THE DIFFERENCE BETWEEN THE CLIQUE NUMBERS OF A GRAPH

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Abstract

Let G be a simple graph. Its clique covering (partition) number cc(G) (cp(G)) is the least number of complete subgraphs needed to cover (partition) its edge-set. We study the function $\sigma(G) \equiv \mathrm{cp}(G) - \mathrm{cc}(G)$ of graphs G.

1. Introduction and Summary

Let G be a simple graph on $n \geqq 1$ vertices. The *clique partition* [*covering*] *number* cp(G) [cc(G)] is the least number of cliques (complete subgraphs of G) needed to partition [cover] the edge-set of G. Evidently

$$(1.1) \qquad \mathrm{cc}(G) \leqq \mathrm{cp}(G) \quad .$$

In a personal communication in 1982, P. Erdös asked how large the difference cp(G) - cc(G) can be as a function of n. Let us call this difference the *spread* of G and denote it by $\sigma(G)$. In [3, Theorem 4] Erdös, Goodman and Pósa proved that the edge-set of G can be partitioned into $\lfloor n^2/4 \rfloor$ or fewer edges and triangles. Thus for $n \geqq 3$

$$(1.2) \qquad \sigma(G) \leqq \lfloor n^2/4 \rfloor - 2 \quad .$$

We obtain a lower bound on σ_n, the maximum spread of all graphs on n vertices $\frac{n^2}{4} - c\, n^{3/2}$, for some constant c. In this paper, we make their bound a little more precise by showing that

$$\sigma_n \geqq \frac{n^2}{4} - \frac{1}{2} n^{3/2} + \frac{n}{4} \quad .$$

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Thus for all $n \geq 3$

$$(1.3) \qquad \frac{n^2}{4} - \frac{1}{2} n^{3/2} + \frac{n}{4} \leq \sigma_n \leq \lfloor n^2/4 \rfloor - 2$$

Let

$$\beta_n = \frac{n^2 - 2n^{3/2} + n}{4} \quad .$$

In Theorem 2, we construct a graph $G(n, j)$ with spread j for each $n \geq 1$ and each integer j between 0 and λ_n, where λ_n agrees with β_n when n is a perfect square. In an earlier preliminary report [2] we showed that for fixed $n \geq 1$, each integer value in the closed interval $[0, (n-1)(n-2)/6]$ is the spread of a connected graph on n vertices.

Let us call a clique partition P of the edge-set of G *minimum* if $|P| = cp(G)$. The graphs that we construct in Theorem 2 have this curious property: each has a minimum clique partition consisting entirely of edges and triangles.

2. Preliminaries

We denote that vertex set of a graph G by $V(G)$ and the edge set of G by $E(G)$. For vertex-disjoint graphs G and H, we use the notation $G \vee H$ to denote the graph whose vertex set is the union of the vertex sets of G and H, such that e is an edge of $G \vee H$ if and only if (i) e is an edge of G or of H, or (ii) one end of e is in G while the other end is in H. As is customary, K_n denotes the complete graph on n vertices and $\overline{K_n}$ denotes the edge-free graph on n vertices. The edge chromatic number of G is denoted by $\chi'(G)$. In particular (see e.g.[1], p. 96):

$$(1.4) \qquad \chi'(K_{2k}) = \chi'(K_{2k-1}) = 2k - 1 \text{ for all } k \geq 1.$$

THEOREM 1. [4, Corollary 1, p210]. Suppose G is a graph on n vertices and e edges having an independent set Z of q vertices, and that H is the subgraph on p vertices and m edges obtained by deleting Z and all its adjacent edges from G.
If at least $\chi'(H)$ vertices of Z are adjacent to every vertex of H,

then $cp(G) = e-2m$ and any clique partition of G using cliques of order exceeding 3 is not minimal. □

LEMMA 1 [4, Inequality(5), p211].
For all $q \geq 1$

$$q \leq cc(H \vee \overline{K}_q) \leq q\, cc(H). \qquad \square$$

The *union* $G \cup H$ of graphs G and H is the graph whose vertex set is $V(G) \cup V(H)$, the union of the vertex sets of G and H and whose edge-set is $E(G) \cup E(H)$, the union of the edge-sets of G and H. When G and H are disjoint, we denote their union by $G + H$. We write pH for the graph consisting of p copies of H, i.e. $pH = \sum_{i=1}^{p} H$. The *intersection* $G \cap H$ of graphs G and H is the graph whose vertex set is $V(G) \cap V(H)$ and whose edge-set is $E(G) \cap E(H)$.

LEMMA 2. If $G \cap H$ has no edges, then

$$\sigma(G \cup H) = \sigma(G) + \sigma(H) \qquad \square$$

3. Main Results

Let S_n denote the set of integers j such that $\sigma(G) = j$ for some connected graph G on n vertices.

LEMMA 3. For all $n \geq 1$ $S_n \subseteq S_{n+1}$

Proof. If $\sigma(H) = j$ and the connected graph H has k vertices, then $j \in S_n$ for all $n \geq k$. This is true because we can augment H by a path of length $n - k$ sharing exactly one vertex with H.
Hence $S_n \subseteq S_{n+1}$ for all $n \geq 1$. □

For $n \geq 1$ we define $p_n = \lceil \sqrt{n} \rceil$, $q_n = \lfloor \sqrt{n}/2 \rfloor$ and

$$(3.1) \qquad \lambda_n = q_n(p_n - 1)(n - p_n(q_n + \tfrac{1}{2})).$$

Note that $\lambda_n = 0$ for $n \leq 3$.
The main objective of this section is to establish the following theorem.

Theorem 2. For each $n \geq 1$ and every integer m in the interval $[0, \lambda_n]$ there is a connected graph on n vertices with spread m having a minimum clique partition consisting solely of edges and triangles. □

Our strategy will be to exhibit for each n, a family of connected graphs G(n) such that each integer m in the closed interval $[\lambda_{n-1}, \lambda_n]$ is the spread of some member of the family G(n). Each of our graphs will have a minimum clique partition consisting solely of edges and triangles. Theorem 2 will then follow because of Lemma 3.

To begin with we note that $\lambda_n = 0$ for $n \leq 3$ and $\sigma(K_n) = 0$ for $n \leq 3$. In describing our graphs for $n \geq 4$ its convienent to suppress the subscripts on p and q. That is, we write p for p_n and q for q_n. Let

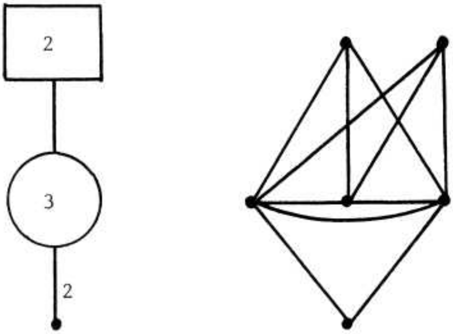
$$G_n = (qK_p) \vee \overline{K}_{n-pq}$$

Then for $n \neq 5$

$$\begin{aligned}\sigma(G_n) &= q\,\sigma(K_p \vee \overline{K}_{n-pq}) && \text{(by Lemma 2)}\\ &= q\,(p-1)(n - p(q + \tfrac{1}{2})) && \text{(by Theorem 1 and Lemma 1)}\\ &= \lambda_n\end{aligned}$$

Note that $\sigma(G_5) = 2$ and $\lambda_5 = 1$.

In the following diagrams a circled k denotes a K_k, a rectangle enclosing m indicates a $\overline{K}_m$. Further, a line with no label joining two graphs indicates that every possible edge between the graphs is present (i.e. every vertex of one graph is joined to all vertices of the other graph), whilst a line labelled i joining a vertex to a complete graph indicates that i edges join the vertex to the graph. Thus the diagram of Figure 1(a) represents the graph of Figure 1(b).



(a) (b)

Figure 1.

Let Q(n, j) denote the graph exhibited in the diagram of Figure 2, where $j = sp + r$ $(0 \leq r \leq p - 1)$. Let Q'(n, j) be the

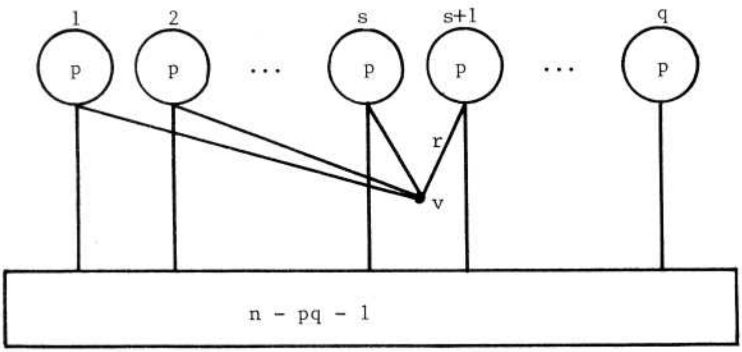


Figure 2. Q(n, j)

graph obtained from Q(n, j) by replacing $q(=q_n)$ by $q' \equiv q_{n-1}$.

Define

$$I(n) = \{\sigma(Q(n, j)): \ 1 \leq j \leq pq\}$$

and

$$I'(n) = \{\sigma(Q'(n, j)): \ 1 \leq j \leq pq'\} .$$

If $p_n = p_{n-1}$ and $q_n = q_{n-1}$, then Q(n, 1) differs from G_{n-1}

by an edge with one end (v) of degree 1. Hence

$$\sigma(Q(n, 1)) = \sigma(G_{n-1}) = \lambda_{n-1}.$$

Also, $Q(n, p_n q_n) = G_n$ and so $\sigma(Q(n, p_n q_n)) = \lambda_n$.

Moreover, if $1 \leqq j < pq$, then

$$(3.2) \qquad \sigma(Q(n, j+1)) = \varepsilon_j + \sigma(Q(n, j))$$

where ε_j is 0 or 1 according to whether or not $j \equiv 0 \pmod p$ (i.e. $r=0$). Thus

$$(3.3) \qquad I(n) = [\lambda_{n-1}, \lambda_{n-1} + 1, \lambda_{n-1} + 2, \ldots, \lambda_n]$$

when $(p_n, q_n) = (p_{n-1}, q_{n-1})$. We denote the set in brackets on the right hand side of (3.3) by $[\lambda_{n-1}, \lambda_n]$.

If $(2k)^2 < n \leqq (2k+1)^2$, then $I(n) = [\lambda_{n-1}, \lambda_n]$, since $\lambda_{n-1} = \lambda_n$ for $n = (2k)^2 + 1$ and for $(2k)^2 + 1 < n \leqq (2k+1)^2$, $(p_n, q_n) = (p_{n-1}, q_{n-1})$.

If $(2k+1)^2 + 1 < n < (2k+2)^2$, then $I(n) = [\lambda_{n-1}, \lambda_n]$ since $(p_n, q_n) = (p_{n-1}, q_{n-1})$. For $n = (2k+1)^2 + 1$, $\sigma(Q(n, 1)) = \lambda_{n-1}$, $\sigma(Q(n, pq)) = \lambda_n$ and, for $1 \leqq j < pq$, $\sigma(Q(n, j+1))$ satisfies (3.2). Hence $I(n) = [\lambda_{n-1}, \lambda_n]$ for $(2k)^2 < n < (2k+2)^2$.

When $n = (2k)^2$, $p_n = 2k$, $q_n = k$, $p_{n-1} = 2k$ and $q_{n-1} = k - 1$. By an argument similar to that used in establishing (3.3) we get

$$I'(n) = [\lambda_{n-1}, \lambda_{n-1} + 1, \lambda_{n-1} + 2, \ldots, x]$$

where

$$x = \lambda_{n-1} + q'(p-1) = \lambda_{n-1} + (k-1)(2k-1).$$

From (3.1) we get

$$\lambda_{n-1} = (k-1)(2k-1)(4k^2 - 1 - 2k(k - 1 + \tfrac{1}{2}))$$

$$= (k-1)(k+1)(2k-1)^2$$

and

$$\lambda_n = k(2k - 1)(4k^2 - 2k(k + \tfrac{1}{2}))$$

$$= k^2(2k - 1)^2$$

Note that $\lambda_n = \lambda_{n-1} + (2k - 1)^2 > x$. □

Now

$$\sigma(Q(n, 1)) = \sigma((k\ K_{2k}) \vee \overline{K}_{2k^2 - 1})$$

$$= k\ \sigma(K_{2k} \vee \overline{K}_{2k^2 - 1}) \qquad \text{(by Lemma 2)}$$

$$= k[2k(2k^2 - 1) - k(2k - 1) - (2k^2 - 1)] \qquad \text{(by Theorem 1 and Lemma 1)}$$

$$= x.$$

Also, $\sigma(Q(n, pq)) = \sigma(G_n) = \lambda_n$ and, for $1 \leq j < pq$, $\sigma(Q(n, j + 1))$ satisfies (3.2).

Hence

$$I(n) = [x, x + 1, x + 2, \ldots, \lambda_n]$$

Consequently, for $n = (2k)^2$

$$I'(n) \cup I(n) = [\lambda_{n-1}, \lambda_n].$$

Thus we have constructed the required graphs for each integer i with $(2k)^2 \leq i < (2k + 2)^2$ for every $k \geq 1$. For $n \leq 3$ we have already noted that $\sigma(K_n) = \lambda_n = 0$. Hence Theorem 2 follows from Lemma 3.

Remark When n is a perfect square

$$\lambda_n = \frac{n^2 - 2n^{3/2} + n}{4} = \beta_n.$$

4. Some Special Constructions

For some n we can construct graphs having a spread greater than λ_n. Table 1 below exhibits such graphs for some small n.

n	λ_n	Graph $G^*(n)$	$\upsilon_n = \sigma(G^*(n))$
13	21	$(2K_3) \vee \overline{K}_7$ or $K_5 \vee \overline{K}_8$	22
14	24	$(K_3 + K_4) \vee \overline{K}_7$ or $K_5 \vee \overline{K}_9$	26
15	27	$(K_3 + K_4) \vee \overline{K}_8$	31
17	36	$(2K_4) \vee \overline{K}_9$	42
18	44	$(2K_4) \vee \overline{K}_{10}$	48
19	52	$(2K_4) \vee \overline{K}_{11}$ or $(K_4 + K_5) \vee \overline{K}_{10}$	54
20	60	$(K_4 + K_5) \vee \overline{K}_{11}$	61
30	150	$(K_6 + K_7) \vee \overline{K}_{17}$	151
34	190	$(K_6 + K_5 + K_5) \vee \overline{K}_{18}$	199
35	200	$(K_6 + K_6 + K_5) \vee \overline{K}_{18}$	212
37	225	$(3K_6) \vee \overline{K}_{19}$	240
38	243	$(3K_6) \vee \overline{K}_{20}$	255

TABLE 1.

In the previous section we observed that $\lambda_n = \lambda_{n-1}$ whenever $n = (2k)^2 + 1$. It is reasonable to expect that one could do better than λ_n in this case. The graph

$$G^*(4k^2 + 1) \equiv (k\ K_{2k}) \vee \overline{K}_{2k^2+1}$$

has, for every $k \geq 1$, spread

$$\begin{aligned} \upsilon_{4k^2+1} &= \sigma(G^*(4k^2+1)) \\ &= k\,\sigma(K_{2k} \vee \overline{K}_{2k^2+1}) \qquad \text{(by Lemma 2)} \\ &= 4k^4 - 4k^3 + 3k^2 - k \qquad \text{(by Theorem 1 and Lemma 1).} \\ &= \lambda_{4k^2+1} + 2k^2 - k. \end{aligned}$$

Moreover, the graph $Q^*(4k^2+1, j)$ exhibited in the diagram of Figure 3, where $j = s(2k) + r$ $(0 \leq r \leq 2k-1)$ provides graphs realizing the spreads

$$[\lambda_{4k^2+1} + 1, \lambda_{4k^2+1} + 2, \lambda_{4k^2+1} + 3, \ldots, \upsilon_{4k^2+1}].$$

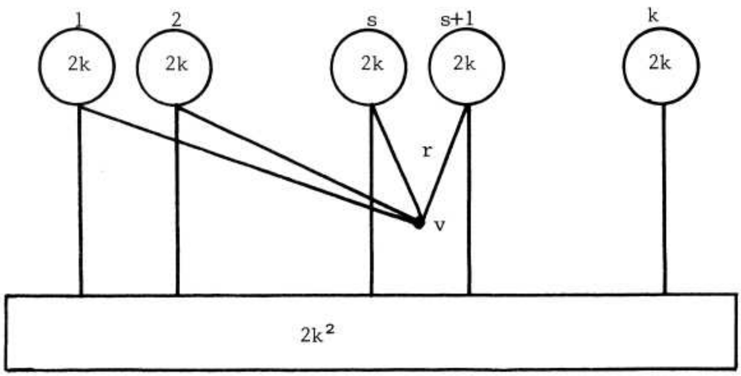


Figure 3. $Q^*(4k^2+1, j)$

We note that $\upsilon_{4k^2+1} \geq \beta_{4k^2+1}$ (In fact, $\upsilon_{4k^2+1} \cong \beta_{4k^2+1}$).

For values of n in the vicinity of $4k^2+1$ we can, by augmenting the graph $G^*(4k^2+1)$, obtain graphs with spread greater than λ_n. For example, the graphs $G^*(34)$, $G^*(35)$ and $G^*(38)$ given in Table 1 are obtainable by augmenting $G^*(37)$. Whilst improvements are possible for some n, we have not been able to improve on λ_n when n is a perfect square.

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