$$

Differentiation shows that the minimum of the quantities

$$\frac{\binom{d}{2}+\binom{e}{2}+1}{de},$$

where d, e are positive integers and $de \geqq 1$, is 3/4. This minimum is achieved at $d=e=2$. Now every edge with one vertex in A and the other in B must be covered by some member of $\mathcal{S}$. Also K^i in $\mathcal{S}$ covers exactly $d_i e_i$ edges joining A to B. Thus

$$\sum_{i=1}^{\sigma} d_i e_i = m^2$$

and hence $cp(H) \geqq 7/4m^2+m$. ∎

Theorem 3. *Let r be the remainder when n is divided by 5. For each $n \geqq 20$*

$$\max\{cp(G)+cp(\bar{G})\} \geqq \frac{7n^2}{25}+\frac{(25+2r)n-41r^2}{50},$$

where the maximum is over all graphs G on n vertices.

Proof. Let L be a replica of K_{m+r} and let K be a replica of K_m. Define G_n to be the graph whose diagram is given in Figure 2 (a). The diagram of $\bar{G}_n$ is given in Figure 2 (b). (We use the same diagrammatic convention here as for Figure 1.)

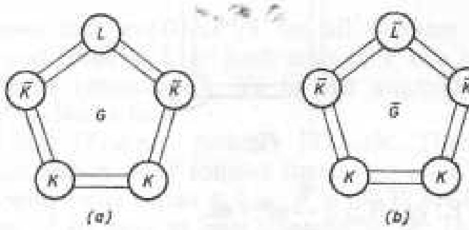

Fig. 2

The graph $G=G_n$ is the edge-disjoint union of $H \equiv H_m$ and $H' \equiv K_{m+r} \vee \bar{K}_{2m}$. Since every clique in G has all its edges in H or all its edges in H', we have $cp(G)= =cp(H)+cp(H')$. Similarly $cp(\bar{G})=cp(H)+2cp(\bar{K} \vee \bar{L})$. Since $n \geqq 20$, $m \geqq 4$ and so equations (1) and (2) imply

$$\text{(3)} \qquad \frac{7n^2}{25}+\frac{25n+2nr-41r^2}{50} \leqq cp(G_n)+cp(\bar{G}_n). \quad ∎$$

When does equality hold in (3)? It is a direct consequence of the following Lemma that equality holds infinitely often.

Lemma 3. [4, proof of Theorem 4, pp. 346, 347]. *Let $K(q, k)$ be the complete k-partite graph defined by k vertex-disjoint replicas of $\overline{K}_q$. Then the edge set of $K(q, k)$ can be partitioned into cliques of order k if there exist $k-2$ mutually orthogonal Latin squares on q symbols.* ∎

With $k=4$, Lemma 3 implies that the edges joining A to B in the graph H_m of Lemma 2 can be covered using edge-disjoint replicas of K_4 for even $m>12$. Therefore when $n>64$ and $(n-r)/5$ is even, equality holds in (3).

Theorem 4. *For each graph G on n vertices, $cp(G)+cp(\overline{G}) \leqq 13n^2/30-n/6$.*

Proof. Let us construct a clique partition of K_n into triangles and edges, each of which is in G or $\overline{G}$. First select as many edge-disjoint triangles as possible. Then the set of s edges uncovered by any of these t triangles cannot contain the edge set of a copy of K_6, for otherwise G or $\overline{G}$ would contain a triangle by an instance of Ramsey's theorem. Therefore, by Turán's theorem (see e.g. [1, Theorem 7.9]), $s \leqq 2n^2/5$. Since $3t+s=\binom{n}{2}$, it follows that the partition has at most $13n^2/30-n/6$ members. ∎

The coefficient of n^2 appearing in the right side of the inequality of Theorem 4 can be reduced by 1/204 by using K_4's as well as K_3's and K_2's in the clique partition, and bounds on higher Ramsey numbers lead to further improvements. However, this approach cannot lead to an exact determination of $\max\{cp(G)+cp(\overline{G})\}$. The bound in Theorem 3 is probably nearer to the actual value.

Theorem 5. *Taking the maximum over all graphs on n vertices,*

$$\frac{39}{2000}n^4+O(n^3) < \max\{cp(G)\,cp(\overline{G})\} < \frac{169}{3600}n^4+O(n^3).$$

Proof. The left inequality is obtained by using the graph G_n of Theorem 3. The right inequality is obtained from the clique partition of K_n constructed in the proof of Theorem 4. It has x of its cliques in G and $\left(\frac{13}{30}n^2-\frac{n}{6}-x\right)$ cliques in $\overline{G}$. ∎

Concluding remarks

L. Pyber proved that the lower bound in Theorem 1 is sharp for n large. Possibly Theorem 3 is close to best possible; that is, $\max\{cp(G)+cp(\overline{G})\} \sim 7n^2/25$ where the maximum is taken over all n-vertex graphs G. Suppose $G_1 \cup G_2 \cup G_3 = K_n$ where the G_i are edge-disjoint. If R is the graph diagrammed in Figure 3 with $A= =\overline{K}_{n/5}$, then we can have $G_1 \cong G_2 \cong R$ and so $cp(G_1)+cp(G_2)=2n^2/5$. (We use the same diagrammatic convention here as in Figure 1.) Probably this is the maximum possible value of $cp(G_1)+cp(G_2)$. The estimate $cc(G_1)+cc(G_2)+cc(G_3)= =2n^2/5\,(1+o(1))$ was proved by L. Pyber (see pp. 393—398 of this issue). Perhaps

$\max\{cc(G_1)+cc(G_2)+cc(G_3)\}=2n^2/5+5$, taking the maximum over all n-vertex graphs.

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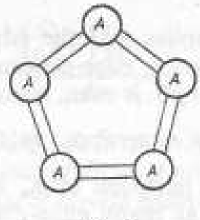

Fig. 3

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